# *Parameters*

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## *Outline*

1. **The Problem** – Matching pattern-book functions to real data
2. **Input Scaling** – Converting quantities to pure numbers
3. **Output Scaling** – Adjusting amplitude and baseline
4. **Complete Parameterization Framework** – The general form $A \cdot g(ax + b) + B$
5. **Parameter Relationships** – Equivalent forms and special cases
6. **Parameterization Conventions** – Standard parameter names
7. **Key Insights** – Why parameters work
8. **Examples** – Applications to real-world modeling

## 1 *Parameters: Scaling Pattern-Book Functions*

### 1.1 *The Problem*

Pattern-book functions are **pure numbers** in, **pure numbers** out.

Real-world quantities have **units**: days, meters, cases, etc.

**Question:** How do we match pattern-book functions to real data?

### 1.2 *Matching Numbers to Quantities*

*Pure Numbers vs. Quantities*

- **Pure numbers:** No units (e.g., 17.32)
- **Quantities:** Have units (e.g., 17.3 days, 34 meters)

Pattern-book functions require pure number inputs.

*Solution: Parameters* Parameters convert quantities to pure numbers before pattern-book evaluation.

# 2 Input Scaling

## 2.1 Definition: Input Scaling

Given a pattern-book function $g(x)$ and a quantity input $t$ with units, we scale the input using parameters:

$$g(at + b)$$

where:

- $a$: scaling parameter (converts units to pure numbers)
- $b$: shift parameter (adjusts starting point)

## 2.2 Example: Exponential with Time Input

Pattern-book: $e^x$ (pure numbers)
Real-world: Cases over time $t$ (in days)
**Model:** $e^{kt}$ where $k$ has units "per day"

*Why this works:*

$$k \cdot t\Big|_{t=10\text{ days}} = (0.2\text{ day}^{-1}) \cdot (10\text{ days}) = 2$$

The "2" is a pure number—units cancel.

## 2.3 COVID-19 Example

**Context:** In early 2020, COVID-19 cases grew rapidly. Data from March 2020 showed case numbers doubling roughly every 3-4 days.[1]

**Why exponential?** The growth rate itself was increasing—each day brought more new cases than the previous day. This is the hallmark of exponential growth.

**Data pattern:** Starting around March 1, cases grew from ∼100 to ∼10,000 over the month. (Note: This is a *hypothetical* example for pedagogical purposes. Actual US case counts on March 1, 2020 were much lower, with only 14 confirmed cases through February 23, 2020 according to CDC data.[2])

**Model:**

$$\text{cases}(t) = 100 \cdot e^{0.19t}$$

where $t$ is days since March 1.

**Why $A = 100$?** This is the *initial value* parameter. At $t = 0$ (March 1), the model gives $\text{cases}(0) = 100 \cdot e^0 = 100$ cases. The value 100 is chosen to match the approximate case count at the start of the time period being modeled (for this hypothetical example).

[1] Schuchat A. Public Health Response to the Initiation and Spread of Pandemic COVID-19 in the United States, February 24–April 21, 2020. *MMWR Morb Mortal Wkly Rep* 2020;69:551–556.

[2] Schuchat A. Public Health Response to the Initiation and Spread of Pandemic COVID-19 in the United States, February 24–April 21, 2020. *MMWR Morb Mortal Wkly Rep* 2020;69:551–556.

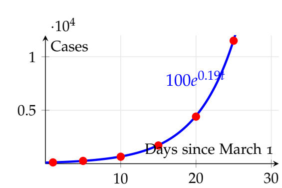


Figure 1: COVID-19 cases: exponential growth model (hypothetical)

*Parameter Interpretation:*

- **Rate parameter** $k = 0.19$ **per day:** Cases increase by about 19% each day. This means the number of cases multiplies by $e^{0.19} \approx 1.21$ each day (21% daily growth).
- **Output scaling** $A = 100$**:** This sets the scale to match the data. At $t = 0$ (March 1), the model predicts $100 \cdot e^0 = 100$ cases.
- **Why no** $B$ **parameter?** Cases start near zero, so no vertical shift needed.

*Key insight:* The exponential form captures the accelerating growth pattern, where the growth rate itself grows over time.

# 3 *Output Scaling*

## 3.1 *Definition: Output Scaling*

After applying pattern-book function, scale and shift the output:

$$A \cdot g(ax + b) + B$$

where:

- $A$: amplitude/scaling parameter
- $B$: vertical shift parameter

## 3.2 *Example: Tide Levels*

**Context:** Ocean tides rise and fall in a predictable pattern. Most locations experience two high tides and two low tides per day, following a roughly sinusoidal pattern.

**Physical situation:** At a particular location, tide levels oscillate between high tide (around 1.5 meters) and low tide (around 0.5 meters) with a period of approximately 12.4 hours (half a lunar day).

**Why sinusoid?** The periodic rise and fall matches the $\sin(x)$ pattern perfectly.

*Step 1: Input scaling* We need to match the 12.4-hour period. Let's work through this step-by-step:

**Step 1a: What do we know?**

- The pattern-book function $\sin(x)$ has period $2\pi$ (one full cycle when $x$ goes from $0$ to $2\pi$).
- We want our model to have period 12.4 hours (one full cycle when $t$ goes from $0$ to $12.4$ hours).

**Step 1b: What do we need?** We need to convert time $t$ (in hours) into the unitless argument that sin requires. We'll use a scaling parameter $a$ so the argument is $a \cdot t$.

**Step 1c: Set up the requirement** For the function $\sin(a \cdot t)$ to have period 12.4 hours, we need:

$$\text{When } t = 12.4 \text{ hours, the argument } a \cdot t \text{ should equal } 2\pi$$

This ensures one complete cycle of sin happens in 12.4 hours.

**Step 1d: Solve for $a$**

$$a \cdot 12.4 = 2\pi$$
$$a = \frac{2\pi}{12.4} = \frac{\pi}{6.2} \approx 0.507 \text{ per hour}$$

**Step 1e: Write the scaled function**

$$\sin(a \cdot t) = \sin\left(\frac{2\pi}{12.4}t\right) = \sin\left(\frac{\pi}{6.2}t\right) \approx \sin(0.507t)$$

**Step 1f: Add phase shift** The phase shift $-1$ adjusts when high tide occurs (shifts the entire pattern horizontally):

$$\sin\left(\frac{2\pi}{12.4}t - 1\right) = \sin\left(\frac{\pi}{6.2}t - 1\right)$$

**Summary:**

- **Scaling parameter $a = \frac{2\pi}{12.4} \approx 0.507$ per hour:** This converts hours into the unitless argument needed for sin. When $t = 12.4$ hours, we get $a \cdot 12.4 = 2\pi$, completing one full cycle.
- **Phase shift $-1$:** This shifts the entire pattern horizontally to match when high/low tide occurs.

*Step 2: Output scaling* The pattern-book $\sin(x)$ oscillates between $-1$ and $+1$. We need:

- Amplitude of 0.5 meters (difference between high and low tide)
- Midline at 1.0 meters (average tide level)

$$\text{tide}(t) = 0.5\sin\left(\frac{2\pi}{12.4}t - 1\right) + 1 = 0.5\sin\left(\frac{\pi}{6.2}t - 1\right) + 1$$

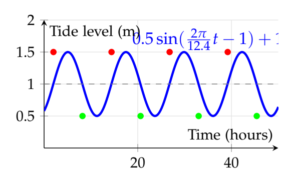

Figure 2: Tide levels: sinusoidal model

*Parameter Interpretation:*

- **Amplitude $A = 0.5$ meters:** Controls the range of tide variation (high tide - low tide = $2 \times 0.5 = 1$ meter total range).

- **Midline $B = 1$ meter:** Sets the average tide level. High tide = $1 + 0.5 = 1.5$ m, low tide = $1 - 0.5 = 0.5$ m.
- **Period parameter $\frac{2\pi}{12.4} \approx 0.507$ per hour:** Creates 12.4-hour period (since $\sin(x)$ has period $2\pi$, we need $\frac{2\pi}{12.4} \cdot 12.4 = 2\pi$ to complete one full cycle).
- **Phase shift $-1$:** Shifts the entire pattern horizontally to match observed timing.

# 4 *Complete Parameterization Framework*

## 4.1 *General Form*

For any pattern-book function $g(x)$:

$$f(x) = A \cdot g(ax + b) + B$$

*Parameters:*

1. **Input parameters:**
   - $a$: Horizontal scaling (stretch/compress)
   - $b$: Horizontal shift (phase/offset)
2. **Output parameters:**
   - $A$: Vertical scaling (amplitude)
   - $B$: Vertical shift (baseline)

## 4.2 *Visualizing Parameter Effects*

**Drawing exercise:** Start with pattern-book function (e.g., $\sin(x)$), then show how each parameter changes the shape:

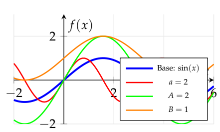

Figure 3: Parameter effects on $\sin(x)$

- **Changing $a$:** Stretches/compresses horizontally (affects period for periodic functions)
- **Changing $b$:** Shifts left/right (affects phase/starting point)
- **Changing $A$:** Stretches/compresses vertically (affects amplitude)
- **Changing $B$:** Shifts up/down (affects baseline/midline)

## 4.3 *Procedure for Building Models*

1. **Choose pattern-book function** whose shape matches data
2. **Find parameter values** $A$, $B$, $a$, $b$ that fit the data

# 5 Parameter Relationships

## 5.1 Input Scaling: Two Equivalent Forms

$$\begin{aligned} g(ax+b) &\quad \text{form 1} \\ g(a(x-x_0)) &\quad \text{form 2} \end{aligned}$$

*Equivalence:*

$$ax + b = a(x - x_0) \quad \text{when} \quad x_0 = -\frac{b}{a}$$

*Interpretation:*

- Form 1: Scale then shift
- Form 2: Shift then scale (often more intuitive)

## 5.2 Exponential Special Case

For exponentials, input scaling can be written:

$$Ae^{at} = e^{\ln(A)}e^{at} = e^{at+\ln(A)} = e^{a(t-t_0)}$$

where $t_0 = -\frac{\ln(A)}{a}$.

*COVID Example (revisited):*

$$\text{cases}(t) = 100e^{0.19t} = e^{0.19(t-t_0)}$$

with $t_0 = -\frac{\ln(100)}{0.19} \approx -24.2$ days.

**Interpretation:** In this hypothetical model, the outbreak would have started ∼24.2 days before March 1 (early February).

# 6 Parameterization Conventions

## 6.1 Common Parameter Forms

Different pattern-book functions use conventional parameter names:

*Exponential:*

$$\begin{aligned} e^{kt} &\quad k\text{: growth rate (per unit time)} \\ e^{t/\tau} &\quad \tau\text{: time constant} \\ 2^{t/\tau_2} &\quad \tau_2\text{: doubling time} \end{aligned}$$

*Sinusoid:*

$$\sin\left(\frac{2\pi}{P}(t - t_0)\right) \quad P\text{: period, } t_0\text{: phase shift}$$
$$\sin(\omega t + \phi) \quad \omega\text{: angular frequency, } \phi\text{: phase}$$

*Power-law:*

$$(x - x_0)^p \quad x_0\text{: shift, } p\text{: exponent}$$

*Straight-line:*

$$mx + b \quad \text{or} \quad m(x - x_0) \quad m\text{: slope}$$

# 7 Key Insights

## 7.1 Why Parameters Work

1. **Unit conversion:** Parameters convert quantities to pure numbers
2. **Shape control:** Parameters adjust function shape to match data
3. **Flexibility:** Same pattern-book function can model many scenarios

## 7.2 The Big Picture

**Remarkable fact:** Many real-world relationships can be modeled by:

1. Choosing one pattern-book function
2. Scaling input and output with parameters

This framework extends to multiple inputs (coming next).

# 8 Examples

## 8.1 Worked Example: Ventilating a classroom ($CO_2$ decay)

**Problem Statement:** A classroom starts out "stuffy" with elevated $CO_2$ concentration. You open a window and $CO_2$ starts dropping toward the outdoor (baseline) level. We want to build a model to predict $CO_2$ levels over time and answer questions like: "When will the air quality be safe?" and "How long until $CO_2$ drops to a specific level?"

**Quantities:** Let $C(t)$ be $CO_2$ concentration in parts per million (ppm), measured $t$ minutes after opening the window.

**Data (example):**

$$C_{\text{out}} = 420 \text{ ppm}, \qquad C(0) = 1400 \text{ ppm}, \qquad C(10) = 980 \text{ ppm}.$$

**Questions we want to answer:**

1. What pattern-book function should we use?
2. How do we parameterize it to match the data?
3. What does the model predict at other times?
4. When will $CO_2$ reach a target level (e.g., 800 ppm)?
5. What is the "half-life" of the decay process?

*Step 1: Choose a pattern-book function.* **Question:** What pattern-book shape matches "approaches a baseline"? **Answer:** Exponential decay of the deviation-from-baseline.

The deviation from baseline is positive and appears to shrink proportionally over time, so we use an exponential decay shape.

*Step 2: Write the parameterized model.* **Question:** What is a reasonable parameterized model for "baseline + decaying deviation"? **Answer:** Use an exponential with a baseline and one rate parameter.

**Why this form?** The pattern-book exponential $e^x$ decays toward zero as $x \to -\infty$. We want the *deviation from baseline* to decay toward zero, so we use $e^{-kt}$ (with $k > 0$). The negative sign ensures decay rather than growth. The baseline $C_{\text{out}}$ is added separately because it's the long-term equilibrium value. The initial deviation $C(0) - C_{\text{out}}$ multiplies the exponential to set the starting point.

Use "baseline + decaying deviation":

$$C(t) = C_{\text{out}} + \big(C(0) - C_{\text{out}}\big)e^{-kt}.$$

Here $k > 0$ is a parameter with units *per minute* so that $kt$ is unitless.

*Step 3: Solve for k from one data point.* **Question:** What value of $k$ makes the model match $C(10) = 980$? **Answer:** Solve for $k$ by isolating the exponential and taking logs.

Subtract the baseline and divide by the initial deviation:

$$\frac{C(t) - C_{\text{out}}}{C(0) - C_{\text{out}}} = e^{-kt}.$$

Plug in $t = 10$ and the numbers:

$$\frac{980 - 420}{1400 - 420} = e^{-10k} \quad \Rightarrow \quad \frac{560}{980} = e^{-10k}.$$

Take logs:

$$-10k = \ln\left(\frac{560}{980}\right) \quad \Rightarrow \quad k = -\frac{1}{10}\ln\left(\frac{560}{980}\right) \approx 0.056 \text{ min}^{-1}.$$

*Step 4: Make predictions.* **Question:** What does the model predict at $t = 5$ and $t = 20$ minutes? **Answer:** Plug $t$ into the fitted model.

With $k \approx 0.056$,

$$C(5) = 420 + 980e^{-0.056 \cdot 5} \approx 1161 \text{ ppm},$$
$$C(20) = 420 + 980e^{-0.056 \cdot 20} \approx 739 \text{ ppm}.$$

*Step 5: Invert the model (solve for time to reach a target).* **Question:** When will $CO_2$ reach 800 ppm? **Answer:** Set $C(t) = 800$ and solve for $t$ (take logs).

$$800 = 420 + 980e^{-kt} \Rightarrow e^{-kt} = \frac{800 - 420}{980} = \frac{380}{980}.$$

So

$$t = -\frac{1}{k} \ln\left(\frac{380}{980}\right) \approx \frac{0.947}{0.056} \approx 16.9 \text{ minutes}.$$

*Step 6: Interpret a parameter via half-life.* **Question:** What is the "half-life" of the deviation-from-baseline? **Answer:** Solve $e^{-kt_{1/2}} = \frac{1}{2}$ to get $t_{1/2} = \ln(2)/k$.

Define the half-life $t_{1/2}$ by cutting the deviation-from-baseline in half:

$$C(t_{1/2}) - C_{\text{out}} = \tfrac{1}{2}\big(C(0) - C_{\text{out}}\big).$$

Then $e^{-kt_{1/2}} = \frac{1}{2}$, so

$$t_{1/2} = \frac{\ln 2}{k} \approx \frac{0.693}{0.056} \approx 12.4 \text{ minutes}.$$

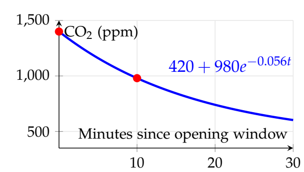

Figure 4: $CO_2$ model with fitted $k$

## 8.2 *Worked Example: "How fast is too fast?" (a sigmoid risk model)*

**Problem Statement:** We want to model the probability of receiving a speeding ticket as a function of driving speed. The relationship should be bounded between 0 and 1 (since it's a probability), and should transition smoothly from "very unlikely" at low speeds to "very likely" at high speeds. We want to answer questions like: "What's the risk at 80 mph?" and "At what speed does the risk reach 80%?"

**Quantity:** Let $p(v)$ be the probability (between 0 and 1) of receiving a speeding ticket, as a function of driving speed $v$ in mph.

**Data (hypothetical calibration):**

$$p(65) = 0.10, \qquad p(75) = 0.50, \qquad p(85) = 0.90.$$

**Questions we want to answer:**

1. What pattern-book function naturally produces values between 0 and 1?

2. How do we parameterize it to match the calibration data?

3. What is the predicted risk at other speeds?

4. At what speed does the risk reach a target level?

*Step 0: Choose pattern-book function and justify the form.* **Question:** What pattern-book function gives values between 0 and 1 and transitions smoothly? **Answer:** The sigmoid function $s(x) = \frac{1}{1+e^{-x}}$.

**Why the sigmoid?**

- **Bounded output:** $s(x)$ is always between 0 and 1, which is exactly what we need for probabilities. As $x \to -\infty$, $s(x) \to 0$; as $x \to +\infty$, $s(x) \to 1$.

- **Smooth transition:** The S-shaped curve provides a gradual transition rather than a sudden jump, which matches how enforcement risk typically behaves.

- **Symmetric around midpoint:** When $x = 0$, $s(0) = 1/2$, giving a natural "50% point" that we can position at any speed using input scaling.

**Model form:** Use input scaling/shift to convert mph into a unitless score:

$$p(v) = s\big(a(v - v_0)\big) = \frac{1}{1 + e^{-a(v-v_0)}}.$$

**Why this parameterization?**

- $v_0$ **(the shift):** This is the speed where $p(v_0) = 0.5$ (the 50% point). We position this at the observed midpoint of the transition.

- $a$ **(the scaling):** This controls how *steep* the transition is. Larger $a$ means a sharper transition from low to high probability. The parameter $a$ has units *per mph* so that $a(v - v_0)$ is unitless.

- **No output scaling needed:** The sigmoid already outputs values in $[0, 1]$, so we don't need $A$ or $B$ parameters—the function is already in the right range.

*Step 1: Use the mid-point to identify $v_0$.* **Question:** What is $v_0$ (the 50% point)? **Answer:** Use the fact that $s(0) = 1/2$, so $p(v_0) = 1/2$.

Since $s(0) = 1/2$, we have

$$p(v_0) = \frac{1}{2}.$$

Given $p(75) = 0.50$, we set $v_0 = 75$.

*Step 2: Solve for a using one more data point.* **Question:** How steep is the transition (what is $a$)? **Answer:** Plug in one calibration point and solve using logs.

Use $p(65) = 0.10$:

$$0.10 = \frac{1}{1 + e^{-a(65-75)}} = \frac{1}{1 + e^{10a}}.$$

Invert:

$$\frac{1}{0.10} - 1 = e^{10a} \quad \Rightarrow \quad 9 = e^{10a} \quad \Rightarrow \quad a = \frac{\ln 9}{10} \approx 0.220 \text{ mph}^{-1}.$$

*Step 3: Sanity check with the symmetric point.* **Question:** Does the fitted model reproduce $p(85) = 0.90$? **Answer:** Yes; the sigmoid is symmetric around $v_0$, so the probabilities mirror.

At $v = 85$,

$$p(85) = \frac{1}{1 + e^{-a(85-75)}} = \frac{1}{1 + e^{-10a}}.$$

With $e^{10a} = 9$, we have $e^{-10a} = 1/9$, so

$$p(85) = \frac{1}{1 + 1/9} = 0.90,$$

which matches the calibration.

*Step 4: Make a prediction.* **Question:** What is the predicted probability at 80 mph? **Answer:** Plug $v = 80$ into $p(v)$.

$$p(80) = \frac{1}{1 + e^{-a(5)}} \approx \frac{1}{1 + e^{-1.10}} \approx 0.75.$$

*Step 5: Invert the model (find a speed for a target risk).* **Question:** At what speed is the risk 80%? **Answer:** Set $p(v) = 0.80$ and solve for $v$ (take logs).

$$0.80 = \frac{1}{1 + e^{-a(v-75)}} \Rightarrow \frac{1}{0.80} - 1 = e^{-a(v-75)}.$$

So

$$v = 75 - \frac{1}{a} \ln\left(\frac{1}{0.80} - 1\right) \approx 75 - \frac{1}{0.220} \ln(0.25) \approx 81.3 \text{ mph}.$$

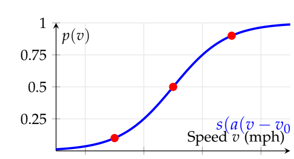

Figure 5: Sigmoid model for $p(v)$

## 8.3 Worked Example: Daily temperature cycle (sinusoid)

**Problem Statement:** Outdoor temperature follows a predictable daily cycle: coolest around sunrise, warmest in the afternoon, then cooling again overnight. We want to model temperature as a function of time of day to answer questions like: "What's the temperature at 2pm?" and "When does it reach 70°F?"

**Quantity:** Let $T(t)$ be temperature in °F, where $t$ is hours since midnight.

**Data (example from a typical day):**

$$T(6) = 55°\text{F}, \qquad T(12) = 75°\text{F}, \qquad T(18) = 80°\text{F}, \qquad T(0) = 58°\text{F}.$$

**Questions we want to answer:**

1. What pattern-book function captures periodic daily cycles?
2. How do we parameterize it to match the observed temperature range and timing?
3. What is the predicted temperature at other times?
4. When does temperature reach a target value?

*Step 0: Choose pattern-book function and justify the form.* **Question:** What pattern-book function repeats every 24 hours? **Answer:** The sinusoid $\sin(x)$ or $\cos(x)$, which are periodic.

**Why a sinusoid?**

- **Periodic:** Temperature cycles repeat daily, matching the periodic nature of $\sin(x)$ and $\cos(x)$.
- **Smooth:** The smooth rise and fall matches how temperature changes gradually throughout the day.
- **Symmetric:** The pattern is roughly symmetric around the daily average, which sinusoids naturally provide.

**Model form:** We'll use the general parameterized form:

$$T(t) = A\sin\big(a(t - t_0)\big) + B.$$

**Why this parameterization?**

- $B$ **(baseline/midline):** This is the *average* daily temperature. The sinusoid oscillates around this value.
- $A$ **(amplitude):** This is half the temperature *range* (difference between max and min). The sinusoid $\sin(x)$ oscillates between $-1$ and $+1$, so $A$ scales it to match the observed temperature swing.
- $a$ **(frequency scaling):** This converts hours into the unitless argument needed for sin. To get a 24-hour period, we need $a = 2\pi/24 = \pi/12$ per hour, since $\sin(x)$ has period $2\pi$.
- $t_0$ **(phase shift):** This shifts the sinusoid horizontally to match when the temperature peaks or troughs occur. For example, if the maximum is at $t = 14$ (2pm), we set $t_0$ accordingly.

*Step 1: Estimate parameters from data.* **Question:** What are reasonable values for $A$, $B$, $a$, and $t_0$? **Answer:** Use the data to identify key features.

From the data:

- **Midline** $B$**:** Average of min and max. Roughly $B \approx (55 + 80)/2 = 67.5°\text{F}$.
- **Amplitude** $A$**:** Half the range. $A \approx (80 - 55)/2 = 12.5°\text{F}$.
- **Period parameter** $a$**:** For 24-hour period, $a = \pi/12 \approx 0.262$ per hour.
- **Phase** $t_0$**:** The maximum appears around $t = 18$ (6pm in this example). For sin, the peak occurs when the argument equals $\pi/2$, so we want $a(18 - t_0) = \pi/2$, giving $t_0 \approx 12$ hours.

*Step 2: Write the complete model.*

$$T(t) = 12.5 \sin\left(\frac{\pi}{12}(t - 12)\right) + 67.5.$$

*Step 3: Verify with data points.* At $t = 6$: $T(6) = 12.5 \sin(\pi/12 \cdot (-6)) + 67.5 = 12.5 \sin(-\pi/2) + 67.5 = -12.5 + 67.5 = 55°\text{F}$. ✓

At $t = 12$: $T(12) = 12.5 \sin(0) + 67.5 = 67.5°\text{F}$. (Close to observed 75°F; model may need refinement.)

At $t = 18$: $T(18) = 12.5 \sin(\pi/2) + 67.5 = 12.5 + 67.5 = 80°\text{F}$. ✓

*Step 4: Make a prediction.* **Question:** What is the temperature at 2pm ($t = 14$)? **Answer:**

$$T(14) = 12.5 \sin\left(\frac{\pi}{12}(14 - 12)\right) + 67.5 = 12.5 \sin\left(\frac{\pi}{6}\right) + 67.5 \approx 12.5(0.5) + 67.5 \approx 73.8°\text{F}.$$

*Step 5: Invert the model.* **Question:** When does temperature reach 70°F? **Answer:** Set $T(t) = 70$ and solve:

$$70 = 12.5 \sin\left(\frac{\pi}{12}(t - 12)\right) + 67.5 \Rightarrow \sin\left(\frac{\pi}{12}(t - 12)\right) = \frac{70 - 67.5}{12.5} = 0.2.$$

So $\frac{\pi}{12}(t - 12) = \arcsin(0.2) \approx 0.201$, giving $t \approx 12 + \frac{12}{\pi}(0.201) \approx 12.8$ hours (about 12:48pm).

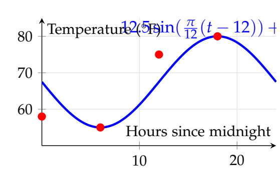


Figure 6: Daily temperature model