

Quantities and Rates

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Overview

We now turn our focus to compounding **quantities**, governed by a given **rate**.

This process is applicable to many parts of life:

- Cellular growth
- Population growth and demographics
- Financial markets (stock returns, IPO pricing, monetary policy)
- **Borrowing, loans, interest rates**

Interest

A fundamental concept in borrowing is interest.

Question: You friend asks to borrow \$1000 for Taylor Swift tickets, and she says she'll pay you back \$1000 in 6 months. Is this a good deal?

Drivers of interest

Interest is born out of the ideas that **time is valuable** and the **future is uncertain**.

Will my friend pay me back?

What else could I do with that money?

Some definitions

Principal is the amount you borrow $\rightarrow P$

Monthly payment is your pay back amount $\rightarrow M$

Interest rate is borrowing cost in percentage $\rightarrow r$

Number of time periods the interest compounds per year $\rightarrow n$

Number of years (if applicable) you're borrowing over $\rightarrow t$

Introduction to interest

Interest is the cost of borrowing money or the reward for saving.

It can be categorized into three types:

- **Simple interest**: Calculated on the principal amount only.
- **Compound interest**: Calculated on the principal and accumulated interest.
- **Continuous compounding**: Interest is compounded infinitely often.

Simple interest

- **Formula:** $\text{Simple Interest} = P \times r \times t$
- Where:
 - P : Principal
 - r : Interest rate
 - t : Time in years
- **Example:** For a \$10,000 loan at 1.68% interest rate for 3 years:

$$\begin{aligned}\text{Simple Interest} &= 10,000 \times 0.0168 \times 3 \\ &= 504\end{aligned}$$

Compound interest

- **Formula:** $A = P \times \left(1 + \frac{r}{n}\right)^{nt}$
- Where:
 - n : Number of compounding periods per year
- Compounding involves earning "interest on interest."
- **Example:** For a \$10,000 loan at 1.68% compounded annually for 3 years:

$$\begin{aligned} A &= 10,000 \times \left(1 + \frac{0.0168}{1}\right)^{1 \times 3} \\ &= 10,512.56 \end{aligned}$$

Continuous compounding

- **Formula:** $A = P \times e^{rt}$ aka PERT
- Where:
 - $e \approx 2.71828$ (Euler's number)
- Compounds interest at every possible moment.
- **Example:** For a \$10,000 loan at 1.68% compounded continuously for 3 years:

$$\begin{aligned} A &= 10,000 \times e^{0.0168 \times 3} \\ &= 10,512.83 \end{aligned}$$

Continuous compounding

Where does the formula come from?

Recall that the compounding formula for n times per year is:

$$A = P \times \left(1 + \frac{r}{n}\right)^{nt}$$

If we show that $\left(1 + \frac{r}{n}\right)^{nt} = e^{rt}$ as $n \rightarrow \infty$, then we've got it!

To the board!

APR and APY

APR is annual percentage rate and equals r .

APY is annual percentage yield .. the amount of interest collected after 1-year equals $\left(1 + \frac{r}{n}\right)^n - 1$.

This means there is a relationship between them!

$$\text{APY} = \left(1 + \frac{\text{APR}}{n}\right)^n - 1$$

Monthly payments

$$M = \frac{\frac{r}{12}P}{1 - \left(1 + \frac{r}{12}\right)^{-N}}$$

see page 43 of Zaslow for the derivation of this quantity

Extensions

We now have a systematic way to calculate how much you require in the **future** to loan out money **today**.

What about the flipped scenario? How much do you value **today** money that is paid out in the **future**?

Remember the following:

Going forwards from **today** to **future** \implies multiply by $(1 + \frac{r}{n})^n$

Going backwards from **future** to **today** \implies divide by $(1 + \frac{r}{n})^n$

$n = \#$ of time periods and $r =$ annual percentage rate