High frequency min-variance portfolios

Jared Fisher and David Puelz

Jared Fisher
e-mail: jared.fisher@utexas.edu

David Puelz
e-mail: david.puelz@utexas.edu

Abstract: We construct minimum-variance portfolios using high-frequency realized variance matrices. Our contributions are two-fold. First, we provide an approach for filling in missing data into realized variance matrices. Second, we derive a penalized minimum variance loss function and ADMM procedure for calculating optimal portfolio weights.

1. Derivation

In this section, we consider a scenario where some covariates are known, or fixed, and the remainder are random. This may occur when when one would like to condition on a particular value of a covariate at some fixed future value.

Let the covariates X be divided into two pieces, those that are considered random: $X_r \in \mathbb{R}^{p_r}$, and those that are considered fixed: $X_f \in \mathbb{R}^{p_f}$, so that the column vector $X = [X_r^T \ X_f^T]^T \in \mathbb{R}^p$ and $p = p_r + p_f$. So, future values of the covariates are given by $\tilde{X} = [\tilde{X}_r^T \ X_f^T]^T$.

Conditioning on the fixed covariates, the distribution of unknowns is: $p(\tilde{Y}, \tilde{X}_r, \Theta | X_f)$ where Θ is a vector of parameters from a specified model. If we assume conditional independence, then we can write:

$$p(\tilde{Y}, \tilde{X}_r, \Theta | X_f) = p(\tilde{Y} | \tilde{X}_r, X_f, \Theta) p(\tilde{X}_r | X_f, \Theta) p(\Theta | X_f). \tag{1}$$

where, as before, $p(\Theta | X_f)$ is the posterior distribution of model parameters conditional on the fixed covariates. Any models may be chosen for the conditional $Y | X_r, X_f$ and the marginal $X_r | X_f$. For example, in the case of X following a multivariate normal distribution implied by a latent factor regression model, we automatically know the conditionals including $X_r | X_f$.

We use the negative log-density of the regression of $p(Y|X)$ as the utility function with an l_0 penalty:

$$\mathcal{L}_\lambda(\tilde{Y}, \tilde{X}, \Theta, \boldsymbol{\gamma}) \equiv \frac{1}{2}(\tilde{Y} - \boldsymbol{\gamma}\tilde{X})^T \Omega (\tilde{Y} - \boldsymbol{\gamma}\tilde{X}) + \lambda \left\| \mathbf{vec}(\boldsymbol{\gamma}) \right\|_0, \tag{2}$$

Define the following block structure for the action, $\boldsymbol{\gamma}$:

$$\boldsymbol{\gamma} = \begin{bmatrix} \boldsymbol{\gamma}_r & \boldsymbol{\gamma}_f \end{bmatrix}, \tag{3}$$

so that $\boldsymbol{\gamma}_r \in \mathbb{R}^{q \times p_r}$ and $\boldsymbol{\gamma}_f \in \mathbb{R}^{q \times p_f}$. We expand out 2 and drop terms that don't involve the action $\boldsymbol{\gamma}$:

$$\mathcal{L}_\lambda(\tilde{Y}, \tilde{X}, \Theta, \boldsymbol{\gamma}) = \frac{1}{2}\left(\tilde{X}_r^T \boldsymbol{\gamma}_r^T \Omega \boldsymbol{\gamma}_r \tilde{X}_r + X_f^T \boldsymbol{\gamma}_f^T \Omega \boldsymbol{\gamma}_f X_f - 2\tilde{X}_r^T \boldsymbol{\gamma}_r^T \Omega \tilde{Y} - 2 X_f^T \boldsymbol{\gamma}_f^T \Omega \tilde{Y} \right) + \lambda \left\| \mathbf{vec}(\boldsymbol{\gamma}) \right\|_0 + \text{constant}. \tag{4}$$

Taking expectations over $p(\tilde{Y}, \tilde{X}_r, \Theta | X_f)$ and dropping the one-half and constant, we obtain the integrated loss function:

$$\mathcal{L}_\lambda(\boldsymbol{\gamma}) = \mathbb{E}\left[\mathrm{tr}[\boldsymbol{\gamma}_r^T \Omega \boldsymbol{\gamma}_r \tilde{X}_r \tilde{X}_r^T]\right] - 2\mathbb{E}\left[\mathrm{tr}[\boldsymbol{\gamma}_r^T \Omega \tilde{Y} \tilde{X}_r^T]\right] + \mathbb{E}\left[\mathrm{tr}[\boldsymbol{\gamma}_f^T \Omega \boldsymbol{\gamma}_f X_f X_f^T]\right] - 2\mathbb{E}\left[\mathrm{tr}[\boldsymbol{\gamma}_f^T \Omega \tilde{Y} X_f^T]\right] + \lambda \left\| \mathbf{vec}(\boldsymbol{\gamma}) \right\|_0 . \tag{5}$$

We simplify the expectations in a similar way to our derivation of the original loss function presented in section 2.

$$\mathcal{L}_\lambda(\boldsymbol{\gamma}) = \mathrm{tr}[M \boldsymbol{\gamma}_r S_r \boldsymbol{\gamma}_r^T] - 2\mathrm{tr}[A_r \boldsymbol{\gamma}_r^T] + \mathrm{tr}[M \boldsymbol{\gamma}_f S_f \boldsymbol{\gamma}_f^T] - 2\mathrm{tr}[A_f \boldsymbol{\gamma}_f^T] + \lambda \left\| \mathbf{vec}(\boldsymbol{\gamma}) \right\|_0 , \tag{6}$$

where,

$$\begin{aligned} A_r &\equiv \mathbb{E}[\Omega \tilde{Y} \tilde{X}_r^T], \quad A_f \equiv \mathbb{E}[\Omega \tilde{Y} \tilde{X}_f^T] \\ S_r &\equiv \mathbb{E}[\tilde{X}_r \tilde{X}_r^T], \quad\ \ S_f = X_f X_f^T \\ M &\equiv \overline{\Omega} \end{aligned} \tag{7}$$

Combining the matrix traces, we simplify the loss function as follows:

$$\mathcal{L}_\lambda(\boldsymbol{\gamma}) = \mathrm{tr}[M \boldsymbol{\gamma} S \boldsymbol{\gamma}^T] - 2\mathrm{tr}[A \boldsymbol{\gamma}^T] + \lambda \left\| \mathbf{vec}(\boldsymbol{\gamma}) \right\|_0 , \tag{8}$$

where,

$$S \equiv \begin{bmatrix} S_r & 0 \\ 0 & S_f \end{bmatrix}, \quad A \equiv \begin{bmatrix} A_r \\ A_f \end{bmatrix}. \tag{9}$$

Then, we proceed exactly as in the appendix to derive the lasso form of loss function 9.