GROUP THEORY

PROBLEM SET 1

Context. Question 1 is a proposition copied from (Sierra, 2020) which I wanted to prove myself; for the same reason, Question 2 is a fact in (Sierra, 2020) which was stated without proof, as an exercise. I attempted in Question 3 to write an interesting, yet vaguely standard exercise which used most of the 'new' concepts that I've learned thus far. Question 4 is loosely based upon a question in an algebra exam which I sat last year. Each question is worth five points.

Question 1. Let G be a finite group.

a. Show that if $|G|$ is prime, then G is cyclic. Is the converse true? Why?

b. Show that if G is a non-trivial p-group, then $Z(G) \neq \{e\}$. Is the converse true? Why?

Question 2. Prove that $D_n \cong \langle a, b \mid a^2, b^n, (ab)^2 \rangle$ for all $n > 2$.

Question 3. Let G be a group with 7514 elements. Show that G is not simple. Deduce therefore that there exist non-trivial distinct normal subgroups $H, K \lhd G$ and maps $\iota_H : G \to G/H$, $\iota_K : G \to G/K$ such that $(G/H)/\iota_H(K) \cong (G/K)/\iota_K(H)$.

Question 4. Let X be a nonempty set of n elements. Denote by $\mathcal{P}(X)$ the power set of X. For which values of n does $\mathcal{P}(X)$ admit a group structure whose identity is $\emptyset$? Does there exist an operation $*$ such that for all n, $(\mathcal{P}(X), *)$ is a group with identity $\emptyset$?