Probability

David Puelz

Outline

The basics and conditional probability

Paradoxes, mixtures, and the rule of total probability

What is probability?

- A measure of **uncertainty**
- Answering the question: "How likely is a given event?"
- As with any mathematical concept, there are a set of axioms setting the "ground rules"
- Separately, there are different ways to interpret probability ...

  (i) **frequentist**: limit of relative frequency after repeating an experiment an infinite number of times (coin flip!)

  (ii) **Bayesian**: subjective belief about the likelihood of an event occurence

Probability basics

If A denotes some event, then $P(A)$ is the probability that this event occurs:

- $P(\text{coin lands heads}) = 0.5$
- $P(\text{rainy day in Ireland}) = 0.85$
- $P(\text{cold day in Hell}) = 0.0000001$

And so on...

Probability basics

Some probabilities are estimated from direct experience over the long run:

- $P(\text{newborn baby is a boy}) = \frac{106}{206}$
- $P(\text{death due to car accident}) = \frac{11}{100{,}000}$
- $P(\text{death due to any cause}) = 1$

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Others are synthesized from our best judgments about unique events:

- $P(\text{Apple stock goes up after next earnings call}) = 0.54$
- $P(\text{Djokovic wins next US Open}) = 0.4$ (6 to 4 odds)
- etc.

Probability basics: conditioning

A conditional probability is the chance that one thing happens, given that some other thing has already happened.

A great example is a weather forecast: if you look outside this morning and see gathering clouds, you might assume that rain is likely and carry an umbrella.

We express this judgment as a conditional probability: e.g. "the conditional probability of rain this afternoon, given clouds this morning, is 60%."

Probability basics: conditioning

In statistics, we write this a bit more compactly:

- $P(\text{rain this afternoon} \mid \text{clouds this morning}) = 0.6$
- That vertical bar means "given" or "conditional upon."
- The thing on the left of the bar is the event we're interested in.
- The thing on the right of the bar is our knowledge, also called the "conditioning event" or "conditioning variable": what we believe or assume to be true.

$P(A \mid B)$: "the probability of A, given that B occurs."

Probability basics: conditioning

Conditional probabilities are how we express judgments in a way that reflects our partial knowledge.

- You just gave *Squid Game* a high rating. What's the conditional probability that you will like *Virgin River* or *Bridgerton*?

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- You follow Donald Trump (@realdonaldtrump) on Instagram. What's the conditional probability that you will respond to a suggestion to follow Kamala Harris (@kamalaharris)?

Probability basics: conditioning

A really important fact is that conditional probabilities are **not symmetric**:

$$P(A \mid B) \neq P(B \mid A)$$

As a quick counter-example, let the events A and B be as follows:

- A: "you can dribble a basketball"
- B: "you play in the NBA"

Probability basics: conditioning

- A: "you can dribble a basketball"
- B: "you play in the NBA"

Clearly $P(A \mid B) = 1$: every NBA player can dribble a basketball!

Probability basics: conditioning

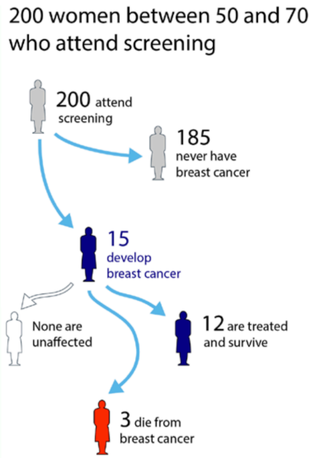
- A: "you can dribble a basketball"
- B: "you play in the NBA"

But $P(B \mid A)$ is nearly zero!

Fine, but what about conditioning in practice?

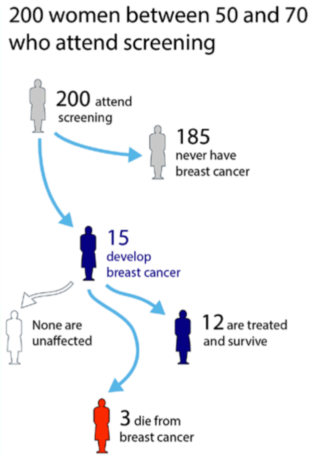
Probability trees are very useful for this task! This involves counting at different levels of the tree.

Conditioning example: mammograms



- $P(\text{cancer}) =$
- $P(\text{die,cancer}) =$
- $P(\text{die} \mid \text{cancer}) =$

Conditioning example: mammograms



- $P(\text{cancer}) = \frac{15}{200}$
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– In general, we can estimate the conditional probability as:

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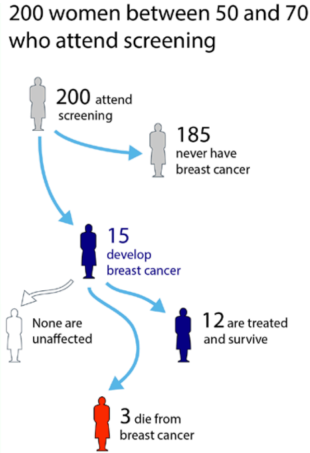

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- $P(\text{die,cancer}) = \frac{3}{200}$
- $P(\text{die} \mid \text{cancer}) = \frac{3}{15}$
  - In general, we can estimate the conditional probability as:

$$P(A \mid B) = \frac{\text{Frequency of } A \text{ and } B \text{ both happening}}{\text{Frequency of } B \text{ happening}}$$

This is actually a new axiom

The multiplication rule – it is an axiom since it can't be derived from the original axioms.

$$P(A \mid B) = \frac{P(A, B)}{P(B)}$$

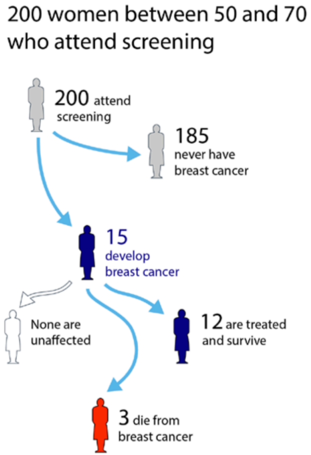
Alternate version

We can also use this alternative version if we want to go in reverse, from a conditional probability to a joint probability.

It says the same thing with the terms rearranged.

$$P(A, B) = P(A \mid B) \cdot P(B)$$

Conditioning example: mammograms (revisited)



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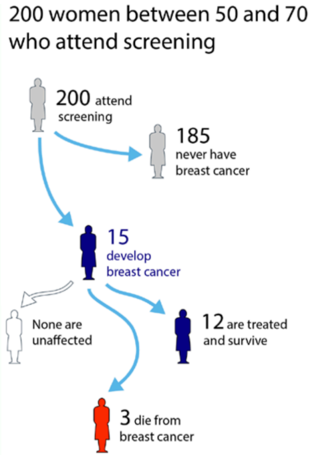

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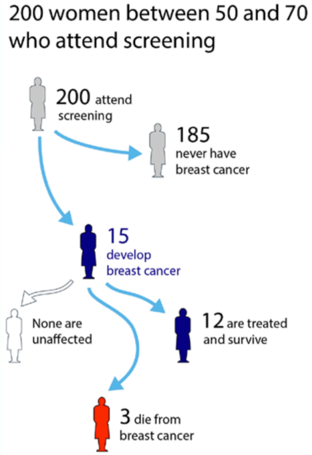
- $P(\text{die} \mid \text{cancer}) = \frac{3}{15}$

– Using the **multiplication rule**, we can estimate the **conditional probability** as:

$$P(\text{die} \mid \text{cancer}) = \frac{P(\text{die,cancer})}{P(\text{cancer})} = \frac{3/200}{15/200} = \frac{3}{15}$$

Conditioning example: mammograms (revisited)



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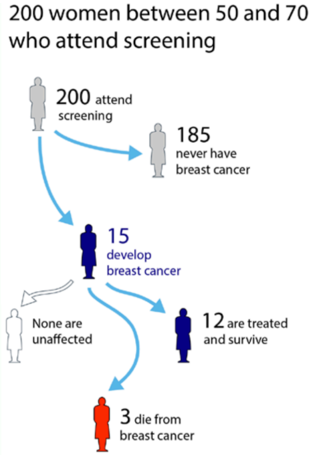

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- $P(\text{die} \mid \text{cancer}) = \frac{3}{15}$
  - Using the multiplication rule, what about computing the **joint probability**?

$$P(\text{die, cancer}) = P(\text{die} \mid \text{cancer}) \cdot P(\text{cancer}) = \frac{3}{15} \cdot \frac{15}{200} = \frac{3}{200}$$

Probabilities from contingency tables

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Probabilities from contingency tables

Suppose you are Netflix ....

You'd like to figure out the chance that Rachel will like Saving Private Ryan, given that she likes Band of Brothers.

- What is unknown (A): Rachel likes Saving Private Ryan
- What is known (B): Rachel likes Band of Brothers
- Key question: What is $P(A \mid B)$?

Go to the data! (and use the multiplication rule)

Subscriber	Liked SPR?	Liked BoB?
1. Nathan Prudeanu	Yes	Yes
2. Gianna McCord	No	Yes
3. Ruby Erickson	Yes	No
4. Ethan Steinfeld	No	No
5. Elizabeth Borodulin	Yes	No
6. Audrey Bolyard	Yes	Yes
⋮	⋮	⋮
1575. Noah Prudeanu	No	Yes
1576. Miranda Xu	No	No

A nice way to look at this data

(check out the `xtabs()` function in `R`)

	Liked SPR	Didn't like it
Liked BoB	743	27
Didn't like it	8	798

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To figure out Rachel's likely preferences:

$$P(\text{Likes SPR} \mid \text{Likes BoB}) = \frac{743}{743+27} \approx 0.96$$

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To figure out Rachel's likely preferences:

$$P(\text{Likes SPR} \mid \text{Likes BoB}) = \frac{743}{743 + 27} \approx 0.96$$

Q: What about $P(\text{Likes BoB} \mid \text{Likes SPR})$, $P(\text{Likes BoB})$, $P(\text{Likes SPR})$?

Conditioning summary

Moral of the story?

Framing problems in terms of conditional probabilities can be immensely useful, whether you are trying to understand individualized preferences or a relationship among uncertain events.

Paradoxes, mixtures, and the rule of total probability

The first paradox

Complication rates across 3,690 deliveries at a large maternity hospital in Cambridge, UK.

	low-risk	high-risk	overall
senior doctor	0.052	0.127	
junior doctor	0.067	0.155	

The first paradox

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Q: What doctor do you want delivering your baby?

The first paradox

- Senior doctors are ...
  $\rightarrow$ better at low-risk
  $\rightarrow$ better at high-risk
  yet, worse overall?!
- This is an example of Simpson's paradox. How is it possible?

The second paradox

Ten richest states and their 2016 electoral college result

Rank	State	Median income	2016 winner
1	Washington, D.C.	$85,203	Clinton
2	Maryland	$83,242	Clinton
3	New Jersey	$81,740	Clinton
4	Hawaii	$80,212	Clinton
5	Massachusetts	$79,835	Clinton
6	Connecticut	$76,348	Clinton
7	California	$75,277	Clinton
8	New Hampshire	$74,991	Clinton
9	Alaska	$74,346	Trump
10	Washington	$74,073	Clinton

The second paradox

Ten poorest states and their 2016 electoral college result

Rank	State	Median income	2016 winner
42	Tennessee	$52,375	Trump
43	South Carolina	$52,306	Trump
44	Oklahoma	$51,924	Trump
45	Kentucky	$50,247	Trump
46	Alabama	$49,861	Trump
47	Louisiana	$47,905	Trump
48	New Mexico	$47,169	Clinton
49	Arkansas	$47,062	Trump
50	Mississippi	$44,717	Trump
51	West Virginia	$44,097	Trump

High-income states vote **blue**
Low-income states vote **red**

"Farmer, factory workers, truck drivers, waitresses..."

vs.

The know-it-alls of Manhattan and Malibu ... who lord over the peasantry with their fancy college degrees

"Average Americans, humble, long-suffering, working hard, who buy their coffee already ground"

vs.

"The wealthy, latte-swilling liberal elite"

"Real Americans, with a lawnmower in the garage and a flag on the front stoop"

vs.

"Wealthy condo-dwellers with contempt for those who feel chills up their spines at 'The Star Spangled Banner"'

And yet ...

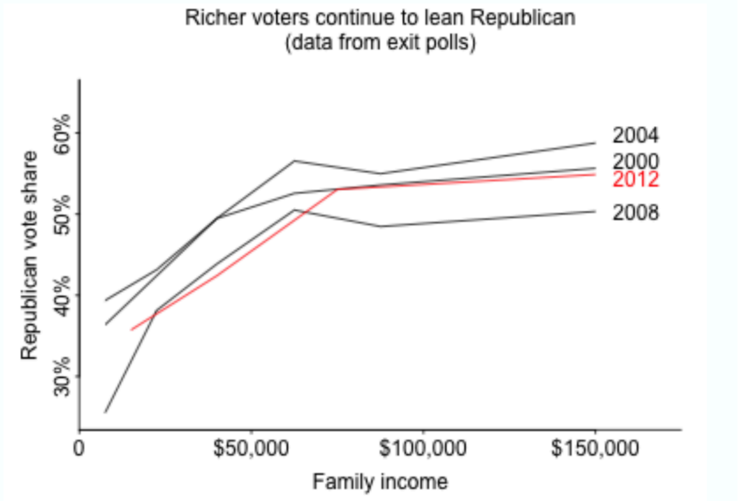
The second paradox

Presidential vote share by personal income

	under $50K		over $50K	
	Dem.	Rep.	Dem.	Rep.
2004	0.55	0.44	0.43	0.56
2008	0.60	0.38	0.49	0.49
2012	0.54	0.44	0.44	0.54
2016	0.52	0.41	0.47	0.49

The second paradox

Presidential vote share by family income



The second paradox

- For states:
  $\rightarrow$ higher income means more likely to vote Democrat
  $\rightarrow$ lower income means more likely to vote Republican

- Yet, for people:
  $\rightarrow$ higher income means more likely to vote Republican
  $\rightarrow$ lower income means more likely to vote Democrat

- How is this possible?

Back to the first paradox

Complication rates and sample sizes across 3,690 deliveries at a large maternity hospital in Cambridge, UK.

	low-risk	**high-risk**	**overall**
senior doctor	0.052 (213)	0.127 (102)	**0.076** (315)
junior doctor	0.067 (3169)	0.155 (206)	**0.072** (3375)

Rule of total probability

The probability of an event is the sum of the probabilities for all of the different ways that event can happen.

$$P(\text{rain}) = P(\text{rain}, \text{wind}) + P(\text{rain}, \text{no wind})$$

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Suppose that $B_1, ...B_N$ are mutually exclusive events whose probabilities sum to 1.

$P(B_i, B_j) = 0 \quad \forall i \neq j \quad$ and $\quad \sum_{i=1}^{N} P(B_i) = 1$

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Then, for any event A:

$$\boxed{P(A) = \sum_{i=1}^{N} P(A, B_i) = \sum_{i=1}^{N} P(A \mid B_i) P(B_i)}$$

Rule of total probability

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The overall (total) probability of a complication is:

$$P(\text{comp}) = P(\text{comp}, \text{low}) + P(\text{comp}, \text{high})$$

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$$\begin{aligned} P(\text{comp}) &= P(\text{comp}, \text{low}) + P(\text{comp}, \text{high}) \\ &= P(\text{low}) \cdot P(\text{comp} \mid \text{low}) + P(\text{high}) \cdot P(\text{comp} \mid \text{high}) \end{aligned}$$

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For senior doctors:

$$P(\text{comp}) = \frac{213}{213+102} \cdot 0.052 + \frac{102}{213+102} \cdot 0.127 = 0.076$$

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For junior doctors:

$$P(\text{comp}) = \frac{3169}{3169+206} \cdot 0.067 + \frac{206}{3169+206} \cdot 0.155 = 0.072$$

First paradox resolved

Senior doctors are...

- better at low-risk *and* high-risk deliveries
- yet worse overall

This is Simpson's paradox in action. Here's what is going on:

- $P(\text{comp} \mid \text{low})$ and $P(\text{comp} \mid \text{high})$ are both lower for senior doctors
- yet senior doctors work fewer low-risk cases: $P(\text{low})$ is smaller in the mixture!

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Moral of the story:

- Make sure you're asking the right question
- Always be sensitive to whether probabilities are conditional or unconditional (marginal, total, overall), and which type makes more sense for your situation.

Back to the second paradox

- For states:
  $\rightarrow$ higher income means more likely to vote Democrat
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- Yet, for people:
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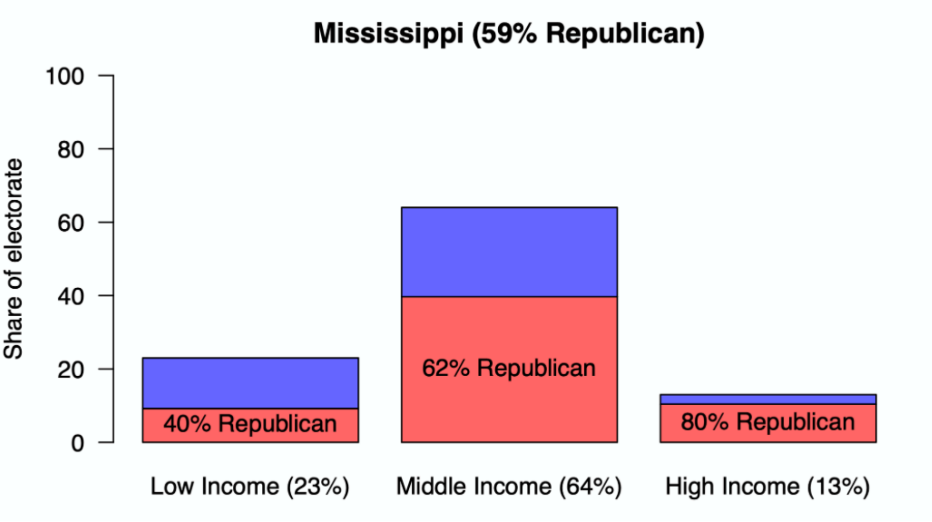
- How is this possible?

Law of total probability, Mississippi



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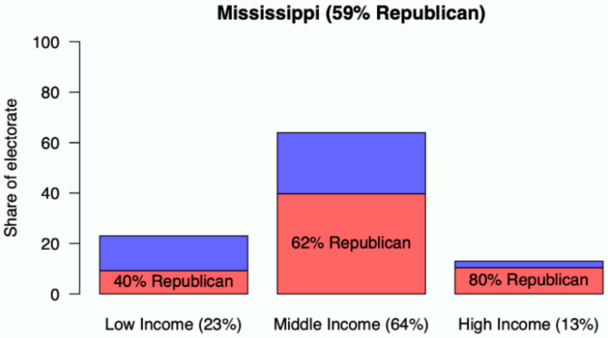

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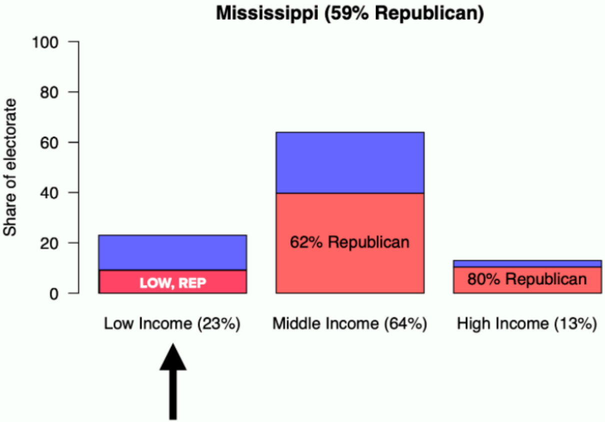

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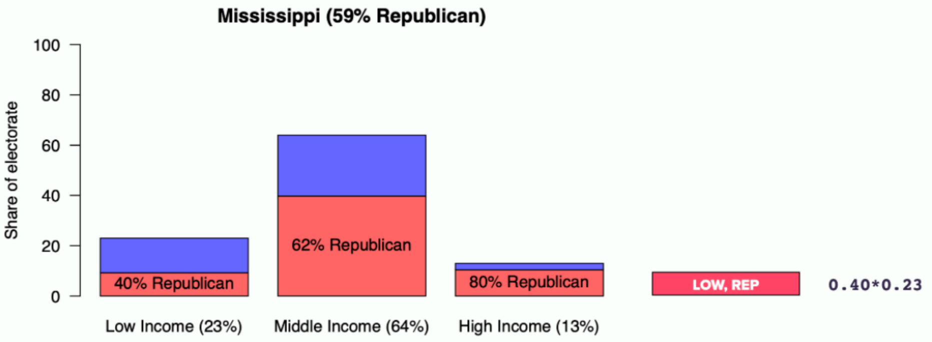

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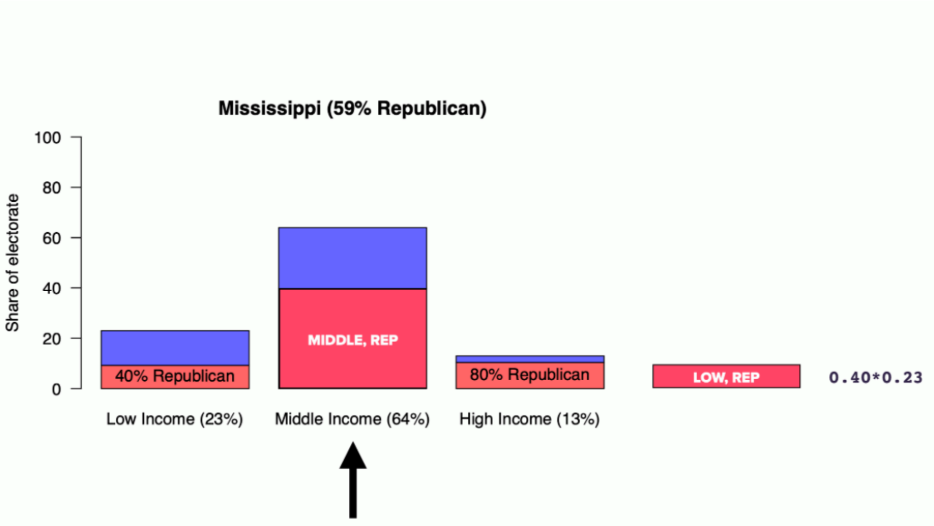

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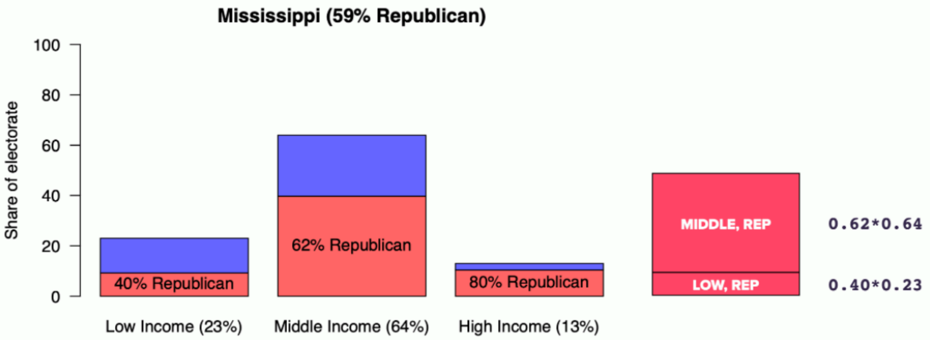

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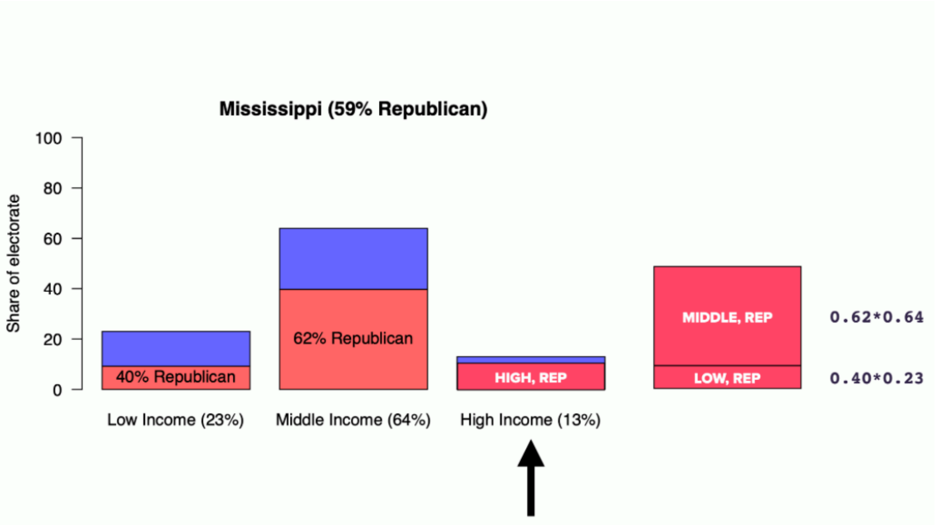

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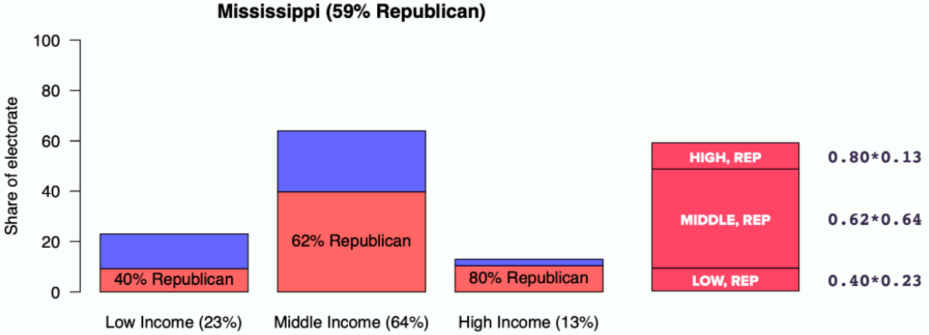
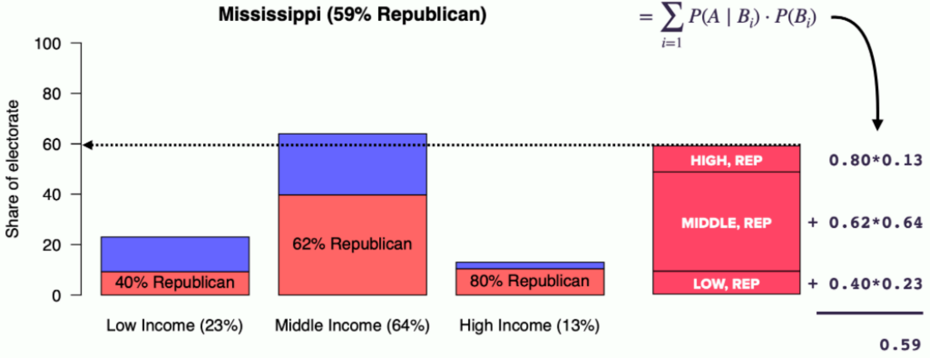


Law of total probability, Mississippi

$$P(A) = \sum_{i=1}^{N} P(A, B_i)$$
$$= \sum_{i=1}^{N} P(A \mid B_i) \cdot P(B_i)$$



Connecticut and Mississippi

Here is $P(\text{Rep} \mid \text{income})$ for each state:

	Low-income	Middle-income	High-income
Connecticut	0.39	0.43	0.47
Mississippi	0.40	0.62	0.80

Connecticut and Mississippi

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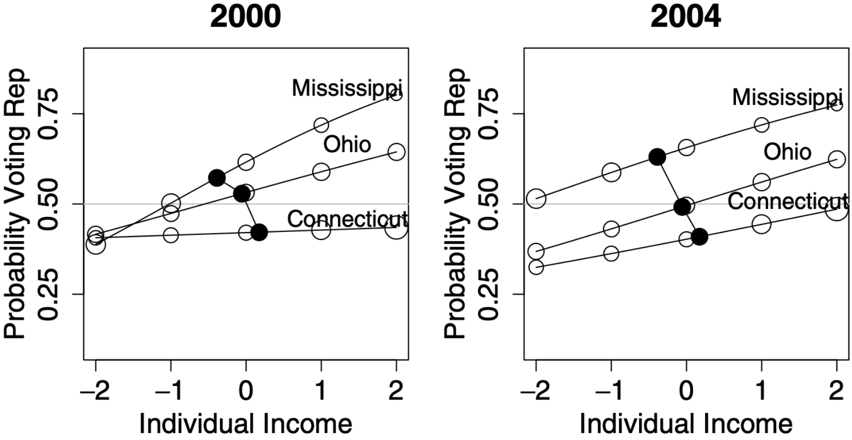
Q: Does income really tell me anything about why CT is blue and MS is red?

Let's look at Mississippi, Ohio, & Connecticut

(from Gelman et. al., Quarterly Journal of Political Science)

- same story, different election years



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Paradox 2 resolved, kind of ...

We've seen how, mechanically, an individual-level effect can be in one direction, and a group-level effect can be in the other direction.

But, conditioning on income alone **cannot** explain why CT is blue and MS is red! What can is the relative positioning of the state lines.

What else (other than income) could be driving this relationship? (homework)

The ecological fallacy

Ecological inference: looking for associations between cause and effect at the level of groups or populations.

Do groups with higher average levels of A tend to have higher B?

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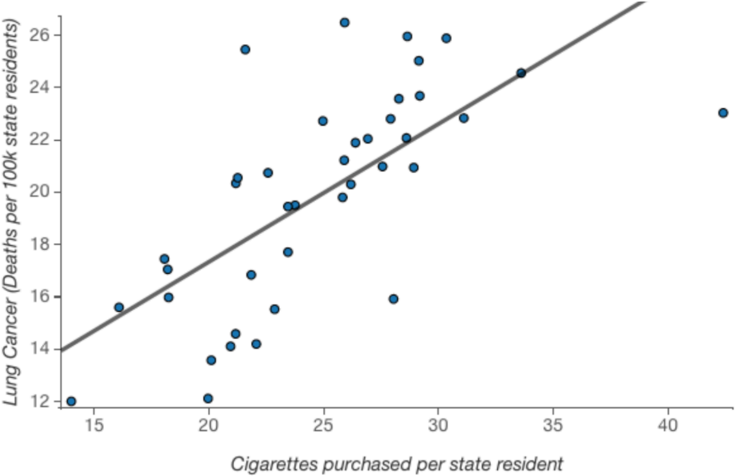
Do groups with higher average levels of A tend to have higher B?

The ecological fallacy: assuming, without further justification, that group-level associations accurately reflect individual level associations.

Groups with higher A have higher B, on average. Therefore, individuals with higher A have higher B, on average. $\leftarrow$ **not necessarily!!**

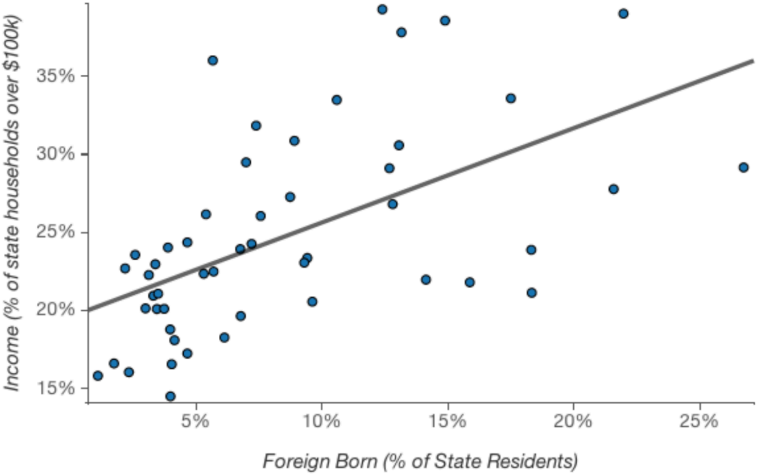
The ecological fallacy

smoking cigarettes really does increase an individual's risk of lung cancer. This ecological association accurately reflects an individual-level trend.



The ecological fallacy

... but this one doesn't. At the individual level, 22.1% of foreign-born residents make more than \$100k, versus 26.1% of US-born residents.



Take-home messages

- A trend that appears when the data are *separated into individuals/smaller groups* can look different, or even reverse entirely, when the data are *aggregated into larger groups*.

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- So what to do? Remember the rule of total probability!

  → Pay attention: the level of grouping matters a lot

  → Ask questions: Do we care about a total or conditional probability? Are we missing any lurking variables?

  → Avoid the ecological fallacy: learn to be skeptical when group-level trends are applied to individuals