Probability

Outline

What is probability?

- A measure of uncertainty
- Answering the question: "How likely is a given event?"
- As with any mathematical concept, there are a set of axioms setting the "ground rules"
- Separately, there are different ways to interpret probability ...

  (i) **frequentist**: limit of relative frequency after repeating an experiment an infinite number of times (coin flip!)

  (ii) **Bayesian**: subjective belief about the likelihood of an event occurence

Probability basics

If A denotes some event, then $P(A)$ is the probability that this event occurs:

- $P(\text{coin lands heads}) = 0.5$
- $P(\text{rainy day in Ireland}) = 0.85$
- $P(\text{cold day in Hell}) = 0.0000001$
- $P(\text{Swift and Kelce stay married}) = 0.00000001$

And so on...

Probability basics

Some probabilities are estimated from direct experience over the long run:

- $P(\text{newborn baby is a boy}) = \frac{106}{206}$
- $P(\text{death due to car accident}) = \frac{11}{100{,}000}$
- $P(\text{death due to any cause}) = 1$

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Others are synthesized from our best judgments about unique events:

- $P(\text{Nvidia stock goes up after next earnings call}) = 0.54$
- $P(\text{Djokovic wins next US Open}) = 0.4$ (6 to 4 odds)
- etc.

Probability basics: conditioning

A conditional probability is the chance that one thing happens, given that some other thing has already happened.

A great example is a weather forecast: if you look outside this morning and see gathering clouds, you might assume that rain is likely and carry an umbrella.

We express this judgment as a conditional probability: e.g. "the conditional probability of rain this afternoon, given clouds this morning, is 60%."

Probability basics: conditioning

In statistics, we write this a bit more compactly:

- $P(\text{rain this afternoon} \mid \text{clouds this morning}) = 0.6$
- That vertical bar means "given" or "conditional upon."
- The thing on the left of the bar is the event we're interested in.
- The thing on the right of the bar is our knowledge, also called the "conditioning event" or "conditioning variable": what we believe or assume to be true.

$P(A \mid B)$: "the probability of A, given that B occurs."

Probability basics: conditioning

Conditional probabilities are how we express judgments in a way that reflects our partial knowledge.

- You just gave *Squid Game* a high rating. What's the conditional probability that you will like *Virgin River* or *Love is Blind*?

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- You just bought organic dog food on Amazon. What's the conditional probability that you will also buy a GPS-enabled dog collar?
- You follow Gavin Newsom (@gavinnewsom) on Instagram. What's the conditional probability that you will respond to a suggestion to follow Greg Abbott (@governorabbott)?

Probability basics: conditioning

A really important fact is that conditional probabilities are **not symmetric**:

$$P(A \mid B) \neq P(B \mid A)$$

As a quick counter-example, let the events A and B be as follows:

- A: "you can dribble a basketball"
- B: "you play in the NBA"

Probability basics: conditioning

- A: "you can dribble a basketball"
- B: "you play in the NBA"

Clearly $P(A \mid B) = 1$: every NBA player can dribble a basketball!

Probability basics: conditioning

- A: "you can dribble a basketball"
- B: "you play in the NBA"

But $P(B \mid A)$ is nearly zero!

Uncertain outcomes and probability models

An uncertain outcome (more formally called a "random process") has two key properties:

1. The set of possible outcomes, called the sample space, *is known* beforehand.

2. The particular outcome that occurs is *not known* beforehand.
We denote the **sample space** as Ω, and some particular element of the sample space as $\omega \in \Omega$

Uncertain outcomes and probability models

Examples:

1. NBA finals, Golden State vs. Toronto:

$$\Omega = \{\text{4-0, 4-1, 4-2, 4-3, 3-4, 2-4, 1-4, 0-4}\}$$

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3. Number of no-shows on an AA flight from Austin to DFW:

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4. Poker hand

$$\Omega = \text{all possible five-card deals from a 52-card deck}$$

Uncertain outcomes and probability

An **event** is a *subset of the sample space*, i.e. $A \subset \Omega$. For example:

1. NBA finals, Golden State vs. Toronto. Let A be the event "Toronto wins". Then

$$A = \{\text{3-4, 2-4, 1-4, 0-4}\} \subset \Omega$$

2. Austin weather. Let A be the event "cooler than 90 degrees". Then

$$A = [10, 90) \subset [10, 115]$$

3. Flight no-shows. Let A be "more than 5 no shows":

$$A = \{6, 7, 8, \ldots, N_{\text{seats}}\}$$

Axioms of probability (Kolmogorov)

These are the ground rules!

Consider an uncertain outcome with sample space Ω. "Probability" $P(\cdot)$ is a set function that maps Ω to the real numbers, such that:

1. Non-negativity: For any event $A \subset \Omega$, $P(A) \geq 0$.
2. Normalization: $P(\Omega) = 1$ and $P(\emptyset) = 0$.
3. Finite additivity: If A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$.

3a. Finite additivity (general): For any sets A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(bonus: prove this with set theory!)

Not that intuitive! Notice no mention of frequencies...

Summary of terms

- Uncertain outcome/"random process": we know the possibilities ahead of time, just not the specific one that occurs
- Sample space: the set of possible outcomes
- Event: a subset of the sample space
- Probability: a function that maps events to real numbers and that obeys Kolmogorov's axioms

OK, so how do we actually *calculate* probabilities?

Counting!

Suppose our sample space Ω is a finite set consisting of N elements $\omega_1, \ldots, \omega_N$.

Suppose further that $P(\omega_i) = 1/N$: each outcome is equally likely, i.e. we have a discrete uniform distribution over possible outcomes.

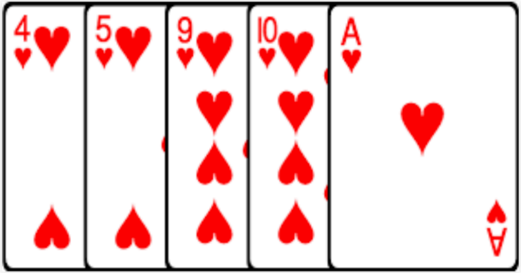
Then for each set $A \subset \Omega$,

$$P(A) = \frac{|A|}{N} = \frac{\text{Number of elements in } A}{\text{Number of elements in } \Omega}$$

That is, to compute $P(A)$, **we just need to count how many elements are in A.**

Counting example

Someone deals you a five-card poker hand from a 52-card deck. What is the probability of a flush (all five cards the same suit)?



Note: this is a very historically accurate illustration of probability, given its origins among bored French aristocrats!

Counting example

- Our sample space has $N = \binom{52}{5} = 2{,}598{,}960$ possible poker hands, each one equally likely.
- How many possible flushes are there? Let's start with hearts:
  $\rightarrow$ There are 13 hearts

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  $\rightarrow$ The same argument works for all four suits, so there are $4 \times 1287 = 5{,}148$ flushes. Thus:

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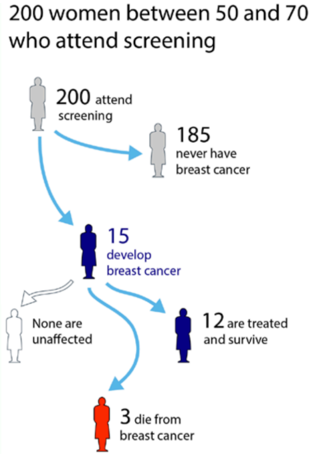
$\rightarrow$ Thus there are $\binom{13}{5} = 1287$ possible flushes with hearts.
$\rightarrow$ The same argument works for all four suits, so there are 4 $\times$ 1287 = 5,148 flushes. Thus:

$$P(\text{flush}) = \frac{|A|}{|\Omega|} = \frac{5148}{2598960} = 0.00198079$$

So we know how to count, but what about conditioning?

Probability trees are very useful for this task! This involves counting at different levels of the tree.

Conditioning example: mammograms



- $P(\text{cancer}) =$
- $P(\text{die,cancer}) =$
- $P(\text{die} \mid \text{cancer}) =$

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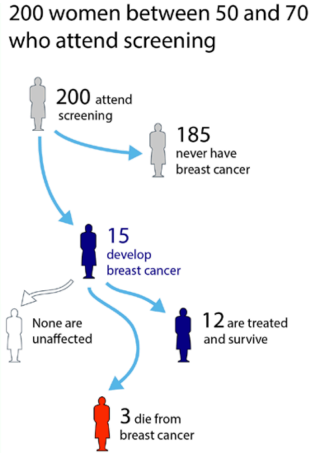

- $P(\text{cancer}) = \frac{15}{200}$
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– In general, we can estimate the conditional probability as:

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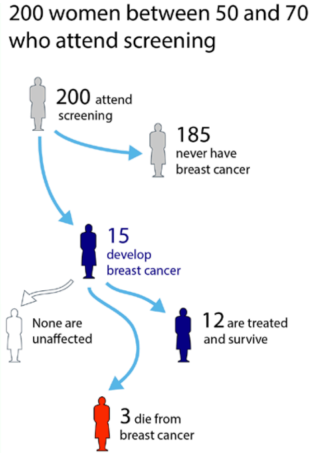

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  - In general, we can estimate the conditional probability as:

$$P(A \mid B) = \frac{\text{Frequency of } A \text{ and } B \text{ both happening}}{\text{Frequency of } B \text{ happening}}$$

This is actually a new axiom

The multiplication rule – it is an axiom since it can't be derived from the original axioms.

$$P(A \mid B) = \frac{P(A, B)}{P(B)}$$

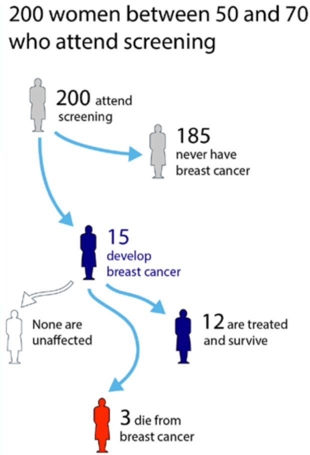
Alternate version

We can also use this alternative version if we want to go in reverse, from a conditional probability to a joint probability.

It says the same thing with the terms rearranged.

$$P(A, B) = P(A \mid B) \cdot P(B)$$

Conditioning example: mammograms (revisited)



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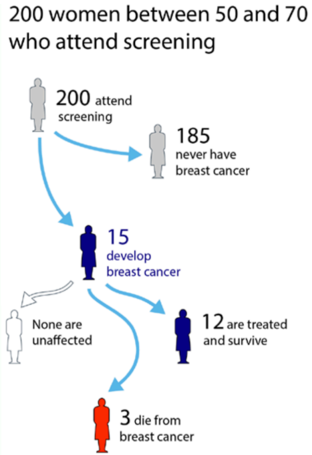

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- $P(\text{die} \mid \text{cancer}) = \frac{3}{15}$
  - Using the **multiplication rule**, we can estimate the **conditional probability** as:

$$P(\text{die} \mid \text{cancer}) = \frac{P(\text{die,cancer})}{P(\text{cancer})} = \frac{3/200}{15/200} = \frac{3}{15}$$

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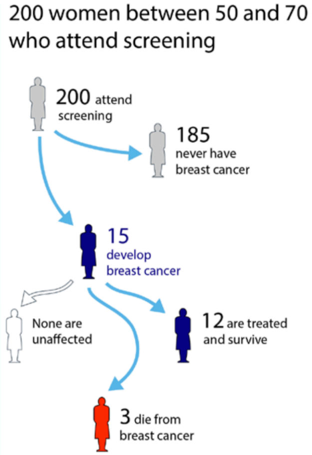

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– Using the **multiplication rule**, what about computing the **joint probability**?

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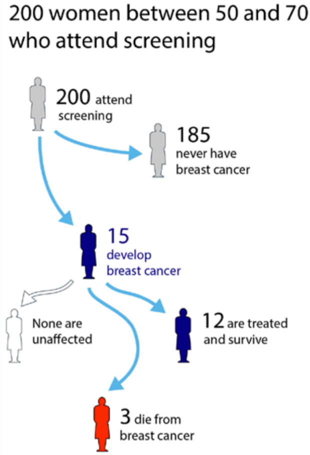

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– Using the multiplication rule, what about computing the **joint probability**?

$$P(\text{die}, \text{cancer}) = P(\text{die} \mid \text{cancer}) \cdot P(\text{cancer}) = \frac{3}{15} \cdot \frac{15}{200} = \frac{3}{200}$$

Probabilities from contingency tables

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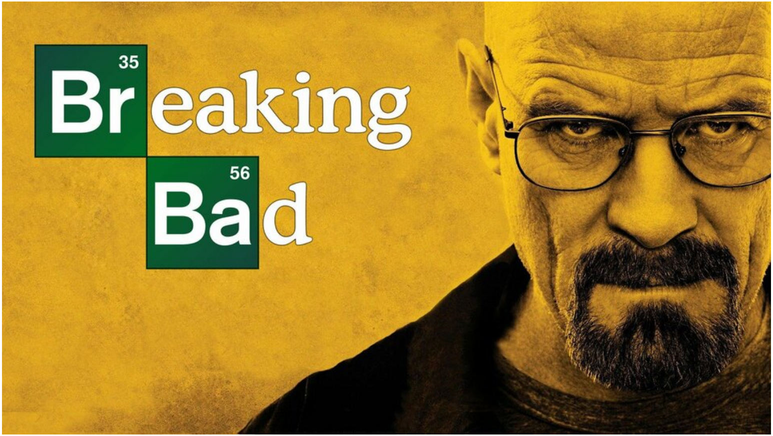

Probabilities from contingency tables

Suppose you are Netflix ....

You'd like to figure out the chance that Eitan will like Ozark, given that he likes Breaking Bad.

- What is unknown (A): Eitan likes Ozark
- What is known (B): Eitan likes Breaking Bad
- Key question: What is $P(A \mid B)$?

Go to the data! (and use the multiplication rule)

Subscriber	Liked OZK?	Liked BB?
1. Jamison Dittmar	Yes	Yes
2. Dotty Laster	No	Yes
3. Nathan Ng	Yes	No
4. Francesca Naugle	No	No
5. Luke Bradley	Yes	No
6. Vasilis Psathas	Yes	Yes
⋮	⋮	⋮
1575. Emerson Lau	No	Yes
1576. Daniel Levins	No	No

A nice way to look at this data

(check out the `xtabs()` function in `R`)

	Liked OZK	**Didn't like it**
Liked BB	743	27
Didn't like it	8	798

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To figure out Eitan's likely preferences:

$$P(\text{Likes OZK} \mid \text{Likes BB}) = \frac{743}{743 + 27} \approx 0.96$$

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Q: What about $P(\text{Likes BB} \mid \text{Likes OZK})$, $P(\text{Likes BB})$, $P(\text{Likes OZK})$?

Conditioning summary

Moral of the story?

Framing problems in terms of conditional probabilities can be immensely useful, whether you are trying to understand individualized preferences or a relationship among uncertain events.

Independence

Two events A and B are independent if

$$P(A \mid B) = P(A)$$

In words: A and B convey no information about each other:

- $P(\text{flip heads second time} \mid \text{flip heads first time}) =$
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- $P(\text{Chiefs go 0-17} \mid \text{Swift's new album underperforms}) = P(\text{Chiefs go 0-17})$

So if A and B are independent, then $P(A, B) = P(A) \cdot P(B)$.

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In some cases, it is sensible:

- $P(\text{flip 1 heads}, \text{flip 2 heads}) = P(\text{flip 1 heads}) \cdot P(\text{flip 2 heads})$
- $P(\text{AAPL up today}, \text{AAPL up tomorrow}) =$
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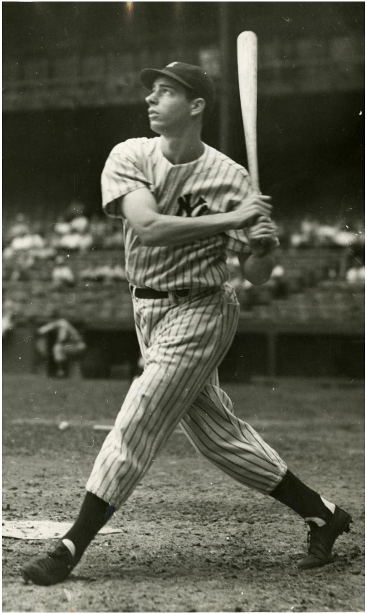
In other cases, it is not sensible:

- $P(\text{rain}, \text{windy}) \neq P(\text{rain}) \cdot P(\text{windy})$
- $P(\text{sibling 1 colorblind}, \text{sibling 2 colorblind}) \neq$
  $P(\text{sibling 1 colorblind}) \cdot P(\text{sibling 2 colorblind})$

Independence $\iff$ ease of calculation



Independence $\iff$ ease of calculation

Independence (or conditional independence) is often something we *choose to assume* for the purpose of making calculations easier.

Example:

Joe DiMaggio got a hit in about 80% of the baseball games he played in.

Suppose that successive games are independent: if JD gets a hit today, it doesn't change the probability he's going to get a hit tomorrow.

Then $P(\text{hit in game 1}, \text{hit in game 2}) = 0.8 \cdot 0.8 = 0.64$.

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This works for more than two events. For example, Joe DiMaggio had a 56-game hitting streak in the 1941 baseball season. This was pretty unlikely!!

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$$
\begin{aligned}
& P(\text{hit game 1, hit game 2, hit game 3, \ldots, hit game 56}) \\
&= P(\text{hit game 1}) \cdot P(\text{hit game 2}) \cdot P(\text{hit game 3}) \cdots P(\text{hit game 56}) \\
&= 0.8 \cdot 0.8 \cdot 0.8 \cdots 0.8 \\
&= 0.8^{56} \\
&\approx \frac{1}{250,000}
\end{aligned}
$$

This is often called the "**compounding rule**."

Independence $\iff$ ease of calculation

Let's compare this with the corresponding probability for Pete Rose, a player who got a hit in 76% of his games. He's only slightly less skillful than DiMaggio! But:

$$
\begin{aligned}
& P(\text{hit game 1, hit game 2, hit game 3, \ldots, hit game 56}) \\
&= 0.76^{56} \\
&\approx \frac{1}{\text{5 million}}
\end{aligned}
$$

Small difference in one game, but a **big difference** over the long run.

Independence $\iff$ ease of calculation

What about an average MLB player who gets a hit in 68% of his games?

$$
\begin{aligned}
& P(\text{hit game 1, hit game 2, hit game 3, \ldots, hit game 56}) \\
&= 0.68^{56} \\
&\approx \frac{1}{2.5 \text{ billion}}
\end{aligned}
$$

Never gonna happen!

Independence summary

Summary:

- Joe DiMaggio: 80% one-game hit probability, 1 in 250,000 streak probability
- Pete Rose: 76% one-game hit probability, 1 in 5 million streak probability
- Average player: 68% one-game hit probability, 1 in 2.5 billion streak probability

A small difference in probabilities becomes an enormous difference over the long term.

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Moral of the story: probability compounds **multiplicatively**, like the interest on your credit cards.

Independence summary

This is a more general assumption that's used in many contexts:

- A mutual-fund manager outperforms the stock market for 15 years straight.
- A World-War II airman completes 25 combat missions without getting shot down, and gets to go home.
- A retired person successfully takes a shower for 1000 days in a row without slipping.
- A child goes 180 school days, or 1 year, without catching a cold from other kids at school. (Good luck!)

However, Many smart folks can make mistakes here .. see the reading on our website about birth control.

Checking independence from data

Suppose we have two random outcomes A and B and we want to know if they're independent or not. **How do we go about this?**

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Solution:

- Check whether B happening seems to change the probability of A happening
- That is, verify using data whether $P(A \mid B) = P(A)$
- These probabilities won't be *exactly* alike because of statistical fluctuations, especially with small samples.
- But with enough data they should be pretty close if A and B are independent.

Paradoxes, mixtures, and the rule of total probability

The first paradox

Complication rates across 3,690 deliveries at a large maternity hospital in Cambridge, UK.

	low-risk	**high-risk**	**overall**
senior doctor	0.052	0.127	
junior doctor	0.067	0.155	

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junior doctor	0.067	0.155	**0.072**

Q: What doctor do you want delivering your baby?

The first paradox

- Senior doctors are ...
  $\rightarrow$ better at low-risk
  $\rightarrow$ better at high-risk
  yet, worse overall?!
- This is an example of Simpson's paradox. How is it possible?

The second paradox

Ten richest states and their 2016 electoral college result

Rank	State	Median income	2016 winner
1	Washington, D.C.	$85,203	Clinton
2	Maryland	$83,242	Clinton
3	New Jersey	$81,740	Clinton
4	Hawaii	$80,212	Clinton
5	Massachusetts	$79,835	Clinton
6	Connecticut	$76,348	Clinton
7	California	$75,277	Clinton
8	New Hampshire	$74,991	Clinton
9	Alaska	$74,346	Trump
10	Washington	$74,073	Clinton

The second paradox

Ten poorest states and their 2016 electoral college result

Rank	State	Median income	2016 winner
42	Tennessee	$52,375	Trump
43	South Carolina	$52,306	Trump
44	Oklahoma	$51,924	Trump
45	Kentucky	$50,247	Trump
46	Alabama	$49,861	Trump
47	Louisiana	$47,905	Trump
48	New Mexico	$47,169	Clinton
49	Arkansas	$47,062	Trump
50	Mississippi	$44,717	Trump
51	West Virginia	$44,097	Trump

High-income states vote **blue**
Low-income states vote **red**

"Farmer, factory workers, truck drivers, waitresses..."

vs.

The know-it-alls of Manhattan and Malibu ... who lord over the peasantry with their fancy college degrees

"Average Americans, humble, long-suffering, working hard, who buy their coffee already ground"

vs.

"The wealthy, latte-swilling liberal elite"

"Real Americans, with a lawnmower in the garage and a flag on the front stoop"

vs.

"Wealthy condo-dwellers with contempt for those who feel chills up their spines at 'The Star Spangled Banner'"

And yet ...

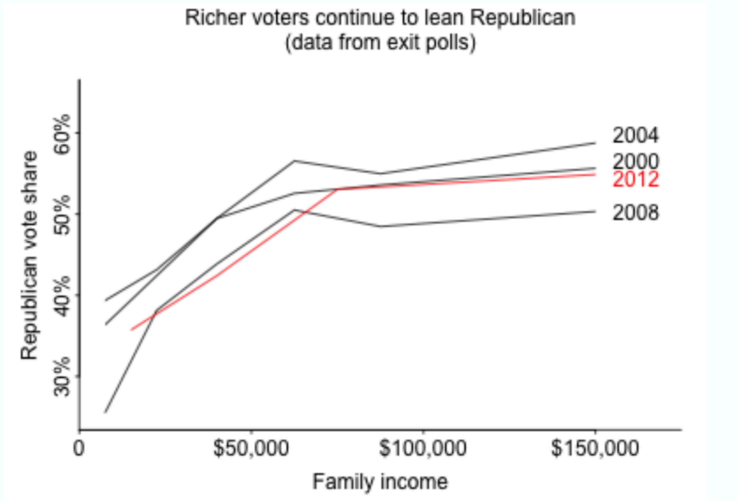
The second paradox

Presidential vote share by personal income

	under $50K		over $50K	
	Dem.	Rep.	Dem.	Rep.
2004	0.55	0.44	0.43	0.56
2008	0.60	0.38	0.49	0.49
2012	0.54	0.44	0.44	0.54
2016	0.52	0.41	0.47	0.49

The second paradox

Presidential vote share by family income



The second paradox

- For states:
  $\rightarrow$ higher income means more likely to vote Democrat
  $\rightarrow$ lower income means more likely to vote Republican

- Yet, for people:
  $\rightarrow$ higher income means more likely to vote Republican
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- How is this possible?

Back to the first paradox

Complication rates and sample sizes across 3,690 deliveries at a large maternity hospital in Cambridge, UK.

	low-risk	**high-risk**	**overall**
senior doctor	0.052 (213)	0.127 (102)	**0.076** (315)
junior doctor	0.067 (3169)	0.155 (206)	**0.072** (3375)

Rule of total probability

The probability of an event is the sum of the probabilities for all of the different ways that event can happen.

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Suppose that $B_1, \ldots B_N$ are mutually exclusive events whose probabilities sum to 1.

$P(B_i, B_j) = 0 \quad \forall i \neq j \quad \text{and} \quad \sum_{i=1}^{N} P(B_i) = 1$

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Then, for any event A:

$$\boxed{P(A) = \sum_{i=1}^{N} P(A, B_i) = \sum_{i=1}^{N} P(A \mid B_i) P(B_i)}$$

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$$\begin{aligned} P(\text{comp}) &= P(\text{comp}, \text{low}) + P(\text{comp}, \text{high}) \\ &= P(\text{low}) \cdot P(\text{comp} \mid \text{low}) + P(\text{high}) \cdot P(\text{comp} \mid \text{high}) \end{aligned}$$

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First paradox resolved

Senior doctors are...

- better at low-risk *and* high-risk deliveries
- yet worse overall

This is Simpson's paradox in action. Here's what is going on:

- $P(\text{comp} \mid \text{low})$ and $P(\text{comp} \mid \text{high})$ are both lower for senior doctors
- yet senior doctors work fewer low-risk cases: $P(\text{low})$ is smaller in the mixture!

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Moral of the story:

- Make sure you're asking the right question
- Always be sensitive to whether probabilities are conditional or unconditional (marginal, total, overall), and which type makes more sense for your situation.

Back to the second paradox

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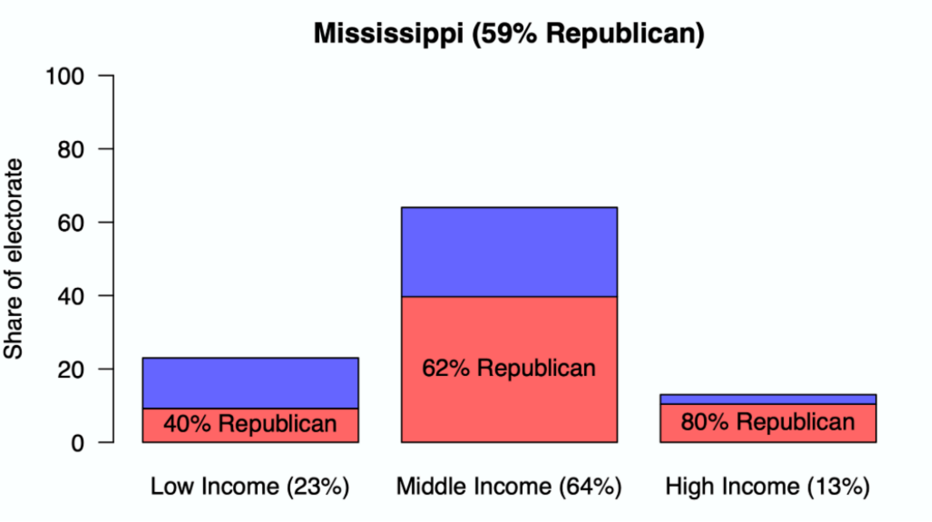
Law of total probability, Mississippi



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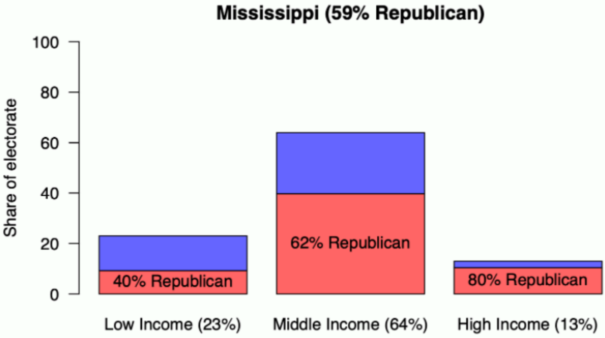

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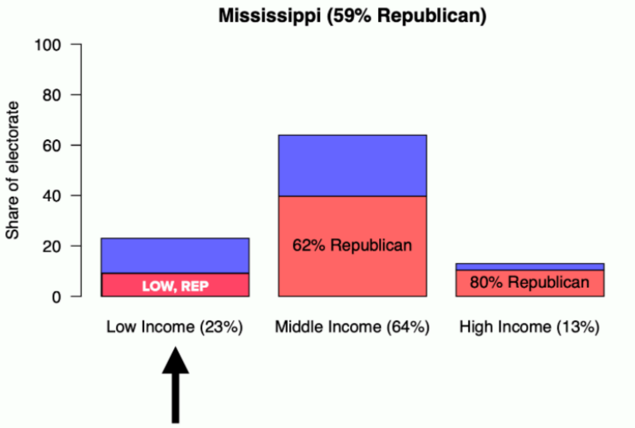

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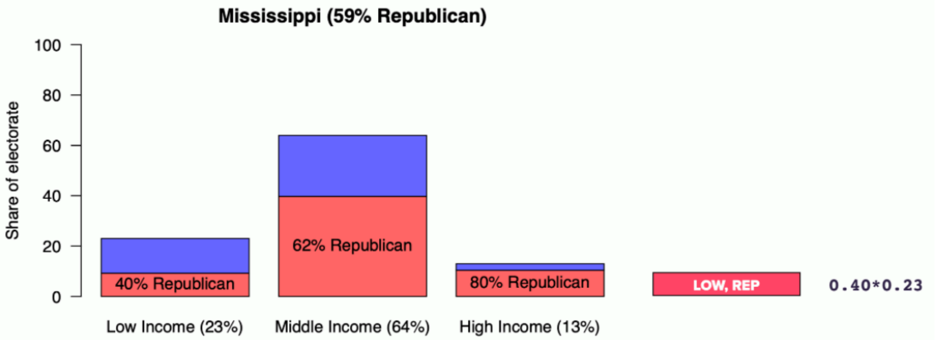

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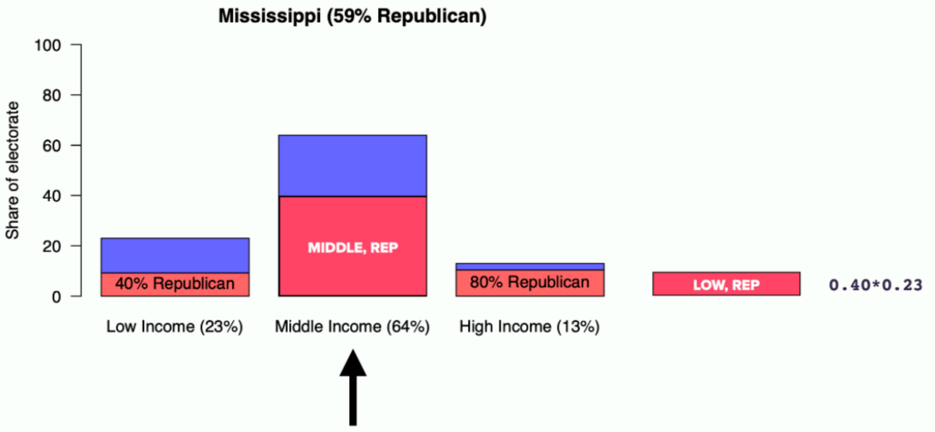

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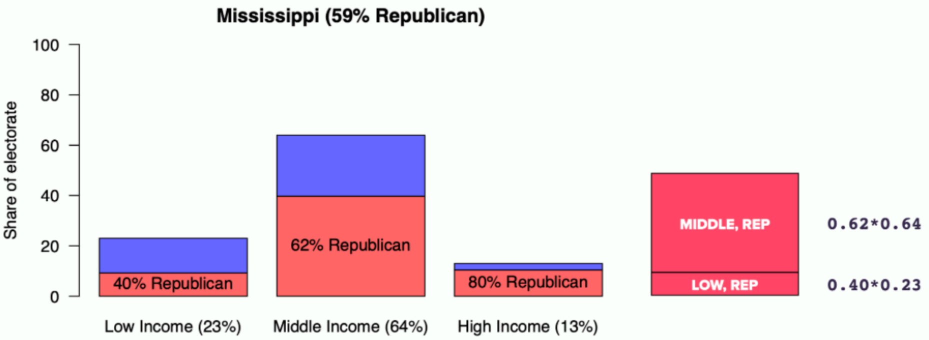

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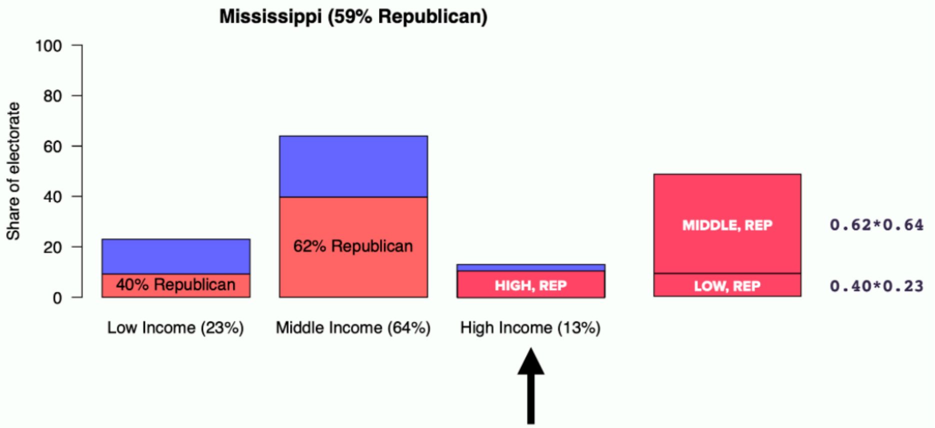

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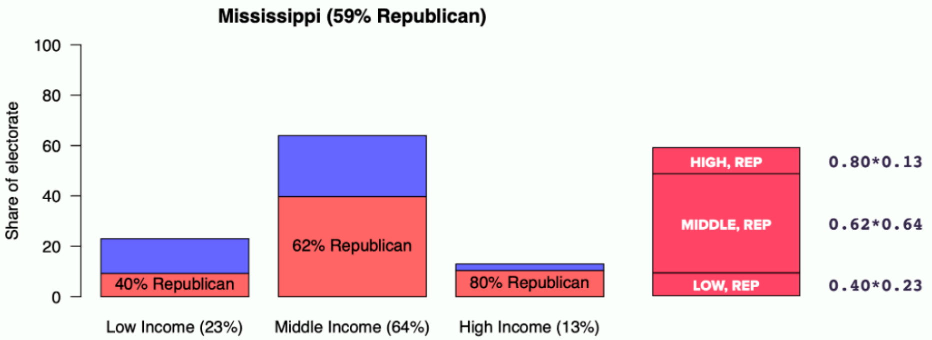

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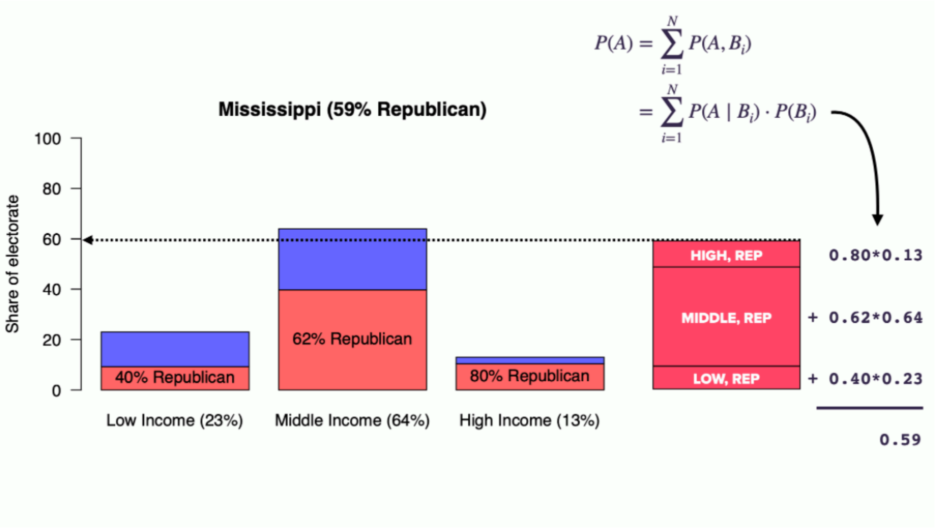
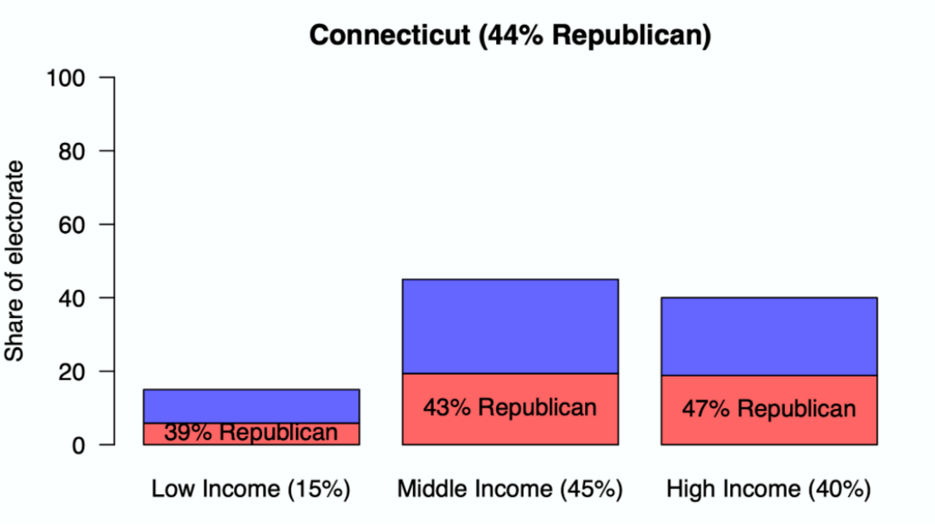


And now Connecticut



Connecticut and Mississippi

Here is $P(\text{Rep} \mid \text{income})$ for each state:

	Low-income	Middle-income	High-income
Connecticut	0.39	0.43	0.47
Mississippi	0.40	0.62	0.80

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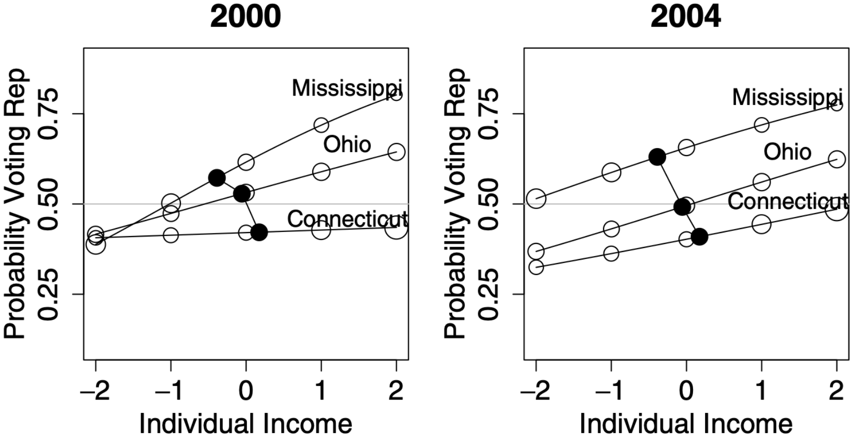
Q: Does income really tell me anything about why CT is blue and MS is red?

Let's look at Mississippi, Ohio, & Connecticut

(from Gelman et. al., Quarterly Journal of Political Science)

- same story, different election years



Let's look at Mississippi, Ohio, & Connecticut

Paradox 2 resolved, kind of ...

We've seen how, mechanically, an individual-level effect can be in one direction, and a group-level effect can be in the other direction.

But, conditioning on income alone **cannot** explain why CT is blue and MS is red! What can is the relative positioning of the state lines.

What else (other than income) could be driving this relationship? (homework)

The ecological fallacy

Ecological inference: looking for associations between cause and effect at the level of groups or populations.

Do groups with higher average levels of A tend to have higher B?

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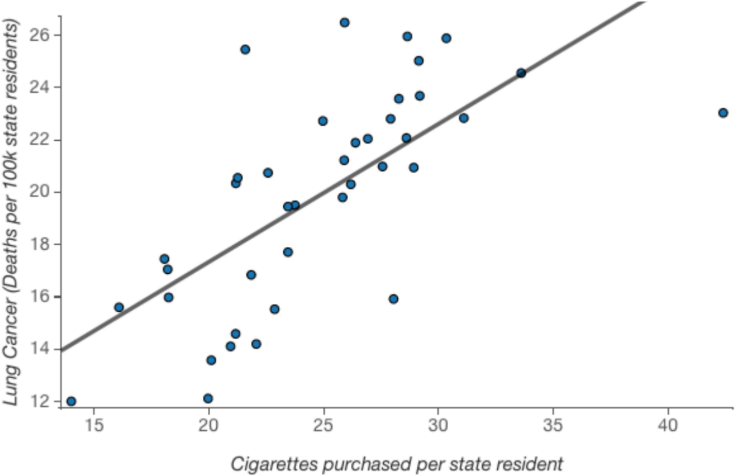
The ecological fallacy: assuming, without further justification, that group-level associations accurately reflect individual level associations.

Groups with higher A have higher B, on average. Therefore, individuals with higher A have higher B, on average. ← **not necessarily!!**

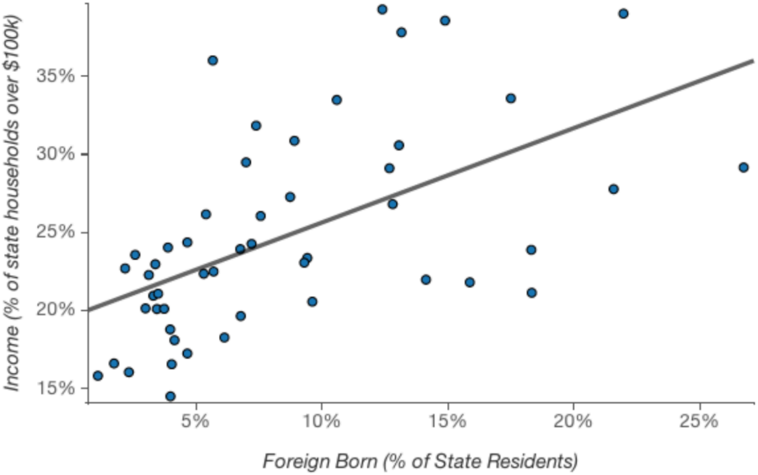
The ecological fallacy

smoking cigarettes really does increase an individual's risk of lung cancer. This ecological association accurately reflects an individual-level trend.



The ecological fallacy

... but this one doesn't. At the individual level, 22.1% of foreign-born residents make more than $100k, versus 26.1% of US-born residents.



Take-home messages

- A trend that appears when the data are *separated into individuals/smaller groups* can look different, or even reverse entirely, when the data are *aggregated into larger groups*.

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- So what to do? Remember the rule of total probability!

  → Pay attention: the level of grouping matters a lot

  → Ask questions: Do we care about a total or conditional probability? Are we missing any lurking variables?

  → Avoid the ecological fallacy: learn to be skeptical when group-level trends are applied to individuals