

Parameters

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Outline

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2. **Input Scaling** – Converting quantities to pure numbers
3. **Output Scaling** – Adjusting amplitude and baseline
4. **Complete Parameterization Framework** – The general form $A \cdot g(ax + b) + B$
5. **Parameter Relationships** – Equivalent forms and special cases
6. **Parameterization Conventions** – Standard parameter names
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8. **Examples** – Applications to real-world modeling

1 Parameters: Scaling Pattern-Book Functions

1.1 The Problem

Pattern-book functions are **pure numbers** in, **pure numbers** out.

Real-world quantities have **units**: days, meters, cases, etc.

Question: How do we match pattern-book functions to real data?

1.2 Matching Numbers to Quantities

Pure Numbers vs. Quantities

- **Pure numbers:** No units (e.g., 17.32)
- **Quantities:** Have units (e.g., 17.3 days, 34 meters)

Pattern-book functions require pure number inputs.

Solution: Parameters Parameters convert quantities to pure numbers before pattern-book evaluation.

2 Input Scaling

2.1 Definition: Input Scaling

Given a pattern-book function $g(x)$ and a quantity input t with units, we scale the input using parameters:

$$g(at + b)$$

where:

- a : scaling parameter (converts units to pure numbers)
- b : shift parameter (adjusts starting point)

2.2 Example: Exponential with Time Input

Pattern-book: e^x (pure numbers)

Real-world: Cases over time t (in days)

Model: e^{kt} where k has units "per day"

Why this works:

$$k \cdot t \Big|_{t=10 \text{ days}} = (0.2 \text{ day}^{-1}) \cdot (10 \text{ days}) = 2$$

The "2" is a pure number—units cancel.

2.3 COVID-19 Example

Data: Confirmed COVID-19 cases in March 2020

Pattern: Exponential growth

$$\text{cases}(t) = 573 \cdot e^{0.19t}$$

where t is in days, 573 is initial scaling

Interpretation:

- Rate parameter: $k = 0.19$ per day
- Output scaling: $A = 573$ (matches initial data)
- Growth pattern: Exponential

3 Output Scaling

3.1 Definition: Output Scaling

After applying pattern-book function, scale and shift the output:

$$A \cdot g(ax + b) + B$$

where:

- A : amplitude/scaling parameter
- B : vertical shift parameter

3.2 Example: Tide Levels

Pattern: Sinusoid $\sin(x)$

Real-world: Tide level over time

Step 1: Input scaling

$$\sin\left(\frac{1}{2}t - 1\right)$$

- $\frac{1}{2}$: Period adjustment (12.4 hour period)
- -1 : Phase shift

Step 2: Output scaling

$$\text{tide}(t) = 0.5 \sin\left(\frac{1}{2}t - 1\right) + 1$$

- 0.5: Amplitude (meters)
- 1: Midline shift (meters)

4 Complete Parameterization Framework

4.1 General Form

For any pattern-book function $g(x)$:

$$f(x) = A \cdot g(ax + b) + B$$

Parameters:

1. Input parameters:

- a : Horizontal scaling (stretch/compress)
- b : Horizontal shift (phase/offset)

2. Output parameters:

- A : Vertical scaling (amplitude)
- B : Vertical shift (baseline)

4.2 Procedure for Building Models

1. Choose pattern-book function whose shape matches data
2. Find parameter values A, B, a, b that fit the data

5 Parameter Relationships

5.1 Input Scaling: Two Equivalent Forms

$$\begin{aligned} g(ax + b) &\quad \text{form 1} \\ g(a(x - x_0)) &\quad \text{form 2} \end{aligned}$$

Equivalence:

$$ax + b = a(x - x_0) \quad \text{when} \quad x_0 = -\frac{b}{a}$$

Interpretation:

- Form 1: Scale then shift
- Form 2: Shift then scale (often more intuitive)

5.2 Exponential Special Case

For exponentials, input scaling can be written:

$$Ae^{at} = e^{\ln(A)}e^{at} = e^{at+\ln(A)} = e^{a(t-t_0)}$$

where $t_0 = -\frac{\ln(A)}{a}$.

COVID Example (revisited):

$$\text{cases}(t) = 573e^{0.19t} = e^{0.19(t-t_0)}$$

$$\text{with } t_0 = -\frac{\ln(573)}{0.19} \approx -33.6 \text{ days.}$$

Interpretation: Outbreak started ~ 33.6 days before March 1 (late January).

6 Parameterization Conventions

6.1 Common Parameter Forms

Different pattern-book functions use conventional parameter names:

Exponential:

$$\begin{aligned} e^{kt} & \quad k: \text{growth rate (per unit time)} \\ e^{t/\tau} & \quad \tau: \text{time constant} \\ 2^{t/\tau_2} & \quad \tau_2: \text{doubling time} \end{aligned}$$

Sinusoid:

$$\begin{aligned} \sin\left(\frac{2\pi}{P}(t - t_0)\right) & \quad P: \text{period}, t_0: \text{phase shift} \\ \sin(\omega t + \phi) & \quad \omega: \text{angular frequency}, \phi: \text{phase} \end{aligned}$$

Power-law:

$$(x - x_0)^p \quad x_0: \text{shift}, p: \text{exponent}$$

Straight-line:

$$mx + b \quad \text{or} \quad m(x - x_0) \quad m: \text{slope}$$

7 Key Insights

7.1 Why Parameters Work

1. **Unit conversion:** Parameters convert quantities to pure numbers
2. **Shape control:** Parameters adjust function shape to match data
3. **Flexibility:** Same pattern-book function can model many scenarios

7.2 The Big Picture

Remarkable fact: Many real-world relationships can be modeled by:

1. Choosing one pattern-book function
2. Scaling input and output with parameters

This framework extends to multiple inputs (coming next).

8 Examples

Example 1: Exponential growth

Pattern: e^x

Model: $P(t) = P_0 e^{rt}$

- P_0 : initial population
- r : growth rate (per unit time)

Example 2: Sinusoidal oscillation

Pattern: $\sin(x)$

Model: $y(t) = A \sin(2\pi ft + \phi) + B$

- A : amplitude
- f : frequency
- ϕ : phase shift
- B : vertical offset

Example 3: Gaussian distribution

Pattern: e^{-x^2}

Model: $\text{dnorm}(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

- μ : mean (center)
- σ : standard deviation (width)