Quantities and Rates

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Overview

We now turn our focus to compounding quantities, governed by a given rate.

This process is applicable to many parts of life:

→ Cellular growth
→ Population growth and demographics
→ Financial markets (stock returns, IPO pricing, monetary policy)
→ Borrowing, loans, interest rates

Interest

A fundamental concept in borrowing is interest.

Question: You friend asks to borrow \$1000 for Taylor Swift tickets, and she says she'll pay you back \$1000 in 6 months. Is this a good deal?

Drivers of interest

Interest is born out of the ideas that time is valuable and the future is uncertain.

Will my friend pay me back?
What else could I do with that money?

Some definitions

Principal is the amount you borrow $\rightarrow P$

Monthly payment is your pay back amount $\rightarrow M$

Interest rate is borrowing cost in percentage $\rightarrow r$

Number of time periods the interest compounds per year $\rightarrow n$

Number of years (if applicable) you're borrowing over $\rightarrow t$

Introduction to interest

Interest is the cost of borrowing money or the reward for saving.

It can be categorized into three types:

- Simple interest: Calculated on the principal amount only.
- Compound interest: Calculated on the principal and accumulated interest.
- Continuous compounding: Interest is compounded infinitely often.

Simple interest

- **Formula:** Simple Interest $= P \times r \times t$
- Where:
  - P: Principal
  - r: Interest rate
  - t: Time in years
- **Example:** For a $10,000 loan at 1.68% interest rate for 3 years:

$$\begin{aligned}\text{Simple Interest} &= 10,000 \times 0.0168 \times 3 \\ &= 504\end{aligned}$$

Compound interest

- **Formula:** $A = P \times \left(1 + \frac{r}{n}\right)^{nt}$
- Where:
  - n: Number of compounding periods per year
- Compounding involves earning "interest on interest."
- **Example:** For a $10,000 loan at 1.68% compounded annually for 3 years:

$$
\begin{aligned}
A &= 10,000 \times \left(1 + \frac{0.0168}{1}\right)^{1\times 3} \\
&= 10,512.56
\end{aligned}
$$

Continuous compounding

- **Formula:** $A = P \times e^{rt}$ .... aka PERT
- Where:
  - $e \approx 2.71828$ (Euler's number)
- Compounds interest at every possible moment.
- **Example:** For a \$10,000 loan at 1.68% compounded continuously for 3 years:

$$\begin{aligned} A &= 10,000 \times e^{0.0168 \times 3} \\ &= 10,512.83 \end{aligned}$$

Continuous componding

Where does the formula come from?

Recall that the compounding formula for n times per year is:

$$A = P \times \left(1 + \frac{r}{n}\right)^{nt}$$

If we show that $\left(1 + \frac{r}{n}\right)^{nt} = e^{rt}$ as n $\to \infty$, then we've got it!

To the board!

APR and APY

APR is annual percentage rate and equals r.

APY is annual percentage yield .. the amount of interest collected after 1-year equals $\left(1 + \frac{r}{n}\right)^n$ - 1.

This means there is a relationship between them!

$$\text{APY} = \left(1 + \frac{\text{APR}}{n}\right)^n - 1$$

Monthly payments

$$M = \frac{\frac{r}{12}P}{1 - \left(1 + \frac{r}{12}\right)^{-N}}$$

see page 43 of Zaslow for the derivation of this quantity. Note that $N = n \times t$, the total number of payment periods.

Extensions

We now have a systematic way to calculate how much you require in the future to loan out money today.

What about the flipped scenario? How much do you value today money that is paid out in the future?

Remember the following:

Going forwards from today to future $\implies$ multiply by $(1 + \frac{r}{n})^n$
Going backwards from future to today $\implies$ divide by $(1 + \frac{r}{n})^n$

$n = \#$ of time periods and $r =$ annual percentage rate