Sequential Loop Closure Based Adaptive Output Feedback

Daniel P. Wiese, *Student Member, IEEE,* Anuradha M. Annaswamy, *Fellow, IEEE,* Jonathan A. Muse, *Fellow, IEEE,* Michael A. Bolender, and Eugene Lavretksy, *Fellow, IEEE*

***Abstract*—This paper presents a new, systematic method of synthesizing an output feedback adaptive controller for a class of uncertain, non-square multi-input/multi-output systems. The control design process consists of first designing an inner-loop controller for a reduced order plant model to enforce command tracking of selected inner-loop variables, with an adaptive element used to accommodate parametric uncertainties in the plant. Once this inner-loop control design is complete, an outer-loop is then designed which prescribes the inner-loop commands to enforce command tracking of selected outer-loop variables.**

The main challenge that needs to be addressed when designing the inner-loop controller is the determination of a corresponding square and strictly positive real transfer function. This is accomplished by appropriate selection of two gain matrices that allow the realization of such a transfer function, thereby allowing a globally stable adaptive output feedback law to be generated. The outer-loop controller is designed around the plant with existing adaptive inner-loop controller such that global stability of the closed-loop system is guaranteed. The design of the outer-loop uses components of a closed-loop reference model in a judicious manner which enables a modular approach, without requiring any re-design of the inner-loop controller. In addition, this architecture facilitates the use of an additional state-limiter to enforce desired limits on the state variables.

A numerical example based on a scramjet powered, generic hypersonic vehicle model is presented, demonstrating the efficacy of the proposed control design.

***Index Terms*—Adaptive control.**

I. Introduction

DIVIDING a control system into hierarchical structure with inner and outer-loops has many benefits, both in aerospace and other control applications. Significant knowledge exists around how to design many of the inner-loop controllers that provide stability and robustness to the closed-loop system, and the limiting of inner-loop commands is facilitated by this hierarchical structure in which these commands are explicitly calculated by the outer-loop.

Obtaining accurate values of the system parameters can be challenging, thus making the process of designing a stabilizing controller more challenging as well. This has led to an increased use of adaptive techniques to solve control problems, with great success [1], [2]. However, many such adaptive controllers have previously focused only on the problem of inner-loop control [3]–[9], enabling the design of lower order controllers to provide stability in the presence of uncertainties. In these and many other cases outer-loops were typically not designed. In aerospace applications, the design of the guidance laws around vehicles with adaptive inner loops is typically accomplished using ad-hoc methods, with stability and performance of the closed-loop system only verified through simulation.

An alternative to the multi-loop design approach described above is to use a higher order model to represent the vehicle dynamics, and design guidance and control laws simultaneously. The result is a more complex controller with a greater number of integrators and adaptive parameters. In Ref. [10] an adaptive controller was designed for a linear system which represents the longitudinal dynamics of a hypersonic vehicle. The controller used feedback from all five state variables to each of the three inputs, with additional feed forward terms, resulting in 24 adaptive parameters.

Other approaches have used sequential loop closure on higher order nonlinear models. In Ref. [11] the non-minimum phase dynamics typically associated with the transfer function from an aircraft's elevator input to the altitude were overcome by the addition of a canard, which would be practically impossible to implement on a hypersonic vehicle due to the effects that aerodynamic heating would have on such a forward control surface. In Ref. [12] a canard is no longer used, and the resulting unstable zero dynamics associated with regulating flight

D.P. Wiese and A.M. Annaswamy are with the Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: dpwiese@mit.edu; aanna@mit.edu).

J.A. Muse and M.A. Bolender are with the Air Force Research Laboratory (Air Vehicles Directorate), Control Design and Analysis Branch, 2210 Eighth Street, WPAFB, OH 45433.

E. Lavretsky is with the Phantom Works Department, Boeing Company, Huntington Beach, CA 92647 USA (e-mail: eugene.lavretsky@boeing.com)

path angle using the elevator input are overcome using a non-adaptive dynamic inversion controller with a low gain outer loop and saturation functions. Reference [13] uses an adaptive dynamic inversion inner-loop control law, with a parameter identification algorithm which requires the state derivative be measurable. The outer-loop is closed using sequential loop closure, but no stability proof is provided to ensure stability of the overall closed-loop system.

The main contribution of this paper is the design of an outer-loop controller which prescribes commands to an inner-loop adaptive controller such as those described in Refs. [2], [4]–[7]. Specifically, this outer-loop controller does not require the inner-loop controller be redesigned, guarantees stability of the closed-loop system, and incorporates a state limiter, allowing the inner and outer-loop command signals to be modified as necessary to limit the evolution of the state trajectories to within a certain prescribed region within the state space.

In the following section the control problem is formulated with a general structure applicable to a wide class of systems.

II. PRELIMINARIES

The following well-known lemma gives necessary and sufficient conditions to ensure that the system $(A, B, C, 0)$ is strictly positive real (SPR).

Lemma 1 (Kalman-Yakubovic) *Given the strictly proper transfer matrix $G(s)$ with stabilizable and detectable realization $(A, B, C, 0)$, where $A \in \mathbb{R}^{n\times n}$ is asymptotically stable, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$, then $G(s)$ is SPR if and only if there exists a $P = P^\top > 0$ such that*

$$A^\top P + PA < 0 \tag{1}$$

$$PB = C^\top \tag{2}$$

PROOF The proof can be found in Ref. [14]. □

Corollary 1 *There exists a matrix $P = P^\top > 0$ that satisfies (2) if and only if*

$$CB = (CB)^\top > 0 \tag{3}$$

Furthermore, when (3) holds, all solutions of (2) are given by

$$P = C^\top (CB)^{-\top} C + B^\perp X B^{\perp\top} \tag{4}$$

where $X = X^\top > 0$ is arbitrary and $B^\perp \in \mathbb{R}^{n\times(n-m)}$.

PROOF The proof can be found in Ref. [15]. □

Lemma 2 (Matrix Elimination) *Given*

$$G + C^\top L^\top P + PLC < 0 \tag{5}$$

where $G \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, and $P = P^\top \in \mathbb{R}^{n\times n}$ is full rank, an $L \in \mathbb{R}^{n\times p}$ exists which satisfies (5) if and only if the following inequality holds

$$C^{\top\perp\top} G C^{\top\perp} < 0$$

where $C^{\top\perp} \in \mathbb{R}^{n\times(n-p)}$ satisfies $CC^{\top\perp} = 0$.

PROOF The proof can be found in Ref. [16]. □

III. CONTROL PROBLEM FORMULATION

Consider the following uncertain linear time-invariant system

$$\begin{aligned}
\begin{bmatrix}\dot{x}_p(t)\\ \dot{x}_g(t)\end{bmatrix} &= \begin{bmatrix}A_p + B_p\Psi_p^\top & B_{gd}\\ B_{gp} & A_g\end{bmatrix}\begin{bmatrix}x_p(t)\\ x_g(t)\end{bmatrix} + \begin{bmatrix}B_p\Lambda\\ 0\end{bmatrix}u(t)\\
\begin{bmatrix}y_p(t)\\ y_g(t)\end{bmatrix} &= \begin{bmatrix}C_p & 0\\ 0 & C_g\end{bmatrix}\begin{bmatrix}x_p(t)\\ x_g(t)\end{bmatrix}\\
\begin{bmatrix}z_p(t)\\ z_g(t)\end{bmatrix} &= \begin{bmatrix}C_{pz} + D_{pz}\Psi_p^\top & 0\\ 0 & C_{gz}\end{bmatrix}\begin{bmatrix}x_p(t)\\ x_g(t)\end{bmatrix} + \begin{bmatrix}D_{pz}\Lambda\\ 0\end{bmatrix}u(t)
\end{aligned} \tag{6}$$

where $A_p \in \mathbb{R}^{n_p\times n_p}$, $A_g \in \mathbb{R}^{n_g\times n_g}$, $B_p \in \mathbb{R}^{n_p\times m}$, $B_{gp} \in \mathbb{R}^{n_g\times n_p}$, $B_{gd} \in \mathbb{R}^{n_p\times n_g}$, $C_p \in \mathbb{R}^{\ell_p\times n_p}$, $C_g \in \mathbb{R}^{\ell_g\times n_g}$, $C_{pz} \in \mathbb{R}^{n_{ep}\times n_p}$, $C_{gz} \in \mathbb{R}^{n_{eg}\times n_g}$, and $D_{pz} \in \mathbb{R}^{n_{ep}\times m}$ are *known* matrices. The nonsingular matrix $\Lambda \in \mathbb{R}^{m\times m}$, and $\Psi_p \in \mathbb{R}^{n_p\times m}$, which represents constant matched uncertainty weights that enter the system through the columns of B_p, are *unknown*. The *measured outputs* are given by $y_p(t)$ and $y_g(t)$, and the *regulated outputs* $z_p(t)$ and $z_g(t)$ correspond to particular outputs for which tracking of command signals $z_{p,\text{cmd}}(t)$ and $z'_{g,\text{cmd}}(t)$, respectively, is desired. The number of regulated outputs cannot exceed the number of inputs, that is $n_{ep} \le m$. Ultimately, the *control goal* is to design $u(t)$ in (6) so that $z_g(t)$ tracks $z'_{g,\text{cmd}}(t)$.

IV. INNER LOOP CONTROL DESIGN

For systems represented by (6), x_p represents the inner-loop, and x_g the outer-loop state variables. An inner-loop controller is designed by neglecting the outer-loop variables by assuming $B_{gd} = 0$ giving

$$\begin{aligned}
\dot{x}_p(t) &= A_p x_p(t) + B_p\big(\Lambda u(t) + \Psi_p^\top x_p(t)\big) + B_{gd} x_g(t)\\
y_p(t) &= C_p x_p(t)\\
z_p(t) &= C_{pz} x_p(t) + D_{pz}\big(\Lambda u(t) + \Psi_p^\top x_p(t)\big)
\end{aligned} \tag{7}$$

To satisfy the control goal, the problem is restated as: first design the input $u(t)$ in (7) so that $z_p(t)$ tracks $z_{p,\text{cmd}}(t)$ with bounded errors in the presence of the uncertainties Λ and Ψ_p. Then re-introduce the outer-loop dynamics and then design $z_{p,\text{cmd}}(t)$ so that $z_g(t)$ tracks $z'_{g,\text{cmd}}(t)$. We make the following assumptions about the form of the system represented in (7).

Assumption 1

A) (A_p, B_p) is controllable.
B) (A_p, C_p) is observable.
C) B_p, C_p, and C_pB_p are full rank.
D) Any finite transmission zeros of $(A_p, B_p, C_p, 0)$ are strictly stable, and the rank of the following matrix is full

$$\text{rank}\left(\begin{bmatrix} A_p & B_p \\ C_{pz} & D_{pz} \end{bmatrix}\right) = n_p + n_{ep}$$

E)(a) Λ is nonsingular and diagonal with entries of known sign
(b) $\|\Psi_p\|_2 < \Psi_{\max} < \infty$, where $\Psi_{\max}$ is known

In order to facilitate command tracking, integral action is introduced, and for this purpose an additional state $x_e(t)$ is defined as

$$\dot{x}_e(t) = z_{p,\text{cmd}}(t) - z_p(t) \tag{8}$$

This error state is appended to the plant in (7) leading to the following integral-augmented open-loop dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\big(\Lambda u(t) + \Psi^\top x(t)\big) + B_{\text{cmd}} z_{p,\text{cmd}}(t) \\ y(t) &= Cx(t) \\ z_p(t) &= C_z x(t) + D_{pz}\big(\Lambda u(t) + \Psi^\top x(t)\big) \end{aligned} \tag{9}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $B_{\text{cmd}} \in \mathbb{R}^{n\times n_e}$, and $C \in \mathbb{R}^{p\times n}$ are the known matrices given by

$$A = \begin{bmatrix} A_p & 0_{n_p\times n_e} \\ -C_{pz} & 0_{n_e\times n_e} \end{bmatrix} \quad B = \begin{bmatrix} B_p \\ -D_{pz} \end{bmatrix} \quad B_{\text{cmd}} = \begin{bmatrix} 0_{n_p\times m} \\ I_{n_e\times n_e} \end{bmatrix}$$
$$C = \begin{bmatrix} C_p & 0_{\ell\times n_e} \\ 0_{n_e\times n_p} & I_{n_e\times n_e} \end{bmatrix} \quad C_z = [C_{pz} \quad 0]$$

the state is given by $x(t) = \begin{bmatrix} x_p^\top(t) & x_e^\top(t) \end{bmatrix}^\top$, and the unknown matrix Ψ is defined as

$$\Psi = [\ \Psi_p{}^\top \quad 0_{m\times n_e}\ ]^\top$$

Note that $p = \ell + n_e$. It can be shown that Assumption 1 regarding the plant in (7) is equivalent to Assumption 1′ regarding the plant in (9), which is stated below.

Assumption 1′

A) (A, B) is controllable.
B) (A, C) is observable.
C) B, C, and CB are full rank.
D) Any finite transmission zeros of $(A, B, C, 0)$ are strictly stable.
E)(a) Λ is nonsingular and diagonal with entries of known sign
(b) $\|\Psi\|_2 < \Psi_{\max} < \infty$, where $\Psi_{\max}$ is known
F) $(A, B, C, 0)$ is tall: $p > m$.

Remark 1 The system in (7) satisfying Assumption 1A-D when augmented with the integral error state as shown in (9) also satisfies Assumption 1′A-D. In other words, under Assumption 1A-D, integral error augmentation does not destroy controllability, observability, or the rank conditions. Nor does it add any transmission zeros [17].

Remark 2 Assumptions 1′A and 1′B are standard. Assumption 1′C implies that inputs and outputs are not redundant, as well as a MIMO equivalent of relative degree one. Assumption 1′D is a standard requirement for output feedback adaptive control. Assumption 1′E implies that there is no control reversal and that the uncertainty is bounded. This bound need not be tight, and in practice can be easily selected. Assumption 1′F can be considered without loss of generality as the case of wide systems $p < m$ holds by duality. The case of square systems has been given in Ref. [15].

A. Inner-Loop Controller

The underlying problem here is to design a control input $u(t)$ in (9) so that the closed-loop system has bounded solutions and $z_p(t)$ tends to $z_{p,\text{cmd}}(t)$ with bounded errors in the presence of the uncertainties Λ and Ψ. As the ultimate goal is to develop an adaptive controller which in turn requires a reference model, a control design where the reference model has components of an observer as well, is proposed. This inner-loop controller includes a Luenberger observer together with LQR feedback control gains. The resulting reference model is referred to as a closed-loop reference model (CRM) and is given by

$$\begin{aligned} \dot{x}_m(t) &= Ax_m(t) + Bu_{\text{bl}}(t) + B_{\text{cmd}} r(t) + L\big(y_m(t) - y(t)\big) \\ y_m(t) &= Cx_m(t) \\ z_{pm}(t) &= C_z x_m(t) + D_{pz} u_{\text{bl}}(t) \end{aligned} \tag{10}$$

Propose the following baseline control law, used to construct the reference model in (10)

$$u_{\text{bl}}(t) = K_x^\top x_m(t) \tag{11}$$

where K_x is chosen such that $A_m = A + BK_x^\top$ is Hurwitz. The reference model in (10) becomes

$$\begin{aligned} \dot{x}_m(t) &= A_m x_m(t) + B_{\text{cmd}} r(t) + L\big(y_m(t) - y(t)\big) \\ y_m(t) &= Cx_m(t) \\ z_{pm}(t) &= (C_z + D_{pz}K_x^\top)x_m(t) \end{aligned} \tag{12}$$

With the reference model constructed using the nominal system, that is (9) with $\Lambda = I$ and $\Psi = 0$, which contains integral action, guarantees that $z_{pm}(t)$ will track $z_{p,\text{cmd}}(t)$ with bounded errors. To accommodate the uncertainty in (7), the nominal controller in (11) is augmented with an adaptive element as

$$u(t) = \big(K_x + \Theta(t)\big)^\top x_m(t) \tag{13}$$

where $\Theta(t)$ is to be determined by a suitable update law. Given a system satisfying Assumption 1′ and the proposed control architecture, the reference tracking control problem is reduced to selecting the CRM gain L in (12) and update law for $\Theta(t)$ in (13).

B. Error Dynamics and Update Law

The state tracking error and parameter error, respectively, are given by $e_x(t) = x(t) - x_m(t)$ and $\widetilde{\Theta}(t) = \Theta(t) - \Theta^*$, where $\Theta^* = (\Lambda^{-1} - I)K_x^\top - \Psi^\top$. The underlying error model can be described as

$$\begin{aligned}\dot{e}_x(t) &= (A + LC + B\Psi^\top)e_x(t) + B\Lambda\widetilde{\Theta}^\top x_m(t) \\ &\quad + B_{\text{cmd}}\big(z_{p,\text{cmd}}(t) - r(t)\big) \\ e_y(t) &= Ce_x(t)\end{aligned} \tag{14}$$

where $e_y(t)$ is the measured output error. Furthermore, select the reference model input $r(t)$ in (14) as

$$r(t) = z_{p,\text{cmd}}(t) \tag{15}$$

Determining a stable adaptive law for an error model as in (14) relies on properties of an underlying transfer function that is SPR [18]. However, the definition of SPR is restricted to square transfer functions. As such, for these properties to be applicable to the error model in (14), a suitable static postcompensator $S_1 \in \mathbb{R}^{m\times p}$ has to be chosen such that

$$S_1C(sI - A - LC - B\Psi^\top)^{-1}B \in \mathbb{R}_p^{m\times m}(s)$$

where $\mathbb{R}_p(s)$ denotes the ring of *proper* rational transfer functions with coefficients in $\mathbb{R}$. It is therefore necessary to introduce a synthetic output error $e_s(t)$ as

$$e_s(t) = S_1Ce_x(t) \tag{16}$$

Using the synthetic output error in (16) in place of the output error, the underlying error model in (14) is modified as

$$\begin{aligned}\dot{e}_x(t) &= \big(A + LC + B\Psi^\top\big)e_x(t) + B\Lambda\widetilde{\Theta}^\top(t)x_m(t) \\ e_s(t) &= S_1Ce_x(t)\end{aligned} \tag{17}$$

Thus, the design of an output feedback adaptive controller is reduced to selecting matrices $S_1 \in \mathbb{R}^{m\times p}$ and $L \in \mathbb{R}^{n\times p}$ such that the error dynamics in (17) are SPR.

C. Selection of S_1 and L

The process for selecting S_1 and L in (17) is provided in Refs. [6], [7]. S_1 is solved analytically, and L is found by solving an LMI, which is guaranteed to be feasible by selection of a matrix P_x. The conditions to ensure $(A + LC + B\Psi^\top, B, S_1C)$ is SPR are given by

$$(A + LC + B\Psi^\top)^\top P_x + P_x(A + LC + B\Psi^\top) < 0 \tag{18}$$

$$P_xB = (S_1C)^\top \tag{19}$$

The matrix P_x in (18) is given by

$$P_x = (S_1C)^\top(S_1CB)^{-\top}S_1C + N^\top XN \tag{20}$$

where the annihilator matrix N satisfies $NB = 0$. Unlike Refs. [6], [7] which selected X in (20) as block diagonal, a more general structure of X is the following.

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^\top & X_{22}\end{bmatrix} \tag{21}$$

The same procedure is followed to determine L, with X_{12} selected such that $X_{12}^\top N_1 B_{\text{cmd}}$ is full rank, where N_1 is given in [6], [7], and X_{11} selected such that $X_{11} > X_{12}X_{22}^{-1}X_{12}^\top$. With S_1 and L selected according to [6], [7], the error dynamics in (17) are made SPR, allowing the following update law be used

$$\dot{\widetilde{\Theta}}(t) = -\Gamma x_m(t)\big(S_1e_y(t)\big)^\top \text{sgn}(\Lambda) \tag{22}$$

Global stability is proved using the following Lyapunov function.

$$V\big(e_x(t), \widetilde{\Theta}(t)\big) = e_x^\top(t)P_xe_x(t) + \text{tr}\big(|\Lambda|\widetilde{\Theta}^\top(t)\Gamma^{-1}\widetilde{\Theta}(t)\big) \tag{23}$$

The system given by the plant in (9), reference model in (12), reference input in (15), control law in (13) and update law in (22) tracks the inner-loop command $z_{p,\text{cmd}}(t)$ with bounded errors.

V. Outer Loop Control Architecture

This section presents an outer-loop control design for uncertain systems represented in (6) which already have an adaptive inner-loop controller designed as described in Section IV. The outer-loop controller presented in this section is designed around the system with closed adaptive inner loop, uses fixed-gains, and guarantees stability of the closed-loop system. The outer-loop uses components of a closed-loop reference model, and generates the appropriate commands for the inner loop $z_{p,\text{cmd}}(t)$ such that the outer-loop regulated output $z_g(t)$ tracks the desired outer-loop command $z'_{g,\text{cmd}}(t)$ with bounded errors. While certain features are added to the inner-loop controller, this outer-loop design does not require any changes to any of the existing inner-loop control gains. This architecture was first presented in Ref. [19] for the case of state feedback. The outer-loop dynamics from (6) are given by

$$\begin{aligned}\dot{x}_g(t) &= A_gx_g(t) + B_{gp}x_p(t) \\ y_g(t) &= C_gx_g(t) \\ z_g(t) &= C_{gz}x_g(t)\end{aligned} \tag{24}$$

where C_g is partitioned as

$$C_g = \begin{bmatrix} C_{gy} \\ C_{gz}\end{bmatrix} \tag{25}$$

A. Outer-Loop Control Architecture

In this section the outer-loop control architecture is presented, and conditions on the selection of the feedback gains to guarantee global stability of the closed-loop system is given. In designing the outer-loop controller, the Assumption that B_{gd} in (7) is zero is relaxed. The inner-loop dynamics in (9) become

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\big(\Lambda u(t) + \Psi^\top x(t)\big)\\ &\quad + B_{\text{cmd}} z_{p,\text{cmd}}(t) + B_d x_g(t)\\ y(t) &= Cx(t)\end{aligned} \tag{26}$$

where $B_d \in \mathbb{R}^{n\times n_g}$ is given by

$$B_d = \begin{bmatrix} B_{gd} \\ 0_{n_{ep}\times n_g}\end{bmatrix}$$

To accommodate the B_d term in (26), the inner-loop reference model in (12) is modified as

$$\begin{aligned}\dot{x}_m(t) &= A_m x_m(t) + B_{\text{cmd}} r(t) - Le_y(t) + B_d x_{gm}(t)\\ y_m(t) &= Cx_m(t)\end{aligned} \tag{27}$$

This modifies the inner-loop error dynamics in (14) as

$$\begin{aligned}\dot{e}_x(t) &= (A + LC + B\Psi^\top)e_x(t) + B\Lambda\widetilde{\Theta}^\top(t)x_m(t)\\ &\quad + B_{\text{cmd}}\big(z_{p,\text{cmd}}(t) - r(t)\big) + B_d e_g(t)\\ e_y(t) &= Ce_x(t)\end{aligned} \tag{28}$$

The reference signals $r(t)$ and $z_{p,\text{cmd}}(t)$ in (28) must be generated so that $z_g(t)$ in (24) tracks $z_{g,\text{cmd}}(t)$. For this additional reference model components are used.

B. Reference Model Design

1) Outer-Loop Reference Model: An additional outer-loop reference model is introduced in addition to the inner-loop reference model in (27) as

$$\begin{aligned}\dot{x}_{gm}(t) &= A_g x_{gm}(t) + B_g x_m(t) - L_y e_y(t) - L_g e_{gy}(t)\\ y_{gm}(t) &= C_g x_{gm}(t)\\ z_{gm}(t) &= C_{gz} x_{gm}(t)\end{aligned} \tag{29}$$

where $L_y \in \mathbb{R}^{n_g\times p}$, and $L_g \in \mathbb{R}^{n_g\times p_g}$. The outer-loop tracking error is given by $e_g(t) = x_g(t) - x_{gm}(t)$ and the measured outer-loop error by $e_{gy}(t) = y_g(t) - y_{gm}(t)$. The goal is to design an outer-loop controller such that $\lim_{t\to\infty} e_g(t) = 0$, which will thus enforce the outer-loop tracking as desired. The outer-loop error dynamics are given by

$$\begin{aligned}\dot{e}_g(t) &= \big(A_g + L_g C_g\big)e_g(t) + \big(B_g + L_y C\big)e_x(t)\\ e_{gy}(t) &= C_g e_g(t)\end{aligned} \tag{30}$$

2) Forward-loop Reference Model: Combining the inner-loop reference model in (27) and the outer-loop reference model in (29), the combined reference model is obtained as

$$\begin{aligned}\begin{bmatrix}\dot{x}_m(t)\\ \dot{x}_{gm}(t)\end{bmatrix} &= \begin{bmatrix}A_m & B_d\\ B_g & A_g\end{bmatrix}\begin{bmatrix}x_m(t)\\ x_{gm}(t)\end{bmatrix} + \begin{bmatrix}B_{\text{cmd}}\\ 0\end{bmatrix} r(t)\\ &\quad + \begin{bmatrix}L\\ L_y\end{bmatrix}\big(y_m(t) - y(t)\big) + \begin{bmatrix}0\\ L_g\end{bmatrix}\big(y_{gm}(t) - y_g(t)\big)\\ z_{gm}(t) &= \begin{bmatrix}0 & C_{gz}\end{bmatrix}\begin{bmatrix}x_m(t)\\ x_{gm}(t)\end{bmatrix}\end{aligned} \tag{31}$$

The forward-loop reference model, which generates the reference model input command $r(t)$ from the outer-loop command signal $z'_{g,\text{cmd}}(t)$ and stabilizes (31), is now designed. Choose

$$\begin{aligned}\dot{x}_{fm}(t) &= A_{fm}x_{fm}(t) + B_{f1}z_{g,\text{cmd}}(t)\\ &\quad + B_{f2}x_{gm}(t) + B_{f3}x_m(t)\\ r_{\text{cmd}}(t) &= C_{fm}x_{fm}(t) + D_{f1}z_{g,\text{cmd}}(t)\\ &\quad + D_{f2}x_{gm}(t) + D_{f3}x_m(t)\end{aligned} \tag{32}$$

where the matrices $A_{fm} \in \mathbb{R}^{n_f\times n_f}$, $B_{f1} \in \mathbb{R}^{n_f\times n_{eg}}$, $B_{f2} \in \mathbb{R}^{n_f\times n_g}$, $B_{f3} \in \mathbb{R}^{n_f\times n}$, $C_{fm} \in \mathbb{R}^{n_{ep}\times n_f}$, $D_{f1} \in \mathbb{R}^{n_{ep}\times n_{eg}}$, $D_{f2} \in \mathbb{R}^{n_{ep}\times n_g}$, and $D_{f3} \in \mathbb{R}^{n_{ep}\times n}$ are selected so the closed loop system given by combining (32) and (31) provides steady-state command tracking of $z_{g,\text{cmd}}(t)$ by $z_{gm}(t)$ when the errors $e_y(t)$ and $e_{gy}(t)$ are zero. Furthermore, set the outer-loop command $z_{g,\text{cmd}}(t)$ in (32) equal to the desired outer-loop command $z'_{g,\text{cmd}}(t)$ as

$$z_{g,\text{cmd}}(t) = z'_{g,\text{cmd}}(t) \tag{33}$$

Set $r(t)$ in (31) using the output from the forward-loop reference model component in (32) as

$$r(t) = r_{\text{cmd}}(t) \tag{34}$$

Substituting the forward-loop controller (32) into (31) gives the following

$$\begin{aligned}\dot{\bar{x}}_m(t) &= \bar{A}\bar{x}_m(t) + \bar{B}r(t) - \bar{L}_y e_y(t) - \bar{L}_g e_{gy}(t)\\ &\quad + \bar{B}_m z_{g,\text{cmd}}(t)\\ r_{\text{cmd}}(t) &= \bar{C}_m\bar{x}_m(t) + D_{f1}z_{g,\text{cmd}}(t)\end{aligned} \tag{35}$$

where the entire reference model state $\bar{x}_m(t) \in \mathbb{R}^{n+n_g+n_f}$ is given by

$$\bar{x}_m(t) = \begin{bmatrix}x_m^\top(t) & x_{gm}^\top(t) & x_{fm}^\top(t)\end{bmatrix}^\top$$

and where $\bar{A} \in \mathbb{R}^{n+n_g+n_f\times n+n_g+n_f}$, $\bar{B} \in \mathbb{R}^{n+n_g+n_f\times n_{ep}}$, $\bar{L}_y \in \mathbb{R}^{n+n_g+n_f\times p}$, $\bar{L}_g \in \mathbb{R}^{n+n_g+n_f\times p_g}$, $\bar{B}_m \in \mathbb{R}^{n+n_g+n_f\times n_{eg}}$, and $\bar{C}_m \in \mathbb{R}^{n_{ep}\times n+n_g+n_f}$ are given by

$$\bar{A} = \begin{bmatrix}A_m & B_d & 0\\ B_g & A_g & 0\\ B_{f3} & B_{f2} & A_{fm}\end{bmatrix} \quad \bar{B} = \begin{bmatrix}B_{\text{cmd}}\\ 0\\ 0\end{bmatrix} \quad \bar{L}_y = \begin{bmatrix}L\\ L_y\\ 0\end{bmatrix}$$

$$\bar{L}_g = \begin{bmatrix}0\\ L_g\\ 0\end{bmatrix} \quad \bar{B}_m = \begin{bmatrix}0\\ 0\\ B_{f1}\end{bmatrix} \quad \bar{C}_m = \begin{bmatrix}D_{f3}^\top\\ D_{f2}^\top\\ C_{fm}^\top\end{bmatrix}^\top$$

Setting the inner-loop reference model command $r(t)$ as in (34) and simplifying (35) gives

$$\begin{aligned}\dot{\bar{x}}_m(t) &= \bar{A}_m\bar{x}_m(t) + \bar{B}_{\text{cmd}}z_{g,\text{cmd}}(t) - \bar{L}_y e_y(t) - \bar{L}_g e_{gy}(t)\\ r_{\text{cmd}}(t) &= \bar{C}_m\bar{x}_m(t) + D_{f1}z_{g,\text{cmd}}(t)\end{aligned} \tag{36}$$

where $\bar{A}_m \in \mathbb{R}^{n+n_g+n_f \times n+n_g+n_f}$ and $\bar{B}_{\text{cmd}} \in \mathbb{R}^{n+n_g+n_f \times n_{eg}}$ are given by

$$\bar{A}_m = \begin{bmatrix} A_m + B_{\text{cmd}} D_{f3} & B_d + B_{\text{cmd}} D_{f2} & B_{\text{cmd}} C_{fm} \\ B_g & A_g & 0 \\ B_{f3} & B_{f2} & A_{fm} \end{bmatrix}$$

$$\bar{B}_{\text{cmd}} = \begin{bmatrix} B_{\text{cmd}} D_{f1} \\ 0 \\ B_{f1} \end{bmatrix}$$

with $\bar{A}_m = \bar{A} + \bar{B}\bar{C}_m$. Appropriate selection of (32) ensures that $\bar{A}_m$ in (36) is Hurwitz.

Remark 3 In addition to (32) selected such that $\bar{A}_m$ in (36) is Hurwitz, it can also be selected such that with with output $z_{gm}(t)$, (36) is a type-1 system with respect to the command $z_{g,\text{cmd}}(t)$.

Combining the integral augmented, uncertain inner-loop dynamics in (26) with the outer-loop guidance dynamics (24) and reference model (36), the following system is obtained

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\big(\Lambda u(t) + \Psi^\top x(t)\big) + B_{\text{cmd}} z_{p,\text{cmd}}(t) \\ &\quad + B_d x_g(t) \\ y(t) &= Cx(t) \\ \dot{x}_g(t) &= A_g x_g(t) + B_g x(t) \\ y_g(t) &= C_g x_g(t) \\ \dot{\bar{x}}_m &= \bar{A}_m \bar{x}_m(t) + \bar{B}_{\text{cmd}} z_{g,\text{cmd}}(t) - \bar{L}_y e_y(t) - \bar{L}_g e_{gy}(t) \\ r_{\text{cmd}} &= \bar{C}_m \bar{x}_m(t) + D_{f1} z_{g,\text{cmd}}(t) \end{aligned} \tag{37}$$

where only the specification of $z_{p,\text{cmd}}(t)$ remains to completely specify the control architecture.

C. Generating the Inner-Loop Command

The command $z_{p,\text{cmd}}(t)$ in (15) with the inner-loop reference model input $r(t) = r_{\text{cmd}}(t)$ as in (34) is modified with an outer-loop error feedback term as

$$z_{p,\text{cmd}}(t) = r(t) + e_{gs}(t) \tag{38}$$

$$e_{gs}(t) = S_g e_{gy}(t) \tag{39}$$

where $S_g \in \mathbb{R}^{n_{ep} \times p_g}$. Combining the inner-loop error dynamics in (28) with $z_{p,\text{cmd}}(t)$ given by (38) and (39) and the outer-loop error dynamics in (30) the following inner and outer-loop error dynamics are obtained

$$\begin{aligned} \dot{e}_x(t) =& \big(A + LC + B\Psi^\top\big)e_x(t) + B\Lambda\widetilde{\Theta}^\top(t)x_m(t) \\ &+ \big(B_{\text{cmd}} S_g C_g + B_d\big)e_g(t) \\ e_y(t) =& Ce_x(t) \\ \dot{e}_g(t) =& \big(A_g + L_g C_g\big)e_g(t) + \big(B_g + L_y C\big)e_x(t) \end{aligned} \tag{40}$$

The CRM feedback gains L_y, L_g, and S_g in (40) need to be selected to guarantee global stability of the closed-loop system. Looking at these error dynamics provides a cue as to how stability may be achieved, with L_g being used to stabilize the outer-loop error dynamics, and S_g and L_y used to cancel the error cross-coupling terms. This reduces the error dynamics to standard adaptive error dynamics on the inner-loop, and stable outer-loop error dynamics. The specific requirements for stability of the error dynamics in (40) and the resulting conditions leading to the solutions for S_g, L_g and L_y are provided in the following subsection.

D. Conditions for Stability

The complete control architecture is specified by the plant and reference model (37), inner-loop command specified by (38) and (39), control input (13), and update law in (22). This control architecture can be represented by the following figure

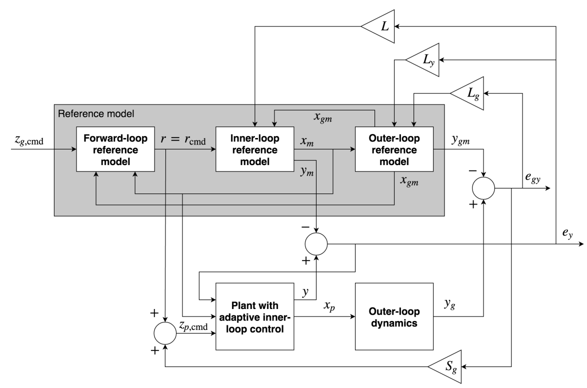

Figure 1. Complete integrated inner and outer-loop design block diagram.

All that remains to complete the control design is to specify solutions to L_y, L_g, and S_g such that the closed-loop system is stable and the control goal of command tracking is satisfied. The problem is restated as: Find the matrices L_y, S_g, L_g and P_g that together satisfy the following conditions

$$C^\top L_y^\top = -P_x B_{\text{cmd}} S_g C_g P_g^{-1} - B_g^\top - B_d P_g^{-1} \tag{41}$$

$$(A_g + L_g C_g)^\top P_g + P_g(A_g + L_g C_g) < -Q_g \tag{42}$$

Rearranging (41) and using Lemma 2, the matrices P_g and S_g must be found satisfying

$$-M^\top N^\top X N B_{\text{cmd}} S_g C_g P_g^{-1} = M^\top (B_g^\top + B_d P_g^{-1}) \tag{43}$$

$$C_g^{\top\perp\top}(A_g^\top P_g + P_g A_g) C_g^{\top\perp} < 0 \tag{44}$$

If S_g and P_g exist satisfying (43) and (44), then the solution to the outer-loop control problem exists. Once solutions S_g and P_g are found analytically, an L_g satisfying (42) is guaranteed to exist, which can be simply found numerically, as was done to obtain L in the inner-loop design [6], [7]. With the solutions S_g and P_g, the solution L_y from (41) can then be calculated. So the problem that remains is to find S_g and P_g satisfying (43) and (44). The existence of solutions to (43) is dependent on the sizes of the matrices, as described in the following cases.

1) Case I: $n-p=n_{ep}$: This case corresponds to the number of inner-loop regulated outputs n_{ep} being equal to the number of unmeasured inner-loop states, given by $n-p$. For S_g to exist satisfying (43), P_g must exist satisfying

$$M^\top B_g^\top P_g C_g^{\top\perp} = -M^\top B_d C_g^{\top\perp} \tag{45}$$

with $M^\top B_g^\top \in \mathbb{R}^{n-p\times n_g}$ and $C_g^{\top\perp} \in \mathbb{R}^{n_g\times n_g-p_g}$. S_g is then calculated as

$$S_g = -(M^\top N^\top XNB_{\text{cmd}})^{-1} M^\top (B_g^\top + B_d P_g^{-1}) P_g C_g^{-1_{\text{right}}} \tag{46}$$

2) Case II: $n-p<n_{ep}$: This case corresponds to the number of inner-loop regulated outputs n_{ep}, being greater than the number of unmeasured inner-loop states, given by $n-p$. For S_g to exist satisfying (43), P_g must exist satisfying

$$M^\top B_g^\top P_g C_g^{\top\perp} = -M^\top B_d C_g^{\top\perp} \tag{47}$$

Once P_g satisfying (47) is found, S_g is determined by

$$S_g = -(M^\top N^\top XNB_{\text{cmd}})^{-1_{\text{right}}} M^\top (B_g^\top + B_d P_g^{-1}) P_g C_g^{-1_{\text{right}}} \tag{48}$$

3) Case III: $n-p>n_{ep}$: This case corresponds to the number of inner-loop regulated outputs n_{ep}, being less than the number of unmeasured inner-loop states, given by $n-p$. In this case $M^\top N^\top XNB_{\text{cmd}}$ in (43) is tall, so S_g doesn't have the degrees of freedom to satisfy (43).

The conditions for the existence of S_g and P_g exist satisfying (43) and (44) is stated in the following theorem.

Theorem 1 *For the existence of S_g and P_g satisfying* (43) *and* (44) *for a stable outer-loop controller, the plant must satisfy $n-p\le n_{ep}$, $n-p<n_g$, the following inequality*

$$C_g^{\top\perp\top} A_g^\top (B_g M)^\perp \left(C_g^{\top\perp\top}(B_g M)^\perp\right)^{-1_{\text{right}}} < 0 \tag{49}$$

and the rank of the following matrix be full

$$\begin{bmatrix} M^\top B_g^\top \\ C_g^{\top\perp} \end{bmatrix}$$

PROOF Finding S_g and P_g satisfying (43) involves first finding the set of all $P_g = P_g^\top > 0$ which satisfy $\Pi_A P_g \Pi_B = \Pi_C$, where Π_A, Π_B, and Π_C are matrices which depend on the state-space plant matrices as in (45) and (47). Finding P_g involves using a generalized singular value decomposition, and fixes certain elements of P_g based on Π_A, Π_B, and Π_C. Then, from this set of P_g, those which also satisfy (44) are found. This involves substituting the form of P_g into (44) and manipulating to obtain (49). These steps are outlined in detail in the following sections. □

Remark 4 The control solution is still possible when the inequality in (49) is not strict, which results in (44) not being strict. The implications this has on tracking are discussed following the stability proof, but it is noted that outer-loop command tracking is still achieved as desired.

Remark 5 For the existence of a stable outer loop as described in Theorem 1 for the case when $n-p\ge n_g$, a solution is still possible, but requires additional constraints to be satisfied.

VI. Solving P_g: Symmetric Solutions to the Matrix Equation $\Pi_A P_g \Pi_B = \Pi_C$

Equations (45) and (47) are in the form $\Pi_A P_g \Pi_B = \Pi_C$. In these two cases, the matrices Π_A, Π_B, and Π_C in (45) and (47) are given by

$$\begin{aligned} \Pi_A &= M^\top B_g^\top \\ \Pi_B &= C_g^{\top\perp} \\ \Pi_C &= -M^\top B_d C_g^{\top\perp} \end{aligned} \tag{50}$$

where $\Pi_A \in \mathbb{R}^{n-p\times n_g}$, $P_g \in \mathbb{R}^{n_g\times n_g}$, $\Pi_B \in \mathbb{R}^{n_g\times n_g-p_g}$ and $\Pi_C \in \mathbb{R}^{n-p\times n_g-p_g}$. Using the definitions in (50), the inequality (44) is rewritten and the problem once again restated as: Find $P_g = P_g^\top > 0$ satisfying

$$\Pi_A P_g \Pi_B = \Pi_C \tag{51}$$

$$\Pi_B^\top (A_g^\top P_g + P_g A_g)\Pi_B < 0 \tag{52}$$

A. A Generalized Singular Value Decomposition

Determining solutions P_g to Eq. (51) involves first decomposing Π_A and Π_B using a generalized singular value decomposition (GSVD) [20]–[23] as follows

$$\begin{aligned} \Pi_A &= U\Sigma_A P \\ \Pi_B^\top &= V\Sigma_B P \end{aligned} \tag{53}$$

where $U \in \mathbb{R}^{n-p\times n-p}$ and $V \in \mathbb{R}^{n_g-p_g\times n_g-p_g}$, $P \in \mathbb{R}^{n_g\times n_g}$, and $\Sigma_A \in \mathbb{R}^{n-p\times n_g}$ and $\Sigma_B \in \mathbb{R}^{n_g-p_g\times n_g}$. The matrices describing the decomposition in Eq. (53) are given by

$$\begin{aligned} \Sigma_A &= \begin{bmatrix} I_{n-p\times n-p} & 0_{n-p\times n_g-p_g} & 0_{n-p\times p-n+p_g} \end{bmatrix} \\ \Sigma_B &= \begin{bmatrix} 0_{n_g-p_g\times n-p} & I_{n_g-p_g\times n_g-p_g} & 0_{n_g-p_g\times p-n+p_g} \end{bmatrix} \end{aligned} \tag{54}$$

and

$$P = P_\Lambda \begin{bmatrix} \Pi_A \\ \Pi_B^\top \\ \begin{bmatrix} \Pi_A^\top & \Pi_B \end{bmatrix}^{\perp\top} \end{bmatrix} \tag{55}$$

where P_Λ is an arbitrary block diagonal matrix, with full rank, given by

$$P_\Lambda = \begin{bmatrix} P_A & 0 & 0 \\ 0 & P_B & 0 \\ 0 & 0 & P_N \end{bmatrix} \tag{56}$$

where $P_A \in \mathbb{R}^{n-p \times n-p}$, $P_B \in \mathbb{R}^{n_g - p_g \times n_g - p_g}$, and $P_N \in \mathbb{R}^{p+p_g-n \times p+p_g-n}$ are each matrices of full rank. Substituting (56) into (55) gives

$$P = \begin{bmatrix} P_A \Pi_A \\ P_B \Pi_B^\top \\ P_N \begin{bmatrix} \Pi_A^\top & \Pi_B \end{bmatrix}^{\perp\top} \end{bmatrix} \tag{57}$$

Selecting U and V in (53) as $U = P_A^{-1}$ and $V = P_B^{-1}$ ensures that the decomposition (53) holds.

B. *Satisfying* $\Pi_A P_g \Pi_B = \Pi_C$ *with* $P_g = P_g^\top > 0$

Substituting Π_A and Π_B decomposed as in (53) into (51) gives

$$U \Sigma_A P P_g P^\top \Sigma_B^\top V^\top = \Pi_C \tag{58}$$

Propose the following solution P_g to (58)

$$P_g = P^{-1} X_D P^{-\top} \tag{59}$$

where $X_D = X_D^\top > 0$ ensures that $P_g = P_g^\top > 0$. Substituting P_g from (59) into (58) results in

$$U \Sigma_A P P^{-1} X_D P^{-\top} P^\top \Sigma_B^\top V^\top = \Pi_C$$

which can be simplified as

$$U \Sigma_A X_D \Sigma_B^\top V^\top = \Pi_C \tag{60}$$

The matrix $X_D = X_D^\top > 0$ is written as

$$X_D = \begin{bmatrix} X_{D11} & X_{D12} & X_{D13} \\ X_{D12}^\top & X_{D22} & X_{D23} \\ X_{D13}^\top & X_{D23}^\top & X_{D33} \end{bmatrix} \tag{61}$$

where $X_{D11} \in \mathbb{R}^{n-p \times n-p}$, $X_{D12} \in \mathbb{R}^{n-p \times n_g-p_g}$, $X_{D13} \in \mathbb{R}^{n-p \times p-n+p_g}$, $X_{D22} \in \mathbb{R}^{n_g-p_g \times n_g-p_g}$, $X_{D23} \in \mathbb{R}^{n_g-p_g \times p-n+p_g}$, and $X_{D33} \in \mathbb{R}^{p-n+p_g \times p-n+p_g}$. Substituting the form of X_D from (61) and Σ_A and Σ_B from (54) into (60) gives

$$U \begin{bmatrix} I & 0 & 0 \end{bmatrix} \begin{bmatrix} X_{D11} & X_{D12} & X_{D13} \\ X_{D12}^\top & X_{D22} & X_{D23} \\ X_{D13}^\top & X_{D23}^\top & X_{D33} \end{bmatrix} \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} V^\top = \Pi_C$$

from which X_{D12} is given as

$$X_{D12} = P_A \Pi_C P_B^\top \tag{62}$$

The choice of X_{D12} in (62) ensures that P_g given by (59) with X_D given by (61) satisfies the equation $\Pi_A P_g \Pi_B = \Pi_C$. However, the remaining degrees of freedom in X_D must be selected to ensure also that $P_g > 0$, and that P_g also satisfies the inequality (52).

C. *Satisfying* $\Pi_B^\top (A_g^\top P_g + P_g A_g) \Pi_B < 0$

With P_g given by (59) dependent on X_D given in (61) with X_{D12} given in (62) and P given in (57), the goal now is to find the remaining elements of X_D so that the resulting P_g satisfies the inequality (52) and ensures $P_g > 0$. Substituting P_g from (59) into (52) gives

$$\Pi_B^\top P^{-1} (P A_g^\top P^{-1} X_D + X_D P^{-\top} A_g P^\top) P^{-\top} \Pi_B < 0 \tag{63}$$

Given P in (55) its inverse P^{-1} must satisfy

$$\begin{bmatrix} P_A \Pi_A \\ P_B \Pi_B^\top \\ P_N \begin{bmatrix} \Pi_A^\top & \Pi_B \end{bmatrix}^{\perp\top} \end{bmatrix} P^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{64}$$

It can be seen from this that

$$\Pi_B^\top P^{-1} = \begin{bmatrix} 0 & P_B^{-1} & 0 \end{bmatrix} \tag{65}$$

Using (65), the inequality (63) becomes

$$\begin{bmatrix} 0 & P_B^{-1} & 0 \end{bmatrix} (P A_g^\top P^{-1} X_D + X_D P^{-\top} A_g P^\top) \begin{bmatrix} 0 \\ P_B^{-\top} \\ 0 \end{bmatrix} < 0 \tag{66}$$

Defining $\bar{P}$ as

$$\bar{P} = P A_g^\top P^{-1} \tag{67}$$

the inequality (66) can be written as

$$\begin{bmatrix} 0 & P_B^{-1} & 0 \end{bmatrix} (\bar{P} X_D + X_D \bar{P}^\top) \begin{bmatrix} 0 \\ P_B^{-\top} \\ 0 \end{bmatrix} < 0 \tag{68}$$

Examining (64) it can be seen that the columns of P^{-1} are given by

$$P^{-1} = \begin{bmatrix} \Pi_B^\perp (P_A \Pi_A \Pi_B^\perp)^{-1_{\text{right}}} & \Pi_A^{\top\perp} (P_B \Pi_B^\top \Pi_A^{\top\perp})^{-1_{\text{right}}} & \times \end{bmatrix} \tag{69}$$

where $\times$ indicates a column of P^{-1} which is to remain unspecified. Expanding P and P^{-1}, the requirement that $PP^{-1} = I$ requires the following conditions be satisfied

$$\begin{aligned} P_A \Pi_A \Pi_B^\perp (P_A \Pi_A \Pi_B^\perp)^{-1_{\text{right}}} &= I \\ P_B \Pi_B^\top \Pi_B^\perp (P_A \Pi_A \Pi_B^\perp)^{-1_{\text{right}}} &= 0 \\ P_N \begin{bmatrix} \Pi_A^\top & \Pi_B \end{bmatrix}^{\perp\top} \Pi_B^\perp (P_A \Pi_A \Pi_B^\perp)^{-1_{\text{right}}} &= 0 \end{aligned} \tag{70}$$

and

$$\begin{aligned} P_A \Pi_A \Pi_A^{\top\perp} (P_B \Pi_B^\top \Pi_A^{\top\perp})^{-1_{\text{right}}} &= 0 \\ P_B \Pi_B^\top \Pi_A^{\top\perp} (P_B \Pi_B^\top \Pi_A^{\top\perp})^{-1_{\text{right}}} &= I \\ P_N \begin{bmatrix} \Pi_A^\top & \Pi_B \end{bmatrix}^{\perp\top} \Pi_A^{\top\perp} (P_B \Pi_B^\top \Pi_A^{\top\perp})^{-1_{\text{right}}} &= 0 \end{aligned} \tag{71}$$

The first two conditions in (70) and (71) are obvious, following from the definition of the right inverse, and properties of the annihilator matrices. With P^{-1} given by (69), $\bar{P}$ in (67) can be written as

$$\bar{P} = \begin{bmatrix} P_A\Pi_A \\ P_B\Pi_B^\top \\ P_N\left[\Pi_A^\top \quad \Pi_B\right]^{\perp\top} \end{bmatrix} A_g^\top \begin{bmatrix} (\Pi_B^\perp\left(P_A\Pi_A\Pi_B^\perp\right)^{-1_{\text{right}}})^\top \\ (\Pi_A^{\top\perp}\left(P_B\Pi_B^\top\Pi_A^{\top\perp}\right)^{-1_{\text{right}}})^\top \\ \times \end{bmatrix}^\top \tag{72}$$

where $\times$ in (72) again represents an element of $\bar{P}$ which remains unspecified. $\bar{P}$ can also be partitioned into a block matrix given by

$$\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} & \bar{P}_{13} \\ \bar{P}_{21} & \bar{P}_{22} & \bar{P}_{23} \\ \bar{P}_{31} & \bar{P}_{32} & \bar{P}_{33} \end{bmatrix} \tag{73}$$

where $P_{11} \in \mathbb{R}^{n-p\times n-p}$, $P_{12} \in \mathbb{R}^{n-p\times n_g-p_g}$, $P_{13} \in \mathbb{R}^{n-p\times p-n+p_g}$, $P_{21} \in \mathbb{R}^{n_g-p_g\times n-p}$, $P_{22} \in \mathbb{R}^{n_g-p_g\times n_g-pg}$, $P_{23} \in \mathbb{R}^{n_g-p_g\times p-n+p_g}$, $P_{31} \in \mathbb{R}^{p-n+p_g\times n-p}$, $P_{32} \in \mathbb{R}^{p-n+p_g\times n_g-p_g}$, and $P_{33} \in \mathbb{R}^{p-n+p_g\times p-n+p_g}$ with $\bar{P}_{21}$ and $\bar{P}_{22}$ given by

$$\begin{aligned} \bar{P}_{21} &= P_B\Pi_B^\top A_g^\top \Pi_B^\perp\left(P_A\Pi_A\Pi_B^\perp\right)^{-1_{\text{right}}} \\ \bar{P}_{22} &= P_B\Pi_B^\top A_g^\top \Pi_A^{\top\perp}\left(P_B\Pi_B^\top\Pi_A^{\top\perp}\right)^{-1_{\text{right}}} \end{aligned} \tag{74}$$

The inequality in (68) with $\bar{P}$ given by (73), where $\bar{P}_{21}$ and $\bar{P}_{22}$ are given by (74), and X_D given by (61) must be satisfied by the selection of the remaining elements of X_D. Plugging these expressions for $\bar{P}$ in (73) and X_D in (61) into the inequality in (68) gives

$$\begin{bmatrix} 0 \\ P_B^{-\top} \\ 0 \end{bmatrix}^\top \left(\begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} & \bar{P}_{13} \\ \bar{P}_{21} & \bar{P}_{22} & \bar{P}_{23} \\ \bar{P}_{31} & \bar{P}_{32} & \bar{P}_{33} \end{bmatrix} \begin{bmatrix} X_{D11} & X_{D12} & X_{D13} \\ X_{D12}^\top & X_{D22} & X_{D23} \\ X_{D13}^\top & X_{D23}^\top & X_{D33} \end{bmatrix} + \begin{bmatrix} X_{D11} & X_{D12} & X_{D13} \\ X_{D12}^\top & X_{D22} & X_{D23} \\ X_{D13}^\top & X_{D23}^\top & X_{D33} \end{bmatrix} \begin{bmatrix} \bar{P}_{11}^\top & \bar{P}_{21}^\top & \bar{P}_{31}^\top \\ \bar{P}_{12}^\top & \bar{P}_{22}^\top & \bar{P}_{32}^\top \\ \bar{P}_{13}^\top & \bar{P}_{23}^\top & \bar{P}_{33}^\top \end{bmatrix} \right) \begin{bmatrix} 0 \\ P_B^{-\top} \\ 0 \end{bmatrix} < 0$$

which is equivalent to

$$\begin{aligned} &\bar{P}_{21}X_{D12} + \bar{P}_{22}X_{D22} + \bar{P}_{23}X_{D23}^\top \\ &\quad + X_{D12}^\top\bar{P}_{21}^\top + X_{D22}\bar{P}_{22}^\top + X_{D23}^\top\bar{P}_{23}^\top < 0 \end{aligned} \tag{75}$$

X_{D12} in (75) is given by (62) thus ensuring $\Pi_A P_g \Pi_B = \Pi_C$. The remaining elements of X_D must be selected so that $X_D > 0$ and so as to satisfy the inequality in Eq. (75). Rearranging the terms in (75) gives

$$\begin{aligned} &(\bar{P}_{22}X_{D22} + X_{D22}\bar{P}_{22}^\top) + (\bar{P}_{21}X_{D12} + X_{D12}^\top\bar{P}_{21}^\top) \\ &\quad < -(\bar{P}_{23}X_{D23}^\top + X_{D23}^\top\bar{P}_{23}^\top) \end{aligned} \tag{76}$$

The problem now is to select the remaining elements of X_D so as to satisfy (76), while also ensuring $X_D > 0$.

D. Solving for X_D

To satisfy (76) and ensure $X_D > 0$ in (61), the solution X_{D12} from (62) is used, and setting $X_{D13} = 0$, $X_{D23} = 0$, and $X_{D33} > 0$ simplifies (76) to

$$(\bar{P}_{22}X_{D22} + X_{D22}\bar{P}_{22}^\top) + (\bar{P}_{21}X_{D12} + X_{D12}^\top\bar{P}_{21}^\top) < 0 \tag{77}$$

and X_D in (61) to

$$X_D = \begin{bmatrix} X_{D11} & X_{D12} & 0 \\ X_{D12}^\top & X_{D22} & 0 \\ 0 & 0 & X_{D33} \end{bmatrix} \tag{78}$$

If $\bar{P}_{22}$ in (77) is stable, this Lyapunov equation (77) can be solved to obtain X_{D22}. Then with X_{D12}, X_{D22}, and X_{D33} fixed, the Schur Complement is then used to selected X_{D11} to ensure $X_D > 0$ in (78). With $\bar{P}_{22}$ given by (74), this provides an easy way to check if the outer-loop control solution exists. However, if $\bar{P}_{22}$ in (77) is not stable, this inequality may still be satisfied, based on the properties of $\bar{P}_{21}$ and X_{D12}. Thus, in this case, these properties must be examined to determine whether the outer-loop control solution exists. These two cases are considered in the following subsections.

1) Case i: $\bar{P}_{22}$ is Stable: If $\bar{P}_{22}$ in (77) is stable, this Lyapunov equation can be solved to obtain X_{D22}. Then, $X_{D11} > 0$ can be selected satisfying the following Schur complement

$$X_{D11} > X_{D12}X_{D22}^{-1}X_{D12}^\top \tag{79}$$

which ensures that P_g satisfies $\Pi_A P_g \Pi_B = \Pi_C$ with $P_g = P_g^\top > 0$, and also satisfies the inequality (52). Thus, satisfaction of the inequality (52) and existence of the outer-loop controller is dependent on $\bar{P}_{22}$ in (74) being stable. Stability of $\bar{P}_{22}$ in (74) is equivalent to the following

$$\Pi_B^\top A_g^\top \Pi_A^{\top\perp}\left(\Pi_B^\top\Pi_A^{\top\perp}\right)^{-1_{\text{right}}} < 0 \tag{80}$$

Using the notation in (50), the requirement in (80) can be written as (49). If (49) is satisfied, then $X_D = X_D^\top > 0$ exists which defines P_g and ensures that $P_g = P_g^\top > 0$, $\Pi_A P_g \Pi_B = \Pi_C$ as in (51), and $\Pi_B^\top(A_g^\top P_g + P_g A_g)\Pi_B < 0$ as in (52).

2) Case ii: $\bar{P}_{22}$ is Not Stable: If $\bar{P}_{22}$ is not stable, the inequality (77) can still be satisfied if

$$(\bar{P}_{21}X_{D12} + X_{D12}^\top\bar{P}_{21}^\top) < 0 \tag{81}$$

In this case, X_{D22} can be selected sufficiently small so that the negative term (81) in (77) ensures the inequality is satisfied. Using the expressions for $\bar{P}_{12}$ from (74) and X_{D12} from (62) to evaluate the quantity $\bar{P}_{21}X_{D12}$ in (81) gives

$$\bar{P}_{21}X_{D12} = P_B\Pi_B^\top A_g^\top \Pi_B^\perp\left(P_A\Pi_A\Pi_B^\perp\right)^{-1_{\text{right}}} P_A\Pi_C P_B^\top \tag{82}$$

The matrix $\bar{P}_{21}X_{D12}$ in (82) is square, with dimensions $n_g - p_g \times n_g - p_g$. Using the expression for $\bar{P}_{21}X_{D12}$ in (82) allows the inequality in (81) to be expressed as

$$\begin{aligned} &P_B\Pi_B^\top A_g^\top \Pi_B^\perp\left(P_A\Pi_A\Pi_B^\perp\right)^{-1_{\text{right}}} P_A\Pi_C P_B^\top \\ &+\left(P_B\Pi_B^\top A_g^\top \Pi_B^\perp\left(P_A\Pi_A\Pi_B^\perp\right)^{-1_{\text{right}}} P_A\Pi_C P_B^\top\right)^\top < 0 \end{aligned} \tag{83}$$

Satisfying the inequality in (83) is independent of the selection of the matrices P_A and P_B. Thus, satisfying the inequality in (83) is equivalent to satisfying

$$\begin{aligned}&\Pi_B^\top A_g^\top \Pi_B^\perp \big(\Pi_A \Pi_B^\perp\big)^{-1_{\text{right}}} \Pi_C \\ &+\big(\Pi_B^\top A_g^\top \Pi_B^\perp \big(\Pi_A \Pi_B^\perp\big)^{-1_{\text{right}}} \Pi_C\big)^\top < 0\end{aligned} \tag{84}$$

Plugging in expressions for Π_A, Π_B and Π_C from (50) in terms of the plant state-space matrices into (84) gives

$$\begin{aligned}&-(C_g^{\top\perp\top} A_g^\top C_g^\top C_g B_g M M^\top B_d C_g^{\top\perp}) \\ &-(C_g^{\top\perp\top} A_g^\top C_g^\top C_g B_g M M^\top B_d C_g^{\top\perp})^\top < 0\end{aligned} \tag{85}$$

Thus, when $\bar{P}_{22}$ in (77) is not stable, a solution still exists if (85) is satisfied. In this case, X_{D22} can be selected sufficiently small so that the negative term (81) in (77) ensures the inequality is satisfied. Thus in this case even if (49) in Theorem 1 does not hold, a control solution still exists if (85) is satisfied.

E. Degrees of Freedom

The degrees of freedom available to the control designer are the matrices P_A, P_B, and P_N in P_Λ and thus P as in (57), that can be selected arbitrarily as long as they are full rank. In addition $X_D > 0$ contains several degrees of freedom. The matrix $X_{D33} = X_{D33}^\top > 0$ is arbitrary, X_{D22} can be selected as desired satisfying (77), and finally X_{D11} can be selected using the Schur complement to ensure $X_D > 0$.

VII. Solving for Remaining Outer-Loop Controller Gains

With the solution P_g determined, S_g can now be determined from (46) or (48), depending on the dimensions. With this P_g, an L_g satisfying the inequality in (43) is guaranteed to exist and can be solved for numerically. The CRM gain L_y can then be solved from (41) as

$$L_y = -\big((P_x B_{\text{cmd}} S_g C_g P_g^{-1})^\top + B_g + P_g^{-\top} B_d^\top\big) C^{-1_{\text{right}}} \tag{86}$$

The matrix L_y modifies the outer-loop guidance portion of the reference model in response to errors within the inner loop. It is this feature which enables stability of the combined inner and outer loops, and provides command tracking at the outer loop. The stability of the complete system using the adaptive inner-loop and sequential loop closure procedure to close the outer loop is given in Theorem 2.

Remark 6 When the outer-loop kinematics do not affect the inner-loop dynamics at all, that is when $B_d = 0$, then P_g changes the solution to S_g as given by (46) and (48), but has no effect on L_y. This can see this by plugging in the solution S_g from (46) or (48) into (86), resulting in P_g canceling out. This is important to note when tuning the outer-loop controller.

VIII. Stability

The inner-loop error dynamics were given in (14) and a Lyapunov function provided in (23), which showed stability of the closed loop system with update law in (22). When the outer-loop dynamics were considered, the assumption that $B_d = 0$ when designing the inner-loop controller was relaxed, giving the modified inner-loop dynamics in (26). This change to the inner-loop plant dynamics modified the inner-loop error dynamics in (14) to those in (28). The inner and outer-loop error dynamics in (40) can be written in matrix form as

$$\begin{aligned}\begin{bmatrix}\dot{e}_x(t)\\ \dot{e}_g(t)\end{bmatrix} &= \begin{bmatrix}A + LC + B\Psi^\top & B_{\text{cmd}} S_g C_g + B_d \\ B_g + L_y C & A_g + L_g C_g\end{bmatrix}\begin{bmatrix}e_x(t)\\ e_g(t)\end{bmatrix} \\ &+ \begin{bmatrix}B\\ 0\end{bmatrix}\Lambda \widetilde{\Theta}^\top(t) x_m(t)\end{aligned} \tag{87}$$

The stability of the closed-loop system with the error dynamics in (87) is proved in the following theorem.

Theorem 2 *The uncertain system in* (6) *with inner-loop controller specified by the control law in* (13)*, update law in* (22)*, and the reference model in* (27) *where S_1 and L are chosen as described in Refs. [6], [7], and the outer-loop controller specified by the outer-loop reference model in* (29)*, forward-loop reference model component in* (32)*, with inner-loop command input prescribed by* (34)*,* (38) *and* (39)*, with S_g, L_g, and L_y selected as described above results in global stability, with* $\lim_{t\to\infty} e_x(t) = 0$ *and* $\lim_{t\to\infty} e_g(t) = 0$.

Proof With a radially unbounded Lyapunov function candidate

$$\begin{aligned}V\big(e_x(t), e_g(t), \widetilde{\Theta}(t)\big) &= e_x^\top(t) P_x e_x(t) + e_g^\top(t) P_g e_g(t) \\ &+ |\Lambda| \widetilde{\Theta}^\top(t) \Gamma^{-1} \widetilde{\Theta}(t)\end{aligned} \tag{88}$$

where P_x is given by (20) and where P_g is the solution to the Lyapunov equation in (42), which is satisfied by the selection of L_g and P_g as described above. The time-derivative of (88) is given by

$$\begin{aligned}\dot{V}\big(e_x(t), e_g(t), \widetilde{\Theta}(t)\big) &= \dot{e}_x^\top(t) P_x e_x(t) + e_x^\top(t) P_x \dot{e}_x(t) \\ &+ \dot{e_g}^\top(t) P_g e_g(t) + e_g^\top(t) P_g \dot{e_g}(t) + 2|\Lambda| \widetilde{\Theta}^\top(t) \Gamma^{-1} \dot{\widetilde{\Theta}}(t)\end{aligned} \tag{89}$$

Substituting the inner and outer-loop error dynamics from (28) and (30), the update law (22), with (42), and with $A_L^\top P_x + P_x A_L = -Q_x < 0$ where $A_L = A + LC + B\Psi^\top$ as assured by the selection of L in (18) and with L_y in (86) simplifies (89) to

$$\dot{V}\big(e_x(t), e_g(t), \widetilde{\Theta}(t)\big) = -e_x^\top(t) Q_x e_x(t) - e_g^\top(t) Q_g e_g(t) \tag{90}$$

which implies that (88) is a Lyapunov function. It can be concluded using Barbalat's Lemma [18] that $\lim_{t\to\infty} e_x(t) = 0$ and $\lim_{t\to\infty} e_g(t) = 0$. Since (88) is radially unbounded stability is global. □

Corollary 2 *The outer-loop regulated output $z_g(t)$ tracks the reference regulated output $z_{gm}(t)$ asymptotically. Furthermore, for piecewise constant outer-loop commands, $z_g(t)$ tracks $z_{g,cmd}(t)$ asymptotically.*

PROOF Using that $\lim_{t\to\infty} e_g = 0$ from Theorem 2 it follows that $z_g(t) \to z_{gm}(t)$ as $t \to \infty$. Furthermore, as stated in Remark 3 the reference model in (36) with output $z_{gm}(t)$ is a type 1 system with respect to the command $z_{g,\text{cmd}}(t)$. Thus, for piecewise constant $z_{g,\text{cmd}}(t)$ it follows that $z_{gm}(t) \to z_{g,\text{cmd}}(t)$ as $t \to \infty$, from which it follows that $z_g(t) \to z_{g,\text{cmd}}(t)$ as $t \to \infty$, which proves the corollary. □

Remark 7 When the inequality (49) is no longer strict, the result is the inability to show $e_g(t) \in \mathcal{L}_2$ and thus it cannot be show that $\lim_{t\to\infty} e_g(t) = 0$. However $\lim_{t\to\infty} e_{gy}(t) = 0$, which gives $y_g(t) \to y_{gm}(t)$ as $t \to \infty$ and with C_g in (25) containing the regulated output, gives $z_g(t) \to z_{gm}(t)$ as $t \to \infty$, providing outer-loop command tracking as desired.

IX. Outer-loop Controller Summary

This section provides a summary of the control design procedure, assuming an adaptive inner-loop control as described in Refs. [6], [7] has already been designed.

1. Design an inner-loop controller as outlined in Refs. [6], [7].
2. Add the B_d term to the inner-loop reference model in (12) resulting in (27)
3. Define the outer-loop reference model in (29), the forward-loop reference model in (32), and the inner-loop command input as in (38)
4. Calculate $\bar{P}_{22}$ from (74) where P_B is an arbitrary full rank matrix.
5. Calculate X_{D12} from (62) where P_A is an arbitrary full rank matrix.
6. Solve the Lyapunov equation (77) to obtain X_{D22}.
7. Assemble X_D in (78), where $X_{D33} > 0$ is arbitrary, X_{D12} is given by (62), X_{D22} satisfies the inequality (77), and X_{D11} satisfies (79)
8. Assemble (56) where P_N is an arbitrary full rank matrix and then calculate P as in (55).
9. Using P in (57) and X_D in (78), calculate P_g from (59).
10. With the solution P_g determined, S_g from (46) or (48) is then solved for, depending on the dimensions.
11. With this P_g, an L_g satisfying the inequality in (43) is guaranteed to exist and can be solved for numerically.
12. Solve for L_y as in (86).

Remark 8 In practice, the synthesis of the proposed controller for both the inner and outer loops is computationally simple, with each case involving basic matrix operations which result in a feasible LMI. The numerical solution of this LMI is trivial with any modern numerical solver.

X. State Limiter

One of the benefits of the proposed architecture is in the explicit calculation of the inner-loop command $r_{\text{cmd}}(t)$, which is used as the input to the inner-loop reference model and plant $r(t)$ as given in (34). In this section a limiter is introduced in the generation of $r_{\text{cmd}}(t)$ so that state variables of interest are curtailed to stay within a certain region. This approach is inspired by the work in [2], [24], [25] and originally developed in [26]. The primary difference is that the limiter proposed here is for the output feedback case, whereas the references above as well as Refs. [27], [28] are for the case of state feedback. However, unlike in [2] where the state limiter is designed to accommodate an bounded, unknown, time-varying disturbance, such disturbances are not considered in this paper.

A. Overview

This approach will generate $r(t)$ by scaling the inner-loop command $r_{\text{cmd}}(t)$, as well as the generating the outer-loop command $z_{g,\text{cmd}}(t)$ by scaling the desired outer-loop command $z'_{g,\text{cmd}}(t)$ based on limits placed on the reference model states. Should the system be command to enter a region in the state-space which would invoke the limiter, these modifications will then affect the outer-loop tracking performance, which is expected. Sacrificing tracking performance to limit the inner-loop command or the system states is an expected trade-off, and also a necessary one.

The reference model (35) with type-1 controller as described in Remark 3 will have no outer-loop command feedthrough, and with $z_{g,\text{cmd}}(t)$ not as in (33) gives

$$\begin{aligned}\dot{\bar{x}}_m(t) &= \bar{A}\bar{x}_m(t) + \bar{B}r(t) - \bar{L}_y e_y(t)\\ &\quad - \bar{L}_g e_{gy}(t) + \bar{B}_m z'_{g,\text{cmd}}(t) \\ r_{\text{cmd}}(t) &= \bar{C}_m \bar{x}_m(t)\end{aligned} \tag{91}$$

However, to facilitate command and state limiting the inner-loop command $r(t)$ and the outer-loop command $z_{g,\text{cmd}}(t)$ in (91) should be modified when certain reference model states become too large. Thus inner-loop command $r(t)$ is no longer set as in (34) but instead as

$$r(t) = r_{\text{cmd}}(t) - r_{\text{lim}}(t) \tag{92}$$

where the inner-loop reference model command limiter $r_{\text{lim}}(t)$ is given by

$$r_{\text{lim}}(t) = -k_r\gamma(\bar{x}_m(t))K_{\text{lim}}\bar{x}_m(t) \tag{93}$$

where $k_r \geq 0$ has dimensions $k_r \in \mathbb{R}^{n_{ep}\times n_{ep}}$ and $K_{\text{lim}} \in \mathbb{R}^{n_{ep}\times n+n_g+n_f}$ is given by

$$K_{\text{lim}} = -R_{\text{lim}}(\bar{B}_m + \bar{B}k_r)^\top \bar{P} \tag{94}$$

where $R_{\text{lim}} = R_{\text{lim}}^\top \geq 0$ has dimensions $R_{\text{lim}} \in \mathbb{R}^{n_{ep}\times n_{ep}}$ and $\bar{P}$ is the solution to the Lyapunov equation

$$\bar{A}_m^\top \bar{P} + \bar{P}\bar{A}_m = -\bar{Q} \tag{95}$$

where $\bar{Q} = \bar{Q}^\top > 0$. The outer-loop command $z_{g,\text{cmd}}(t)$ is no longer selected as in (33), but is instead generated from the desired outer-loop command $z'_{g,\text{cmd}}(t)$ as

$$z_{g,\text{cmd}}(t) = s\big(\gamma(\bar{x}_m(t))\big)z'_{g,\text{cmd}}(t) - z_{g,\text{lim}}(t) \tag{96}$$

where

$$s\big(\gamma(\bar{x}_m(t))\big) = 1 - \gamma(\bar{x}_m(t)) \tag{97}$$

and $z_{g,\text{lim}}(t)$ is given by

$$z_{g,\text{lim}}(t) = -\gamma(\bar{x}_m(t))K_{\text{lim}}\bar{x}_m(t) \tag{98}$$

The scalar quantity $\gamma(\bar{x}_m(t))$ is the modulation function, which is a function of the entire reference model state $\bar{x}_m(t)$, and is selected such that $\gamma(\bar{x}_m(t)) \in [0, 1]$. For $\bar{x}_m(t) \in \Omega_\delta$ the modulation function $\gamma(\bar{x}_m(t)) = 0$. This corresponds to no state limiting, and when $\gamma(\bar{x}_m(t)) = 1$ this corresponds to the state limiter being fully active. Thus, $\gamma(\bar{x}_m(t))$ is selected using several regions within the reference model state space such that within an inner region $\gamma(\bar{x}_m(t)) = 0$, an annulus region within which $\gamma(\bar{x}_m(t))$ varies between 0 and 1, and an outer region for which $\gamma(\bar{x}_m(t)) = 1$. See, for example the modulation function in Ref. [2]. This modifies the block diagram in Fig. 1 as shown in Fig. 2.

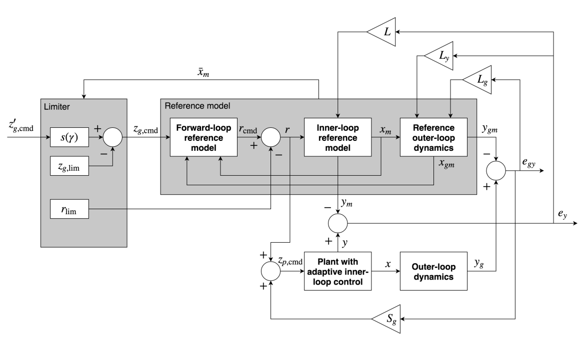

Figure 2. Expanded outer-loop block diagram with limiter.

Using the outer-loop command $z_{g,\text{cmd}}(t)$ as generated by (96) into (91) gives

$$\begin{aligned}\dot{\bar{x}}_m(t) =& \big(\bar{A}_m + \bar{B}k_r\gamma(\bar{x}_m(t))K_{\text{lim}} + \bar{B}_m\gamma(\bar{x}_m(t))K_{\text{lim}}\big)\bar{x}_m(t)\\ &+ \bar{B}_m\big(1-\gamma(\bar{x}_m(t))\big)z'_{g,\text{cmd}}(t)\\ &- \bar{L}_y e_y(t) - \bar{L}_g e_{gy}(t)\\ r_{\text{cmd}}(t) =& \bar{C}_m\bar{x}_m(t)\end{aligned} \tag{99}$$

B. Stability

Because $r(t)$ and $z_{g,\text{cmd}}(t)$ do not appear in the error dynamics (40), the state-limiter modification does not require any change to the Lyapunov function in (88) to prove boundedness of the errors $e_x(t)$ and $e_g(t)$. However, in the stability proof without the state limiter, the boundedness of $z_{g,\text{cmd}}(t)$, $e_y(t)$, and $e_{gy}(t)$ and stability of $\bar{A}_m$ in (36) imply boundedness of the reference model states $x_m(t)$, $x_{gm}(t)$, and $x_{fm}(t)$, from which boundedness of the plant states $x(t)$ and $x_g(t)$ is concluded. However, showing boundedness of the reference model states is less obvious when using the state limiter, which modifies the entire reference model dynamics in (36) to obtain the limited reference model dynamics in (99). Thus it is necessary to ensure that with the limiting modifications the reference model state $\bar{x}_m(t)$ is still bounded, and global stability is still proved, as stated in the following theorem.

Theorem 3 *The uncertain system in* (6) *with inner-loop controller specified by the control law in* (13)*, update law in* (22)*, and the reference model in* (27) *where S_1 and L are chosen as described in Section IV, and the outer-loop controller specified by the outer-loop reference model in* (29)*, forward-loop reference model component in* (32)*, with inner-loop command input $z_{p,cmd}(t)$ is prescribed by* (38) *and* (39)*, where $r(t)$ is given by* (99)*,* (92)*, and* (93)*, and outer-loop command generated by* (96)*,* (97)*, and* (98)*, with S_g, L_g, and L_y selected as described above, results in global stability, with $\lim_{t\to\infty} e_x(t) = 0$ and $\lim_{t\to\infty} e_g(t) = 0$.*

PROOF This proof follows from the proof of Theorem 2 by proposing the same candidate Lyapunov function as in (88) and differentiating to obtain (90) from which it can be concluded that $e_x(t),\ e_g(t),\ \widetilde{\Theta}(t) \in \mathcal{L}_\infty$. Bounds on $e_x(t)$ and $e_g(t)$ can be found as follows

$$\begin{aligned}\|e_x(t)\| &\leq \sqrt{\frac{V(0)}{\lambda_{\min}(P_x)}}\\ \|e_g(t)\| &\leq \sqrt{\frac{V(0)}{\lambda_{\min}(P_g)}}\end{aligned} \tag{100}$$

giving the following bounds on their respective measured output errors $e_y(t)$ and $e_{gy}(t)$ as

$$\begin{aligned}\|e_y(t)\| &\leq e_{y,\max} = \|C\|\sqrt{\frac{V(0)}{\lambda_{\min}(P_x)}} \\ \|e_{gy}(t)\| &\leq e_{gy,\max} = \|C_g\|\sqrt{\frac{V(0)}{\lambda_{\min}(P_g)}}\end{aligned} \tag{101}$$

Propose the following additional candidate Lyapunov function to prove boundedness of the reference model state

$$\bar{V}\big(\bar{x}_m(t)\big) = \bar{x}_m^\top(t)\bar{P}\bar{x}_m(t) \tag{102}$$

Differentiating (102) gives

$$\dot{\bar{V}}\big(\bar{x}_m(t)\big) = \dot{\bar{x}}_m^\top(t)\bar{P}\bar{x}_m(t) + \bar{x}_m^\top(t)\bar{P}\dot{\bar{x}}_m(t) \tag{103}$$

Using $\bar{Q}$ from (95) and defining the following

$$\bar{Q}_{\lim}(\gamma) = 2\bar{P}(\bar{B}_m + \bar{B}k_r)R_{\lim}^\top\gamma(k_r^\top\bar{B}^\top + \bar{B}_m^\top)\bar{P} \geq 0 \tag{104}$$

allowing (103) to be rewritten as

$$\begin{aligned}\dot{\bar{V}}\big(\bar{x}_m(t)\big) = &-\bar{x}_m^\top(\bar{Q} + \bar{Q}_{\lim}(\gamma))\bar{x}_m(t) \\ &+ 2\bar{x}_m^\top(t)\bar{P}\big(\bar{B}_m(1-\gamma)z'_{g,\text{cmd}}(t) \\ &- \bar{L}_y e_y(t) - \bar{L}_g e_{gy}(t)\big)\end{aligned} \tag{105}$$

Note that the bounds on $e_y(t)$ and $e_{gy}(t)$ in (101) are independent of $\bar{x}_m(t)$. Eq. (105) contains a negative quadratic term in $\bar{x}_m(t)$, and a sign indefinite term which is linear in $\bar{x}_m(t)$. Thus, for sufficiently large $\bar{x}_m(t)$, the derivative $\dot{\bar{V}}\big(x_m(t)\big)$ in (105) becomes strictly negative. This is quantified precisely by the following statement: $\dot{\bar{V}}\big(\bar{x}_m(t)\big) < 0$ outside the compact set

$$E_\delta = \left\{\bar{x}_m(t) \in \mathbb{R}^n \ : \ \|\bar{x}_m(t)\| \leq \frac{2\lambda_{\max}(\bar{P})\big(\|\bar{B}_m\|(1-\gamma)z'_{g,\text{cmd,max}} + \|\bar{L}_y\|e_{y,\max} + \|\bar{L}_g\|e_{gy,\max}\big)}{\lambda_{\min}(\bar{Q} + \bar{Q}_{\lim}(\gamma))}\right\} \tag{106}$$

for all $\gamma(\bar{x}_m(t)) \in [0,1]$. Thus the entire reference model state $\bar{x}_m(t)$ is bounded [18] which, with the boundedness of the errors $e_x(t)$ and $e_g(t)$, implies that $x(t),\ x_g(t) \in \mathcal{L}_\infty$. With this, it can be concluded using Barbalat's Lemma [18] that $\lim_{t\to\infty} e_x(t) = 0$ and $\lim_{t\to\infty} e_g(t) = 0$. □

In the absence of the state limiter, satisfaction of the control goal of outer-loop command tracking was discussed in Corollary 2. When using the state limiter, Theorem 2, like Theorem 3, provided $z_g(t) \to z_{gm}(t)$ as $t \to \infty$. However, without the limiter, the reference model in (36) produced $z_{gm}(t)$ that was a filtered version of $z'_{g,\text{cmd}}(t)$. When using the limiter this is no longer true; $z_{gm}(t)$ is the output of (99). Thus asymptotic tracking of $z'_{g,\text{cmd}}(t)$ by $z_g(t)$ doesn't hold in general. However, if a desired outer-loop command $z'_{g,\text{cmd}}(t)$ is given such that the limiter is inactive and $\gamma\big(\bar{x}_m(t)\big) = 0$, the same conclusion as in Corollary 2 can be made, with $z_{g,\text{cmd}}(t) = z'_{g,\text{cmd}}(t)$. This statement is formalized in the following corollary to Theorem 3.

Corollary 3 *For all piecewise constant outer-loop command inputs $z'_{g,\text{cmd}}(t)$ which satisfy $\|z'_{g,\text{cmd}}(t)\|_\infty \leq z'_{g,\text{cmd,max}}$, the outer-loop regulated output $z_g(t)$ tracks $z'_{g,\text{cmd}}(t)$ asymptotically, where $z'_{g,\text{cmd,max}}$ is given by*

$$z'_{g,\text{cmd,max}} = \frac{\bar{x}_{m,\max}}{\|h_m\|_1} \tag{107}$$

where h_m is the impulse response of the nominal reference model, given by (99) *with $\gamma(\bar{x}_m(t)) = 0$, $\bar{L}_y = 0$ and $\bar{L}_g = 0$, and $\bar{x}_{m,\max} = \max_{\bar{x}_m(t)\in\Omega_\delta}\|\bar{x}_m(t)\|$.*

PROOF For all $\bar{x}_m(t) \in \Omega_\delta$ the state limiter is inactive, and the evolution of the reference model state $\bar{x}_m(t)$ is governed by (99) with $\gamma(\bar{x}_m(t)) = 0$, while $e_y(t)$ and $e_{gy}(t)$ tend to zero asymptotically. Thus, the reference model state $\bar{x}_m(t)$ ultimately depends only on the command input $z'_{g,\text{cmd}}(t)$. The following bound on the reference model state holds, where h_m is the impulse response of the nominal reference model, (99) with $\gamma(\bar{x}_m(t)) = 0$.

$$\|\bar{x}_m(t)\|_\infty = \bar{x}_{m,\max} \leq \|h_m\|_1\|z'_{g,\text{cmd}}(t)\|_\infty$$

From this, the bound $z'_{g,\text{cmd,max}}$ that ensures the reference model state $\bar{x}_m(t) \in \Omega_\delta$, thus not invoking the state limiter, and providing the tracking properties given in Corollary 2. □

Remark 9 Corollary 3 states that if the desired outer-loop command $z'_{g,\text{cmd}}(t)$ is such that the system is not driven to enter the limiting region, that the limiter will not impact tracking performance of the system. This is due to the fact that the convergence of the tracking errors $e_y(t)$ and $e_{gy}(t)$ to zero is obtained regardless of whether the limiter is invoked or not. In other words, as these errors tend to zero, only the desired outer-loop command $z'_{g,\text{cmd}}(t)$ can drive the reference model state $\bar{x}_m(t)$ out of Ω_δ, as governed by (36). Thus, if the desired outer-loop command is such that it does not force $\bar{x}_m(t)$ outside of Ω_δ, the limiter will become inactive. Corollary 3 then finds the maximum value of $z'_{g,\text{cmd}}(t)$ such that $\bar{x}_m(t) \in \Omega_\delta$ using the impulse response of the reference model.

Remark 10 The benefits of the state limiter are apparent from the compact set in (106) outside of which $\dot{V}(\bar{x}_m(t)) < 0$. The size of E_δ monotonically decreases in size as $\gamma(\bar{x}_m(t))$ increases, hence shrinking the bound on $\bar{x}_m(t)$ when the limiter is invoked, versus without the limiter.

1) Degrees of Freedom: The limiter described above has several degrees of freedom which can be chosen by the designer to achieve the desired performance. These degrees of freedom are the gains k_r, R_{lim}, $\bar{Q}$, and the modulation function $\gamma(\bar{x}_m(t))$ and the corresponding sets Ω and Ω_δ. The limiter components $r_{\text{lim}}(t)$ and $z_{g,\text{lim}}(t)$ enter through the input matrices $\bar{B}$ and $\bar{B}_m$, respectively, of the reference model in (91). The matrix R_{lim} scales K_{lim} in (94), which is the gain used in both of the limiting components $r_{\text{lim}}(t)$ and $z_{g,\text{lim}}(t)$, whereas k_r scales only $r_{\text{lim}}(t)$. Thus, by adjusting R_{lim} and k_r, the relative influence of the limiter through $\bar{B}$ and $\bar{B}_m$ can be changed. This alters the the effective reference model matrix in (99) when the limiter becomes active, and thus $\bar{Q}_{\text{lim}}(\gamma(\bar{x}_m(t)))$ in (104). This, along with the matrix $\bar{Q}$, alters the region outside of which $\dot{\bar{V}} < 0$, and thus affects the time response of the system when the state limiter is active. With $k_r = 0$ and $R_{\text{lim}} = 0$ the limiter would still be stable, however the only adjustment would come through the reduction of the outer-loop command $z'_{g,\text{cmd}}(t)$ in (105). The modulation function $\gamma(\bar{x}_m(t))$ simply defines based on $\bar{x}_m$ when the limiter becomes active, and can be selected so as to depend on the various elements of $\bar{x}_m(t)$ as desired.

C. Complete Controller Summary with Limiter

The uncertain plant (26), outer-loop dynamics (24), inner-loop reference model (27), outer-loop reference model (29), forward-loop reference model component (32), inner-loop command (38), (39), (92) and (93), outer-loop command (96), (97), and (98), control law (13), and update law (22) are summarized as follows.

$$
\begin{aligned}
\textbf{Plant:} \quad & \dot{x}(t) = Ax(t) + B\big(\Lambda u(t) + \Psi^\top x(t)\big) \\
& \qquad + B_{\text{cmd}} z_{p,\text{cmd}}(t) + B_d x_g(t) \\
& \dot{x}_g(t) = A_g x_g(t) + B_g x(t) \\
\textbf{Reference model:} \quad & \dot{x}_m(t) = A_m x_m(t) + B_{\text{cmd}} r(t) \\
& \qquad - L e_y(t) + B_d x_{gm}(t) \\
& \dot{x}_{gm}(t) = B_g x_m(t) + A_g x_{gm}(t) \\
& \qquad - L_y e_y(t) - L_g e_{gy}(t) \\
& \dot{x}_{fm}(t) = B_{f3} x_m(t) + B_{f2} x_{gm}(t) \\
& \qquad + A_{fm} x_{fm}(t) + B_{f1} z_{g,\text{cmd}}(t) \\
\textbf{Command:} \quad & r_{\text{cmd}}(t) = C_{fm} x_{fm}(t) + D_{f1} z_{g,\text{cmd}}(t) \\
& \qquad + D_{f2} x_{gm}(t) + D_{f3} x_m(t) \\
& r(t) = r_{\text{cmd}}(t) - r_{\text{lim}}(t) \\
& r_{\text{lim}}(t) = -k_r \gamma\big(\bar{x}_m(t)\big) K_{\text{lim}} \bar{x}_m(t) \\
& z_{g,\text{cmd}}(t) = s(\gamma) z'_{g,\text{cmd}}(t) - z_{g,\text{lim}}(t) \\
& z_{p,\text{cmd}}(t) = r(t) + S_g e_{gy}(t) \\
\textbf{Errors:} \quad & e_y(t) = C\big(x(t) - x_m(t)\big) \\
& e_{gy}(t) = C_g\big(x_g(t) - x_{gm}(t)\big) \\
\textbf{Control:} \quad & u(t) = \big(K_x + \Theta(t)\big)^\top x_m(t) \\
& \dot{\Theta}(t) = -\Gamma x_m(t)\big(S_1 e_y(t)\big)^\top \text{sgn}(\Lambda)
\end{aligned}
$$

XI. Simulation Results

This section contains simulations comparing the performance of the baseline and adaptive controller, as well as the state limiter, on the nonlinear 6-DOF Generic Hypersonic Vehicle model [5]–[7], [29]. The equations of motion were linearized about a Mach 5 flight condition at an altitude of 80,000 feet. Modal analysis was then used to decouple the linearized equations of motion into three reduced order subsystems consisting of the first-order velocity, fourth-order longitudinal, and fifth-order lateral-directional dynamics. This allowed three decoupled controllers to be designed: a velocity controller with single loop, and controllers for the longitudinal and lateral-directional subsystems with both inner and outer loops, as described above. The longitudinal subsystem inner-loop state variables are angle-of-attack and pitch rate, with the outer-loop state variables pitch angle and altitude. The lateral-directional inner-loop state variables are sideslip angle, roll rate, and yaw rate, with outer-loop state variables roll angle and heading angle. Uncertainty consisting of control effectiveness on all surfaces reduced to 20% of the nominal value, center-of-gravity shifted 0.7 feet rearward, and the rolling moment coefficient C_l reduced to 10% of the nominal value is considered. The performance of the nominal baseline controller is compared to that of the adaptive controller both with and without the state limiter, for a heading change of 5 degrees, on the nonlinear, uncertain GHV model. For each subsystem $\Psi_{\max}$ in Assumption 1E-b was selected by acknowledging physical constraints of the plant. For example, the center-of-gravity must lie within the physical extents of the vehicle, and values of aerodynamic coefficients are bounded based on the flight envelop as determined by the propulsive and structural limitations of the vehicle.

Figure 3 shows that the baseline controller is not able to maintain stability when applied to the uncertain vehicle model. In this figure, both the reference and plant state can seen be seen to be diverging, showing the instability caused by the introduction of the uncertainty, and demonstrating a case in which an adaptive controller can be used to stabilize the uncertain system. Figure 4 shows the response of the uncertain plant with the adaptive controller summarized in Sec. IX. In this case, the adaptive controller is able to accommodate the uncertainty and maintain stability, but with large oscillations and sideslip angle occuring, both of which are undesirable. Figure 5 shows the response of the uncertain plant with adaptive controller and limiter as described in Sec. X used to suppress the large oscillations in sideslip. Here, the modulation function $\gamma(\bar{x}(t))$ as described in Ref. [2] in (98) (93) is chosen only as a function of sideslip angle, and the constraints on sideslip as given by Ω set at 0.2 degrees. In this case the adaptive controller not only maintains stability, but the use of the limiter confines the oscillations in sideslip angle to a maximum magnitude of less than 0.2 degrees as desired.

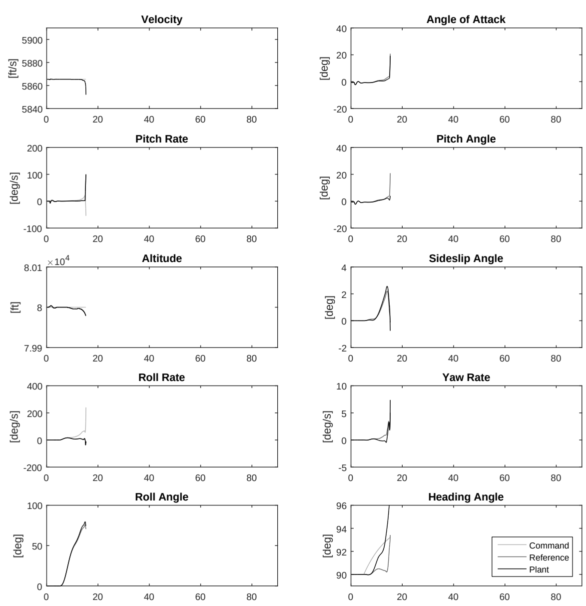

Figure 3. Plant states for baseline controller applied to uncertain plant in response to a 5 degree heading turn.

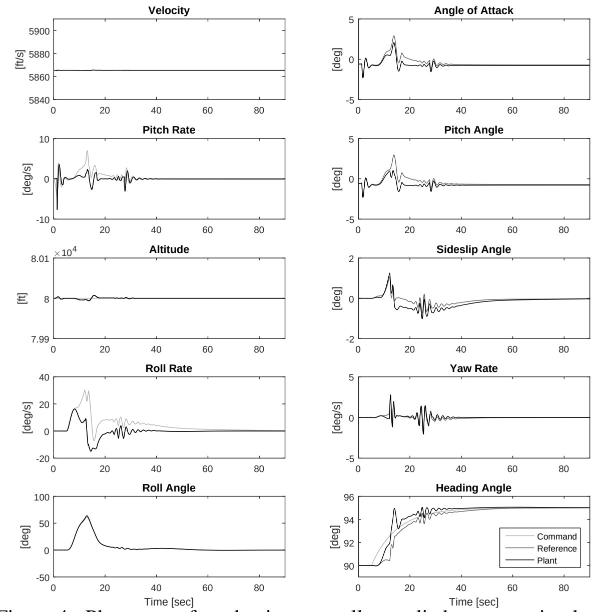

Figure 4. Plant states for adaptive controller applied to uncertain plant in response to a 5 degree heading turn.

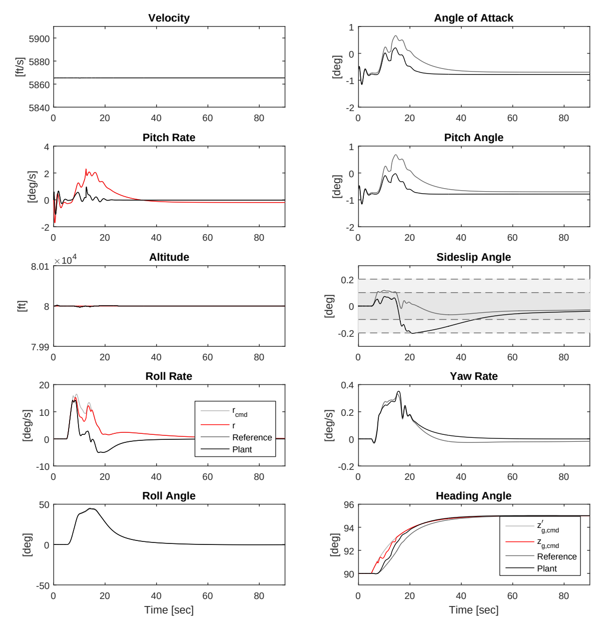

Figure 5. Plant states for adaptive controller applied to uncertain plant in response to a 5 degree heading turn with sideslip angle Limiter.

ACKNOWLEDGMENT

This research is funded by the Universal Technology Corporation (UTC) contract FA8650-10-D-3037 subcontract agreement 15-S2606-04-C19 for Adaptive Flight Control for Hypersonic Vehicles with Integrated Loops.

References

[1] K. J. Åström, "Adaptive feedback control," *Proceedings of the IEEE*, vol. 75, no. 2, pp. 185–217, February 1987.

[2] E. Lavretsky and K. A. Wise, *Robust and Adaptive Control*. Springer, 2013.

[3] M. B. McFarland and A. J. Calise, "Neural-adaptive nonlinear autopilot design for an agile anti-air missile," in *AIAA Guidance, Navigation and Control Conference*. San Diego, CA: AIAA, July 1996, pp. 29–31.

[4] Z. Qu, E. Lavretsky, and A. M. Annaswamy, "An adaptive controller for very flexible aircraft," in *AIAA Guidance, Navigation, and Control (GNC) Conference*. Boston, MA: AIAA, August 2013.

[5] D. P. Wiese, A. M. Annaswamy, J. A. Muse, and M. A. Bolender, "Adaptive control of a generic hypersonic vehicle," in *AIAA Guidance, Navigation, and Control Conference*, AIAA. Boston, MA: American Institute of Aeronautics and Astronautics, August 2013.

[6] D. P. Wiese, A. M. Annaswamy, J. A. Muse, M. A. Bolender, and E. Lavretsky, "Adaptive output feedback based on closed-loop reference models for hypersonic vehicles," in *AIAA Scitech: Guidance, Navigation, and Control Conference*, AIAA. Kissimmee, FL: American Institute of Aeronautics and Astronautics, January 2015.

[7] ——, "Adaptive output feedback based on closed-loop reference models for hypersonic vehicles," *Journal of Guidance, Control, and Dynamics*, vol. 38, no. 12, pp. 2429–2440, 2015.

[8] K. Wise, E. Lavretsky, J. Zimmerman, J. Francis, D. Dixon, and B. Whitehead, "Adaptive flight control of a sensor guided munition," in *AIAA Guidance, Navigation and Control Conference*. San Francisco, CA: AIAA, August 2005.

[9] K. A. Wise, "Robust stability analysis of adaptive missile autopilots," in *AIAA Guidance, Navigation and Control Conference*. Honolulu, HI: AIAA, August 2008, pp. 18–21.

[10] T. E. Gibson and A. M. Annaswamy, "Adaptive control of hypersonic vehicles in the presence of thrust and actuator uncertainties," in *AIAA Guidance, Navigation and Control Conference and Exhibit*. Honolulu, HI: American Institute of Aeronautics and Astronautics, August 2008.

[11] L. Fiorentini, A. Serrani, M. A. Bolender, and D. B. Doman, "Robust nonlinear sequential loop closure control design for an air-breathing hypersonic vehicle model," in *American Control Conference (ACC)*, IEEE. Seattle, WA: Institute of Electrical and Electronics Engineers, June 2008, pp. 3458–3463.

[12] ——, "Nonlinear control of non-minimum phase hypersonic vehicle models," in *American Control Conference (ACC)*, IEEE. St. Louis, MO: Institute of Electrical and Electronics Engineers, June 2009, pp. 3160–3165.

[13] M. Bodson, "Reconfigurable nonlinear autopilot," *Journal of Guidance, Control, and Dynamics*, vol. 26, no. 5, pp. 719–727, 2003.

[14] K. S. Narendra and J. H. Taylor, *Frequency Domain Criteria for Absolute stability*, ser. Electrical science series. Elsevier Science, 1973.

[15] C. Huang, P. A. Ioannou, J. Maroulas, and M. G. Safonov, "Design of strictly positive real systems using constant output feedback," *IEEE Transactions on Automatic Control*, vol. 44, no. 3, pp. 569–573, March 1999.

[16] S. P. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. Studies in Applied Mathematics. Society for Industrial and Applied Mathematics, 1994.

[17] E. Lavretsky, "Adaptive output feedback design using asymptotic properties of LQG/LTR controllers," in *AIAA Guidance, Navigation, and Control Conference*. Toronto, Ontario, Canada: AIAA, August 2010.

[18] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*, ser. Dover Books on Electrical Engineering Series. Dover, 2005.

[19] D. P. Wiese, A. M. Annaswamy, J. A. Muse, M. A. Bolender, and E. Lavretsky, "Sequential loop closure based adaptive autopilot design for a hypersonic vehicle," in *AIAA Scitech: Guidance, Navigation, and Control Conference*, AIAA. San Diego, CA: American Institute of Aeronautics and Astronautics, January 2016.

[20] G. H. Golub and C. F. Van Loan, *Matrix Computations*. Johns Hopkins University Press, Baltimore, MD,, 1996.

[21] H. Dai, "Symmetric positive definite solutions of matrix equations (AX, XB)=(C,D) and AXB=C," *Nanjing University of Aeronautics and Astronautics*, vol. 13, no. 2, pp. 175–179, 1996.

[22] C. C. Paige and M. A. Saunders, "Towards a generalized singular value decomposition," *SIAM Journal on Numerical Analysis*, vol. 18, no. 3, pp. 398–405, 1981.

[23] D. Hua, "On the symmetric solutions of linear matrix equations," *Linear Algebra and its Applications*, vol. 131, pp. 1–7, 1990.

[24] R. Gadient, E. Lavretsky, and D. Hyde, "State limiter for model following control systems," in *AIAA Guidance, Navigation and Control Conference*, AIAA. Portland, OR: American Institute of Aeronautics and Astronautics, August 2011.

[25] E. Lavretsky and R. Gradient, "Robust adaptive design for aerial vehicles with state-limiting constraints," *Journal of Guidance, Control, and Dynamics*, vol. 33, no. 6, pp. 1743–1752, 2013/05/14 2010.

[26] R. M. Sanner and J.-J. E. Slotine, "Gaussian networks for direct adaptive control," *Neural Networks, IEEE Transactions on*, vol. 3, no. 6, pp. 837–863, 1992.

[27] D. Famularo, J. Valasek, J. A. Muse, and M. A. Bolender, "Enforcing state constraints on a model of a hypersonic vehicle," in *AIAA Scitech: Guidance, Navigation, and Control Conference*, AIAA. San Diego, CA: American Institute of Aeronautics and Astronautics, January 2016, p. 1865.

[28] J. A. Muse, "A method for enforcing state constraints in adaptive control," in *AIAA Guidance, Navigation and Control Conference*, AIAA. Portland, OR: American Institute of Aeronautics and Astronautics, August 2011.

[29] E. Rollins, J. Valasek, J. A. Muse, and M. A. Bolender, "Nonlinear adaptive dynamic inversion applied to a generic hypersonic vehicle," in *AIAA Guidance, Navigation, and Control Conference*. American Institute of Aeronautics and Astronautics, 2013.