Pattern-Book Functions

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Table of Pattern-Book Functions

Name	**Traditional notation**	**R notation**
exponential	e^x	`exp(x)`
logarithm ("natural log")	$\ln(x)$	`log(x)`
sinusoid	$\sin(x)$	`sin(x)`
square	x^2	`x^2`
identity	x	`x`
one	1	`1`
reciprocal	$1/x$ or x^{-1}	`1/x`
gaussian	e^{-x^2}	`exp(-x^2)`
sigmoid	$\frac{1}{1+e^{-x}}$	`1/(1 + exp(-x))`

Table 1: The pattern-book functions used in *MOSAIC Calculus*.

Definitions

Before examining the pattern-book functions, we need definitions for the properties we'll use to describe them.

Domain: The set of all possible input values x for which the function $f(x)$ is defined.

Range: The set of all possible output values $f(x)$ that the function can produce.

Increasing function: A function f is *increasing* if whenever $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

Decreasing function: A function f is *decreasing* if whenever $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

Even function: A function f is *even* if $f(-x) = f(x)$ for all x in the domain. Even functions are symmetric about the y-axis.

Odd function: A function f is *odd* if $f(-x) = -f(x)$ for all x in the domain. Odd functions are symmetric about the origin.

Periodic function: A function f is *periodic* with period P if $f(x + P) = f(x)$ for all x in the domain. The function repeats itself every P units.

Vertical asymptote: A function has a *vertical asymptote* at $x = a$ if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$ (or both).

Horizontal asymptote: A function has a *horizontal asymptote* at $y = L$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ (or both).

1. Exponential Function **Definition:** $f(x) = e^x$ (natural exponential, base e), where $e \approx 2.71828$ is Euler's number.

In R notation: `exp(x)`

Properties: Domain: $\mathbb{R}$, Range: $(0, \infty)$, Always increasing, $\lim_{x \to -\infty} e^x = 0$, $\lim_{x \to \infty} e^x = \infty$

Range $(0, \infty)$**:**

Since $e > 0$, we have $e^x > 0$ for all x.

For example: $e^{-2} \approx 0.135$, $e^0 = 1$, $e^2 \approx 7.39$, $e^5 \approx 148$.

For any $y > 0$, solving $e^x = y$ gives $x = \ln(y)$.

Therefore every positive number is in the range.

Increasing:

Since $e > 1$, if $x_1 < x_2$ then $e^{x_1} < e^{x_2}$.

For example: $1 < 2$ gives $e^1 \approx 2.72 < e^2 \approx 7.39$.

Applications:

- Population growth: $P(t) = P_0 e^{rt}$
- Radioactive decay: $N(t) = N_0 e^{-\lambda t}$
- Compound interest: $A(t) = Pe^{rt}$
- Bacterial growth: exponential phase models
- Exponential cooling/heating: Newton's law

2. Logarithm Function **Definition:** $f(x) = \ln(x)$ (natural logarithm, base e), where $x > 0$.

In R notation: `log(x)` (in R, `log()` is the natural logarithm)

Properties: Domain: $(0, \infty)$, Range: $\mathbb{R}$, Vertical asymptote at $x = 0$, Always increasing

Domain $(0, \infty)$**:**

By definition, $\ln(x) = y$ means $e^y = x$.

Since $e^y > 0$ always (e.g., $e^{-10} \approx 0.000045$, $e^0 = 1$, $e^{10} \approx 22026$), we need $x > 0$.

If $x \leq 0$, there is no real y such that $e^y = x$.

Vertical asymptote at $x = 0$**:**

As $x \to 0^+$, we need $e^y \to 0$.

Trying values:

- $y = -1$ gives $e^{-1} \approx 0.368$
- $y = -2$ gives $e^{-2} \approx 0.135$
- $y = -5$ gives $e^{-5} \approx 0.0067$
- $y = -10$ gives $e^{-10} \approx 0.000045$

As $y \to -\infty$, $e^y \to 0$.
Therefore $\lim_{x\to 0^+} \ln(x) = -\infty$.
Applications:

- pH scale: $\text{pH} = -\log_{10}([\text{H}^+])$
- Richter scale: $M = \log_{10}(A/A_0)$
- Information content (bits): $I = -\log_2(p)$
- Converting exponential growth to linear scale
- Solving exponential equations: $a^x = b \Rightarrow x = \frac{\ln(b)}{\ln(a)}$

3. *Sinusoid Function* **Definition:** $f(x) = \sin(x)$ or $f(x) = \cos(x)$.
Properties: Domain: $\mathbb{R}$, Range: $[-1, 1]$, Periodic with period 2π
Range $[-1, 1]$**:** On the unit circle (radius 1), the coordinates satisfy $-1 \le \sin(x) \le 1$ and $-1 \le \cos(x) \le 1$ for all x. Examples: $\sin(0) = 0$, $\sin(\pi/2) = 1$, $\sin(\pi) = 0$, $\sin(3\pi/2) = -1$; $\cos(0) = 1$, $\cos(\pi/2) = 0$, $\cos(\pi) = -1$.
Period 2π**:** 2π radians is one complete rotation on the unit circle. For example: $\sin(0) = 0$ and $\sin(0 + 2\pi) = \sin(2\pi) = 0$; $\sin(\pi/4) = \frac{\sqrt{2}}{2}$ and $\sin(\pi/4 + 2\pi) = \frac{\sqrt{2}}{2}$. Therefore $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$.
Applications: Sound waves, light waves, alternating current (AC), tides, seasonal patterns, simple harmonic motion, Fourier analysis.

4. *Square Function* **Definition:** $f(x) = x^2$.
In R notation: `x^2`
Properties: Domain: $\mathbb{R}$, Range: $[0, \infty)$, Even function: $f(-x) = f(x)$ (symmetric about y-axis)
Range $[0, \infty)$**:**
Since $x^2 \ge 0$ for all x (e.g., $(-3)^2 = 9$, $0^2 = 0$, $3^2 = 9$), we have Range $\subseteq [0, \infty)$.
For any $y \ge 0$, we can solve $x^2 = y$:

- if $y = 4$, then $x = 2$ or $x = -2$ both give $x^2 = 4$
- if $y = 9$, then $x = 3$ or $x = -3$ both give $x^2 = 9$

In general, $x = \sqrt{y}$ or $x = -\sqrt{y}$ works.
Even function:
For any x, we have $f(-x) = (-x)^2 = (-x)(-x) = x^2 = f(x)$.
For example:

- $f(-3) = (-3)^2 = 9 = 3^2 = f(3)$

- $f(-5) = 25 = f(5)$

Applications:

- Area of squares: $A = s^2$ where s is side length
- Kinetic energy: $E_k = \frac{1}{2}mv^2$
- Variance in statistics: $\sigma^2 = \mathbb{E}[(X - \mu)^2]$
- Distance squared in physics: $r^2 = x^2 + y^2 + z^2$
- Optimization problems: minimizing/maximizing quadratic forms

5. Identity Function **Definition:** $f(x) = x$.

In R notation: `x`

Properties: Domain: $\mathbb{R}$, Range: $\mathbb{R}$, Odd function: $f(-x) = -f(x)$ (symmetric about origin)

Odd function:

For any x, we have $f(-x) = -x = -(x) = -f(x)$.

For example:

- $f(-3) = -3 = -(3) = -f(3)$
- $f(-5) = -5 = -f(5)$

Applications:

- Linear relationships with slope 1: $y = x$
- Reference line for transformations: scaling, shifting
- Building blocks for more complex functions: compositions

6. One Function **Definition:** $f(x) = 1$ (constant function with value 1).

In R notation: `1`

Properties: Domain: $\mathbb{R}$, Range: $\{1\}$, Even function: $f(-x) = f(x)$ (symmetric about y-axis)

Range $\{1\}$**:**

$f(x) = 1$ for all x, so Range $= \{1\}$.

Applications:

- Normalization constants: $f(x) = 1$ as unit element
- Baseline values: reference level
- Building blocks for constant functions: $c = c \cdot 1$
- Unit scaling: maintaining scale factors

7. *Reciprocal Function* **Definition:** $f(x) = \frac{1}{x} = x^{-1}$.

In R notation: `1/x`

Properties: Domain: $\mathbb{R} \setminus \{0\}$, Range: $\mathbb{R} \setminus \{0\}$, Vertical asymptote at $x = 0$, Horizontal asymptote at $y = 0$, Odd function: $f(-x) = -f(x)$

Range $\mathbb{R} \setminus \{0\}$:

First, show 0 is not in range (proof by contradiction):

suppose $\frac{1}{x} = 0$ for some x.

Multiplying both sides by x gives $1 = 0 \cdot x = 0$, which is impossible.

Therefore $f(x) \neq 0$ for any x.

Now show every nonzero number is in range:

for any $y \neq 0$, solving $\frac{1}{x} = y$ gives $x = \frac{1}{y}$.

For example:

- if $y = 2$, then $x = \frac{1}{2}$ and $f(1/2) = 2$
- if $y = -3$, then $x = -\frac{1}{3}$ and $f(-1/3) = -3$

Vertical asymptote at $x = 0$:

Division by zero is undefined, so $x = 0$ not in domain.

Trying values as $x \to 0^+$:

- $x = 0.1$ gives $\frac{1}{0.1} = 10$
- $x = 0.01$ gives $\frac{1}{0.01} = 100$
- $x = 0.001$ gives $\frac{1}{0.001} = 1000$

As $x \to 0^+$, $\frac{1}{x} \to +\infty$.

Trying values as $x \to 0^-$:

- $x = -0.1$ gives $\frac{1}{-0.1} = -10$
- $x = -0.01$ gives -100
- $x = -0.001$ gives -1000

As $x \to 0^-$, $\frac{1}{x} \to -\infty$.

Odd function:

For any x, we have $f(-x) = \frac{1}{-x} = -\frac{1}{x} = -f(x)$.

For example:

- $f(-2) = \frac{1}{-2} = -0.5 = -\frac{1}{2} = -f(2)$
- $f(-5) = -0.2 = -f(5)$

Applications:

- Inverse relationships: time vs. speed for fixed distance $d = vt \Rightarrow t = d/v$
- Gravitational force: $F = G\frac{m_1 m_2}{r^2}$

- Electrical resistance in parallel: $R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$
- Hyperbolic relationships: inverse proportionality

8. Gaussian Function **Definition:** $f(x) = e^{-x^2}$.

In R notation: `exp(-x^2)`

Properties: Domain: $\mathbb{R}$, Range: $(0, 1]$, Maximum at $x = 0$ where $f(0) = 1$, Symmetric about $x = 0$, $\lim_{x \to \pm\infty} e^{-x^2} = 0$

Maximum at $x = 0$:

At $x = 0$: $f(0) = e^{-0^2} = e^0 = 1$.

For $x \neq 0$:

- if $x = 1$, then $f(1) = e^{-1^2} = e^{-1} \approx 0.368 < 1$
- if $x = 2$, then $f(2) = e^{-4} \approx 0.018 < 1$
- if $x = -1$, then $f(-1) = e^{-1} \approx 0.368 < 1$

In general, for $x \neq 0$, we have $x^2 > 0$, so $-x^2 < 0$, and thus $e^{-x^2} < e^0 = 1$.

Symmetry about $x = 0$:

For any x, we have $f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$.

For example:

- $f(-2) = e^{-4} = f(2)$
- $f(-1) = e^{-1} = f(1)$

Applications:

- Probability distributions: normal distribution kernel $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$
- Measurement errors: error distributions
- Heat diffusion: $u(x, t) \propto e^{-x^2/(4kt)}$
- Wave packets in quantum mechanics: Gaussian wavefunctions
- Image blur filters: Gaussian smoothing kernels

9. Sigmoid Function **Definition:** $f(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$ (logistic sigmoid).

In R notation: `1/(1 + exp(-x))`

Properties: Domain: $\mathbb{R}$, Range: $(0, 1)$, Always increasing, Horizontal asymptotes: $y = 0$ (as $x \to -\infty$), $y = 1$ (as $x \to \infty$), $f(0) = \frac{1}{2}$

Range $(0, 1)$:

Since $e^{-x} > 0$ for all x, we have $1 + e^{-x} > 1$.

Trying values:

- at $x = 0$, $1 + e^0 = 2$, so $f(0) = \frac{1}{2} = 0.5$
- at $x = 1$, $1 + e^{-1} \approx 1.368$, so $f(1) \approx 0.731$
- at $x = -1$, $1 + e^1 \approx 3.718$, so $f(-1) \approx 0.269$

Since $1 + e^{-x} > 1$, we have $\frac{1}{1+e^{-x}} < 1$.
Also, $1 + e^{-x} > 0$ implies $\frac{1}{1+e^{-x}} > 0$.
The function is increasing and has horizontal asymptotes at 0 and 1, so Range $= (0, 1)$.
Horizontal asymptotes:
As $x \to -\infty$:
trying $x = -5$, we have $-x = 5$, so $e^5 \approx 148$, giving $f(-5) = \frac{1}{1+148} \approx 0.0067$
at $x = -10$, $e^{10} \approx 22026$, so $f(-10) \approx 0.000045$
As $x \to -\infty$, $e^{-x} \to \infty$, so $\frac{1}{1+e^{-x}} \to 0$.
As $x \to \infty$:
trying $x = 5$, we have $e^{-5} \approx 0.0067$, so $f(5) = \frac{1}{1.0067} \approx 0.993$
at $x = 10$, $e^{-10} \approx 0.000045$, so $f(10) \approx 0.999955$
As $x \to \infty$, $e^{-x} \to 0$, so $\frac{1}{1+e^{-x}} \to 1$.
Applications:

- Probability (logistic regression): $P(Y = 1|X) = \frac{1}{1+e^{-(\beta_0+\beta_1 X)}}$
- Classification models: binary classification
- Growth with saturation limits: logistic growth $N(t) = \frac{K}{1+e^{-rt}}$
- Neural network activation functions: sigmoid activation
- Dose-response curves: $E(d) = \frac{E_{\max}}{1+e^{-k(d-d_0)}}$

Constant Function $f(x) = c$ (where $c \neq 1$): A special case of the constant function; when $c = 1$, this is the "one" function in the pattern-book list above. The constant function $f(x) = c$ has domain $\mathbb{R}$, range $\{c\}$, and represents fixed quantities that don't depend on the input.

Line Function $f(x) = mx + b$: When $m = 1$ and $b = 0$, this is the identity function $f(x) = x$; when $m = 0$, this is the constant function $f(x) = b$. The line function represents proportional relationships and has domain $\mathbb{R}$, range $\mathbb{R}$ (when $m \neq 0$), and constant slope m.

Power Function $f(x) = x^n$: When $n = 2$, this is the square function $f(x) = x^2$; when $n = 1$, this is the identity function $f(x) = x$; when $n = -1$, this is the reciprocal function $f(x) = 1/x$. Power functions represent scaling relationships and have various domains depending on the exponent n.