Functions, Computing & Pattern-Book Functions

David Puelz
The University of Austin

Course goals

- Understand calculus as the mathematics of change and accumulation
- Build mathematical models using functions, derivatives, and optimization
- Use computing (R) to explore calculus concepts and solve real-world problems

Who am I?

- Educator: Teaching at UATX since 2024, UT Austin from 2020-2024, graduate student from 2013-2018. Previously at the University of Chicago Booth School of Business and Goldman Sachs.
- Researcher: I develop new statistical/machine learning methods for analyzing complex data, with applications in policy, economics, and the social sciences.
- Human: Grew up in Dallas. Husband and dad with 2 girls + 1 boy + 1 dog. I like working out, smoking meat, golf, and cycling.

Answer two questions for me:

What is one thing you're *excited* about for this class?

What is one thing you're *nervous or concerned* about for this class?

Pass around your answers. Another student will read your responses, so everyone stays anonymous.

Populi & Github

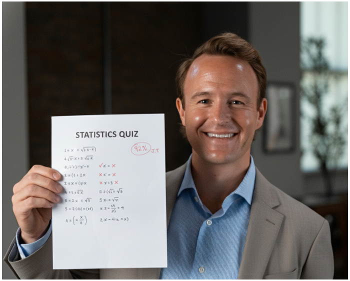
- Access at `uaustin.populiweb.com`
- Access at `https://github.com/dpuelz/Calculus`
- Github pages will contain schedule, homeworks, data, code, ...
- Github will be our main landing page. Populi will be used for grades and administrative tasks.

Class structure

- Understanding the concepts really only comes from practice
- Class time will be mostly lecture and discussion. We will also have “in-class reps” on Quiz reviews and exercises.

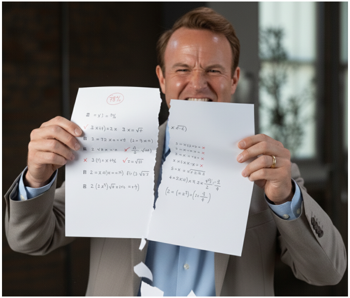
Quizzes

- 5 quizzes, taken during class on the 2nd, 4th, 6th, 8th, and 10th Friday
- Assessing your knowledge of lecture material up to that point
- Concepts assessed will come directly from the homeworks



Homework

- Why homework?
- 5 homework assignments, due on the 2nd, 4th, 6th, 8th, and 10th Friday (like the quizzes)
- Homeworks can be completed individually or in groups of up to three people. They must be turned in by the start of each Friday class. Names of all group members must be included on the writeups.
- I expect professional writeups submitted in pdf format



A pedagogical note on AI

LLMs can be useful for the homeworks and learning concepts. I encourage their use! And ... I will not police their use. I only ask that you cite when LLMs are used.

However, if overused & relied on too heavily, you will not learn anything.

The quizzes are designed from the homeworks and will assess how well you understand the concepts. Through the quizzes, you will get feedback quickly on whether you've overused AI! Use this feedback to optimize your learning with AI.

Exam

- The exam will be during the 11th week (final exam week). There are no make-ups or alternative dates, so please plan accordingly!
- It will be a free response, written exam taken with pen and paper (exactly like our quizzes)

Grading

Homework	**20%**
Exam	**30%**
Quizzes	**50%**

Computing with R

- We will use R for mathematical computation throughout the course
- You are definitely welcome to use Python, but the course materials will (mostly) be in R
- This is industrial-strength, state-of-the-art, and free software for mathematical computing
- We will access R through the RStudio graphical interface; make sure both are installed on your laptop and bring it to every class

Getting help

- My office hours
- Schedule an one-on-one time with me to chat (Zoom or in-person)

What is calculus really about?

Calculus is the mathematics of **change** and **accumulation**.

What is calculus really about?

Calculus is the mathematics of **change** and **accumulation**.

- How things grow, decay, and oscillate
- The language of science, engineering, economics, and AI
- A toolkit for building **models** of the real world

What is calculus really about?

Calculus is the mathematics of **change** and **accumulation**.

- How things grow, decay, and oscillate
- The language of science, engineering, economics, and AI
- A toolkit for building **models** of the real world

Calculus is more than just computing derivatives and integrals.
It's about understanding relationships between quantities.

The two big ideas

1. **Differentiation**: How fast is something changing *right now*?
    - Instantaneous rates of change
    - Slopes, velocities, sensitivities

The two big ideas

1. **Differentiation**: How fast is something changing *right now*?
   - Instantaneous rates of change
   - Slopes, velocities, sensitivities

2. **Optimization**: Where are the peaks, valleys, and best choices?
   - Finding maxima and minima
   - Applications in science, engineering, economics

The two big ideas

1. **Differentiation**: How fast is something changing *right now*?
   - Instantaneous rates of change
   - Slopes, velocities, sensitivities

2. **Optimization**: Where are the peaks, valleys, and best choices?
   - Finding maxima and minima
   - Applications in science, engineering, economics

Calculus gives us the tools to find optimal solutions and understand how systems behave.

Why this course is different

Traditional calculus: Symbolic manipulation, memorization

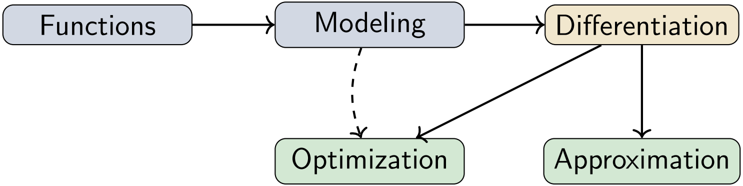
Why this course is different

Traditional calculus: Symbolic manipulation, memorization

This course (MOSAIC approach): Modeling, computing, understanding

- Functions as the **central object**—not formulas
- Emphasis on why and when, not just how
- Computing (R) as a tool for exploration
- Real applications

Course roadmap



Block I: Functions and Modeling
Block II: Differentiation
Block III: Optimization and Approximation

Part I

Functions: The Language of Relationships

What is a function?

Definition: Function

A **function** f is a rule that assigns to each element x in a set D (the *domain*) exactly one element $f(x)$ in a set R (the *range*).

What is a function?

Definition: Function

A **function** f is a rule that assigns to each element x in a set D (the *domain*) exactly one element $f(x)$ in a set R (the *range*).

Key insight: A function is a **relationship**, not just a formula.

Examples from your daily life:

- `temperature`(time of day) $\rightarrow$ degrees Fahrenheit
- `Uber_price`(distance, surge_multiplier) $\rightarrow$ dollars
- `GPA`(hours_studied, natural_ability) $\rightarrow$ number

What is NOT a function?

A relation is not a function if one input maps to **multiple outputs**.

What is NOT a function?

A relation is not a function if one input maps to **multiple outputs**.

Examples of non-functions:

- $x^2 + y^2 = 1$ (circle: for $x = 0$, we get $y = \pm 1$)
- $y^2 = x$ (parabola opening right: for $x = 4$, we get $y = \pm 2$)
- A relation where one person has multiple phone numbers

What is NOT a function?

A relation is not a function if one input maps to **multiple outputs**.

Examples of non-functions:

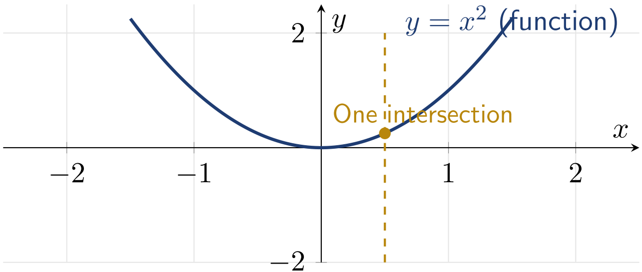
- $x^2 + y^2 = 1$ (circle: for $x = 0$, we get $y = \pm 1$)
- $y^2 = x$ (parabola opening right: for $x = 4$, we get $y = \pm 2$)
- A relation where one person has multiple phone numbers

Key Principle

For a function, each input must produce **exactly one** output.

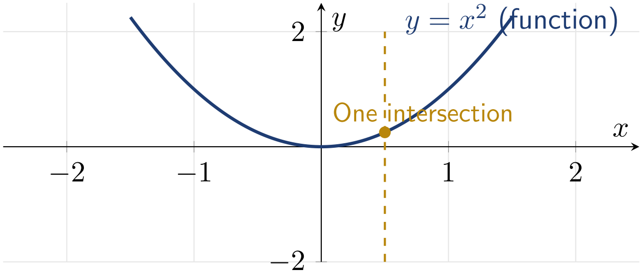
The vertical line test

A graphical way to check if a relation is a function:



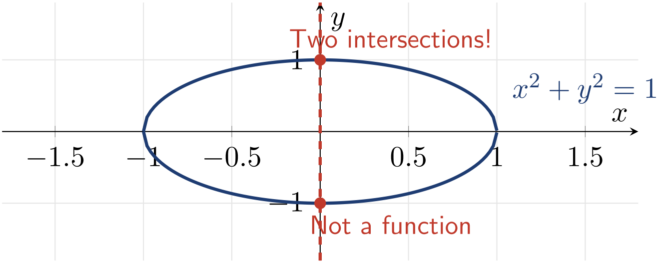
The vertical line test

A graphical way to check if a relation is a function:

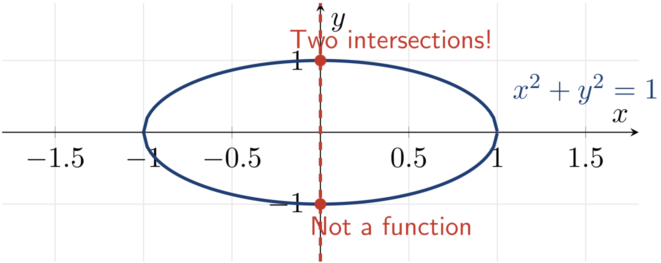


Vertical Line Test: If any vertical line intersects the graph in **more than one point**, the relation is not a function.

Non-function example: Circle

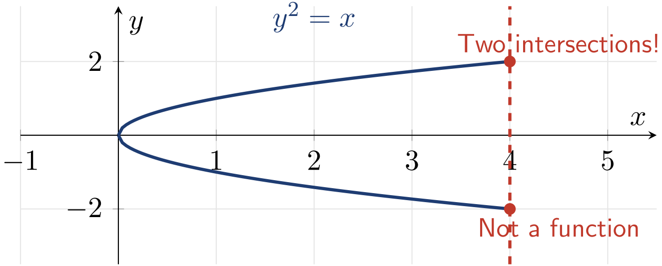


Non-function example: Circle

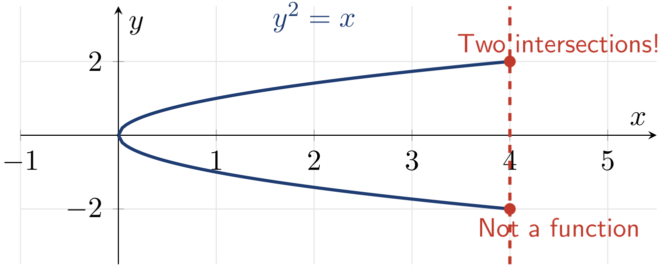


Why it's not a function: For $x = 0$, we get $y = 1$ **and** $y = -1$. One input $\to$ two outputs!

Non-function example: Sideways parabola



Non-function example: Sideways parabola



Why it's not a function: For $x = 4$, we get $y = 2$ **and** $y = -2$. One input $\rightarrow$ two outputs!

Making a non-function into a function

Sometimes we can restrict the domain to make a relation into a function:

Making a non-function into a function

Sometimes we can restrict the domain to make a relation into a function:

Example: The circle $x^2 + y^2 = 1$ is not a function, but we can define:

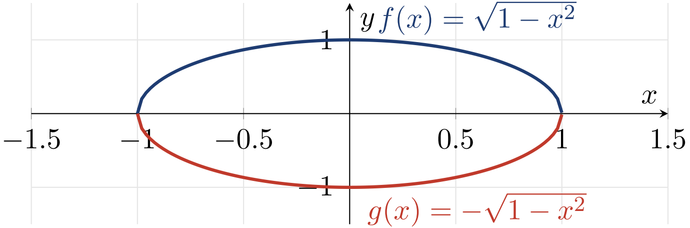
- $f(x) = \sqrt{1 - x^2}$ (upper half, domain: $[-1, 1]$)
- $g(x) = -\sqrt{1 - x^2}$ (lower half, domain: $[-1, 1]$)

Making a non-function into a function

Sometimes we can restrict the domain to make a relation into a function:

Example: The circle $x^2 + y^2 = 1$ is not a function, but we can define:

- $f(x) = \sqrt{1 - x^2}$ (upper half, domain: $[-1, 1]$)
- $g(x) = -\sqrt{1 - x^2}$ (lower half, domain: $[-1, 1]$)

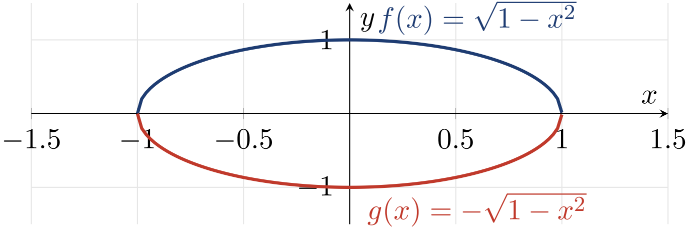


Making a non-function into a function

Sometimes we can restrict the domain to make a relation into a function:

Example: The circle $x^2 + y^2 = 1$ is not a function, but we can define:

- $f(x) = \sqrt{1 - x^2}$ (upper half, domain: $[-1, 1]$)
- $g(x) = -\sqrt{1 - x^2}$ (lower half, domain: $[-1, 1]$)



Both f and g are functions! Each input gives exactly one output.

Function notation: the details

Standard notation:

$$y = f(x)$$

Function notation: the details

Standard notation:

$$y = f(x)$$

- x is the **input** (also: *argument*, *independent variable*)
- y is the **output** (also: *value*, *dependent variable*)
- f is the **name** of the function

Function notation: the details

Standard notation:

$$y = f(x)$$

- x is the **input** (also: *argument*, *independent variable*)
- y is the **output** (also: *value*, *dependent variable*)
- f is the **name** of the function

Important: f and $f(x)$ are different!

- f is the function itself (the "machine")
- $f(x)$ is the output when you feed in x

Evaluating functions

Example: Let $f(x) = x^2 - 3x + 2$

Evaluating functions

Example: Let $f(x) = x^2 - 3x + 2$

Evaluate at specific points:

$$f(0) = 0^2 - 3(0) + 2 = 2$$
$$f(1) = 1^2 - 3(1) + 2 = 0$$
$$f(2) = 2^2 - 3(2) + 2 = 0$$
$$f(-1) = (-1)^2 - 3(-1) + 2 = 6$$

Evaluating functions

Example: Let $f(x) = x^2 - 3x + 2$

Evaluate at specific points:

$$f(0) = 0^2 - 3(0) + 2 = 2$$
$$f(1) = 1^2 - 3(1) + 2 = 0$$
$$f(2) = 2^2 - 3(2) + 2 = 0$$
$$f(-1) = (-1)^2 - 3(-1) + 2 = 6$$

Evaluate at an expression:

$$f(a + h) = (a + h)^2 - 3(a + h) + 2$$

$$= a^2 + 2ah + h^2 - 3a - 3h + 2$$

Functions with multiple inputs

Real-world relationships often involve **multiple inputs**:

A function of two variables: $z = f(x, y)$
A function of n variables: $w = f(x_1, x_2, \ldots, x_n)$

Functions with multiple inputs

Real-world relationships often involve **multiple inputs**:

A function of two variables: $z = f(x, y)$
A function of n variables: $w = f(x_1, x_2, \ldots, x_n)$

Examples:

- Area of a rectangle: $A(w, h) = w \cdot h$
- Ideal gas law: $P(V, T, n) = \frac{nRT}{V}$
- Mortgage payment: $M(P, r, n) = P \cdot \frac{r(1+r)^n}{(1+r)^n - 1}$

Example: the Cobb-Douglas production function

In economics, how do labor (L) and capital (K) combine to produce output (Y)?

$$Y = A \cdot L^{\alpha} \cdot K^{1-\alpha}$$

Example: the Cobb-Douglas production function

In economics, how do labor (L) and capital (K) combine to produce output (Y)?

$$Y = A \cdot L^{\alpha} \cdot K^{1-\alpha}$$

- $A =$ total factor productivity (technology level)
- $\alpha =$ labor's share of output (typically ≈ 0.7)
- $1 - \alpha =$ capital's share

Example: the Cobb-Douglas production function

In economics, how do labor (L) and capital (K) combine to produce output (Y)?

$$Y = A \cdot L^{\alpha} \cdot K^{1-\alpha}$$

- $A =$ total factor productivity (technology level)
- $\alpha =$ labor's share of output (typically ≈ 0.7)
- $1 - \alpha =$ capital's share

Key property: Constant returns to scale

$$f(tL, tK) = A(tL)^{\alpha}(tK)^{1-\alpha} = t \cdot AL^{\alpha}K^{1-\alpha} = t \cdot f(L, K)$$

Double both inputs $\Rightarrow$ double the output!

Domain and range: formal definitions

Definition: Domain

The **domain** of a function f is the set of all inputs x for which $f(x)$ is defined.

Definition: Range

The **range** of a function f is the set of all possible outputs: $\{f(x) : x \in \text{domain}\}$.

Domain and range: formal definitions

Definition: Domain

The **domain** of a function f is the set of all inputs x for which $f(x)$ is defined.

Definition: Range

The **range** of a function f is the set of all possible outputs: $\{f(x) : x \in \text{domain}\}$.

Common domain restrictions:

- Division by zero: $f(x) = \frac{1}{x}$ excludes $x = 0$
- Square roots of negatives: $f(x) = \sqrt{x}$ requires $x \geq 0$
- Logarithms: $f(x) = \ln(x)$ requires $x > 0$

Domain and range: examples

Example 1: $f(x) = \sqrt{x-2}$

- Domain: $x - 2 \geq 0 \Rightarrow x \geq 2$, i.e., $[2, \infty)$
- Range: $[0, \infty)$

Domain and range: examples

Example 1: $f(x) = \sqrt{x-2}$

- Domain: $x - 2 \geq 0 \Rightarrow x \geq 2$, i.e., $[2, \infty)$
- Range: $[0, \infty)$

Example 2: $g(x) = \frac{1}{x^2-4}$

- Domain: $x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2$
- Domain $= (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
- Range: $(-\infty, -1/4] \cup (0, \infty)$

Domain and range: examples

Example 1: $f(x) = \sqrt{x-2}$

- Domain: $x - 2 \geq 0 \Rightarrow x \geq 2$, i.e., $[2, \infty)$
- Range: $[0, \infty)$

Example 2: $g(x) = \frac{1}{x^2-4}$

- Domain: $x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2$
- Domain $= (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
- Range: $(-\infty, -1/4] \cup (0, \infty)$

Example 3: $h(x) = \ln(x^2)$

- Domain: $x^2 > 0 \Rightarrow x \neq 0$
- Domain $= (-\infty, 0) \cup (0, \infty)$
- Range: $(-\infty, \infty)$ (since x^2 can be any positive number, and $\ln(x^2)$ can be any real number)

Domain and range: practice problems

Find the domain and range for each function:

1. $f(x) = \sqrt{3x+1}$
2. $g(x) = \frac{1}{x-5}$
3. $h(x) = \ln(x+2)$
4. $j(x) = \frac{x}{x^2-9}$
5. $k(x) = \sqrt{4-x^2}$
6. $m(x) = \frac{1}{\sqrt{x-1}}$
7. $n(x) = \ln(x^2-4)$
8. $p(x) = \frac{x+1}{x^2+1}$
9. $q(x) = \sqrt{x^2-5x+6}$
10. $r(x) = \frac{1}{\ln(x)}$

Domain and range: solutions

1. $f(x) = \sqrt{3x+1}$
Domain: $3x + 1 \geq 0 \Rightarrow x \geq -1/3$, so $[-1/3, \infty)$
Range: $[0, \infty)$

2. $g(x) = \frac{1}{x-5}$
Domain: $x \neq 5$, so $(-\infty, 5) \cup (5, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

3. $h(x) = \ln(x+2)$
Domain: $x + 2 > 0 \Rightarrow x > -2$, so $(-2, \infty)$
Range: $(-\infty, \infty)$

4. $j(x) = \frac{x}{x^2-9}$
Domain: $x^2 - 9 \neq 0 \Rightarrow x \neq \pm 3$, so $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Range: All real numbers

5. $k(x) = \sqrt{4 - x^2}$
Domain: $4 - x^2 \geq 0 \Rightarrow -2 \leq x \leq 2$, so $[-2, 2]$
Range: $[0, 2]$

Domain and range: solutions (continued)

6. $m(x) = \frac{1}{\sqrt{x-1}}$
Domain: $x - 1 > 0 \Rightarrow x > 1$, so $(1, \infty)$
Range: $(0, \infty)$

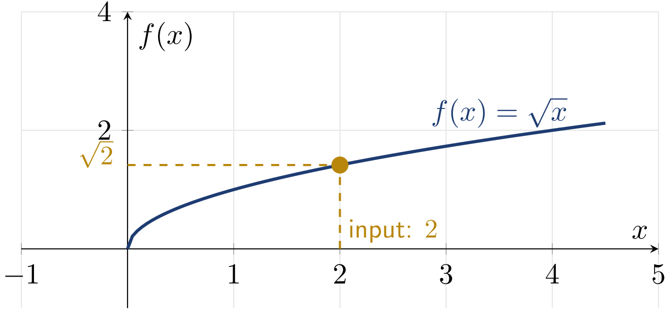
7. $n(x) = \ln(x^2 - 4)$
Domain: $x^2 - 4 > 0 \Rightarrow |x| > 2$, so $(-\infty, -2) \cup (2, \infty)$
Range: $(-\infty, \infty)$

8. $p(x) = \frac{x+1}{x^2+1}$
Domain: $x^2 + 1 \neq 0$ for all x, so $(-\infty, \infty)$
Range: $[-1/2, 1/2]$ (requires calculus to find exactly)

9. $q(x) = \sqrt{x^2 - 5x + 6}$
Domain: Factor: $(x-2)(x-3) \geq 0$, so $(-\infty, 2] \cup [3, \infty)$
Range: $[0, \infty)$

10. $r(x) = \frac{1}{\ln(x)}$
Domain: $\ln(x) \neq 0$ and $x > 0 \Rightarrow x \neq 1$ and $x > 0$, so $(0, 1) \cup (1, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

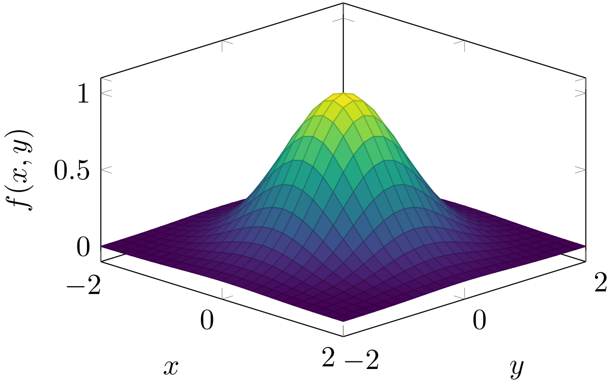
Visualizing functions: one input



A function of **one input** is visualized as a curve in 2D.

Visualizing functions: two inputs

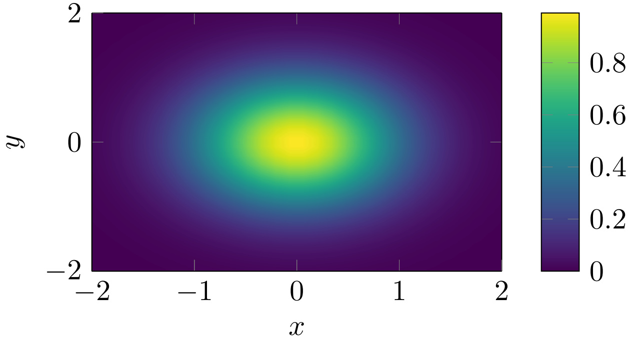
A function of **two inputs** is a surface in 3D space:



This is the **Gaussian** $f(x, y) = e^{-(x^2+y^2)}$ — a 2D bell curve!

Contour plots: flattening 3D to 2D

A **contour plot** shows level curves where $f(x, y) = c$ (constant).



Interpretation: Like a topographic map—each color band represents a height.

Real-world contour example: weather maps

Pressure isobars are contour lines of atmospheric pressure!

- Each line connects points of equal pressure
- Closely spaced lines = rapid pressure change = strong winds
- **Gradients** (next topic) point from low to high pressure

Other examples of contours:

- Elevation on hiking maps
- Temperature isotherms
- Indifference curves in economics

Part II

Computing with R

Why we compute

Paper-and-pencil calculus was designed for the 1800s.

Why we compute

Paper-and-pencil calculus was designed for the 1800s.

- Computers let us **visualize** complex relationships instantly
- Computers let us **explore** “what if” scenarios
- Computers let us work with **real data**
- Modern science and industry *require* computational thinking

Why we compute

Paper-and-pencil calculus was designed for the 1800s.

- Computers let us **visualize** complex relationships instantly
- Computers let us **explore** "what if" scenarios
- Computers let us work with **real data**
- Modern science and industry *require* computational thinking

Philosophy: We use R not to avoid mathematics, but to *deepen* our mathematical understanding through exploration.

Why R?

- **Free and open source**—runs anywhere
- Designed for **data analysis and statistics**
- Excellent **graphics** capabilities
- Used by scientists, economists, data analysts worldwide
- The `mosaicCalc` package provides calculus-specific tools

Why R?

- **Free and open source**—runs anywhere
- Designed for **data analysis and statistics**
- Excellent **graphics** capabilities
- Used by scientists, economists, data analysts worldwide
- The `mosaicCalc` package provides calculus-specific tools

Alternatives exist: Python (with NumPy/SciPy), MATLAB, Julia
The concepts transfer—learning one makes learning others easy.

Defining functions in R

In R, we define functions with the `makeFun()` command:

```
# Define a quadratic function
f <- makeFun(x^2 - 3*x + 2 ~ x)

# Evaluate it at specific points
f(0)    # returns 2
f(1)    # returns 0
f(5)    # returns 12
```

Defining functions in R

In R, we define functions with the `makeFun()` command:

```
# Define a quadratic function
f <- makeFun(x^2 - 3*x + 2 ~ x)

# Evaluate it at specific points
f(0)    # returns 2
f(1)    # returns 0
f(5)    # returns 12
```

The `~` (tilde) is read as "*is a function of*":

$$\underbrace{x^2 - 3x + 2}_{\text{output expression}} \quad \sim \quad \underbrace{x}_{\text{input variable}}$$

Functions with parameters

Parameters are values we might want to change:

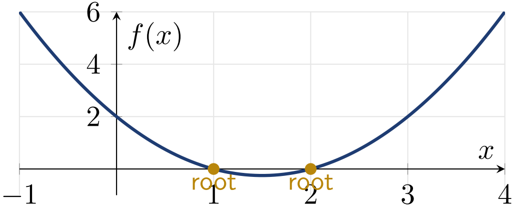
```
# Exponential growth with parameters
growth <- makeFun(A * exp(k*t) ~ t, A = 100, k =
    0.1)

# Use default parameters
growth(5)     # A=100, k=0.1, t=5

# Override parameters
growth(5, A = 200)          # different starting
    value
growth(5, k = 0.2)          # faster growth rate
growth(5, A = 50, k = 0.05) # both changed
```

This lets us explore: "What if the growth rate were different?"

Plotting functions: `slice_plot`

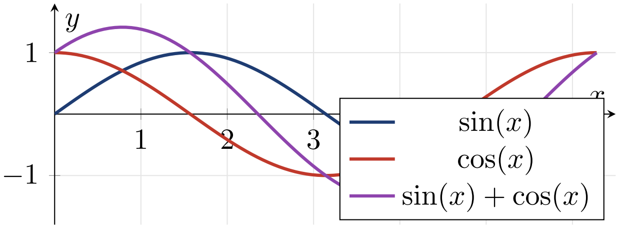
Visualization is **essential** for understanding:

```
# Plot the function over a domain
slice_plot(x^2 - 3*x + 2 ~ x, domain(x = -1:4))
```



Multiple functions on one plot

```
slice_plot(sin(x) ~ x, domain(x = 0:2*pi),
           color = "blue") %>%
slice_plot(cos(x) ~ x, color = "red") %>%
slice_plot(sin(x) + cos(x) ~ x, color = "purple")
```



Functions of two variables

```r
# Define Cobb-Douglas production function
production <- makeFun(A * L^alpha * K^(1-alpha)
                      ~ L & K,
                      A = 1, alpha = 0.7)

# Evaluate: 100 workers, 50 machines
production(L = 100, K = 50)
```

Functions of two variables

```
# Define Cobb-Douglas production function
production <- makeFun(A * L^alpha * K^(1-alpha)
                      ~ L & K,
                      A = 1, alpha = 0.7)

# Evaluate: 100 workers, 50 machines
production(L = 100, K = 50)
```

Note: Use & to separate multiple input variables.

The **domain** for 2D functions:

```
domain(L = 1:100, K = 1:100)
```

Surface plots and contour plots

```r
# 3D surface plot
surface_plot(exp(-(x^2 + y^2)) ~ x & y,
             domain(x = -2:2, y = -2:2))

# 2D contour plot
contour_plot(exp(-(x^2 + y^2)) ~ x & y,
             domain(x = -2:2, y = -2:2))
```

Contour plot interpretation:

- Lines close together = steep surface (rapid change)
- Lines far apart = flat surface (slow change)
- Closed loops = peaks or valleys

Finding special values

R can help us find roots, maxima, and other special points:

```
# Find where f(x) = 0
f <- makeFun(x^2 - 3*x + 2 ~ x)
Zeros(f, domain(x = -5:5))
# Returns: x = 1, x = 2

# Find maximum of a function
g <- makeFun(-x^2 + 4*x ~ x)
argM(g, domain(x = -5:5))    # x value at maximum
# Returns: x = 2
```

These numerical tools complement (not replace) analytical thinking!

Part III

The Pattern-Book Functions

The modeling toolkit

Just as writers use a vocabulary of words, modelers use a vocabulary of **basic functions**.

Key Insight: Almost any real-world relationship can be modeled by combining just **nine basic functions**.

The modeling toolkit

Just as writers use a vocabulary of words, modelers use a vocabulary of **basic functions**.

> **Key Insight**: Almost any real-world relationship can be modeled by combining just **nine basic functions**.

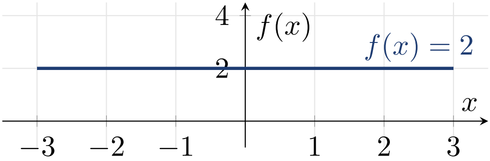
These pattern-book functions form your modeling toolkit:

- Constant
- Line
- Power
- Exponential
- Logarithm
- Sinusoid
- Hump
- Sigmoid
- Reciprocal

1. The constant function

Definition

$$f(x) = c \quad \text{where } c \text{ is a constant}$$



Properties:

- Domain: all real numbers; Range: $\{c\}$
- Derivative: $f'(x) = 0$ (no change)

Uses: Baselines, fixed quantities, asymptotes

Constant function: examples

Physics: Speed of light $c \approx 3 \times 10^8$ m/s (constant in vacuum)

Economics: Fixed cost—cost that doesn't depend on production quantity

Chemistry: Avogadro's number $N_A = 6.022 \times 10^{23}$

Constant function: examples

Physics: Speed of light $c \approx 3 \times 10^8$ m/s (constant in vacuum)

Economics: Fixed cost—cost that doesn't depend on production quantity

Chemistry: Avogadro's number $N_A = 6.022 \times 10^{23}$

Mathematical role: Constants appear in:

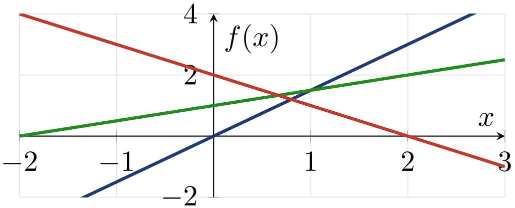
- Initial conditions: $f(0) = c$
- Asymptotes: $\lim_{x \to \infty} g(x) = c$
- Vertical shifts: $g(x) = f(x) + c$

2. The line

Definition

$$f(x) = mx + b$$

where m is the **slope** and b is the y-**intercept**



The slope $m = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ measures the **rate of change**.

Linear functions: deep dive

Key formula: Given two points (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Linear functions: deep dive

Key formula: Given two points (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope form: Line through (x_0, y_0) with slope m:

$$y - y_0 = m(x - x_0)$$

Linear functions: deep dive

Key formula: Given two points (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope form: Line through (x_0, y_0) with slope m:

$$y - y_0 = m(x - x_0)$$

Example: Find the line through $(1, 3)$ and $(4, 9)$.

$$m = \frac{9 - 3}{4 - 1} = 2$$

$$y - 3 = 2(x - 1) \quad \Rightarrow \quad y = 2x + 1$$

Linear functions: applications

Hooke's Law (springs):

$$F = -kx$$

Force is proportional to displacement.

Ohm's Law (electricity):

$$V = IR$$

Voltage equals current times resistance.

Linear depreciation:

$$V(t) = V_0 - dt$$

Value decreases at constant rate d per year.

Linear functions: applications

Hooke's Law (springs):

$$F = -kx$$

Force is proportional to displacement.

Ohm's Law (electricity):

$$V = IR$$

Voltage equals current times resistance.

Linear depreciation:

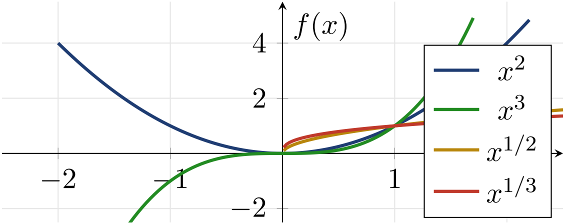
$$V(t) = V_0 - dt$$

Value decreases at constant rate d per year.

Linear models are the foundation—even nonlinear functions look linear when you zoom in close enough!

Definition

$$f(x) = x^n \quad \text{where } n \text{ is any real number}$$



Power functions: classification

Integer powers:

- $n > 0$: Polynomial behavior, $f(0) = 0$
- Even n: Symmetric about y-axis (even function)
- Odd n: Symmetric about origin (odd function)

Power functions: classification

Integer powers:

- $n > 0$: Polynomial behavior, $f(0) = 0$
- Even n: Symmetric about y-axis (even function)
- Odd n: Symmetric about origin (odd function)

Fractional powers (roots):

- $x^{1/2} = \sqrt{x}$: Square root
- $x^{1/3} = \sqrt[3]{x}$: Cube root
- $x^{1/n} = \sqrt[n]{x}$: nth root

Power functions: classification

Integer powers:

- $n > 0$: Polynomial behavior, $f(0) = 0$
- Even n: Symmetric about y-axis (even function)
- Odd n: Symmetric about origin (odd function)

Fractional powers (roots):

- $x^{1/2} = \sqrt{x}$: Square root
- $x^{1/3} = \sqrt[3]{x}$: Cube root
- $x^{1/n} = \sqrt[n]{x}$: nth root

Negative powers:

- $x^{-1} = 1/x$: Reciprocal
- $x^{-2} = 1/x^2$: Inverse square

Power functions: key examples

Kinetic Energy:

$$KE = \frac{1}{2}mv^2$$

Energy grows as the *square* of velocity.

Gravitational Force (inverse square law):

$$F = G\frac{m_1 m_2}{r^2} = Gm_1m_2 \cdot r^{-2}$$

Kepler's Third Law:

$$T^2 \propto r^3 \quad \Leftrightarrow \quad T = k \cdot r^{3/2}$$

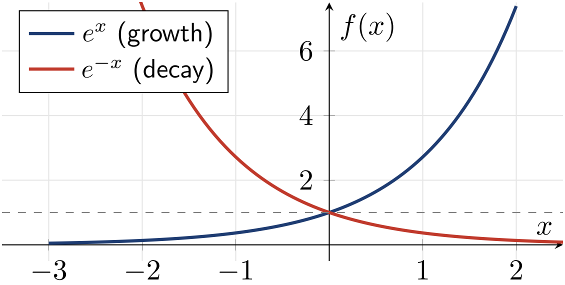
Orbital period T vs. orbital radius r.

4. The exponential function

Definition

$$f(x) = e^x \quad \text{where } e \approx 2.71828...$$

More generally: $f(x) = a^x$ for any base $a > 0$



Why e? The magic number

What makes $e \approx 2.71828...$ special?

Why e? The magic number

What makes $e \approx 2.71828...$ special?

Key Property

The exponential e^x is **its own derivative**:

$$\frac{d}{dx}e^x = e^x$$

Why e? The magic number

What makes $e \approx 2.71828...$ special?

Key Property

The exponential e^x is **its own derivative**:

$$\frac{d}{dx}e^x = e^x$$

Definition via compound interest:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

If you invest \$1 at 100% interest compounded n times per year, as $n \to \infty$, you get \$$e$.

The exponential law of growth

Fundamental Principle

If something grows at a rate *proportional to its current size*, it grows exponentially:

$$\frac{dN}{dt} = kN \quad \Rightarrow \quad N(t) = N_0 e^{kt}$$

Growth $(k > 0)$: Population, compound interest, viral spread

Decay $(k < 0)$: Radioactive decay, drug metabolism, cooling

The exponential law of growth

Fundamental Principle

If something grows at a rate *proportional to its current size*, it grows exponentially:

$$\frac{dN}{dt} = kN \quad \Rightarrow \quad N(t) = N_0 e^{kt}$$

Growth $(k > 0)$: Population, compound interest, viral spread

Decay $(k < 0)$: Radioactive decay, drug metabolism, cooling

Doubling time: $T_d = \frac{\ln 2}{k} \approx \frac{0.693}{k}$

Half-life: $T_{1/2} = \frac{\ln 2}{|k|}$ (for decay)

Doubling time: concept and formula

Doubling time T_d: The time it takes for a quantity to double in size.

Doubling time: concept and formula

Doubling time T_d: The time it takes for a quantity to double in size.

Formula

If $N(t) = N_0 e^{kt}$ with $k > 0$, then:

$$T_d = \frac{\ln(2)}{k} \approx \frac{0.693}{k}$$

Doubling time: concept and formula

Doubling time T_d: The time it takes for a quantity to double in size.

Formula

If $N(t) = N_0 e^{kt}$ with $k > 0$, then:

$$T_d = \frac{\ln(2)}{k} \approx \frac{0.693}{k}$$

Why this formula?

$$\begin{aligned} 2N_0 &= N_0 e^{kT_d} \\ 2 &= e^{kT_d} \\ \ln(2) &= kT_d \\ T_d &= \frac{\ln(2)}{k} \end{aligned}$$

Doubling time example 1: COVID-19 cases

Scenario: Early pandemic data showed cases doubling every 3 days.

Doubling time example 1: COVID-19 cases

Scenario: Early pandemic data showed cases doubling every 3 days.

Find the growth rate k:

$$T_d = 3 \text{ days}$$
$$k = \frac{\ln(2)}{T_d} = \frac{0.693}{3} \approx 0.231 \text{ per day}$$

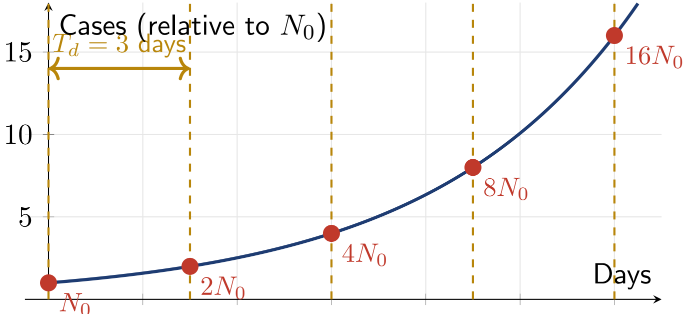
Doubling time example 1: COVID-19 cases

Scenario: Early pandemic data showed cases doubling every 3 days.

Find the growth rate k:

$$T_d = 3 \text{ days}$$

$$k = \frac{\ln(2)}{T_d} = \frac{0.693}{3} \approx 0.231 \text{ per day}$$

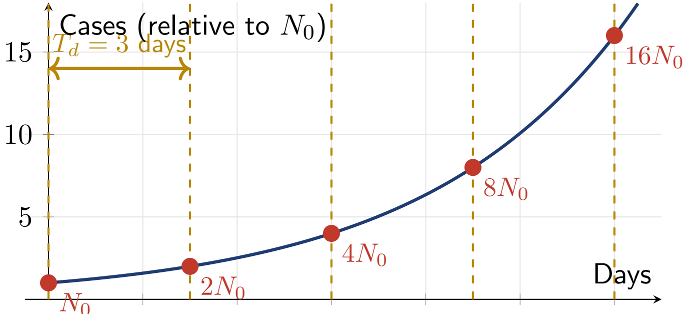


Doubling time example 1: COVID-19 cases

Scenario: Early pandemic data showed cases doubling every 3 days.

Find the growth rate k:

$$T_d = 3 \text{ days}$$
$$k = \frac{\ln(2)}{T_d} = \frac{0.693}{3} \approx 0.231 \text{ per day}$$



Doubling time example 2: Bacterial growth

Scenario: A bacterial culture doubles every 20 minutes. Starting with 1000 cells:

Doubling time example 2: Bacterial growth

Scenario: A bacterial culture doubles every 20 minutes. Starting with 1000 cells:

Find the growth rate k:

$$T_d = 20 \text{ minutes} = \frac{1}{3} \text{ hour}$$

$$k = \frac{\ln(2)}{1/3} = 3\ln(2) \approx 2.08 \text{ per hour}$$

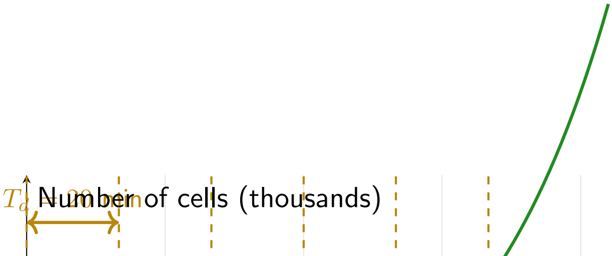
Doubling time example 2: Bacterial growth

Scenario: A bacterial culture doubles every 20 minutes. Starting with 1000 cells:

Find the growth rate k:

$$T_d = 20 \text{ minutes} = \frac{1}{3} \text{ hour}$$

$$k = \frac{\ln(2)}{1/3} = 3\ln(2) \approx 2.08 \text{ per hour}$$



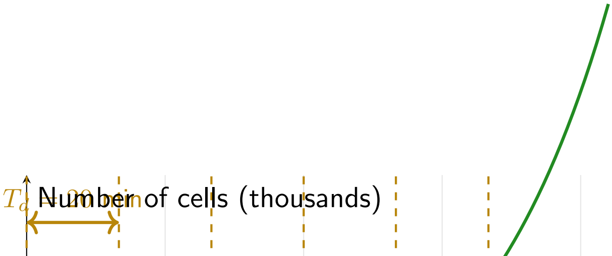
Doubling time example 2: Bacterial growth

Scenario: A bacterial culture doubles every 20 minutes. Starting with 1000 cells:

Find the growth rate k:

$$T_d = 20 \text{ minutes} = \frac{1}{3} \text{ hour}$$

$$k = \frac{\ln(2)}{1/3} = 3\ln(2) \approx 2.08 \text{ per hour}$$



Doubling time: key insights

Three ways to specify exponential growth:

- **Growth rate** k: "Grows at 23% per day"
- **Doubling time** T_d: "Doubles every 3 days"
- **Percentage growth**: "Grows by 100% every T_d"

Doubling time: key insights

Three ways to specify exponential growth:

- **Growth rate** k: "Grows at 23% per day"
- **Doubling time** T_d: "Doubles every 3 days"
- **Percentage growth**: "Grows by 100% every T_d"

Relationship: $T_d = \frac{\ln(2)}{k}$ and $k = \frac{\ln(2)}{T_d}$

Doubling time: key insights

Three ways to specify exponential growth:

- **Growth rate** k: "Grows at 23% per day"
- **Doubling time** T_d: "Doubles every 3 days"
- **Percentage growth**: "Grows by 100% every T_d"

Relationship: $T_d = \frac{\ln(2)}{k}$ and $k = \frac{\ln(2)}{T_d}$

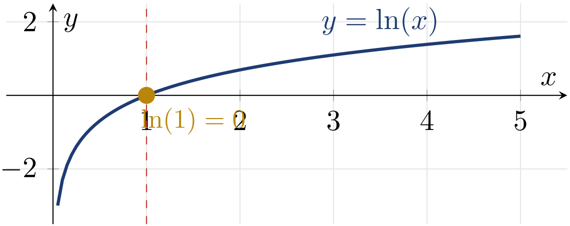
Key insight: Doubling time is independent of the starting value N_0! The time to double depends only on the growth rate k.

5. The logarithm

Definition

The **natural logarithm** $\ln(x)$ is the *inverse* of e^x:

$$y = \ln(x) \quad \Leftrightarrow \quad x = e^y$$



Logarithm properties

Key Properties

$$
\begin{aligned}
\ln(ab) &= \ln(a) + \ln(b) \quad \text{(products become sums)} \\
\ln(a/b) &= \ln(a) - \ln(b) \quad \text{(quotients become differences)} \\
\ln(a^n) &= n \ln(a) \quad \text{(powers become products)} \\
\ln(e) &= 1, \quad \ln(1) = 0
\end{aligned}
$$

Logarithm properties

Key Properties

$$\ln(ab) = \ln(a) + \ln(b) \quad \text{(products become sums)}$$
$$\ln(a/b) = \ln(a) - \ln(b) \quad \text{(quotients become differences)}$$
$$\ln(a^n) = n\ln(a) \quad \text{(powers become products)}$$
$$\ln(e) = 1, \quad \ln(1) = 0$$

Key insight: Logarithms "undo" exponentiation.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln(x)} = x$$

Why we use log scales

Earthquake Magnitude (Richter Scale)

Magnitude	Energy (Joules)	Example
3.0	2×10^9	Minor tremor
5.0	2×10^{12}	Moderate damage
7.0	2×10^{15}	Major (Haiti 2010)
9.0	2×10^{18}	Great (Japan 2011)

Each unit increase = **31.6× more energy** (because $10^{1.5} \approx 31.6$)!

Why we use log scales

Earthquake Magnitude (Richter Scale)

Magnitude	Energy (Joules)	Example
3.0	2×10^9	Minor tremor
5.0	2×10^{12}	Moderate damage
7.0	2×10^{15}	Major (Haiti 2010)
9.0	2×10^{18}	Great (Japan 2011)

Each unit increase = **31.6× more energy** (because $10^{1.5} \approx 31.6$)!

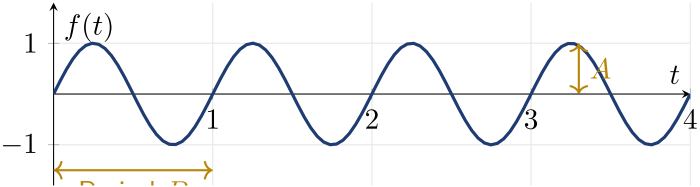
Other log scales: Decibels (sound), pH (acidity), stellar magnitude

6. The sinusoid

Definition

$$f(t) = A\sin\left(\frac{2\pi t}{P}\right) + C$$

or equivalently $f(t) = A\sin(\omega t + \phi) + C$



Sinusoid parameters

$$f(t) = A \sin\left(\frac{2\pi}{P}(t - t_0)\right) + C$$

Parameter	Meaning
A (amplitude)	Height of oscillation from center
P (period)	Time for one complete cycle
t_0 (phase shift)	Horizontal shift
C (vertical shift)	Baseline/center line

Sinusoid parameters

$$f(t) = A \sin\left(\frac{2\pi}{P}(t - t_0)\right) + C$$

Parameter	**Meaning**
A (amplitude)	Height of oscillation from center
P (period)	Time for one complete cycle
t_0 (phase shift)	Horizontal shift
C (vertical shift)	Baseline/center line

Frequency: $f = 1/P$ (cycles per unit time)
Angular frequency: $\omega = 2\pi/P = 2\pi f$ (radians per unit time)

Sinusoids in nature

Sound waves:

$$p(t) = A\sin(2\pi f t)$$

Middle C: $f = 262$ Hz (cycles per second)

Alternating current:

$$V(t) = V_0 \sin(2\pi \cdot 60 \cdot t)$$

US household electricity: 60 Hz

Tides:

$$h(t) = \bar{h} + A\sin\left(\frac{2\pi}{12.42}t\right)$$

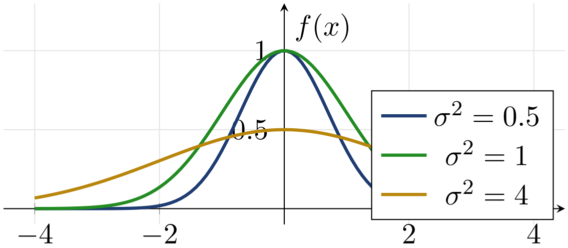
Period $\approx$ 12.42 hours (lunar day / 2)

Seasons: Temperature varies sinusoidally with period $=$ 1 year

Definition

$$f(x) = e^{-x^2}$$

General form: $f(x) = A \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$



The normal distribution

When properly normalized, the Gaussian becomes the **Normal Distribution**:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The normal distribution

When properly normalized, the Gaussian becomes the **Normal Distribution**:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Parameters:

- μ = mean (center of the bell)
- σ = standard deviation (width)
- σ^2 = variance

The normal distribution

When properly normalized, the Gaussian becomes the **Normal Distribution**:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Parameters:

- μ = mean (center of the bell)
- σ = standard deviation (width)
- σ^2 = variance

The 68-95-99.7 Rule:

- 68% of data within $\mu \pm \sigma$
- 95% of data within $\mu \pm 2\sigma$
- 99.7% of data within $\mu \pm 3\sigma$

Why Gaussians appear everywhere

Central Limit Theorem

The sum (or average) of many independent random variables tends toward a Normal distribution, regardless of the original distributions.

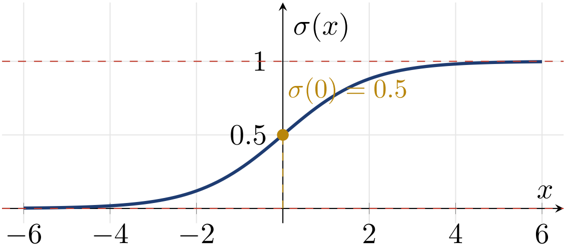
This explains:

- Heights of people (sum of many genetic factors)
- Measurement errors (sum of many small errors)
- Stock price changes (sum of many trades)
- Test scores (sum of many question responses)

8. The sigmoid

Definition: Logistic Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$$



Sigmoid properties

Key properties:

- **Range**: $(0, 1)$—perfect for probabilities!
- **Symmetry**: $\sigma(-x) = 1 - \sigma(x)$
- **Center**: $\sigma(0) = 0.5$
- **Limits**: $\lim_{x \to -\infty} \sigma(x) = 0$, $\lim_{x \to \infty} \sigma(x) = 1$

Sigmoid properties

Key properties:

- **Range**: $(0, 1)$—perfect for probabilities!
- **Symmetry**: $\sigma(-x) = 1 - \sigma(x)$
- **Center**: $\sigma(0) = 0.5$
- **Limits**: $\lim_{x \to -\infty} \sigma(x) = 0$, $\lim_{x \to \infty} \sigma(x) = 1$

Derivative (important for ML!):

$$\sigma'(x) = \sigma(x)(1 - \sigma(x))$$

Maximum slope at $x = 0$ where $\sigma'(0) = 0.25$.

Sigmoids in machine learning

The sigmoid is **fundamental to modern AI**:

Logistic Regression:

$$P(\text{spam}|\text{email}) = \sigma(\beta_0 + \beta_1 x_1 + \cdots + \beta_n x_n)$$

Convert any linear combination into a probability!

Sigmoids in machine learning

The sigmoid is **fundamental to modern AI**:

Logistic Regression:

$$P(\text{spam}|\text{email}) = \sigma(\beta_0 + \beta_1 x_1 + \cdots + \beta_n x_n)$$

Convert any linear combination into a probability!

Neural Networks:

- Sigmoid was the classic activation function
- Squashes any input to $(0, 1)$
- Modern alternatives: ReLU, tanh, softmax

Sigmoids in machine learning

The sigmoid is **fundamental to modern AI**:

Logistic Regression:

$$P(\text{spam}|\text{email}) = \sigma(\beta_0 + \beta_1 x_1 + \cdots + \beta_n x_n)$$

Convert any linear combination into a probability!

Neural Networks:

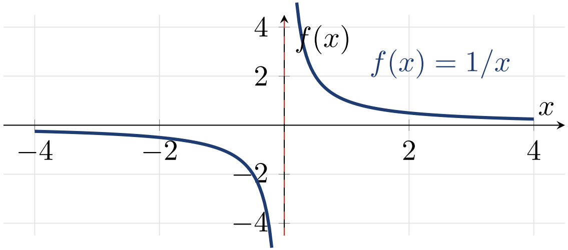
- Sigmoid was the classic activation function
- Squashes any input to $(0, 1)$
- Modern alternatives: ReLU, tanh, softmax

When ChatGPT predicts the next word, it uses softmax (a multi-class generalization of sigmoid) to convert scores into probabilities.

9. The reciprocal

Definition

$$f(x) = \frac{1}{x} = x^{-1}$$



Reciprocal: properties and applications

Properties:

- Domain: $x \neq 0$
- Vertical asymptote at $x = 0$
- Horizontal asymptote: $y = 0$
- Odd function: $f(-x) = -f(x)$

Reciprocal: properties and applications

Properties:

- Domain: $x \neq 0$
- Vertical asymptote at $x = 0$
- Horizontal asymptote: $y = 0$
- Odd function: $f(-x) = -f(x)$

Applications:

- **Speed-time tradeoff**: If $d = st$, then $t = d/s = d \cdot s^{-1}$
- **Parallel resistors**: $\frac{1}{R_{\text{total}}} = \frac{1}{R_1} + \frac{1}{R_2}$
- **Lens equation**: $\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$
- **Harmonic series**: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

Pattern-book summary

Function	Formula	Key Feature	Models...
Constant	c	Flat line	Baselines
Line	$mx + b$	Straight line	Proportional change
Power	x^n	Curvature varies	Scaling, geometry
Exponential	e^{kx}	Constant % growth	Growth, decay
Logarithm	$\ln(x)$	Inverse of exp	Compressed scales
Sinusoid	$\sin(\omega t)$	Periodic	Oscillation, cycles
Hump	e^{-x^2}	Bell shape	Distributions, peaks
Sigmoid	$1/(1 + e^{-x})$	S-curve	Transitions
Reciprocal	$1/x$	Hyperbola	Inverse relations

These are your **building blocks**. Future: how to combine them!

Part IV

Describing Functions

Function-shape vocabulary

Before building customized functions, we need a vocabulary for describing function features.

Function-shape vocabulary

Before building customized functions, we need a vocabulary for describing function features.

Seven function-shape concepts:

- Slope
- Concavity
- Continuity
- Monotonicity
- Periodicity
- Asymptotic behavior
- Local extrema

Function-shape vocabulary

Before building customized functions, we need a vocabulary for describing function features.

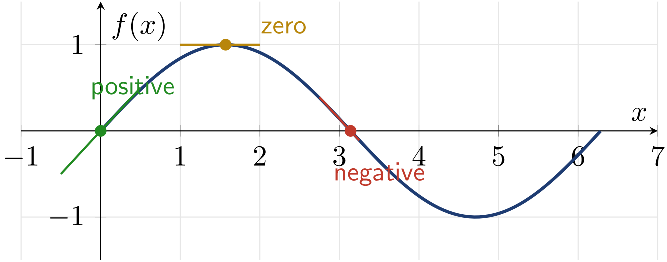
Seven function-shape concepts:

- Slope
- Concavity
- Continuity
- Monotonicity
- Periodicity
- Asymptotic behavior
- Local extrema

Each concept describes how the function output changes as the input changes.

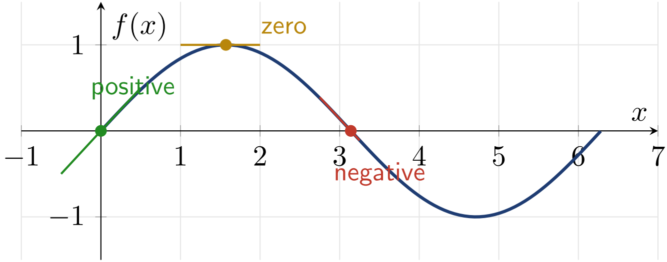
1. Slope

Slope describes whether the output goes up or down, and the extent of this rise or fall, as the input changes.



1. Slope

Slope describes whether the output goes up or down, and the extent of this rise or fall, as the input changes.

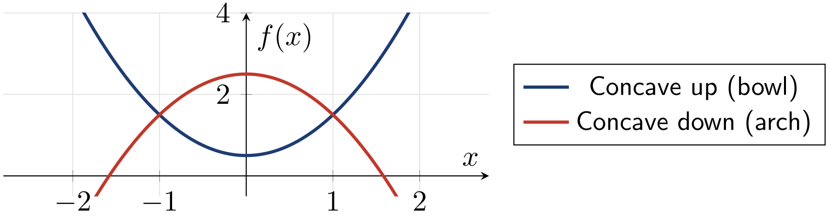


Key points:

- Slope can be positive (increasing), negative (decreasing), or zero (flat)
- For most functions, slope varies with the input value
- Slope measures the rate of change of the function
- Slope $= \frac{\Delta y}{\Delta x} =$ rise over run

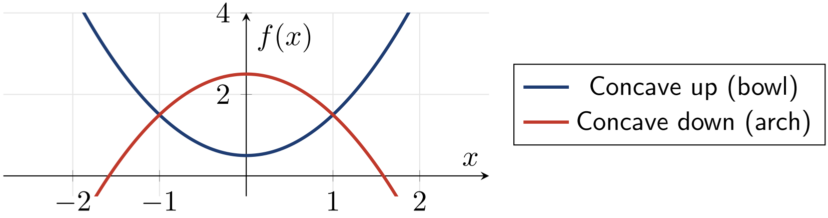
2. Concavity

Concavity describes the change in slope—whether the slope is increasing or decreasing.



2. Concavity

Concavity describes the change in slope—whether the slope is increasing or decreasing.

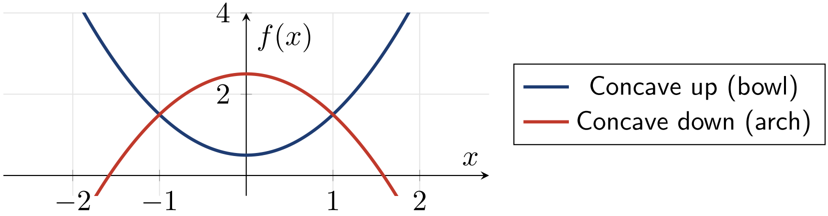


Two types:

- **Concave up**: Slope is increasing (function curves upward like a bowl)
- **Concave down**: Slope is decreasing (function curves downward like an arch)

2. Concavity

Concavity describes the change in slope—whether the slope is increasing or decreasing.



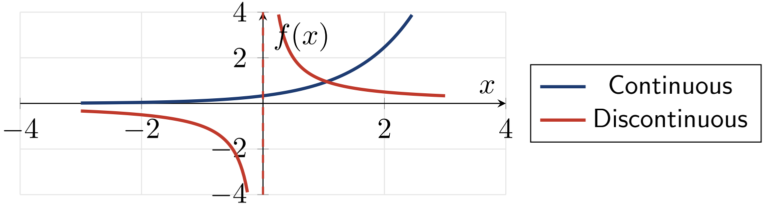
Two types:

- **Concave up**: Slope is increasing (function curves upward like a bowl)
- **Concave down**: Slope is decreasing (function curves downward like an arch)

Key insight: Concavity is about the *change* in slope, not the slope

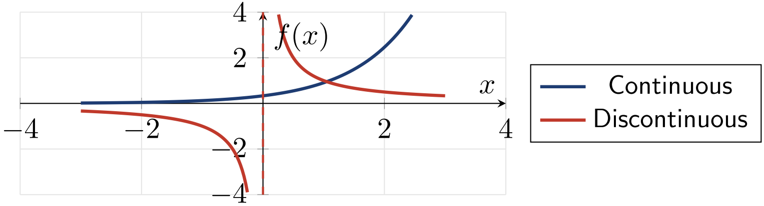
3. Continuity

A function is **continuous** if you can trace its graph without lifting your pencil.



3. Continuity

A function is **continuous** if you can trace its graph without lifting your pencil.

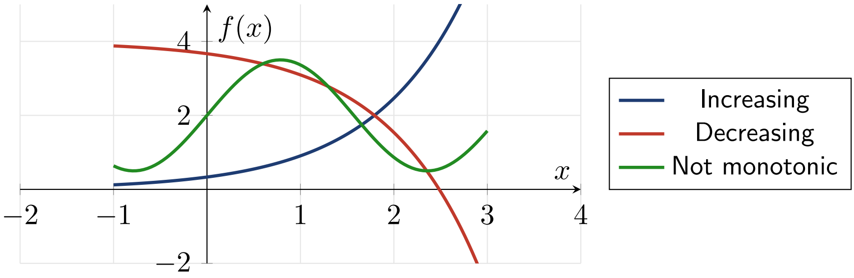


Key points:

- All pattern-book functions are continuous on their domains
- Exception: Power functions with negative exponents (like $1/x = x^{-1}$) are not defined at $x = 0$
- On any interval that doesn't include 0, the reciprocal function is continuous

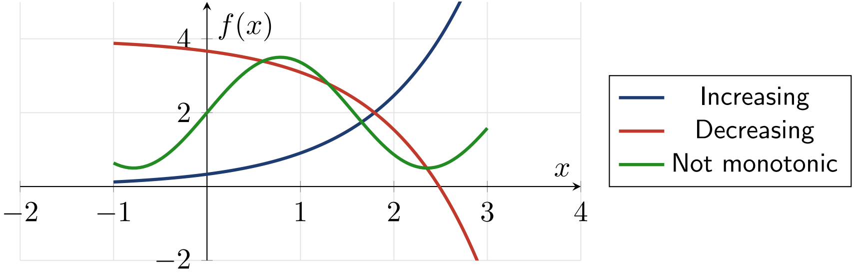
4. Monotonicity

A function is **monotonic** on a domain when the *sign* of the slope never changes.



4. Monotonicity

A function is **monotonic** on a domain when the *sign* of the slope never changes.

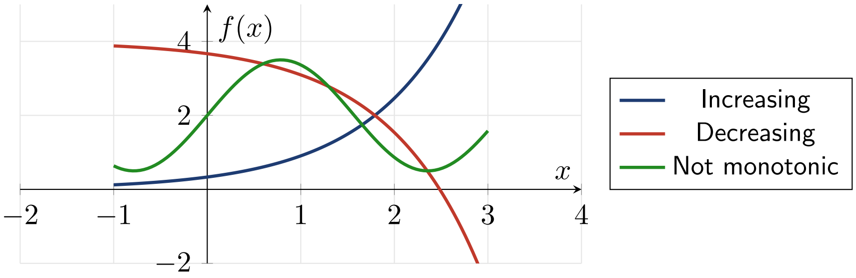


Two types:

- **Monotonically increasing**: Function steadily increases (slope always positive or zero)
- **Monotonically decreasing**: Function steadily decreases (slope always negative or zero)

4. Monotonicity

A function is **monotonic** on a domain when the *sign* of the slope never changes.

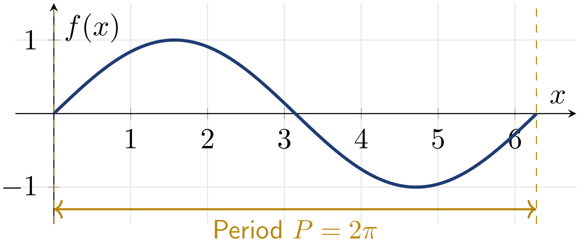


Two types:

- **Monotonically increasing**: Function steadily increases (slope always positive or zero)
- **Monotonically decreasing**: Function steadily decreases (slope always negative or zero)

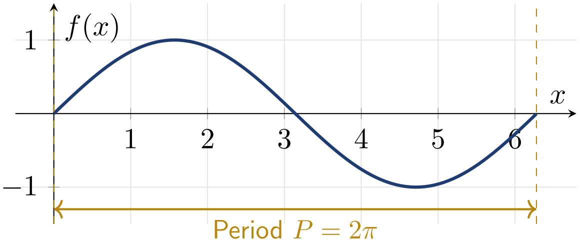
5. Periodicity

A phenomenon is **periodic** if it repeats a pattern over and over again.



5. Periodicity

A phenomenon is **periodic** if it repeats a pattern over and over again.

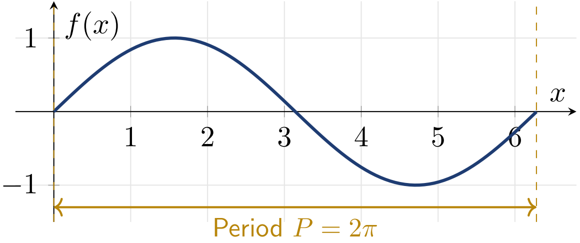


Key concepts:

- **Cycle**: The pattern that is repeated
- **Period**: The duration of one complete cycle
- **Frequency**: Number of cycles per unit time $(f = 1/P)$

5. Periodicity

A phenomenon is **periodic** if it repeats a pattern over and over again.



Key concepts:

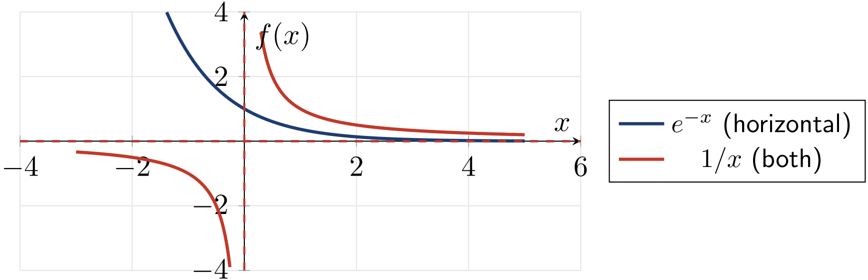
- **Cycle**: The pattern that is repeated
- **Period**: The duration of one complete cycle
- **Frequency**: Number of cycles per unit time ($f = 1/P$)

Examples:

- Day-night cycle: Period $=$ 24 hours
- Seasons: Period $=$ 1 year

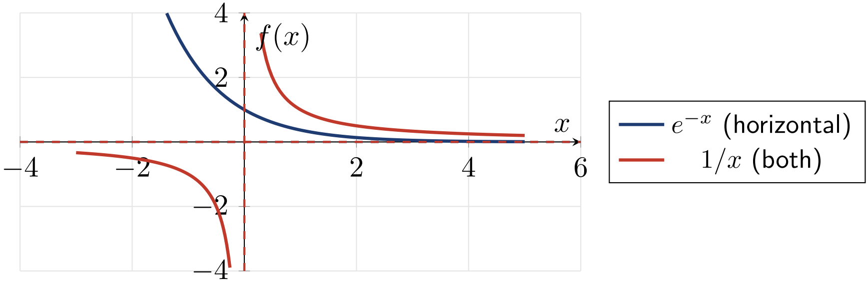
6. Asymptotic behavior

Asymptotic behavior describes what happens as inputs or outputs get very large.



6. Asymptotic behavior

Asymptotic behavior describes what happens as inputs or outputs get very large.

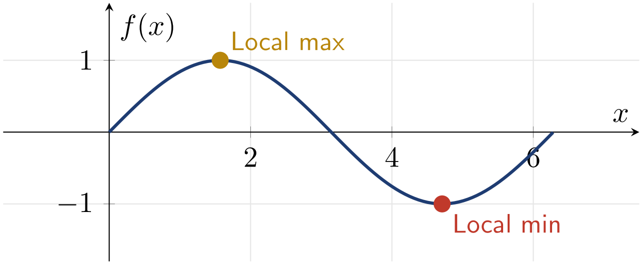


Two types:

- **Horizontal asymptote**: As input $\to \pm\infty$, output approaches a specific value
- **Vertical asymptote**: Output $\to \pm\infty$ while input changes only a little

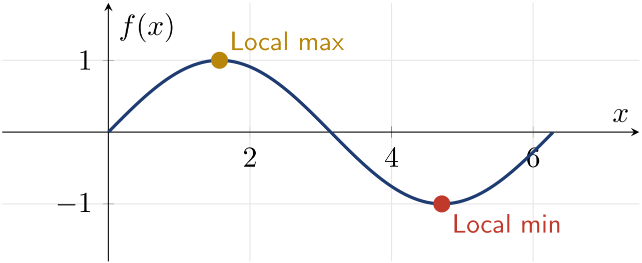
7. Local extrema

Local extrema are peaks (maxima) and valleys (minima) in a function.



7. Local extrema

Local extrema are peaks (maxima) and valleys (minima) in a function.

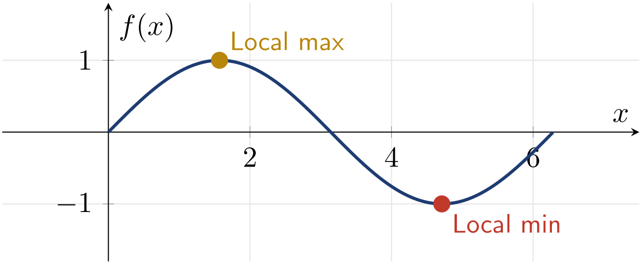


Key concepts:

- **Local maximum**: A peak where output is larger than nearby values
- **Local minimum**: A valley where output is smaller than nearby values
- **Argmax**: The input value where the function reaches its maximum
- **Argmin**: The input value where the function reaches its minimum

7. Local extrema

Local extrema are peaks (maxima) and valleys (minima) in a function.



Key concepts:

- **Local maximum**: A peak where output is larger than nearby values
- **Local minimum**: A valley where output is smaller than nearby values
- **Argmax**: The input value where the function reaches its maximum
- **Argmin**: The input value where the function reaches its minimum

Optimization

The process of finding an argmin or argmax is called **optimization**.

Optimization

The process of finding an argmin or argmax is called **optimization**.

Applications:

- **Business**: Set prices to maximize profit
- **Engineering**: Find speed that minimizes fuel consumption
- **Science**: Find conditions that maximize efficiency

Optimization

The process of finding an argmin or argmax is called **optimization**.

Applications:

- **Business**: Set prices to maximize profit
- **Engineering**: Find speed that minimizes fuel consumption
- **Science**: Find conditions that maximize efficiency

Collectively, maxima and minima are called **extrema**.

Optimization

The process of finding an argmin or argmax is called **optimization**.

Applications:

- **Business**: Set prices to maximize profit
- **Engineering**: Find speed that minimizes fuel consumption
- **Science**: Find conditions that maximize efficiency

Collectively, maxima and minima are called **extrema**.

Key principle: At an extremum, the slope is zero (or undefined).

Describing functions: summary

Concept	Describes...
Slope	Whether output goes up or down
Concavity	Change in slope (curvature)
Continuity	Whether graph can be drawn without lifting pencil
Monotonicity	Whether slope sign never changes
Periodicity	Whether pattern repeats
Asymptotic behavior	Behavior as inputs/outputs get very large
Local extrema	Peaks and valleys

These concepts help us communicate about function behavior and design functions for modeling.

What comes next

Next: Modeling and Assembling Functions

- How to combine pattern-book functions (sums, products, composition)
- Parameters and what they control
- Fitting models to real data
- Dimensional analysis: Why units matter

"All models are wrong, but some are useful."
— George Box, statistician

Practice: explore in R

```r
# Compare exponential and power growth
slice_plot(exp(x) ~ x, domain(x = 0:5), color="blue"
    ) %>%
slice_plot(x^2 ~ x, color="red") %>%
slice_plot(x^3 ~ x, color="green")
# Which dominates for large x?

# Explore the sigmoid
s <- makeFun(1/(1 + exp(-k*x)) ~ x, k = 1)
slice_plot(s(x, k=0.5) ~ x, domain(x=-10:10)) %>%
slice_plot(s(x, k=1) ~ x) %>%
slice_plot(s(x, k=2) ~ x)
# What does k control?
```

Practice problems

1. Find the domain of $f(x) = \ln(x^2 - 4)$.

2. If a population doubles every 5 years, write it as $P(t) = P_0 e^{kt}$ and find k.

3. A sinusoid has maximum value 10, minimum value 2, and period 6. Write its equation in the form $f(t) = A \sin\left(\frac{2\pi}{P} t\right) + C$.

4. Sketch $f(x) = e^{-x^2}$ and $g(x) = e^{-|x|}$. How do they differ?

5. Show that $\sigma(x) + \sigma(-x) = 1$ for the sigmoid $\sigma(x) = \frac{1}{1+e^{-x}}$.