McCOMBS SCHOOL OF BUSINESS
Salem Center for Policy

Probability

David Puelz

September 16, 2021

Outline

The basics and conditional probability

Independence

Paradoxes, mixtures, and the rule of total probability

Random variables, distributions, and simulation

What is probability?

- A measure of uncertainty
- Answering the question: "How likely is a given event?"
- As with any mathematical concept, there are a set of axioms setting the "ground rules"
- Separately, there are different ways to interpret probability ...

  (i) **frequentist**: limit of relative frequency after repeating an experiment an infinite number of times (coin flip!)

  (ii) **Bayesian**: subjective belief about the likelihood of an event occurence

Probability basics

If A denotes some event, then $P(A)$ is the probability that this event occurs:

- $P(\text{coin lands heads}) = 0.5$
- $P(\text{rainy day in Ireland}) = 0.85$
- $P(\text{cold day in Hell}) = 0.0000001$

And so on...

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Probability basics

Some probabilities are estimated from direct experience over the long run:

- $P(\text{newborn baby is a boy}) = \frac{106}{206}$
- $P(\text{death due to car accident}) = \frac{11}{100{,}000}$
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Others are synthesized from our best judgments about unique events:

- $P(\text{Apple stock goes up after next earnings call}) = 0.54$
- $P(\text{Djokovic wins next US Open}) = 0.4$ (6 to 4 odds)
- etc.

Probability basics: conditioning

A conditional probability is the chance that one thing happens, given that some other thing has already happened.

A great example is a weather forecast: if you look outside this morning and see gathering clouds, you might assume that rain is likely and carry an umbrella.

We express this judgment as a conditional probability: e.g. "the conditional probability of rain this afternoon, given clouds this morning, is 60%."

Probability basics: conditioning

In statistics, we write this a bit more compactly:

- $P(\text{rain this afternoon} \mid \text{clouds this morning}) = 0.6$
- That vertical bar means "given" or "conditional upon."
- The thing on the left of the bar is the event we're interested in.
- The thing on the right of the bar is our knowledge, also called the "conditioning event" or "conditioning variable": what we believe or assume to be true.

$P(A \mid B)$: "the probability of A, given that B occurs."

Probability basics: conditioning

Conditional probabilities are how we express judgments in a way that reflects our partial knowledge.

- You just gave *Manifest* a high rating. What's the conditional probability that you will like *The Bourne Identity* or *Saving Private Ryan*?

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Probability basics: conditioning

A really important fact is that conditional probabilities are **not symmetric**:

$$P(A \mid B) \neq P(B \mid A)$$

As a quick counter-example, let the events A and B be as follows:

- A: "you can dribble a basketball"
- B: "you play in the NBA"

Probability basics: conditioning

- A: "you can dribble a basketball"
- B: "you play in the NBA"

Clearly $P(A \mid B) = 1$: every NBA player can dribble a basketball!

Probability basics: conditioning

- A: "you can dribble a basketball"
- B: "you play in the NBA"

But $P(B \mid A)$ is nearly zero!

Uncertain outcomes and probability models

An uncertain outcome (more formally called a "random process") has two key properties:

1. The set of possible outcomes, called the sample space, *is known* beforehand.

2. The particular outcome that occurs is *not known* beforehand.
We denote the **sample space** as Ω, and some particular element of the sample space as $\omega \in \Omega$

Uncertain outcomes and probability models

Examples:

1. NBA finals, Golden State vs. Toronto:

$$\Omega = \{\text{4-0, 4-1, 4-2, 4-3, 3-4, 2-4, 1-4, 0-4}\}$$

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2. Temperature in degrees F in Austin on a random day:

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3. Number of no-shows on an AA flight from Austin to DFW:

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3. Number of no-shows on an AA flight from Austin to DFW:

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4. Poker hand

$$\Omega = \text{all possible five-card deals from a 52-card deck}$$

Uncertain outcomes and probability

An **event** is a *subset of the sample space*, i.e. $A \subset \Omega$. For example:

1. NBA finals, Golden State vs. Toronto. Let A be the event "Toronto wins". Then

$$A = \{\text{3-4, 2-4, 1-4, 0-4}\} \subset \Omega$$

2. Austin weather. Let A be the event "cooler than 90 degrees". Then

$$A = [10, 90) \subset [10, 115]$$

3. Flight no-shows. Let A be "more than 5 no shows":

$$A = \{6, 7, 8, \ldots, N_{\text{seats}}\}$$

Some set theory concepts

We need some basic set-theory concepts to make sense of probability, since the sample space Ω is a set, and since "events" are subsets of Ω.

Union: $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$

Intersection: $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

Complement: $A^C = \tilde{A} = \{\omega : \omega \notin A\}$

Difference: $A \setminus B = \{\omega : \omega \in A, \omega \notin B\}$

Disjointness: A and B are disjoint if $A \cap B = \emptyset$ (the empty set).

Axioms of probability (Kolmogorov)

These are the ground rules!

Consider an uncertain outcome with sample space Ω. "Probability" $P(\cdot)$ is a set function that maps Ω to the real numbers, such that:

1. Non-negativity: For any event $A \subset \Omega$, $P(A) \geq 0$.
2. Normalization: $P(\Omega) = 1$ and $P(\emptyset) = 0$.
3. Finite additivity: If A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$.

3a. Finite additivity (general): For any sets A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(bonus: prove this with set theory!)

Not that intuitive! Notice no mention of frequencies...

Summary of terms

- Uncertain outcome/"random process": we know the possibilities ahead of time, just not the specific one that occurs
- Sample space: the set of possible outcomes
- Event: a subset of the sample space
- Probability: a function that maps events to real numbers and that obeys Kolmogorov's axioms

OK, so how do we actually *calculate* probabilities?

Calculation

Now that we have an understanding of the axioms, notation, and interpretation, how do we calculate probabilities?

Counting!

(review 6.1.3-6.1.4 in the QSS book ... ways to count objects in structured sets are discussed)

Suppose our sample space Ω is a finite set consisting of N elements $\omega_1, \ldots, \omega_N$.

Suppose further that $P(\omega_i) = 1/N$: each outcome is equally likely, i.e. we have a discrete uniform distribution over possible outcomes.

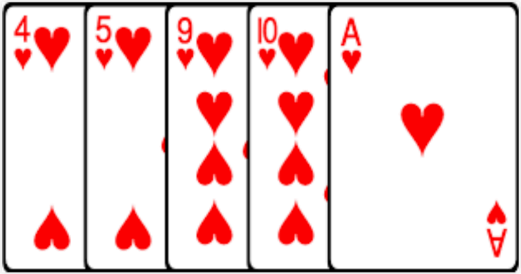
Then for each set $A \subset \Omega$,

$$P(A) = \frac{|A|}{N} = \frac{\text{Number of elements in } A}{\text{Number of elements in } \Omega}$$

That is, to compute $P(A)$, **we just need to count how many elements are in** A.

Counting example

Someone deals you a five-card poker hand from a 52-card deck. What is the probability of a flush (all five cards the same suit)?



Note: this is a very historically accurate illustration of probability, given its origins among bored French aristocrats!

Counting example

- Our sample space has $N = \binom{52}{5} = 2{,}598{,}960$ possible poker hands, each one equally likely.
- How many possible flushes are there? Let's start with hearts:
  $\rightarrow$ There are 13 hearts

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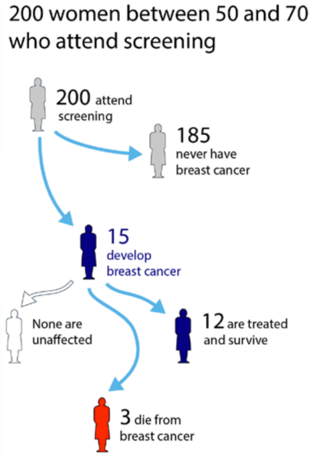
$\rightarrow$ Thus there are $\binom{13}{5} = 1287$ possible flushes with hearts.
$\rightarrow$ The same argument works for all four suits, so there are 4 $\times$ 1287 = 5,148 flushes. Thus:

$$P(\text{flush}) = \frac{|A|}{|\Omega|} = \frac{5148}{2598960} = 0.00198079$$

So we know how to count, but what about conditioning?

Probability trees are very useful for this task! This involves counting at different levels of the tree.

Conditioning example: mammograms



- $P(\text{cancer}) = \frac{15}{200}$
- $P(\text{die,cancer}) = \frac{3}{200}$
- $P(\text{die} \mid \text{cancer}) = \frac{3}{15}$

– In general, we can estimate the conditional probability as:

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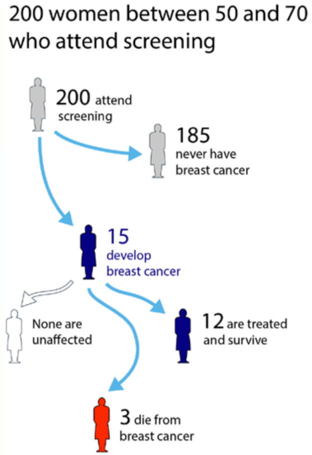

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  - In general, we can estimate the conditional probability as:

$$P(A \mid B) = \frac{\text{Frequency of } A \text{ and } B \text{ both happening}}{\text{Frequency of } B \text{ happening}}$$

This is actually a new axiom

The multiplication rule – it is an axiom since it can't be derived from the original axioms.

$$P(A \mid B) = \frac{P(A, B)}{P(B)}$$

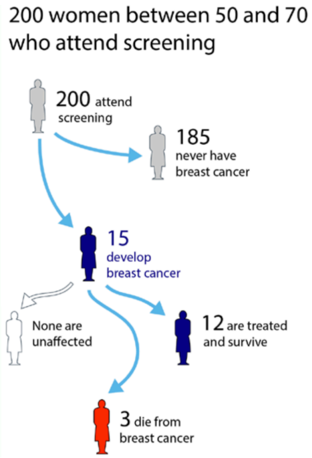
Alternate version

We can also use this alternative version if we want to go in reverse, from a conditional probability to a joint probability.

It says the same thing with the terms rearranged.

$$P(A, B) = P(A \mid B) \cdot P(B)$$

Conditioning example: mammograms (revisited)



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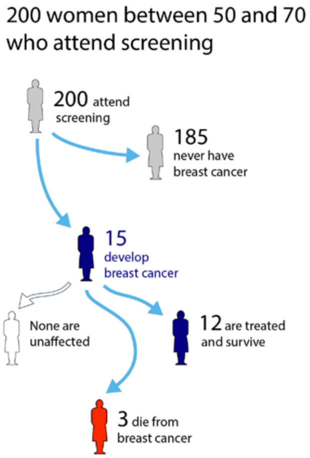

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- $P(\text{die} \mid \text{cancer}) = \frac{3}{15}$
  - Using the **multiplication rule**, we can estimate the **conditional probability** as:

$$P(\text{die} \mid \text{cancer}) = \frac{P(\text{die,cancer})}{P(\text{cancer})} = \frac{3/200}{15/200} = \frac{3}{15}$$

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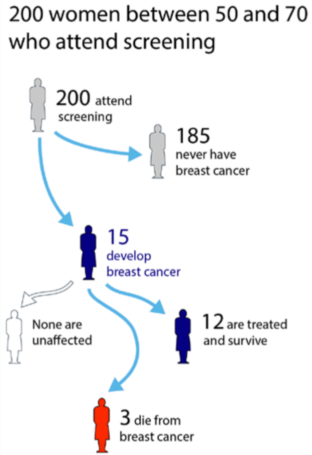

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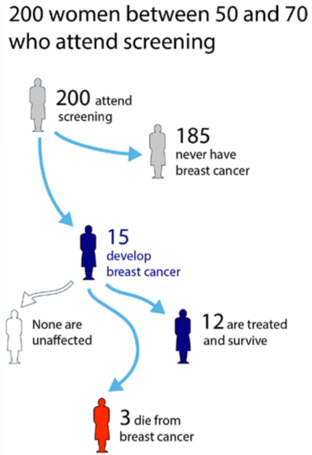

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$$P(\text{die, cancer}) = P(\text{die} \mid \text{cancer}) \cdot P(\text{cancer}) = \frac{3}{15} \cdot \frac{15}{200} = \frac{3}{200}$$

Probabilities from contingency tables

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Suppose you are Netflix ....

You'd like to figure out the chance that Ashna will like Saving Private Ryan, given that she likes Band of Brothers.

- What is unknown (A): Ashna likes Saving Private Ryan
- What is known (B): Ashna likes Band of Brothers
- Key question: What is $P(A \mid B)$?

Go to the data! (and use the multiplication rule)

Subscriber	Liked SPR?	Liked BoB?
1. Vivan Bui	Yes	Yes
2. Chetan Goenka	No	Yes
3. Utsavi Manglik	Yes	No
4. Anna Pate	No	No
5. Hugo Sandoval	Yes	No
6. Andrew Sall	Yes	Yes
⋮	⋮	⋮
1575. Serena Chiu	No	Yes
1576. Evelyn Cai	No	No

A nice way to look at this data

(check out the `xtabs()` function in `R`)

	Liked SPR	**Didn't like it**
Liked BoB	743	27
Didn't like it	8	798

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To figure out Ashna's likely preferences:

$$P(\text{Likes SPR} \mid \text{Likes BoB}) = \frac{743}{743 + 27} \approx 0.96$$

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Q: What about $P(\text{Likes BoB} \mid \text{Likes SPR})$, $P(\text{Likes BoB})$, $P(\text{Likes SPR})$?

Conditioning summary

Moral of the story?

Framing problems in terms of conditional probabilities can be immensely useful, whether you are trying to understand individualized preferences or a relationship among uncertain events.

Independence

Two events A and B are independent if

$$P(A \mid B) = P(A)$$

In words: A and B convey no information about each other:

- $P(\text{flip heads second time} \mid \text{flip heads first time}) = P(\text{flip heads second time})$

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- $P(\text{stock market up} \mid \text{bird poops on your car}) = P(\text{stock market up})$
- $P(\text{God exists}|\text{Longhorns win title}) = P(\text{God exists})$

So if A and B are independent, then $P(A, B) = P(A) \cdot P(B)$.

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In some cases, it is sensible:

- $P(\text{flip 1 heads}, \text{flip 2 heads}) = P(\text{flip 1 heads}) \cdot P(\text{flip 2 heads})$
- $P(\text{AAPL up today}, \text{AAPL up tomorrow}) =$
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- $P(\text{AAPL up today}, \text{AAPL up tomorrow}) = P(\text{AAPL up today}) \cdot P(\text{AAPL up tomorrow})$

In other cases, it is not sensible:

- $P(\text{rain}, \text{windy}) \neq P(\text{rain}) \cdot P(\text{windy})$
- $P(\text{sibling 1 colorblind}, \text{sibling 2 colorblind}) \neq P(\text{sibling 1 colorblind}) \cdot P(\text{sibling 2 colorblind})$

Conditional independence

Two events A and B are conditionally independent, given C, if

$$P(A, B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

A and B convey no information about each other, once we know C: $P(A \mid B, C) = P(A \mid C)$.

Neither independence nor conditional independence implies the other.

It is possible for two outcomes to be dependent and yet conditionally independent. Less intuitively, it is possible for two outcomes to be independent and yet conditionally dependent.

Conditional independence

Let's see an example. Alice and Brianna live next door to each other and both commute to work on the same metro line.

A = Alice is late for work.

B = Brianna is late for work.

A and B are dependent: if Brianna is late for work, we might infer that the metro line was delayed or that their neighborhood had bad weather. This means Alice is more likely to be late for work, so in terms of conditional probabilities:

$$P(A \mid B) > P(A)$$

Conditional independence

Now let's add some additional information:

A = Alice is late for work.

B = Brianna is late for work.

C = The metro is running on time and the weather is clear.

A and B are conditionally independent, given C. If Brianna is late for work but we know that the metro is running on time and the weather is clear, then we don't really learn anything about Alice's commute:

$$P(A \mid B, C) = P(A \mid C)$$

Conditional independence

Same characters, different story:

A = Alice has blue eyes.

B = Brianna has blue eyes.

A and B are independent: Alice's eye color can't give us information about Brianna's.

Conditional independence

Again, let's add some additional information.

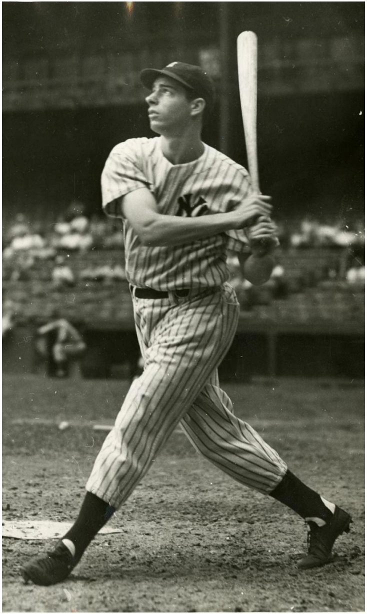
A = Alice has blue eyes.

B = Brianna has blue eyes.

C = Alice and Brianna are sisters.

A and B are conditionally dependent, given C: if Alice blue eyes, and we know that Brianna is her sister, then we know something about Brianna's genes. It is now more likely that Brianna has blue eyes.

Independence $\iff$ ease of calculation



Independence $\iff$ ease of calculation

Independence (or conditional independence) is often something we *choose to assume* for the purpose of making calculations easier.

Example:

Joe DiMaggio got a hit in about 80% of the baseball games he played in.

Suppose that successive games are independent: if JD gets a hit today, it doesn't change the probability he's going to get a hit tomorrow.

Then $P(\text{hit in game 1}, \text{hit in game 2}) = 0.8 \cdot 0.8 = 0.64$.

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$$
\begin{aligned}
& P(\text{hit game 1, hit game 2, hit game 3, } \ldots \text{, hit game 56}) \\
&= P(\text{hit game 1}) \cdot P(\text{hit game 2}) \cdot P(\text{hit game 3}) \cdots P(\text{hit game 56}) \\
&= 0.8 \cdot 0.8 \cdot 0.8 \cdots 0.8 \\
&= 0.8^{56} \\
&\approx \frac{1}{250,000}
\end{aligned}
$$

This is often called the "**compounding rule**."

Let's compare this with the corresponding probability for Pete Rose, a player who got a hit in 76% of his games. He's only slightly less skillful than DiMaggio! But:

$$
\begin{aligned}
&P(\text{hit game 1, hit game 2, hit game 3, \ldots, hit game 56}) \\
&= 0.76^{56} \\
&\approx \frac{1}{5 \text{ million}}
\end{aligned}
$$

Small difference in one game, but a big difference over the long run.

Independence $\iff$ ease of calculation

What about an average MLB player who gets a hit in 68% of his games?

$$
\begin{aligned}
&P(\text{hit game 1, hit game 2, hit game 3, \ldots, hit game 56})\\
&= 0.68^{56}\\
&\approx \frac{1}{\text{2.5 billion}}
\end{aligned}
$$

Never gonna happen!

Independence summary

Summary:

- Joe DiMaggio: 80% one-game hit probability, 1 in 250,000 streak probability
- Pete Rose: 76% one-game hit probability, 1 in 5 million streak probability
- Average player: 68% one-game hit probability, 1 in 2.5 billion streak probability

A small difference in probabilities becomes an enormous difference over the long term.

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Moral of the story: probability compounds **multiplicatively**, like the interest on your credit cards.

Independence summary

This is a more general assumption that's used in many contexts:

- A mutual-fund manager outperforms the stock market for 15 years straight.
- A World-War II airman completes 25 combat missions without getting shot down, and gets to go home.
- A retired person successfully takes a shower for 1000 days in a row without slipping.
- A child goes 180 school days, or 1 year, without catching a cold from other kids at school. (Good luck!)

However, Many smart folks can make mistakes here .. see the reading on our website about birth control.

Checking independence from data

Suppose we have two random outcomes A and B and we want to know if they're independent or not. **How do we go about this?**

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Suppose we have two random outcomes A and B and we want to know if they're independent or not. **How do we go about this?**

Solution:

- Check whether B happening seems to change the probability of A happening
- That is, verify using data whether $P(A \mid B) = P(A)$
- These probabilities won't be *exactly* alike because of statistical fluctuations, especially with small samples.
- But with enough data they should be pretty close if A and B are independent.