A PRECIS ON A SUMMER OF GROUP THEORY.

Contents (in order of appearance)

References

Sue Sierra. Group Theory. University of Edinburgh. 2020.
J. J. Rotman. An Introduction to the Theory of Groups. Springer New York, NY. 1994.
Ahmed Ayache, Khalid Amin. Introduction to Group Theory. Springer Singapore. 2025

A group $(G, *)$ is a set G equipped with an associative map $* : G \times G \to G$ so that G has an identity and inverses with respect to $*$. If $*$ is commutative, then G is abelian. A subgroup H of a group G is a nonempty subset $H \subseteq G$ which is closed under $*$ and taking inverses, and is denoted $H \leq G$. For any subgroup $H \leq G$, then $e_H = e_G$ and inverses in H and G are identical.

Subgroup test. If G is a group and $H \subseteq G$, then $H \leq G$ if and only if H is nonempty and $xy^{-1} \in H$ for all $x, y \in H$.

The subgroup of a group G generated by a nonempty set $X \subseteq G$ is

$$\langle X \rangle := \{x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} \mid a_i \in \{\pm 1\}, x_i \in X \text{ for all } 1 \leq i \leq k,\ k \geq 0\}$$

and can equivalently be characterised as the intersection of all subgroups of G which contain X. In this sense, $\langle X \rangle$ is the 'smallest' subgroup of G which contains X. A group G is finitely generated if there exists a finite set $X \subseteq G$ such that $G = \langle X \rangle$. A group G is cyclic if there exists $g \in G$ such that $G = \langle \{g\} \rangle := \langle g \rangle$.

The order of a group G is the number of elements of the group, and is denoted $|G|$. If G contains infinitely many elements, then $|G| := \infty$. The order of an element $a \in G$ is the order of $\langle a \rangle \leq G$.

The cyclic group of order $n \in \mathbb{Z}_{>0}$ is $C_n := \langle g \mid g^n = e \rangle$. (Strictly speaking, I'm cheating here: this is a group presentation of C_n which we'll see later.)

Let G be a group and $H \leq G$. A left coset of H in G is $gH := \{gh : h \in H\}$ for some $g \in G$. A right coset of H in G is $Hg := \{hg : h \in H\}$ for some $g \in G$. There is an equivalence relation on G given by $g_1 \sim g_2$ iff $g_1 \in g_2 H$ for all $g_1, g_2 \in G$. An analogous equivalence relation exists on G for right cosets of H in G.

Lagrange's theorem. If G is a group and $H \leq G$, then $|G| = [G : H]|H|$ where $[G : H]$ is the index, meaning the number of left (or equivalently, right) cosets, of H in G.

GROUP ACTIONS AND SYLOW THEOREMS.

The converse of Lagrange's theorem doesn't generally hold, but there exist partial converses, such as:

Cauchy's theorem. If G is a group and p is a prime which divides $|G|$, then there exists $H \leq G$ such that $|H| = p$.

A p-group is a group in which every element is of order p^k. A p-subgroup is a subgroup which is a p-group. A finite group G is a p-group if and only if $|G| = p^k$ for some $k \in \mathbb{Z}$.

A left group action of a group G on a set X is a map $\alpha : G \times X \to X$ such that $\alpha(e, x) = x$ and $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $g, h \in G$ and $x \in X$. A right group action is defined analogously. It's common to write $g.x := \alpha(g, x)$ for a group action α, $g \in G$ and $x \in X$. Given a group action, the orbit of an element $x \in X$ is $G.x := \{g.x : g \in G\} \subseteq X$ and the stabiliser of $x \in X$ is $\mathrm{Stab}_G(x) := \{g : g.x = x\} \subseteq G$. A fixed point of a group action is $x \in X$ such that $G.x = \{x\}$.

If G is a finite group which acts upon a set X, then $\mathrm{Stab}_G(x) < G$ for each $x \in X$. Moreover, there exists an equivalence relation $\sim$ on X such that for all $x, y \in X$, $x \sim y$ iff there exists $g.x = y$. That is, a group action partitions a set into orbits. Lastly, we have:

Orbit-stabiliser theorem. If G is a finite group which acts on a set X, then $|G| = |G.x||\mathrm{Stab}_G(x)|$ for all $x \in X$.

A pair of elements $a, b \in G$ are conjugate if there exists $g \in G$ such that $a = gbg^{-1}$. A group G acts on itself by conjugation via the map $G \times G \to G$, $(g, a) \mapsto gag^{-1}$. The conjugacy class $\mathrm{Cl}(a)$ of an element $a \in G$ is the orbit of a under conjugation; hence, conjugation partitions a group into conjugacy classes so that the class equation $|G| = \sum_i |\mathrm{Cl}(a_i)|$ for a finite set $\{a_1, \ldots, a_k\} \subset G$.

The centraliser $C_G(a)$ of an element $a \in G$ is the stabiliser of a under conjugation, given explicitly as the subset of G with respect to which a is invariant under conjugation, $C_G(a) := \{g : a = gag^{-1}\}$.

Immediately, $C_G(a) < G$. Also, $|G| = |Cl(a)||C_G(a)|$ for each $a \in G$ and thus $|\mathrm{Cl}(a)| = [G : C_G(a)]$ by Lagrange's theorem.

A pair of subsets $T, S \subseteq G$ are conjugate if there exists $g \in G$ such that $T = gSg^{-1}$. That is, conjugation by g induces a one-to-one correspondence between T and S. The centraliser $C_G(S)$ of a subset $S \subseteq G$ is $C_G(S) := \{g \ : \ s = gsg^{-1} \text{ for all } s \in S\} \leq G$. The centre $Z(G)$ of a group G is $Z(G) := C_G(G)$. The normaliser $N_G(S)$ of a subset $S \subseteq G$ is $N_G(S) := \{g \ : \ S = gSg^{-1}\} \leq G$ such that $C_G(S) \subset N_G(S)$.

A subgroup $H \leq G$ is normal if $H = gHg^{-1}$ for all $g \in G$, and is denoted $H \lhd G$. Equivalently, $H \lhd G$ iff $gHg^{-1} \leq H$ iff $G = N_G(H)$ iff left and right cosets of H in G coincide. Given an arbitary subgroup $K \leq G$, the normaliser $N_G(K)$ is equal to the union of all subgroups (*i.e.* the 'largest' subgroup) which contain K as a normal subgroup.

Our group action technology and a pair of lemmas, which are:

1. If p is a prime and G is a finite-p group which acts on a finite set X, then the number of fixed points in X is congruent to $|X|$ modulo p,

2. If G is a finite group and $H \leq G$, then $[G : N_G(H)]$ equals the number of distinct conjugate subgroups of H in G and moreover, for each prime p which divides $|G|$ and Sylow p-subgroup $P \leq G$, we have $n_p = [G : N_G(P)]$,

allow us to prove:

Sylow theorems. Let G be a group and p be a prime such that k is the largest exponent for which p^k divides $|G|$.

1. There exists a subgroup of G whose order is p^k. Such a subgroup is called a Sylow p-subgroup.
2. For each Sylow p-subgroup $P \leq G$ and p-subgroup $H \leq G$, there exists $g \in G$ such that $H \subseteq gPg^{-1}$. Hence, any two Sylow p-subgroups of G are conjugate.
3. The number n_p of Sylow p-subgroups of G divides $|G|/p^k$, is congruent to 1 modulo p and equals $[G : N_G(P)]$ for each Sylow p-subgroup $P \leq G$.

As a consequence of the Sylow theorems, a Sylow p-subgroup of G is normal iff $n_p = 1$. A subgroup $H \leq G$ is simple if H contains only $\{e\}$ and H as normal subgroups; that is, H contains no 'non-trivial' normal subgroups.

ISOMORPHISM THEOREMS.

Given a pair of groups G and H, a group homomorphism is a map $\phi : G \to H$ such that for all $g, h \in G$, $\phi(gh) = \phi(g)\phi(h)$. A group isomorphism is a bijective group homomorphism. If $\phi : G \to H$ is a group homomorphism, then $\phi(e_G) = e_H$ and $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$. The kernel $\ker\phi$ of a group homomorphism $\phi : G \to H$ is $\ker\phi := \{g \ : \ \phi(g) = e_H\} \subseteq G$. A subgroup $N \leq G$ is normal in G iff there exists a group H and a group homomorphism $\phi : G \to H$ such that $\ker\phi = N$.

A factor group G/N is the set of left (or equivalently, right) cosets of a normal subgroup $N \lhd G$ in G equipped with the group operation $(gN) * (hN) = (gh)N$ for each $g, h \in G$. There is a canonical surjective group homomorphism $\mathrm{can} : G \to G/N, g \mapsto gN$.

Universal property of factor groups. Let G be a group and $N \lhd G$. For each group H and group homomorphism $\psi : G \to H$ with $N \subseteq \ker\psi$, there exists a unique group homomorphism $\overline{\psi} : G/N \to H$ such that $\psi = \overline{\psi} \circ \mathrm{can}$.

As a useful corollary, if $\phi : G \to K$ is a surjective group homomorphism and $\psi : G \to H$ is a group homomorphism with $\ker\phi \subseteq \ker\psi$, then there exists a unique group homomorphism $\overline{\psi} : K \to H$ such that $\psi = \overline{\psi} \circ \phi$.

First isomorphism theorem. If G, H are groups and $\phi : G \to H$ is a group homomorphism, then $\ker\phi \lhd G$, $\mathrm{im}\phi \leq H$ and there exists a group isomorphism $\overline{\phi} : G/\ker\phi \to \mathrm{im}\phi$ given by $gN \mapsto \phi(g)$ with $N := \ker\phi$. If ϕ is surjective, then $G/\ker\phi \cong H$.

If G is a group and $N \lhd G$, then we make two observations: first, if $K \leq G/N$, then $\mathrm{can}^{-1}(K) \leq G$ with $N \subseteq \mathrm{can}^{-1}(K)$ and moreover, $\mathrm{can}^{-1}(K) \lhd G$ if and only if $K \lhd G/N$; second, if $N \leq H \leq G$, then $H = \mathrm{can}^{-1}(\mathrm{can}(H))$. Combining these allows us to prove:

Correspondence theorem. Let G be a group, $N \lhd G$ and $\mathrm{can} : G \to G/N$ denote the canonical surjection. There is a bijection between subgroups of G which contain N and subgroups of G/N given by $H \mapsto \mathrm{can}(H)$ which restricts to a bijection between normal subgroups of G which contain N and normal subgroups of G/N.
Furthermore, if $A, B \leq G$, then $A \subseteq B$ iff $\mathrm{can}(A) \subseteq \mathrm{can}(B)$.

which in turn allows us to prove:

Third isomorphism theorem. If G is a group and $N \leq H \leq G$ are such that $N, H \lhd G$, then $(G/N)/(H/N) \cong G/H$.

Given a pair of subsets $X, Y \subseteq G$, define $XY := \{xy \,:\, x \in X, y \in Y\}$.

Second isomorphism theorem. If G is a group, $N \lhd G$ and $H \leq G$, then

1. $HN \leq G$,
2. $N \lhd HN$,
3. $H \cap N \lhd H$,
4. $HN/N \cong H/(H \cap N)$.

GROUP PRESENTATIONS.

The free group on generators $x_1, \ldots, x_n$ is the set of all words in the symbols $x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$ equipped with concatenation, and is denoted $\langle x_1, \ldots, x_n \rangle$.

Given an arbitrary group G, the normal closure $\mathrm{ncl}_G(S)$ of a subset $S \subseteq G$ is the intersection of all normal subgroups of G which contain S; that is, $\mathrm{ncl}_G(S)$ is the 'smallest' normal subgroup of G which contains S. The group generated by $x_1, \ldots, x_n$ subject to 'relations' $r_1, \ldots, r_k \in \langle x_1, \ldots, x_n \rangle$ is

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_k \rangle := \langle x_1, \ldots, x_n \rangle / \mathrm{ncl}_G(\{r_1, \ldots, r_k\}).$$

If a group G is isomorphic to $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_k \rangle$, then the latter is a 'presentation' of G. Also, $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_k \rangle$ is equivalently the free group $\langle x_1, \ldots, x_n \rangle$ subjected to the additional conditions $r_1 = \cdots = r_k = e$ and all logical consequences thereof.

The n-th dihedral group D_n is the group of symmetries of a regular n-sided polygon; formally, D_n is defined for each $n \in \mathbb{Z}_{\geq 3}$ as the group generated by two elements, g and h, where g denotes reflection across a line through a specified, fixed vertex and h denotes rotation by $2\pi/n$ radians. There is a group presentation for D_n given by $D_n \cong \left\langle g, h \mid g^2, h^n, (gh)^2 \right\rangle$ for each $n \geq 3$.

Universal property of free groups. Let G be a group generated by a set $\{s_1, \ldots, s_n\}$ and $F = \langle S_1, \ldots, S_n \rangle$ be the free group generated by the letters $S_1, \ldots, S_n$. There exists a unique surjective group homomorphism $\pi : F \to G$ such that $\pi(S_i) = s_i$ for each $i \in \{1, \ldots, n\}$.

FINITELY-GENERATED ABELIAN GROUPS.

In an abelian group, all subgroups are normal because left and right cosets must coincide; in a finite abelian group, all Sylow p-subgroups are unique. It follows that a finite abelian group is isomorphic to the direct product of its Sylow p-subgroups which implies:

Fundamental theorem of finite abelian groups. 1. A finite abelian group is isomorphic to a direct product of cyclic groups of prime power order uniquely up to reordering.

2. A finite abelian group of order n is isomorphic to a direct product of cyclic groups $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_s}$ where $n_i | n_{i+1}$ for all $1 \leq i \leq s-1$ and $n_1 n_2 \cdots n_s = n$ uniquely up to reordering the factors.

where the latter is implied by:

Chinese remainder theorem. If $m, n \in \mathbb{Z} - \{0\}$ are coprime, then $C_{mn} \cong C_m \times C_n$.

Given a finite abelian group, the first and second decompositions are clearly isomorphic.

The exponent $e(G)$ of any finite group is the least common multiple of the orders of elements in G; that is, the smallest n such that $g^n = e$ for all $g \in G$. Any finite abelian group G contains an element of order $e(G)$ so that if $e(G) = |G|$ then G is cyclic since such an element generates G.

The set of nonzero elements of a field form a group under multiplication in the field; that is, the *group of units* K^* of a field K given a field K. If A is a finite subgroup of K^* for a field K, then A is cyclic; as a consequence, the group of units of any finite field is cyclic.

If R is a ring, then an R-module is an abelian group $(M, +)$ equipped with a map $R \times M \to M$, $(r, m) \mapsto rm$ which is distributive, associative and unital. Given any ring R, the free R-module of rank s is the set of s-tuples R^s equipped with 'pointwise' addition and a map $R \times R^s \to R^s$ given by $r(r_1, \ldots, r_s) \mapsto (rr_1, \ldots, rr_s)$.

An abelian group G is a $\mathbb{Z}$-module equipped with the 'obvious' map $\mathbb{Z} \times G \to G, (n, a) \mapsto na$ where the latter is additive notation for $+$ in G. This observation leads to:

Fundamental theorem of finitely-generated abelian groups. If A is a finitely-generated abelian group, then there exist $k, l \in \mathbb{N}$ and nonzero $r_1, \ldots, r_k \in \mathbb{Z}$ with $r_i | r_{i+1}$ for each $1 \leq i \leq k-1$ such that $A \cong \mathbb{Z}/r_1\mathbb{Z} \times \cdots \times \mathbb{Z}/r_k\mathbb{Z} \times \mathbb{Z}^l$ uniquely up to isomorphism.

whose proof is constructive as follows: given a $\mathbb{Z}$-module A finitely generated by s elements $r_1, \ldots, r_s$, there exists an 'obvious' surjection $\varphi : \mathbb{Z}^s \to A, e_i \mapsto r_i$ for standard basis vectors e_i within $\mathbb{Z}^s$ so that by the first isomorphism theorem for modules $\mathbb{Z}^s / \ker \varphi \cong \mathrm{im}\varphi = A$; however, factor modules of $\mathbb{Z}^s$ by a submodule such as $\ker \varphi$ are invariant up to isomorphism with respect to $\mathbb{Z}$-module automorphisms so as invertible row and column operations on the matrix 'corresponding to' a finitely generated submodule[1] are instances of such automorphisms, then $\mathbb{Z}^s / \ker \varphi$ is invariant under such operations upon the matrix 'corresponding to' $\ker \varphi$; we thereby build a matrix of the form $\mathrm{diag}(r_1, \ldots, r_k, 0, \ldots 0)$ which corresponds to a submodule of the form $K := \mathbb{Z}(r_1, 0, \ldots, 0) + \cdots + \mathbb{Z}(0, \ldots, 0, r_k, 0, \ldots, 0)$ such that $A \cong \mathbb{Z}^s / K$ is exactly as desired where $l := s - k$ and we choose our operations so that $r_i | r_{i+1}\ \forall\ i$.

ALTERNATING GROUPS.

[1] *i.e.* given a submodule $K \subseteq \mathbb{Z}^s$ finitely generated by $x_1, \ldots, x_k$, writing each $x_i = \sum_{j=1}^{s} a_{ij} e_j$ implies an $r \times s$-matrix $(a_{ij}) \in \mathrm{Mat}_{r \times s}(\mathbb{Z})$ associated to K so that $(a_{ij})(e_j) = (x_i)$.