# *Parameters*

*David Puelz*
*The University of Austin*

## *Outline*

1. **The Problem** – Matching pattern-book functions to real data
2. **Input Scaling** – Converting quantities to pure numbers
3. **Output Scaling** – Adjusting amplitude and baseline
4. **Complete Parameterization Framework** – The general form $A \cdot g(ax + b) + B$
5. **Parameter Relationships** – Equivalent forms and special cases
6. **Parameterization Conventions** – Standard parameter names
7. **Key Insights** – Why parameters work
8. **Examples** – Applications to real-world modeling

## 1 *Parameters: Scaling Pattern-Book Functions*

### 1.1 *The Problem*

Pattern-book functions are **pure numbers** in, **pure numbers** out.

Real-world quantities have **units**: days, meters, cases, etc.

**Question:** How do we match pattern-book functions to real data?

### 1.2 *Matching Numbers to Quantities*

*Pure Numbers vs. Quantities*

- **Pure numbers:** No units (e.g., 17.32)
- **Quantities:** Have units (e.g., 17.3 days, 34 meters)

Pattern-book functions require pure number inputs.

*Solution: Parameters* Parameters convert quantities to pure numbers before pattern-book evaluation.

# 2 *Input Scaling*

## 2.1 *Definition: Input Scaling*

Given a pattern-book function $g(x)$ and a quantity input $t$ with units, we scale the input using parameters:

$$g(at + b)$$

where:

- $a$: scaling parameter (converts units to pure numbers)
- $b$: shift parameter (adjusts starting point)

## 2.2 *Example: Exponential with Time Input*

Pattern-book: $e^x$ (pure numbers)
Real-world: Cases over time $t$ (in days)
**Model:** $e^{kt}$ where $k$ has units "per day"

*Why this works:*

$$k \cdot t \Big|_{t=10 \text{ days}} = (0.2 \text{ day}^{-1}) \cdot (10 \text{ days}) = 2$$

The "2" is a pure number—units cancel.

## 2.3 *COVID-19 Example*

**Context:** In early 2020, COVID-19 cases grew rapidly. Data from March 2020 showed case numbers doubling roughly every 3-4 days.

**Why exponential?** The growth rate itself was increasing—each day brought more new cases than the previous day. This is the hallmark of exponential growth.

**Data pattern:** Starting around March 1, cases grew from ∼100 to ∼10,000 over the month.

**Model:**

$$\text{cases}(t) = 573 \cdot e^{0.19t}$$

where $t$ is days since March 1.

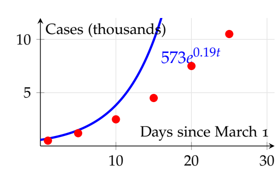

Figure 1: COVID-19 cases: exponential growth model

*Parameter Interpretation:*

- **Rate parameter** $k = 0.19$ **per day:** Cases increase by about 19% each day. This means the number of cases multiplies by $e^{0.19} \approx 1.21$ each day (21% daily growth).

- **Output scaling** $A = 573$**:** This sets the scale to match the data. At $t = 0$ (March 1), the model predicts $573 \cdot e^0 = 573$ cases (in thousands).
- **Why no** $B$ **parameter?** Cases start near zero, so no vertical shift needed.

*Key insight:* The exponential form captures the accelerating growth pattern, where the growth rate itself grows over time.

## 3 *Output Scaling*

### 3.1 *Definition: Output Scaling*

After applying pattern-book function, scale and shift the output:

$$A \cdot g(ax + b) + B$$

where:

- $A$: amplitude/scaling parameter
- $B$: vertical shift parameter

### 3.2 *Example: Tide Levels*

**Context:** Ocean tides rise and fall in a predictable pattern. Most locations experience two high tides and two low tides per day, following a roughly sinusoidal pattern.

**Physical situation:** At a particular location, tide levels oscillate between high tide (around 1.5 meters) and low tide (around 0.5 meters) with a period of approximately 12.4 hours (half a lunar day).

**Why sinusoid?** The periodic rise and fall matches the $\sin(x)$ pattern perfectly.

*Step 1: Input scaling* We need to match the 12.4-hour period. The pattern-book $\sin(x)$ has period $2\pi$, so:

$$\sin\left(\frac{2\pi}{12.4}t\right) = \sin\left(\frac{1}{2}t\right)$$

The phase shift $-1$ adjusts when high tide occurs:

$$\sin\left(\frac{1}{2}t - 1\right)$$

- $\frac{1}{2}$: Converts time to match 12.4-hour period
- $-1$: Phase shift (adjusts timing of high/low tide)

*Step 2: Output scaling* The pattern-book $\sin(x)$ oscillates between $-1$ and $+1$. We need:

- Amplitude of 0.5 meters (difference between high and low tide)
- Midline at 1.0 meters (average tide level)

$$\text{tide}(t) = 0.5 \sin\left(\frac{1}{2}t - 1\right) + 1$$

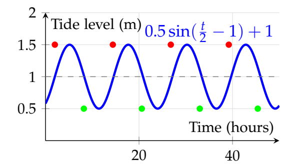

Figure 2: Tide levels: sinusoidal model

*Parameter Interpretation:*

- **Amplitude** $A = 0.5$ **meters:** Controls the range of tide variation (high tide - low tide = $2 \times 0.5 = 1$ meter total range).
- **Midline** $B = 1$ **meter:** Sets the average tide level. High tide = $1 + 0.5 = 1.5$ m, low tide = $1 - 0.5 = 0.5$ m.
- **Period parameter** $\frac{1}{2}$**:** Creates 12.4-hour period (since $\sin(\frac{1}{2} \cdot 12.4) = \sin(6.2) \approx \sin(2\pi)$).
- **Phase shift** $-1$**:** Shifts the entire pattern horizontally to match observed timing.

# 4 *Complete Parameterization Framework*

## 4.1 *General Form*

For any pattern-book function $g(x)$:

$$f(x) = A \cdot g(ax + b) + B$$

*Parameters:*

1. **Input parameters:**
   - $a$: Horizontal scaling (stretch/compress)
   - $b$: Horizontal shift (phase/offset)
2. **Output parameters:**
   - $A$: Vertical scaling (amplitude)
   - $B$: Vertical shift (baseline)

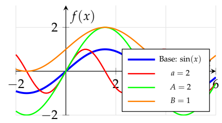

Figure 3: Parameter effects on $\sin(x)$

### 4.2 Visualizing Parameter Effects

**Drawing exercise:** Start with pattern-book function (e.g., $\sin(x)$), then show how each parameter changes the shape:

- **Changing** $a$**:** Stretches/compresses horizontally (affects period for periodic functions)
- **Changing** $b$**:** Shifts left/right (affects phase/starting point)
- **Changing** $A$**:** Stretches/compresses vertically (affects amplitude)
- **Changing** $B$**:** Shifts up/down (affects baseline/midline)

### 4.3 Procedure for Building Models

1. **Choose pattern-book function** whose shape matches data
2. **Find parameter values** $A, B, a, b$ that fit the data

## 5 Parameter Relationships

### 5.1 Input Scaling: Two Equivalent Forms

$$\begin{aligned} g(ax + b) &\quad \text{form 1} \\ g(a(x - x_0)) &\quad \text{form 2} \end{aligned}$$

*Equivalence:*

$$ax + b = a(x - x_0) \quad \text{when} \quad x_0 = -\frac{b}{a}$$

*Interpretation:*

- Form 1: Scale then shift
- Form 2: Shift then scale (often more intuitive)

### 5.2 Exponential Special Case

For exponentials, input scaling can be written:

$$Ae^{at} = e^{\ln(A)}e^{at} = e^{at+\ln(A)} = e^{a(t-t_0)}$$

where $t_0 = -\frac{\ln(A)}{a}$.

*COVID Example (revisited):*

$$\text{cases}(t) = 573e^{0.19t} = e^{0.19(t-t_0)}$$

with $t_0 = -\frac{\ln(573)}{0.19} \approx -33.6$ days.

**Interpretation:** Outbreak started ∼33.6 days before March 1 (late January).

# 6 Parameterization Conventions

## 6.1 Common Parameter Forms

Different pattern-book functions use conventional parameter names:

*Exponential:*

$$\begin{array}{rl} e^{kt} & k\text{: growth rate (per unit time)} \\ e^{t/\tau} & \tau\text{: time constant} \\ 2^{t/\tau_2} & \tau_2\text{: doubling time} \end{array}$$

*Sinusoid:*

$$\begin{array}{rl} \sin\left(\frac{2\pi}{P}(t-t_0)\right) & P\text{: period, } t_0\text{: phase shift} \\ \sin(\omega t+\phi) & \omega\text{: angular frequency, } \phi\text{: phase} \end{array}$$

*Power-law:*

$$(x-x_0)^p \quad x_0\text{: shift, } p\text{: exponent}$$

*Straight-line:*

$$mx+b \quad \text{or} \quad m(x-x_0) \quad m\text{: slope}$$

# 7 Key Insights

## 7.1 Why Parameters Work

1. **Unit conversion:** Parameters convert quantities to pure numbers
2. **Shape control:** Parameters adjust function shape to match data
3. **Flexibility:** Same pattern-book function can model many scenarios

## 7.2 The Big Picture

**Remarkable fact:** Many real-world relationships can be modeled by:

1. Choosing one pattern-book function
2. Scaling input and output with parameters

This framework extends to multiple inputs (coming next).

# 8 Examples

**Example 1:** Population Growth

**Context:** A bacterial culture grows exponentially when resources are unlimited. Starting with $P_0$ bacteria, the population doubles every $\frac{\ln(2)}{r}$ time units.

**Pattern:** $e^x$ (exponential growth)

**Model:** $P(t) = P_0 e^{rt}$

- $P_0$**:** Initial population at $t = 0$
- $r$**:** Growth rate (per unit time). If $r = 0.05$ per hour, population increases by 5% per hour.

**Why this works:** Each bacterium reproduces independently, so total growth rate is proportional to current population—the definition of exponential growth.

**Example 2:** AC Voltage Signal

**Context:** Alternating current (AC) electricity oscillates sinusoidally. Household AC in the US oscillates at 60 Hz (60 cycles per second).

**Pattern:** $\sin(x)$ (periodic oscillation)

**Model:** $V(t) = A\sin(2\pi f t + \phi) + B$

- $A$**:** Peak voltage amplitude (e.g., 120V for US household)
- $f$**:** Frequency in Hz (60 Hz for US AC)
- $\phi$**:** Phase shift (determines starting point of cycle)
- $B$**:** DC offset (usually 0 for pure AC)

**Why this works:** The sinusoidal pattern naturally arises from rotating generators and matches the physical behavior of AC circuits.

**Example 3:** Measurement Error Distribution

**Context:** When measuring a quantity repeatedly, measurement errors often follow a bell-shaped (Gaussian) distribution centered at zero.

**Pattern:** $e^{-x^2}$ (bell curve)

**Model:** $\text{dnorm}(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

- $\mu$**:** Mean (center of distribution). For measurement errors, typically $\mu = 0$ (unbiased measurements).
- $\sigma$**:** Standard deviation (width/spread). Larger $\sigma$ means more variable measurements.

**Why this works:** The Central Limit Theorem shows that many independent small errors combine to create a Gaussian distribution. This is why measurement errors, test scores, and many natural phenomena follow this pattern.

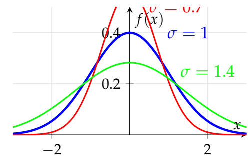


Figure 4: Gaussian distributions with different $\sigma$