Functions, Computing & Pattern-Book Functions

David Puelz
The University of Austin

Course goals

- Understand calculus as the mathematics of change and accumulation
- Build mathematical models using functions, derivatives, and optimization
- Use computing (R) to explore calculus concepts and solve real-world problems

Who am I?

- Educator: Teaching at UATX since 2024, UT Austin from 2020-2024, graduate student from 2013-2018. Previously at the University of Chicago Booth School of Business and Goldman Sachs.
- Researcher: I develop new statistical/machine learning methods for analyzing complex data, with applications in policy, economics, and the social sciences.
- Human: Grew up in Dallas. Husband and dad with 2 girls + 1 boy + 1 dog. I like working out, smoking meat, golf, and cycling.

Answer two questions for me:

What is one thing you're *excited* about for this class?

What is one thing you're *nervous or concerned* about for this class?

Pass around your answers. Another student will read your responses, so everyone stays anonymous.

Populi & Github

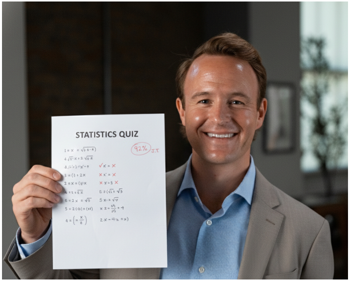
- Access at `uaustin.populiweb.com`
- Access at `https://github.com/dpuelz/Calculus`
- Github pages will contain schedule, homeworks, data, code, ...
- Github will be our main landing page. Populi will be used for grades and administrative tasks.

Class structure

- Understanding the concepts really only comes from practice
- Class time will be mostly lecture and discussion. We will also have "in-class reps" on Quiz reviews and exercises.

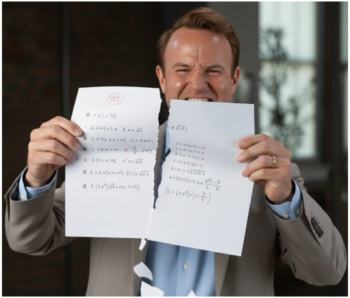
Quizzes

- 5 quizzes, taken during class on the 2nd, 4th, 6th, 8th, and 10th Friday
- Assessing your knowledge of lecture material up to that point
- Concepts assessed will come directly from the homeworks



Homework

- Why homework?
- 5 homework assignments, due on the 2nd, 4th, 6th, 8th, and 10th Friday (like the quizzes)
- Homeworks can be completed individually or in groups of up to three people. They must be turned in by the start of each Friday class. Names of all group members must be included on the writeups.
- I expect professional writeups submitted in pdf format



A pedagogical note on AI

LLMs can be useful for the homeworks and learning concepts. I encourage their use! And ... I will not police their use. I only ask that you cite when LLMs are used.

However, if overused & relied on too heavily, you will not learn anything.

The quizzes are designed from the homeworks and will assess how well you understand the concepts. Through the quizzes, you will get feedback quickly on whether you've overused AI! Use this feedback to optimize your learning with AI.

Exam

- The exam will be during the 11th week (final exam week). There are no make-ups or alternative dates, so please plan accordingly!
- It will be a free response, written exam taken with pen and paper (exactly like our quizzes)

Grading

Homework	**20%**
Exam	**30%**
Quizzes	**50%**

Computing with R

- We will use R for mathematical computation throughout the course
- You are definitely welcome to use Python, but the course materials will (mostly) be in R
- This is industrial-strength, state-of-the-art, and free software for mathematical computing
- We will access R through the RStudio graphical interface; make sure both are installed on your laptop and bring it to every class

Getting help

- My office hours
- Schedule an one-on-one time with me to chat (Zoom or in-person)

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- How things grow, decay, and oscillate
- The language of science, engineering, economics, and AI
- A toolkit for building **models** of the real world

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Calculus is more than just computing derivatives and integrals.
It's about understanding relationships between quantities.

The two big ideas

1. **Differentiation**: How fast is something changing *right now*?
   - Instantaneous rates of change
   - Slopes, velocities, sensitivities

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   - Applications in science, engineering, economics

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Calculus gives us the tools to find optimal solutions and understand how systems behave.

Why this course is different

Traditional calculus: Symbolic manipulation, memorization

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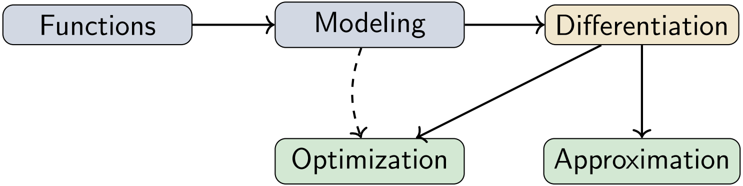
This course (MOSAIC approach): Modeling, computing, understanding

- Functions as the **central object**—not formulas
- Emphasis on why and when, not just how
- Computing (R) as a tool for exploration
- Real applications

Course roadmap



Block I: Functions and Modeling
Block II: Differentiation
Block III: Optimization and Approximation

Part I

Functions: The Language of Relationships

What is a function?

Definition: Function

A **function** f is a rule that assigns to each element x in a set D (the *domain*) exactly one element $f(x)$ in a set R (the *range*).

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Key insight: A function is a **relationship**, not just a formula.

Examples from your daily life:

- `temperature`(time of day) $\to$ degrees Fahrenheit
- `Uber_price`(distance, surge_multiplier) $\to$ dollars
- `GPA`(hours_studied, natural_ability) $\to$ number

What is NOT a function?

A relation is not a function if one input maps to **multiple outputs**.

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Examples of non-functions:

- $x^2 + y^2 = 1$ (circle: for $x = 0$, we get $y = \pm 1$)
- $y^2 = x$ (parabola opening right: for $x = 4$, we get $y = \pm 2$)
- A relation where one person has multiple phone numbers

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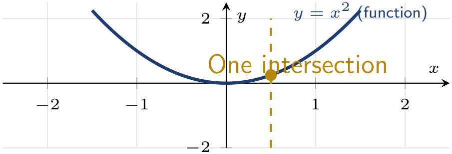
Key Principle

For a function, each input must produce **exactly one** output.

The vertical line test

A graphical way to check if a relation is a function:



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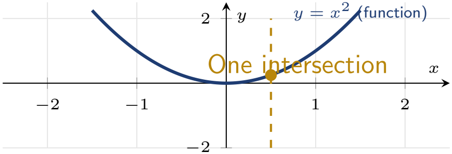
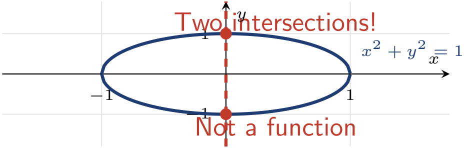


Vertical Line Test: If any vertical line intersects the graph in **more than one point**, the relation is not a function.

Non-function example: Circle



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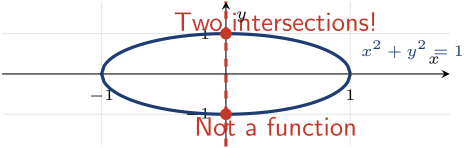
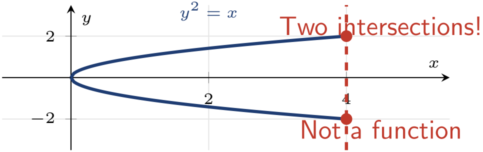


Why it's not a function: For $x = 0$, we get $y = 1$ **and** $y = -1$. One input $\rightarrow$ two outputs!

Non-function example: Sideways parabola



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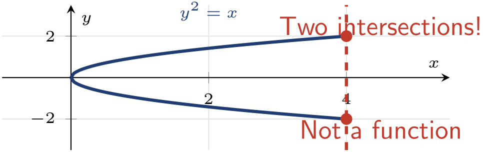

Why it's not a function: For $x = 4$, we get $y = 2$ **and** $y = -2$. One input $\to$ two outputs!

Making a non-function into a function

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Example: The circle $x^2 + y^2 = 1$ is not a function, but we can define:

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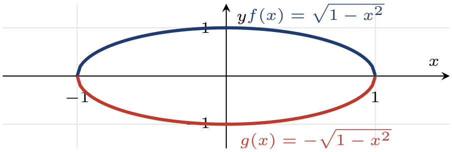

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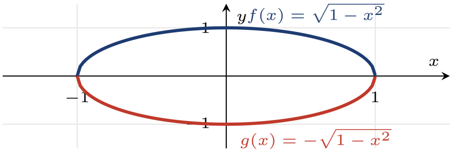

Both f and g are functions! Each input gives exactly one output.

Function notation: the details

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Important: f and $f(x)$ are different!

- f is the function itself (the "machine")
- $f(x)$ is the output when you feed in x

Evaluating functions

Example: Let $f(x) = x^2 - 3x + 2$

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Evaluate at specific points:

$$\begin{aligned} f(0) &= 0^2 - 3(0) + 2 = 2 \\ f(1) &= 1^2 - 3(1) + 2 = 0 \\ f(2) &= 2^2 - 3(2) + 2 = 0 \\ f(-1) &= (-1)^2 - 3(-1) + 2 = 6 \end{aligned}$$

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\end{aligned}
$$

Evaluate at an expression:

$$f(a+h) = (a+h)^2 - 3(a+h) + 2$$

$$= a^2 + 2ah + h^2 - 3a - 3h + 2$$

Functions with multiple inputs

Real-world relationships often involve **multiple inputs**:

A function of two variables: $z = f(x, y)$
A function of n variables: $w = f(x_1, x_2, \ldots, x_n)$

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Examples:

- Area of a rectangle: $A(w, h) = w \cdot h$
- Ideal gas law: $P(V, T, n) = \frac{nRT}{V}$
- Mortgage payment: $M(P, r, n) = P \cdot \frac{r(1+r)^n}{(1+r)^n - 1}$

Example: the Cobb-Douglas production function

In economics, how do labor (L) and capital (K) combine to produce output (Y)?

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- $1 - \alpha$ = capital's share

Key property: Constant returns to scale

$$f(tL, tK) = A(tL)^{\alpha}(tK)^{1-\alpha} = t \cdot AL^{\alpha}K^{1-\alpha} = t \cdot f(L, K)$$

Double both inputs $\Rightarrow$ double the output!

Domain and range: formal definitions

Definition: Domain

The **domain** of a function f is the set of all inputs x for which $f(x)$ is defined.

Definition: Range

The **range** of a function f is the set of all possible outputs: $\{f(x) : x \in \text{domain}\}$.

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Common domain restrictions:

- Division by zero: $f(x) = \frac{1}{x}$ excludes $x = 0$
- Square roots of negatives: $f(x) = \sqrt{x}$ requires $x \geq 0$
- Logarithms: $f(x) = \ln(x)$ requires $x > 0$

Domain and range: examples

Example 1: $f(x) = \sqrt{x-2}$

- Domain: $x - 2 \geq 0 \Rightarrow x \geq 2$, i.e., $[2, \infty)$
- Range: $[0, \infty)$

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Example 2: $g(x) = \frac{1}{x^2-4}$

- Domain: $x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2$
- Domain $= (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
- Range: $(-\infty, -1/4] \cup (0, \infty)$

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Example 3: $h(x) = \ln(x^2)$

- Domain: $x^2 > 0 \Rightarrow x \neq 0$
- Domain $= (-\infty, 0) \cup (0, \infty)$
- Range: $(-\infty, \infty)$ (since x^2 can be any positive number, and $\ln(x^2)$ can be any real number)

Domain and range: practice problems

Find the domain and range for each function:

1. $f(x) = \sqrt{3x+1}$
2. $g(x) = \frac{1}{x-5}$
3. $h(x) = \ln(x+2)$
4. $j(x) = \frac{x}{x^2-9}$
5. $k(x) = \sqrt{4-x^2}$
6. $m(x) = \frac{1}{\sqrt{x-1}}$
7. $n(x) = \ln(x^2-4)$
8. $p(x) = \frac{x+1}{x^2+1}$
9. $q(x) = \sqrt{x^2-5x+6}$
10. $r(x) = \frac{1}{\ln(x)}$

Domain and range: solutions

1. $f(x) = \sqrt{3x+1}$
Domain: $3x + 1 \geq 0 \Rightarrow x \geq -1/3$, so $[-1/3, \infty)$
Range: $[0, \infty)$

2. $g(x) = \frac{1}{x-5}$
Domain: $x \neq 5$, so $(-\infty, 5) \cup (5, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

3. $h(x) = \ln(x+2)$
Domain: $x + 2 > 0 \Rightarrow x > -2$, so $(-2, \infty)$
Range: $(-\infty, \infty)$

4. $j(x) = \frac{x}{x^2-9}$
Domain: $x^2 - 9 \neq 0 \Rightarrow x \neq \pm 3$, so $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Range: All real numbers

5. $k(x) = \sqrt{4 - x^2}$
Domain: $4 - x^2 \geq 0 \Rightarrow -2 \leq x \leq 2$, so $[-2, 2]$
Range: $[0, 2]$

Domain and range: solutions (continued)

6. $m(x) = \frac{1}{\sqrt{x-1}}$
Domain: $x - 1 > 0 \Rightarrow x > 1$, so $(1, \infty)$
Range: $(0, \infty)$

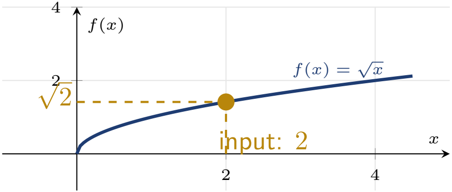
7. $n(x) = \ln(x^2 - 4)$
Domain: $x^2 - 4 > 0 \Rightarrow |x| > 2$, so $(-\infty, -2) \cup (2, \infty)$
Range: $(-\infty, \infty)$

8. $p(x) = \frac{x+1}{x^2+1}$
Domain: $x^2 + 1 \neq 0$ for all x, so $(-\infty, \infty)$
Range: $[-1/2, 1/2]$ (requires calculus to find exactly)

9. $q(x) = \sqrt{x^2 - 5x + 6}$
Domain: Factor: $(x-2)(x-3) \geq 0$, so $(-\infty, 2] \cup [3, \infty)$
Range: $[0, \infty)$

10. $r(x) = \frac{1}{\ln(x)}$
Domain: $\ln(x) \neq 0$ and $x > 0 \Rightarrow x \neq 1$ and $x > 0$, so $(0, 1) \cup (1, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

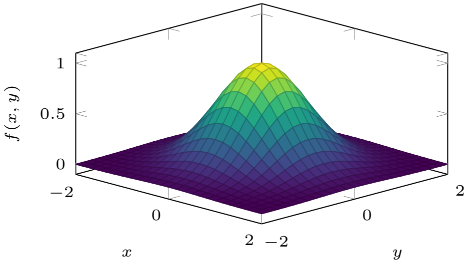
Visualizing functions: one input



A function of **one input** is visualized as a curve in 2D.

Visualizing functions: two inputs

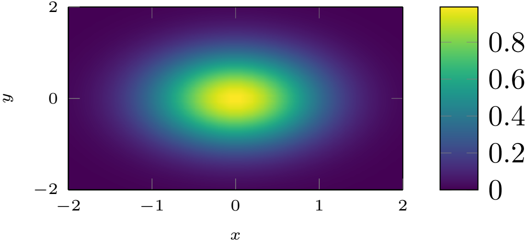
A function of **two inputs** is a surface in 3D space:



This is the **Gaussian** $f(x, y) = e^{-(x^2+y^2)}$ — a 2D bell curve!

Contour plots: flattening 3D to 2D

A **contour plot** shows level curves where $f(x, y) = c$ (constant).



Interpretation: Like a topographic map—each color band represents a height.

Real-world contour example: weather maps

Pressure isobars are contour lines of atmospheric pressure!

- Each line connects points of equal pressure
- Closely spaced lines = rapid pressure change = strong winds
- **Gradients** (next topic) point from low to high pressure

Other examples of contours:

- Elevation on hiking maps
- Temperature isotherms
- Indifference curves in economics

Part II

Computing with R

Why we compute

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- Computers let us work with **real data**
- Modern science and industry *require* computational thinking

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Philosophy: We use R not to avoid mathematics, but to *deepen* our mathematical understanding through exploration.

Why R?

- **Free and open source**—runs anywhere
- Designed for **data analysis and statistics**
- Excellent **graphics** capabilities
- Used by scientists, economists, data analysts worldwide
- The `mosaicCalc` package provides calculus-specific tools

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Alternatives exist: Python (with NumPy/SciPy), MATLAB, Julia
The concepts transfer—learning one makes learning others easy.

Defining functions in R

In R, we define functions with the `makeFun()` command:

```
# Define a quadratic function
f <- makeFun(x^2 - 3*x + 2 ~ x)

# Evaluate it at specific points
f(0)    # returns 2
f(1)    # returns 0
f(5)    # returns 12
```

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f(5)    # returns 12
```

The `~` (tilde) is read as "*is a function of*":

$$\underbrace{x^2 - 3x + 2}_{\text{output expression}} \quad \sim \quad \underbrace{x}_{\text{input variable}}$$

Functions with parameters

Parameters are values we might want to change:

```
# Exponential growth with parameters
growth <- makeFun(A * exp(k*t) ~ t, A = 100, k =
    0.1)

# Use default parameters
growth(5)     # A=100, k=0.1, t=5

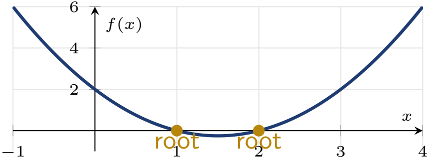
# Override parameters
growth(5, A = 200)          # different starting
    value
growth(5, k = 0.2)          # faster growth rate
growth(5, A = 50, k = 0.05) # both changed
```

This lets us explore: "What if the growth rate were different?"

Plotting functions: `slice_plot`

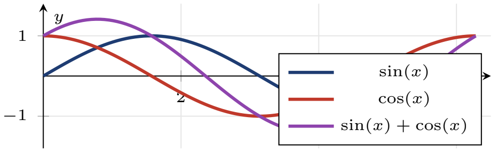
Visualization is **essential** for understanding:

```
# Plot the function over a domain
slice_plot(x^2 - 3*x + 2 ~ x, domain(x = -1:4))
```



Multiple functions on one plot

```
slice_plot(sin(x) ~ x, domain(x = 0:2*pi),
           color = "blue") %>%
slice_plot(cos(x) ~ x, color = "red") %>%
slice_plot(sin(x) + cos(x) ~ x, color = "purple")
```



Functions of two variables

```r
# Define Cobb-Douglas production function
production <- makeFun(A * L^alpha * K^(1-alpha)
                      ~ L & K,
                      A = 1, alpha = 0.7)

# Evaluate: 100 workers, 50 machines
production(L = 100, K = 50)
```

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# Define Cobb-Douglas production function
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                      ~ L & K,
                      A = 1, alpha = 0.7)

# Evaluate: 100 workers, 50 machines
production(L = 100, K = 50)
```

Note: Use & to separate multiple input variables.

The **domain** for 2D functions:

```
domain(L = 1:100, K = 1:100)
```

Surface plots and contour plots

```r
# 3D surface plot
surface_plot(exp(-(x^2 + y^2)) ~ x & y,
             domain(x = -2:2, y = -2:2))

# 2D contour plot
contour_plot(exp(-(x^2 + y^2)) ~ x & y,
             domain(x = -2:2, y = -2:2))
```

Contour plot interpretation:

- Lines close together = steep surface (rapid change)
- Lines far apart = flat surface (slow change)
- Closed loops = peaks or valleys

Finding special values

R can help us find roots, maxima, and other special points:

```
# Find where f(x) = 0
f <- makeFun(x^2 - 3*x + 2 ~ x)
Zeros(f, domain(x = -5:5))
# Returns: x = 1, x = 2

# Find maximum of a function
g <- makeFun(-x^2 + 4*x ~ x)
argM(g, domain(x = -5:5))   # x value at maximum
# Returns: x = 2
```

These numerical tools complement (not replace) analytical thinking!

Part III

The Pattern-Book Functions

The modeling toolkit

Just as writers use a vocabulary of words, modelers use a vocabulary of **basic functions**.

Key Insight: Almost any real-world relationship can be modeled by combining just **nine basic functions**.

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Key Insight: Almost any real-world relationship can be modeled by combining just **nine basic functions**.

These pattern-book functions form your modeling toolkit:

- Exponential
- Logarithm
- Sinusoid
- Square
- Identity
- One
- Reciprocal
- Gaussian
- Sigmoid

Table of Pattern-Book Functions

Name	Traditional notation	R notation
exponential	e^x	`exp(x)`
logarithm ("natural log")	$\ln(x)$	`log(x)`
sinusoid	$\sin(x)$	`sin(x)`
square	x^2	`x^2`
identity	x	`x`
one	1	`1`
reciprocal	$1/x$ or x^{-1}	`1/x`
gaussian	e^{-x^2}	`exp(-x^2)`
sigmoid	$\frac{1}{1+e^{-x}}$	`1/(1 + exp(-x))`

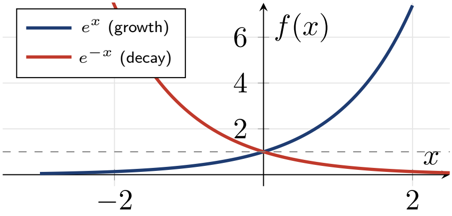
These are the pattern-book functions used in *MOSAIC Calculus*.

1. The exponential function

Definition

$$f(x) = e^x \quad \text{where } e \approx 2.71828...$$

More generally: $f(x) = a^x$ for any base $a > 0$



Why e? The magic number

What makes $e \approx 2.71828...$ special?

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Key Property

The exponential e^x is **its own derivative**:

$$\frac{d}{dx}e^x = e^x$$

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Definition via compound interest:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

If you invest \$1 at 100% interest compounded n times per year, as $n \to \infty$, you get \$$e$.

The exponential law of growth

Fundamental Principle

If something grows at a rate *proportional to its current size*, it grows exponentially:

$$\frac{dN}{dt} = kN \quad \Rightarrow \quad N(t) = N_0 e^{kt}$$

Growth $(k > 0)$: Population, compound interest, viral spread

Decay $(k < 0)$: Radioactive decay, drug metabolism, cooling

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Doubling time: $T_d = \frac{\ln 2}{k} \approx \frac{0.693}{k}$

Half-life: $T_{1/2} = \frac{\ln 2}{|k|}$ (for decay)

Doubling time: concept and formula

Doubling time T_d: The time it takes for a quantity to double in size.

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Why this formula?

$$\begin{aligned} 2N_0 &= N_0 e^{kT_d} \\ 2 &= e^{kT_d} \\ \ln(2) &= kT_d \\ T_d &= \frac{\ln(2)}{k} \end{aligned}$$

Doubling time example 1: COVID-19 cases

Scenario: Early pandemic data showed cases doubling every 3 days.

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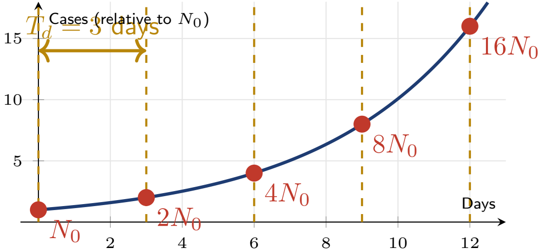

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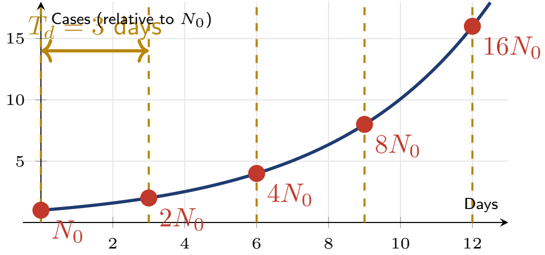

After 30 days: $N(30) = N_0 e^{0.231 \times 30} \approx 1024 N_0$ (over 1000× increase!)

Doubling time example 2: Bacterial growth

Scenario: A bacterial culture doubles every 20 minutes. Starting with 1000 cells:

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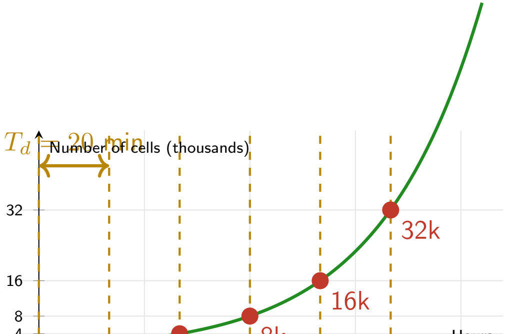

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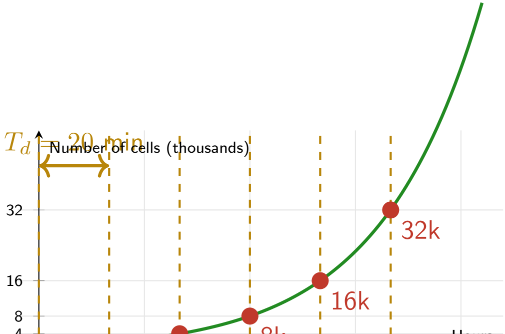

Doubling time: key insights

Three ways to specify exponential growth:

- **Growth rate** k: "Grows at 23% per day"
- **Doubling time** T_d: "Doubles every 3 days"
- **Percentage growth**: "Grows by 100% every T_d"

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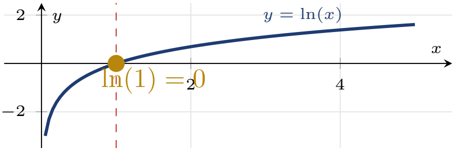
Key insight: Doubling time is independent of the starting value N_0! The time to double depends only on the growth rate k.

2. The logarithm

Definition

The **natural logarithm** $\ln(x)$ is the *inverse* of e^x:

$$y = \ln(x) \quad \Leftrightarrow \quad x = e^y$$



Logarithm properties

Key Properties

$$\ln(ab) = \ln(a) + \ln(b) \quad \text{(products become sums)}$$
$$\ln(a/b) = \ln(a) - \ln(b) \quad \text{(quotients become differences)}$$
$$\ln(a^n) = n\ln(a) \quad \text{(powers become products)}$$
$$\ln(e) = 1, \quad \ln(1) = 0$$

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\ln(e) &= 1, \quad \ln(1) = 0
\end{aligned}$$

Key insight: Logarithms “undo” exponentiation.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln(x)} = x$$

Why we use log scales

Earthquake Magnitude (Richter Scale)

Magnitude	Energy (Joules)	Example
3.0	2×10^9	Minor tremor
5.0	2×10^{12}	Moderate damage
7.0	2×10^{15}	Major (Haiti 2010)
9.0	2×10^{18}	Great (Japan 2011)

Each unit increase $=$ **31.6$\times$ more energy** (because $10^{1.5} \approx 31.6$)!

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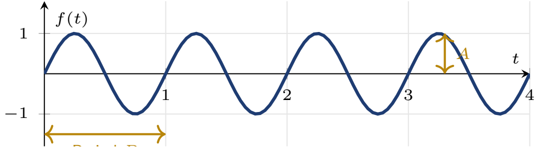
Other log scales: Decibels (sound), pH (acidity), stellar magnitude

3. The sinusoid

Definition

$$f(t) = A \sin\left(\frac{2\pi t}{P}\right) + C$$

or equivalently $f(t) = A\sin(\omega t + \phi) + C$



Sinusoid parameters

$$f(t) = A \sin\left(\frac{2\pi}{P}(t - t_0)\right) + C$$

Parameter	**Meaning**
A (amplitude)	Height of oscillation from center
P (period)	Time for one complete cycle
t_0 (phase shift)	Horizontal shift
C (vertical shift)	Baseline/center line

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Frequency: $f = 1/P$ (cycles per unit time)
Angular frequency: $\omega = 2\pi/P = 2\pi f$ (radians per unit time)

Sinusoids in nature

Sound waves:

$$p(t) = A\sin(2\pi f t)$$

Middle C: $f = 262$ Hz (cycles per second)

Alternating current:

$$V(t) = V_0 \sin(2\pi \cdot 60 \cdot t)$$

US household electricity: 60 Hz

Tides:

$$h(t) = \bar{h} + A\sin\left(\frac{2\pi}{12.42}t\right)$$

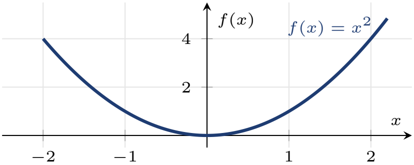
Period $\approx$ 12.42 hours (lunar day / 2)

Seasons: Temperature varies sinusoidally with period = 1 year

4. The square function

Definition

$$f(x) = x^2$$



Properties:

- Domain: $\mathbb{R}$; Range: $[0, \infty)$
- Even function: $f(-x) = f(x)$ (symmetric about y-axis)

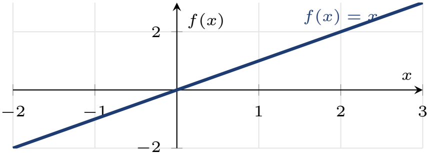
Applications:

- Area of squares: $A = s^2$ where s is side length
- Kinetic energy: $E_k = \frac{1}{2}mv^2$
- Variance in statistics: $\sigma^2 = \mathbb{E}[(X - \mu)^2]$

5. The identity function

Definition

$$f(x) = x$$



Properties:

- Domain: $\mathbb{R}$; Range: $\mathbb{R}$
- Odd function: $f(-x) = -f(x)$ (symmetric about origin)

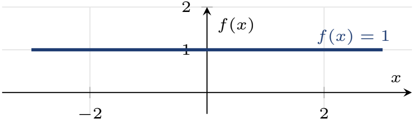
Applications:

- Linear relationships with slope 1: $y = x$
- Reference line for transformations: scaling, shifting
- Building blocks for more complex functions: compositions

6. The one function

Definition

$$f(x) = 1 \quad \text{(constant function with value 1)}$$



Properties:

- Domain: $\mathbb{R}$; Range: $\{1\}$
- Even function: $f(-x) = f(x)$ (symmetric about y-axis)

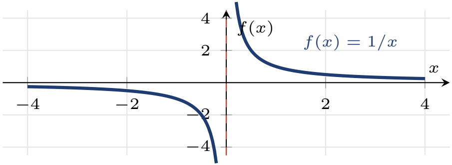
Applications:

- Normalization constants: $f(x) = 1$ as unit element
- Baseline values: reference level
- Building blocks for constant functions: $c = c \cdot 1$

7. The reciprocal function

Definition

$$f(x) = \frac{1}{x} = x^{-1}$$



Properties:

- Domain: $\mathbb{R} \setminus \{0\}$; Range: $\mathbb{R} \setminus \{0\}$
- Vertical asymptote at $x = 0$; Horizontal asymptote at $y = 0$
- Odd function: $f(-x) = -f(x)$

Applications:

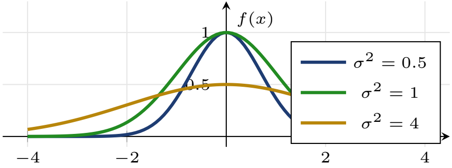
- Inverse relationships: time vs. speed for fixed distance
  $d = vt \Rightarrow t = d/v$
- Gravitational force: $F = G\frac{m_1 m_2}{}$

8. The Gaussian function

Definition

$$f(x) = e^{-x^2}$$

General form: $f(x) = A \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$



Properties:

- Domain: $\mathbb{R}$; Range: $(0, 1]$
- Maximum at $x = 0$ where $f(0) = 1$
- Symmetric about $x = 0$
- $\lim_{x\to\pm\infty} e^{-x^2} = 0$

Applications:

The normal distribution

When properly normalized, the Gaussian becomes the **Normal Distribution**:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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Parameters:

- μ = mean (center of the bell)
- σ = standard deviation (width)
- σ^2 = variance

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Parameters:

- $\mu =$ mean (center of the bell)
- $\sigma =$ standard deviation (width)
- $\sigma^2 =$ variance

The 68-95-99.7 Rule:

- 68% of data within $\mu \pm \sigma$
- 95% of data within $\mu \pm 2\sigma$
- 99.7% of data within $\mu \pm 3\sigma$

Why Gaussians appear everywhere

Central Limit Theorem

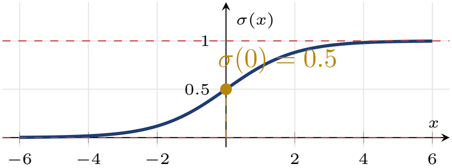
The sum (or average) of many independent random variables tends toward a Normal distribution, regardless of the original distributions.

This explains:

- Heights of people (sum of many genetic factors)
- Measurement errors (sum of many small errors)
- Stock price changes (sum of many trades)
- Test scores (sum of many question responses)

Definition: Logistic Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$$



Sigmoid properties

Key properties:

- **Range**: $(0, 1)$—perfect for probabilities!
- **Symmetry**: $\sigma(-x) = 1 - \sigma(x)$
- **Center**: $\sigma(0) = 0.5$
- **Limits**: $\lim_{x \to -\infty} \sigma(x) = 0$, $\lim_{x \to \infty} \sigma(x) = 1$

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Derivative (important for ML!):

$$\sigma'(x) = \sigma(x)(1 - \sigma(x))$$

Maximum slope at $x = 0$ where $\sigma'(0) = 0.25$.

Sigmoids in machine learning

The sigmoid is **fundamental to modern AI**:

Logistic Regression:

$$P(\text{spam}|\text{email}) = \sigma(\beta_0 + \beta_1 x_1 + \cdots + \beta_n x_n)$$

Convert any linear combination into a probability!

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When ChatGPT predicts the next word, it uses softmax (a multi-class generalization of sigmoid) to convert scores into probabilities.

Pattern-book summary

Function	Formula	Key Feature	Models...
Exponential	e^x	Constant % growth	Growth, decay
Logarithm	$\ln(x)$	Inverse of exp	Compressed scales
Sinusoid	$\sin(x)$	Periodic	Oscillation, cycles
Square	x^2	Parabola	Scaling, geometry
Identity	x	Straight line (slope 1)	Proportional change
One	1	Flat line	Baselines
Reciprocal	$1/x$	Hyperbola	Inverse relations
Gaussian	e^{-x^2}	Bell shape	Distributions, peaks
Sigmoid	$1/(1+e^{-x})$	S-curve	Transitions

These are your **building blocks**. Future: how to combine them!

Part IV

Describing Functions

Function-shape vocabulary

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Seven function-shape concepts:

- Slope
- Concavity
- Continuity
- Monotonicity
- Periodicity
- Asymptotic behavior
- Local extrema

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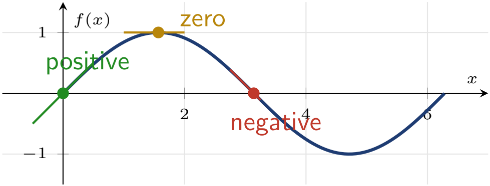
- Concavity
- Continuity
- Monotonicity
- Periodicity
- Asymptotic behavior
- Local extrema

Each concept describes how the function output changes as the input changes.

1. Slope

Slope describes whether the output goes up or down, and the extent of this rise or fall, as the input changes.



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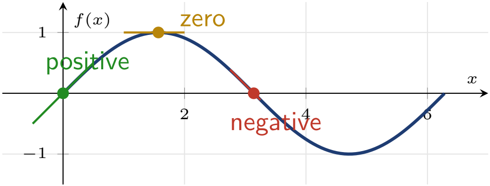
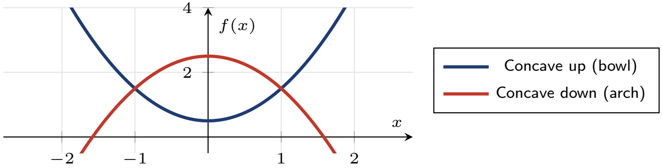


Key points:

- Slope can be positive (increasing), negative (decreasing), or zero (flat)
- For most functions, slope varies with the input value
- Slope measures the rate of change of the function
- Slope $= \frac{\Delta y}{\Delta x} =$ rise over run

2. Concavity

Concavity describes the change in slope—whether the slope is increasing or decreasing.



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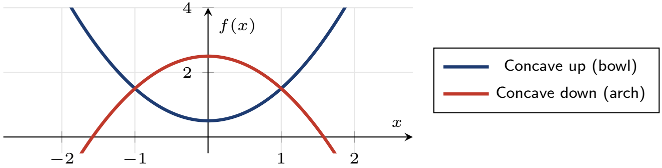

Two types:

- **Concave up**: Slope is increasing (function curves upward like a bowl)
- **Concave down**: Slope is decreasing (function curves downward like an arch)

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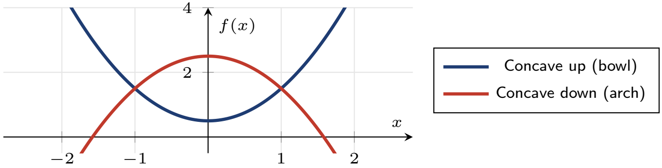


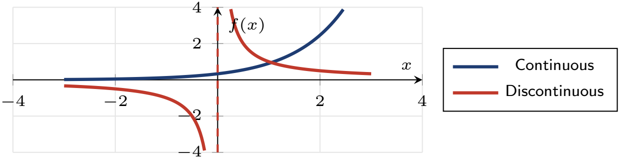
Two types:

- **Concave up**: Slope is increasing (function curves upward like a bowl)
- **Concave down**: Slope is decreasing (function curves downward like an arch)

Key insight: Concavity is about the *change* in slope, not the slope itself.

3. Continuity

A function is **continuous** if you can trace its graph without lifting your pencil.



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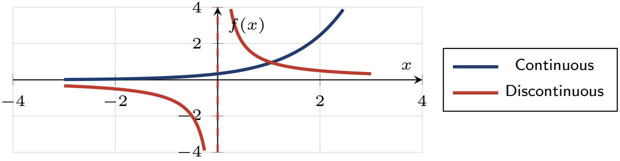
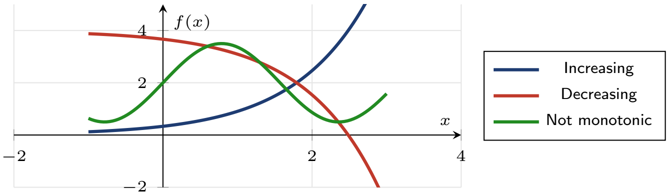


Key points:

- All pattern-book functions are continuous on their domains
- Exception: Power functions with negative exponents (like $1/x = x^{-1}$) are not defined at $x = 0$
- On any interval that doesn't include 0, the reciprocal function is continuous

4. Monotonicity

A function is **monotonic** on a domain when the *sign* of the slope never changes.



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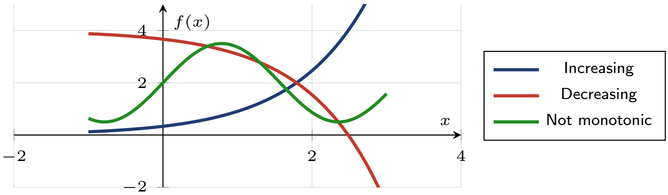

Two types:

- **Monotonically increasing**: Function steadily increases (slope always positive or zero)
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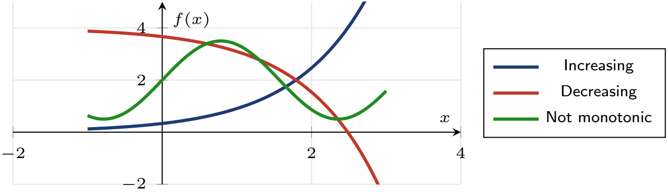

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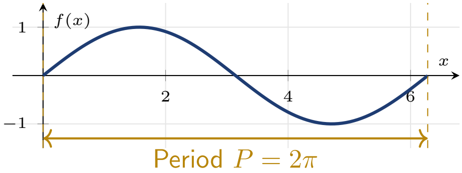
Key property: Monotonic functions preserve (or reverse) the order of inputs when mapping to outputs

5. Periodicity

A phenomenon is **periodic** if it repeats a pattern over and over again.



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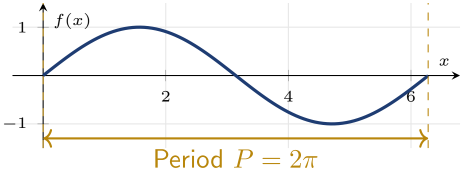

Key concepts:

- **Cycle**: The pattern that is repeated
- **Period**: The duration of one complete cycle
- **Frequency**: Number of cycles per unit time ($f = 1/P$)

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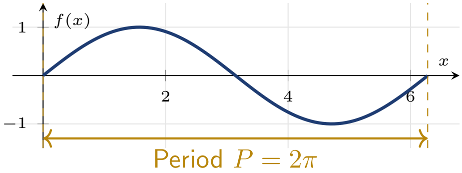

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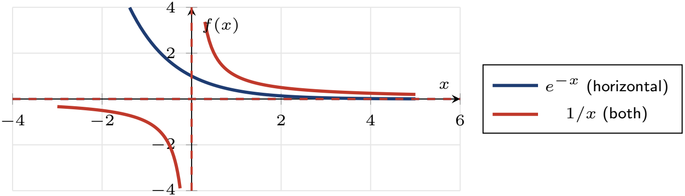
Examples:

- Day-night cycle: Period = 24 hours
- Seasons: Period = 1 year
- Sinusoid: Period = 2π (for the basic $\sin(x)$)

6. Asymptotic behavior

Asymptotic behavior describes what happens as inputs or outputs get very large.



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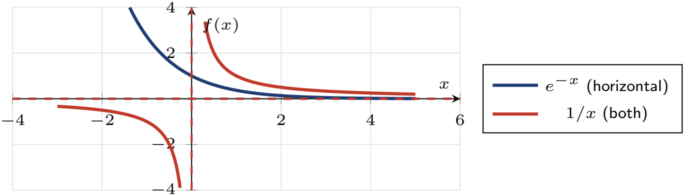
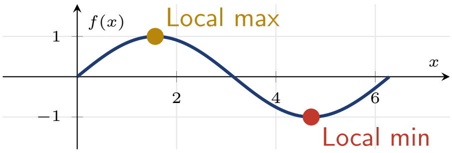


Two types:

- **Horizontal asymptote**: As input $\to \pm\infty$, output approaches a specific value
- **Vertical asymptote**: Output $\to \pm\infty$ while input changes only a little

7. Local extrema

Local extrema are peaks (maxima) and valleys (minima) in a function.



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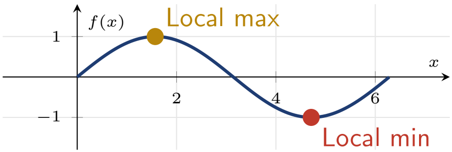

Key concepts:

- **Local maximum**: A peak where output is larger than nearby values
- **Local minimum**: A valley where output is smaller than nearby values
- **Argmax**: The input value where the function reaches its maximum
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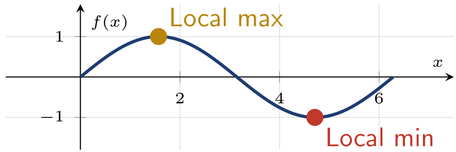

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Key insight: Functions with extrema cannot be monotonic (slope changes from positive to negative or vice versa).

Optimization

The process of finding an argmin or argmax is called **optimization**.

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Applications:

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- **Engineering**: Find speed that minimizes fuel consumption
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Key principle: At an extremum, the slope is zero (or undefined).

Describing functions: summary

Concept	Describes...
Slope	Whether output goes up or down
Concavity	Change in slope (curvature)
Continuity	Whether graph can be drawn without lifting pencil
Monotonicity	Whether slope sign never changes
Periodicity	Whether pattern repeats
Asymptotic behavior	Behavior as inputs/outputs get very large
Local extrema	Peaks and valleys

These concepts help us communicate about function behavior and design functions for modeling.

What comes next

Next: Modeling and Assembling Functions

- How to combine pattern-book functions (sums, products, composition)
- Parameters and what they control
- Fitting models to real data
- Dimensional analysis: Why units matter

"All models are wrong, but some are useful."
— George Box, statistician

Practice: explore in R

```r
# Compare exponential and power growth
slice_plot(exp(x) ~ x, domain(x = 0:5), color="blue"
  ) %>%
slice_plot(x^2 ~ x, color="red") %>%
slice_plot(x^3 ~ x, color="green")
# Which dominates for large x?

# Explore the sigmoid
s <- makeFun(1/(1 + exp(-k*x)) ~ x, k = 1)
slice_plot(s(x, k=0.5) ~ x, domain(x=-10:10)) %>%
slice_plot(s(x, k=1) ~ x) %>%
slice_plot(s(x, k=2) ~ x)
# What does k control?
```

Practice problems

1. Find the domain of $f(x) = \ln(x^2 - 4)$.

2. If a population doubles every 5 years, write it as $P(t) = P_0 e^{kt}$ and find k.

3. A sinusoid has maximum value 10, minimum value 2, and period 6. Write its equation in the form $f(t) = A \sin\left(\frac{2\pi}{P} t\right) + C$.

4. Sketch $f(x) = e^{-x^2}$ and $g(x) = e^{-|x|}$. How do they differ?

5. Show that $\sigma(x) + \sigma(-x) = 1$ for the sigmoid $\sigma(x) = \frac{1}{1+e^{-x}}$.