Homework 2

Calculus I (STM 1001)

due: **Friday, January 30, 2026 at 11:30a**

Instructions: Please submit solutions on Populi as a knitted markdown pdf. For handwritten work, please scan your solution pages and include in the final pdf. Homeworks can be completed individually or in groups of up to three people. Names of all group members must be included on the writeups.

Problem 1: Fitting by feature (cooling coffee)

Let $T(t)$ be the temperature (in °C) of a cup of coffee t minutes after it is poured. A common model is Newton's law of cooling,

$$T(t) = T_{\text{room}} + (T_0 - T_{\text{room}})e^{-kt},$$

where T_{room} is the ambient room temperature, T_0 is the initial coffee temperature, and $k > 0$ is a parameter.

Suppose a student measures:

$$T_{\text{room}} = 22, \qquad T_0 = 95, \qquad T(10) = 69.$$

a. Solve for the parameter k using the measurement at $t = 10$.

b. Using your fitted model, predict $T(5)$ and $T(20)$.

c. At what time t will the coffee reach 60°C? (Show algebra.)

d. In the model, what is the meaning of the parameter k? What happens to the cooling curve when k is larger?

e. Define the "half-life relative to room temperature" to be the time $t_{1/2}$ such that

$$T(t_{1/2}) - T_{\text{room}} = \frac{1}{2}(T_0 - T_{\text{room}}).$$

Solve for $t_{1/2}$ in terms of k, then compute its numerical value using your fitted k.

f. Show that the model implies a straight-line relationship

$$\log\left(\frac{T(t) - T_{\text{room}}}{T_0 - T_{\text{room}}}\right) = -kt.$$

Use $t = 10$ and the given measurement to numerically verify your value of k from part (a).

g. Suppose that right after measuring $T(10) = 69$, the student adds a small amount of cold milk and the temperature instantly drops by 3°C. Assume the room temperature remains 22°C and the cooling rate parameter k stays the same after adding milk.

   i. Write down a piecewise formula for the new temperature curve $\widetilde{T}(t)$ for $t \geq 0$.

   ii. Under this new model, at what time will the coffee reach 60°C?

Problem 2: Parameters + assembling functions (two daily peaks)

The "hump" pattern-book function can be written as $h(x) = e^{-x^2}$. Suppose we want a simple model of gym occupancy (in arbitrary units) over a day with two peaks: a morning peak around 7am and an evening peak around 6pm.

Let t denote time in hours after midnight (so $t \in [0, 24]$). Consider the assembled model

$$G(t) = B + A_1\, h\left(\frac{t - \mu_1}{s_1}\right) + A_2\, h\left(\frac{t - \mu_2}{s_2}\right),$$

with parameters $B, A_1, A_2, \mu_1, \mu_2, s_1, s_2$.

a. Explain what each parameter does (baseline, peak heights, peak locations, peak widths).

b. Choose reasonable values for the parameters so that:

- there is a morning peak near $t = 7$ and an evening peak near $t = 18$,
- the evening peak is higher than the morning peak,
- the baseline is not zero.

Write down your final formula $G(t)$ with your chosen parameter values.

c. Plot $G(t)$ over $t \in [0, 24]$. Label the two peak times on your plot.

d. Using your model, compute $G(6)$, $G(12)$, and $G(18)$ and interpret them in context.

Problem 3: Dimensions and units (pendulum)

a. The period P of a simple pendulum is believed to depend on the length L of the pendulum and gravitational acceleration g.

i. Use dimensional analysis to show that any formula of the form $P = f(L, g)$ must have the structure

$$P = C\sqrt{\frac{L}{g}},$$

where C is a dimensionless constant.

ii. Using $C = 2\pi$, $L = 0.75$ meters, and $g = 9.8$ m/s^2, compute P in seconds.

b. Now suppose someone claims that the period also depends on the mass m of the pendulum bob (with dimension $[m] = M$). Use dimensional analysis on the form

$$P = C\, L^a g^b m^c$$

to show that necessarily $c = 0$ (so mass cannot affect P in any dimensionally consistent model built only from L, g, and m).

c. Scaling. Suppose we keep g fixed and change the length from L to $4L$.

i. Using $P = C\sqrt{L/g}$, what is the ratio $\frac{P(4L)}{P(L)}$?

ii. Using your numerical value from part (a.ii), compute the predicted period for a pendulum of length 3.0 meters.

d. Units sanity check. Recompute the period for $L = 0.75$ meters using centimeters instead (so $L = 75$ cm). Convert $g = 9.8$ m/s^2 into cm/s^2 and verify that you get the same period in seconds.

Problem 4: Multiple inputs + low-order polynomials (Austin restaurants)

In this problem you'll build and interpret a simple function of multiple inputs using the file `Calculus/data/austin_restaurants.csv`.

Let r denote the number of reviews and y denote the year opened. Define a centered year variable $v = y - 2015$.

You can fit parameters from data using `fitModel()`. Below is example code for you to try.

```
library(mosaic)
d <- read.csv("austin_restaurants.csv")
d$v <- d$year_opened - 2015
mod <- fitModel(rating ~ b0 + b1*reviews + b2*v,
                data = d,
                start = list(b0 = 4, b1 = 0, b2 = 0))
coef(mod)                  # b0, b1, b2 estimates
mod(reviews = 100, v = -5) # prediction (y=2010)
```

a. Load the dataset into `R` and create the variable $v = y - 2015$.

b. Fit the two-input linear model

$$\widehat{\text{rating}} = \beta_0 + \beta_1 r + \beta_2 v.$$

Report your fitted coefficients $\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2$.

c. Interpret $\widehat{\beta}_1$ and $\widehat{\beta}_2$ in words using comparisons:

- $\widehat{\beta}_1$: compare two restaurants opened in the same year (y the same) but with different numbers of reviews; what does an increase of 100 reviews do to the predicted rating?
- $\widehat{\beta}_2$: compare two restaurants with the same number of reviews (r the same) but opened one year apart; what does a 1-year increase in y do to the predicted rating?

Then use your fitted model to compute the predicted rating for each case:

i. $r = 100$ reviews and $y = 2010$

ii. $r = 1000$ reviews and $y = 2020$