

Single Particle

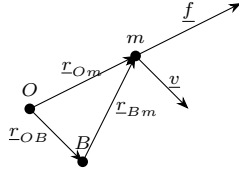


Figure 1: Point mass m under action of force f . Point O is fixed in inertial space, and point B is a general point, not necessarily fixed in inertial space.

$$\begin{aligned} \underline{p} &= m\underline{v} \\ \underline{f} &= \frac{d\underline{p}}{dt} \\ \tau_O &= \underline{r}_{Om} \times \underline{f} \\ \boxed{\underline{h}_O} &= \underline{r}_{Om} \times \underline{p} \end{aligned} \quad \begin{aligned} \tau_O &= \frac{d}{dt} \underline{h}_O \\ \tau_B &= \tau_O - \underline{r}_{OB} \times \underline{f} \\ \underline{h}_B &= \underline{h}_O - \underline{r}_{OB} \times \underline{p} \\ \tau_B &= \frac{d}{dt} \underline{h}_B + \underline{v}_B \times \underline{p} \end{aligned}$$

Kinematics of Rigid Bodies

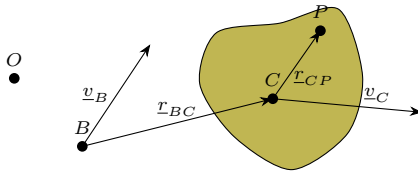


Figure 2: Rigid body. O is inertially fixed in space, and point B is a general point, which can be moving, about which we may take moments. Point C is the body center of mass, and point P is some other point fixed on the body

Note that the vector notation \underline{r}_{BC} means a vector from point B to point C .

$$\begin{aligned} \underline{P} &= m\underline{v}_C \\ \tau_O^{\text{ext}} &= \frac{d}{dt} \underline{H}_O \\ \boxed{\tau_B^{\text{ext}}} &= \frac{d}{dt} \underline{H}_B + \underline{v}_B \times \underline{P} \\ \underline{H}_B &= [I]_B \underline{\omega} \end{aligned} \quad \begin{aligned} \boxed{\underline{F}^{\text{ext}}} &= \frac{d}{dt} \underline{P} \\ \boxed{\underline{H}_B} &= \underline{H}_C + \underline{r}_{BC} \times \underline{P} \\ \underline{H}_B &= \underline{H}_O + \underline{r}_{BO} \times \underline{P} \\ \boxed{\underline{v}_P} &= \underline{v}_C + \underline{\omega} \times \underline{r}_{CP} \end{aligned}$$

For non-holonomic (rolling coin):

$$\tau_B^{\text{ext}} = \frac{d}{dt} \underline{H}_C + \underline{r}_{BC} \times \frac{d}{dt} \underline{P}$$

To find principal axes

$$I_{\text{principal}} - \lambda I = 0$$

Impulse

Take linear and angular momentum principle for a rigid body

$$\underline{F}^{\text{ext}} = \frac{d}{dt} \underline{P}$$

$$\tau_B^{\text{ext}} = \frac{d}{dt} \underline{H}_B + \underline{v}_B \times \underline{P}$$

and separate them and integrate them over a short period of time

$$\begin{aligned} \int_{t=0^-}^{t=0^+} \underline{F}^{\text{ext}} dt &= \int_{\underline{P}(0^-)}^{\underline{P}(0^+)} d\underline{P} \\ \int_{t=0^-}^{t=0^+} \tau_B^{\text{ext}} dt &= \int_{\underline{H}_B(0^-)}^{\underline{H}_B(0^+)} d\underline{H}_B + \int_{t=0^-}^{t=0^+} \underline{v}_B \times \underline{P} dt \\ \Delta \underline{P} &= \int_{t=0^-}^{t=0^+} \underline{F}^{\text{ext}} dt \end{aligned}$$

Remember when integrating from $t = 0^-$ to $t = 0^+$ constant forces like gravity integrate to zero

Work and Energy Principles

KE of rigid body rotating about CM:

$$T = \frac{1}{2} M v_G^2 + \frac{1}{2} (I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2)$$

Finding Center of Mass and Moment of Inertia

P is a general point.

Parallel axis theorem: $I_P = I_C + Mh^2$

$$[I]_P = [I]_C + M \begin{bmatrix} b^2 + c^2 & -ab & -ac \\ -ab & a^2 + c^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{bmatrix}$$

Moments of inertia $I_x = \int_m (y^2 + z^2) dm$

Products of inertia $I_{xy} = \int_V \rho xy dV$

$$I_x = \rho \int_V (y^2 + z^2) dV$$

x CM: $x_{cm} = \frac{\sum_i A_i r_i}{\sum_i A_i}$

y CM: $z_{cm} = \frac{\int_m z dm}{\int_m dm}$

z CM: $z_{cm} = \frac{\int_V z dV}{\int_V dV}$

Areas, Volumes, Centroids, Moments of Inertia

$$A_{\text{sphere}} = 4\pi r^2$$

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3$$

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

$$I_{\text{cylinder}, z} = \frac{1}{2} m r^2$$

Through axis of rot.

Through center $I_{\text{cylinder}, x, y} = \frac{1}{12} m (3r^2 + h^2)$

Rod length L about end: $I_{\text{rod}, \text{end}} = \frac{1}{3} m L^2$

Rod length L about center: $I_{\text{rod}, \text{center}} = \frac{1}{12} m L^2$

Sphere radius r : $I_{\text{sphere}} = \frac{2}{5} m r^2$

Cone $I_{\text{cone}, z} = \frac{3}{10} m r^2$

Cube thru cent $l(x), w(y), h(z)$ $I_{\text{cube}, x} = \frac{1}{12} (w^2 + h^2)$

Axes at tip of cone $I_{\text{cone}, x, y} = \frac{3}{80} m (4r^2 + h^2)$

Axes at base of cone $I_{\text{cone}, x, y} = \frac{3}{20} m r^2 + \frac{1}{10} m h^2$

through center of hoop $I_{\text{hoop}, z} = m r^2$

Centroid of cone up from base $z = \frac{h}{4}$

Lagrange's Method to find EOM

1. Identify number of generalized coordinates and any generalized forces
2. Choose generalized coordinates ξ_1, ξ_2, \dots
3. Find kinetic energy T and potential energy V in terms of these generalized coordinates
4. Assemble Lagrangian

$$\mathcal{L} = T - V$$

5. Express generalized forces in terms of the generalized coordinates

$$\delta W = F_x \delta x$$

$$\delta W = \Xi_j \delta \xi_j$$

- When finding generalized forces which require solving δx in terms of $\delta \xi$, sometimes it is easiest to find velocities, then cancel dt and make dx into $d\xi$ and $d\xi$ into $\delta \xi$.
- When breaking force F into components F_x , don't forget the sign
- Measure springs deflections from static equilibrium and gravity won't appear in equations of motion
- When finding kinetic energy of rigid bodies, place coordinate system at CG and such that it is a set of principal axes, then the moment of inertia is *about the CG* not the physical point of rotation.

6. Evaluate

$$\frac{\partial \mathcal{L}}{\partial \xi_j} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{\xi}_j}$$

7. Use the formula

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_j} \right) - \frac{\partial \mathcal{L}}{\partial \xi_j} = \Xi_j$$

That gives us the equations of motion

Stability Analysis of Discrete Systems

1. Use Lagrange's method to get EOM
2. Identify steady motions
 - If ξ_j does not show up explicitly in the Lagrangian \mathcal{L} , it is *ignorable*, or *cyclic*. Set

$$\dot{\xi}_{\text{ignorable}} = \text{constant}$$

- If ξ_j does show up in the Lagrangian \mathcal{L} , it is *non-ignorable*. Set

$$\xi_{\text{non-ignorable}} = \xi_s = \text{constant}$$

3. Linearize the equations of motion. The form is

$$[M]\ddot{x} + [K]x = 0$$

From here there are two options

4. Solve for the natural frequencies and mode shapes

- Let $\dot{x} = sx$

- Solve $([M]s^2 + [K])x = 0 \Rightarrow \det([K] - \omega_i^2[M]) = 0$

- This is an eigenvalue problem where the eigenvalues are the natural frequencies, and the eigenvectors are the mode shapes

Alternatively This way requires $[M]$ and $[K]$

5. Guess as many modes $\{a\}_i$ as possible
6. Use orthogonality to verify guessed modes, and find new modes

$$\begin{aligned} \{a\}_i^\top [M] \{a\}_j &= 0 \\ \{a\}_i^\top [K] \{a\}_j &= 0 \end{aligned}$$

The orthogonality condition comes from left multiplying $([K] - \omega_i^2[M])\{a\}_i = 0$ for two cases with i and j by two modes which are orthogonal, $\{a\}_j^\top$ and $\{a\}_i^\top$.

7. Use **Rayleigh quotient** to find ω_i

$$\omega_i^2 = \frac{\{a\}_i^\top [K] \{a\}_i}{\{a\}_i^\top [M] \{a\}_i}$$

which comes from $([K] - \omega_i^2[M])\{a\}_i = 0$ and left multiplying by $\{a\}_i^\top$

For the system $[M]\{\ddot{x}\} + [K]\{x\} = \{F\} \sin \omega t$, after we find the modes, we can put the modes into a matrix $[\Phi]$ and use this matrix to come up with a new system with vector $\{u\}$, where the mass and spring matrix are diagonal. Let $\{x\} = [\Phi]\{u\}$. Plugging this in we get

$$\underbrace{[\Phi]^\top [M] [\Phi]}_{[M]_D} \{\ddot{u}\} + \underbrace{[\Phi]^\top [K] [\Phi]}_{[K]_D} \{u\} = [\Phi]^\top \{F\} \sin \omega t$$

Derivations for Continuous Systems

Wave Equation for a String

String with mass/length ρ under tension T . So mass of a little piece is $dm = \rho dx$. String has length s with angle α on left side, $\alpha + \frac{\partial \alpha}{\partial x} dx$ on the right side, and the angle is small, so the string is approximately length dx .

Conservation of momentum in x-direction: the string does not move in the x -direction

$$T(x + dx) \cos \left(\alpha + \frac{\partial \alpha}{\partial x} dx \right) - T(x) \cos(\alpha) = 0$$

Expanding $\cos \left(\alpha + \frac{\partial \alpha}{\partial x} dx \right)$

$$\cos \left(\alpha + \frac{\partial \alpha}{\partial x} dx \right) = \cos(\alpha) \cos \left(\frac{\partial \alpha}{\partial x} dx \right) - \sin(\alpha) \sin \left(\frac{\partial \alpha}{\partial x} dx \right)$$

Substituting this in

$$T(x + dx) \left[\cos(\alpha) \cos \left(\frac{\partial \alpha}{\partial x} dx \right) - \sin(\alpha) \sin \left(\frac{\partial \alpha}{\partial x} dx \right) \right] - T(x) \cos(\alpha) = 0$$

Divide both sides by dx and take the limit as $dx \rightarrow 0$

$$T(x) = T = \text{constant}$$

Conservation of momentum in y-direction:

$$T(x + dx) \sin \left(\alpha + \frac{\partial \alpha}{\partial x} dx \right) - T(x) \sin(\alpha) = \frac{d^2 y}{dt^2} \rho dx$$

Expanding $\sin \left(\alpha + \frac{\partial \alpha}{\partial x} dx \right)$

$$\sin \left(\alpha + \frac{\partial \alpha}{\partial x} dx \right) = \sin(\alpha) \cos \left(\frac{\partial \alpha}{\partial x} dx \right) + \cos(\alpha) \sin \left(\frac{\partial \alpha}{\partial x} dx \right)$$

Plugging in, and using the fact that $T(x) = T$ we have

$$T \left[\sin(\alpha) \cos \left(\frac{\partial \alpha}{\partial x} dx \right) + \cos(\alpha) \sin \left(\frac{\partial \alpha}{\partial x} dx \right) \right] - T \sin(\alpha) = \frac{d^2 y}{dt^2} \rho dx$$

Using small angle approximations we get

$$T \left[\sin(\alpha) + \frac{\partial \alpha}{\partial x} dx \right] - T \sin(\alpha) = \frac{d^2 y}{dt^2} \rho dx$$

Simplifying, we get

$$T \frac{\partial \alpha}{\partial x} = \frac{d^2 y}{dt^2} \rho$$

Using small angle assumption again where $\alpha \approx \tan(\alpha) = \frac{dy}{dx}$ we have

$$T \frac{\partial^2 y}{\partial x^2} = \frac{d^2 y}{dt^2} \rho$$

Can add forcing as

$$\frac{d^2 y}{dt^2} \rho = T \frac{\partial^2 y}{\partial x^2} + f(x, t)$$

Euler-Bernoulli Beam Equation

Beam with **mass/length** ρA , with **internal shear force** Q , **bending moment** M_b , **height** $y(x, t)$. Square piece of block with shear force Q down on left side, $Q + \frac{\partial Q}{\partial x} dx$ pointing up on the right side, and moments M_b going up on the left side and $M_b + \frac{\partial M_b}{\partial x} dx$ going up on the right side. The **constitutive law** for a bending beam relates moment to curvature as

$$M_b = EI \frac{\partial^2 y}{\partial x^2}$$

Conservation of angular momentum about right side gives

$$M_b + \frac{\partial M_b}{\partial x} - M_b + Q dx = 0$$

gives

$$Q = -\frac{\partial M_b}{\partial x}$$

Substituting the constitutive law

$$Q = -EI \frac{\partial^3 y}{\partial x^3}$$

Conservation of linear momentum in y-direction gives

$$(\rho A dx) \frac{\partial^2 y}{\partial t^2} = \frac{\partial Q}{\partial x} dx$$

Evaluating $\frac{\partial Q}{\partial x}$ using the expression for Q derived using conservation of angular momentum

$$\frac{\partial Q}{\partial x} = -EI \frac{\partial^4 y}{\partial x^4}$$

Substituting in

$$\rho A \frac{\partial^2 y}{\partial t^2} = -EI \frac{\partial^4 y}{\partial x^4}$$

Self adjoint means solution is separable.

Longitudinal Displacement (Stretching) of a Rod

$$\rho A \frac{\partial^2 \xi}{\partial t^2} = EA \frac{\partial^2 \xi}{\partial x^2}$$

Axial Displacement (Twisting) of a Shaft

$$\rho J \frac{\partial^2 \phi}{\partial t^2} = GJ \frac{\partial^2 \phi}{\partial x^2}$$

Solving Continuous Systems

Solving String Problems with Forcing

To solve the forced response, always solve the unforced problem first.

1. Write down governing equation

$$\rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} + f(x, t)$$

2. Propose a solution of the form

$$y(x, t) = a(x) \cos(\omega t)$$

where the time varying harmonic function matches that of the forcing function (including frequency)

3. Plug this solution in, and obtain a simplified ODE in $a(x)$

$$-\rho \omega^2 a(x) \cos(\omega t) = T \frac{d^2 a}{dx^2} \cos(\omega t) + f(x, t)$$

Take for example the forcing function to be $f(x, t) = F_0 \cos(\omega t)$ then

$$-\rho \omega^2 a(x) = T \frac{d^2 a}{dx^2} + F_0$$

rearranging

$$\frac{d^2 a}{dx^2} + \frac{\rho \omega^2}{T} a(x) = -F_0$$

using $\lambda^2 = \frac{\rho \omega^2}{T}$

$$\frac{d^2 a}{dx^2} + \lambda^2 a(x) = -F_0$$

4. Find the homogeneous solution $a_h(x)$ and particular solution $a_p(x)$. Propose the homogeneous solution

$$a(x) = Ae^{Bx}$$

which has second derivative

$$\frac{d^2 a}{dx^2} = B^2 A e^{Bx}$$

plugging in we get

$$B = \pm \lambda i$$

So the solutions are

$$a_1(x) = A_1 e^{\lambda i x}$$

$$a_2(x) = A_2 e^{\lambda - i x}$$

giving

$$a_h(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$$

- Solution to forced equation should be constant
- Form the total solution by adding the homogeneous and particular solutions, and apply BCs

Answering the question

- Determine the steady-state vibration just means find $y(x, t)$
- Identifying resonances is to find values of ω where the solution blows up

Solving String Problems with a Mass on them

The governing equation for this is the same as a regular string, but the solution is not valid across mass. Will need to use two solutions, one valid on each side of the mass.

- Write down governing equation

$$\rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

- Consider solving this equation between the left end at $x = -L$ and the mass, and then between the mass and the right end at $x = L$. The general form of the solution is

$$y_L(x, t) = a_L(x) \sin(\omega t)$$

$$y_R(x, t) = a_R(x) \sin(\omega t)$$

Leads to

$$y_L(x, t) = C_L \sin(\lambda x + \phi_L) \sin(\omega t)$$

$$y_R(x, t) = C_R \sin(\lambda x + \phi_R) \sin(\omega t)$$

where $\lambda^2 = \frac{\rho \omega^2}{T}$. Should get $\phi_L = \lambda L$ and $\phi_R = -\lambda L$

- Apply 4 boundary conditions: each end of the string, and the matching conditions at the mass

$$y_L(x_m, t) = y_R(x_m, t)$$

and the following, which comes from **linear momentum of mass in y-direction**

$$M \frac{\partial^2 y}{\partial t^2} \Big|_{x_m} = T \left(\frac{\partial y_R}{\partial x} \Big|_{x_m} - \frac{\partial y_L}{\partial x} \Big|_{x_m} \right)$$

Where either $y_L(x, t)$ or $y_R(x, t)$ can be used to evaluate the second derivative. Use $y_R(x, t)$.

- Arranging these two conditions in matrix form

$$\begin{bmatrix} -\sin(\lambda(x_m - L)) & \sin(\lambda(x_m + L)) \\ -M\lambda^2 \sin(\lambda(x_m - L)) - T\lambda \cos(\lambda(x_m - L)) & -T\lambda \cos(\lambda(x_m - L)) \end{bmatrix} \begin{bmatrix} C_R \\ C_L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Special case is when the mass is in the middle of the string at x_m . This reduces the above by taking the determinant to

$$\sin(\lambda L) [M\lambda^2 \sin(\lambda L) - 2T\lambda \cos(\lambda L)] = 0$$

This equation can be solved to find the frequencies ω_n from each λ_n .

- Mode shapes are those when the mass is stationary

Solving Beam Problems

- Write down governing equation

$$\rho A \frac{\partial^2 y}{\partial t^2} = -EI \frac{\partial^4 y}{\partial x^4}$$

- Propose following solution for beam problems. Can show such a separable solution works.

$$y(x, t) = a(x) \sin(\omega t)$$

The second and fourth derivatives of general solution to beam equation are

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 a(x) \sin(\omega t)$$

$$\frac{d^4 y}{dx^4} = \frac{d^4 a(x)}{dx^4} \sin(\omega t)$$

- Plugging the proposed solution into the governing equation we get

$$\frac{d^4 a}{dx^4} - \lambda^4 a(x) = 0 \quad \text{where} \quad \lambda^4 = \frac{\rho A \omega^2}{EI}$$

- The general solution to this ODE is

$$a(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \sinh \lambda x + C_4 \cosh \lambda x$$

- Apply boundary conditions to solve. This may reduce the number of constants, e.g. $C_2 = C_4 = 0$. Then put the remaining equations into matrix form, and solve for the constants, either by row operations or by taking the determinant

Damping Problems

W is energy loss/cycle, V is peak potential energy (of whole system), η is the loss factor. Some formulas are:

$$W = \int_0^{2\pi/\omega} f_d dx$$

where, for a linear dashpot

$$\text{Linear dashpot:} \quad f_d = c\dot{x}$$

where x is the compression of the damper. This gives

$$W = \int_0^{2\pi/\omega} c \frac{dx}{dt} dx$$

$$W = \int_0^{2\pi/\omega} c \left(\frac{dx}{dt} \right)^2 dt$$

The loss factor is calculated as

$$\eta = \frac{W}{2\pi V}$$

To solve damping problems

- Do lagrangian to get EOM and find the natural frequencies and mode shapes *assuming there is no damping*
- Propose a solution of the form

$$\underline{x}(t) = \underline{a} \sin(\omega t)$$

- Use modes to break this into components
- Differentiate this solution to get $\frac{d\underline{x}}{dt}$ and plug into the integral to evaluate W .

Rigid Symmetric Body EOM

$$\underline{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$$

$$\underline{H}_c = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$$

$$\underline{\tau}^{\text{ext}} = \frac{d}{dt} \underline{H}_c$$

$$\begin{aligned} \frac{d}{dt} \underline{H}_c &= I_1 (\dot{\omega}_1 \hat{e}_1 - \omega_1 \omega_2 \hat{e}_3 + \omega_1 \omega_3 \hat{e}_2) \\ &\quad + I_2 (\dot{\omega}_2 \hat{e}_2 + \omega_2 \omega_1 \hat{e}_3 - \omega_2 \omega_3 \hat{e}_1) \\ &\quad + I_3 (\dot{\omega}_3 \hat{e}_3 - \omega_3 \omega_1 \hat{e}_2 + \omega_3 \omega_2 \hat{e}_1) \end{aligned}$$

Set this equal to $\underline{\tau}^{\text{ext}}$ and group components together to get EOM. Can simplify with symmetry, e.g. $I_1 = I_2$.

$$\tau_1 = I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2)$$

$$\tau_2 = I_2 \dot{\omega}_2 + \omega_1 \omega_3 (I_1 - I_3)$$

$$\tau_3 = I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1)$$

Euler Angles

Start with coordinate system C_{XYZ} . Rotate ϕ about Z and get C_{abc} axes, then rotate θ about a and get C_{xyz} , and finally rotate ψ about z to get C_{123} axes, which are the body fixed axes.

$$\begin{aligned} \underline{\omega} &= \dot{\phi} \hat{e}_Z + \dot{\theta} \hat{e}_x + \dot{\psi} \hat{e}_3 \\ &= \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3 \end{aligned}$$

where

$$\omega_1 = \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi$$

$$\omega_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi$$

$$\omega_3 = \dot{\psi} + \dot{\phi} \cos \theta$$

Torque-Free Precession

When there is **no torque** acting on the system, angular momentum principle $\frac{d}{dt} \underline{H}_C = 0$ tells us that \underline{H}_C is **constant**, and since it is a vector this means its magnitude and direction are constant. So, we can **choose the coordinate system C_{XYZ} such that the Z axis is aligned with \underline{H}_C** .

$$\begin{aligned} \underline{H}_C &= H_C \hat{e}_Z \\ &= H_C (\sin \theta \hat{e}_y + \cos \theta \hat{e}_z) \\ &= H_C (\sin \theta \sin \psi \hat{e}_1 + \sin \theta \cos \psi \hat{e}_2 + \cos \theta \hat{e}_3) \end{aligned}$$

Compare this expression for \underline{H}_C to the earlier one

$$\underline{H}_C = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$$

Components have to match exactly. This gives

$$\begin{aligned} \dot{\phi} &= H_C \left(\frac{\sin^2 \psi}{I_1} + \frac{\cos^2 \psi}{I_2} \right) \\ \dot{\theta} &= H_C \left(\frac{1}{I_1} - \frac{1}{I_2} \right) \sin \theta \cos \psi \sin \psi \\ \dot{\psi} &= H_C \cos \theta \left(\frac{1}{I_3} - \frac{\sin^2 \psi}{I_1} - \frac{\cos^2 \psi}{I_2} \right) \end{aligned}$$

If the body has **symmetry**, say **about 3 axis**, then $I_1 = I_2 = I$ and we can see from these expressions that $\dot{\theta} = \theta_s = \text{constant}$, that $\dot{\phi} = \frac{H_C}{I} = \Omega = \text{constant}$, and $\dot{\psi} = [(I - I_3)/I_3] \Omega \cos \theta_s = \text{constant}$. **This is torque-free precession.**

Spinning Top with Euler Angles

- Use LaGrange to get EOM, neglecting \underline{v}_C
- Using Euler angles for $\underline{\omega}$ in Lagrange $\underline{\omega} = \dot{\phi} \hat{e}_Z + \dot{\theta} \hat{e}_x + \dot{\psi} \hat{e}_3$ Describe in terms of C_{xyz} with $\hat{e}_Z = \sin \theta \hat{e}_z + \cos \theta \hat{e}_y$ $\hat{e}_3 = \hat{e}_z$ $\underline{\omega} = \dot{\phi} \sin \theta \hat{e}_z + \dot{\phi} \cos \theta \hat{e}_y + \dot{\theta} \hat{e}_x + \dot{\psi} \hat{e}_z$
- Do Lagrange, identify from $\delta\psi$ equation $I_3 \omega_z = \text{constant}$ $\omega_z \approx \dot{\psi}$

Torque-Free Motion of a Rigid Body

$$\begin{aligned}
\omega &= \phi \hat{e}_Z + \dot{\psi} \hat{e}_z \\
&= \Omega \sin \theta_s \hat{e}_y + (\dot{\psi} + \Omega \cos \theta_s) \hat{e}_z \\
\underline{H}_C &= I \omega_y \hat{e}_y + I_3 \omega_z \hat{e}_z \\
&= I \Omega \sin \theta_s \hat{e}_y + I_3 (\dot{\psi} + \Omega \cos \theta_s) \hat{e}_z \\
&= I \Omega (\sin \theta_s \hat{e}_y + \cos \theta_s \hat{e}_z) \\
&= I \Omega \hat{e}_Z
\end{aligned}$$

If we evaluate $\frac{d}{dt} \underline{H}_C$ we can solve for the rate of change of the angular rates as

$$\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1} \omega_2 \omega_3 \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2} \omega_3 \omega_1 \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3} \omega_1 \omega_2
\end{aligned}$$

We have with no torque that $\underline{H}_C = I_3(\dot{\psi} + \dot{\phi})$ is constant and $\omega = (\dot{\psi} + \dot{\phi})\hat{e}_z$. Choose \underline{H}_C to be parallel to \hat{e}_Z . Use Euler equations to examine stability of steady rotation. $\omega = \omega_3 \hat{e}_3$. So $\dot{\omega}_1$ and $\dot{\omega}_2$ are basically constant, giving

$$\ddot{\omega}_1 + \frac{(I_1 - I_3)(I_2 - I_3)}{I_1 I_2} \omega_3^2 \omega_1 = 0$$

and this is stable when $(I_1 - I_3)(I_2 - I_3) > 0$ and unstable when $(I_1 - I_3)(I_2 - I_3) < 0$. So stable when I_3 is either a maximum or minimum moment of inertia.

General Math Stuff

Taylor series

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

Using this for sine and cosine, small angles

$$\begin{aligned}
\cos(x) &= 1 - \frac{1}{2}x^2 \\
\sin(x) &= x - \frac{x^3}{6}
\end{aligned}$$

Identities

$$\begin{aligned}
\sin(u \pm v) &= \sin u \cos v \pm \cos u \sin v \\
\cos(u \pm v) &= \cos u \cos v \mp \sin u \sin v \\
e^{ix} &= \cos x + i \sin x \\
\sinh(x) &= \frac{1}{2}(e^x - e^{-x}) \\
\cosh(x) &= \frac{1}{2}(e^x + e^{-x}) \\
\sin(x) &= \frac{1}{2i}(e^{ix} - e^{-ix}) \\
\cos(x) &= \frac{1}{2}(e^{ix} + e^{-ix}) \\
\sinh(x) + \cosh(x) &= e^x \\
\frac{d}{dx} \sinh(x) &= \cosh(x) \\
\frac{d}{dx} \cosh(x) &= \sinh(x) \\
\sinh(0) &= 0 \quad \cosh(0) = 1
\end{aligned}$$

Integrals

$$\begin{aligned}
\int \sin^2(ax) dx &= \frac{x}{2} - \frac{\sin(2ax)}{4a} \\
\int \cos^2(ax) dx &= \frac{x}{2} + \frac{\sin(2ax)}{4a}
\end{aligned}$$

Writing sum of sin and cos as sin with a phase shift

$$A_1 \sin(\omega x) + A_2 \cos(\omega x) = \sqrt{A_1^2 + A_2^2} \left(\underbrace{\frac{A_1}{\sqrt{A_1^2 + A_2^2}}}_{\cos \phi} \sin(\omega x) + \underbrace{\frac{A_2}{\sqrt{A_1^2 + A_2^2}}}_{\sin \phi} \cos(\omega x) \right)$$

Can check that $\sin^2 \phi + \cos^2 \phi = 1$. Then use identity $\sin(\omega x + \phi) = \sin(\omega x) \cos \phi + \cos(\omega x) \sin \phi$ giving

$$\begin{aligned}
A_1 \sin(\omega x) + A_2 \cos(\omega x) &= A_3 \sin(\omega x + \phi) \\
\frac{\sin \phi}{\cos \phi} = \frac{A_2}{A_1} &\Rightarrow \phi = \tan^{-1} \frac{A_2}{A_1} \quad \text{and} \quad A_3 = \sqrt{A_1^2 + A_2^2}
\end{aligned}$$

Wave Equation on String

This page gives an outline of the general procedure to **derive the equations of motion**, **propose a general solution**, and **solve for constants using boundary and initial conditions** (here we assume the boundary conditions are both ends fixed, and zero initial conditions) in order to get the **mode shapes** and **natural frequencies**.

Physical assumptions: *homogenous string* $\rho A = \text{constant}$, *the string is perfectly elastic* (no resistance to bending), *the tension is way more than gravity*, and *string only vibrates perfectly up and down*.

Non dispersive waves: anything that obeys the wave equation, e.g. a string, the speed of wave propagation is constant and independent of frequency. All energy travels the same speed independent of frequency. $\frac{T}{\rho A}$ is the wave speed. $\sqrt{\frac{T}{\rho A}}$ is phase velocity. In beams non dispersive waves, the high frequency waves travel ahead of the lower frequency waves.

1. Derive governing equation

- Momentum in x -direction gives $T(x)$ is constant
- Do momentum in the y -direction
- Use small angles: $\sin(\alpha + \frac{\partial \alpha}{\partial x} dx) = \alpha + \frac{\partial \alpha}{\partial x} dx$ and $\tan(\alpha) = \alpha$

The governing equation is
$$T \frac{\partial^2 y}{\partial x^2} = \rho A \frac{\partial^2 y}{\partial t^2}$$

2. Propose a general separable solution

$$y(x, t) = a(x)f(t)$$

- Rearrange the governing equation as $C^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ where $C^2 = \frac{T}{\rho A}$ and propose $f(t) = Ae^{i\omega_n t}$ giving $y(x, t) = a(x)Ae^{i\omega_n t}$ and plug in
- The governing equation becomes $C^2 \frac{\partial^2 a}{\partial x^2} + \omega_n^2 a(x) = 0$ *note: C is always positive*
- Propose $a(x) = Be^{i\lambda x}$ and get $\omega_n = C\lambda$
- The total solution is then $y(x, t) = Be^{i\frac{\omega_n}{C}x} Ae^{i\omega_n t}$ which can be decomposed into sine and cosine as $y(x, t) = (B_1 \sin(\lambda x) + B_2 \cos(\lambda x))(A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t))$

3. Apply boundary and initial conditions to get the constants

- Apply boundary conditions $y(x=0, t) = y(x=L, t) = 0$ gives $B_2 = 0$ and $B_1 \sin(\frac{\omega_n}{C}L) = 0$ so $\frac{\omega_n}{C}L = n\pi$ where $n = 1, 2, 3, \dots$. So $\omega_n = \frac{Cn\pi}{L}$. The solution becomes $y(x, t) = B_1 \sin(\frac{\omega_n}{C}x)(A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t))$
- Apply initial conditions $y(x, t=0) = 0$ gives $A_2 = 0$ reducing solution to $y(x, t) = B_1 \sin(\frac{\omega_n}{C}x)A_1 \sin(\omega_n t)$ or by combining the constants
$$y(x, t) = C_n \sin(\frac{\omega_n}{C}x) \sin(\omega_n t)$$

4. Now we have the governing equation, now we see if it is **self-adjoint** if it satisfies the following conditions

- $\int u \rho A v dx = \int v \rho A u dx$
- $\int v \left(-T \frac{\partial^2}{\partial x^2} \right) u dx = \int u \left(-T \frac{\partial^2}{\partial x^2} \right) v dx$

The first condition is satisfied automatically, since u and v (in our case $a(x)$ and $f(t)$) commute. We show that the second condition holds by doing integration by parts twice.

$$\int_0^L \underbrace{a_i}_{u} \underbrace{\left(-T \frac{\partial^2}{\partial x^2} \right) a_j}_{dv} dx = \underbrace{a_i}_{u} \underbrace{\left(-T \frac{\partial}{\partial x} (a_j) \right)}_v \bigg|_0^L + \int_0^L \underbrace{T \frac{\partial}{\partial x} (a_j)}_v \underbrace{\frac{da_i}{dx}}_{du} dx$$

one more integration by parts

$$\int_0^L \underbrace{\frac{da_i}{dx}}_u \underbrace{T \frac{\partial}{\partial x} (a_j)}_{dv} dx = \underbrace{\frac{da_i}{dx}}_u \underbrace{(-T a_j)}_v \bigg|_0^L + \int_0^L \underbrace{T a_j}_v \underbrace{\frac{d^2 a_i}{dx^2}}_{du} dx$$

gives

$$\int_0^L a_i \left(-T \frac{\partial^2}{\partial x^2} \right) a_j dx = a_i \left(-T \frac{\partial}{\partial x} (a_j) \right) \bigg|_0^L - \left(\frac{da_i}{dx} (-T a_j) \right) \bigg|_0^L + \int_0^L a_j \left(-T \frac{\partial^2}{\partial x^2} (a_i) \right) dx$$

and since we evaluate the first two terms on the right hand side at $x=0$ and $x=L$, the boundary conditions dictate that $a_i = a_j = 0$ here, thus proving the system is **self-adjoint**. *Self-adjointness depends on the boundary conditions.*

5. Now we use the self adjoint property to show that the modes are **orthogonal**, where a_j and a_i are orthogonal functions if they satisfy

$$\int_0^L a_j a_i dx = 0$$

- Start with the **governing equation** for the **spatial function** for two different modes a_i and a_j , where $i \neq j$.

$$C^2 \frac{\partial^2 a}{\partial x^2} + \omega_n^2 a(x) = 0$$

- Since the governing equation equals zero, we can post-multiply each of these expressions by the *other mode*, sum them, and it is still zero.

$$\left(C^2 \frac{\partial^2 a_i}{\partial x^2} + \omega_n^2 a_i(x) \right) a_j + \left(C^2 \frac{\partial^2 a_j}{\partial x^2} + \omega_n^2 a_j(x) \right) a_i = 0$$

- Expand this expression and **integrate from 0 to L**. Using the self-adjoint property which we just showed, we get

$$\frac{1}{C^2} \left(\omega_i^2 - \omega_j^2 \right) \int_0^L a_i a_j dx = 0$$

- Since the natural frequencies corresponding to each of these modes is different, the integral must be zero, satisfying the definition and showing the modes are orthogonal.
6. To find the i -th **modal mass** and i -th **modal stiffness**, write down the governing *spatial* differential equation $T \frac{\partial^2 a}{\partial x^2} + \rho A \omega_n^2 a(x) = 0$ for a mode a_i and multiply both sides by a_i and dx , rearrange and integrate from 0 to L.

$$a_i \left(T \frac{\partial^2 a_i}{\partial x^2} \right) dx + \rho A \omega_n^2 a_i^2(x) dx = 0$$

giving the i -th modal mass and i -th modal stiffness as

$$\underbrace{\omega_n^2 \rho A \int_0^L a_i^2(x) dx}_{m_i \delta_{ij}} = \underbrace{\int_0^L a_i \left(-T \frac{\partial^2 a_i}{\partial x^2} \right) dx}_{k_i \delta_{ij}}$$

And we can divide and solve for ω_R , which are regular modes? which gives the Rayleigh quotient.

$$\omega_R^2 = \frac{\int_0^L a_i \left(-T \frac{\partial^2 a_i}{\partial x^2} \right) dx}{\rho A \int_0^L a_i^2(x) dx} = \frac{k_i \delta_{ij}}{m_i \delta_{ij}}$$

Modal Decomposition

When the string problem is forced, the governing equation is

$$\rho A \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = f_0(x) \sin \Omega t$$

We propose the same separable solution as before, where we make explicit that there are an infinite number of solutions, indexed by i , and the total solution is the sum

$$y(x, t) = \sum_i a_i(x) f_i(t)$$

Plugging this into the governing equation we get

$$\sum_i \left(\rho A a_i(x) \frac{\partial^2 f_i}{\partial t^2} - f_i(t) T \frac{\partial^2 a_i}{\partial x^2} \right) = f_0(x) \sin \Omega t$$

Left multiply by a_i and integrate across the beam from 0 to L

$$\sum_i \left(\int_0^L \rho A a_i^2(x) \frac{\partial^2 f_i}{\partial t^2} dx - \int_0^L a_i(x) T \frac{\partial^2 a_i}{\partial x^2} f_i(t) dx \right) = \int_0^L a_i f_0(x) \sin \Omega t dx$$

giving

$$\sum_i \left(\frac{\partial^2 f_i}{\partial t^2} \underbrace{\int_0^L \rho A a_i^2(x) dx}_{m_i \delta_{ij}} + f_i(t) \underbrace{\int_0^L -a_i(x) T \frac{\partial^2 a_i}{\partial x^2} dx}_{k_i \delta_{ij}} \right) = \int_0^L a_i f_0(x) \sin \Omega t dx$$

giving

$$\frac{\partial^2 f_i}{\partial t^2} m_i \delta_{ij} + k_i \delta_{ij} f_i(t) = \sin \Omega t \int_0^L a_i f_0(x) dx$$

Solving for $f_i(t)$

$$k_i f_i(t) = \sin \Omega t \int_0^L a_i f_0(x) dx - \frac{\partial^2 f_i}{\partial t^2} m_i$$

$$f_i(t) = \frac{\int_0^L a_i(x) f_0(x) dx}{k_i} \sin \Omega t - \frac{\partial^2 f_i}{\partial t^2} \frac{m_i}{k_i}$$

Boundary Conditions

Rollers at End with Spring

$$T \frac{dy}{dx} = ky$$

Euler-Bernoulli Beam Equation

Proposing Separable Solution

Given the governing PDE for a bending beam

$$\rho A \frac{\partial^2 y}{\partial t^2} = -EI \frac{\partial^4 y}{\partial x^4}$$

we propose a solution of the form

$$y(x, t) = a(x)y(t)$$

Evaluating the necessary derivatives given this solution form we get

$$\frac{\partial^2 y}{\partial t^2} = a(x) \frac{d^2 f}{dt^2} \quad \text{and} \quad \frac{\partial^4 y}{\partial x^4} = \frac{d^4 a}{dx^4} f(t)$$

substituting in

$$\rho A a(x) \frac{d^2 f}{dt^2} = -EI \frac{d^4 a}{dx^4} f(t)$$

which can be separated as

$$\rho A \frac{1}{f(t)} \frac{d^2 f}{dt^2} = -EI \frac{1}{a(x)} \frac{d^4 a}{dx^4} = \text{constant}$$

And so now we can solve each ODE separately now. This leads to a solution of the form

$$y(x, t) = a(x) \sin \omega(t)$$

Plugging this back into the governing equation, we reduce the equation to an ODE and then only have to find $a(x)$

$$\frac{d^4 a}{dx^4} - \lambda^4 a(x) = 0 \quad \text{where} \quad \lambda^4 = \frac{\rho A \omega^2}{EI}$$

Self-Adjointness

Now we have the governing equation for beams, now we see if it is **self-adjoint** if it satisfies the following conditions

$$\begin{aligned} \text{i)} \quad & \int u \rho A v dx = \int v \rho A u dx \\ \text{ii)} \quad & EI \int u \frac{d^4 v}{dx^4} dx = EI \int v \frac{d^4 u}{dx^4} dx \end{aligned}$$

where the integrals are evaluated from one end of the beam to the other, usually 0 to L . The first condition is trivial, and we can show the second condition holds by applying integration by parts. Additionally, from doing this integration, we also find the following relationship

$$\int_0^L v \frac{d^4 u}{dx^4} dx = \int_0^L \frac{d^2 u}{dx^2} \frac{d^2 v}{dx^2} dx$$

Orthogonality

To show **orthogonality** of the modes, start with the spatial governing equation for two modes a_i and a_j with $i \neq j$

$$\begin{aligned} EI \frac{d^4 a_i}{dx^4} - \rho A \omega^2 a_i(x) &= 0 \\ EI \frac{d^4 a_j}{dx^4} - \rho A \omega^2 a_j(x) &= 0 \end{aligned}$$

Left multiply the first equation by a_j and the second by a_i . Integrate across the beam (from 0 to L) and subtract

$$EI \int_0^L a_j \frac{d^4 a_i}{dx^4} dx - \rho A \omega^2 \int_0^L a_j a_i dx - EI \int_0^L a_i \frac{d^4 a_j}{dx^4} dx + \rho A \omega^2 \int_0^L a_i a_j dx = 0$$

Use self-adjointness to cancel out terms, giving

$$\rho A \omega^2 \int_0^L a_i a_j dx = \rho A \omega^2 \int_0^L a_j a_i dx$$

Thus showing the modes are orthogonal.

Finding i -th Modal Mass and Stiffness

To find the i -th modal mass and stiffness, again use the spatial governing ODE for mode a_i

$$EI \frac{d^4 a_i}{dx^4} - \rho A \omega^2 a_i(x) = 0$$

left multiply by a_i , and integrate across the beam from 0 to L

$$EI \int_0^L a_i \frac{d^4 a_i}{dx^4} dx - \rho A \omega^2 \int_0^L a_i^2(x) dx = 0$$

use the additional property from self-adjointness to replace the fourth derivative as the product of two second derivatives

$$EI \underbrace{\int_0^L \left(\frac{d^2 a_i}{dx^2} \right)^2 dx}_{k_i \delta_{ij}} - \omega^2 \underbrace{\rho A \int_0^L a_i^2(x) dx}_{m_i \delta_{ij}} = 0$$

where k_i and m_i are the i -th modal stiffness and mass, respectively. From this we can find ω as

$$\omega^2 = \frac{k_i \delta_{ij}}{m_i \delta_{ij}}$$

Boundary Conditions

Free End No moment, no shear.

Fixed or Clamped End Displacement and slope are zero.

Pinned End No displacement, no moment. Remember $M_b = EI \frac{\partial^2 y}{\partial x^2}$ so for pinned end this means the second derivative must be zero.

Point Mass at End No moment, external shear due to mass boundary condition, from conservation of linear momentum in y -direction.

$$-Q = m \frac{\partial^2 y}{\partial t^2}$$

Applied moment From $M_b = EI \frac{\partial^2 y}{\partial x^2}$, the boundary condition due to M_{applied} is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{EI} M_{\text{applied}}$$

Forced Beam Response

To solve the forced response, always solve the unforced problem first.

$$y(x, t) = \sum_i a_i(x) f_i(t)$$

$$\sum_i \left(EI \frac{d^4 a_i}{dx^4} f_i + \rho A a_i \frac{d^2 f_i}{dt^2} \right) = f_0 \sin(\Omega t)$$

$$\int_0^L a_j \left[\sum_i \left(EI \frac{d^4 a_i}{dx^4} f_i + \rho A a_i \frac{d^2 f_i}{dt^2} \right) = f_0 \sin(\Omega t) \right] dx$$

using orthogonality of modes (ith modal mass and stiffness?)

$$k_j f_j + m_j \frac{d^2 f_j}{dt^2} = \int_0^L a_j f_0 \sin(\Omega t) dx$$

The solution is

$$f_j(t) = \frac{\int_0^L a_j(x) dx}{m_j(\omega_j^2 - \Omega^2)} f_0 \sin(\Omega t) + C_j \sin(\omega_j t + B_j)$$

Find C_j and B_j by applying initial conditions.