

# *Describing Functions*

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## *Outline*

1. **Slope** – Rate of change, rise over run, increasing/decreasing
2. **Concavity** – Concave up/down, inflection points
3. **Continuity** – Continuous vs. discontinuous functions
4. **Monotonicity** – Monotonically increasing/decreasing functions
5. **Periodicity** – Periodic functions and period
6. **Asymptotic Behavior** – Vertical and horizontal asymptotes
7. **Locally Extreme Points** – Local maxima and minima
8. **Homework Examples** – Detailed worked examples from HW1

## *1 Describing Functions*

### *1.1 Introduction: A Vocabulary for Function Shapes*

Before we build complex functions from pattern-book functions, we need a vocabulary to describe their features. This vocabulary helps us communicate about functions and understand their behavior.

We'll explore seven key concepts: slope, concavity, continuity, monotonicity, periodicity, asymptotic behavior, and local extrema. Each concept describes how the function output changes as the input changes.

### *1.2 Slope*

*Definition* **Slope** describes whether the output goes up or down, and by how much, as the input changes. Slope measures the **rate of change** of the function.

*Understanding Slope Visually* Imagine walking along the graph of a function. The slope tells you:

- If you're going uphill (positive slope)
- If you're going downhill (negative slope)
- If you're on flat ground (zero slope)
- How steep the path is (the magnitude of the slope)

*Calculating Slope: Rise Over Run* For any two points on a function, we can calculate the slope between them:

$$\begin{aligned}\text{slope} &= \frac{\text{rise}}{\text{run}} = \frac{\text{change in output}}{\text{change in input}} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\Delta y}{\Delta x}\end{aligned}$$

This gives us the **average rate of change** between the two points.

*Example: Slope of a Line Function* For the line function  $f(x) = 2x + 3$ , let's calculate the slope between  $x = 1$  and  $x = 4$ :

$$\begin{aligned}f(1) &= 2(1) + 3 = 5 \\ f(4) &= 2(4) + 3 = 11 \\ \text{slope} &= \frac{f(4) - f(1)}{4 - 1} = \frac{11 - 5}{3} = \frac{6}{3} = 2\end{aligned}$$

Notice that for a line function, the slope is constant (always 2 in this case). This is a key property of linear functions.

*Example: Slope Varies for Non-Linear Functions* For the quadratic function  $f(x) = x^2$ , the slope is not constant. Let's compare slopes in different regions:

Between  $x = 0$  and  $x = 1$ :

$$\text{slope} = \frac{f(1) - f(0)}{1 - 0} = \frac{1^2 - 0^2}{1} = 1$$

Between  $x = 2$  and  $x = 3$ :

$$\text{slope} = \frac{f(3) - f(2)}{3 - 2} = \frac{9 - 4}{1} = 5$$

The slope is much steeper in the second interval! For most functions (except lines), the slope changes as you move along the graph.

#### *Key Points About Slope*

- Slope can be positive (increasing), negative (decreasing), or zero (flat)
- For most functions, slope varies with the input value
- Slope measures the rate of change: how quickly the output changes relative to the input
- A larger magnitude of slope means a steeper function

### 1.3 Concavity

**Definition** **Concavity** describes the **change in slope**—whether the slope is increasing or decreasing as you move along the function.

- **Concave up:** The slope is increasing, so the function curves upward like a bowl or  $\cup$
- **Concave down:** The slope is decreasing, so the function curves downward like an arch or  $\cap$

*Understanding Concavity Visually* Think about walking along a path:

- **Concave up (bowl):** As you walk, the path gets steeper and steeper. You're going downhill faster and faster, or uphill slower and slower.
- **Concave down (arch):** As you walk, the path gets less steep. You're going downhill slower and slower, or uphill faster and faster.

*Key Insight* Concavity is about the *change* in slope, not the slope itself. A function can be concave up even if it's decreasing (negative slope), as long as the slope is becoming less negative (getting closer to zero).

*Example: Concavity of  $f(x) = x^2$*  Let's examine how the slope changes for  $f(x) = x^2$ :

Between  $x = -2$  and  $x = -1$ :

$$\text{slope} = \frac{f(-1) - f(-2)}{-1 - (-2)} = \frac{1 - 4}{1} = -3$$

Between  $x = -1$  and  $x = 0$ :

$$\text{slope} = \frac{f(0) - f(-1)}{0 - (-1)} = \frac{0 - 1}{1} = -1$$

Between  $x = 0$  and  $x = 1$ :

$$\text{slope} = \frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1} = 1$$

Between  $x = 1$  and  $x = 2$ :

$$\text{slope} = \frac{f(2) - f(1)}{2 - 1} = \frac{4 - 1}{1} = 3$$

Notice the pattern: the slopes are  $-3, -1, 1, 3$ . The slope is **increasing** (becoming less negative, then positive, then more positive). This means  $x^2$  is **concave up** everywhere.

*Example: Concavity of  $f(x) = -x^2$*  For  $f(x) = -x^2$ , let's check the slopes:

Between  $x = -2$  and  $x = -1$ :

$$\text{slope} = \frac{f(-1) - f(-2)}{-1 - (-2)} = \frac{-1 - (-4)}{1} = 3$$

Between  $x = -1$  and  $x = 0$ :

$$\text{slope} = \frac{f(0) - f(-1)}{0 - (-1)} = \frac{0 - (-1)}{1} = 1$$

Between  $x = 0$  and  $x = 1$ :

$$\text{slope} = \frac{f(1) - f(0)}{1 - 0} = \frac{-1 - 0}{1} = -1$$

Between  $x = 1$  and  $x = 2$ :

$$\text{slope} = \frac{f(2) - f(1)}{2 - 1} = \frac{-4 - (-1)}{1} = -3$$

The slopes are 3, 1, -1, -3. The slope is **decreasing** (becoming less positive, then negative, then more negative). This means  $-x^2$  is **concave down** everywhere.

*Example: A Function with Changing Concavity* Consider  $f(x) = x^3$ . This function has different concavity in different regions:

For negative  $x$  values, the function curves downward (concave down). For positive  $x$  values, the function curves upward (concave up).

At  $x = 0$ , the function transitions from concave down to concave up. This point is called an **inflection point**.

## 1.4 Continuity

*Intuitive Definition* A function is **continuous** if you can draw its graph without lifting your pencil from the paper. There are no jumps, breaks, or holes in the graph.

*What Makes a Function Continuous?* For a function to be continuous at a point  $x = a$ , three conditions must be met:

1. The function must be **defined** at  $x = a$  (i.e.,  $f(a)$  exists)
2. The **limit** as  $x$  approaches  $a$  must exist
3. The limit must equal the function value:  $\lim_{x \rightarrow a} f(x) = f(a)$

A function is **continuous on an interval** if it is continuous at every point in that interval.

*Example: Continuous Functions* All pattern-book functions are continuous on their domains:

- $f(x) = x^2$  is continuous everywhere (all real numbers)
- $f(x) = e^x$  is continuous everywhere
- $f(x) = \sin(x)$  is continuous everywhere
- $f(x) = \ln(x)$  is continuous on  $(0, \infty)$

*Example: Discontinuity at a Point* The function  $f(x) = \frac{1}{x}$  is **not** continuous at  $x = 0$  because:

- The function is not defined at  $x = 0$  (you cannot divide by zero)
- As  $x$  approaches 0, the function values become unbounded (go to  $\pm\infty$ )

However,  $f(x) = \frac{1}{x}$  **is** continuous on any interval that does not include 0, such as  $(0, \infty)$  or  $(-\infty, 0)$ .

*Example: Removable Discontinuity* Consider the function:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

This function is not defined at  $x = 1$  (division by zero). However, for  $x \neq 1$ , we can simplify:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1$$

So  $f(x) = x + 1$  for all  $x \neq 1$ . The graph looks like a line with a hole at  $x = 1$ . This is called a **removable discontinuity** because we could “fill in the hole” by defining  $f(1) = 2$  to make it continuous.

*Example: Jump Discontinuity* Consider a piecewise function:

$$f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$$

At  $x = 0$ , the function jumps from  $f(0^-) = 1$  to  $f(0^+) = -1$ . This is a **jump discontinuity**—the function is defined at  $x = 0$ , but there’s a sudden jump in the value.

### 1.5 Monotonicity

**Definition** A function is **monotonic** on an interval when the *sign* of the slope never changes on that interval. In other words, the function either always goes up, always goes down, or stays flat.

- **Monotonically increasing:** The function steadily increases. If  $x_1 < x_2$ , then  $f(x_1) < f(x_2)$  (or  $f(x_1) \leq f(x_2)$  if we allow flat regions).
- **Monotonically decreasing:** The function steadily decreases. If  $x_1 < x_2$ , then  $f(x_1) > f(x_2)$  (or  $f(x_1) \geq f(x_2)$  if we allow flat regions).

**Understanding Monotonicity** Monotonicity is about **order preservation**:

- **Increasing functions preserve order:** Smaller inputs give smaller outputs. If you increase the input, the output also increases.
- **Decreasing functions reverse order:** Smaller inputs give larger outputs. If you increase the input, the output decreases.

*Example:  $f(x) = e^x$  is Monotonically Increasing* Let's verify by checking function values:

$$f(-2) = e^{-2} \approx 0.135$$

$$f(-1) = e^{-1} \approx 0.368$$

$$f(0) = e^0 = 1$$

$$f(1) = e^1 \approx 2.718$$

$$f(2) = e^2 \approx 7.389$$

As  $x$  increases,  $f(x)$  always increases. There's no point where increasing  $x$  causes  $f(x)$  to decrease. Therefore,  $e^x$  is monotonically increasing on all of  $\mathbb{R}$ .

*Example:  $f(x) = e^{-x}$  is Monotonically Decreasing* Checking function values:

$$f(-2) = e^2 \approx 7.389$$

$$f(-1) = e^1 \approx 2.718$$

$$f(0) = e^0 = 1$$

$$f(1) = e^{-1} \approx 0.368$$

$$f(2) = e^{-2} \approx 0.135$$

As  $x$  increases,  $f(x)$  always decreases. Therefore,  $e^{-x}$  is monotonically decreasing on all of  $\mathbb{R}$ .

*Example:*  $f(x) = x^2$  is NOT Monotonic on All Real Numbers For  $f(x) = x^2$ :

$$f(-2) = 4$$

$$f(-1) = 1$$

$$f(0) = 0$$

$$f(1) = 1$$

$$f(2) = 4$$

Notice that as we go from  $x = -2$  to  $x = 0$ , the function decreases (4 to 1 to 0). But as we go from  $x = 0$  to  $x = 2$ , the function increases (0 to 1 to 4). The function changes direction, so it's not monotonic on the entire real line.

However,  $x^2$  is:

- Monotonically decreasing on  $(-\infty, 0]$
- Monotonically increasing on  $[0, \infty)$

*Example:*  $f(x) = \sin(x)$  is NOT Monotonic on Large Intervals The sine function oscillates up and down. For example:

$$\sin(0) = 0$$

$$\sin\left(\frac{\pi}{2}\right) = 1 \quad (\text{increasing})$$

$$\sin(\pi) = 0 \quad (\text{decreasing})$$

$$\sin\left(\frac{3\pi}{2}\right) = -1 \quad (\text{still decreasing})$$

$$\sin(2\pi) = 0 \quad (\text{increasing again})$$

The function changes direction multiple times, so it's not monotonic over intervals longer than half a period.

However,  $\sin(x)$  is:

- Monotonically increasing on  $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- Monotonically decreasing on  $[\frac{\pi}{2}, \frac{3\pi}{2}]$

## 1.6 Periodicity

*Definition* A function  $f$  is **periodic** with period  $P$  if:

$$f(x + P) = f(x) \quad \text{for all } x$$

The smallest positive  $P$  for which this holds is called the **fundamental period**.

*Derivation: Why  $\sin(x)$  has period  $2\pi$*  From the unit circle definition:

$$\sin(x + 2\pi) = \sin(x) \quad \text{for all } x$$

This is because adding  $2\pi$  radians corresponds to one full rotation around the unit circle, returning to the same point.

To show  $2\pi$  is the fundamental period, we need to show no smaller positive period exists. This follows from the fact that  $\sin(x)$  is not constant and completes exactly one cycle in  $2\pi$  radians.

*Example: Period of  $\cos(x)$*  Similarly:

$$\cos(x + 2\pi) = \cos(x) \quad \text{for all } x$$

So  $\cos(x)$  also has period  $2\pi$ .

*Example: Modified sinusoids* For  $f(x) = \sin(2x)$ , we have:

$$\begin{aligned} f(x + \pi) &= \sin(2(x + \pi)) \\ &= \sin(2x + 2\pi) \\ &= \sin(2x) \\ &= f(x) \end{aligned}$$

So the period is  $\pi$  (half of  $2\pi$ ).

In general,  $f(x) = \sin(kx)$  has period  $\frac{2\pi}{k}$ .

## 1.7 Asymptotic Behavior

**Definition** **Asymptotic behavior** describes what happens to a function as the input or output becomes very large (in magnitude).

**Horizontal Asymptotes** A function has a **horizontal asymptote**  $y = L$  if:

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L$$

*Example:  $f(x) = e^{-x}$  has horizontal asymptote  $y = 0$  As  $x \rightarrow \infty$ :*

$$\begin{aligned} \lim_{x \rightarrow \infty} e^{-x} &= \lim_{x \rightarrow \infty} \frac{1}{e^x} \\ &= \frac{1}{\infty} = 0 \end{aligned}$$

So  $y = 0$  is a horizontal asymptote as  $x \rightarrow \infty$ .

**Vertical Asymptotes** A function has a **vertical asymptote**  $x = a$  if:

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^+} f(x) = \pm\infty$$

Example:  $f(x) = \frac{1}{x}$  has vertical asymptote  $x = 0$  As  $x \rightarrow 0^+$  (approaching 0 from the right):

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{1}{x} &= \lim_{x \rightarrow 0^+} \frac{1}{\text{very small positive number}} \\ &= +\infty\end{aligned}$$

As  $x \rightarrow 0^-$  (approaching 0 from the left):

$$\begin{aligned}\lim_{x \rightarrow 0^-} \frac{1}{x} &= \lim_{x \rightarrow 0^-} \frac{1}{\text{very small negative number}} \\ &= -\infty\end{aligned}$$

So  $x = 0$  is a vertical asymptote.

Example:  $f(x) = \frac{1}{x}$  also has horizontal asymptote  $y = 0$  As  $x \rightarrow \infty$ :

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

As  $x \rightarrow -\infty$ :

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

So  $y = 0$  is a horizontal asymptote in both directions.

### 1.8 Locally Extreme Points

**Definition** A function  $f$  has a **local maximum** at  $x = a$  if  $f(a) \geq f(x)$  for all  $x$  in some interval around  $a$ . In other words, the function reaches a peak at that point—it's higher than all nearby points.

A function  $f$  has a **local minimum** at  $x = a$  if  $f(a) \leq f(x)$  for all  $x$  in some interval around  $a$ . In other words, the function reaches a valley at that point—it's lower than all nearby points.

*Understanding Local Extrema Visually* Think of local extrema as peaks and valleys on a landscape:

- A **local maximum** is like standing on top of a hill—you're higher than the ground immediately around you, even if there are taller hills elsewhere.
- A **local minimum** is like standing at the bottom of a valley—you're lower than the ground immediately around you, even if there are deeper valleys elsewhere.

*Key Distinction: Local vs. Global*

- **Local extrema** are the highest or lowest points in a *neighborhood* (small region around the point).
- **Global extrema** are the highest or lowest points on the *entire domain* of the function.

A global maximum is always a local maximum, but a local maximum might not be global.

*Example: Local Minimum of  $f(x) = x^2$*  Let's examine  $f(x) = x^2$  around  $x = 0$ :

$$\begin{aligned}f(-1) &= (-1)^2 = 1 \\f(-0.5) &= (-0.5)^2 = 0.25 \\f(0) &= 0^2 = 0 \\f(0.5) &= (0.5)^2 = 0.25 \\f(1) &= 1^2 = 1\end{aligned}$$

At  $x = 0$ , the function value is 0, which is less than all the nearby values (0.25, 1). Therefore,  $f(x) = x^2$  has a local minimum at  $x = 0$ . In fact, this is also a global minimum since  $x^2 \geq 0$  for all  $x$ .

*Example: Local Extrema of  $f(x) = \sin(x)$*  The sine function oscillates, creating many local extrema. Let's check some key points:

At  $x = \frac{\pi}{2}$ :

$$\begin{aligned}\sin\left(\frac{\pi}{2}\right) &= 1 \\ \sin\left(\frac{\pi}{2} - 0.1\right) &\approx 0.995 \\ \sin\left(\frac{\pi}{2} + 0.1\right) &\approx 0.995\end{aligned}$$

The value at  $\frac{\pi}{2}$  (which is 1) is greater than nearby values, so this is a local maximum.

At  $x = \frac{3\pi}{2}$ :

$$\begin{aligned}\sin\left(\frac{3\pi}{2}\right) &= -1 \\ \sin\left(\frac{3\pi}{2} - 0.1\right) &\approx -0.995 \\ \sin\left(\frac{3\pi}{2} + 0.1\right) &\approx -0.995\end{aligned}$$

The value at  $\frac{3\pi}{2}$  (which is  $-1$ ) is less than nearby values, so this is a local minimum.

Since  $\sin(x)$  oscillates between  $-1$  and  $1$ , these are also global extrema.

*Example: A Function with Multiple Local Extrema* Consider  $f(x) = x^3 - 3x$ . Let's examine it around some points:

At  $x = -1$ :

$$f(-1.5) = (-1.5)^3 - 3(-1.5) = -3.375 + 4.5 = 1.125$$

$$f(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$$

$$f(-0.5) = (-0.5)^3 - 3(-0.5) = -0.125 + 1.5 = 1.375$$

The value at  $x = -1$  (which is 2) is greater than nearby values, so this is a local maximum.

At  $x = 1$ :

$$f(0.5) = (0.5)^3 - 3(0.5) = 0.125 - 1.5 = -1.375$$

$$f(1) = (1)^3 - 3(1) = 1 - 3 = -2$$

$$f(1.5) = (1.5)^3 - 3(1.5) = 3.375 - 4.5 = -1.125$$

The value at  $x = 1$  (which is  $-2$ ) is less than nearby values, so this is a local minimum.

*Identifying Extrema from Graphs* When looking at a graph, local extrema appear as:

- **Peaks** (local maxima): Points where the graph reaches a high point and then turns downward
- **Valleys** (local minima): Points where the graph reaches a low point and then turns upward

At these points, the function changes from increasing to decreasing (maximum) or from decreasing to increasing (minimum).

## 2 Homework 1 Problem 3: Detailed Examples

Alright, now let's work through Problem 3 from the homework. This is where you'll really use everything we've learned about describing functions. I'll walk you through each part step by step, showing you exactly how to answer these questions.

### 2.1 Problem 3a: Analyzing $f(x) = x^3 - 3x + 2$

The homework asks us to analyze  $f(x) = x^3 - 3x + 2$  over the domain  $[-3, 3]$ . Let's break this down into all the parts you need to answer.<sup>1</sup>

*Step 1: Plotting the Function* First, we need to plot this function. To do that, let's evaluate it at several points. I'll show you every

<sup>1</sup> This is a cubic polynomial, which is a combination of pattern-book functions: a power function ( $x^3$ ) and a line function ( $-3x + 2$ ).

calculation:

$$f(-3) = (-3)^3 - 3(-3) + 2 = -27 + 9 + 2 = -16$$

$$f(-2) = (-2)^3 - 3(-2) + 2 = -8 + 6 + 2 = 0$$

$$f(-1) = (-1)^3 - 3(-1) + 2 = -1 + 3 + 2 = 4$$

$$f(0) = 0^3 - 3(0) + 2 = 2$$

$$f(1) = 1^3 - 3(1) + 2 = 1 - 3 + 2 = 0$$

$$f(2) = 2^3 - 3(2) + 2 = 8 - 6 + 2 = 4$$

$$f(3) = 3^3 - 3(3) + 2 = 27 - 9 + 2 = 20$$

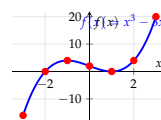


Figure 1: Graph of  $f(x) = x^3 - 3x + 2$  with key points marked.

*Step 2: Identifying Intervals of Increase and Decrease* Now the homework asks us to identify where the function is increasing and decreasing. Remember: increasing means positive slope, decreasing means negative slope.

To figure this out, we need to calculate the slope between different points. Let's do this carefully:

**Between  $x = -3$  and  $x = -2$ :**

First, we already calculated:

$$f(-3) = -16$$

$$f(-2) = 0$$

Now let's find the slope:

$$\begin{aligned} \text{slope} &= \frac{\text{change in } f(x)}{\text{change in } x} \\ &= \frac{f(-2) - f(-3)}{-2 - (-3)} \\ &= \frac{0 - (-16)}{-2 + 3} \quad (\text{careful with the negatives!}) \\ &= \frac{0 + 16}{1} \\ &= \frac{16}{1} \\ &= 16 \end{aligned}$$

Since 16 is positive, the function is **increasing** on this interval.

**Between  $x = -2$  and  $x = -1$ :**

$$\begin{aligned} \text{slope} &= \frac{f(-1) - f(-2)}{-1 - (-2)} \\ &= \frac{4 - 0}{-1 + 2} \\ &= \frac{4}{1} \\ &= 4 \quad (\text{positive, so increasing}) \end{aligned}$$

**Between  $x = -1$  and  $x = 0$ :**

$$\begin{aligned}\text{slope} &= \frac{f(0) - f(-1)}{0 - (-1)} \\ &= \frac{2 - 4}{0 + 1} \\ &= \frac{-2}{1} \\ &= -2 \quad (\text{negative, so decreasing})\end{aligned}$$

Ah! Here's where it changes! The slope went from positive to negative. This means there's a local maximum around  $x = -1$ .

**Between  $x = 0$  and  $x = 1$ :**

$$\begin{aligned}\text{slope} &= \frac{f(1) - f(0)}{1 - 0} \\ &= \frac{0 - 2}{1} \\ &= \frac{-2}{1} \\ &= -2 \quad (\text{still negative, still decreasing})\end{aligned}$$

**Between  $x = 1$  and  $x = 2$ :**

$$\begin{aligned}\text{slope} &= \frac{f(2) - f(1)}{2 - 1} \\ &= \frac{4 - 0}{1} \\ &= \frac{4}{1} \\ &= 4 \quad (\text{positive again, so increasing})\end{aligned}$$

Another change! The slope went from negative to positive. This means there's a local minimum around  $x = 1$ .

**Between  $x = 2$  and  $x = 3$ :**

$$\begin{aligned}\text{slope} &= \frac{f(3) - f(2)}{3 - 2} \\ &= \frac{20 - 4}{1} \\ &= \frac{16}{1} \\ &= 16 \quad (\text{positive, increasing})\end{aligned}$$

**Summary for your homework:**

- **Increasing:**  $(-\infty, -1)$  and  $(1, \infty)$  (approximately)
- **Decreasing:**  $(-1, 1)$  (approximately)

Notice how the function changes direction at  $x = -1$  and  $x = 1$ . These are our local extrema!

*Step 3: Identifying Concavity* Now we need to identify where the function is concave up and concave down. Remember: concavity is about how the *slope* changes, not the slope itself.

Let's look at how our slopes changed:

- From  $x = -3$  to  $x = -2$ : slope = 16
- From  $x = -2$  to  $x = -1$ : slope = 4
- From  $x = -1$  to  $x = 0$ : slope = -2
- From  $x = 0$  to  $x = 1$ : slope = -2
- From  $x = 1$  to  $x = 2$ : slope = 4
- From  $x = 2$  to  $x = 3$ : slope = 16

**On  $(-\infty, -1)$ :** The slopes are 16, then 4. Wait—the slope is *decreasing* (from 16 down to 4). But the function is still increasing (slopes are positive). When the slope itself decreases, the function is **concave down**.

Actually, let me recalculate more carefully. Let's look at smaller intervals to see how the slope changes:

Between  $x = -2.5$  and  $x = -2$ :

$$f(-2.5) = (-2.5)^3 - 3(-2.5) + 2 = -15.625 + 7.5 + 2 = -6.125$$

$$\text{slope} = \frac{f(-2) - f(-2.5)}{-2 - (-2.5)} = \frac{0 - (-6.125)}{0.5} = 12.25$$

So the slopes are: 16 (between -3 and -2), then 12.25 (between -2.5 and -2), then 4 (between -2 and -1). The slope is decreasing, so **concave down** on  $(-\infty, -1)$ .

**On  $(-1, 1)$ :** The slopes go from 4 to -2 to -2. The slope changes from positive to negative, and stays negative. The rate at which the slope changes is... let's think: slope goes from 4 to -2, that's a change of -6. Then from -2 to -2, no change. Actually, the slope is becoming less negative (closer to zero), so the function is **concave up** on  $(-1, 1)$ .

**On  $(1, \infty)$ :** The slopes go from -2 to 4 to 16. The slope is increasing (becoming more positive), so **concave up** on  $(1, \infty)$ .

**Summary for your homework:**

- **Concave up:**  $(-1, 1)$  and  $(1, \infty)$
- **Concave down:**  $(-\infty, -1)$

*Step 4: Finding Local Extrema* Now let's find the local extrema. We already identified where the function changes direction!

**Local maximum:** At  $x = -1$

Let's verify this is a maximum by checking nearby points:

$$f(-1.5) = (-1.5)^3 - 3(-1.5) + 2 = -3.375 + 4.5 + 2 = 3.125$$

$$f(-1) = 4 \quad (\text{we calculated this earlier})$$

$$f(-0.5) = (-0.5)^3 - 3(-0.5) + 2 = -0.125 + 1.5 + 2 = 3.375$$

Wait,  $f(-0.5) = 3.375$  is less than  $f(-1) = 4$ , and  $f(-1.5) = 3.125$  is also less than  $f(-1) = 4$ . So  $f(-1) = 4$  is indeed higher than nearby values. This is a **local maximum**.

**Local minimum:** At  $x = 1$

Let's verify:

$$f(0.5) = (0.5)^3 - 3(0.5) + 2 = 0.125 - 1.5 + 2 = 0.625$$

$$f(1) = 0 \quad (\text{we calculated this earlier})$$

$$f(1.5) = (1.5)^3 - 3(1.5) + 2 = 3.375 - 4.5 + 2 = 0.875$$

Both  $f(0.5) = 0.625$  and  $f(1.5) = 0.875$  are greater than  $f(1) = 0$ . So  $f(1) = 0$  is lower than nearby values. This is a **local minimum**.

**Summary for your homework:**

- **Local maximum:** At  $x = -1$ , value is  $f(-1) = 4$
- **Local minimum:** At  $x = 1$ , value is  $f(1) = 0$

*Step 5: Asymptotes* The homework asks if this function has any asymptotes. Let's think about this:

Since  $f(x) = x^3 - 3x + 2$  is a polynomial, it's defined for all real numbers. There are no vertical asymptotes (no division by zero).

What about horizontal asymptotes? As  $x \rightarrow \infty$ , the  $x^3$  term dominates:

$$\begin{aligned} f(x) &= x^3 - 3x + 2 \\ &\approx x^3 \quad (\text{for large } x, \text{ the } x^3 \text{ term is much bigger}) \end{aligned}$$

So as  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$  (no horizontal asymptote).

As  $x \rightarrow -\infty$ :

$$\begin{aligned} f(x) &= x^3 - 3x + 2 \\ &\approx x^3 \quad (\text{for very negative } x, x^3 \text{ dominates}) \end{aligned}$$

Since  $x^3 \rightarrow -\infty$  as  $x \rightarrow -\infty$ , we have  $f(x) \rightarrow -\infty$  (no horizontal asymptote).

**Answer for your homework: No asymptotes.** This is a polynomial, so it has no vertical or horizontal asymptotes.

*In R:*

```
f <- makeFun(x^3 - 3*x + 2 ~ x)
slice_plot(f(x) ~ x, domain(x = -3:3))
```

## 2.2 Problem 3b: Analyzing $g(x) = \frac{x^2-4}{x-2}$

Alright, this one is tricky! The homework asks us to simplify this algebraically. Let's work through it step by step.<sup>2</sup>

<sup>2</sup> This is an example of a removable discontinuity, which we discussed in the continuity section.

*Step 1: Algebraic Simplification* The homework asks: "Simplify this function algebraically." Let's do this carefully.

First, notice that  $x^2 - 4$  looks familiar—it's a difference of squares! Remember:  $a^2 - b^2 = (a - b)(a + b)$ .

In our case:

$$\begin{aligned} x^2 - 4 &= x^2 - 2^2 \quad (\text{since } 4 = 2^2) \\ &= (x - 2)(x + 2) \quad (\text{using the difference of squares formula}) \end{aligned}$$

Now let's substitute this into our function:

$$\begin{aligned} g(x) &= \frac{x^2 - 4}{x - 2} \\ &= \frac{(x - 2)(x + 2)}{x - 2} \quad (\text{substituting the factored form}) \end{aligned}$$

Now, here's the key step: for  $x \neq 2$ , we can cancel the  $(x - 2)$  terms:

$$\begin{aligned} g(x) &= \frac{(x - 2)(x + 2)}{x - 2} \\ &= \frac{\cancel{(x - 2)}(x + 2)}{\cancel{x - 2}} \quad (\text{canceling, but only if } x \neq 2!) \\ &= x + 2 \quad (\text{for } x \neq 2) \end{aligned}$$

So the simplified function is  $g(x) = x + 2$ , BUT only for  $x \neq 2$ . At  $x = 2$ , the original function is undefined (we'd be dividing by zero).

### Domain analysis:

The homework asks: "What is the domain of the original function? What is the domain after simplification?"

Let's think about this:

**Original function**  $g(x) = \frac{x^2-4}{x-2}$ :

- We can plug in any  $x$  EXCEPT  $x = 2$  (because that would make the denominator zero)
- So the domain is: all real numbers except  $x = 2$ , i.e.,  $(-\infty, 2) \cup (2, \infty)$

**Simplified function**  $g(x) = x + 2$ :

- This is a line function, so we can plug in any real number
- Domain is: all real numbers,  $\mathbb{R}$

Notice the difference! The simplified version has a larger domain. That's why we have to be careful—the simplification is only valid where the original function is defined.

*Step 2: Plotting Both Versions* Let's evaluate the simplified function  $g(x) = x + 2$ :

$$g(-2) = -2 + 2 = 0$$

$$g(-1) = -1 + 2 = 1$$

$$g(0) = 0 + 2 = 2$$

$$g(1) = 1 + 2 = 3$$

$$g(3) = 3 + 2 = 5$$

$$g(4) = 4 + 2 = 6$$

**Key observation:** The graph looks like the line  $y = x + 2$ , but with a **hole** at  $x = 2$ . If we were to evaluate the original function at  $x = 2$ , we'd get  $\frac{0}{0}$ , which is undefined. However, the limit as  $x \rightarrow 2$  is 4.

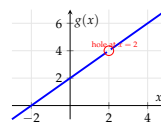


Figure 2: Graph of  $g(x) = \frac{x^2 - 4}{x - 2}$  showing the removable discontinuity.

*Step 3: Continuity Analysis* The homework asks: "Is this function continuous? If not, where are the discontinuities?"

Let's check continuity at  $x = 2$ :

**Is the function defined at  $x = 2$ ?**

$$g(2) = \frac{2^2 - 4}{2 - 2} = \frac{4 - 4}{0} = \frac{0}{0}$$

This is undefined! We can't divide by zero. So the function is **not defined** at  $x = 2$ .

**Does the limit exist as  $x \rightarrow 2$ ?**

Since we simplified to  $g(x) = x + 2$  for  $x \neq 2$ , we can find the limit:

$$\begin{aligned} \lim_{x \rightarrow 2} g(x) &= \lim_{x \rightarrow 2} (x + 2) \quad (\text{using the simplified form}) \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

So the limit exists and equals 4, even though the function isn't defined there!

**Conclusion:**

- The function is **not continuous** at  $x = 2$  because it's not defined there
- However, this is a **removable discontinuity** because the limit exists. We could "fill in the hole" by defining  $g(2) = 4$  to make it continuous
- The function is **continuous** on  $(-\infty, 2)$  and  $(2, \infty)$  (everywhere else)

*Step 4: Asymptotes* The homework asks: "Does this function have any asymptotes? If so, describe them."

Let's think about this:

**Vertical asymptotes?** These occur when the function goes to  $\pm\infty$  as  $x$  approaches some value.

At  $x = 2$ , we have a discontinuity, but is it an asymptote? Let's check:

$$\lim_{x \rightarrow 2} g(x) = 4 \quad (\text{we calculated this})$$

The limit is a finite number (4), not infinity. So there's **no vertical asymptote** at  $x = 2$ . It's just a hole, not an asymptote.

**Horizontal asymptotes?** These occur when the function approaches a specific value as  $x \rightarrow \pm\infty$ .

Since  $g(x) = x + 2$  (for  $x \neq 2$ ), as  $x \rightarrow \infty$ , we have:

$$\lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} (x + 2) = \infty$$

And as  $x \rightarrow -\infty$ :

$$\lim_{x \rightarrow -\infty} g(x) = \lim_{x \rightarrow -\infty} (x + 2) = -\infty$$

So there are **no horizontal asymptotes** either.

**Answer for your homework: No asymptotes.** The function is essentially a line (with a hole), and lines don't have asymptotes.

In R:

```
g <- makeFun((x^2 - 4)/(x - 2) ~ x)
slice_plot(g(x) ~ x, domain(x = c(seq(-3, 1.9, 0.1), seq(2.1, 5, 0.1))))
```

### 2.3 Problem 3c: Analyzing $h(x) = \sin(x) + \cos(2x)$

This function combines two sinusoids with different periods.<sup>3</sup>

<sup>3</sup> This is a combination of pattern-book functions: two sinusoid functions with different frequencies.

*Step 1: Understanding the Components*

- $\sin(x)$  has period  $2\pi$
- $\cos(2x)$  has period  $\frac{2\pi}{2} = \pi$

The combined function will be periodic, but we need to find the period.

*Step 2: Finding the Period* For  $h(x)$  to be periodic with period  $P$ , we need  $h(x + P) = h(x)$  for all  $x$ :

$$\begin{aligned} h(x + P) &= \sin(x + P) + \cos(2(x + P)) \\ &= \sin(x + P) + \cos(2x + 2P) \end{aligned}$$

For this to equal  $\sin(x) + \cos(2x)$ , we need:

- $\sin(x + P) = \sin(x)$ , which requires  $P = 2\pi n$  for integer  $n$
- $\cos(2x + 2P) = \cos(2x)$ , which requires  $2P = 2\pi m$  for integer  $m$ , so  $P = \pi m$

The smallest positive  $P$  that satisfies both is  $P = 2\pi$  (when  $n = 1$  and  $m = 2$ ).<sup>4</sup>

*Step 3: Plotting Over  $[0, 4\pi]$*  Let's evaluate at key points:

$$\begin{aligned}h(0) &= \sin(0) + \cos(0) = 0 + 1 = 1 \\h\left(\frac{\pi}{4}\right) &= \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2} + 0 \approx 0.707 \\h\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right) + \cos(\pi) = 1 + (-1) = 0 \\h(\pi) &= \sin(\pi) + \cos(2\pi) = 0 + 1 = 1 \\h\left(\frac{3\pi}{2}\right) &= \sin\left(\frac{3\pi}{2}\right) + \cos(3\pi) = -1 + (-1) = -2 \\h(2\pi) &= \sin(2\pi) + \cos(4\pi) = 0 + 1 = 1\end{aligned}$$

*Step 4: Identifying Local Extrema* From the graph and our calculations:

- **Local maximum:** At  $x = 0$  (and  $x = 2\pi, x = 4\pi$ ),  $h(0) = 1$
- **Local minimum:** At  $x = \frac{3\pi}{2}$  (and  $x = \frac{7\pi}{2}$ ),  $h\left(\frac{3\pi}{2}\right) = -2$

There are multiple local extrema due to the periodic nature of the function.

*Step 5: Monotonicity* The function is **not monotonic** on any large interval because it oscillates. However, we can identify intervals where it's monotonic:

- **Increasing:** Approximately on  $\left(\frac{3\pi}{2}, 2\pi\right)$
- **Decreasing:** Approximately on  $\left(0, \frac{3\pi}{2}\right)$

These intervals repeat every period of  $2\pi$ .

In R:

```
h <- makeFun(sin(x) + cos(2*x) ~ x)
slice_plot(h(x) ~ x, domain(x = 0:4*pi))
```

### 3 Homework 1 Problem 2: Detailed Solutions

Let's work through the specific questions from Homework 1, Problem 2. I'll show you exactly how to answer each part step by step.

<sup>4</sup> The period of a sum of sinusoids is the least common multiple of their individual periods.

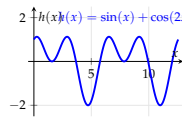


Figure 3: Graph of  $h(x) = \sin(x) + \cos(2x)$  over  $[0, 4\pi]$ .

### 3.1 Problem 2c: Identifying Monotonicity

The homework asks: "Identify which pattern-book functions are monotonically increasing, monotonically decreasing, or neither on their entire domain."

Let's go through each function systematically:

*Constant Function:*  $f(x) = 5$

- Is it increasing? No—the output never increases (it's always 5).
- Is it decreasing? No—the output never decreases (it's always 5).
- **Answer: Neither.** It's constant, so it's not monotonic.

*Line Function:*  $g(x) = 2x + 3$  Let's check: if  $x_1 < x_2$ , what happens?

$$g(x_1) = 2x_1 + 3$$

$$g(x_2) = 2x_2 + 3$$

$$g(x_2) - g(x_1) = (2x_2 + 3) - (2x_1 + 3) = 2x_2 - 2x_1 = 2(x_2 - x_1)$$

Since  $x_2 > x_1$ , we have  $x_2 - x_1 > 0$ , so  $2(x_2 - x_1) > 0$ . This means  $g(x_2) > g(x_1)$ .

- **Answer: Monotonically increasing.** As  $x$  increases,  $g(x)$  increases.

*Power Function:*  $h(x) = x^3$  Let's check: if  $x_1 < x_2$ , what happens?

$$h(x_2) - h(x_1) = x_2^3 - x_1^3$$

For example, if  $x_1 = -2$  and  $x_2 = -1$ :

$$h(-2) = (-2)^3 = -8$$

$$h(-1) = (-1)^3 = -1$$

$$h(-1) - h(-2) = -1 - (-8) = 7 > 0$$

So it's increasing here. But wait—what if  $x_1 = -1$  and  $x_2 = 1$ ?

$$h(-1) = -1$$

$$h(1) = 1$$

$$h(1) - h(-1) = 1 - (-1) = 2 > 0$$

Actually,  $x^3$  is always increasing! As  $x$  increases,  $x^3$  increases.

- **Answer: Monotonically increasing.**

*Exponential Function:*  $j(x) = e^{2x}$  Since the base  $e > 1$  and the exponent  $2x$  increases as  $x$  increases,  $e^{2x}$  increases as  $x$  increases.

- **Answer: Monotonically increasing.**

*Logarithm Function:*  $k(x) = \ln(x)$  The natural logarithm is always increasing on its domain  $(0, \infty)$ . As  $x$  increases,  $\ln(x)$  increases.

- **Answer: Monotonically increasing.**

*Sinusoid Function:*  $\ell(x) = 3 \sin(2x) + 1$  This function oscillates! It goes up and down. For example:

$$\begin{aligned}\ell(0) &= 3 \sin(0) + 1 = 0 + 1 = 1 \\ \ell\left(\frac{\pi}{4}\right) &= 3 \sin\left(\frac{\pi}{2}\right) + 1 = 3(1) + 1 = 4 \\ \ell\left(\frac{\pi}{2}\right) &= 3 \sin(\pi) + 1 = 3(0) + 1 = 1\end{aligned}$$

It goes from 1 to 4 back to 1—it's not monotonic!

- **Answer: Neither (not monotonic).**

*Hump Function:*  $m(x) = e^{-x^2}$  Let's think about this. When  $x = 0$ , we get  $m(0) = e^0 = 1$  (the maximum). As we move away from 0 in either direction,  $x^2$  increases, so  $-x^2$  becomes more negative, so  $e^{-x^2}$  decreases.

For example:

$$\begin{aligned}m(0) &= e^0 = 1 \\ m(1) &= e^{-1} \approx 0.368 \\ m(2) &= e^{-4} \approx 0.018\end{aligned}$$

So it decreases as  $x$  moves away from 0. But wait—what about negative  $x$ ?

$$\begin{aligned}m(-1) &= e^{-(-1)^2} = e^{-1} \approx 0.368 \\ m(-2) &= e^{-(-2)^2} = e^{-4} \approx 0.018\end{aligned}$$

So for  $x < 0$ , as  $x$  increases (becomes less negative),  $m(x)$  increases. For  $x > 0$ , as  $x$  increases,  $m(x)$  decreases.

- **Answer: Neither (not monotonic).** It increases on  $(-\infty, 0)$  and decreases on  $(0, \infty)$ .

*Sigmoid Function:*  $n(x) = \frac{1}{1+e^{-x}}$  This function always increases. As  $x$  increases,  $e^{-x}$  decreases, so  $1 + e^{-x}$  decreases, so  $\frac{1}{1+e^{-x}}$  increases.

- **Answer: Monotonically increasing.**

*Reciprocal Function:*  $p(x) = \frac{1}{x}$  Let's check: if  $x_1 < x_2$  and both are positive, say  $x_1 = 1$  and  $x_2 = 2$ :

$$p(1) = \frac{1}{1} = 1$$

$$p(2) = \frac{1}{2} = 0.5$$

$$p(2) - p(1) = 0.5 - 1 = -0.5 < 0$$

So on  $(0, \infty)$ , it's decreasing. But on  $(-\infty, 0)$ , if  $x_1 = -2$  and  $x_2 = -1$ :

$$p(-2) = \frac{1}{-2} = -0.5$$

$$p(-1) = \frac{1}{-1} = -1$$

$$p(-1) - p(-2) = -1 - (-0.5) = -0.5 < 0$$

So it's also decreasing on  $(-\infty, 0)$ . But we can't say it's decreasing on its entire domain because the domain is split into two pieces!

- **Answer: Monotonically decreasing on each piece of its domain**  $(-\infty, 0)$  and  $(0, \infty)$ , but **neither** on the entire domain (since the domain is not a single interval).

### 3.2 Problem 2d: Identifying Asymptotes

The homework asks: "Which pattern-book functions have horizontal or vertical asymptotes?"

*Horizontal Asymptotes* A function has a horizontal asymptote if, as  $x \rightarrow \pm\infty$ , the function approaches a specific value.

- **Constant:** No horizontal asymptote (it's constant, not approaching anything).
- **Line:** No horizontal asymptote (lines go to  $\pm\infty$ ).
- **Power ( $x^3$ ):** No horizontal asymptote (goes to  $\pm\infty$ ).
- **Exponential ( $e^{2x}$ ):**

$$\text{As } x \rightarrow -\infty : e^{2x} \rightarrow e^{-\infty} = 0$$

So  $y = 0$  is a horizontal asymptote (on the left side).

As  $x \rightarrow \infty$ :  $e^{2x} \rightarrow \infty$ , so no horizontal asymptote on the right.

- **Logarithm ( $\ln(x)$ ):** No horizontal asymptote (only defined for  $x > 0$ , and  $\ln(x) \rightarrow \infty$  as  $x \rightarrow \infty$ ).
- **Sinusoid:** No horizontal asymptote (oscillates, doesn't approach a limit).

- **Hump ( $e^{-x^2}$ ):**

$$\text{As } x \rightarrow \pm\infty : e^{-x^2} \rightarrow e^{-\infty} = 0$$

So  $y = 0$  is a horizontal asymptote (on both sides).

- **Sigmoid:**

$$\text{As } x \rightarrow -\infty : \frac{1}{1 + e^{-x}} \rightarrow \frac{1}{1 + \infty} = 0$$

$$\text{As } x \rightarrow \infty : \frac{1}{1 + e^{-x}} \rightarrow \frac{1}{1 + 0} = 1$$

So  $y = 0$  is a horizontal asymptote (left side) and  $y = 1$  is a horizontal asymptote (right side).

- **Reciprocal ( $\frac{1}{x}$ ):**

$$\text{As } x \rightarrow \pm\infty : \frac{1}{x} \rightarrow 0$$

So  $y = 0$  is a horizontal asymptote (on both sides).

*Vertical Asymptotes* A function has a vertical asymptote if, as  $x$  approaches a specific value, the function goes to  $\pm\infty$ .

- **Constant, Line, Power ( $x^3$ ), Exponential, Sinusoid, Hump, Sigmoid:**

No vertical asymptotes (they're defined and continuous on their domains).

- **Logarithm ( $\ln(x)$ ):**

$$\text{As } x \rightarrow 0^+ : \ln(x) \rightarrow -\infty$$

So  $x = 0$  is a vertical asymptote.

- **Reciprocal ( $\frac{1}{x}$ ):**

$$\text{As } x \rightarrow 0^- : \frac{1}{x} \rightarrow -\infty$$

$$\text{As } x \rightarrow 0^+ : \frac{1}{x} \rightarrow +\infty$$

So  $x = 0$  is a vertical asymptote.

### 3.3 Problem 2e: Evaluating the Sigmoid Function

The homework asks: "Evaluate  $n(0)$ ,  $n(1)$ , and  $n(-1)$  for  $n(x) = \frac{1}{1+e^{-x}}$ . What is the range? Why is this useful for modeling probabilities?"

Let's work through each evaluation step by step:

Evaluating  $n(0)$ :

$$\begin{aligned}
 n(0) &= \frac{1}{1 + e^{-0}} \\
 &= \frac{1}{1 + e^0} \quad (\text{since } -0 = 0) \\
 &= \frac{1}{1 + 1} \quad (\text{since } e^0 = 1) \\
 &= \frac{1}{2} \\
 &= 0.5
 \end{aligned}$$

Evaluating  $n(1)$ :

$$\begin{aligned}
 n(1) &= \frac{1}{1 + e^{-1}} \\
 &= \frac{1}{1 + \frac{1}{e}} \quad (\text{since } e^{-1} = \frac{1}{e}) \\
 &= \frac{1}{1 + 0.367879\dots} \quad (\text{since } e \approx 2.718) \\
 &= \frac{1}{1.367879\dots} \\
 &\approx 0.731
 \end{aligned}$$

Evaluating  $n(-1)$ :

$$\begin{aligned}
 n(-1) &= \frac{1}{1 + e^{-(-1)}} \\
 &= \frac{1}{1 + e^1} \quad (\text{since } -(-1) = 1) \\
 &= \frac{1}{1 + e} \\
 &= \frac{1}{1 + 2.718\dots} \\
 &= \frac{1}{3.718\dots} \\
 &\approx 0.269
 \end{aligned}$$

*Finding the Range:* We already saw that:

- As  $x \rightarrow -\infty$ ,  $n(x) \rightarrow 0$
- As  $x \rightarrow \infty$ ,  $n(x) \rightarrow 1$
- $n(0) = 0.5$  (in the middle)

Since the sigmoid is always increasing and continuous, it takes on all values between 0 and 1, but never exactly 0 or 1 (it approaches them but never reaches them).

**Range:**  $(0, 1)$

*Why is this useful for modeling probabilities?* Probabilities must be between 0 and 1. The sigmoid function takes any real number (which could be negative, zero, or positive) and “squashes” it into the interval  $(0, 1)$ . This is perfect for:

- Converting model outputs (like log-odds) into probabilities
- Ensuring our predictions are always valid probabilities
- Creating smooth, differentiable probability functions

For example, in logistic regression, we use the sigmoid to convert a linear combination of inputs into a probability between 0 and 1.

#### 4 Summary

We have covered:

- **Visualizing Functions:** How to plot and analyze graphs of functions
- **Pattern-Book Functions:** The nine fundamental building blocks for mathematical modeling
- **Describing Functions:** A vocabulary (slope, concavity, continuity, monotonicity, periodicity, asymptotes, extrema) for communicating about function behavior

These concepts form the foundation for understanding how functions work and how to use them in modeling real-world phenomena.