Visualizing Functions

David Puelz
The University of Austin

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Introduction to Function Visualization

To visualize functions, we use graphs: visual representations showing how outputs change with inputs. We first introduce limits to analyze function behavior, then develop methods for plotting and analyzing graphs.

Introduction to Limits

Definition: Limit at a Point Let f be a function defined on an open interval containing a (except possibly at a). We say

$$\lim_{x \to a} f(x) = L$$

if for every sequence of values x_n approaching a (with $x_n \neq a$), the sequence $f(x_n)$ approaches L.

Definition: One-Sided Limits

$$\lim_{x \to a^-} f(x) = L \quad \text{(left-hand limit)}$$
$$\lim_{x \to a^+} f(x) = L \quad \text{(right-hand limit)}$$

The limit $\lim_{x \to a} f(x)$ exists if and only if both one-sided limits exist and are equal:

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Definition: Limits at Infinity

$$\lim_{x \to \infty} f(x) = L \quad \text{if } f(x) \to L \text{ as } x \to +\infty$$
$$\lim_{x \to -\infty} f(x) = L \quad \text{if } f(x) \to L \text{ as } x \to -\infty$$

Definition: Infinite Limits

$$\lim_{x \to a} f(x) = +\infty \quad \text{if } f(x) \text{ grows without bound as } x \to a$$
$$\lim_{x \to a} f(x) = -\infty \quad \text{if } f(x) \text{ becomes arbitrarily negative as } x \to a$$

Examples **Example 1:** $\lim_{x \to 2}(x+1)$

$$f(1.9) = 2.9$$
$$f(1.99) = 2.99$$
$$f(1.999) = 2.999$$
$$f(2.001) = 3.001$$
$$f(2.01) = 3.01$$
$$f(2.1) = 3.1$$

As $x \to 2$, we have $f(x) \to 3$. Therefore:

$$\lim_{x \to 2}(x+1) = 3$$

Example 2: $\lim_{x \to 2} \frac{x^2-4}{x-2}$
The function is undefined at $x = 2$. For $x \neq 2$:

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$$

Evaluating near $x = 2$:

$$g(1.9) = 3.9$$
$$g(1.99) = 3.99$$
$$g(2.01) = 4.01$$
$$g(2.1) = 4.1$$

Therefore:

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$$

Example 3: $\lim_{x \to \infty} \frac{1}{x}$

$$f(1) = 1$$
$$f(10) = 0.1$$
$$f(100) = 0.01$$
$$f(1000) = 0.001$$

As $x \to \infty$, we have $\frac{1}{x} \to 0$. Therefore:

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Example 4: $\lim_{x \to 0} \frac{1}{x}$
Left-hand limit:

$$f(-0.1) = -10$$
$$f(-0.01) = -100$$
$$f(-0.001) = -1000$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Right-hand limit:

$$f(0.1) = 10$$
$$f(0.01) = 100$$
$$f(0.001) = 1000$$

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

Since the one-sided limits are not equal, $\lim_{x \to 0} \frac{1}{x}$ does not exist.
Example 5: $\lim_{x \to 0} \frac{\sin(x)}{x}$
Evaluating near $x = 0$:

$$f(0.1) \approx 0.9983$$
$$f(0.01) \approx 0.99998$$
$$f(0.001) \approx 0.9999998$$
$$f(-0.1) \approx 0.9983$$
$$f(-0.01) \approx 0.99998$$

Therefore:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Example 6: $\lim_{x \to -\infty} e^x$

$$f(-1) = e^{-1} \approx 0.368$$
$$f(-2) = e^{-2} \approx 0.135$$
$$f(-3) = e^{-3} \approx 0.050$$
$$f(-10) = e^{-10} \approx 0.000045$$

As $x \to -\infty$, we have $e^x \to 0$. Therefore:

$$\lim_{x \to -\infty} e^x = 0$$

The Graph of a Function

Definition: The graph of a function f is the set of all ordered pairs $(x, f(x))$ where x is in the domain of f.

In other words, the graph consists of all points (x, y) such that $y = f(x)$.

Plotting Functions: Step-by-Step

Procedure

1. **Choose domain:** Select x-values that show key features.
2. **Evaluate:** Compute $f(x)$ at multiple points.
3. **Plot points:** Mark $(x, f(x))$ on coordinate plane.
4. **Connect:** Draw smooth curve through points (for continuous functions).

Examples: Plotting Functions

Example 1: Plot $f(x) = x^2$ on $[-2, 2]$

Step 1: Choose domain $[-2, 2]$.

Step 2: Evaluate at sample points:

$$\begin{aligned} f(-2) &= 4 \\ f(-1.5) &= 2.25 \\ f(-1) &= 1 \\ f(-0.5) &= 0.25 \\ f(0) &= 0 \\ f(0.5) &= 0.25 \\ f(1) &= 1 \\ f(1.5) &= 2.25 \\ f(2) &= 4 \end{aligned}$$

Step 3: Points: $(-2, 4), (-1.5, 2.25), (-1, 1), (-0.5, 0.25), (0, 0), (0.5, 0.25), (1, 1), (1.5, 2.25), (2, 4)$.

Step 4: The function is symmetric: $f(-x) = f(x)$ for all x.

Notice how the points form a smooth curve. The function is symmetric about the y-axis because $f(-x) = (-x)^2 = x^2 = f(x)$.[1]

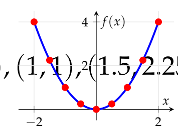

Figure 1: Graph of $f(x) = x^2$ with sample points marked.

[1] This property is called *even symmetry*. Functions with even symmetry satisfy $f(-x) = f(x)$ for all x in their domain.

Key Features to Identify in Graphs

Definitions

- **x-intercepts:** Points where $f(x) = 0$. Found by solving $f(x) = 0$.
- **y-intercept:** Point $(0, f(0))$ if $0 \in \text{dom}(f)$. Found by evaluating $f(0)$.
- **Behavior at extremes:** Determined by $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
- **Asymptotes:**
  - Vertical: $x = a$ if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$
  - Horizontal: $y = L$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$
- **Special points:** Local extrema, inflection points (covered in Describing Functions).

- **Regions:** Intervals of increase/decrease, concavity (covered in Describing Functions).

Example 1: Analyze $g(x) = \frac{1}{x}$
Domain: $(-\infty, 0) \cup (0, \infty)$
Evaluating at sample points:

$$\begin{aligned} g(-2) &= -0.5 \\ g(-1) &= -1 \\ g(-0.5) &= -2 \\ g(0.5) &= 2 \\ g(1) &= 1 \\ g(2) &= 0.5 \end{aligned}$$

Limits:

$$\begin{aligned} \lim_{x \to 0^-} \frac{1}{x} &= -\infty \\ \lim_{x \to 0^+} \frac{1}{x} &= +\infty \\ \lim_{x \to \infty} \frac{1}{x} &= 0 \\ \lim_{x \to -\infty} \frac{1}{x} &= 0 \end{aligned}$$

Asymptotes: Vertical asymptote at $x = 0$, horizontal asymptote at $y = 0$.

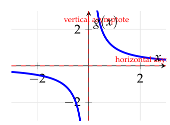

Figure 2: Graph of $g(x) = \frac{1}{x}$ showing vertical and horizontal asymptotes.

Example 2: Analyze $h(x) = x^2 - 4$

Domain: $(-\infty, \infty)$

x-intercepts: Solve $h(x) = 0$:

$$\begin{aligned} x^2 - 4 &= 0 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

x-intercepts at $(-2, 0)$ and $(2, 0)$.

y-intercept:

$$h(0) = 0^2 - 4 = -4$$

y-intercept at $(0, -4)$.

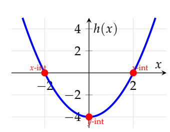

Figure 3: Graph of $h(x) = x^2 - 4$ showing intercepts.

Behavior at extremes:

$$\begin{aligned} \lim_{x \to \infty} (x^2 - 4) &= \infty \\ \lim_{x \to -\infty} (x^2 - 4) &= \infty \end{aligned}$$

Asymptotes: No vertical or horizontal asymptotes.

Example 3: Analyze $k(x) = x^3 - x$

Domain: $(-\infty, \infty)$

x**-intercepts:** Solve $k(x) = 0$:

$$\begin{aligned} x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ x(x-1)(x+1) &= 0 \end{aligned}$$

x-intercepts at $(-1, 0)$, $(0, 0)$, and $(1, 0)$.

y**-intercept:**

$$k(0) = 0^3 - 0 = 0$$

y-intercept at $(0, 0)$.

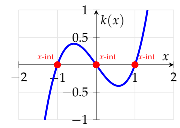

Figure 4: Graph of $k(x) = x^3 - x$ showing three x-intercepts.

Behavior at extremes:

$$\lim_{x \to \infty} (x^3 - x) = \infty$$
$$\lim_{x \to -\infty} (x^3 - x) = -\infty$$

Asymptotes: No vertical or horizontal asymptotes.

Example 4: Analyze $m(x) = \frac{x^2-1}{x-1}$

Domain: $(-\infty, 1) \cup (1, \infty)$

Simplification: For $x \neq 1$:

$$\begin{aligned} m(x) &= \frac{x^2 - 1}{x - 1} \\ &= \frac{(x-1)(x+1)}{x-1} \\ &= x + 1 \end{aligned}$$

x**-intercepts:** Solve $m(x) = 0$:

$$\begin{aligned} x + 1 &= 0 \\ x &= -1 \end{aligned}$$

x-intercept at $(-1, 0)$.

y**-intercept:**

$$m(0) = 0 + 1 = 1$$

y-intercept at $(0, 1)$.

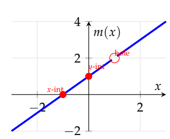

Figure 5: Graph of $m(x) = \frac{x^2-1}{x-1}$ showing removable discontinuity at $x = 1$.

Behavior at extremes:

$$\lim_{x \to \infty} m(x) = \lim_{x \to \infty} (x+1) = \infty$$
$$\lim_{x \to -\infty} m(x) = \lim_{x \to -\infty} (x+1) = -\infty$$

Discontinuity: Removable discontinuity at $x = 1$. The limit exists:

$$\lim_{x \to 1} m(x) = \lim_{x \to 1} (x+1) = 2$$

but $m(1)$ is undefined.

Asymptotes: No vertical or horizontal asymptotes.