A PRECIS ON A SUMMER OF GROUP THEORY.[1]

Contents (in order of appearance)

References

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[1]Version of 2025/07/19. Notes taken by David Punton.

A group $(G, *)$ is a set G equipped with an associative map $* : G \times G \to G$ so that G has an identity and inverses with respect to $*$. If $*$ is commutative, then G is abelian. A subgroup H of a group G is a nonempty subset $H \subseteq G$ which is closed under $*$ and taking inverses, and is denoted $H \leq G$. For any subgroup $H \leq G$, then $e_H = e_G$ and inverses in H and G are identical.

Subgroup test.
If G is a group and $H \subseteq G$, then $H \leq G$ iff H is nonempty and $xy^{-1} \in H$ for all $x, y \in H$.

The subgroup of a group G generated by a nonempty set $X \subseteq G$ is

$$\langle X \rangle := \{x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} \mid a_i \in \{\pm 1\}, x_i \in X \text{ for all } 1 \leq i \leq k,\ k \geq 0\}$$

and can equivalently be characterised as the intersection of all subgroups of G which contain X. In this sense, $\langle X \rangle$ is the 'smallest' subgroup of G which contains X. A group G is finitely generated if there exists a finite set $X \subseteq G$ such that $G = \langle X \rangle$. A group G is cyclic if there exists $g \in G$ such that $G = \langle \{g\} \rangle := \langle g \rangle$.

The order of a group G is the number of elements of the group, and is denoted $|G|$. If G contains infinitely many elements, then $|G| := \infty$. The order of an element $a \in G$ is the order of $\langle a \rangle \leq G$.

The cyclic group of order $n \in \mathbb{Z}_{>0}$ is $C_n := \langle g \mid g^n = e \rangle$. (Strictly speaking, I'm cheating here: this is a group presentation of C_n which we'll see later.)

Let G be a group and $H \leq G$. A left coset of H in G is $gH := \{gh : h \in H\}$ for some $g \in G$. A right coset of H in G is $Hg := \{hg : h \in H\}$ for some $g \in G$. There is an equivalence relation on G given by $g_1 \sim g_2$ iff $g_1 \in g_2 H$ for all $g_1, g_2 \in G$. An analogous equivalence relation exists on G for right cosets of H in G.

Lagrange's theorem.
If G is a group and $H \leq G$, then $|G| = [G : H]|H|$ where $[G : H]$ is the *index*, meaning the number of left (or equivalently, right) cosets, of H in G.

GROUP ACTIONS AND SYLOW THEOREMS.

The converse of Lagrange's theorem doesn't generally hold, but there exist partial converses, such as:

Cauchy's theorem.
If G is a group and p is a prime which divides $|G|$, then there exists $H \leq G$ such that $|H| = p$.

A p-group is a group in which every element is of order p^k. A p-subgroup is a subgroup which is a p-group. A finite group G is a p-group if and only if $|G| = p^k$ for some $k \in \mathbb{Z}$.

A left group action of a group G on a set X is a map $\alpha : G \times X \to X$ such that $\alpha(e, x) = x$ and $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $g, h \in G$ and $x \in X$. A right group action is defined analogously. It's common to write $g.x := \alpha(g, x)$ for a group action α, $g \in G$ and $x \in X$. Given a group action, the orbit of an element $x \in X$ is $G.x := \{g.x : g \in G\} \subseteq X$ and the stabiliser of $x \in X$ is $\mathrm{Stab}_G(x) := \{g : g.x = x\} \subseteq G$. A fixed point of a group action is $x \in X$ such that $G.x = \{x\}$.

If G is a finite group which acts upon a set X, then $\mathrm{Stab}_G(x) < G$ for each $x \in X$. Moreover, there exists an equivalence relation $\sim$ on X such that for all $x, y \in X$, $x \sim y$ iff there exists $g.x = y$. That is, a group action partitions a set into orbits. Lastly, we have:

Orbit-stabiliser theorem.
If G is a finite group which acts on a set X, then $|G| = |G.x||\mathrm{Stab}_G(x)|$ for all $x \in X$.

A pair of elements $a, b \in G$ are conjugate if there exists $g \in G$ such that $a = gbg^{-1}$. A group G acts on itself by conjugation via the map $G \times G \to G$, $(g, a) \mapsto gag^{-1}$. The conjugacy class $\mathrm{Cl}(a)$ of an element $a \in G$ is the orbit of a under conjugation; hence, conjugation partitions a group into conjugacy classes so that the class equation $|G| = \sum_i |\mathrm{Cl}(a_i)|$ for a finite set $\{a_1, \ldots, a_k\} \subset G$.

The centraliser $C_G(a)$ of an element $a \in G$ is the stabiliser of a under conjugation, given explicitly as the subset of G with respect to which a is invariant under conjugation, $C_G(a) := \{g : a = gag^{-1}\}$.

Immediately, $C_G(a) < G$. Also, $|G| = |Cl(a)||C_G(a)|$ for each $a \in G$ and thus $|\mathrm{Cl}(a)| = [G : C_G(a)]$ by Lagrange's theorem.

A pair of subsets $T, S \subseteq G$ are conjugate if there exists $g \in G$ such that $T = gSg^{-1}$. That is, conjugation by g induces a one-to-one correspondence between T and S. The centraliser $C_G(S)$ of a subset $S \subseteq G$ is $C_G(S) := \{g \ : \ s = gsg^{-1} \text{ for all } s \in S\} \leq G$. The centre $Z(G)$ of a group G is $Z(G) := C_G(G)$. The normaliser $N_G(S)$ of a subset $S \subseteq G$ is $N_G(S) := \{g \ : \ S = gSg^{-1}\} \leq G$ such that $C_G(S) \subset N_G(S)$.

A subgroup $H \leq G$ is normal if $H = gHg^{-1}$ for all $g \in G$, and is denoted $H \lhd G$. Equivalently, $H \lhd G$ iff $gHg^{-1} \leq H$ iff $G = N_G(H)$ iff left and right cosets of H in G coincide. Given an arbitary subgroup $K \leq G$, the normaliser $N_G(K)$ is equal to the union of all subgroups (*i.e.* the 'largest' subgroup) which contain K as a normal subgroup.

Our group action technology and a pair of lemmas, which are:

1. If p is a prime and G is a finite-p group which acts on a finite set X, then the number of fixed points in X is congruent to $|X|$ modulo p,

2. If G is a finite group and $H \leq G$, then $[G : N_G(H)]$ equals the number of distinct conjugate subgroups of H in G and moreover, for each prime p which divides $|G|$ and Sylow p-subgroup $P \leq G$, we have $n_p = [G : N_G(P)]$,

allow us to prove:

Sylow theorems.

Let G be a group and p be a prime such that k is the largest exponent for which p^k divides $|G|$.

1. There exists a subgroup of G whose order is p^k. Such a subgroup is called a Sylow p-subgroup.

2. For each Sylow p-subgroup $P \leq G$ and p-subgroup $H \leq G$, there exists $g \in G$ such that $H \subseteq gPg^{-1}$. Hence, any two Sylow p- subgroups of G are conjugate.

3. The number n_p of Sylow p-subgroups of G divides $|G|/p^k$, is congruent to 1 modulo p and equals $[G : N_G(P)]$ for each Sylow p-subgroup $P \leq G$.

As a consequence of the Sylow theorems, a Sylow p-subgroup of G is normal iff $n_p = 1$. A subgroup $H \leq G$ is simple if H contains only $\{e\}$ and H as normal subgroups; that is, H contains no 'non-trivial' normal subgroups.

ISOMORPHISM THEOREMS.

Given a pair of groups G and H, a group homomorphism is a map $\phi : G \to H$ such that for all $g, h \in G$, $\phi(gh) = \phi(g)\phi(h)$. A group isomorphism is a bijective group homomorphism. If $\phi : G \to H$ is a group homomorphism, then $\phi(e_G) = e_H$ and $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$. The kernel $\ker\phi$ of a group homomorphism $\phi : G \to H$ is $\ker\phi := \{g \ : \ \phi(g) = e_H\} \subseteq G$. A subgroup $N \leq G$ is normal in G iff there exists a group H and a group homomorphism $\phi : G \to H$ such that $\ker\phi = N$.

A factor group G/N is the set of left (or equivalently, right) cosets of a normal subgroup $N \lhd G$ in G equipped with the group operation $(gN) * (hN) = (gh)N$ for each $g, h \in G$. There is a canonical surjective group homomorphism $\mathrm{can} : G \to G/N, g \mapsto gN$.

Universal property of factor groups.

Let G be a group and $N \lhd G$. For each group H and group homomorphism $\psi : G \to H$ with $N \subseteq \ker\psi$, there exists a unique group homomorphism $\overline{\psi} : G/N \to H$ such that $\psi = \overline{\psi} \circ \mathrm{can}$.

As a useful corollary, if $\phi : G \to K$ is a surjective group homomorphism and $\psi : G \to H$ is a group homomorphism with $\ker\phi \subseteq \ker\psi$, then there exists a unique group homomorphism $\overline{\psi} : K \to H$ such that $\psi = \overline{\psi} \circ \phi$.

First isomorphism theorem.

If G, H are groups and $\phi : G \to H$ is a group homomorphism, then $\ker\phi \lhd G$, $\mathrm{im}\phi \leq H$ and there exists a group isomorphism $\overline{\phi} : G/\ker\phi \to \mathrm{im}\phi$ given by $gN \mapsto \phi(g)$ with $N := \ker\phi$. If ϕ is surjective, then $G/\ker\phi \cong H$.

If G is a group and $N \lhd G$, then we make two observations: first, if $K \leq G/N$, then $\mathrm{can}^{-1}(K) \leq G$ with $N \subseteq \mathrm{can}^{-1}(K)$ and moreover, $\mathrm{can}^{-1}(K) \lhd G$ if and only if $K \lhd G/N$; second, if $N \leq H \leq G$, then $H = \mathrm{can}^{-1}(\mathrm{can}(H))$. Combining these allows us to prove:

Correspondence theorem.
Let G be a group, $N \lhd G$ and $\mathrm{can} : G \to G/N$ denote the canonical surjection. There is a bijection between subgroups of G which contain N and subgroups of G/N given by $H \mapsto \mathrm{can}(H)$ which restricts to a bijection between normal subgroups of G which contain N and normal subgroups of G/N.
Furthermore, if $A, B \leq G$, then $A \subseteq B$ iff $\mathrm{can}(A) \subseteq \mathrm{can}(B)$.

which in turn allows us to prove:

Third isomorphism theorem.
If G is a group and $N \leq H \leq G$ are such that $N, H \lhd G$, then $(G/N)/(H/N) \cong G/H$.

Given a pair of subsets $X, Y \subseteq G$, define $XY := \{xy \,:\, x \in X, y \in Y\}$.

Second isomorphism theorem.
If G is a group, $N \lhd G$ and $H \leq G$, then
1. $HN \leq G$,
2. $N \lhd HN$,
3. $H \cap N \lhd H$,
4. $HN/N \cong H/(H \cap N)$.

GROUP PRESENTATIONS.

The free group on generators $x_1, \ldots, x_n$ is the set of all words in the symbols $x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$ equipped with concatenation, and is denoted $\langle x_1, \ldots, x_n \rangle$.

Given an arbitrary group G, the normal closure $\mathrm{ncl}_G(S)$ of a subset $S \subseteq G$ is the intersection of all normal subgroups of G which contain S; that is, $\mathrm{ncl}_G(S)$ is the 'smallest' normal subgroup of G which contains S. The group generated by $x_1, \ldots, x_n$ subject to 'relations' $r_1, \ldots, r_k \in \langle x_1, \ldots, x_n \rangle$ is

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_k \rangle := \langle x_1, \ldots, x_n \rangle / \mathrm{ncl}_G(\{r_1, \ldots, r_k\}).$$

If a group G is isomorphic to $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_k \rangle$, then the latter is a 'presentation' of G. Also, $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_k \rangle$ is equivalently the free group $\langle x_1, \ldots, x_n \rangle$ subjected to the additional conditions $r_1 = \cdots = r_k = e$ and all logical consequences thereof.

The n-th dihedral group D_n is the group of symmetries of a regular n-sided polygon; formally, D_n is defined for each $n \in \mathbb{Z}_{\geq 3}$ as the group generated by two elements, g and h, where g denotes reflection across a line through a specified, fixed vertex and h denotes rotation by $2\pi/n$ radians. There is a group presentation for D_n given by $D_n \cong \left\langle g, h \mid g^2, h^n, (gh)^2 \right\rangle$ for each $n \geq 3$.

Universal property of free groups.
Let G be a group generated by a set $\{s_1, \ldots, s_n\}$ and $F = \langle S_1, \ldots, S_n \rangle$ be the free group generated by the letters $S_1, \ldots, S_n$. There exists a unique surjective group homomorphism $\pi : F \to G$ such that $\pi(S_i) = s_i$ for each $i \in \{1, \ldots, n\}$.

FINITELY-GENERATED ABELIAN GROUPS.

In an abelian group, all subgroups are normal because left and right cosets must coincide; in a finite abelian group, all Sylow p-subgroups are unique. It follows that a finite abelian group is isomorphic to the direct product of its Sylow p-subgroups which implies:

Fundamental theorem of finite abelian groups.
1. A finite abelian group is isomorphic to a direct product of cyclic groups of prime power order uniquely up to reordering.

2. A finite abelian group of order n is isomorphic to a direct product of cyclic groups $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_s}$ where $n_i | n_{i+1}$ for all $1 \leq i \leq s-1$ and $n_1 n_2 \cdots n_s = n$ uniquely up to reordering the factors.

where the latter is implied by:

Chinese remainder theorem.
If $m, n \in \mathbb{Z} - \{0\}$ are coprime, then $C_{mn} \cong C_m \times C_n$.

Given a finite abelian group, the first and second decompositions are clearly isomorphic.

The exponent $e(G)$ of any finite group is the least common multiple of the orders of elements in G; that is, the smallest n such that $g^n = e$ for all $g \in G$. Any finite abelian group G contains an element of order $e(G)$ so that if $e(G) = |G|$ then G is cyclic since such an element generates G.

The set of nonzero elements of a field form a group under multiplication in the field; that is, the *group of units* K^* of a field K given a field K. If A is a finite subgroup of K^* for a field K, then A is cyclic; as a consequence, the group of units of any finite field is cyclic.

If R is a ring, then an R-module is an abelian group $(M, +)$ equipped with a map $R \times M \to M$, $(r, m) \mapsto rm$ which is distributive, associative and unital. Given any ring R, the free R-module of rank s is the set of s-tuples R^s equipped with 'pointwise' addition and a map $R \times R^s \to R^s$ given by $r(r_1, \ldots, r_s) \mapsto (rr_1, \ldots, rr_s)$.

An abelian group G is a $\mathbb{Z}$-module equipped with the 'obvious' map $\mathbb{Z} \times G \to G, (n, a) \mapsto na$ where the latter is additive notation for $+$ in G. This observation leads to:

Fundamental theorem of finitely-generated abelian groups.
If A is a finitely-generated abelian group, then there exist $k, l \in \mathbb{N}$ and nonzero $r_1, \ldots, r_k \in \mathbb{Z}$ with $r_i | r_{i+1}$ for each $1 \leq i \leq k-1$ such that $A \cong \mathbb{Z}/r_1\mathbb{Z} \times \cdots \times \mathbb{Z}/r_k\mathbb{Z} \times \mathbb{Z}^l$uniquely up to isomorphism.

whose proof is constructive as follows: given a $\mathbb{Z}$-module A finitely generated by s elements $r_1, \ldots, r_s$, there exists an 'obvious' surjection $\varphi : \mathbb{Z}^s \to A, e_i \mapsto r_i$ for standard basis vectors e_i within $\mathbb{Z}^s$ so that by the first isomorphism theorem for modules $\mathbb{Z}^s / \ker \varphi \cong \mathrm{im}\varphi = A$; however, factor modules of $\mathbb{Z}^s$ by a submodule such as $\ker \varphi$ are invariant up to isomorphism with respect to $\mathbb{Z}$-module automorphisms so as invertible row and column operations on the matrix 'corresponding to' a finitely generated submodule[2] are instances of such automorphisms, then $\mathbb{Z}^s / \ker \varphi$ is invariant under such operations upon the matrix 'corresponding to' $\ker \varphi$; we thereby build a matrix of the form $\mathrm{diag}(r_1, \ldots, r_k, 0, \ldots 0)$ which corresponds to a submodule of the form $K := \mathbb{Z}(r_1, 0, \ldots, 0) + \cdots + \mathbb{Z}(0, \ldots, 0, r_k, 0, \ldots, 0)$ such that $A \cong \mathbb{Z}^s / K$ is exactly as desired where $l := s - k$ and we choose our operations so that $r_i | r_{i+1}$ $\forall$ i.

ALTERNATING GROUPS.

Given a set X, the set of bijections $X \to X$ forms a group under composition called the *symmetric group* $S(X)$ on X. The n-th finite symmetric group is $S_n := S(\{1, \ldots, n\})$ so that for any finite set X of n elements, $S(X) \cong S_n$ via. the 'obvious' set bijection $\{1, \ldots, n\} \to X$, with $|S_n| = n!$ for each n.

An element of S_n is often called a *permutation* and written either as an $2 \times n$ array or using cycle notation. Any permutation can be written uniquely (up to reordering) as a product of k disjoint cycles, or <u>non-uniquely</u> either as a product of transpositions (2-cycles) or adjacent transpositions (2-cycles of the form $(i\ i+1)$ for some i). The former implies for each $\sigma \in S_n$ a well-defined k-tuple called the *cycle type* of σ whose entries are the lengths of each disjoint cycle in decreasing order. A pair of permutations in S_n are conjugate iff they have the same cycle type; thus, conjugacy classes and cycle types in S_n are in exact one-to-one correspondence. Of explicit note also is that conjugation of a cycle $(i_1, \cdots i_k)$ by an element $\tau \in S_n$ is given by $\tau(i_1 \cdots i_k)\tau^{-1} = (\tau(i_1) \cdots \tau(i_k))$ which generalises to any $\sigma \in S_n$ by writing $\sigma = c_1 c_2 \cdots c_k$ uniquely as disjoint cycles so that

$$\tau\sigma\tau^{-1} = \tau c_1 c_2 \cdots c_k \tau^{-1} = \tau c_1 (\tau^{-1}\tau) c_2 (\tau^{-1}\tau) \cdots (\tau^{-1}\tau) c_k \tau^{-1} = (\tau c_1 \tau^{-1})(\tau c_2 \tau^{-1}) \cdots (\tau c_k \tau^{-1})$$

[2] *i.e.* given a submodule $K \subseteq \mathbb{Z}^s$ finitely generated by $x_1, \ldots, x_k$, writing each $x_i = \sum_{j=1}^{s} a_{ij} e_j$ implies an $r \times s$-matrix $(a_{ij}) \in \mathrm{Mat}_{r \times s}(\mathbb{Z})$ associated to K so that $(a_{ij})(e_j) = (x_i)$.

via associativity in S_n.

Informally, the n-th alternating group A_n 'is' the subgroup of S_n consisting of even permutations, where an even/odd permutation is one which can be written as a product of an even/odd number of transpositions; for the latter to be formally well-defined, one approach constructs a group action of S_n a stabiliser of which is A_n.

Let $x_1, \ldots, x_n$ be indeterminates. Define $P := \prod_{1 \leq i < j \leq n}(x_i - x_j)$ so that S_n acts on $X := \{P, -P\}$ by permuting the indices. Observe that P consists of all possible distinct pairs $i < j$ within $\{1, \ldots, n\}$; as each $\sigma \in S_n$ is a bijection, $\sigma.P = (-1)^k P$ where k is the number of pairs whose order σ 'flips.' A permutation $\sigma \in S_n$ is even if $\sigma.P = P$ and odd if $\sigma.P = -P$ and the n-th alternating group is the subset of S_n consisting of even permutations; immediately, $A_n = \mathrm{Stab}_{S_n}(P) \leq S_n$.

The product of two even or two odd permutations is even, whereas that of an even and an odd permutation in either order is odd; thus, an n-cycle is even/odd iff n is odd/even. Moreover, $A_n \lhd S_n$ of index 2 so that $|A_n| = n!/2$ by Lagrange's theorem.

Whereas $A_1, A_2 \cong 1$ and $A_3 \cong C_3$ are both simple, A_4 contains a unique non-trivial normal subgroup N of order 4 such that $A_4/N \cong C_3$ whereas $S_4/N \cong S_3$, meaning that A_4 is not simple. The observation that for any finite group G, a subgroup $N \leq G$ is normal iff it is a union of conjugacy classes (one of which is necessarily that of the identity) allows to prove by induction that A_n is simple for all $n \geq 5$ by exploiting five key facts:

1. $\mathrm{Cl}_{A_n}(\sigma) \subseteq \mathrm{Cl}_{S_n}(\sigma)$ for all $\sigma \in A_n$ (possibly a strict inclusion!),
2. any pair of 3-cycles are conjugate in A_n for all $n \geq 5$ (*i.e.* equality holds in 1. for 3-cycles),
3. A_n is generated by 3-cycles for all $n \geq 3$,
4. for $H \leq S_n$ every non-trivial element of which is *fixed point free* (*i.e.* $\sigma(i) \neq i$ for all i), $|H| \leq n$.
5. $|\mathrm{Cl}_{A_n}(\sigma)| \geq n$ for all $n \geq 6$ and non-trivial $\sigma \in A_n$,

and so conclude that A_n is not simple iff $n = 4$.

COMPOSITION SERIES AND THE JORDAN-HÖLDER THEOREM.

A composition series of a group G is a chain of normal subgroups

$$\{e\} =: G_0 \lhd G_1 \lhd \cdots \lhd G_{l-1} \lhd G_l := G \quad (*)$$

such that $G_i \neq G_{i-1}$ and G_{i-1}/G_i is simple for all $0 \leq i \leq l-1$. The observation that $\{e\} \lhd G$ is a composition series for G iff $G/\{e\} \cong G$ is simple implies a common analogy that composition series are to groups as prime factorisation is to the integers wherein simple groups are alike primes: those groups which cannot be 'decomposed' any further by means of a composition series.

The *composition factors* of the series are the factor groups G_{i+1}/G_i for all i and the *length* is the number of composition factors, in $(*)$ denoted l.

Jordan-Hölder theorem.

If G is a finite group, then there exists at least one composition series for G and any two composition series for G have the same length and composition factors up to isomorphism and reordering.

The Jordan-Hölder theorem states essentially that any finite group has one and 'only' one composition series; it does <u>not</u> however state that these uniquely *determine* a finite group. For instance, C_6 and S_3 both have composition series of length 2 and factors $\{C_2, C_3\}$, given by $\{e\} \lhd C_3 \lhd C_6$ and $\{e\} \lhd A_3 \lhd S_3$ respectively which are 'the same' in the Jordan-Hölder sense; only, C_6 and S_3 are not isomorphic since C_6 is abelian whereas S_3 is not abelian. Our simple group-prime number analogy here breaks down and also in that finite simple groups are entirely known as a result of the

Classification of finite simple groups.

If G is a finite simple group, then G is isomorphic to one of C_p for a prime p, A_n for $n \geq 5$, a group of Lie type[3], or a sporadic group[4].

[3]There are 16 infinite familes together called groups of Lie type, such as the projective special linear groups over a finite field.

[4]These are 26 groups which don't fit into any of the previous eighteen infinite families.

Proving Jordan-Hölder is informally summarised as follows: in both cases, we induct on $|G|$...

1. The base case $|G| = 1$ is trivial, as then $G = \{e\}$ which is a composition series (c.s.) for itself. Otherwise, let $|G| = k$ and suppose that any group with fewer than k elements has a composition series. If G is simple, then $\{e\} \lhd G$ is a c.s. for G as above; else, there exists a non-trivial $N \lhd G$. As $N \neq G$, then $|N| < k$ so N has a c.s.; as $N \neq \{e\}$, then $|N| > 1$ so by Lagrange's theorem $|G/N| < k$ so G/N also has a c.s. Using the canonical map can : $G \to G/N$ to pull back the latter to a chain of normal subgroups within G terminating at $N = \text{can}^{-1}(\{e_{G/N}\})$ and attaching this in the 'obvious' way to our c.s. for N yields a composition series for G.

2. The base case $|G| = 1$ is again trivial since $\{e\}$ has only one possible composition series, itself. Similarly, let $|G| = k$ and suppose that the latter part of the theorem holds for any group fewer than k elements. One takes a pair of arbitary composition series for G and distinguishes between whether the penultimate elements of either chain are equal or distinct as normal subgroups of G, in the former case immediately using our assumption to obtain the result whereas in the latter case, one (more labouriously!) constructs four composition series for G which provide the result[5].

SOLVABLE GROUPS.

A subnormal series for a group G is a chain of normal subgroups $\{e\} =: G_0 \lhd G_1 \lhd G_{l-1} \lhd G_l := G$. If $G_i \neq G_{i+1}$ for all $0 \leq i \leq l-1$, the chain is a series 'without repetition.' Every composition series is a subnormal series subject to the additional condition that each factor G_i/G_{i-1} is simple.

A group G is solvable if there exists a subnormal series for G such that each factor G_i/G_{i-1} is abelian. A group G is abelian iff it is solvable via. $\{e\} \lhd G$. This implies informally that a group is solvable if it can be decomposed into abelian groups via. a subnormal series.

Properties of solvable groups. If G is a finite abelian group of order $p_1^{n_1} \cdots p_k^{n_k}$, then the composition factors of G are $C_i, \cdots, C_i$ p_i-times for all $1 \leq i \leq k$ in some order; as a consequence, a finite group is solvable iff all composition factors are cyclic; for any group G and $N \lhd G$, it follows that G is solvable iff N and G/N are solvable.

If a group G is solvable and $H \leq G$, then H is solvable. For instance, A_n is solvable iff $n \leq 4$ meaning that S_n is not solvable for all $n \geq 5$. Any simple non-abelian group H is generally not solvable, since $\{e\} \lhd H$ is the only subnormal series of H and $H/\{e\} \cong H$ is not abelian.

Derived subgroups. The commutator of $a, b \in G$ is $[a, b] := aba^{-1}b^{-1}$ so that $[a, b] = e$ iff $ab = ba$ in G (*i.e.* a and b commute). The derived (or, commutator) subgroup is $G' := \langle [a, b] \mid a, b \in G \rangle$ so that $[a, b]^{-1} = [b, a]$ and $z[a, b]z^{-1} = [zaz^{-1}, zbz^{-1}]$ for all $z \in G$, meaning every element of G' is a product of commutators and thus $G' \lhd G$.

Since for all $N \lhd G$, G/N is abelian iff $G' \subseteq N$ and in particular taking $N = G'$ implies that G/G' is abelian, we interpret G' as the smallest normal subgroup N of G such that G/N is abelian.

Derived series. Iterating this construction implies that the *derived series* of a group G is the chain $G := G^{(0)} \rhd G^{(1)} \rhd G^{(2)} \rhd \cdots$ where $G^{(i+1)} = (G^{(i)})'$ for each i so that G is solvable iff there exists n for which $G^{(n)} = \{e\}$. Of note is that if there exists i such that $G^{(i+1)} = G^{(i)}$ then $G^{(j)} = G^{(i)}$ for all $j \geq i$ where if G is a finite group, such an i always exists: for instance, if G is simple and not abelian, then $G^{(i)} = G$ for all $i \geq 1$. The derived length of a solvable group G is the least n so that $G^{(n)} = \{e\}$. As an example, D_n has derived length 2 for all $n \geq 3$.

Feit-Thompson theorem.
If G is a finite group of odd order, then G is solvable.

[5] Each pair of c.s. for G correspond to c.s. one of the two distinct subgroups which allows us to similarly apply our assumption in either case.