All Notes

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

NOTES
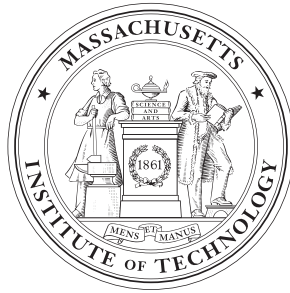
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Contents

List of Symbols

$\underline{p}$ Linear momentum

ρ Density

μ Dynamic, shear, or ordinary viscosity

ν Kinematic viscosity

λ Second or bulk viscosity

$\underline{\tau}$ Newtonian stress tensor (Including normal and shear)

$\underline{\sigma}$ Viscous stress tensor

$\underline{v}_c$ Control volume velocity

$\underline{e}$ Strain rate tensor

Part I

Fluid Mechanics

Chapter 1

Introduction and Preliminaries

These notes were made while in MIT's Fluid Mechanics class, 2.25 taught by Professor Gareth McKinley in the Fall semester of 2013.

1.1 Gibb's and Index Notation

There are two ways to denote math used in vector and tensor calculus. One is *symbolic* or *Gibb's* notation, the other is *index* or *Cartesian* notation. Gibb's notation handles scalars, vectors, and tensors as different types of things, and so we need to be careful when conducting operations between these different things, and keep in mind what operations are allowed, and which aren't. Index notation always operates on the scalar entries within the vectors and tensors, so the problem of what operations are legal or not is simplified.

A free index occurs once and only once in each and every term in an equation, where term means product of multiple quantities. The free index means the equation can be written three times, where the free index is substituted for 1, 2, and 3. The free index can be changed to any letter, as long as it is changed everywhere it appears, and not to a letter which is already being used in an index. The dummy or summation index are those which are not a free index. That is, the dummy indexes are those which occur more than once in every term. Like the the free index, the dummy index can be changed to a different letter, as long as it is changed everywhere, and not to a letter that is already used as an index.

Example 1 Free Index

$$a_k = b_i c_{ik} + d_{ijk} e_{ij}$$

In this equation the free index is k. So this equation can be written out as

$$a_1 = b_i c_{i1} + d_{ij1} e_{ij}$$
$$a_2 = b_i c_{i2} + d_{ij2} e_{ij}$$
$$a_3 = b_i c_{i3} + d_{ij3} e_{ij}$$

The free index can also be changed fro k to n and the equation is the same.

$$a_n = b_i c_{in} + d_{ijn} e_{ij}$$

1.2 Review

A vector $\underline{a}$ and $\underline{b}$ can be written

$$\underline{a} = a_1\hat{\underline{e}}_1 + a_2\hat{\underline{e}}_2 + a_3\hat{\underline{e}}_3$$
$$\underline{b} = b_1\hat{\underline{e}}_1 + b_2\hat{\underline{e}}_2 + b_3\hat{\underline{e}}_3$$

1.2.1 Dot Product

Operation between two vectors which produces a scalar. The dot product of a unit vector with itself is one. The dot product of any perpendicular vectors is zero. When evaluating the dot product of two vectors $\underline{a}$ and $\underline{b}$, write out the two vectors in their component form, and then distribute

$$\begin{aligned}
\underline{a} \cdot \underline{b} &= (a_1\hat{\underline{e}}_1 + a_2\hat{\underline{e}}_2 + a_3\hat{\underline{e}}_3) \cdot (b_1\hat{\underline{e}}_1 + b_2\hat{\underline{e}}_2 + b_3\hat{\underline{e}}_3) \\
&= a_1b_1(\hat{\underline{e}}_1 \cdot \hat{\underline{e}}_1) + a_1b_2(\hat{\underline{e}}_1 \cdot \hat{\underline{e}}_2) + \ldots \\
&= a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}$$

1.2.2 Cross Product

Operation between two vectors that produces a vector perpendicular to the first two.

$$\begin{aligned}
\underline{a} \times \underline{b} &= (a_1\hat{\underline{e}}_1 + a_2\hat{\underline{e}}_2 + a_3\hat{\underline{e}}_3) \times (b_1\hat{\underline{e}}_1 + b_2\hat{\underline{e}}_2 + b_3\hat{\underline{e}}_3) \\
&= a_1b_1(\hat{\underline{e}}_1 \times \hat{\underline{e}}_1) + a_1b_2(\hat{\underline{e}}_1 \times \hat{\underline{e}}_2) + \ldots
\end{aligned}$$

1.3 Index Notation

$$\begin{aligned}
\underline{a} &= a_1\hat{\underline{e}}_1 + a_2\hat{\underline{e}}_2 + a_3\hat{\underline{e}}_3 \\
&= \sum_{i=1}^{3} a_i\hat{\underline{e}}_i
\end{aligned}$$

in index notation the summation is dropped, and the vector $\underline{a}$ is expressed

$$\underline{a} = a_i\hat{\underline{e}}_i$$

1.3.1 Dot Product

$$\begin{aligned}
\underline{a} \cdot \underline{b} &= \left(\sum_{i=1}^{3} a_i\hat{\underline{e}}_i\right) \cdot \left(\sum_{j=1}^{3} b_i\hat{\underline{e}}_i\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (a_i\hat{\underline{e}}_i) \cdot (b_j\hat{\underline{e}}_j) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} a_ib_j(\hat{\underline{e}}_i \cdot \hat{\underline{e}}_j)
\end{aligned}$$

And then we have the quantity in parentheses is zero for $i \neq j$ and one if $i = j$. This gets a special quantity called δ_{ij}. But since $\delta_{ij} = 1$ only when $i = j$ we can simplify the above expression to

$$\underline{a} \cdot \underline{b} = a_i b_j \delta_{ij}$$

$$\boxed{\underline{a} \cdot \underline{b} = a_i b_i}$$

1.3.2   Cross Product

$$\underline{a} \times \underline{b} = \left(\sum_{i=1}^{3} a_i \hat{\underline{e}}_i \right) \times \left(\sum_{j=1}^{3} b_j \hat{\underline{e}}_j \right)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} (a_i \hat{\underline{e}}_i) \times (b_j \hat{\underline{e}}_j)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j (\hat{\underline{e}}_i \times \hat{\underline{e}}_j)$$

$$(\hat{\underline{e}}_i \times \hat{\underline{e}}_j) = \varepsilon_{ijk} \hat{\underline{e}}_k$$

$$\boxed{\underline{a} \times \underline{b} = a_i b_j \varepsilon_{ijk} \hat{\underline{e}}_k}$$

ε_{ijk} is a scalar, with value ± 1 or 0. ε_{ijk} is determined by the values of the first two indices, and the third one is the result of right hand rule cross product on right handed coordinate system. If $i = j$, then $\varepsilon_{ijk} = 0$. Also notice that

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$$

$$\varepsilon_{kji} = \varepsilon_{jik} = \varepsilon_{ikj}$$

Note that when writing out the combined summations for the operations on the vectors that the order of the summations does not matter. Show vector triple products

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c} = (\underline{c} \times \underline{a}) \cdot \underline{b}$$

Evaluate the first quantity using index notation

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \left(\sum_{i=1}^{3} a_i \hat{\underline{e}}_i \right) \cdot \left\{ \left(\sum_{j=1}^{3} b_j \hat{\underline{e}}_j \right) \times \left(\sum_{k=1}^{3} c_k \hat{\underline{e}}_k \right) \right\}$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_i b_j c_k \hat{\underline{e}}_i \cdot (\hat{\underline{e}}_j \times \hat{\underline{e}}_k)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_i b_j c_k \varepsilon_{jkl} \hat{\underline{e}}_i \cdot \hat{\underline{e}}_l$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_i b_j c_k \varepsilon_{jkl} \delta_{il}$$

$$= a_i b_j c_k \varepsilon_{ijk}$$

The second quantity

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = \left\{ \left(\sum_{i=1}^{3} a_i \hat{\underline{e}}_i \right) \times \left(\sum_{j=1}^{3} b_j \hat{\underline{e}}_j \right) \right\} \cdot \left(\sum_{k=1}^{3} c_k \hat{\underline{e}}_k \right)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_i b_j c_k (\hat{\underline{e}}_i \times \hat{\underline{e}}_j) \cdot \hat{\underline{e}}_k$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_i b_j c_k \varepsilon_{ijl} \hat{\underline{e}}_l \cdot \hat{\underline{e}}_k$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_i b_j c_k \varepsilon_{ijl} \delta_{lk}$$

$$= a_i b_j c_k \varepsilon_{ijk}$$

The third quantity

$$(\underline{c} \times \underline{a}) \cdot \underline{b} = \left\{ \left(\sum_{k=1}^{3} c_k \hat{\underline{e}}_k \right) \times \left(\sum_{i=1}^{3} a_i \hat{\underline{e}}_i \right) \right\} \cdot \left(\sum_{j=1}^{3} b_j \hat{\underline{e}}_j \right)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_i b_j c_k (\hat{\underline{e}}_k \times \hat{\underline{e}}_i) \cdot \hat{\underline{e}}_j$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_i b_j c_k \varepsilon_{kil} \hat{\underline{e}}_l \cdot \hat{\underline{e}}_j$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_i b_j c_k \varepsilon_{kil} \delta_{lj}$$

$$= a_i b_j c_k \varepsilon_{ijk}$$

1.3.3 Del Operator

In Cartesian coordinate systems, the del operator is

$$\underline{\nabla} = \frac{\partial}{\partial x_1}\hat{e}_1 + \frac{\partial}{\partial x_2}\hat{e}_2 + \frac{\partial}{\partial x_3}\hat{e}_3$$

In cylindrical coordinate systems the del operator is

$$\underline{\nabla} = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z$$

In spherical

$$\underline{\nabla} = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{e}_\phi$$

Del Operator

Cartesian:
$$\underline{\nabla} = \frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z$$

Cylindrical:
$$\underline{\nabla} = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z$$

Spherical:
$$\underline{\nabla} = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{e}_\phi$$

In index notation

$$\underline{\nabla} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\hat{e}_i$$

$$\boxed{\underline{\nabla} = \frac{\partial}{\partial x_i}\hat{e}_i}$$

1.3.4 Gradient

The gradient is basically a derivative with respect to position.

$$\boxed{\underline{\nabla}\Phi = \frac{\partial \Phi}{\partial x_i}\hat{e}_i}$$

In cylindrical coordinates

$$\underline{\nabla}\Phi = \frac{\partial \Phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\hat{e}_\theta + \frac{\partial \Phi}{\partial z}\hat{e}_z$$

1.3.5 Divergence

$$\boxed{\underline{\nabla} \cdot \underline{v} = \frac{\partial v_i}{\partial x_i}}$$

In cylindrical coordinates

$$\underline{\nabla} \cdot \underline{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

1.3.6 Curl

Curl is the operation of the del operator acting on a vector with the cross product.

$$\boxed{\underline{\nabla} \times \underline{v} = \frac{\partial v_j}{\partial x_i}\varepsilon_{ijk}\hat{\underline{e}}_k}$$

The curl of the velocity field is known as the vorticity, $\underline{\omega}$. In Gibbs notation, the curl, in cylindrical coordinates is given by

$$\underline{\nabla} \times \underline{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\underline{e}}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\underline{e}}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(v_\theta r) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\hat{\underline{e}}_z$$

In cartesian coordinates it is

$$\underline{\nabla} \times \underline{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\underline{e}}_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\underline{e}}_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\underline{e}}_z$$

1.4 Tensors

Dot product of vector with tensor produces a vector.

$$\underline{\underline{T}} = T_{ij}(\hat{\underline{e}}_i\hat{\underline{e}}_j)$$

1.4.1 Symmetric Tensors

$$T_{ij} = T_{ji}$$

1.4.2 Antisymmetric Tensors

$$T_{ij} = -T_{ji}$$

1.4.3 Tensor Products

note the following *IS THIS TRUE?*

$$T_{ij}S_{ji} = T_{ij}S_{ij}$$

1.4.4 Dyadic Product

Pretty much just outer product.

$$\underline{\nabla}\,\underline{v} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

1.4.5 Unit Vector Derivatives
Cartesian Coordinates

$$\underline{\hat{e}}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{\hat{e}}_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{\hat{e}}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From these, we can see that any derivative of a unit vector in a cartesian coordinate system is zero.

$$\frac{\partial \underline{\hat{e}}_x}{\partial x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Cylindrical Coordinates

$$\underline{\hat{e}}_r = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \qquad \underline{\hat{e}}_\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \qquad \underline{\hat{e}}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Taking the derivatives of these unit vectors with respect to the different directions we have

$$\frac{\partial}{\partial r}\underline{\hat{e}}_r = 0 \qquad\qquad \frac{\partial}{\partial r}\underline{\hat{e}}_\theta = 0 \qquad\qquad \frac{\partial}{\partial r}\underline{\hat{e}}_z = 0$$

$$\frac{\partial}{\partial \theta}\underline{\hat{e}}_r = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} = \underline{\hat{e}}_\theta \qquad \frac{\partial}{\partial \theta}\underline{\hat{e}}_\theta = \begin{bmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{bmatrix} = -\underline{\hat{e}}_r \qquad \frac{\partial}{\partial \theta}\underline{\hat{e}}_z = 0$$

$$\frac{\partial}{\partial z}\underline{\hat{e}}_r = 0 \qquad\qquad \frac{\partial}{\partial z}\underline{\hat{e}}_\theta = 0 \qquad\qquad \frac{\partial}{\partial z}\underline{\hat{e}}_z = 0$$

Components:

$$\underline{\nabla}\,\underline{v} = \begin{bmatrix} rr & r\theta & rz \\ \theta r & \theta\theta & \theta z \\ zr & z\theta & zz \end{bmatrix}$$

Chapter 2

Basic Conservation Laws

2.1 Continuity: Conservation of Mass

Conservation of mass is also called the *continuity equation* to emphasize that the continuum assumptions are prerequisites [1]. The continuity equation states that the time rate of change of the mass of a material region is zero. The continuity equation can typically be found in integral and derivative form, and can be derived in several different ways. The integral form of conservation of mass is

Compressible mass conservation (integral): $\qquad \underbrace{\dfrac{\partial}{\partial t}\int_V \rho dV}_{\text{Rate of change of mass}} = \underbrace{-\oint_S \rho \underline{v}\cdot \underline{n}dS}_{\text{Net inflow of mass}}$

2.1.1 Deriving the Differential Form of Continuity

Using Gauss' theorem the right hand side of the integral form of conservation of mass equation can be written

$$\oint_S \rho \underline{v}\cdot \underline{n}dS = \int_V \underline{\nabla}\cdot(\rho\underline{v})dV$$

using this definition the integral form of mass conservation can be written

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int_V \underline{\nabla}\cdot(\rho\underline{v})dV$$

Leibnitz's theorem allows the integral to be moved inside giving

$$\int_V \left(\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{v})\right)dV = 0$$

Then, since the choice of the material region that we chose was arbitrary, the only way this equation can be true is if the integrand is zero, which is the differential form of conservation of mass.

Compressible mass conservation (differential): $\qquad \dfrac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{v}) = 0$

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2.1.2 Conservation of Mass for an Incompressible Fluid

This section provides a first-principles, control volume derivation of conservation of mass for an incompressible fluid. This is an alternative derivation of the differential form of continuity, and is a little bit easier and more intuitive to derive using an infinitesimal control volume, as opposed to starting with the integral form and using Gauss' theorem.
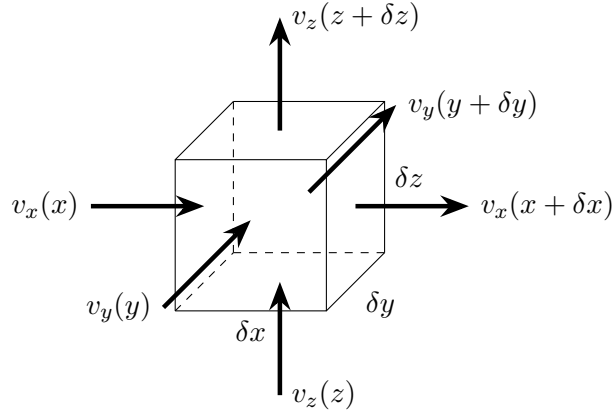
Figure 2.1: Infinitesimal control volume

The box is a fixed volume. Continuity represents the rate of accumulation of mass within the box, minus the net flow out of the box. Conservation of mass is written as

$$\frac{\partial \rho}{\partial t}\delta x \delta y \delta z + \rho v_x(x)\delta y \delta z + \rho v_y(y)\delta x \delta z + \rho v_z(z)\delta x \delta y$$
$$= (\rho + \delta \rho_x)v_x(x + \delta x)\delta y \delta z + (\rho + \delta \rho_y)v_y(y + \delta y)\delta x \delta z + (\rho + \delta \rho_z)v_z(z + \delta z)\delta x \delta y$$

Note now that ρ is a constant and thus all derivatives of ρ are zero, giving

$$v_x(x)\delta y \delta z + v_y(y)\delta x \delta z + v_z(z)\delta x \delta y$$
$$= v_x(x + \delta x)\delta y \delta z + v_y(y + \delta y)\delta x \delta z + v_z(z + \delta z)\delta x \delta y$$

Multiplying each velocity term by unity gives

$$v_x(x)\frac{\delta y \delta z \delta x}{\delta x} + v_y(y)\frac{\delta x \delta z \delta y}{\delta y} + v_z(z)\frac{\delta x \delta y \delta z}{\delta z}$$
$$= v_x(x + \delta x)\frac{\delta y \delta z \delta x}{\delta x} + v_y(y + \delta y)\frac{\delta x \delta z \delta y}{\delta y} + v_z(z + \delta z)\frac{\delta x \delta y \delta z}{\delta z}$$

Dividing both sides by $\delta x \delta y \delta z$ gives

$$\frac{v_x(x)}{\delta x} + \frac{v_y(y)}{\delta y} + \frac{v_z(z)}{\delta z} = \frac{v_x(x + \delta x)}{\delta x} + \frac{v_y(y + \delta y)}{\delta y} + \frac{v_z(z + \delta z)}{\delta z}$$

Combining terms

$$\frac{v_x(x) - v_x(x + \delta x)}{\delta x} + \frac{v_y(y) - v_y(y + \delta y)}{\delta y} + \frac{v_z(z) - v_z(z + \delta z)}{\delta z} = 0$$

Taking the limit as the fluid element gets small

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

In vector form

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} v_x & v_y & v_z \end{bmatrix} = 0$$

And of course this can be written using the del operator as

> **Incompressible mass conservation (differential):** $\qquad \underline{\nabla} \cdot \underline{v} = 0$

2.1.3 Conservation of Mass for a Compressible Fluid

This section provides a control volume derivation of conservation of mass for a compressible fluid. It follows the same derivation as the one above for incompressible fluids, except that the density is not constant and thus has, in general, nonzero derivatives. Furthermore, the result above can be obtained from the one below when density is constant.
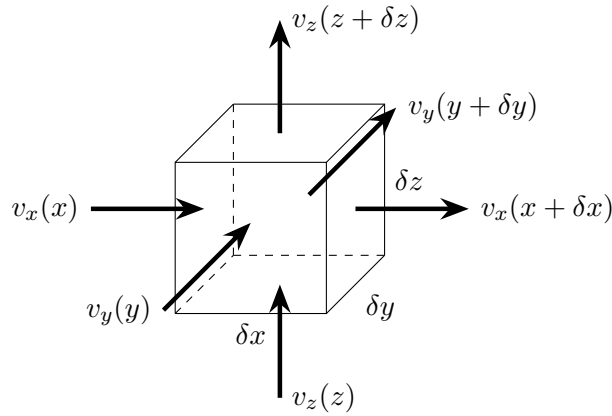
Figure 2.2: Infinitesimal control volume

Conservation of mass

$$\frac{\partial \rho}{\partial t} \delta x \delta y \delta z + \rho v_x(x) \delta y \delta z + \rho v_y(y) \delta x \delta z + \rho v_z(z) \delta x \delta y$$
$$= (\rho + \delta \rho_x) v_x(x + \delta x) \delta y \delta z + (\rho + \delta \rho_y) v_y(y + \delta y) \delta x \delta z + (\rho + \delta \rho_z) v_z(z + \delta z) \delta x \delta y$$

Looking at the terms on the right hand side with

$$\delta\rho_x = \frac{\partial\rho}{\partial x}\delta x \qquad \delta\rho_y = \frac{\partial\rho}{\partial y}\delta y \qquad \delta\rho_z = \frac{\partial\rho}{\partial z}\delta z$$

and using a first order Taylor series approximation on the following

$$v_x(x + \delta x) = v_x(x) + \frac{\partial v_x}{\partial x}\delta x$$

$$v_y(y + \delta y) = v_y(y) + \frac{\partial v_y}{\partial y}\delta y$$

$$v_z(z + \delta z) = v_z(z) + \frac{\partial v_z}{\partial z}\delta z$$

we get

$$\rho v_x(x)\delta y\delta z + \rho v_y(y)\delta x\delta z + \rho v_z(z)\delta x\delta y = \left(\rho + \frac{\partial\rho}{\partial x}\delta x\right)\left(v_x(x) + \frac{\partial v_x}{\partial x}\delta x\right)\delta y\delta z$$

$$+ \left(\rho + \frac{\partial\rho}{\partial y}\delta y\right)\left(v_y(y) + \frac{\partial v_y}{\partial y}\delta y\right)\delta x\delta z$$

$$+ \left(\rho + \frac{\partial\rho}{\partial z}\delta z\right)\left(v_z(z) + \frac{\partial v_z}{\partial z}\delta z\right)\delta x\delta y$$

Multiplying out the left hand side and neglecting second order terms

$$\rho v_x(x)\delta y\delta z = \left(\rho v_x(x) + \rho\frac{\partial v_x}{\partial x}\delta x + v_x(x)\frac{\partial\rho}{\partial x}\delta x\right)\delta y\delta z$$

$$\rho v_y(y)\delta x\delta z = \left(\rho v_y(y) + \rho\frac{\partial v_y}{\partial y}\delta y + v_y(y)\frac{\partial\rho}{\partial y}\delta y\right)\delta x\delta z$$

$$\rho v_z(z)\delta x\delta y = \left(\rho v_z(z) + \rho\frac{\partial v_z}{\partial z}\delta z + v_z(z)\frac{\partial\rho}{\partial z}\delta z\right)\delta x\delta y$$

simplifying

$$0 = \left(\rho\frac{\partial v_x}{\partial x}\delta x + v_x(x)\frac{\partial\rho}{\partial x}\delta x\right)\delta y\delta z$$

$$0 = \left(\rho\frac{\partial v_y}{\partial y}\delta y + v_y(y)\frac{\partial\rho}{\partial y}\delta y\right)\delta x\delta z$$

$$0 = \left(\rho\frac{\partial v_z}{\partial z}\delta z + v_z(z)\frac{\partial\rho}{\partial z}\delta z\right)\delta x\delta y$$

Recognizing these quantities as from product rule, and combining the components back onto a single equation

$$0 = \frac{\partial(\rho v_x)}{\partial x}\delta x + \frac{\partial(\rho v_y)}{\partial y}\delta y + \frac{\partial(\rho v_z)}{\partial z}\delta z$$

$$0 = \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}$$

In vector form

$$\frac{\partial \rho}{\partial t} + \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \rho v_x & \rho v_y & \rho v_z \end{bmatrix} = 0$$

And again this can be written as follows using the del operator.

> **Compressible mass conservation (differential):** $\qquad \dfrac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$

And we can see that if ρ is a constant, this expression reduces to conservation of mass for an incompressible fluid.

2.2 Conservation of Momentum

2.2.1 Cauchy Momentum Equation

This is the most basic form, where the surface and pressure forces are very general. Fluid doesn't have to be Newtonian, or compressible, etc.

2.2.2 Conservation of Momentum with Euler's Equation

This section is about conservation of momentum for an incompressible fluid with pressure and gravity, deriving Euler's Equation.

Conservation of momentum when only surface force is pressure and only body force is gravity. Consider a differential element of fluid, and consider only pressure and gravity forces acting upon it. We also assume the density is constant. To derive Euler's equation, by assuming that these are the only forces acting on the fluid, this version of Euler's equation is only valid for inviscid, incompressible flows.
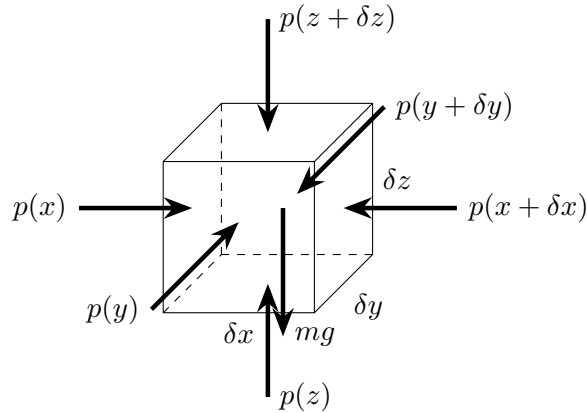
Figure 2.3: Fluid element with pressure and gravity acting on it

Summing the forces in the x, y, and z directions we have

$$p\delta y\delta z - \left(p + \frac{\partial p}{\partial x}\right)\delta y\delta z = \rho\delta x\delta y\delta z a_x$$

$$p\delta x\delta z - \left(p + \frac{\partial p}{\partial y}\right)\delta x\delta z = \rho\delta x\delta y\delta z a_y$$

$$p\delta x\delta y - \left(p + \frac{\partial p}{\partial z}\right)\delta x\delta y - \rho g_z\delta x\delta y\delta z = \rho\delta x\delta y\delta z a_z$$

simplifying

$$-\frac{\partial p}{\partial x} = \rho a_x$$

$$-\frac{\partial p}{\partial y} = \rho a_y$$

$$-\frac{\partial p}{\partial z} - \rho g_z = \rho a_z$$

we obtain

$$\boxed{\rho\underline{a} = -\underline{\nabla}p + \rho\underline{g}}$$

Substituting the following

$$\underline{a} = \frac{D\underline{v}}{Dt}$$

$$= \frac{\partial\underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}$$

into above to obtain

Euler's Equation: $\quad\boxed{\rho\left(\frac{\partial\underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) = -\underline{\nabla}p + \rho\underline{g}}$ $\qquad$ (2.1)

The assumptions of Euler's equation are: inviscid flow (neglects stress tensor $\underline{\tau}$). For inviscid flow, if it is irrotational at any instant in time, it remains irrotational for all subsequent time.

2.2.3 Conservation of Momentum including Viscous Forces

The above derivation can be repeated including surface forces (but still assuming density is constant), resulting in the following.

$$\boxed{\rho\left(\frac{\partial\underline{v}}{\partial t} + \underline{\nabla}\cdot(\underline{vv})\right) = -\underline{\nabla}p + \underline{\nabla}\cdot\tau + \rho\underline{g}}$$

Chapter 3

Material Derivative

Local change plus convective change, where $\underline{f}$ can be a vector or scalar quantity. The left hand side of the equation is the Lagrangian side: it is following a particular material element through the flow, and the right hand side is the Eulerian side.

$$\textbf{Material Derivative:} \qquad \frac{D\underline{f}}{Dt} = \underbrace{\frac{\partial \underline{f}}{\partial t}}_{\text{local rate of change}} + \underbrace{\underline{f} \cdot \nabla \underline{f}}_{\text{convective change}}$$

3.1 Hydrostatics

Note that in hydrostatic problems the fluid is at rest, and so viscosity can be ignored, we set $\underline{a} = 0$ in Euler's equation to obtain the following

$$\textbf{Hydrostatic Equation:} \qquad \nabla p = \rho \underline{g}$$

In a lot of cases, pressure variations occur only in the vertical direction due to gravity. In these cases we have

$$\nabla p = \rho \underline{g}$$
$$\begin{bmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \end{bmatrix} = \rho \begin{bmatrix} g_x & g_y & g_z \end{bmatrix}$$

If we have a coordinate system where the z axis points upwards and gravity points downwards, the z component of gravity is $g_z = -g$ where g is gravitational acceleration constant, e.g. 9.81 m/s^2 on Earth. This gives

$$\frac{\partial p}{\partial z} = -\rho g$$

Separating and integrating back

$$\int \partial p = -\int \rho g \partial z$$
$$p_2 - p_1 = -\rho g(z_2 - z_1)$$

3.1.1 Force and Moment

To find force and moment about a point on say, a door or floodgate holding back hydrostatic fluid

$$F = \int_A p(z)dA$$

$$\tau = \int_A rp(z)dA$$

where r is the distance from the point about which we are taking moments along the door or floodgate.

3.2 Motion of a Fluid Element Along a Streamline

3.2.1 Streamline Coordinates

The *osnl* coordinate is orthogonal but not always Cartesian. For example for the rigid body problem streamlines are circles and the *osnl* coordinate system becomes similar to cylindrical (e_s is e_θ and e_n is e_r and e_l is e_z) whereas in other arbitrary flows it may be something else. What people call these is a subset of "curvilinear coordiante systems" See Wikipedia.

Assume constant density? Think about looking at Euler's equation along a streamline in order to simplify it. This will allow the velocity components to be simplified, since by definition the velocity vector along a streamline is always tangent to the streamline. Recall Euler's equation in (2.1).

Start generally though with with the velocity vector given by the following components

$$\underline{v} = v_s\underline{\hat{e}}_s + v_n\,\underline{\hat{e}}_n + v_l\,\underline{\hat{e}}_l = \begin{bmatrix} v_s \\ v_n \\ v_l \end{bmatrix}$$

Evaluating Euler's equation starting with the term $\underline{v} \cdot \nabla$

$$\underline{v} \cdot \underline{\nabla} = \begin{bmatrix} v_s \\ v_n \\ v_l \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial n} \\ \frac{\partial}{\partial l} \end{bmatrix} = v_s\frac{\partial}{\partial s} + v_n\frac{\partial}{\partial n} + v_l\frac{\partial}{\partial l}$$

Now evaluating $(\underline{v} \cdot \underline{\nabla})\underline{v}$

$$\begin{aligned}
(\underline{v} \cdot \underline{\nabla})\underline{v} &= \left(v_s\frac{\partial}{\partial s} + v_n\frac{\partial}{\partial n} + v_l\frac{\partial}{\partial l} \right)\underline{v} \\
&= v_s\frac{\partial}{\partial s}\underline{v} + v_n\frac{\partial}{\partial n}\underline{v} + v_l\frac{\partial}{\partial l}\underline{v} \\
&= v_s\frac{\partial}{\partial s}(v_s\underline{\hat{e}}_s + v_n\,\underline{\hat{e}}_n + v_l\,\underline{\hat{e}}_l) \\
&\quad + v_n\frac{\partial}{\partial n}(v_s\underline{\hat{e}}_s + v_n\,\underline{\hat{e}}_n + v_l\,\underline{\hat{e}}_l) \\
&\quad + v_l\frac{\partial}{\partial l}(v_s\underline{\hat{e}}_s + v_n\,\underline{\hat{e}}_n + v_l\,\underline{\hat{e}}_l) \\
&= v_s\frac{\partial}{\partial s}(v_s\underline{\hat{e}}_s) + v_n\frac{\partial}{\partial n}(v_s\underline{\hat{e}}_s) + v_l\frac{\partial}{\partial l}(v_s\underline{\hat{e}}_s) \\
&\quad + v_s\frac{\partial}{\partial s}(v_n\underline{\hat{e}}_n) + v_n\frac{\partial}{\partial n}(v_n\underline{\hat{e}}_n) + v_l\frac{\partial}{\partial l}(v_n\underline{\hat{e}}_n) \\
&\quad + v_s\frac{\partial}{\partial s}(v_l\underline{\hat{e}}_l) + v_n\frac{\partial}{\partial n}(v_l\underline{\hat{e}}_l) + v_l\frac{\partial}{\partial l}(v_l\underline{\hat{e}}_l)
\end{aligned}$$

Now using the following simplification since we are considering flow along a streamline we have $v_n = v_l = 0$. Using this, we simplify the above to

$$(\underline{v} \cdot \underline{\nabla})\underline{v} = v_s \frac{\partial}{\partial s}(v_s \underline{\hat{e}}_s)$$

$$= v_s \left(\frac{\partial}{\partial s}(v_s)\underline{\hat{e}}_s + v_s \frac{\partial}{\partial s}(\underline{\hat{e}}_s) \right)$$

Since $\underline{\hat{e}}_s$ has unit magnitude, it can change only in direction. This change must be perpendicular to $\underline{\hat{e}}_s$ itself. Therefore $\underline{\hat{e}}_n$ is defined by

$$\underline{\hat{e}}_n = -R \frac{\partial \underline{\hat{e}}_s}{\partial s}$$

giving

$$\frac{\partial \underline{\hat{e}}_s}{\partial s} = -\frac{1}{R}\underline{\hat{e}}_n$$

$$(\underline{v} \cdot \underline{\nabla})\underline{v} = v_s \frac{\partial v_s}{\partial s}\underline{\hat{e}}_s - \frac{v_s^2}{R}\underline{\hat{e}}_n$$

3.2.2 Motion Tangent to the Streamline

We can see in this expression that there is a component of acceleration tangential to the streamline, as well as the centripetal component of acceleration. If we neglect the centripetal acceleration, we have

$$\boxed{(\underline{v} \cdot \underline{\nabla})\underline{v} = v_s \frac{\partial v_s}{\partial s}\underline{\hat{e}}_s}$$

The next term we are looking at is $\frac{\partial \underline{v}}{\partial t}$ at a fixed point in space, and we are considering steady flow here, so

$$\boxed{\frac{\partial \underline{v}}{\partial t} = 0}$$

Looking at the terms on the right hand side we have

$$\underline{\nabla}p = \frac{\partial p}{\partial s}\underline{\hat{e}}_s + \frac{\partial p}{\partial n}\underline{\hat{e}}_n + \frac{\partial p}{\partial l}\underline{\hat{e}}_l$$

And if we consider only pressure variations along the streamline, this expression is simplified to

$$\boxed{\underline{\nabla}p = \frac{\partial p}{\partial s}\underline{\hat{e}}_s}$$

With $\underline{g}$ a vector pointing down, and considering only the component of gravity along the streamline g_s we have

$$\boxed{\rho \underline{g} = \rho g_s \underline{\hat{e}}_s}$$

And sothe total simplified Euler equation along a streamline as

$$\rho v_s \frac{\partial v_s}{\partial s}\underline{\hat{e}}_s = -\frac{\partial p}{\partial s}\underline{\hat{e}}_s + \rho g_s \underline{\hat{e}}_s$$

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Dropping the unit vector showing its along the streamline

$$\rho v_s \frac{\partial v_s}{\partial s} = -\frac{\partial p}{\partial s} + \rho g_s$$

but we can express g_s as

$$g_s = -g\frac{dz}{ds}$$

giving

$$\rho v_s \frac{\partial v_s}{\partial s} = -\frac{\partial p}{\partial s} - \rho g\frac{dz}{ds}$$

moving stuff over

$$\rho v_s \frac{\partial v_s}{\partial s} + \frac{\partial p}{\partial s} + \rho g\frac{dz}{ds} = 0$$

$$\rho v_s \partial v_s + \partial p + \rho g dz = 0$$

3.2.3   Bernoulli's Equation

integrating, but only in the s-direction

$$\int_{s_1}^{s_2} \left(\rho v_s \frac{\partial v_s}{\partial s} + \frac{\partial p}{\partial s} + \rho g\frac{dz}{ds} \right) = 0$$

$$\int_{s_1}^{s_2} \left(\rho v_s \partial v_s + \partial p + \rho g dz \right) = 0$$

$$\left. \frac{1}{2}\rho v_s^2(s) + p(s) + \rho g z(s) \right|_{s_1}^{s_2} = 0$$

$$\left(\frac{1}{2}\rho v_s^2(s_2) + p(s_2) + \rho g z(s_2) \right) - \left(\frac{1}{2}\rho v_s^2(s_1) + p(s_1) + \rho g z(s_1) \right) = 0$$

finally

Bernoulli's along streamline: $\qquad \dfrac{1}{2}\rho v_{s2}^2 + p_2 + \rho g z_2 = \dfrac{1}{2}\rho v_{s1}^2 + p_1 + \rho g z_1$

Bernoulli's equation is for steady incompressible flow of a fluid in the absence of viscous effects along a streamline.

3.2.4 Motion Normal to the Streamline

$$(\underline{v} \cdot \underline{\nabla})\underline{v} = \frac{v_s^2}{R}\hat{\underline{e}}_n$$

The next term we are looking at is $\frac{\partial v}{\partial t}$ at a fixed point in space, and we are considering steady flow here, so

$$\frac{\partial \underline{v}}{\partial t} = 0$$

Looking at the terms on the right hand side we have

$$\underline{\nabla}p = \frac{\partial p}{\partial s}\hat{\underline{e}}_s + \frac{\partial p}{\partial n}\hat{\underline{e}}_n + \frac{\partial p}{\partial l}\hat{\underline{e}}_l$$

And if we consider only pressure variations normal to the streamline, this expression is simplified to

$$\underline{\nabla}p = \frac{\partial p}{\partial n}\hat{\underline{e}}_n$$

With g a vector pointing down, and considering only the component of gravity along the streamline g_s we have

$$\rho \underline{g} = \rho g_n \hat{\underline{e}}_n$$

And sothe total simplified Euler equation along a streamline as

$$-\rho \frac{v_s^2}{R}\hat{\underline{e}}_n = -\frac{\partial p}{\partial n}\hat{\underline{e}}_n + \rho g_n \hat{\underline{e}}_n$$

> **Bernoulli's equation normal to streamline:** $\qquad -\rho \frac{v_s^2}{R} = -\frac{\partial p}{\partial n} + \rho g_n$

From this expression we can see that the change in pressure with respect to the normal direction is always positive with respect to R, so pressure increases in the n direction.

3.3 Solid Body Rotation

Rotating bucket of water. The velocity is dependent only on the radial direction r. This cylindrical coordinate system is inertially fixed. Bucket spinning with angular velocity Ω.
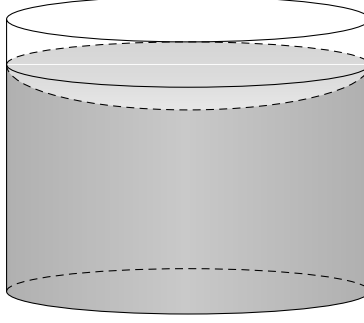
Figure 3.1: Solid body rotation

Thus, the total velocity in cylindrical coordinates is

$$\underline{v} = v_r \hat{\underline{e}}_r + v_\theta \hat{\underline{e}}_\theta + v_z \hat{\underline{e}}_z = \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} = \begin{bmatrix} 0 \\ r\Omega \\ 0 \end{bmatrix}$$

Apply Euler's equation

$$\rho \underline{a} = -\underline{\nabla} p + \rho \underline{g}$$

$$\underline{a} = \frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \, \underline{v}$$

3.3.1 Cylindrical Coordinates

The partial derivative with respect to time, also known as the local rate of change, is zero, because at a fixed point in the fluid, the velocity is not changing with time.

$$\frac{\partial \underline{v}}{\partial t} = 0$$

Looking at the next term

$$\underline{\nabla} = \frac{\partial}{\partial r} \hat{\underline{e}}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\underline{e}}_\theta + \frac{\partial}{\partial z} \hat{\underline{e}}_z$$

Now evaluating $(\underline{v} \cdot \underline{\nabla})\underline{v}$

$$(\underline{v} \cdot \underline{\nabla})\underline{v} = \left(v_r \frac{\partial}{\partial r} + \frac{1}{r} v_\theta \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \right) \underline{v}$$

$$= v_r \frac{\partial}{\partial r} \underline{v} + \frac{1}{r} v_\theta \frac{\partial}{\partial \theta} \underline{v} + v_z \frac{\partial}{\partial z} \underline{v}$$

$$= v_r \frac{\partial}{\partial r} (v_r \hat{\underline{e}}_r + v_\theta \, \hat{\underline{e}}_\theta + v_z \, \hat{\underline{e}}_z)$$

$$+ \frac{1}{r} v_\theta \frac{\partial}{\partial \theta} (v_r \hat{\underline{e}}_r + v_\theta \, \hat{\underline{e}}_\theta + v_z \, \hat{\underline{e}}_z)$$

$$+ v_z \frac{\partial}{\partial z} (v_r \hat{\underline{e}}_r + v_\theta \, \hat{\underline{e}}_\theta + v_z \, \hat{\underline{e}}_z)$$

Now substitute in that $v_r = v_z = 0$ and get

$$
\begin{aligned}
(\underline{v} \cdot \underline{\nabla})\, \underline{v} &= \frac{1}{r} v_\theta \frac{\partial}{\partial \theta}(v_\theta \hat{\underline{e}}_\theta) \\
&= \frac{1}{r} v_\theta \frac{\partial v_\theta}{\partial \theta} \hat{\underline{e}}_\theta + \frac{1}{r} v_\theta^2 \frac{\partial \hat{\underline{e}}_\theta}{\partial \theta} \\
&= 0 + \frac{1}{r} v_\theta^2 \frac{\partial \hat{\underline{e}}_\theta}{\partial \theta} \\
&= -\frac{v_\theta^2}{r} \hat{\underline{e}}_r
\end{aligned}
$$

since

$$
\frac{\partial \hat{\underline{e}}_\theta}{\partial \theta} = -\hat{\underline{e}}_r
$$

And note that we have

$$
\underline{a} = \frac{D\underline{v}}{Dt} = -\frac{v_\theta^2}{r} \hat{\underline{e}}_r
$$

So we can see this as centripetal acceleration. The surface force term

$$
\nabla p = \frac{\partial p}{\partial r} \hat{\underline{e}}_r + \frac{1}{r} \frac{\partial p}{\partial \theta} \hat{\underline{e}}_\theta + \frac{\partial p}{\partial z} \hat{\underline{e}}_z
$$

body force term

$$
\rho \underline{g} = -\rho g_z \hat{\underline{e}}_z
$$

Forming the whole equation we have

$$
-\rho \frac{v_\theta^2}{r} \hat{\underline{e}}_r = -\frac{\partial p}{\partial r} \hat{\underline{e}}_r - \frac{1}{r} \frac{\partial p}{\partial \theta} \hat{\underline{e}}_\theta - \frac{\partial p}{\partial z} \hat{\underline{e}}_z - \rho g_z \hat{\underline{e}}_z
$$

Equating the components we have

$$
-\rho \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} \qquad 0 = -\frac{1}{r} \frac{\partial p}{\partial \theta} \qquad 0 = -\frac{\partial p}{\partial z} - \rho g_z
$$

This gives

$$
\frac{\partial p}{\partial r} = \rho r \Omega^2 \qquad \frac{\partial p}{\partial \theta} = 0 \qquad \frac{\partial p}{\partial z} = -\rho g_z
$$

separating and integrating these back

$$
\int \partial p = \int \rho r \Omega^2 \partial r \qquad \frac{\partial p}{\partial \theta} = 0 \qquad \int \partial p = -\int \rho g_z \partial z
$$

The second equation shows that the pressure is not a function of θ. Integrating back the first and third equations we have the following

$$p = \frac{1}{2}\rho r^2 \Omega^2 + f(z) + c$$

$$p = -\rho g_z z + g(r) + c$$

and we have

$$f(z) = -\rho g_z z$$

$$g(r) = \frac{1}{2}\rho r^2 \Omega^2$$

and so the pressure in the rotating cylinder is given by

$$p(r, z) = \frac{1}{2}\rho r^2 \Omega^2 - \rho g_z z + c$$

And we can find the constant c by setting $r = 0$ and looking at the interface height at the center of the cylinder, at the vertex of the parabaloid shape, and call this height z_0. At this point, the pressure is equal to atmospheric pressure.

$$p(r = 0, z_0) = -\rho g_z z_0 + c = p_{\text{atm}}$$

giving the following value of the constant

$$c = \rho g_z z_0 + p_{\text{atm}}$$

substituting this into the pressure equation we obtain the following

Pressure for solid body rotation: $\qquad p(r, z) = \frac{1}{2}\rho r^2 \Omega^2 + \rho g_z(z_0 - z) + p_{\text{atm}}$

To determine the shape that the air-water interface makes, we apply the boundary condition at this interface. That is

$$p(r, z)\big|_{\text{interface}} = p_{\text{atm}}$$

And then we have

$$0 = \frac{1}{2}\rho r_{\text{interface}}^2 \Omega^2 - \rho g_z z_{\text{interface}} + \rho g_z z_0$$

$$z_{\text{interface}} = \frac{\Omega^2}{2g_z} r_{\text{interface}}^2 + z_0$$

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Isobars

Can also find out the shape of the fluid by solving for isobars by evaluating dp and setting it equal to zero. Since $p = p(r, z)$ chain rule gives us the following for dp.

$$dp = \left(\frac{\partial p}{\partial r}\right) dr + \left(\frac{\partial p}{\partial z}\right) dz$$

$$dp = \rho r \Omega^2 dr - \rho g_z dz = 0$$

$$g_z dz = r \Omega^2 dr$$

Integrating back and solving for the constant of integration when $r = 0$ we have

$$z = \frac{\Omega^2}{2 g_z} r^2 + h_0$$

Vorticity

The vorticity of a fluid element in our bucket is given by the curl of the velocity vector. That is,

$$\underline{\omega} = \underline{\nabla} \times \underline{v}$$
$$= \left(\hat{\underline{e}}_r \frac{\partial}{\partial r} + \hat{\underline{e}}_\theta \frac{1}{r} \frac{\partial}{\partial r} + \hat{\underline{e}}_z \frac{\partial}{\partial z} \right) \times (0 \hat{\underline{e}}_r + r \Omega \hat{\underline{e}}_\theta + 0 \hat{\underline{e}}_z)$$
$$= \hat{\underline{e}}_r \times \frac{\partial}{\partial r}(r\Omega \hat{\underline{e}}_\theta) + \hat{\underline{e}}_\theta \frac{1}{r} \times \frac{\partial}{\partial \theta}(r\Omega \hat{\underline{e}}_\theta) + \hat{\underline{e}}_z \times \frac{\partial}{\partial z}(r\Omega \hat{\underline{e}}_\theta)$$
$$= \hat{\underline{e}}_r \times (\Omega \hat{\underline{e}}_\theta) + \hat{\underline{e}}_\theta \frac{1}{r} \times (-r\Omega \hat{\underline{e}}_r)$$
$$= 2\Omega \hat{\underline{e}}_z$$

And from this expression we can see that the vorticity everywhere in the fluid is the same.

3.3.2 Cartesian Coordinates

Velocity is given in cylindrical coordinates, so we convert it to cartesian coordinates, using an inertially fixed coordinate system

$$\underline{v} = \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} = \begin{bmatrix} 0 \\ r\omega \\ 0 \end{bmatrix}$$

$$x = r \cos \theta$$
$$y = r \sin \theta$$

$$v_x = -v \sin \theta$$
$$v_y = v \cos \theta$$

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$$\underline{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} -\omega y \\ \omega x \\ 0 \end{bmatrix}$$

$$\frac{\partial \underline{v}}{\partial t} = 0$$

Looking at the next term

$$(\underline{v} \cdot \underline{\nabla}) \, \underline{v} =$$

$$\rho v_\theta \frac{\partial}{\partial \theta} (v_\theta \hat{\underline{e}}_\theta) = -\frac{dp}{dz} \hat{\underline{e}}_z - \rho g_z \hat{\underline{e}}_z$$

$$\underline{\nabla} \, \underline{v} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} v_x & v_y \end{bmatrix} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} \end{bmatrix}$$

and now

$$\underline{v} \cdot \underline{\nabla} \, \underline{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} \end{bmatrix}$$

3.4   Ideal, Irrotational, or Free Vortex

An ideal, or free vortex is one in which the flow is irrotational.

3.4.1   Derivation

Deriving the shape of an ideal vortex. We will use this assumption to first derive the velocity distribution of the ideal vortex, and then show the shape that the free surface makes with the air is a hyperboloid.
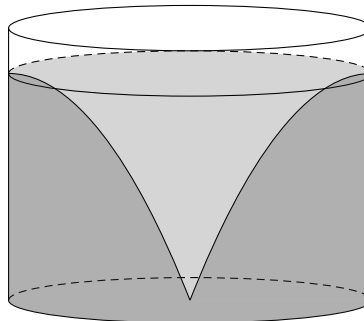
Figure 3.2: Ideal vortex

Derivation of Velocity Profile

The irrotational vortex has zero vorticity ω, so all of the components of $\omega = \underline{\nabla} \times \underline{v}$ must be zero. From the definition for curl in cylindrical coordinates, these components are the following

$$\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} = 0$$

$$\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(v_\theta r) - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = 0$$

The vortex is azimuthally symmetric, so all the θ derivatives are zero, that is $\frac{\partial}{\partial \theta} = 0$, allowing the equations above to simplify to

$$\frac{\partial v_\theta}{\partial z} = 0$$

$$\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0$$

$$\frac{\partial}{\partial r}(v_\theta r) = 0$$

From the fact that the vortex is azimuthally symmetric, no properties are functions of θ. Furthermore, from the first equation we can see that v_θ is not a function of z. That is

$$v_\theta = v_\theta(r)$$

From the third equation, applying product rule we have

$$r\frac{\partial v_\theta}{\partial r} + v_\theta = 0$$

$$-\int \frac{\partial r}{r} = \int \frac{\partial v_\theta}{v_\theta}$$

$$-\ln(r) = \ln(v_\theta) + c_1$$

$$\ln(v_\theta) = \ln(c) - \ln(r)$$

$$\ln(v_\theta) = \ln\left(\frac{c}{r}\right)$$

$$v_\theta = \frac{c}{r}$$

Derivation of Pressure Distribution

Now write down Euler's equation in each direction. The partial derivative with respect to time, also known as the local rate of change, is zero, because at a fixed point in the fluid, the velocity is not changing with time.

$$\frac{\partial \underline{v}}{\partial t} = 0$$

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Looking at the next term

$$\underline{\nabla} = \frac{\partial}{\partial r}\hat{\underline{e}}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\underline{e}}_\theta + \frac{\partial}{\partial z}\hat{\underline{e}}_z$$

Now evaluating $(\underline{v} \cdot \underline{\nabla})\underline{v}$

$$
\begin{aligned}
(\underline{v} \cdot \underline{\nabla})\underline{v} &= \left(v_r \frac{\partial}{\partial r} + \frac{1}{r}v_\theta \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \right) \underline{v} \\
&= v_r \frac{\partial}{\partial r}\underline{v} + \frac{1}{r}v_\theta \frac{\partial}{\partial \theta}\underline{v} + v_z \frac{\partial}{\partial z}\underline{v} \\
&= v_r \frac{\partial}{\partial r}(v_r \hat{\underline{e}}_r + v_\theta\ \hat{\underline{e}}_\theta + v_z\ \hat{\underline{e}}_z) + \frac{1}{r}v_\theta \frac{\partial}{\partial \theta}(v_r \hat{\underline{e}}_r + v_\theta\ \hat{\underline{e}}_\theta + v_z\ \hat{\underline{e}}_z) + v_z \frac{\partial}{\partial z}(v_r \hat{\underline{e}}_r + v_\theta\ \hat{\underline{e}}_\theta + v_z\ \hat{\underline{e}}_z)
\end{aligned}
$$

Now substitute in that $v_r = v_z = 0$ and get

$$
\begin{aligned}
(\underline{v} \cdot \underline{\nabla})\,\underline{v} &= \frac{1}{r}v_\theta \frac{\partial}{\partial \theta}(v_\theta \hat{\underline{e}}_\theta) \\
&= \frac{1}{r}v_\theta \frac{\partial v_\theta}{\partial \theta}\hat{\underline{e}}_\theta + \frac{1}{r}v_\theta^2 \frac{\partial \hat{\underline{e}}_\theta}{\partial \theta} \\
&= 0 + \frac{1}{r}v_\theta^2 \frac{\partial \hat{\underline{e}}_\theta}{\partial \theta} \\
&= -\frac{v_\theta^2}{r}\hat{\underline{e}}_r
\end{aligned}
$$

The surface force term

$$\nabla p = \frac{\partial p}{\partial r}\hat{\underline{e}}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{\underline{e}}_\theta + \frac{\partial p}{\partial z}\hat{\underline{e}}_z$$

body force term

$$\rho \underline{g} = -\rho g_z \hat{\underline{e}}_z$$

Forming the whole equation we have

$$-\rho \frac{v_\theta^2}{r}\hat{\underline{e}}_r = -\frac{\partial p}{\partial r}\hat{\underline{e}}_r - \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{\underline{e}}_\theta - \frac{\partial p}{\partial z}\hat{\underline{e}}_z - \rho g_z \hat{\underline{e}}_z$$

Equating the components we have

$$-\rho \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} \qquad\qquad 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} \qquad\qquad 0 = -\frac{\partial p}{\partial z} - \rho g_z$$

This gives

$$\frac{\partial p}{\partial r} = \frac{\rho c^2}{r^3} \qquad\qquad \frac{\partial p}{\partial \theta} = 0 \qquad\qquad \frac{\partial p}{\partial z} = -\rho g_z$$

separating and integrating these back

$$\int \partial p = \int \frac{\rho c^2}{r^3}\partial r \qquad\qquad \frac{\partial p}{\partial \theta} = 0 \qquad\qquad \int \partial p = -\int \rho g_z \partial z$$

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The second equation shows that the pressure is not a function of θ. Integrating back the first and third equations we have the following

$$p = -\frac{\rho c^2}{2r^2} + f(z) + k$$
$$p = -\rho g_z z + g(r) + k$$

and we have

$$f(z) = -\rho g_z z$$
$$g(r) = -\frac{\rho c^2}{2r^2}$$

and so the pressure in the rotating cylinder is given by

$$p(r, z) = -\frac{\rho c^2}{2r^2} - \rho g_z z + k$$

From this equation for the pressure in the vortex, we can see that when pressure is a constant, i.e. on an isobar, that $z \approx -\frac{1}{r^2}$. We can solve for the constant of integration c by saying when $r = R$, the radius of the tank, p_{atm} is achieved at a height h_0. Plugging this in

$$p_{atm} = -\frac{\rho c^2}{2R^2} - \rho g_z h_0 + k$$

$$k = p_{atm} + \frac{\rho c^2}{2R^2} + \rho g_z h_0$$

The final equation is

> **Pressure in ideal vortex:** $\qquad p(r, z) = \frac{\rho c^2}{2} \left(\frac{1}{R^2} - \frac{1}{r^2} \right) + p_{atm} + \rho g_z (h_0 - z)$

To determine the shape that the air-water interface makes, we apply the boundary condition at this interface. That is

$$p(r, z)\big|_{\text{interface}} = p_{atm}$$

And then we have

$$0 = \frac{\rho c^2}{2} \left(\frac{1}{R^2} - \frac{1}{r_{\text{interface}}^2} \right) + \rho g_z (h_0 - z_{\text{interface}})$$

$$z_{\text{interface}} = \frac{c^2}{2g_z} \left(\frac{1}{R^2} - \frac{1}{r_{\text{interface}}^2} \right) + h_0$$

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Isobars

Can also find out the shape of the fluid by solving for isobars by evaluating dp and setting it equal to zero. Since $p = p(r, z)$ chain rule gives us the following for dp.

$$dp = \left(\frac{\partial p}{\partial r}\right) dr + \left(\frac{\partial p}{\partial z}\right) dz$$

$$dp = \frac{\rho c_2^2}{r^3} dr - \rho g_z dz = 0$$

$$dz = \frac{c_2^2}{g_z r^3} dr$$

Integrating back and solving for the constant of integration when $r = 0$ we have

$$z_2 - z_1 = -\frac{c_2^2}{2g_z}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)$$

$$z = -\frac{c_2^2}{2g_z}\left(\frac{1}{r^2} - \frac{1}{R^2}\right) + h_0$$

$$z = \frac{c_2^2}{2g_z}\left(\frac{1}{R^2} - \frac{1}{r^2}\right) + h_0$$

Pretty sure this c_2 is just supposed to be c.

Problem 10.11

Has thin inlet in the outer edge of tank to supply water for vortex. From original equation where we solve for v_θ

$$c = VR$$

$$z = \frac{(VR)^2}{2g_z}\left(\frac{1}{R^2} - \frac{1}{r^2}\right) + h_0$$

3.5 Hydrostatics

Hydrostatics comes from simplifying Euler's equation by making the acceleration of the fluid element zero, that is $\underline{a} = 0$, giving the following equation

$$-\underline{\nabla}p + \rho\underline{g} = 0$$

Chapter 4

Gauss' and Stokes' Theorem

4.1 Gauss' Theorem

> **Gauss' Theorem:** $$\int_V (\underline{\nabla} \cdot \underline{v}) dV = \oint_S \underline{v} \cdot \underline{dA}$$

4.2 Stokes' Theorem

> **Stokes' Theorem:** $$\int_A \underbrace{(\underline{\nabla} \times \underline{v})}_{\text{vorticity}} \cdot dA = \underbrace{\oint_C \underline{v} \cdot \underline{dl}}_{\text{circulation}}$$

And Γ is called the circulation. Stokes' theorem relates the area integral of vorticity to circulation.

$$\Gamma = \oint_C \underline{v} \cdot \underline{dl}$$

Chapter 5

Control Surfaces, Volumes, and Masses

5.1   More on Conservation Equations: Forms A and B

<div>

Mass Conservation

Form A:
$$\frac{d}{dt}\int_{CV(t)}\rho dV + \int_{CS(t)}\rho(\overline{v}-\overline{v}_c)\cdot\overline{n}dA = 0$$

Form B:
$$\int_{CV(t)}\frac{\partial\rho}{\partial t}dV + \int_{CS(t)}\rho v_n dA = 0$$

</div>

$$v_{rn} = (\overline{v}-\overline{v}_c)\cdot\overline{n}$$

$\overline{v}$ is the velocity across the control surface.

<div>

Momentum Conservation

Form A:
$$\frac{d}{dt}\int_{CV(t)}\rho\overline{v}dV + \int_{CS(t)}\rho\overline{v}(\overline{v}-\overline{v}_c)\cdot\overline{n}dA = \overline{F}_{CV}(t)$$

Form B:
$$\int_{CV(t)}\frac{\partial(\rho\overline{v})}{\partial t}dV + \int_{CS(t)}\rho\overline{v}v_n dA = \overline{F}_{CV}(t)$$

</div>

Chapter 6

Viscous Flow

The Newtonian stress tensor $\underline{\underline{\tau}}$ contains contributions of normal stress from pressure, as well as shear stresses in the form of the viscous stress tensor $\underline{\underline{\sigma}}$. Thus, the Newtonian stress tensor can be written as the following, where for an inviscid fluid $\underline{\underline{\sigma}} = 0$. The viscous stress tensor is a tensor used in continuum mechanics to model the part of the stress at a point within some material that can be attributed to the strain rate, the rate at which it is deforming around that point.

Inviscid	$\underline{\underline{\tau}} = -p\underline{\underline{I}}$
Viscous	$\underline{\underline{\tau}} = -p\underline{\underline{I}} + \underline{\underline{\sigma}}$

Definition 1   Newton's viscosity law (Viscosity)

$$viscosity = -\frac{shear\ stress}{rate\ of\ shear\ deformation\ or\ strain}$$

$$\mu = -\frac{\tau}{\frac{d\gamma}{dt}}$$

This is what it is to be a Newtonian fluid, one where the shear stress is proportional to the shear strain rate, where the constant of proportionality is called the fluid's viscosity.

Definition 2   Reynolds number   *The Reynolds number is the ratio between the inertial viscous forces in a fluid.*

$$Re = \frac{\rho U l}{\mu}$$

Airplanes flying at high altitude (rarefied gas) the density is very low, which means the Reynolds number is very small. This is the same effect as highly viscous flow. Basically $\underline{\nabla}\,\underline{v}$ is the divergence of the velocity field. Thinking about this in terms of a square fluid element in some velocity field, $\underline{\nabla}\,\underline{v}$ describes how this fluid element will move with time. This movement can cause the fluid element to translate, rotate, and deform. We would like to split these three parts up.

$$\underline{\nabla}\,\underline{u} = \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + (\underline{\nabla}\,\underline{u})^{\top}\right) + \frac{1}{2}\left(\underline{\nabla}\,\underline{u} - (\underline{\nabla}\,\underline{u})^{\top}\right)$$
$$= \underline{\underline{e}} + \underline{\underline{\Omega}}$$

| **Strain rate tensor:** | $\underline{\underline{e}} = \dfrac{1}{2}\left(\nabla\,\underline{u} + (\nabla\,\underline{u})^{\top}\right)$ |
| **Rotation rate tensor:** | $\underline{\underline{\Omega}} = \dfrac{1}{2}\left(\nabla\,\underline{u} - (\nabla\,\underline{u})^{\top}\right)$ |

Can derive from pictures

$$\underline{\underline{\sigma}} = 2\mu\underline{\underline{e}}$$

For incompressible flows with constant viscosity

$$\nabla \cdot \underline{v} = 0$$

The strain rate tensor represents shearing/stretching of the fluid element.
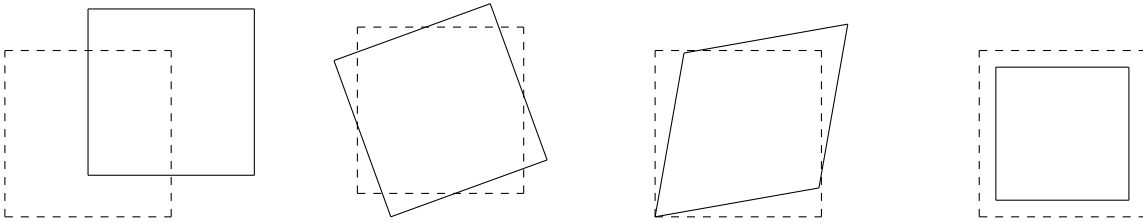
Figure 6.1: a. Translation, b. rotation, c. shearing, d. Pure compression

6.1   Derivation of Incompressible Navier-Stokes' Equations

The Navier Stokes' Equation describes conservation of linear momentum for isothermal flow of an incompressible newtonian fluid.
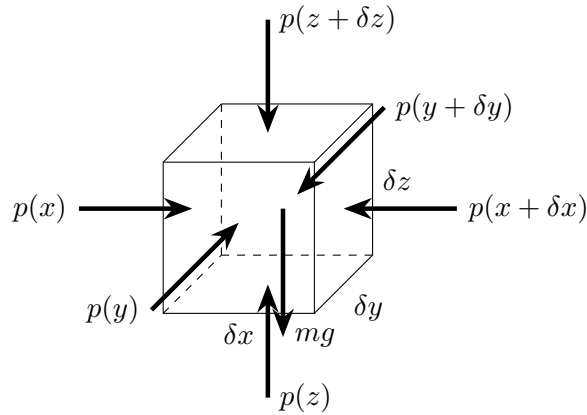
Figure 6.2: Fluid element with pressure and gravity acting on it

Summing the forces in the x, y, and z directions we have

$$p\delta y\delta z - \left(p + \frac{\partial p}{\partial x}\right)\delta y\delta z = \rho\delta x\delta y\delta z a_x$$

$$p\delta x\delta z - \left(p + \frac{\partial p}{\partial y}\right)\delta x\delta z = \rho\delta x\delta y\delta z a_y$$

$$p\delta x\delta y - \left(p + \frac{\partial p}{\partial z}\right)\delta x\delta y - \rho g_z\delta x\delta y\delta z = \rho\delta x\delta y\delta z a_z$$

simplifying

Incompressible Navier-Stokes
$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\underline{v}\right) = -\underline{\nabla}p + \mu\nabla^2\underline{v} + \rho\underline{g}$$

$$\rho\frac{D\underline{v}}{Dt} = -\underline{\nabla}p + \underline{\nabla}\cdot\underline{\underline{\sigma}} + \rho\underline{g}$$

$$\rho\frac{D\underline{v}}{Dt} = -\underline{\nabla}p + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Fully developed flow implies that the velocity profile does not change in the fluid flow direction hence the momentum also does not change in the flow direction. In such a case, the pressure in the flow direction will balance the shear stress near the wall.

The assumptions of the equation are that the fluid is incompressible and newtonian; the flow is laminar through a pipe of constant circular cross-section that is substantially longer than its diameter; and there is no acceleration of fluid in the pipe. For velocities and pipe diameters above a threshold, actual fluid flow is not laminar but turbulent, leading to larger pressure drops than calculated by the Hagen−Poiseuille equation.

6.2   Exact Solutions to the Navier-Stokes Equations

Some cases where an exact analytic solution to the Navier-Stokes equations exist are for the steady case Poisseiulle flow (viscous flow through a circular pipe, or between two long parallel plates) and Coutte flow (laminar flow between two parallel plates where one is moving). In other time dependent cases we have Stokes' first and second problems.

6.2.1   Poiseuille Flow in Circular Pipe

In this section we derive of Poiseuille flow in a circular pipe from Navier-Stokes equation. The laminar flow through a pipe of uniform (circular) cross-section is one of two cases known as Hagen−Poiseuille flow. In this case, we make the following assumptions:

In a large pipe we can have hydrostatic pressure variations, but usually these are very small and can be neglected. *Use the Navier-Stokes' equation sheet to obtain the expanded equations in cylindrical coordinates.* Simplifying the Navier-Stokes equations using the above assumptions we have

$$0 = -\frac{\partial p}{\partial r}$$

$$0 = -\frac{\partial p}{\partial \theta}$$

$$0 = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right] - \frac{\partial p}{\partial z}$$

The first two equations show that the pressure in the tube is only a function of z. Because of this, the partial derivative in the third equation can be made a full derivative, and then we can integrate this equation back to find the an expression for the velocity along the pipe, v_z.

$$0 = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right] - \frac{dp}{dz}$$

$$\frac{dp}{dz} = \frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right)$$

$$\int \frac{dp}{dz} r \partial r = \int \mu \partial \left(r \frac{\partial v_z}{\partial r} \right)$$

$$\frac{1}{2} \frac{dp}{dz} r^2 + c_3 = \mu r \frac{\partial v_z}{\partial r}$$

$$\frac{1}{2} \frac{dp}{dz} r + \frac{c_3}{r} = \mu \frac{\partial v_z}{\partial r}$$

$$\int \left(\frac{1}{2} \frac{dp}{dz} r + \frac{c_3}{r} \right) \partial r = \int \mu \partial v_z$$

$$\frac{1}{4} \frac{dp}{dz} r^2 + c_3 \ln r + c_4 = \mu v_z$$

$$v_z = \frac{1}{4\mu} \frac{dp}{dz} r^2 + c_1 \ln r + c_2$$

$$v_z = \frac{1}{4\mu}\frac{dp}{dz}r^2 + c_1 \ln r + c_2$$

Applying Boundary Conditions

Here we need to use the boundary conditions to find the constants c_1 and c_2.

Regular Pipe
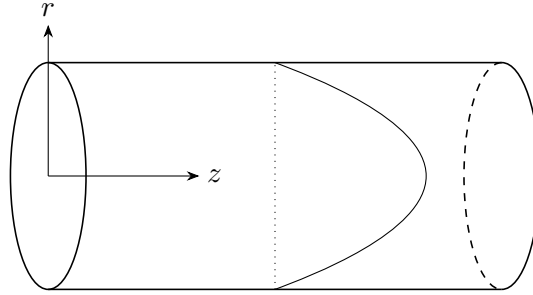
Figure 6.3: Parabolic velocity profile for Poiseuille flow through a circular pipe

Finite velocity along center of pipe:	$v_z(r = 0) = \text{finite}$
No slip at pipe wall:	$v_z(R) = 0$

From the first boundary condition, when $r = 0$, v_z must be finite. Looking at $c_1 \ln r$ when $r = 0$ we see that this term goes to infinity when r goes to zero. To prevent the velocity from going to infinity c_1 must be zero. Now look at the second boundary condition

$$0 = \frac{1}{4\mu}\frac{dp}{dz}R^2 + c_2$$

$$c_2 = -\frac{1}{4\mu}\frac{dp}{dz}R^2$$

so we have

$$v_z = \frac{1}{4\mu}\frac{dp}{dz}r^2 - \frac{1}{4\mu}\frac{dp}{dz}R^2$$

Velocity for Poiseuille flow in regular pipe:	$v_z = -\frac{1}{4\mu}\frac{dp}{dz}(R^2 - r^2)$

Now we can integrate the velocity across the pipe to get the average velocity and flow rate. Furthermore, we can see that the velocity decreases away from the center of the pipe, where the velocity is maximum. The maximum velocity is

Maximum velocity for Poiseuille flow in regular pipe: $\qquad v_{z,\max} = -\dfrac{R^2}{4\mu}\dfrac{dp}{dz}$

Integrating to find the average velocity and flow rate, which are related by $Q = v_{z,\max}A$ we have the following, where $dA = 2\pi r dr$.

$$Q = \int_A -\frac{1}{4\mu}\frac{dp}{dz}(R^2 - r^2)dA$$

$$Q = -\frac{\pi}{2\mu}\frac{dp}{dz}\int_0^R (R^2 - r^2)rdr$$

$$Q = -\frac{\pi}{4\mu}\frac{dp}{dz}\left[\frac{1}{2}R^2r^2 - \frac{1}{4}r^4\right]_0^R$$

Volume flow rate for Poiseuille flow in regular pipe: $\qquad Q = -\dfrac{\pi}{8\mu}\dfrac{dp}{dz}R^4$

The average velocity is then given by

$$v_{z,\text{avg}} = \frac{Q}{A}$$

Average velocity for Poiseuille flow in regular pipe: $\qquad v_{z,\text{avg}} = -\dfrac{1}{8\mu}\dfrac{dp}{dz}R^2$

Conservation of mass, and the fully developed assumption gave the condition that the velocity distribution along the pipe is constant. Once the Navier Stokes' equations are solved, the equation for velocity distribution is expressed in terms of the pressure gradient down the pipe. Since we know for any given value of r that the velocity is constant along the z direction, we can see that the pressure gradient is constant. This allows us to replace the pressure gradient in all of the above equations with the pressure drop along a length of pipe

$$-\frac{dp}{dz} = \frac{\Delta p}{L}$$

In particular, we can use this expression with the flow rate equation, and solve for the pressure drop as a function of pipe length, flow rate, and diameter, among others. From this we can see that to minimize pressure drop we want a pipe with a large diameter.

Pipe Poiseuille Flow		

Velocity $\qquad\qquad\qquad\qquad v_z = \dfrac{1}{4\mu}\dfrac{\Delta p}{L}(R^2 - r^2)$

Maximum velocity $\qquad\quad v_{z,\text{max}} = \dfrac{R^2}{4\mu}\dfrac{\Delta p}{L}$

Average velocity $\qquad\quad v_{z,\text{avg}} = \dfrac{1}{8\mu}\dfrac{\Delta p}{L}R^2$

Flow rate $\qquad\qquad\qquad\quad Q = \dfrac{\pi}{8\mu}\dfrac{\Delta p}{L}R^4$

Pressure drop $\qquad\qquad\quad \Delta p = \dfrac{128\mu L Q}{\pi D^4}$

Annulus Pipe

This problem is from 6.04 of Shapiro and Sonin problems.
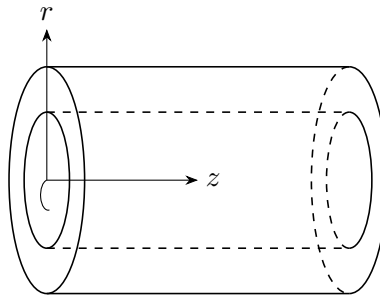
No slip at pipe wall: $\qquad v_z(R_1) = 0$
No slip at pipe wall: $\qquad v_z(R_2) = 0$

$$0 = \frac{1}{4\mu}\frac{dp}{dz}R_1^2 + c_1 \ln R_1 + c_2$$

$$0 = \frac{1}{4\mu}\frac{dp}{dz}R_2^2 + c_1 \ln R_2 + c_2$$

Let $A = \frac{1}{4\mu}\frac{dp}{dz}$

$$0 = AR_1^2 + c_1 \ln R_1 + c_2$$

$$0 = AR_2^2 + c_1 \ln R_2 + c_2$$

$$AR_1^2 + c_1 \ln R_1 = AR_2^2 + c_1 \ln R_2$$

$$A(R_1^2 - R_2^2) = c_1(\ln R_2 - \ln R_1)$$

$$c_1 = \frac{A(R_1^2 - R_2^2)}{\ln\left(\frac{R_2}{R_1}\right)}$$

$$c_1 = \frac{\frac{1}{4\mu}\frac{dp}{dz}(R_1^2 - R_2^2)}{\ln\left(\frac{R_2}{R_1}\right)}$$

$$c_2 = -\frac{1}{4\mu}\frac{dp}{dz}R_2^2 - c_1 \ln R_2$$

$$c_2 = -\frac{1}{4\mu}\frac{dp}{dz}R_2^2 - \frac{\frac{1}{4\mu}\frac{dp}{dz}(R_1^2 - R_2^2)}{\ln\left(\frac{R_2}{R_1}\right)}\ln R_2$$

$$c_2 = -\frac{1}{4\mu}\frac{dp}{dz}\left(R_2^2 + \frac{(R_1^2 - R_2^2)\ln R_2}{\ln\left(\frac{R_2}{R_1}\right)}\right)$$

$$v_z = \frac{1}{4\mu}\frac{dp}{dz}r^2 + c_1 \ln r + c_2$$

$$\boxed{v_z = \frac{1}{4\mu}\frac{dp}{dz}r^2 + \left[\frac{\frac{1}{4\mu}\frac{dp}{dz}(R_1^2 - R_2^2)}{\ln\left(\frac{R_2}{R_1}\right)}\right]\ln r + \left[-\frac{1}{4\mu}\frac{dp}{dz}\left(R_2^2 + \frac{(R_1^2 - R_2^2)\ln R_2}{\ln\left(\frac{R_2}{R_1}\right)}\right)\right]}$$

6.2.2   Derivation of Plane Poiseuille Flow from Navier-Stokes

Consider the following parallel plates
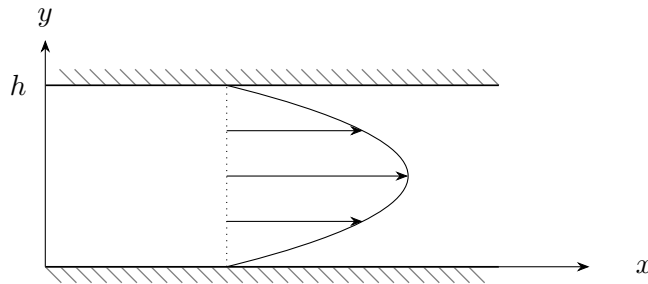
Figure 6.4: Parabolic velocity profile for plane Poisseiulle flow

$$\textbf{Steady:} \qquad \frac{\partial}{\partial t} = 0$$

$$\textbf{No vertical velocity:} \qquad v_y = 0$$

$$\textbf{2-D flow:} \qquad \frac{\partial}{\partial z} = 0$$

$$v_z = 0$$

$$\textbf{Fully developed:} \qquad \frac{\partial v_x}{\partial x} = 0$$

$$0 = \mu \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial p}{\partial x}$$

$$0 = -\frac{\partial p}{\partial y} + \rho g_y$$

$$0 = -\frac{\partial p}{\partial z} + \rho g_z$$

Basically need to solve

$$0 = \mu \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial p}{\partial x}$$

$$\frac{\partial p}{\partial x} \partial y = \mu \partial \frac{\partial v_x}{\partial y}$$

$$\left[\frac{\partial p}{\partial x} y \right] + C_1 = \mu \frac{\partial v_x}{\partial y}$$

$$\left[\frac{\partial p}{\partial x} y + C_1 \right] \partial y = \mu \partial v_x$$

$$\frac{1}{2} \frac{\partial p}{\partial x} y^2 + C_1 y + C_2 = \mu v_x$$

And using the boundary conditions, we have that $v_x(0) = 0$ and $v_x(h) = 0$ so

$$C_2 = 0$$

$$C_1 = -\frac{1}{2} \frac{\partial p}{\partial x} h$$

Velocity for plane Poisseiulle flow: $\qquad v_x(y) = -\frac{h^2}{2\mu} \frac{\partial p}{\partial x} \frac{y}{h} \left(1 - \frac{y}{h} \right)$

Finding volume flow rate

6.2.3 Plane Coutte Flow
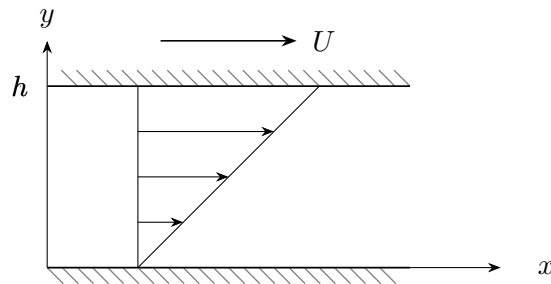
Figure 6.5: Linear velocity profile for plane Couette flow

Couette flow is flow driven by moving one plate relative to another.

Steady:	$\dfrac{\partial}{\partial t} = 0$
Laminar:	$v_y = v_z = 0$
Symmetric:	$\dfrac{\partial}{\partial z} = 0$
Fully developed:	$\dfrac{\partial v_x}{\partial x} = 0$

Use the Navier-Stokes' equation sheet to obtain the expanded equations in cartesian coordinates. Simplifying the Navier-Stokes equations using the above assumptions we have

$$0 = \mu \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial p}{\partial x}$$

$$0 = -\frac{\partial p}{\partial y} + \rho g_y$$

$$0 = -\frac{\partial p}{\partial z} + \rho g_z$$

Assume the plates are really long, so

$$\frac{\partial p}{\partial x} = 0$$

so pretty much need to solve the following, where v_x on depends on the y position, so the partial derivative can be made a full derivative

$$0 = \frac{d^2 v_x}{dy^2}$$

$$\int 0 \, dy = \int d \frac{dv_x}{dy}$$

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$$C_1 = \frac{dv_x}{dy}$$

$$\int C_1 dy = \int dv_x$$

$$C_1 y + C_2 = v_x$$

Using boundary conditions $v_x(y = h) = U$ and $v_x(y = 0) = 0$ we get $C_2 = 0$ and $C_1 = \frac{U}{h}$. So the final solution is

> **Velocity for plane Couette flow:** $\qquad v_x(y) = \dfrac{U}{h} y$

6.2.4 Rotational Couette Flow

Steady:	$\dfrac{\partial}{\partial t} = 0$
Laminar:	$v_r = v_z = 0$
Azimuthally symmetric:	$\dfrac{\partial}{\partial \theta} = 0$
Fully developed:	$\dfrac{\partial v_x}{\partial x} = 0$
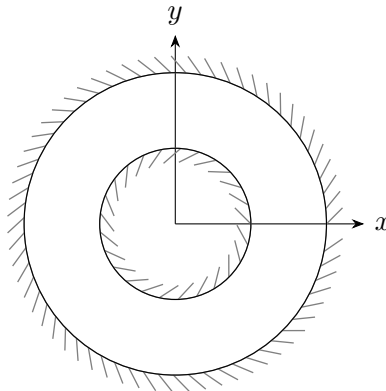
6.2.5 Rayleigh Problem: Stoke's First Problem

Abrupt movement of a flat plate in fluid at rest. Assuming parallel flow with no instabilities.
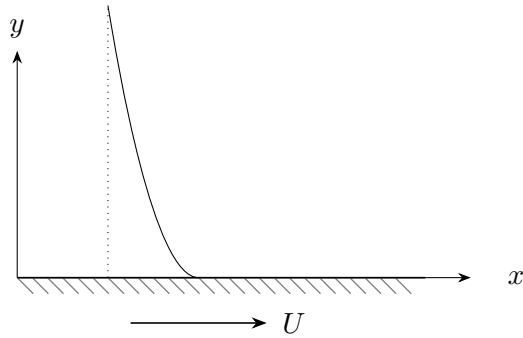
Steady: $\dfrac{\partial}{\partial t} = 0$

Laminar or parallel flow: $v_y = v_z = 0$

2-D flow: $\dfrac{\partial}{\partial z} = 0$

Infinite plate: $\dfrac{\partial v_x}{\partial x} = 0$

Reducing the Navier-Stokes' equations as presented in component form on the handout by applying the simplifying assumptions listed above gives

$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

$$0 = -\frac{\partial p}{\partial y} + \rho g_y$$

$$0 = -\frac{\partial p}{\partial z}$$

These equations can be rearranged to give

$$\frac{\partial v_x}{\partial t} = \frac{\mu}{\rho} \frac{\partial^2 v_x}{\partial y^2}$$

$$0 = -\frac{\partial p}{\partial y} + \rho g_y$$

$$0 = 0$$

The quantity $\frac{\mu}{\rho} = \nu$ is the dynamic viscosity, and we can simplify the first equation with this. The second equation is just hydrostatic pressure in the y-direction. So we want to solve the first equation.

$$\boxed{\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}}$$

This is also known as the "Heat Equation". So basically we have a PDE, but we want to make it an ODE somehow, so I guess we try to non-dimensionalize?

Solution of Stokes' First Problem: Method 1

$$f(\nu, y, t, \frac{v_x}{U}) = 0$$

$$\nu : \frac{L^2}{T}$$
$$y : L$$
$$t : T$$
$$\frac{v_x}{U} : 1$$

and we have

$$n = 4$$
$$k = 2$$
$$j = n - k = 2$$

Pick y and t to be the primary variables

$$\Pi_1 = \nu y^a t^b$$

$$\left(\frac{L^2}{T}\right) L^a T^b = 1$$

$a = -2, b = 1$

$$\boxed{\Pi_1 = \frac{\nu t}{y^2}}$$

However, we can make new Pi groups from any of the "original" Pi groups by operating on them by a function. So, in this case it is convention (and simplifies the solution of the problem) if we pick the first Pi group instead to be

$$\Pi_1^* = \frac{y}{\sqrt{\nu t}}$$

However, while this simplifies the solution, we will show first the case using our original Pi group. The second Pi group is

$$\Pi_2 = \frac{v_x}{U} y^c t^d$$

$$1 L^c T^d = 1$$

$c = 0, d = 0$

$$\boxed{\Pi_2 = \frac{v_x}{U}}$$

and so we have

$$\boxed{\frac{v_x}{U} = \phi\left(\frac{\nu t}{y^2}\right)}$$

or

$$v_x = U\phi\left(\frac{\nu t}{y^2}\right)$$

letting

$$\boxed{\eta = \frac{\nu t}{y^2}}$$

we have

$$v_x = U\phi(\eta)$$

Now we want to use this non-dimensionalized expression to help us solve the PDE, and reduce it to an ODE.

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

So we need to evaluate using the function ϕ the following

$$\frac{\partial v_x}{\partial t} = U\frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial t}$$

$$\frac{\partial v_x}{\partial y} = U\frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 v_x}{\partial y^2} = U\left[\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial \eta}\right)\frac{\partial \eta}{\partial y} + \frac{\partial \phi}{\partial \eta}\frac{\partial}{\partial y}\left(\frac{\partial \eta}{\partial y}\right)\right]$$

$$\frac{\partial^2 v_x}{\partial y^2} = U\left[\frac{\partial^2 \phi}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial \phi}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}\right]$$

And now we need to evaluate

$$\frac{\partial \eta}{\partial t} = \frac{\nu}{y^2}$$

$$\frac{\partial \eta}{\partial y} = -2\frac{\nu t}{y^3}$$

$$\frac{\partial^2 \eta}{\partial y^2} = 6\frac{\nu t}{y^4}$$

$$\left(\frac{\partial \eta}{\partial y}\right)^2 = 4\frac{\nu^2 t^2}{y^6}$$

This gives

$$U\frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial t} = \nu U\left[\frac{\partial^2 \phi}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial \phi}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}\right]$$

canceling out the U and substituting in the known derivatives we get

$$\frac{\partial \phi}{\partial \eta}\frac{\nu}{y^2} = \nu\left[\frac{\partial^2 \phi}{\partial \eta^2}\left(4\frac{\nu^2 t^2}{y^6}\right) + \frac{\partial \phi}{\partial \eta}6\frac{\nu t}{y^4}\right]$$

$$\frac{\partial \phi}{\partial \eta}\left(\frac{\nu}{y^2} - 6\nu\frac{\nu t}{y^4}\right) = \nu\frac{\partial^2 \phi}{\partial \eta^2}\left(4\frac{\nu^2 t^2}{y^6}\right)$$

$$\frac{\partial \phi}{\partial \eta}\left(\frac{y^2 - 6\nu t}{y^2}\right) = \frac{\partial^2 \phi}{\partial \eta^2}\left(4\frac{\nu^2 t^2}{y^4}\right)$$

$$\frac{\partial \phi}{\partial \eta}\left(1 - 6\eta\right) = \frac{\partial^2 \phi}{\partial \eta^2}\left(4\eta^2\right)$$

$$\frac{\partial^2 \phi}{\partial \eta^2} = \left(\frac{1 - 6\eta}{4\eta^2}\right)\frac{\partial \phi}{\partial \eta}$$

Now we separate and integrate twice

$$\frac{\frac{\partial^2 \phi}{\partial \eta^2}}{\frac{\partial \phi}{\partial \eta}} = \frac{1 - 6\eta}{4\eta^2}$$

$$\frac{\frac{\partial}{\partial \eta}\frac{\partial \phi}{\partial \eta}}{\frac{\partial \phi}{\partial \eta}} = \frac{1 - 6\eta}{4\eta^2}$$

$$\int\frac{1}{\frac{\partial \phi}{\partial \eta}}\partial\left(\frac{\partial \phi}{\partial \eta}\right) = \int\frac{1 - 6\eta}{4\eta^2}\partial\eta$$

$$\int\frac{1}{\frac{\partial \phi}{\partial \eta}}\partial\left(\frac{\partial \phi}{\partial \eta}\right) = \frac{1}{4}\int\frac{1}{\eta^2}\partial\eta - \frac{6}{4}\int\frac{1}{\eta}\partial\eta$$

$$\ln\left(\frac{\partial \phi}{\partial \eta}\right) = -\frac{1}{4\eta} - \frac{6}{4}\ln(\eta) + f_1(\text{not }\eta)$$

$$\frac{\partial \phi}{\partial \eta} = \exp\left(-\frac{1}{4\eta} - \frac{6}{4}\ln(\eta) + C_1\right)$$

$$\int \partial\phi = \int \exp\left(-\frac{1}{4\eta} - \frac{6}{4}\ln(\eta) + C_1\right)\partial\eta$$

$$\phi = C_2 \int_0^\eta \exp\left(-\frac{1}{4\eta} - \frac{6}{4}\ln(\eta)\right)\partial\eta + C_3$$

Evaluate this integral to get

$$\phi = C_2 \left[-2\sqrt{\pi}\,\mathrm{erf}\left(\frac{1}{2\sqrt{\eta}}\right)\right]_0^\eta + C_3$$

$$\phi = 2\sqrt{\pi}C_2 \left[1 - \mathrm{erf}\left(\frac{1}{2\sqrt{\eta}}\right)\right] + C_3$$

Now apply boundary conditions to determine C_2 and C_3. Looking at η in terms of y we have

$$y \to \infty \Rightarrow \eta \to 0 \text{ and } \frac{v_x(y \to \infty)}{U} = 0 \Rightarrow \phi(\eta = 0) = 0$$

$$y = 0 \Rightarrow \eta = \infty \text{ and } \frac{v_x(y = 0)}{U} = 1 \Rightarrow \phi(\eta = \infty) = 1$$

Continuing to apply these boundary conditions, with the following

$$\mathrm{erf}(0) = 0$$

$$\mathrm{erf}(\infty) = 1$$

First boundary condition $\eta = 0$

$$0 = 2\sqrt{\pi}C_2 \left[1 - \mathrm{erf}(\infty)\right] + C_3$$

$$C_3 = 0$$

Next boundary condition $\eta \to \infty$

$$1 = 2\sqrt{\pi}C_2 \left[1 - \mathrm{erf}(0)\right] + C_3$$

$$1 = 2\sqrt{\pi}C_2$$

$$C_2 = \frac{1}{2\sqrt{\pi}}$$

$$\phi(\eta) = \left[-\mathrm{erf}\left(\frac{1}{2\sqrt{\eta}}\right)\right]_0^\eta$$

$$\phi(\eta) = \left[-\mathrm{erf}\left(\frac{1}{2\sqrt{\eta}}\right) + \mathrm{erf}(\infty)\right]$$

$$\phi(\eta) = \left[1 - \mathrm{erf}\left(\frac{1}{2\sqrt{\eta}} \right) \right]$$

Now relate this function with the non-dimensional variables, or Pi groups, back to the physical variables

$$\frac{v_x(y,t)}{U} = \left[1 - \mathrm{erf}\left(\frac{1}{2\sqrt{\frac{\nu t}{y^2}}} \right) \right]$$

Velocity for Stokes' first problem: $\qquad \dfrac{v_x(y,t)}{U} = \left[1 - \mathrm{erf}\left(\dfrac{y}{2\sqrt{\nu t}} \right) \right]$

Now, if at the beginning we had picked a different non dimensional variable, or Pi group, η the result would have been the same, but the solution would have been easier, in particular in the evaluation of the integral that resulted in the error function. Some extra stuff. Look at the dimensional analysis sheet and see that the time scale for the diffusion of viscous effects into the fluid are like

$$t_c \sim \frac{L^2}{\nu}$$

and we need to pick a characteristic length scale, which is usually the boundary layer thickness δ. This gives

$$t_c \sim \frac{\delta^2}{\nu}$$

Solving for δ we have

$$\delta \sim \sqrt{\nu t}$$

And we can approximate the shear stress as a linear velocity profile over the boundary layer

$$\tau_w \sim \frac{\mu U}{\delta} \sim \frac{\mu U}{\sqrt{\nu t}}$$

Solution of Stokes' First Problem: Method 2

Using different η

6.2.6 Stokes' Second Problem

Stokes apparently had many problems. In this problem, at first I thought we could just reuse most of the solution from Stokes' first problem, but change the boundary condition and somehow take into account the oscillating boundary condition because the constants of integration that we found last time are not actually constants, but functions of not eta. Should clear up the notation on how to express arbitrary constants that are "not a function" of some variable. This didn't work though, and I think it is something like we fundamentally ignored the variable omega (plate oscillation frequency) when we non-dimensionalized, so our solution won't work. This basically means the whole non-dimensionalizing part to turn the governing PDE into an ODE needs to be redone.

$U \cos \omega t$
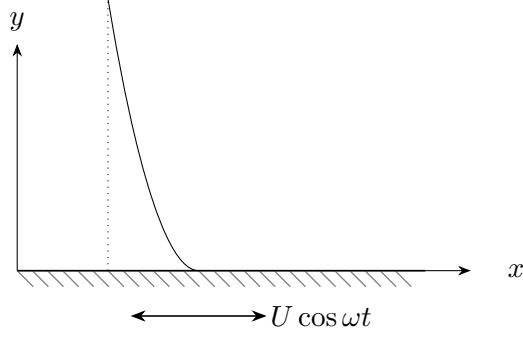
Example 2 Viscometer A motor with a cylinder attached is submerged in a viscous fluid. The motor is set to rotate at a known rpm, and a device is used to measure the torque required to rotate the cylinder. From this, we can figure out the viscosity of the fluid.

6.3 Nondimensionalizing the Navier-Stokes' Equation

By nondimensionalizing the Navier-Stokes' Equations, we can understand better what contributions like viscosity are "large" or "small". To express in dimensionless variables, we have to scale all the variables in the problem using characteristic scales for the problem of interest.

$$\underline{x}^* = \frac{x}{l} \qquad \underline{v}^* = \frac{v}{V} \qquad p^* = \frac{p}{\rho V^2} \qquad t^* = \frac{t}{\left(\frac{l}{V}\right)} = \frac{Vt}{l}$$

solving

$$\underline{x} = \underline{x}^* l \qquad \underline{v} = \underline{v}^* V \qquad p = p^* \rho V^2 \qquad t = \frac{l t^*}{V}$$

Where the time is called the convective time scale. And the del operator $\underline{\nabla}$ takes a different form when nondimensionalized as well.

$$\begin{aligned}
\underline{\nabla} &= \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial}{\partial(x^* l)} & \frac{\partial}{\partial(y^* l)} & \frac{\partial}{\partial(z^* l)} \end{bmatrix} \\
&= \frac{1}{l} \begin{bmatrix} \frac{\partial}{\partial x^*} & \frac{\partial}{\partial y^*} & \frac{\partial}{\partial z^*} \end{bmatrix} \\
&= \frac{1}{l} \underline{\nabla}^*
\end{aligned}$$

The Navier-Stokes' Equation is

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} \right) = -\underline{\nabla} p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

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$$\rho\left(\frac{\partial(\underline{v}^*V)}{\partial(\frac{lt^*}{V})} + (\underline{v}^*V)\cdot\frac{1}{l}\underline{\nabla}^*(\underline{v}^*V)\right) = -\frac{1}{l}\underline{\nabla}^*(p^*\rho V^2) + \frac{\mu}{l^2}\nabla^{*2}(\underline{v}^*V) + \rho\underline{g}$$

$$\rho\left(\frac{V^2}{l}\frac{\partial\underline{v}^*}{\partial t^*} + \frac{V^2}{l}\underline{v}^*\cdot\underline{\nabla}^*\underline{v}^*\right) = -\frac{\rho V^2}{l}\underline{\nabla}^*p^* + \frac{\mu V}{l^2}\nabla^{*2}\underline{v}^* + \rho\underline{g}$$

Divide through by $\rho\frac{V^2}{l}$ and get

$$\frac{\partial\underline{v}^*}{\partial t^*} + \underline{v}^*\cdot\underline{\nabla}^*\underline{v}^* = -\underline{\nabla}^*p^* + \underbrace{\left(\frac{\mu}{\rho Vl}\right)}_{\frac{1}{\mathrm{Re}}}\nabla^{*2}\underline{v}^* + \underbrace{\left(\frac{gl}{V^2}\right)}_{\frac{1}{\mathrm{Fr}^2}}\hat{e}_z$$

The Froude number plays an analogous role to the Mach number in a compressible flow. From this non-dimensionalized version of Navier-Stokes' Equation, we can see if $\mathrm{Re} = \frac{\rho Vl}{\mu}$ is very large, the viscous terms in the equation of motion become very small and negligible compared to the inertial terms in the equation. The *flow* is inviscid, not the *fluid*.

$$\frac{\partial\underline{v}^*}{\partial t^*} + \underline{v}^*\cdot\underline{\nabla}^*\underline{v}^* = -\underline{\nabla}^*p^* + \frac{1}{\mathrm{Re}}\nabla^{*2}\underline{v}^* + \frac{gl}{V^2}\hat{e}_z$$

$$\frac{D\underline{v}^*}{Dt^*} = -\underline{\nabla}^*p^* + \frac{1}{\mathrm{Re}}\nabla^{*2}\underline{v}^* + \frac{gl}{V^2}\hat{e}_z$$

$$\frac{D\underline{v}^*}{Dt^*} = -\underline{\nabla}^*p^* + \frac{1}{\mathrm{Re}}\nabla^{*2}\underline{v}^* + \frac{1}{\mathrm{Fr}^2}\hat{e}_z$$

To solve such problems, use the following procedure. Nondimensionalize the problem we are interested in: the skinny gap considered for lubrication theory problems. Do this just like all the other dimensional analysis problems before. Take those non dimensional quantities or Pi groups and make new ones just so they look like one we are familiar with. Write a few inequalities based on the prescribed geometry. That is: $h << L$ and $\frac{dh}{dx} << 1$ which says gap is small and doesn't expand too quickly. Solve for the dimensional variables in terms of the nondomensional ones. Plug these dimensional variables into NSE and obtain a non dimensionalized version of the equation. From this equation and using our assumptions of flow geometry, we can simplify terms within the non dimensional NSE.

- why in lubrication theory can we show that inertia effects are small?

- How do we say that two non dimensional derivatives are of the same order of magnitude? To allow us to compare terms like mckinley did in notes comparing 5/6...

- integration of multivariable functions required for the solution to Stokes' first problem $f(notz)$

- Stokes flow from original nondimensionalization (see sphere moving in fluid example below) and in general how to ignore inertial effects? $\rho = 0$?

Chapter 7

Dimensional Analysis

To solve a dimensional analysis problem, use the following steps.

1. Pick the fundamental variables which describe the problem, and note the number n of these fundamental variables

   - In picking the fundamental variables, we typically want to pick one quantity which describe the fluid, the flow, and the geometry.

2. Write the fundamental dimensions for each fundamental variable, and note the number k of independent fundamental dimensions.

3. Subtracting $j = n - k$ we need j Pi groups. We need to pick k primary variables to use in making these Pi groups.

4. Make the j Pi groups

5. Identify as many of the Pi groups as known dimensionless quantities, such as Reynolds number, Weber number, etc.

Example 3 Sphere falling in a viscous fluid
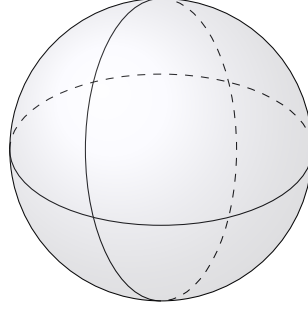
Fundamental variables F_D, U, ρ, μ, D $n = 5$. Fundamental dimensions are

$$F_D : \frac{ML}{T^2}$$

$$U : \frac{L}{T}$$

$$\rho : \frac{M}{L^3}$$

$$\mu : \frac{M}{LT}$$

$$D : L$$

From this we see that the number of independent dimensions is $k = 3$. So $n - k = 2$ and we need 2 Pi groups. Choose ρ, U, and D as the primary variables.

$$\Pi_1 = F_D \rho^a U^b D^c \qquad \Pi_2 = \mu \rho^d U^e D^f$$

$$\left(\frac{ML}{T^2}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b (L)^c = M^0 L^0 T^0$$

$$\left(\frac{M}{LT}\right)\left(\frac{M}{L^3}\right)^d \left(\frac{L}{T}\right)^e (L)^f = M^0 L^0 T^0$$

$$T^{-2}T^{-b} = T^0 \qquad T^{-1}T^{-f} = T^0$$

$$MM^a = M^0 \qquad MM^d = M^0$$

$$LL^{-3a}L^bL^c = L^0 \qquad L^{-1}L^{-3d}L^eL^f = L^0$$

$$b = -2 \qquad f = -1$$

$$a = -1 \qquad d = -1$$

$$LL^3L^{-2}L^c = L^0 \qquad L^{-1}L^3L^eL^{-1} = L^0$$

$$c = -2 \qquad e = -1$$

$$\Pi_1 = F_D \rho^{-1}U^{-2}D^{-2} \qquad \Pi_2 = \mu \rho^{-1}U^{-1}D^{-1}$$

$$\Pi_1 = \frac{F_D}{\rho D^2 U^2} \qquad\qquad \Pi_2 = \frac{\mu}{\rho D U} = \frac{1}{\text{Re}}$$

Π_1 is essentially the drag coefficient. Since we know the Pi groups can always be off by a constant factor, the drag coefficient usually looks like

$$C_D = \frac{F_d}{\frac{1}{2}\rho U^2 A}$$

where A is the projected area. And we can see that

$$\Pi_1 = \phi\left(\Pi_2\right)$$

$$C_D = \phi\left(\frac{1}{\text{Re}}\right)$$

And when the Reynolds number is very large, inertia dominates and viscous forces are negligible. In this case, we can redo the dimensional analysis, but this time without μ. This gives $n = 4$, but with $k = 3$ still, and so there is only one Pi group. In this case we know that that one Pi group, which is shown below, must be a constant.

$$\textbf{High Re:} \qquad \frac{F_D}{\rho U^2 D^2} = \text{constant}$$

What about when Reynolds number is very very small? Essentially this means inertia is negligible, so that means ρ is small?

$$\textbf{Low Re:} \qquad \frac{F_D}{xyz} = \text{constant}$$

Basically the drag coefficient is given as a dimensionless drag force. So take the drag force and divide it by something that has units of force. To get a force we do pressure times area, so depending on whether viscous or inertial pressure dominates, pick the correct one, multiply it by an area, and we have a force.

$$\textbf{High Re:} \qquad C_D = \text{const}\frac{F_D}{\rho U^2 D^2}$$

$$\textbf{Low Re:} \qquad C_D = \text{const}\frac{F_D}{\frac{\mu U}{R}R^2}$$

so the drag forces in each of these cases are

$$\textbf{High Re:} \qquad F_D = \text{const}\rho U^2 D^2$$

$$\textbf{Low Re:} \qquad F_D = \text{const}\mu U R$$

The drag on a sphere... when viscosity negligible:

$$F_D = K_2 \rho v^2 R^2$$

when inertia is negligible

$$F_D = K_3 \mu v R$$

Stokes Creeping Flow

Can get the following equations by simplifying and solving Navier-Stokes' equations.

$$v_r = -V \cos(\theta) \left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3} \right)$$

$$v_\theta = V \sin(\theta) \left(1 - \frac{3R}{4r} + \frac{R^3}{4r^3} \right)$$

$$p = p_\infty - \frac{3}{2} \left(\frac{\mu v}{R} \right) \frac{R^2}{r^2} \cos(\theta)$$

To find the drag force on a sphere, need the pressure gradient and the shear stress at the surface, $\tau_{r\theta}$ and integrate all the terms over the surface of a sphere

Stokes drag on a sphere $\qquad \underline{F}_{D,x} = \int_A \underline{e}_x \cdot \underline{\tau} dA = 6\pi\mu RV = 3\pi\mu VD$

and so C_D of the sphere simplifies to

$$C_D = \frac{24}{\text{Re}}$$

Example 4 How long for a falling sphere to get to steady state velocity?

Chapter 8

Lubrication Theory

The key requirement for lubrication theory is that the ratio $h/l << 1$ is small, where h is gap between surfaces and L is the length. More than one readily identifiable characteristic length scale. If there is a clear separation of scales $h << l$ then we can use lubrication analysis to simplify the Navier-Stokes' equation. (Slender body analysis).

8.1 Cartesian Coordinates

$$\underline{x}^* = \frac{x}{l} \qquad \underline{y}^* = \frac{y}{h} \qquad \underline{v}_x^* = \frac{v_x}{U} \qquad \underline{v}_y^* = \frac{v_y}{V_c} \qquad t^* = \frac{t}{t_c}$$

$$p^* = \frac{p}{p_c} \qquad p^* = \frac{p}{\rho V^2} \qquad t^* = \frac{t}{\left(\frac{l}{V}\right)} = \frac{Vt}{l}$$

where V_c is a characteristic velocity. Solving for the dimensional quantities in terms of the dimensionless ones, we have

$$\underline{x} = \underline{x}^* l \qquad \underline{y} = \underline{y}^* h \qquad \underline{v}_x = \underline{v}_x^* U \qquad \underline{v}_y = \underline{v}_y^* V_c \qquad t = t^* t_c \qquad p = p^* p_c$$

8.1.1 Non-dimensionalization of Conservation of Mass

Starting with conservation of mass for 2-D in cartesian coordinates

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

we substitute these into conservation of mass and get

$$\frac{\partial(v_x^* U)}{\partial(x^* l)} + \frac{\partial(v_y^* V_c)}{\partial(y^* h)} = 0$$

Pulling out the characteristic terms we have

$$\frac{U}{l} \frac{\partial v_x^*}{\partial x^*} + \frac{V_c}{h} \frac{\partial v_y^*}{\partial y^*} = 0$$

and so from this both of the dimensionless groups are are order one dimensionless quantities, and so we see that the characteristic velocity must scale as

$$V_c \sim \frac{Uh}{l}$$

8.1.2 Simplification of Navier-Stokes for Lubrication Theory
x-direction

The x-component of the Navier-Stokes equation in cartesian coordinates is the following

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right] - \frac{\partial p}{\partial x} + \rho g_x$$

Simplifying these equations for 2-D flow, we get

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{x} + v_y \frac{\partial v_x}{\partial y} \right) = \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right] - \frac{\partial p}{\partial x} + \rho g_x$$

Substituting in the dimensional variables in terms of the non dimensional variables and characteristic values, we get

$$\rho \left(\frac{\partial (v_x^* U)}{\partial (t^* t_c)} + v_x^* U \frac{\partial (v_x^* U)}{\partial (x^* l)} + v_y^* V \frac{\partial (v_x^* U)}{\partial (y^* h)} \right) = \mu \left[\frac{\partial^2 (v_x^* U)}{\partial (x^* l)^2} + \frac{\partial^2 (v_x^* U)}{\partial (y^* h)^2} \right] - \frac{\partial (p^* p_c)}{\partial (x^* l)} + \rho g_x$$

Pull out characteristic values to leave differential equation in dimensionless form

$$\rho \left(\frac{U}{t_c} \frac{\partial v_x^*}{\partial t^*} + \frac{U^2}{l} v_x^* \frac{\partial v_x^*}{\partial x^*} + \frac{VU}{h} v_y^* \frac{\partial v_x^*}{\partial y^*} \right) = \mu \left[\frac{U}{l^2} \frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{U}{h^2} \frac{\partial^2 v_x^*}{\partial y^{*2}} \right] - \frac{p_c}{l} \frac{\partial p^*}{\partial x^*} + \rho g_x$$

Plug in the scaling for V which came from continuity and dividing by μ

$$\frac{\rho}{\mu} \left(\frac{U}{t_c} \frac{\partial v_x^*}{\partial t^*} + \frac{U^2}{l} v_x^* \frac{\partial v_x^*}{\partial x^*} + \frac{U^2}{l} v_y^* \frac{\partial v_x^*}{\partial y^*} \right) = \left[\frac{U}{l^2} \frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{U}{h^2} \frac{\partial^2 v_x^*}{\partial y^{*2}} \right] - \frac{p_c}{\mu l} \frac{\partial p^*}{\partial x^*} + \frac{\rho}{\mu} g_x$$

Multiply both sides by $\frac{h^2}{U}$

$$\frac{\rho h^2}{\mu t_c} \frac{\partial v_x^*}{\partial t^*} + \frac{\rho U h^2}{\mu l} v_x^* \frac{\partial v_x^*}{\partial x^*} + \frac{\rho U h^2}{\mu l} v_y^* \frac{\partial v_x^*}{\partial y^*} = \frac{h^2}{l^2} \frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{\partial^2 v_x^*}{\partial y^{*2}} - \frac{h^2 p_c}{\mu U l} \frac{\partial p^*}{\partial x^*} + \frac{h^2 \rho}{\mu U} g_x$$

Notice now that to keep the left side of the same order, we must pick the characteristic time as

$$t_c = \frac{l}{U}$$

Substituting this in to get

$$\frac{\rho U h^2}{\mu l} \frac{\partial v_x^*}{\partial t^*} + \frac{\rho U h^2}{\mu l} v_x^* \frac{\partial v_x^*}{\partial x^*} + \frac{\rho U h^2}{\mu l} v_y^* \frac{\partial v_x^*}{\partial y^*} = \frac{h^2}{l^2} \frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{\partial^2 v_x^*}{\partial y^{*2}} - \frac{h^2 p_c}{\mu U l} \frac{\partial p^*}{\partial x^*} + \frac{h^2 \rho}{\mu U} g_x$$

Recognize the Reynolds number terms

$$\text{Re}_l \frac{h^2}{l^2}\left(\frac{\partial v_x^*}{\partial t^*} + v_x^*\frac{\partial v_x^*}{\partial x^*} + v_y^*\frac{\partial v_x^*}{\partial y^*}\right) = \frac{h^2}{l^2}\frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{\partial^2 v_x^*}{\partial y^{*2}} - \frac{h^2 p_c}{\mu U l}\frac{\partial p^*}{\partial x^*} + \frac{h^2 \rho}{\mu U}g_x$$

If $\text{Re}_l \frac{h^2}{l^2} \ll 1$ and $\frac{h^2}{l^2} \ll 1$ then we can simplify this expression

$$0 = \frac{\partial^2 v_x^*}{\partial y^{*2}} - \frac{h^2 p_c}{\mu U l}\frac{\partial p^*}{\partial x^*} + \frac{h^2 \rho}{\mu U}g_x$$

Rearranging terms we can redimensionalize

$$0 = \frac{\mu U}{h^2}\frac{\partial^2 v_x^*}{\partial y^{*2}} - \frac{p_c}{l}\frac{\partial p^*}{\partial x^*} + \rho g_x$$

$$\mu \frac{\partial^2 (U v_x^*)}{\partial (h y^{*2})} - \frac{\partial (p_c p^*)}{\partial (l x^*)} + \rho g_x = 0$$

So the governing equation of motion for lubrication theory in the x-direction is the following, where g_x is the component of gravity along the x-axis.

> **Lubrication theory x-direction:** $\qquad \mu\frac{\partial^2 v_x}{\partial^2} - \frac{\partial p}{\partial x} + \rho g_x = 0$

We can now separate this equation and integrate it back to obtain an expression for the velocity. However, we require two boundary conditions, and these depend on the problem being solved. A free surface corresponds to Neumann boundary conditions, those where the shear stress at the free surface are zero, and so the relationship of velocity and shear stress says that the derivative of the velocity with respect to the perpendicular direction are zero. Will clear this up later. Dirichlet boundary conditions are those where the velocity itself takes a certain value rather than derivative. Integrate the governing equation back.

$$\mu \int d\frac{dv_x}{dy} = \int \left(\frac{\partial p}{\partial x} - \rho g_x\right) dy$$

$$\mu \frac{dv_x}{dy} = \left(\frac{\partial p}{\partial x} - \rho g_x\right) y + C_1$$

$$\mu \int dv_x = \int \left(\frac{\partial p}{\partial x} - \rho g_x\right) y \, dy + \int C_1 dy$$

$$\mu v_x = \frac{1}{2}\left(\frac{\partial p}{\partial x} - \rho g_x\right) y^2 + C_1 y + C_2$$

y-direction

Taking the Navier-Stokes equation in the y-direction and substituting in the dimensional terms we have

$$\rho\left(\frac{\partial (v_y^* V)}{\partial (t^* t_c)} + v_x^* U\frac{\partial (v_y^* V)}{\partial (x^* l)} + v_y^* V\frac{\partial (v_y^* V)}{\partial (y^* h)}\right) = \mu\left[\frac{\partial^2 (v_y^* V)}{\partial (x^* l)^2} + \frac{\partial^2 (v_y^* V)}{\partial (y^* h)^2}\right] - \frac{\partial (p^* p_c)}{\partial (y^* h)} + \rho g_y$$

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Pulling out the characteristic values so the derivatives are dimensionless

$$\rho\left(\frac{V}{t_c}\frac{\partial v_y^*}{\partial t^*} + \frac{UV}{l}v_x^*\frac{\partial v_y^*}{\partial x^*} + \frac{V^2}{h}v_y^*\frac{\partial v_y^*}{\partial y^*}\right) = \mu\left[\frac{V}{l^2}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{V}{h^2}\frac{\partial^2 v_y^*}{\partial y^{*2}}\right] - \frac{p_c}{h}\frac{\partial p^*}{\partial y^*} + \rho g_y$$

First substitute in the expression from conservation of mass that gives the scaling of V, and the characteristic time

$$V \sim \frac{Uh}{l}$$

this gives

$$\rho\left(\frac{Uh}{l}\frac{1}{t_c}\frac{\partial v_y^*}{\partial t^*} + \frac{U^2h}{l^2}v_x^*\frac{\partial v_y^*}{\partial x^*} + \frac{U^2h}{l^2}v_y^*\frac{\partial v_y^*}{\partial y^*}\right) = \mu\left[\frac{Uh}{l^3}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{U}{hl}\frac{\partial^2 v_y^*}{\partial y^{*2}}\right] - \frac{p_c}{h}\frac{\partial p^*}{\partial y^*} + \rho g_y$$

Something here about if the flow is slowly varying, the timescale in the y-direction is roughly the same as timescale in the x-direction, so we use again

$$t_c = \frac{l}{U}$$

$$\frac{\rho U^2 h}{l^2}\frac{\partial v_x^*}{\partial t^*} + \frac{\rho U^2 h}{l^2}v_x^*\frac{\partial v_x^*}{\partial x^*} + \frac{\rho U^2 h}{l^2}v_y^*\frac{\partial v_x^*}{\partial y^*} = \frac{\mu Uh}{l^3}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{\mu U}{hl}\frac{\partial^2 v_y^*}{\partial y^{*2}} - \frac{p_c}{h}\frac{\partial p^*}{\partial y^*} + \rho g_y$$

$$\frac{\rho U^2 h}{l^2}\left(\frac{\partial v_y^*}{\partial t^*} + v_x^*\frac{\partial v_y^*}{\partial x^*} + v_y^*\frac{\partial v_y^*}{\partial y^*}\right) = \frac{\mu Uh}{l^3}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{\mu U}{hl}\frac{\partial^2 v_y^*}{\partial y^{*2}} - \frac{p_c}{h}\frac{\partial p^*}{\partial y^*} + \rho g_y$$

multiply both sides by

$$\frac{l^3}{U\mu h}\frac{h^3}{l^3} = \frac{h^2}{U\mu}$$

Where the first term we can see would make the coefficient on the left side become the Reynolds number $Re = \frac{\rho Ul}{\mu}$. Then, the second term would make the left hand side be $Re\left(\frac{h}{l}\right)^3$. That way if we can say that $Re\left(\frac{h}{l}\right)^3 << 1$, then the whole left side goes away. We will see what happens to the right hand side.

$$Re\left(\frac{h}{l}\right)^3\left(\frac{\partial v_y^*}{\partial t^*} + v_x^*\frac{\partial y_x^*}{\partial x^*} + v_y^*\frac{\partial y_x^*}{\partial y^*}\right) = \left(\frac{h}{l}\right)^3\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{h}{l}\frac{\partial^2 v_y^*}{\partial y^{*2}} - \frac{p_c h}{U\mu}\frac{\partial p^*}{\partial y^*} + \frac{h^2}{U\mu}\rho g_y$$

And so if $\frac{h}{l} << 1$ and $Re\frac{h}{l} << 1$ then this equation can be simplified to

$$\frac{h^2}{U\mu}\rho g_y - \frac{p_c h}{U\mu}\frac{\partial p^*}{\partial y^*} = 0$$

Now we redimensionalize again, remembering that

$$\frac{1}{U} \sim \frac{h}{Vl}$$

$$\frac{h^3}{Vl\mu}\rho g_y - \frac{p_c h^2}{Vl\mu}\frac{\partial p^*}{\partial y^*} = 0$$

$$\frac{h^3}{Vl\mu}\rho g_y - \frac{h^3}{Vl\mu}\frac{\partial(p_c p^*)}{\partial(hy^*)} = 0$$

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Finally we have

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Lubrication theory y-direction: $\qquad \rho g_y - \dfrac{\partial p}{\partial y} = 0$

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And integrating this expression back, we can have the following function for pressure, where we have to apply boundary conditions to solve for the constant

$$p = \rho g_y y + C_1$$

Lubrication Theory Equations: Cartesian

x-**momentum**	$\mu \dfrac{d^2 v_x}{dy^2} - \dfrac{\partial p}{\partial x} + \rho g_x = 0$
y-**momentum**	$\rho g_y - \dfrac{\partial p}{\partial y} = 0$
Continuity	$\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0$

8.2 Cylindrical Polar Coordinates

$$v_r = v_r^* U \qquad v_z = v_z^* V \qquad t = t^* \frac{R}{U} \qquad z = z^* h \qquad r = r^* R \qquad p = p^* p_c$$

8.2.1 Conservation of Mass

$$\frac{1}{r} \frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{z} = 0$$

Plut in quantities expressed in terms of the dimensionless and characteristic

$$\frac{U}{R} \frac{1}{r^*} \frac{\partial (r^* v_r^*)}{\partial r^*} + \frac{V}{h} \frac{\partial (v_z^*)}{z^*} = 0$$

and so from this we see

$$\frac{U}{R} \sim \frac{V}{h}$$

$$\boxed{V \sim \frac{Uh}{R}}$$

8.2.2 Navier-Stokes Equation

Neglect gravity and azimuthally symmetric (fully developed in the θ direction)

r-direction

This equation is for azimuthally symmetric flow on a spinning disk.

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{\partial^2 v_r}{\partial z^2} \right] - \frac{\partial p}{\partial r}$$

The velocity v_θ was moved over to the right side and $v_\theta = r\omega$. Then we get

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right) = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{\partial^2 v_r}{\partial z^2} \right] - \frac{\partial p}{\partial r} + \rho \omega^2 r$$

Now plugging in the non dimensional quantities we get

$$\rho \left(\frac{\partial (v_r^* U)}{\partial (t^* \frac{R}{U})} + v_r^* U \frac{\partial (v_r^* U)}{\partial (r^* R)} + v_z^* V \frac{\partial (v_r^* U)}{\partial (z^* h)} \right)$$
$$= \mu \left[\frac{\partial}{\partial (r^* R)} \left(\frac{1}{r^* R} \frac{\partial}{\partial (r^* R)} (r^* R v_r^* U) \right) + \frac{\partial^2 (v_r^* U)}{\partial (z^* h)^2} \right] - \frac{\partial (p^* p_c)}{\partial (r^* R)} + \rho \omega^2 r^* R$$

$$\rho \left(\frac{U^2}{R} \frac{\partial v_r^*}{\partial t^*} + \frac{U^2}{R} v_r^* \frac{\partial v_r^*}{\partial r^*} + \frac{VU}{h} v_z^* \frac{\partial v_r^*}{\partial z^*} \right)$$
$$= \mu \left[\frac{U}{R^2} \frac{\partial}{\partial r^*} \left(\frac{1}{r^*} \frac{\partial (r^* v_r^*)}{\partial r^*} \right) + \frac{U}{h^2} \frac{\partial^2 v_r^*}{\partial z^{*2}} \right] - \frac{p_c}{R} \frac{\partial p^*}{\partial r^*} + R\rho \omega^2 r^*$$

Using the relationship from continuity

$$V \sim \frac{Uh}{R}$$

we get

$$\frac{\rho U^2}{R} \left(\frac{\partial v_r^*}{\partial t^*} + v_r^* \frac{\partial v_r^*}{\partial r^*} + v_z^* \frac{\partial v_r^*}{\partial z^*} \right) = \mu \left[\frac{U}{R^2} \frac{\partial}{\partial r^*} \left(\frac{1}{r^*} \frac{\partial (r^* v_r^*)}{\partial r^*} \right) + \frac{U}{h^2} \frac{\partial^2 v_r^*}{\partial z^{*2}} \right] - \frac{p_c}{R} \frac{\partial p^*}{\partial r^*} + R\rho \omega^2 r^*$$

We want to get $\mathrm{Re}_R \left(\frac{h}{R} \right)^2 << 1$ on the left hand side to cancel all those terms, so want the left hand side to have coefficient

$$\mathrm{Re}_R \left(\frac{h}{R} \right)^2 = \frac{\rho U R}{\mu} \frac{h^2}{R^2} = \frac{\rho U h^2}{\mu R} << 1$$

so to do this, multiply both sides by

$$\frac{h^2}{\mu U}$$

giving

$$\text{Re}_R \left(\frac{h}{R}\right)^2 \left(\frac{\partial v_r^*}{t^*} + v_r^* \frac{\partial v_r^*}{\partial r^*} + v_z^* \frac{\partial v_r^*}{\partial z^*}\right)$$
$$= \frac{h^2}{U} \left[\frac{U}{R^2} \frac{\partial}{\partial r^*} \left(\frac{1}{r^*} \frac{\partial (r^* v_r^*)}{\partial r^*}\right) + \frac{U}{h^2} \frac{\partial^2 v_r^*}{\partial z^{*2}}\right] - \frac{p_c h^2}{R\mu U} \frac{\partial p^*}{\partial r^*} + \frac{h^2 R\rho}{\mu U} \omega^2 r^*$$

simplifying

$$\text{Re}_R \left(\frac{h}{R}\right)^2 \left(\frac{\partial v_r^*}{t^*} + v_r^* \frac{\partial v_r^*}{\partial r^*} + v_z^* \frac{\partial v_r^*}{\partial z^*}\right)$$
$$= \left[\frac{h^2}{R^2} \frac{\partial}{\partial r^*} \left(\frac{1}{r^*} \frac{\partial (r^* v_r^*)}{\partial r^*}\right) + \frac{\partial^2 v_r^*}{\partial z^{*2}}\right] - \frac{p_c h^2}{R\mu U} \frac{\partial p^*}{\partial r^*} + \frac{h^2 R\rho}{\mu U} \omega^2 r^*$$

simplifying

$$0 = \frac{\partial^2 v_r^*}{\partial z^{*2}} - \frac{p_c h^2}{R\mu U} \frac{\partial p^*}{\partial r^*} + \frac{h^2 R\rho}{\mu U} \omega^2 r^*$$

we want to keep the pressure term, so the characteristic pressure should scale as

$$p_c = \frac{\mu U R}{h^2}$$

giving

$$0 = \frac{\partial^2 v_r^*}{\partial z^{*2}} - \frac{\partial p^*}{\partial r^*} + \frac{h^2 R\rho}{\mu U} \omega^2 r^*$$

Redimensionalizing

$$0 = \frac{\partial^2 \left(\frac{v_r}{U}\right)}{\partial \left(\frac{z}{h}\right)^2} - \frac{\partial \left(\frac{p}{p_c}\right)}{\partial \left(\frac{r}{R}\right)} + \frac{h^2 R\rho}{\mu U} \omega^2 \frac{r}{R}$$

$$0 = \frac{h^2}{U} \frac{\partial^2 v_r}{\partial z^2} - \frac{R}{p_c} \frac{\partial p}{\partial r} + \frac{h^2 r\rho}{\mu U} \omega^2$$

$$0 = \frac{h^2}{U} \frac{\partial^2 v_r}{\partial z^2} - \frac{Rh^2}{\mu U R} \frac{\partial p}{\partial r} + \frac{h^2 r\rho}{\mu U} \omega^2$$

$$\frac{\partial^2 v_r}{\partial z^2} - \frac{1}{\mu} \frac{\partial p}{\partial r} + \frac{r\rho}{\mu} \omega^2 = 0$$

And if $\frac{\partial p}{\partial r}$ is very small this reduces to

$$\boxed{\mu \frac{\partial^2 v_r}{\partial z} + \rho \omega^2 r = 0}$$

Chapter 9

Potential Flows

In fluid dynamics, potential flow describes the velocity field as the gradient of a scalar function: the velocity potential. As a result, a potential flow is characterized by an irrotational velocity field, which is a valid approximation for several applications. The irrotationality of a potential flow is due to the curl of a gradient always being equal to zero. A velocity potential is used in fluid dynamics, when a fluid occupies a simply-connected region and is irrotational. In such a case

$$\underline{\nabla} \times \underline{u} = 0$$

where $\underline{u}$ denotes the flow velocity of the fluid. As a result, $\underline{u}$ can be represented as the gradient of a scalar function Φ:

$$\underline{u} = \underline{\nabla}\Phi$$

Φ is known as a velocity potential for $\underline{u}$. Unlike a stream function, a velocity potential can exist in three-dimensional flow. The stream function is defined for two-dimensional flows of various kinds. The stream function can be used to plot streamlines In most cases, the stream function is the imaginary part of the complex potential, while the potential function is the real part.

The general procedure for solving a potential flow problem is:

1. to guess a proper potential function Φ

2. check that it satisfies the Laplace equation $\underline{\nabla}^2\Phi = 0$

3. check whether the corresponding velocity field $\underline{v} = \underline{\nabla}\Phi$ satisfies the boundary conditions

9.1   Stream Function

The stream function is defined in general only for 2-D flows. There are some special cases of 3-D flows where the stream function is used, but for this class we will consider only the stream function for 2-D flow. Let stream function be defined so that

$$v_x = \frac{\partial \psi}{\partial y} \quad v_y = -\frac{\partial \psi}{\partial x}$$

Definition 3   Streamline   *Curve that is everywhere tangent to velocity field*

Looking at a velocity vector and the streamline, we can see there are similar triangles.

So along a streamline we have

$$\frac{dy}{dx} = \frac{v_y}{v_x}$$

which can be written

$$v_x dy - v_y dx = 0$$

Substituting the definition for stream function in

$$\frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = 0$$

Notice that this is the derivative $d\psi$ as evaluated using chain rule, giving

$$d\psi = 0$$

So this shows the important fact that *the stream function is constant along stream lines*. Schwarz's theorem: if ψ has continuous 2nd order partial derivatives (*when is this true for the stream function?*) over all space, then

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)$$

$$\frac{\partial}{\partial x} v_x = -\frac{\partial}{\partial y} v_y$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

And so stream function is defined in a way that automatically satisfies continuity for incompressible flow. Units of stream function, as mass flow between two streamlines.

Example 5 Volumetric flow rate between two streamlines

Definition 4 Pathline *Locus of points through which a particle of fixed identity has traveled*

For steady flow path lines and streamlines are identical. See wikipedia streamline page video.

Chapter 10

Vorticity

The vorticity ω is defined as the curl of the velocity field:

$$\underline{\omega} = \underline{\nabla} \times \underline{u}$$

Kelvin's circulation theorem states that the circulation Γ does not change with respect to time.

$$\Gamma = \oint_C \underline{v} \cdot \underline{dl}$$

Recall Euler's equation (2.1).

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \, \underline{v} \right) = -\underline{\nabla} p + \rho \underline{g}$$

Simplify for steady flow $\partial \underline{v}/\partial t = 0$ gives

$$\rho(\underline{v} \cdot \underline{\nabla} \, \underline{v}) = -\underline{\nabla} p + \rho \underline{g}$$

Use the identity, which is a general form of $\underline{\nabla}(A \cdot B)$

$$\frac{1}{2}\underline{\nabla}(\underline{v} \cdot \underline{v}) = \underline{v} \times \underline{\nabla} \times \underline{v} + \underline{v} \cdot \underline{\nabla} \, \underline{v}$$

$$\rho \left(\frac{1}{2}\underline{\nabla}(\underline{v} \cdot \underline{v}) - \underline{v} \times \underline{\nabla} \times \underline{v} \right) = -\underline{\nabla} p + \rho \underline{g}$$

$$\rho(\underline{v} \times \underline{\nabla} \times \underline{v}) = \rho \frac{1}{2}\underline{\nabla}(\underline{v} \cdot \underline{v}) + \underline{\nabla} p - \rho \underline{g}$$

And we can substitute in for gravity $\underline{g} = -g\underline{\nabla}h$ giving

$$\rho(\underline{v} \times \underline{\nabla} \times \underline{v}) = \rho \frac{1}{2}\underline{\nabla}(\underline{v} \cdot \underline{v}) + \underline{\nabla} p + \rho g \underline{\nabla} h$$

$$\rho(\underline{v} \times \underline{\nabla} \times \underline{v}) = \underline{\nabla} \left(\rho \frac{1}{2} v^2 + p + \rho g h \right)$$

To look at this along a streamline take the unit vector s along a streamline and dot it onto both sides. The directional derivative by definition is $\frac{\partial f}{\partial s} = \underline{\nabla} f \cdot s$. And so we have the change in Bernoulli's along a

streamline. On the left hand side, orthogonality of the vectors dotted into $\underline{s}$ gives zero. Looking at how this quantity changes normal to a streamline, we take a unit vector $\underline{n}$ normal to the streamline.

$$\rho(\underline{v} \times \underline{\omega}) = \underline{\nabla}\left(\rho\frac{1}{2}v^2 + p + \rho gh\right)$$

Integrate both sides from 1 to 2.

Chapter 11

Boundary Layers

The essential characteristics of regions described by boundary layer theory are that they are thin and that they have steep velocity gradients that make the viscous effects important. See Panton page 418.

11.1 Derivation of the Boundary Layer Equations: Cartesian Coordinates

Derivation Outline and Assumptions

- Start with Navier-Stokes equation in the $x-$ and y-direction, and conservation of mass

- Assume **steady, 2-D flow** and **neglect gravity** to simplify the Navier-Stokes equations

- Nondimensionalize to get the boundary layer equations

  - Assume during the non-dimensionalization that **Re**$>> 1$
  - $\left(\frac{L}{\delta}\right)^2 \sim \text{Re} >> 1$ so $\left(\frac{L}{\delta}\right)^2 >> 1$
  - **Laminar**

Start with Navier-Stokes equation in the x-direction and conservation of mass as shown below

$$\textbf{NSE } (\boldsymbol{x})\textbf{:} \qquad \rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right)$$

$$= \mu\left[\frac{\partial^2}{\partial x^2}v_x + \frac{\partial^2}{\partial y^2}v_x + \frac{\partial^2}{\partial z^2}v_x\right] - \frac{\partial p}{\partial x} + \rho g_x$$

$$\textbf{Continuity:} \qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

The x Navier-Stokes equation can be simplified by only considering 2-D steady flow, where gravity is in the y-direction, reducing this equation to

$$\rho\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = \mu\left[\frac{\partial^2}{\partial x^2}v_x + \frac{\partial^2}{\partial y^2}v_x\right] - \frac{\partial p}{\partial x}$$

11.1.1 Non-Dimensionalize to Get Boundary Layer Equations

Define the following dimensionless quantities

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{\delta} \qquad v_x^* = \frac{v_x}{U_\infty} \qquad v_y^* = \frac{v_y}{V_c} \qquad p^* = \frac{p}{p_c}$$

where V_c is a characteristic velocity. Solving for the dimensional quantities in terms of the dimensionless ones, we have

$$x = x^* L \qquad y = y^* \delta \qquad v_x = v_x^* U_\infty \qquad v_y = v_y V_c \qquad p = p^* p_c$$

Non-dimensionalizing Conservation of Mass

Substituting in the dimensional variables

$$\frac{U_\infty}{L} \frac{\partial v_x^*}{\partial x^*} + \frac{V_c}{\delta} \frac{\partial v_y^*}{\partial y^*} = 0$$

And so from this we see

$$\frac{U_\infty}{L} \sim \frac{V_c}{\delta}$$

and so our characteristic length scale is

$$L \sim \frac{U_\infty \delta}{V_c}$$

Nondimensionalizing Navier-Stokes Equations in x-Direction

Plugging all the dimensional variables into the simplified Navier-Stokes equation we have

$$\rho \left(v_x^* U_\infty \frac{\partial (v_x^* U_\infty)}{\partial (x^* L)} + v_y^* V_c \frac{\partial (v_x^* U_\infty)}{\partial (y^* \delta)} \right) = \mu \left[\frac{\partial^2 (v_x^* U_\infty)}{\partial (x^* L)^2} + \frac{\partial^2 (v_x^* U_\infty)}{\partial (y^* \delta)^2} \right] - \frac{\partial (p^* p_c)}{\partial (x^* L)}$$

Pulling out the dimensional quantities from the derivative terms

$$\frac{U_\infty^2}{L} v_x^* \frac{\partial v_x^*}{\partial x^*} + \frac{V_c U_\infty}{\delta} v_y^* \frac{\partial v_x^*}{\partial y^*} = \frac{\mu}{\rho} \left[\frac{U_\infty}{L^2} \frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{U_\infty}{\delta^2} \frac{\partial^2 v_x^*}{\partial y^{*2}} \right] - \frac{p_c}{L\rho} \frac{\partial p^*}{\partial x^*}$$

Dividing through

$$v_x^* \frac{\partial v_x^*}{\partial x^*} + \frac{V_c L}{\delta U_\infty} v_y^* \frac{\partial v_x^*}{\partial y^*} = \nu \left[\frac{1}{L U_\infty} \frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{L}{\delta^2 U_\infty} \frac{\partial^2 v_x^*}{\partial y^{*2}} \right] - \frac{p_c}{\rho U_\infty^2} \frac{\partial p^*}{\partial x^*}$$

Using the result of non-dimensionalizing conservation of mass, and recognizing the Reynolds number term this becomes

$$v_x^* \frac{\partial v_x^*}{\partial x^*} + v_y^* \frac{\partial v_x^*}{\partial y^*} = \frac{1}{\text{Re}} \frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{\nu L}{\delta^2 U_\infty} \frac{\partial^2 v_x^*}{\partial y^{*2}} - \frac{p_c}{\rho U_\infty^2} \frac{\partial p^*}{\partial x^*}$$

Manipulating one of the terms on the right hand side this gives

$$v_x^* \frac{\partial v_x^*}{\partial x^*} + v_y^* \frac{\partial v_x^*}{\partial y^*} = \frac{1}{\text{Re}} \frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{\nu}{L U_\infty} \frac{L^2}{\delta^2} \frac{\partial^2 v_x^*}{\partial y^{*2}} - \frac{p_c}{\rho U_\infty^2} \frac{\partial p^*}{\partial x^*}$$

Now we recognize this as another coefficient with the Reynolds number in it

$$v_x^* \frac{\partial v_x^*}{\partial x^*} + v_y^* \frac{\partial v_x^*}{\partial y^*} = \frac{1}{\text{Re}} \frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{1}{\text{Re}} \frac{L^2}{\delta^2} \frac{\partial^2 v_x^*}{\partial y^{*2}} - \frac{p_c}{\rho U_\infty^2} \frac{\partial p^*}{\partial x^*}$$

We can also see what the characteristic pressure needs to be to satisfy this non dimensional equation

$$p_c = \rho U_\infty^2$$

giving

$$v_x^* \frac{\partial v_x^*}{\partial x^*} + v_y^* \frac{\partial v_x^*}{\partial y^*} = \frac{1}{\text{Re}} \frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{1}{\text{Re}} \frac{L^2}{\delta^2} \frac{\partial^2 v_x^*}{\partial y^{*2}} - \frac{\partial p^*}{\partial x^*}$$

So when the Reynolds number gets big, the first term with Reynolds number in the denominator goes away. But in order for the second viscous term to remain when the Reynolds number gets big, we need the following scaling relationship to hold

$$\frac{L^2}{\delta^2} \sim \text{Re}$$

So this gives that δ needs to scale as

$$\delta \sim \frac{L}{\sqrt{\text{Re}}}$$

With the assumption that Reynolds number is large, and with δ scaling as above, the boundary layer equation becomes

$$v_x^* \frac{\partial v_x^*}{\partial x^*} + v_y^* \frac{\partial v_x^*}{\partial y^*} = \frac{1}{\text{Re}} \frac{L^2}{\delta^2} \frac{\partial^2 v_x^*}{\partial y^{*2}} - \frac{\partial p^*}{\partial x^*}$$

The dimensionless quantities can then be substituted back in for, yielding the dimensional form of the x-momentum equation for a boundary layer.

Nondimensionalizing Navier-Stokes Equations in y-Direction

11.1.2   Boundary Layer Equations in Cartesian Summary

To summarize, the simplified equations are shown below.

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Boundary Layer Equations: Cartesian

x-**momentum**	$\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = \mu \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial p}{\partial x}$
y-**momentum**	$\frac{\partial p}{\partial y} = 0$
Continuity	$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

</div>

We can see the shape that the boundary layer makes over a surface depending on the pressure gradient in the x-direction. That is, at the surface we have $v_x = v_y = 0$, so for $\frac{\partial p}{\partial x} < 0$, $\frac{\partial p}{\partial x} = 0$, and $\frac{\partial p}{\partial x} > 0$ we can see the $\frac{\partial v_x}{\partial y}$, that is, the curvature of the velocity profile.

Displacement thickness	$\delta^* = \displaystyle\int_0^\infty \left(1 - \frac{v_x(y)}{U_\infty} \right) dy$
Momentum thickness	$\theta = \displaystyle\int_0^\infty \frac{v_x}{U_\infty} \left(1 - \frac{v_x(y)}{U_\infty} \right) dy$
Momentum Integral Equation	$\dfrac{d}{dx}(U^2\theta) + \delta^* U \dfrac{dU}{dx} = \dfrac{\tau_0}{\rho}$

$$\delta \sim \sqrt{\nu t^*}$$

$$t^* = \frac{x}{U_\infty}$$

Chapter 12

Surface Tension

Surface tension is not a property of materials but of interfaces between two (or more) materials. It is implicit in its definition that the interface separates two kinds of materials that behave differently (otherwise the interface would be just some imaginary surface inside the one material with no physical meaning) and so there must always be some surface tension that sustains the physical interface. First of all, Marek is right that a surface tension exists only between two different materials (well, I would say between two different phases, for example water and ice)

σ is the surface tension.
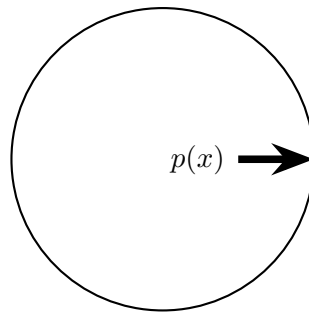
Surface tension for cylinder: $\qquad \Delta p = \dfrac{\sigma}{r}$

Surface tension for sphere: $\qquad \Delta p = \dfrac{2\sigma}{r}$

Young-Laplace equation: $\qquad \Delta p = \sigma \left(\dfrac{1}{r_x} + \dfrac{1}{r_y} \right)$

Contact angle α is a property of the fluid, the material it is touching, and the third fluid it is in (like air).

Young's equation: $\qquad \cos\theta_E = \dfrac{\sigma_{sv} - \sigma_{sl}}{\sigma_{lv}}$

where s, l, and v are solid, liquid, and vapor, respectively, and θ_E is the equilibrium contact angle.

Chapter 13

Appendix

Volume of sphere:	$V = \dfrac{4}{3}\pi r^3$

Surface area of sphere:	$A = 4\pi r^2$

Power of fan or pump where $\dot{W}$ is the time derivative of work, aka power, and η is the fan or pump efficiency.

$$\eta\dot{W} = \frac{1}{2}\rho v^2 Q$$

Work of fluid with velocity U applying force F

$$W_{\text{ext}} = FU$$

Power is

$$P = F_{\text{ext}}U$$

Work is

$$W = Q\Delta p$$

13.1 Equation Summary Sheet

Bernoulli's along streamline:	$\dfrac{1}{2}\rho v_{s2}^2 + p_2 + \rho g z_2 = \dfrac{1}{2}\rho v_{s1}^2 + p_1 + \rho g z_1$

Reynolds transport theorem (form A)

$$\frac{d}{dt}\int_{CV(t)}\phi dV + \int_{CS}\phi(\underline{v} - \underline{v}_c)\cdot\underline{n}dA = \frac{d}{dt}\int_{MV}\phi dV$$

Material Derivative

$$\rho \frac{D\underline{v}}{Dt} = \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right)$$

Navier Stokes Equation

$$\rho \frac{D\underline{v}}{Dt} = -\underline{\nabla} p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Incompressible Navier-Stokes	$\rho \left(\dfrac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\underline{\nabla} p + \mu \nabla^2 \underline{v} + \rho \underline{g}$
	$\rho \dfrac{D\underline{v}}{Dt} = -\underline{\nabla} p + \underline{\nabla} \cdot \underline{\underline{\sigma}} + \rho \underline{g}$
	$\rho \dfrac{D\underline{v}}{Dt} = -\underline{\nabla} p + \mu \nabla^2 \underline{v} + \rho \underline{g}$

Mass Conservation

Form A: $\dfrac{d}{dt} \displaystyle\int_{CV(t)} \rho dV + \int_{CS(t)} \rho(\overline{v} - \overline{v}_c) \cdot \overline{n} dA = 0$

Form B: $\displaystyle\int_{CV(t)} \dfrac{\partial \rho}{\partial t} dV + \int_{CS(t)} \rho v_n dA = 0$

$$v_{rn} = (\overline{v} - \overline{v}_c) \cdot \overline{n}$$

$\overline{v}$ is the velocity across the control surface.

Momentum Conservation

Form A: $\dfrac{d}{dt} \displaystyle\int_{CV(t)} \rho \overline{v} dV + \int_{CS(t)} \rho \overline{v}(\overline{v} - \overline{v}_c) \cdot \overline{n} dA = \overline{F}_{CV}(t)$

Form B: $\displaystyle\int_{CV(t)} \dfrac{\partial (\rho \overline{v})}{\partial t} dV + \int_{CS(t)} \rho \overline{v} v_n dA = \overline{F}_{CV}(t)$

Vorticity:	$\underline{\omega} = \underline{\nabla} \times \underline{v}$

Derivative of Error Function:	$\dfrac{d}{dz} \mathrm{erf}(z) = \dfrac{2}{\sqrt{\pi}} e^{-z^2}$

13.2 Stokes

13.2.1 From Stokes First Problem

look at the dimensional analysis sheet and see that the time scale for the diffusion of viscous effects into the fluid are like

$$t_c \sim \frac{L^2}{\nu}$$

and we need to pick a characteristic length scale, which is usually the boundary layer thickness δ. This gives

$$t_c \sim \frac{\delta^2}{\nu}$$

Solving for δ we have

$$\delta \sim \sqrt{\nu t_c}$$

And we can approximate the shear stress as a linear velocity profile over the boundary layer

$$\tau_w \sim \frac{\mu U}{\delta} \sim \frac{\mu U}{\sqrt{\nu t}}$$

To find boundary layer growth, we have the characteristic time scale for convection

$$t_c = \frac{L}{U}$$

so Blasius boundary layer grows like

$$\delta \sim \sqrt{\frac{\nu L}{U}}$$

which is the solution for the growth over a flat plate. But the boundary layer is usually really small compared to the radius of curvature of non-flat surfaces, so we can pretty much use this always.

Lubrication Theory Equations: Cartesian

x-**momentum** $\qquad \mu \dfrac{d^2 v_x}{dy^2} - \dfrac{\partial p}{\partial x} + \rho g_x = 0$

y-**momentum** $\qquad \rho g_y - \dfrac{\partial p}{\partial y} = 0$

Continuity $\qquad \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0$

13.2.2 Boundary Layer Equations in Cartesian Summary

To summarize, the simplified equations are shown below.

Boundary Layer Equations: Cartesian

x-**momentum** $\qquad \rho \left(v_x \dfrac{\partial v_x}{\partial x} + v_y \dfrac{\partial v_x}{\partial y} \right) = \mu \dfrac{\partial^2 v_x}{\partial y^2} - \dfrac{\partial p}{\partial x}$

y-**momentum** $\qquad \dfrac{\partial p}{\partial y} = 0$

Continuity $\qquad \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0$

We can see the shape that the boundary layer makes over a surface depending on the pressure gradient in the x-direction. That is, at the surface we have $v_x = v_y = 0$, so for $\frac{\partial p}{\partial x} < 0$, $\frac{\partial p}{\partial x} = 0$, and $\frac{\partial p}{\partial x} > 0$ we can see the $\frac{\partial v_x}{\partial y}$, that is, the curvature of the velocity profile.

Displacement thickness
$$\delta^* = \int_0^\infty \left(1 - \frac{v_x(y)}{U_\infty}\right) dy$$

Momentum thickness
$$\theta = \int_0^\infty \frac{v_x}{U_\infty}\left(1 - \frac{v_x(y)}{U_\infty}\right) dy$$

Momentum Integral Equation
$$\frac{d}{dx}(U^2\theta) + \delta^* U \frac{dU}{dx} = \frac{\tau_0}{\rho}$$

$$\delta \sim \sqrt{\nu t^*}$$

$$t^* = \frac{x}{U_\infty}$$

Young-Laplace equation:
$$\Delta p = \sigma\left(\frac{1}{r_x} + \frac{1}{r_y}\right)$$

Surface Tension Energy:
$$dE = \sigma dA$$
$$E_{\text{surf}} = \sigma A_{\text{surf}}$$

13.3   What Formulas to use When

If the question says, show that a given functional form satisfies some governing equation, rather than trying to derive the given functional form from some governing equation, instead pick a governing equation and plug the given functional form in just to check that it satisfies it. Relating u to v: think continuity! When asked to compare gravity and surface tension forces, compare hydrostatic pressure of the whole column of fluid to the pressure due to surface tension as calculated from the Young-Laplace equation.

13.4   Stuff to Remember for Quals

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] - \frac{\partial p}{\partial x} + \rho g_x$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z - \frac{\partial p}{\partial z}$$

$$\Delta p = \sigma \left(\frac{1}{r_x} + \frac{1}{r_y} \right)$$

Boundary layer:

- Start with Navier-Stokes equation in the $x-$ and y-direction, and conservation of mass
- Assume **steady, 2-D flow** and **neglect gravity** to simplify the Navier-Stokes equations
- Nondimensionalize to get the boundary layer equations

  – Assume during the non-dimensionalization that **Re$>> 1$**
  – $\left(\frac{L}{\delta} \right)^2 \sim$ Re $>> 1$ so $\left(\frac{L}{\delta} \right)^2 >> 1$
  – **Laminar**

$$\frac{d}{dt} \int_{CV(t)} \rho \bar{v} dV + \int_{CS(t)} \rho \bar{v} (\bar{v} - \bar{v}_c) \cdot \bar{n} dA = \overline{F}_{CV}(t)$$

Part II

Dynamics and Modeling

Chapter 14

Linear and Angular Momentum Principles

14.1 Single Particle
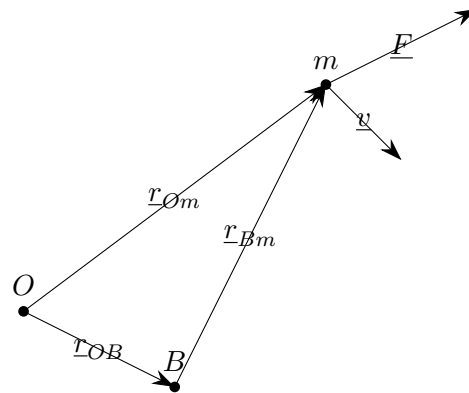
Figure 14.1: **Point mass m under action of force $\underline{f}$. Point O is fixed in inertial space, and point B is a general point, not necessarily fixed in inertial space.**

Linear momentum:		$\underline{p} = m\underline{v}$

Linear momentum principle:		$\underline{f} = \dfrac{d\underline{p}}{dt}$

Torque about point O:		$\underline{\tau}_O = \underline{r}_{Om} \times \underline{f}$

Moment of momentum about point O:		$\underline{h}_O = \underline{r}_{Om} \times \underline{p}$

differentiate expression for h_O

$$\frac{d}{dt}\underline{h}_O = \frac{d}{dt}\underline{r}_{Om} \times \underline{p} + \underline{r}_{Om} \times \frac{d}{dt}\underline{p}$$

but

$$\frac{d}{dt}\underline{r}_{Om} = \underline{v}$$

and since $\underline{p} = m\underline{v}$ the cross product $\frac{d}{dt}\underline{r}_{Om} \times \underline{p} = 0$ and so

$$\frac{d}{dt}\underline{h}_O = \underline{r}_{Om} \times \frac{d}{dt}\underline{p}$$

and substituting the linear momentum principle $\frac{dp}{dt} = \underline{f}$ this gives

$$\frac{d}{dt}\underline{h}_O = \underline{r}_{Om} \times \underline{f}$$

But the right hand side is just the torque about point O, so we have

<div style="border:1px solid red;">

Angular momentum principle of particle about point O: $\qquad \underline{\tau}_O = \frac{d}{dt}\underline{h}_O$

</div>

About a general point B the torque and moment of momentum can be expressed as

$$\underline{\tau}_B = \underline{r}_{Bm} \times \underline{f}$$
$$\underline{h}_B = \underline{r}_{Bm} \times \underline{p}$$

Using $\underline{r}_{Bm} = \underline{r}_{Om} - \underline{r}_{OB}$ and substituting in we have

$$\begin{aligned}\underline{\tau}_B &= (\underline{r}_{Om} - \underline{r}_{OB}) \times \underline{f} \\ &= \underline{r}_{Om} \times \underline{f} - \underline{r}_{OB} \times \underline{f} \\ &= \underline{\tau}_O - \underline{R}_{OB} \times \underline{f}\end{aligned}$$

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Torque on particle about general point B: $\qquad \underline{\tau}_B = \underline{\tau}_O - \underline{r}_{OB} \times \underline{f}$

</div>

and

$$\begin{aligned}\underline{h}_B &= (\underline{r}_{Om} - \underline{r}_{OB}) \times \underline{p} \\ &= \underline{r}_{Om} \times \underline{p} - \underline{r}_{OB} \times \underline{p} \\ &= \underline{h}_O - \underline{r}_{OB} \times \underline{p}\end{aligned}$$

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Moment of momentum of particle about gen. point B: $\qquad \underline{h}_B = \underline{h}_O - \underline{r}_{OB} \times \underline{p}$

</div>

Evaluate the following

$$\frac{d}{dt}\underline{h}_B = \frac{d}{dt}\underline{h}_O - \frac{d}{dt}\underline{r}_{OB} \times \underline{p} - \underline{r}_{OB} \times \frac{d}{dt}\underline{p}$$
$$= \underline{\tau}_O - \underline{v}_B \times \underline{p} - \underline{r}_{OB} \times \underline{f}$$

From above we have $\underline{r}_{OB} \times \underline{f} = \underline{\tau}_O - \underline{\tau}_B$ and so we can write

$$\frac{d}{dt}\underline{h}_B = \underline{\tau}_O - \underline{v}_B \times \underline{p} - \underline{\tau}_O + \underline{\tau}_B$$
$$= -\underline{v}_B \times \underline{p} + \underline{\tau}_B$$

solving for $\underline{\tau}_B$ this gives

> **Angular momentum principle of particle about gen. point B:** $\qquad \underline{\tau}_B = \dfrac{d}{dt}\underline{h}_B + \underline{v}_B \times \underline{p}$

14.2   General System (System of Particles)
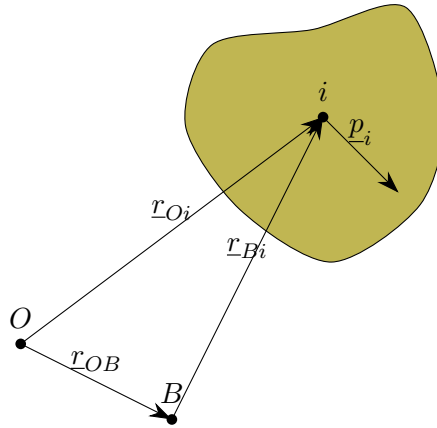
Figure 14.2: **General system which is made up of many masses m_i, with total mass M. There are forces $\underline{f}_i$ acting on the i particles. Point O is fixed in inertial space, and point B is a general point, not necessarily fixed in inertial space.**

> **Linear momentum principle for general system:** $\qquad \underline{F}^{\text{ext}} = \dfrac{d}{dt}\underline{P}$

$$\underline{H}_O = [I]_O \omega$$

To find the angular momentum principle for general system about general point B, we take the derivative of above expression

$$\frac{d}{dt}\underline{H}_B = \frac{d}{dt}\underline{H}_O - \frac{d}{dt}\underline{r}_{OB} \times \underline{P} - \underline{r}_{OB} \times \frac{d}{dt}\underline{P}$$
$$= \underline{\tau}_O^{\text{ext}} - \underline{v}_B \times \underline{P} - \underline{r}_{OB} \times \underline{F}^{\text{ext}}$$

but we have that

$$\underline{F}^{\text{ext}} = \sum_i \underline{f}_i^{\text{ext}}$$

The angular momentum principle of a system about a general point is the following

If we have another general point A we can write

14.3   Kinematics of Rigid Bodies
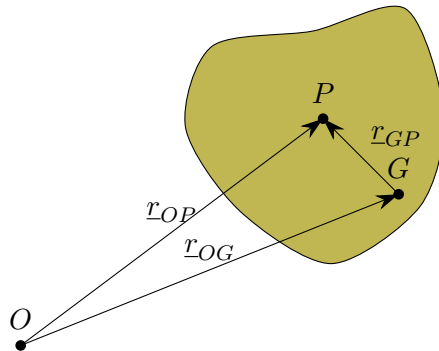
Figure 14.3: **Rigid Body. Point O is fixed in inertial space, and point B is a general point, not necessarily fixed in inertial space.**

Velocity of point P on rigid body relative to point G on body	$\underline{v}_P = \underline{v}_G + \underline{\omega} \times \underline{r}_{GP}$

14.3.1 Moment of Inertia

Moment of inertia	$\underline{H}_B = [I]_B \omega$

To find principle axes

$$I_{\text{principal}} - \lambda I = 0$$

14.4 Impulse

Take linear and angular momentum principle for a rigid body. The linear and angular momentum principles for a system about a general point are given as follows.

Linear momentum principle for general system:	$\underline{F}^{\text{ext}} = \dfrac{d}{dt}\underline{P}$
Angular momentum principle about general point B:	$\underline{\tau}_B^{\text{ext}} = \dfrac{d}{dt}\underline{H}_B + \underline{v}_B \times \underline{P}$

and separate them and integrate them over a short period of time

$$\int_{t=0^-}^{t=0^+} \underline{F}^{\text{ext}} dt = \int_{\underline{P}(0^-)}^{\underline{P}(0^+)} d\underline{P}$$

$$\int_{t=0^-}^{t=0^+} \underline{\tau}_B^{\text{ext}} dt = \int_{\underline{H}_B(0^-)}^{\underline{H}_B(0^+)} d\underline{H}_B + \int_{t=0^-}^{t=0^+} \underline{v}_B \times \underline{P} dt$$

$$\Delta \underline{P} = \int_{t=0^-}^{t=0^+} \underline{F}^{\text{ext}} dt$$

Chapter 15

Work and Energy Principles
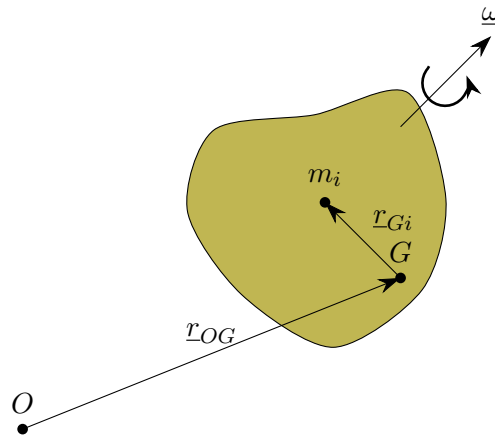
Figure 15.1: **Rigid Body. Point O is fixed in inertial space, and point B is a general point, not necessarily fixed in inertial space.**

The kinetic energy of this rigid body is

$$T = \sum_i \frac{1}{2} m_i \underline{v}_i \cdot \underline{v}_i$$

In a rigid body $\underline{v}_i$ is given by

$$\underline{v}_i = \underline{v}_G + \underline{\omega} \times \underline{r}_i$$

Substituting this in we get the following expression

$$T = \frac{1}{2} \left(\sum_i m_i \right) \underline{v}_G \cdot \underline{v}_G + \underline{v}_G \cdot \left(\underline{\omega} \times \sum_i m_i \underline{r}_i \right) + \frac{1}{2} \sum_i m_i (\underline{\omega} \times \underline{r}_i) \cdot (\underline{\omega} \times \underline{r}_i)$$

The middle term drops out if $\underline{v}_G = 0$ or if G is at the center of mass of the body, allowing the kinetic energy to be written as

KE of rigid body about CM:
$$T = \frac{1}{2}M|\underline{v}_G|^2 + \frac{1}{2}\sum_i m_i(\underline{\omega} \times \underline{r}_i) \cdot (\underline{\omega} \times \underline{r}_i)$$

KE of rigid body rotating about CM:
$$T = \frac{1}{2}Mv_G^2 + \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right)$$

15.1   Finding Center of Mass and Moment of Inertia

Finding CM:
$$x_{cm} = \frac{\sum_i A_i r_i}{\sum_i A_i}$$

Finding CM:
$$z_{cm} = \frac{\int_m z\,dm}{\int_m dm}$$

Finding CM:
$$z_{cm} = \frac{\int_V z\,dV}{\int_V dV}$$

Parallel axis theorem:
$$I_B = I_0 + Mh^2$$

$$I_x = \int_m (y^2 + z^2)dm$$
$$I_y = \int_m (x^2 + z^2)dm$$
$$I_z = \int_m (x^2 + y^2)dm$$

constant density

$$I_x = \rho \int_V (y^2 + z^2)dV$$
$$I_y = \rho \int_V (x^2 + z^2)dV$$
$$I_z = \rho \int_V (x^2 + y^2)dV$$

Example 6 Cylinder
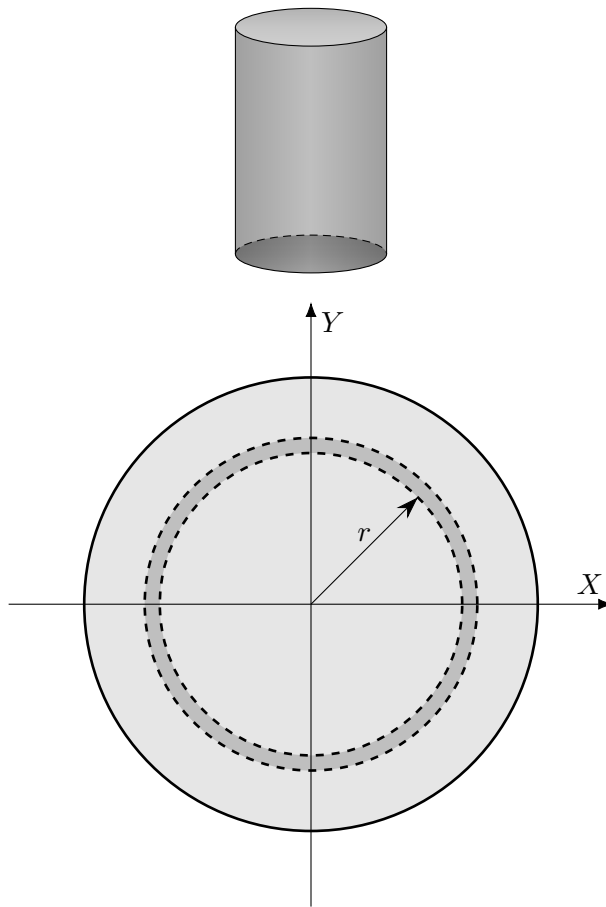
By definition, the moment of inertia is

$$I_{\text{cylinder},z} = \int_V \rho(x^2 + y^2)dV$$

The volume of a cylinder is given by

$$V = \pi r^2 L$$

A differential volume element is given by

$$dV = 2\pi r L dr$$

where $r^2 = x^2 + y^2$. Substituting this into the expression for moment of inertia

$$I_{\text{cylinder}} = \int_0^R \rho r^2 2\pi r L dr$$

$$= 2\pi L \rho \int_0^R r^3 dr$$

$$I_{\text{cylinder}} = \frac{\pi L \rho R^4}{2}$$

But, for a cylinder, $M = \rho V = \rho \phi R^2 L$ giving

$$\boxed{I_{\text{cylinder}} = \frac{1}{2}MR^2}$$

Example 7 Rectangular Cube
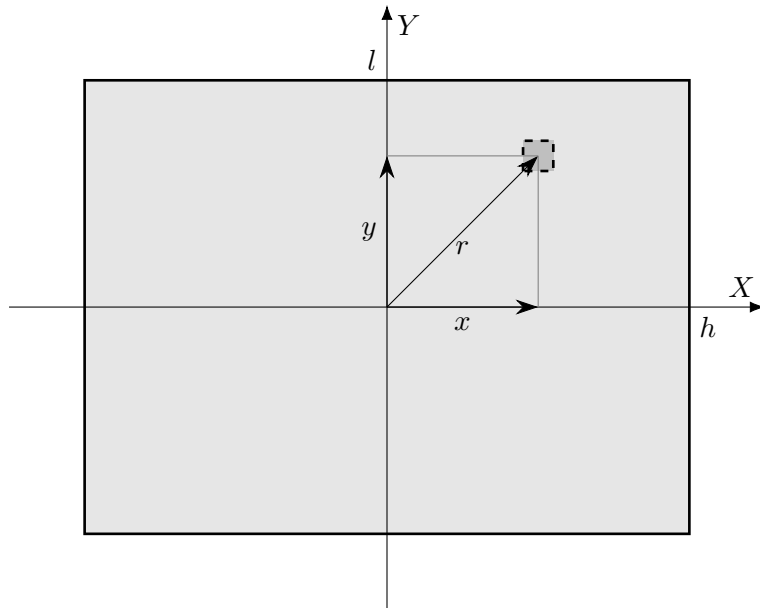
This cube has depth w into the page. z axis is coming out of the page. By definition, the moment of inertia is

$$I_{\text{cube},z} = \int_V \rho(x^2 + y^2)dV$$

The volume of a cube is given by

$$V = wxy$$

A differential volume element is given by

$$dV = wdxdy$$

Substituting this into the expression for moment of inertia

$$I_{\text{cube},z} = \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(x^2 + y^2) w \, dx \, dy$$

$$I_{\text{cube},z} = \rho w \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} (yx^2 + \frac{y^3}{3}) dx \right]_{y=-\frac{l}{2}}^{y=\frac{l}{2}}$$

$$= \rho w \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{l}{2}x^2 + \frac{l^3}{24} \right) - \left(-\frac{l}{2}x^2 - \frac{l^3}{24} \right) dx$$

$$= \rho w \int_{-\frac{h}{2}}^{\frac{h}{2}} lx^2 + \frac{l^3}{12} dx$$

$$= \rho w \left[\frac{l}{3}x^3 + \frac{l^3}{12}x \right]_{x=-\frac{h}{2}}^{x=\frac{h}{2}}$$

$$= \rho w \left[\left(\frac{lh^3}{24} + \frac{hl^3}{24} \right) - \left(-\frac{lh^3}{24} - \frac{hl^3}{24} \right) \right]_{x=-\frac{h}{2}}^{x=\frac{h}{2}}$$

$$= \rho w \left(\frac{lh^3}{12} + \frac{hl^3}{12} \right)$$

$$= \rho w l h \left(\frac{h^2 + l^2}{12} \right)$$

$$\boxed{I_{\text{cube},z} = m \frac{h^2 + l^2}{12}}$$

Example 8 Sphere

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3$$

Example 9 Cone

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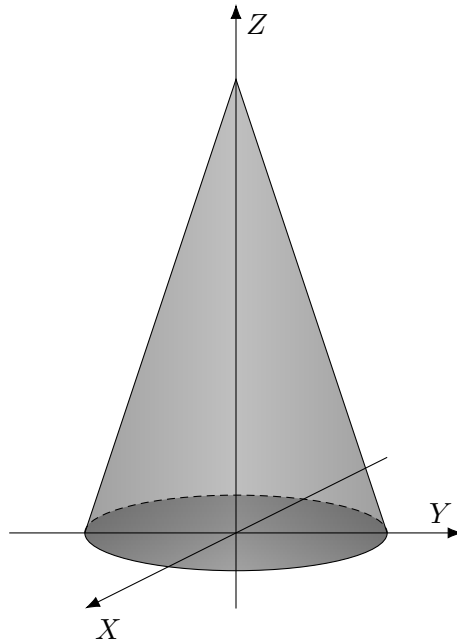
Find the centroid. The cone is symmetric about the z axis, so the centroid is along the z axis, and we just need to find where.

$$z_{cm} = \frac{\int_m z\, dm}{\int_m dm}$$

Since the cone has constant density we have that $dm = \rho dV$ and so this equation becomes the following, where the constant density is pulled out of the integral and cancels

$$z_{cm} = \frac{\int_V z\, dV}{\int_V dV}$$

We integrate over this volume by using a stack of thin disks, and integrating from $z = 0$ to $z = h$. Each disk has radius r. We find the formula for the disk radius with z as

$$r = R - \frac{R}{h}z$$

A differential volume element is given by

$$dV = \pi r^2 dz$$
$$= \pi \left(R - \frac{R}{h}z \right)^2 dz$$
$$= \pi R \left(1 - \frac{1}{h}z \right)^2 dz$$
$$= \pi R \left(1 - \frac{2z}{h} + \frac{z^2}{h^2} \right) dz$$

integrating

Cylinder radius r:	$I_{\text{cylinder}} = \dfrac{1}{2}mr^2$
Sphere radius r:	$I_{\text{sphere}} = \dfrac{2}{5}mr^2$
Rod length L about end:	$I_{\text{rod,end}} = \dfrac{1}{3}mL^2$
Rod length L about center:	$I_{\text{rod,center}} = \dfrac{1}{12}mL^2$

Chapter 16

Lagrange

Holonomic system: the number of independent coordinates in the large is equal to the number of independent admissible variations. Non-holonomic system is not holonomic. Usually the number of admissible variations are less than the number of generalized coordinates. Usually we can see nonholonomic systems by allowing the system to undergo large motions. Remember example of coin rolling without slipping on table.

1. choose generalized coordinates in the large $\xi_1, \ldots, \xi_n$

   - A complete set of coordinates should be able to exactly define the orientation of the system without ambiguity

   - An independent set of coordinates should not have any redundancy. In other words, none of the generalized coordinates should be able to be written in terms of the others. Should be the minimal amount of coordinates that can describe the orientation of the system

2. Consider displacement about each of these coordinates one at a time, while holding all the others fixed. Calculate the incremental work δW done by the external forces under the incremental displacement $\delta \xi_j$. Express this incremental work as $\delta W = \Xi_j \delta \xi_j$ to see what the generalized force is. Repeat this for each of the generalized coordinates.

3. Find kinetic and potential energy T and V for the system.

4. Make lagrangian $\mathscr{L} = T - V$

5. Use the formula
$$\frac{d}{dt} \left(\frac{\partial \mathscr{L}}{\partial \dot{\xi}_j} \right) - \frac{\partial \mathscr{L}}{\partial \xi_j} = \Xi_j$$

That gives us the equations of motion

Chapter 17

Problems
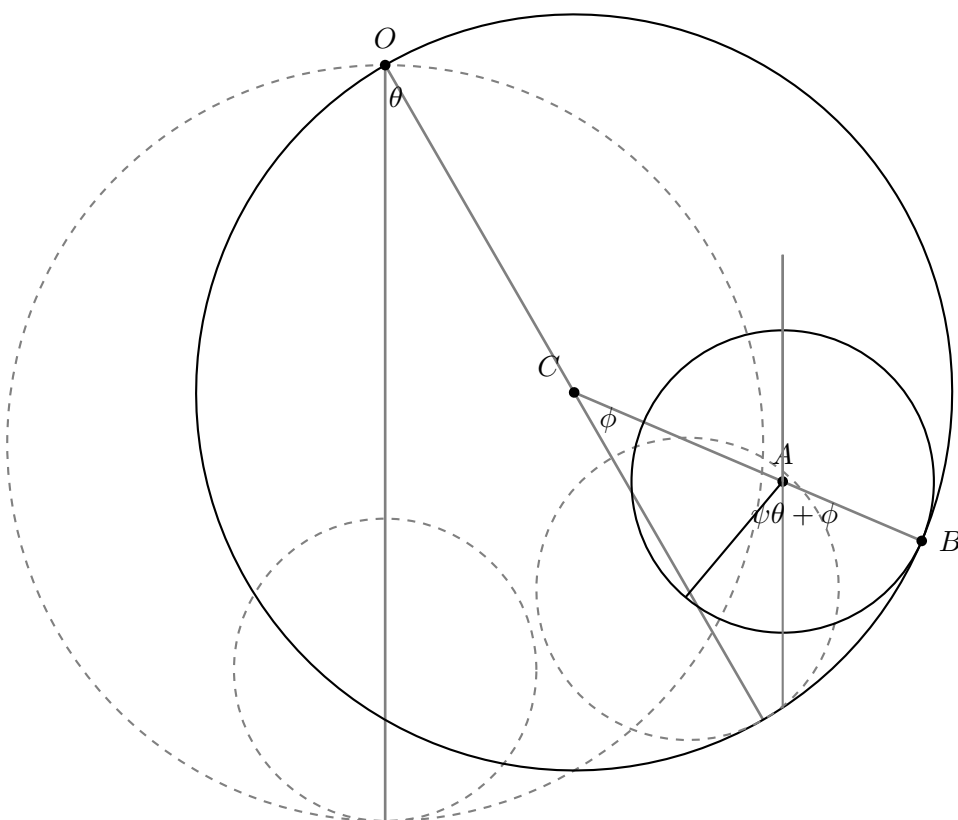
Looking at the arc lengths, and how the cylinder rolled inside the ring, with ψ the absolute angular displacement of the cylinder, we have

$$(\psi + \theta + \phi)r = \phi R$$
$$\psi = \frac{\phi(R - r) - \theta r}{r}$$
$$\psi = \phi\left(\frac{R}{r} - 1\right) - \theta$$

and so differentiating to get the angular velocity of the cylinder

$$\dot{\psi} = \dot{\phi}\left(\frac{R}{r} - 1\right) - \dot{\theta}$$

But we can also find this relationship using first principles: equations for motions of rigid bodies. Since there is no slip, we will find expressions for the velocity of the material point of the ring where it touches the cylinder (point B) and do the same for the material point of the cylinder where it touches the ring (again point B). By equating these two expressions, we should be able to solve for $\dot{\psi}$ in terms of the two generalized coordinates θ and ϕ and their derivatives and constants.

$$\underline{v}_B\big|_{\text{ring}} = \underline{\omega}_{\text{ring}} \times \underline{r}_{OB}$$

$$\boxed{\underline{\omega}_{\text{ring}} = \dot{\theta}\hat{\underline{e}}_Z}$$

$$\underline{r}_{OB} = \underline{r}_{OC} + \underline{r}_{CB}$$

$$\underline{r}_{OC} = R\sin\theta\hat{\underline{e}}_X - R\cos\theta\hat{\underline{e}}_Y$$

$$\underline{r}_{CB} = R\sin(\theta+\phi)\hat{\underline{e}}_X - R\cos(\theta+\phi)\hat{\underline{e}}_Y$$

$$\underline{v}_B\big|_{\text{ring}} = \dot{\theta}R(\sin\theta + \sin(\theta+\phi))\hat{\underline{e}}_Y + \dot{\theta}R(\cos\theta + \cos(\theta+\phi))\hat{\underline{e}}_X$$

$$\underline{v}_B\big|_{\text{disk}} = \underline{v}_A + \underline{\omega}_{\text{disk}} \times \underline{r}_{AB}$$

$$\underline{v}_A = \underline{v}_C + (\dot{\theta} + \dot{\phi})\hat{\underline{e}}_Z \times \underline{r}_{OC}$$

$$\underline{v}_C = \dot{\theta}\hat{\underline{e}}_Z \times \underline{r}_{OC}$$

$$\boxed{\underline{v}_C = \dot{\theta}R\sin\theta\hat{\underline{e}}_Y + \dot{\theta}R\cos\theta\hat{\underline{e}}_X}$$

$$\underline{r}_{CA} = (R-r)\sin(\theta+\phi)\hat{\underline{e}}_X - (R-r)\cos(\theta+\phi)\hat{\underline{e}}_Y$$

$$\underline{v}_A = \dot{\theta}R\cos\theta\hat{\underline{e}}_X + \dot{\theta}R\sin\theta\hat{\underline{e}}_Y + (\dot{\theta}+\dot{\phi})(R-r)\sin(\theta+\phi)\hat{\underline{e}}_Y + (\dot{\theta}+\dot{\phi})(R-r)\cos(\theta+\phi)\hat{\underline{e}}_X$$

$$\boxed{\underline{v}_A = \big(\dot{\theta}R\cos\theta + (\dot{\theta}+\dot{\phi})(R-r)\cos(\theta+\phi)\big)\hat{\underline{e}}_X + \big(\dot{\theta}R\sin\theta + (\dot{\theta}+\dot{\phi})(R-r)\sin(\theta+\phi)\big)\hat{\underline{e}}_Y}$$

$$\underline{v}_B\big|_{\text{disk}} = \underline{v}_A - \omega_{\text{disk}}\hat{\underline{e}}_Z \times (r\sin(\theta+\phi)\hat{\underline{e}}_X - r\cos(\theta+\phi)\hat{\underline{e}}_Y)$$

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Equating the two velocities of disk and ring, and separating into X and Y components we have

$$\dot{\theta}R\cos\theta + \dot{\theta}R\cos(\theta + \phi) = \dot{\theta}R\cos\theta + (\dot{\theta} + \dot{\phi})(R - r)\cos(\theta + \phi) - \omega_{\text{disk}}r\cos(\theta + \phi)$$
$$\dot{\theta}R(\sin\theta + \sin(\theta + \phi)) = \dot{\theta}R\sin\theta + (\dot{\theta} + \dot{\phi})(R - r)\sin(\theta + \phi) = \omega_{\text{disk}}r\sin(\theta + \phi)$$

and both these equations simplify to

$$\dot{\theta}R = (\dot{\theta} + \dot{\phi})(R - r) - \omega_{\text{disk}}r$$

giving

$$\omega_{\text{disk}} = \dot{\phi}\left(\frac{R}{r} - 1\right) - \dot{\theta}$$

$$\boxed{\underline{\omega}_{\text{disk}} = \left\{\dot{\phi}\left(\frac{R}{r} - 1\right) - \dot{\theta}\right\}\hat{\underline{e}}_Z}$$

Now start doing Lagrangian stuff. Start by writing an expression for the kinetic energy T and potential energy V.

$$I_{\text{ring}} = MR^2$$

$$I_{\text{disk}} = \frac{1}{2}mr^2$$

$$\boxed{V = -MgR\cos\theta - mg(R\cos\theta + \cos(\theta + \phi)(R - r))}$$

$$T = \frac{1}{2}M\underline{v}_C \cdot \underline{v}_C + \frac{1}{2}I_{\text{ring}}\omega_{\text{ring}}^2 + \frac{1}{2}m\underline{v}_A \cdot \underline{v}_A + \frac{1}{2}I_{\text{disk}}\omega_{\text{disk}}^2$$

$$\underline{v}_C \cdot \underline{v}_C = (\dot{\theta}R\sin\theta)^2 + (\dot{\theta}R\cos\theta)^2$$
$$= \dot{\theta}^2 R^2$$

$$\omega_{\text{ring}}^2 = \dot{\theta}^2$$

$$\underline{v}_A \cdot \underline{v}_A = (\dot{\theta}R\cos\theta + (\dot{\theta} + \dot{\phi})(R - r)\cos(\theta + \phi))^2$$
$$+ (\dot{\theta}R\sin\theta + (\dot{\theta} + \dot{\phi})(R - r)\sin(\theta + \phi))^2$$

$$\underline{v}_A \cdot \underline{v}_A = \dot{\theta}^2 R^2 \cos^2\theta + 2\dot{\theta}R\cos\theta(\dot{\theta}$$
$$+ \dot{\phi})(R - r)\cos(\theta + \phi) + (\dot{\theta} + \dot{\phi})^2(R - r)^2\cos^2(\theta + \phi)$$
$$+ \dot{\theta}^2 R^2 \sin^2\theta + 2\dot{\theta}R\sin\theta(\dot{\theta} + \dot{\phi})(R - r)\sin(\theta + \phi)$$
$$+ (\dot{\theta} + \dot{\phi})^2(R - r)^2\sin^2(\theta + \phi)$$

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$$\underline{v}_A \cdot \underline{v}_A = \dot{\theta}^2 R^2 + (\dot{\theta} + \dot{\phi})^2 (R - r)^2$$
$$+ 2\dot{\theta}R\cos\theta(\dot{\theta} + \dot{\phi})(R - r)\cos(\theta + \phi)$$
$$+ 2\dot{\theta}R\sin\theta(\dot{\theta} + \dot{\phi})(R - r)\sin(\theta + \phi)$$

$$\underline{v}_A \cdot \underline{v}_A = \dot{\theta}^2 R^2 + (\dot{\theta} + \dot{\phi})^2 (R - r)^2$$
$$+ 2\dot{\theta}R(\dot{\theta} + \dot{\phi})(R - r)\big(\cos\theta\cos(\theta + \phi) + \sin\theta\sin(\theta + \phi)\big)$$

$$\cos\theta\cos(\theta + \phi) + \sin\theta + \sin(\theta + \phi) = \cos(-\phi) = \cos\phi$$

$$\underline{v}_A \cdot \underline{v}_A = \dot{\theta}^2 R^2 + (\dot{\theta} + \dot{\phi})^2 (R - r)^2 + 2\dot{\theta}R(\dot{\theta} + \dot{\phi})(R - r)\cos\phi$$

$$\omega_{\text{disk}}^2 = \left\{ \dot{\phi}\left(\frac{R}{r} - 1\right) - \dot{\theta} \right\}^2$$

$$T = \frac{1}{2}M\dot{\theta}^2 R^2 + \frac{1}{2}MR^2\dot{\theta}^2$$
$$+ \frac{1}{2}m\left(\dot{\theta}^2 R^2 + (\dot{\theta} + \dot{\phi})^2(R - r)^2 + 2\dot{\theta}R(\dot{\theta} + \dot{\phi})(R - r)\cos\phi\right)$$
$$+ \frac{1}{4}mr^2\left\{ \dot{\phi}\left(\frac{R}{r} - 1\right) - \dot{\theta} \right\}^2$$

$$T = M\dot{\theta}^2 R^2$$
$$+ \frac{1}{2}m\left(\dot{\theta}^2 R^2 + (\dot{\theta} + \dot{\phi})^2(R - r)^2 + 2\dot{\theta}R(\dot{\theta} + \dot{\phi})(R - r)\cos\phi\right)$$
$$+ \frac{1}{4}mr^2\left\{ \dot{\phi}\left(\frac{R}{r} - 1\right) - \dot{\theta} \right\}^2$$

$$\boxed{\begin{aligned} T &= M\dot{\theta}^2 R^2 + \frac{1}{2}m\dot{\theta}^2 R^2 \\ &+ \frac{1}{2}m(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta} + \dot{\phi}^2)(R - r)^2 \\ &+ mR(\dot{\theta}^2 + \dot{\phi}\dot{\theta})(R - r)\cos\phi \\ &+ \frac{1}{4}mr^2\left\{ \dot{\phi}\left(\frac{R}{r} - 1\right) - \dot{\theta} \right\}^2 \end{aligned}}$$

$$V = -MgR\cos\theta - mg(R\cos\theta + \cos(\theta + \phi)(R - r))$$

$$\mathscr{L} = T - V$$

$$\frac{\partial \mathscr{L}}{\partial \dot{\theta}} = 2M\dot{\theta}R^2 + m\dot{\theta}R^2 + m(\dot{\theta} + \dot{\phi})(R - r)^2$$

$$+ mR(2\dot{\theta} + \dot{\phi})(R - r)\cos\phi + \frac{1}{2}mr^2\left\{\dot{\phi}\left(\frac{R}{r} - 1\right) - \dot{\theta}\right\}$$

$$\frac{\partial \mathscr{L}}{\partial \theta} = MgR\sin\theta + mgR\sin\theta + mg\sin(\theta + \phi)(R - r)$$

$$\frac{\partial \mathscr{L}}{\partial \dot{\phi}} = m(\dot{\theta} + \dot{\phi})(R - r)^2 + mR\dot{\theta}(R - r)\cos\phi + \frac{1}{2}mr^2\left\{\dot{\phi}\left(\frac{R}{r} - 1\right) - \dot{\theta}\right\}\left(\frac{R}{r} - 1\right)$$

$$\frac{\partial \mathscr{L}}{\partial \phi} =$$

17.1 Wave Equation on String

This page gives an outline of the general procedure to derive the equation of motion, propose a general solution, and solve for constants using boundary and initial conditions (here we assume the boundary conditions are both ends fixed, and zero initial conditions just to get the mode shapes and natural frequencies.)

Physical assumptions: *homogenous string ρA = constant, the string is perfectly elastic (no resistance to bending), the tension is way more than gravity, and string only vibrates perfectly up and down.*

1. **Derive governing equation**

   (a) Momentum in x-direction gives $T(x)$ is constant
   (b) Do momentum in the y-direction
   (c) Use small angles: $\sin(\alpha + \frac{\partial \alpha}{\partial x}dx) = \alpha + \frac{\partial \alpha}{\partial x}dx$ and $\tan(\alpha) = \alpha$

   The governing equation is $\boxed{T\dfrac{\partial^2 y}{\partial x^2} = \rho A\dfrac{\partial^2 y}{\partial t^2}}$

2. **Propose a general separable solution** $\boxed{y(x, t) = a(x)f(t)}$

   (a) Rearrange the governing equation as $C^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ and propose $f(t) = Ae^{i\omega_n t}$ giving $y(x, t) = a(x)Ae^{i\omega_n t}$ and plug in
   (b) The governing equation becomes $C^2\frac{\partial^2 a}{\partial x^2} + \omega_n^2 a(x) = 0$
   (c) Propose $a(x) = Be^{i\lambda x}$ and get $\omega_n = C\lambda$
   (d) The total solution is then $\boxed{y(x, t) = Be^{i\frac{\omega_n}{C}x}Ae^{i\omega_n t}}$ which can be decomposed into sine and cosine as $y(x, t) = (B_1\sin(\lambda x) + B_2\cos(\lambda x))(A_1\sin(\omega_n t) + A_2\cos(\omega_n t))$

3. **Apply boundary and initial conditions** to get the constants

   (a) Apply boundary conditions $y(x = 0, t) = y(x = L, t) = 0$ gives $B_2 = 0$ and $B_1\sin(\frac{\omega_n}{C}L) = 0$ so $\frac{\omega_n}{C}L = n\pi$ where $n = 1, 2, 3 \ldots$. So $\omega_n = \frac{Cn\pi}{L}$. The solution becomes $y(x, t) = B_1\sin(\frac{\omega_n}{C}x)(A_1\sin(\omega_n t) + A_2\cos(\omega_n t))$

(b) Apply initial conditions $y(x, t = 0) = 0$ gives $A_2 = 0$ reducing solution to $y(x, t) = B_1 \sin(\frac{\omega_n}{C} x) A_1 \sin(\omega_n t)$ or by combining the constants

$$\boxed{y(x, t) = C_n \sin(\frac{\omega_n}{C} x) \sin(\omega_n t)}$$

4. Now we have the governing equation, now we see if it is **self-adjoint** if it satisfies the following conditions

   (i) $\int u \rho A v \, dx = \int v \rho A u \, dx$

   (ii) $\int v \left(-T \frac{\partial^2}{\partial x^2} \right) u \, dx = \int u \left(-T \frac{\partial^2}{\partial x^2} \right) v \, dx$

   The first condition is satisfied automatically, since u and v (in our case $a(x)$ and $f(t)$ commute.) We show that the second condition holds by doing integration by parts twice.

   $$\int_0^L a_i \left(-T \frac{\partial^2}{\partial x^2} \right) a_j \, dx$$
   $$= a_i \left(-T \frac{\partial}{\partial x} (a_j) \right) \Big|_0^L + \int_0^L T \frac{\partial}{\partial x} (a_j) \frac{da_i}{dx} \, dx$$

   one more integration by parts

   $$\int_0^L a_i \left(-T \frac{\partial^2}{\partial x^2} \right) a_j \, dx$$
   $$= a_i \left(-T \frac{\partial}{\partial x} (a_j) \right) \Big|_0^L - \left(\frac{da_i}{dx} (-T a_j) \right) \Big|_0^L + \int_0^L a_j \left(-T \frac{\partial^2}{\partial x^2} (a_i) \right) dx$$

   and since we evaluate the first two terms on the right hand side at $x = 0$ and $x = L$, the boundary conditions dictate that $a_i = a_j = 0$ here, thus proving the system is **self-adjoint**. *Self-adjointness depends on the boundary conditions.*

5. Now we use the self adjoint property to show that the modes are **orthogonal**, where a_j and a_i are orthogonal functions if they satisfy

   $$\int_0^L a_j a_i \, dx = 0$$

Chapter 18

Aircraft Equations of Motion

18.1 Introduction

The equations of rigid body motion are expanded and expressed in state space form. The expression of the equations in this form assumes the Earth is flat, and inertially fixed. The atmosphere is stationary with respect to the Earth.

- Show linear equations represented in stability axes ...

- Show Taylor series expansion

- Stability derivative has dimensions. Stability coefficient does not (Nelson pg 109).

- Express stability derivatives in terms of stability coefficients

- How to know before linearization that longitudinal and lateral equations can be decoupled?

18.1.1 Notation

I use R for rotation matrix, and T for the transformation matrix from body axes to Euler axes. Etkin uses L for rotation matrices, and $\boldsymbol{R}$ for the transformation matrix from body axes to Euler axes. Bilimoria and Schmidt use $[T]$ for rotation matrices and $[L]$ for the transformation matrix from body axes to Euler axes. Some of the standard notation describing the expression of vectors in various reference frames is outlined below.

- F_a denotes reference frame a in Etkin. I will use lower-case f_a to denote reference frame a.

- v_a describes vector v of a point along axes of reference frame a, when the referred point is obvious.

- v_0 indicates the velocity of point 0.

- v_{0_a} indicates the velocity of point 0 along the axes of reference frame a.

- SUPERSCRIPT BASICALLY MEANS RELATIVE TO. Bilimoria and Schmidt use $|\cdot$ instead of just a superscript, and often when there is no superscript, it is implied that it is actually relative to body axes.

- A superscript indicates motion relative to a certain reference frame. v^a is the velocity of a point relative to frame f_a, when the referred point is obvious.

- The notation $v_b{}^a$ gives the velocity of a point relative to reference frame f_a described along the axes of f_b.

- To be clear, when the point of interest is not obvious, or there are multiple points, the notation $v_{0_b}{}^a$ would describe the velocity of point 0 relative to reference frame f_a, described along the axes of f_b.

- ω is typically reserved to describe the angular velocity of a reference frame relative to inertial axes f_I. Making use of the notation above, ω^E represents the angular velocity of reference frame f_E relative to f_I (pretty sure this is from Etkin). Bilimoria and Schmidt: $\omega_{1,2}$ is the angular velocity of reference frame 1 relative to reference frame 2. This implies vector is expressed in the coordinates of reference frame 2?

- In Bilimoria and Schmidt V_I is the inertial velocity of the vehicle

  - I is inertial frame
  - EC is Earth-centered, Earth-fixed frame
  - E is Earth-surface frame
  - V is vehicle carrying frame
  - V is vehicle-carried frame
  - A is atmosphere-fixed frame
  - W is air-trajectory frame (wind axes)
  - B is body-fixed frame (body axes)
  - S stability axes (special set of body axes)

- The transformation R_{ab} describes a vector transformation from being expressed in reference frame b to being expressed in reference frame a.

- Typically capital letter denotes vector quantity

18.2   Equations of Motion

In this problem, the rigid body equations of motion shown in Equations (18.1-18.6) below were expanded and expressed in a state space representation. See Steven's and Lewis page 44 for moment equation derivation. Poisson orientation equations page 28, and list of different kinematic equations page 46.

- Flat earth, this defined the reference frames. A constant velocity in the flat earth does not lead to moments, but on a spherical earth it does.

- The aircraft is a rigid body thus has no rotating terms due to rotating turbo machinery

Force		$\dot{V}_B = -\omega_{B,I} \times V_B + R_{IB}{}^\top g + F_B/m$	(18.1)
Moment		$\dot{\omega}_B = J^{-1}(M_B - \omega_{B,I} \times J\omega_B)$	(18.2)
Orientation	Poisson:	$\dot{R}_{IB} = R_{IB}\hat{\omega}_{B,I}$	(18.3)
	Euler's:		(18.4)
	Quaternion:		(18.5)
Location		$\dot{\Delta} = R_{IB}v_B$	(18.6)

Where the gravity vector in inertial coordinates g is given by

$$g = \begin{bmatrix} 0 & 0 & g_0 \end{bmatrix}^\top$$

Making use of the "hat" operator, the cross product operations in equations (18.1-18.2) can be written

$$\dot{V}_B = -\hat{\omega}_B V_B + R_{IB}{}^\top g + F_B/m$$
$$\dot{\omega}_B = J^{-1}(-\hat{\omega}_B J \omega_B + \tau_B)$$
$$\dot{R}_{IB} = R_{IB}\hat{\omega}_B$$
$$\dot{\Delta} = R_{IB}v_B$$

where the hat operator is defined as follows

$$\omega = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \Rightarrow \qquad \hat{\omega} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

The rotation matrix R_{IB} is given by

$$R_{IB} = \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta & \cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\phi\cos\theta \end{bmatrix}$$

with its transpose

$$R_{IB}{}^\top = \begin{bmatrix} \cos\psi\cos\theta & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta & \cos\theta\sin\phi \\ \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta & \cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi & \cos\phi\cos\theta \end{bmatrix}$$

The body linear and angular velocity components are given by the following:

$$\omega_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad V_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Force Equations

Writing equation (18.1) out using the hat operator and the rotation matrix transpose

$$\dot{V}_B = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} -g_D\sin(\theta) \\ g_D\sin(\phi)\cos(\theta) \\ g_D\cos(\phi)\cos(\theta) \end{bmatrix} + \frac{F_B}{m} \qquad (18.7)$$

where the force vector that represents all non-gravitational forces acting on the body, in body axes is given by:

$$F_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

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The forces included in F_B are all forces other than gravitational forces. For an aircraft these forces are aerodynamic and propulsive. Expanding equation (18.7) gives the following represent the force equations of a generalized rigid body. These equations describe the the motion of its cg since the origin of the axis system was placed at the cg.

$$\begin{aligned}
\dot{u} &= rv - qw - g_0 \sin(\theta) + X/m \\
\dot{v} &= -ru + pw + g_0 \sin(\phi) \cos(\theta) + Y/m \\
\dot{w} &= qu - pv + g_0 \cos(\phi) \cos(\theta) + Z/m
\end{aligned} \tag{18.8}$$

Moment equations

To expand equation (18.2), the moment of inertia matrix J is needed, as defined below. Defined (McLean pg 21) (Stevens and Lewis pg 43)

$$J = \begin{bmatrix} J_{xx} & -J_{xy} & -J_{xz} \\ -J_{xy} & J_{yy} & -J_{yz} \\ -J_{xz} & -J_{yz} & J_{zz} \end{bmatrix}$$

Without any simplification, expansion of the moment equations would become very cumbersome. In general, aircraft are symmetric about the $x - z$ plane, mass is uniformly distributed, and the body coordinate system is oriented such that $J_{xy} = J_{yz} = 0$. This allows the moment of inertia matrix to be simplified to:

$$J = \begin{bmatrix} J_{xx} & 0 & -J_{xz} \\ 0 & J_{yy} & 0 \\ -J_{xz} & 0 & J_{zz} \end{bmatrix}$$

Sometimes the product of inertia J_{xz} is sufficiently small, allowing this term to be neglected. This is the case when the aircraft body axes are aligned with the principle axes (McLean 23). A further simplification can be made if it is assumed that the aircraft body axes are aligned to be principal inertial axes. In this special case the remaining product of inertia J_{xz} is also zero. This simplification is not often used owing to the difficulty of precisely determining the principal inertia axes. However, the symmetry of the aircraft determines that J_{xz} is generally very much smaller than J_{xx}, J_{yy} and J_{zz} and can often be neglected (Cook pg 72). The inverse of J is given by

$$J^{-1} = \begin{bmatrix} \dfrac{J_{zz}}{J_{xx}J_{zz}-J_{xz}{}^2} & 0 & \dfrac{J_{xz}}{J_{xx}J_{zz}-J_{xz}{}^2} \\ 0 & \dfrac{1}{J_{yy}} & 0 \\ \dfrac{J_{xz}}{J_{xx}J_{zz}-J_{xz}{}^2} & 0 & \dfrac{J_{xx}}{J_{xx}J_{zz}-J_{xz}{}^2} \end{bmatrix}$$

where sometimes $\Gamma = J_{xx}J_{zz} - J_{xz}{}^2$ is used to simplify this expression (Stevens and Lewis pg 45, 110). The input torque in body axes is given by

$$M_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

Writing equation (18.2) out using the hat operator and simplified moment of inertia matrix gives, where $\omega_B = \begin{bmatrix} p & q & r \end{bmatrix}^\top$:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = J^{-1} \left(- \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} J_{xx} & 0 & -J_{xz} \\ 0 & J_{yy} & 0 \\ -J_{xz} & 0 & J_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix} \right)$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = J^{-1} \left(\begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix} \begin{bmatrix} J_{xx}p - J_{xz}r \\ J_{yy}q \\ -J_{xz}p + J_{zz}r \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix} \right)$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = J^{-1} \left(\begin{bmatrix} qrJ_{yy} + pqJ_{xz} - qrJ_{zz} \\ -prJ_{xx} + r^2 J_{xz} - p^2 J_{xz} + prJ_{zz} \\ pqJ_{xx} - qrJ_{xz} - pqJ_{yy} \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix} \right)$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = J^{-1} \left(\begin{bmatrix} qr(J_{yy} - J_{zz}) + pqJ_{xz} \\ (r^2 - p^2)J_{xz} + pr(J_{zz} - J_{xx}) \\ pq(J_{xx} - J_{yy}) - qrJ_{xz} \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix} \right)$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = J^{-1} \begin{bmatrix} qr(J_{yy} - J_{zz}) + pqJ_{xz} \\ (r^2 - p^2)J_{xz} + pr(J_{zz} - J_{xx}) \\ pq(J_{xx} - J_{yy}) - qrJ_{xz} \end{bmatrix} + J^{-1} \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

Evaluating the first term on the right hand side

$$\begin{bmatrix} \frac{J_{zz}}{J_{xx}J_{zz} - J_{xz}^2} & 0 & \frac{J_{xz}}{J_{xx}J_{zz} - J_{xz}^2} \\ 0 & \frac{1}{J_{yy}} & 0 \\ \frac{J_{xz}}{J_{xx}J_{zz} - J_{xz}^2} & 0 & \frac{J_{xx}}{J_{xx}J_{zz} - J_{xz}^2} \end{bmatrix} \begin{bmatrix} qr(J_{yy} - J_{zz}) + pqJ_{xz} \\ (r^2 - p^2)J_{xz} + pr(J_{zz} - J_{xx}) \\ pq(J_{xx} - J_{yy}) - qrJ_{xz} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{J_{zz}[qr(J_{yy} - J_{zz}) + pqJ_{xz}] + J_{xz}[pq(J_{xx} - J_{yy}) - qrJ_{xz}]}{J_{xx}J_{zz} - J_{xz}^2} \\ \frac{(r^2 - p^2)J_{xz} + pr(J_{zz} - J_{xx})}{J_{yy}} \\ \frac{J_{xz}[qr(J_{yy} - J_{zz}) + pqJ_{xz}] + J_{xx}[pq(J_{xx} - J_{yy}) - qrJ_{xz}]}{J_{xz}^2 - J_{xx}J_{zz}} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{J_{zz}(J_{yy} - J_{zz})qr + pqJ_{xz}J_{zz} + pqJ_{xz}(J_{xx} - J_{yy}) - qrJ_{xz}^2}{J_{xx}J_{zz} - J_{xz}^2} \\ \frac{(r^2 - p^2)J_{xz} + pr(J_{zz} - J_{xx})}{J_{yy}} \\ \frac{J_{xz}(J_{yy} - J_{zz})qr + J_{xz}^2 pq + J_{xx}(J_{xx} - J_{yy})pq - J_{xx}J_{xz}qr}{J_{xz}^2 - J_{xx}J_{zz}} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{pq[J_{xz}J_{zz} + J_{xz}(J_{xx} - J_{yy})] + qr[J_{zz}(J_{yy} - J_{zz}) - J_{xz}^2]}{J_{xx}J_{zz} - J_{xz}^2} \\ \frac{(r^2 - p^2)J_{xz} + pr(J_{zz} - J_{xx})}{J_{yy}} \\ \frac{qr[J_{xz}(J_{yy} - J_{zz}) - J_{xx}J_{xz}] + pq[J_{xz}^2 + J_{xx}(J_{xx} - J_{yy})]}{J_{xz}^2 - J_{xx}J_{zz}} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{J_{xz}(J_{xx} - J_{yy} + J_{zz})pq + [J_{zz}(J_{yy} - J_{zz}) - J_{xz}^2]qr}{J_{xx}J_{zz} - J_{xz}^2} \\ \frac{(r^2 - p^2)J_{xz} + pr(J_{zz} - J_{xx})}{J_{yy}} \\ \frac{[(J_{xx} - J_{yy})J_{xx} + J_{xz}^2]pq + J_{xz}[-J_{xx} + J_{yy} - J_{zz}]qr}{J_{xz}^2 - J_{xx}J_{zz}} \end{bmatrix}$$

Evaluating the second term on the right hand side

$$
\begin{bmatrix}
\frac{J_{zz}}{J_{xx}J_{zz}-J_{xz}^2} & 0 & \frac{J_{xz}}{J_{xx}J_{zz}-J_{xz}^2} \\
0 & \frac{1}{J_{yy}} & 0 \\
\frac{J_{xz}}{J_{xx}J_{zz}-J_{xz}^2} & 0 & \frac{J_{xx}}{J_{xx}J_{zz}-J_{xz}^2}
\end{bmatrix}
\begin{bmatrix} L \\ M \\ N \end{bmatrix}
=
\begin{bmatrix}
\frac{J_{zz}L+J_{xz}N}{J_{xx}J_{zz}-J_{xz}^2} \\
\frac{M}{J_{yy}} \\
\frac{J_{xz}L+J_{xx}N}{J_{xx}J_{zz}-J_{xz}^2}
\end{bmatrix}
$$

Putting everything together:

$$
\begin{aligned}
\dot{p} &= \frac{J_{xz}(J_{xx}-J_{yy}+J_{zz})pq + [J_{zz}(J_{yy}-J_{zz})-J_{xz}^2]qr}{J_{xx}J_{zz}-J_{xz}^2} + \frac{J_{zz}L+J_{xz}N}{J_{xx}J_{zz}-J_{xz}^2} \\
\dot{q} &= \frac{(J_{zz}-J_{xx})pr - J_{xz}(p^2-r^2)}{J_{yy}} + \frac{M}{J_{yy}} \\
\dot{r} &= \frac{[(J_{xx}-J_{yy})J_{xx}+J_{xz}^2]pq - J_{xz}[J_{xx}-J_{yy}+J_{zz}]qr}{J_{xz}^2-J_{xx}J_{zz}} + \frac{J_{xz}L+J_{xx}N}{J_{xx}J_{zz}-J_{xz}^2}
\end{aligned}
\tag{18.9}
$$

Kinematic Equations

The orientation, or kinematic equations describe the orientation of the aircraft body axes with respect to the inertial axes. Expanding Equation (18.3) *Converting from the $\dot{R}_{IB}$ and R_{IB} equation to whats below? Also, make clear what subscript means exactly.* The relationship between Euler rates and body angular velocities is

$$
\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}
=
\begin{bmatrix}
1 & \tan(\theta)\sin(\phi) & \tan(\theta)\cos(\phi) \\
0 & \cos(\phi) & -\sin(\phi) \\
0 & \sin(\phi)/\cos(\theta) & \cos(\phi)/\cos(\theta)
\end{bmatrix}
\begin{bmatrix} p \\ q \\ r \end{bmatrix}
$$

where the following is the transformation matrix T

$$
T =
\begin{bmatrix}
1 & \tan(\theta)\sin(\phi) & \tan(\theta)\cos(\phi) \\
0 & \cos(\phi) & -\sin(\phi) \\
0 & \sin(\phi)/\cos(\theta) & \cos(\phi)/\cos(\theta)
\end{bmatrix}
$$

Expanding

$$
\begin{aligned}
\dot{\phi} &= p + \tan(\theta)[q\sin(\phi) + r\cos(\phi)] \\
\dot{\theta} &= q\cos(\phi) - r\sin(\phi) \\
\dot{\psi} &= [q\sin(\phi) + r\cos(\phi)]/\cos(\theta)
\end{aligned}
\tag{18.10}
$$

Navigation Equations

The location, or navigation equations describe the location of the origin of the body fixed coordinate system with respect to the inertial axes. Writing out Equation (18.6) where $\Delta = \begin{bmatrix} x & y & z \end{bmatrix}^\top$

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}
=
\begin{bmatrix}
\cos\psi\cos\theta & \cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta \\
\cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta & \cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi \\
-\sin\theta & \cos\theta\sin\phi & \cos\phi\cos\theta
\end{bmatrix}
\begin{bmatrix} u \\ v \\ w \end{bmatrix}
$$

Expanding

$$\dot{x} = u\cos\psi\cos\theta + v[\cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi] + w[\sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta]$$
$$\dot{y} = u\cos\theta\sin\psi + v[\cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta] + w[\cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi] \qquad (18.11)$$
$$\dot{z} = -u\sin\theta + v\cos\theta\sin\phi + w\cos\phi\cos\theta$$

Equation Summary

The flat-earth, nonlinear 6-DOF equations which describe the motion of an aircraft in body axes are summarized below, and are consistent with (Stevens and Lewis pg 110). The nonlinear equations above started by assuming

- Flat earth, this defined the reference frames. A constant velocity in the flat earth does not lead to moments, but on a spherical earth it does.

- The aircraft is a rigid body, no rotating terms due to rotating turbo machinery

- The products of inertia $J_{xy} = J_{yz} = 0$ due to symmetry of the aircraft

FORCE EQUATIONS

$$\dot{u} = rv - qw - g_D\sin(\theta) + X/m$$
$$\dot{v} = -ru + pw + g_D\sin(\phi)\cos(\theta) + Y/m \qquad (18.8)$$
$$\dot{w} = qu - pv + g_D\cos(\phi)\cos(\theta) + Z/m$$

$$\dot{U} = RV - QW - g\sin(\Theta) + X/m$$
$$\dot{V} = -RU + PW + g\sin(\Phi)\cos(\Theta) + Y/m \qquad (18.12)$$
$$\dot{W} = QU - PV + g\cos(\Phi)\cos(\Theta) + Z/m$$

MOMENT EQUATIONS

$$\dot{p} = \frac{J_{xz}(J_{xx} - J_{yy} + J_{zz})pq + [J_{zz}(J_{yy} - J_{zz}) - J_{xz}^2]qr}{J_{xx}J_{zz} - J_{xz}^2} + \frac{J_{zz}L + J_{xz}N}{J_{xx}J_{zz} - J_{xz}^2}$$
$$\dot{q} = \frac{(J_{zz} - J_{xx})pr - J_{xz}(p^2 - r^2)}{J_{yy}} + \frac{M}{J_{yy}} \qquad (18.9)$$
$$\dot{r} = \frac{[(J_{xx} - J_{yy})J_{xx} + J_{xz}^2]pq - J_{xz}[J_{xx} - J_{yy} + J_{zz}]qr}{J_{xz}^2 - J_{xx}J_{zz}} + \frac{J_{xz}L + J_{xx}N}{J_{xx}J_{zz} - J_{xz}^2}$$

$$\dot{P} = \frac{J_{xz}(J_{xx} - J_{yy} + J_{zz})PQ + [J_{zz}(J_{yy} - J_{zz}) - J_{xz}{}^2]QR}{J_{xx}J_{zz} - J_{xz}{}^2} + \frac{J_{zz}L + J_{xz}N}{J_{xx}J_{zz} - J_{xz}{}^2}$$

$$\dot{Q} = \frac{(J_{zz} - J_{xx})PR - J_{xz}(P^2 - R^2)}{J_{yy}} + \frac{M}{J_{yy}}$$

$$\dot{R} = \frac{[(J_{xx} - J_{yy})J_{xx} + J_{xz}{}^2]PQ - J_{xz}[J_{xx} - J_{yy} + J_{zz}]QR}{J_{xz}{}^2 - J_{xx}J_{zz}} + \frac{J_{xz}L + J_{xx}N}{J_{xx}J_{zz} - J_{xz}{}^2}$$

KINEMATIC EQUATIONS

$$\dot{\phi} = p + \tan(\theta)[q\sin(\phi) + r\cos(\phi)]$$
$$\dot{\theta} = q\cos(\phi) - r\sin(\phi) \qquad\qquad (18.10)$$
$$\dot{\psi} = [q\sin(\phi) + r\cos(\phi)]/\cos(\theta)$$

$$\dot{\Phi} = P + \tan(\Theta)[Q\sin(\Phi) + R\cos(\Phi)]$$
$$\dot{\Theta} = Q\cos(\Phi) - R\sin(\Phi)$$
$$\dot{\Psi} = [Q\sin(\Phi) + R\cos(\Phi)]/\cos(\Theta)$$

NAVIGATION EQUATIONS

$$\dot{x} = u\cos\psi\cos\theta + v[\cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi] + w[\sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta]$$
$$\dot{y} = u\cos\theta\sin\psi + v[\cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta] + w[\cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi] \qquad (18.11)$$
$$\dot{z} = -u\sin\theta + v\cos\theta\sin\phi + w\cos\phi\cos\theta$$

The nonlinear, rigid body equations of motion are now expressed in the state space representation $\dot{X} = f(X, U)$, where the state vector X and input vector U are given by

$$X = \begin{bmatrix} u & v & w & p & q & r & \phi & \theta & \psi & x & y & z \end{bmatrix}^\top$$
$$U = \begin{bmatrix} X & Y & Z & L & M & N \end{bmatrix}^\top$$

The equations as represented in this form turn out to not be very useful. When considering the motion of a spacecraft tumbling in space, forces and moments in the body axes may be able to be directly applied using thrusters and reaction wheels. However, Euler angles are not the best way to represent the orientation of a tumbling spacecraft, due to singularities that exist. For the motion of an aircraft, the body forces and moments are not simply system inputs, as they are functions of the aircraft's current motion. The forces and moments in body axes would have to be calculated based on the current state and control surface deflections to be of any use.

$$
\begin{aligned}
U &= U_{\text{eq}} + u & P &= P_{\text{eq}} + p & \Phi &= \Phi_{\text{eq}} + \phi \\
V &= V_{\text{eq}} + v & Q &= Q_{\text{eq}} + q & \Theta &= \Theta_{\text{eq}} + \theta \\
W &= W_{\text{eq}} + w & R &= R_{\text{eq}} + R & \Psi &= \Psi_{\text{eq}} + \psi
\end{aligned}
$$

Stability Axes

$$\sin(\alpha) = \frac{w}{V_{\text{eq}}}$$

approximating

$$\alpha \approx \frac{w}{V_{\text{eq}}}$$

18.3   Linearizing

Consider the following system

$$\dot{X} = f(X, U) \tag{18.13}$$

The equilibrium, or trim state X_{eq} and input U_{eq} satisfy

$$\dot{X}_{\text{eq}} = f(X_{\text{eq}}, U_{\text{eq}}) = 0 \tag{18.14}$$

The equilibrium state and input are found for the nominal steady, level cruise condition, and Equation (18.13) is linearized about this trim condition as follows. Defining x and u to be state and input perturbations about equilibrium, the state and input can be expressed as

$$X = X_{\text{eq}} + x$$
$$U = U_{\text{eq}} + u$$

Using this representation for X and U we have

$$\dot{X} = \dot{x} = f(X, U)$$
$$= f(X_{\text{eq}} + x, U_{\text{eq}} + u)$$

Performing a Taylor series expansion, neglecting second order terms and higher

$$f(X, U) = f(X_{\text{eq}}, U_{\text{eq}}) + \left.\frac{\partial f(X, U)}{\partial X}\right|_{\text{eq}} x + \left.\frac{\partial f(X, U)}{\partial U}\right|_{\text{eq}} u + \epsilon$$

where the subscript $(\cdot)_{\text{eq}}$ indicates these quantities be evaluated at the equilibrium point. With $f(X_{\text{eq}}, U_{\text{eq}}) = 0$, the linearization results in the state-space system given by

$$\dot{x} = Ax + Bu \tag{18.15}$$

where

$$A = \left.\frac{\partial f(X, U)}{\partial X}\right|_{\text{eq}} \qquad B = \left.\frac{\partial f(X, U)}{\partial U}\right|_{\text{eq}} \tag{18.16}$$

So now we need to see how to actually evaluate (18.16).

$$\frac{\partial f(X,U)}{\partial X}\bigg|_{\text{eq}} = \frac{\partial f(X,U)}{\partial X_1}\bigg|_{\text{eq}} + \frac{\partial f(X,U)}{\partial X_2}\bigg|_{\text{eq}} + \frac{\partial f(X,U)}{\partial X_3}\bigg|_{\text{eq}} + \dots$$

Let's do an example from the X-force equation. The use of U as either control input vector or velocity can be deduced by context, as can X as either the force vector in the X-direction, or to mean the state vector.

$$\dot{U} = RV - QW - g\sin(\Theta) + X/m$$

So in this example, $f(X,U) = RV - QW - g\sin(\Theta) + X/m$. First term.

$$\frac{\partial f}{\partial (X_1 = U)}\bigg|_{\text{eq}} = 0$$

18.3.1 Stability and Control Derivatives

The equations in the form $\dot{X} = f(X,U)$ will do little to help solve control problems for aircraft using the current input U (total forces and moments). For a spacecraft in outer space, the input vector U is quite reasonable: forces and torques in body axes could be generated using thrusters and/or reaction wheels. When the thrusters are switched off, no other forces will act on the spacecraft. However, for an aircraft, the "input" U is not so much an input, as it is itself a function of the state X, as well as other terms (such as $\dot{w}$). That is, the forces and moments generated during flight depend on the state of the aircraft; the aircraft's current velocity, in addition to control surface deflections, determines the total body force.

Under steady straight and level flight, the longitudinal and lateral equations can be decoupled.

18.3.2 Longitudinal Equations

Grouping the longitudinal equations below, and dropping the dependency on the lateral variables:

$$\dot{u} = qw - g_0\sin(\theta) + X/m$$
$$\dot{w} = qu + g_0\cos(\phi)\cos(\theta) + Z/m$$
$$\dot{q} = \frac{M}{J_{yy}}$$
$$\dot{\theta} = q$$

These equations are to be linearized about a trim point X_{eq}. The state X and input U are given by: $X = X_{\text{eq}} + \Delta X$ and $U = U_{\text{eq}} + \Delta U$. The perturbation state and input ΔX and ΔU, respectively, are given by:

$$\Delta X = \begin{bmatrix} \Delta u & \Delta v & \Delta w & \Delta p & \Delta q & \Delta r & \Delta\phi & \Delta\theta & \Delta\psi & \Delta x & \Delta y & \Delta z \end{bmatrix}^\top$$
$$\Delta U = \begin{bmatrix} \delta_{\text{th}} & \delta_e & \delta_a & \delta_r \end{bmatrix}^\top$$

Linearizing

Linearizing for the x force equation:

$$m\dot{u} = -mqw - mg_0\sin(\theta) + X \tag{18.17}$$

where, for an aircraft the force X is a function given by $X(u, \dot{u}, w, \dot{w}, q, \dot{q}, \delta_e, \dot{\delta}_e, \delta_{\text{th}})$. The linearization yields:

$$m\Delta\dot{u} = \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial \dot{u}}\Delta\dot{u} + \left(\frac{\partial X}{\partial w} - mq_{\text{eq}}\right)\Delta w + \frac{\partial X}{\partial \dot{w}}\Delta\dot{w}$$
$$+ \left(\frac{\partial X}{\partial q} - mw_{\text{eq}}\right)\Delta q + \frac{\partial X}{\partial \dot{q}}\Delta\dot{q} - g_D\cos(\theta_{\text{eq}})\Delta\theta$$
$$+ \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \dot{\delta}_e}\Delta\dot{\delta}_e + \frac{\partial X}{\partial \delta_{\text{th}}}\Delta\delta_{\text{th}}$$

Dividing through by m and using the following definition

$$\boxed{X_i = \frac{1}{m}\frac{\partial X}{\partial i}}$$

gives

$$\Delta\dot{u} = X_u\Delta u + X_{\dot{u}}\Delta\dot{u} + \left(X_w - q_{\text{eq}}\right)\Delta w + X_{\dot{w}}\Delta\dot{w} + \left(X_q - w_{\text{eq}}\right)\Delta q + X_{\dot{q}}\Delta\dot{q} - g_D\cos(\theta_{\text{eq}})\Delta\theta$$
$$+ X_{\delta_e}\Delta\delta_e + X_{\dot{\delta}_e}\Delta\dot{\delta}_e + X_{\delta_{\text{th}}}\Delta\delta_{\text{th}}$$

By studying aircraft aerodynamic data it is found that many of the stability derivatives under most flight conditions can be neglected. These typically are (McLean pg 33, Nelson pg 149):

$$\boxed{X_{\dot{u}},\ X_q\ X_{\dot{w}}\ X_{\delta_e}\ Z_{\dot{u}}\ Z_{\dot{w}}\ M_{\dot{u}}\ Z_{\dot{\delta}_e}\ M_{\dot{\delta}_e}}$$

giving:

$$\Delta\dot{u} = X_u\Delta u + \left(X_w - q_{\text{eq}}\right)\Delta w + \left(X_q - w_{\text{eq}}\right)\Delta q + X_{\dot{q}}\Delta\dot{q} - g_D\cos(\theta_{\text{eq}})\Delta\theta + X_{\delta_e}\Delta\delta_e + X_{\delta_{\text{th}}}\Delta\delta_{\text{th}}$$

evaluating this linearization about steady wings level flight:

$$\Delta\dot{u} = X_u\Delta u + X_w\Delta w + \left(X_q - w_{\text{eq}}\right)\Delta q - g_D\cos(\theta_{\text{eq}})\Delta\theta + X_{\delta_e}\Delta\delta_e + X_{\delta_{\text{th}}}\Delta\delta_{\text{th}}$$

For the w equation

$$\boxed{Z_i = \frac{1}{m}\frac{\partial Z}{\partial i}}$$

$$\Delta\dot{w} = Z_u\Delta u + Z_w\Delta w + \left(Z_q + u_{\text{eq}}\right)\Delta q - g_D\sin(\theta_{\text{eq}})\Delta\theta + Z_{\delta_e}\Delta\delta_e + Z_{\delta_{\text{th}}}\Delta\delta_{\text{th}}$$

For the q equation

$$\Delta\dot{q} = M_u\Delta u + M_{\dot{u}}\Delta\dot{u} + M_w\Delta w + M_{\dot{w}}\Delta\dot{w} + M_q\Delta q + M_{\dot{q}}\Delta\dot{q} + M_{\delta_e}\Delta\delta_e + M_{\dot{\delta}_e}\Delta\dot{\delta}_e + M_{\delta_{\text{th}}}\Delta\delta_{\text{th}}$$

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Where the following definition is used

$$M_i = \frac{1}{J_{yy}} \frac{\partial M}{\partial i}$$

Dropping small terms gives

$$\Delta \dot{q} = M_u \Delta u + M_w \Delta w + M_{\dot{w}} \Delta \dot{w} + M_q \Delta q + M_{\dot{q}} \Delta \dot{q} + M_{\delta_e} \Delta \delta_e + M_{\delta_{th}} \Delta \delta_{th}$$

writing these equations in the form $E\dot{x} = Ax + Bu$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - Z_{\dot{w}} & 0 & 0 \\ 0 & -M_{\dot{w}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{w} \\ \Delta \dot{q} \\ \Delta \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & X_q - w_{eq} & -g_0 \cos(\theta_{eq}) \\ Z_u & Z_w & Z_q + u_{eq} & -g_0 \sin(\theta_{eq}) \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix} + \begin{bmatrix} X_{\delta_{th}} & X_{\delta_e} \\ Z_{\delta_{th}} & Z_{\delta_e} \\ M_{\delta_{th}} & M_{\delta_e} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \\ \Delta \\ \Delta \end{bmatrix}$$

where E is given by

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - Z_{\dot{w}} & 0 & 0 \\ 0 & -M_{\dot{w}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with inverse given by

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1 - Z_{\dot{w}}} & 0 & 0 \\ 0 & \frac{M_{\dot{w}}}{1 - Z_{\dot{w}}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying stuff out

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1 - Z_{\dot{w}}} & 0 & 0 \\ 0 & \frac{M_{\dot{w}}}{1 - Z_{\dot{w}}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_u & X_w & X_q - w_{eq} & -g_0 \cos(\theta_{eq}) \\ Z_u & Z_w & Z_q + u_{eq} & -g_0 \sin(\theta_{eq}) \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} X_u & X_w & X_q - w_{eq} \\ \frac{Z_u}{1 - Z_{\dot{w}}} & \frac{Z_w}{1 - Z_{\dot{w}}} & \frac{Z_q + u_{eq}}{1 - Z_{\dot{w}}} \\ \frac{Z_u M_{\dot{w}}}{1 - Z_{\dot{w}}} + M_u & \frac{Z_w M_{\dot{w}}}{1 - Z_{\dot{w}}} + M_w & \frac{(Z_q + u_{eq}) M_{\dot{w}}}{1 - Z_{\dot{w}}} + M_q \\ 0 & 0 & 1 \end{bmatrix}$$

IN STABILITY AXES

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{V_0} & 1 + \frac{Z_q}{V_0} \\ M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} \frac{Z_{\delta_e}}{V_0} \\ M_{\delta_e} \end{bmatrix} \delta_e$$

Linearizing the altitude equation to get $\dot{h}$ for the body-fixed axis system.

$$\dot{H} = U\sin(\Theta) - W\cos(\Theta)$$

$$\dot{H} = \dot{h} = f(U, W, \Theta) = U\sin(\Theta) - W\cos(\Theta)$$

$$\dot{h} = \left.\frac{\partial f}{\partial(U)}\right|_{\text{eq}} u + \left.\frac{\partial f}{\partial(W)}\right|_{\text{eq}} w + \left.\frac{\partial f}{\partial(\Theta)}\right|_{\text{eq}} \theta$$

$$\dot{h} = \sin(\Theta_{\text{eq}})u - \cos(\Theta_{\text{eq}})w + U_{\text{eq}}\cos(\Theta_{\text{eq}})\theta + W_{\text{eq}}\sin(\Theta_{\text{eq}})\theta$$

A bit of an abuse of notation, but $U_{\text{eq}} = u_{\text{eq}}$, $W_{\text{eq}} = w_{\text{eq}}$, and $\Theta_{\text{eq}} = \theta_{\text{eq}}$.

$$\dot{h} = \sin(\theta_{\text{eq}})u - \cos(\theta_{\text{eq}})w + u_{\text{eq}}\cos(\theta_{\text{eq}})\theta + w_{\text{eq}}\sin(\theta_{\text{eq}})\theta$$

Applying small angles $\sin(\theta_{\text{eq}}) = 0$ and $\cos(\theta_{\text{eq}}) = 1$

$$\dot{h} = u_{\text{eq}}\theta - w$$

18.3.3 Lateral-Directional Equations

Now the lateral equations are grouped, and the dependency on the longitudinal variables under steady wings level flight is dropped:

$$\dot{v} = -ru + pw + g_D\sin(\phi)\cos(\theta) + Y/m$$

$$\dot{p} = \frac{J_{xz}(-J_{xx} + J_{yy} - J_{zz})pq + [J_{zz}(J_{yy} - J_{zz}) - J_{xz}{}^2]qr}{J_{xx}J_{zz} - J_{xz}{}^2} + \frac{J_{zz}L - J_{xz}N}{J_{xx}J_{zz} - J_{xz}{}^2}$$

$$\dot{r} = \frac{[(J_{xx} - J_{yy})J_{xx} + J_{xz}{}^2]pq + J_{xz}[J_{xx} - J_{yy} + J_{zz}]qr}{J_{xx}J_{zz} - J_{xz}{}^2} + \frac{-J_{xz}L + J_{xx}N}{J_{xx}J_{zz} - J_{xz}{}^2}$$

$$\dot{\phi} = p$$

Following the same procedure as was done for the longitudinal equations, and using the following definitions

$$\boxed{Y_i = \frac{1}{m}\frac{\partial Y}{\partial i} \quad L_i = \frac{1}{J_{xx}}\frac{\partial Y}{\partial i} \quad N_i = \frac{1}{J_{zz}}\frac{\partial N}{\partial i}}$$

The lateral-directional linearized equations of motion can be written in the form $E\dot{x} = Ax + Bu$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{J_{xz}}{J_{xx}} & 0 \\ 0 & -\frac{J_{xz}}{J_{zz}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & Y_r - u_{\text{eq}} & -g_0\cos(\theta_{\text{eq}}) \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

From Yechout page 291

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -\frac{J_{xz}}{J_{xx}} & 0 \\
0 & -\frac{J_{xz}}{J_{zz}} & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi}
\end{bmatrix}
=
\begin{bmatrix}
Y_v & Y_p & Y_r - U_{\text{eq}} & g\cos(\Theta_{\text{eq}}) \\
L_v & L_p & L_r & 0 \\
N_v & N_p & N_r & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v \\ p \\ r \\ \phi
\end{bmatrix}
+
\begin{bmatrix}
Y_{\delta_a} & Y_{\delta_r} \\
L_{\delta_a} & L_{\delta_r} \\
N_{\delta_a} & N_{\delta_r} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_a \\ \delta_r
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -\frac{J_{xz}}{J_{xx}} & 0 \\
0 & -\frac{J_{xz}}{J_{zz}} & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}^{-1}
=
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{J_{xx}J_{zz}}{J_{xx}J_{zz}-J_{xz}^2} & \frac{J_{xz}J_{zz}}{J_{xz}^2-J_{xx}J_{zz}} & 0 \\
0 & \frac{J_{xz}J_{xx}}{J_{xz}^2-J_{xx}J_{zz}} & \frac{J_{xx}J_{zz}}{J_{xx}J_{zz}-J_{xz}^2} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

See also McLean page 37, where he defines primed stability derivatives, and then makes the linear model as follows, as shown on page 49. The primed notation just takes into account coupling.

$$
\begin{bmatrix}
\dot{\beta} \\ \dot{p} \\ \dot{r} \\ \dot{\phi}
\end{bmatrix}
=
\begin{bmatrix}
Y_\beta & 0 & -1 & \frac{g}{U_{\text{eq}}} \\
L_\beta & L_p & L_r & 0 \\
N_\beta & N_p & N_r & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\beta \\ p \\ r \\ \phi
\end{bmatrix}
+
\begin{bmatrix}
0 & Y_{\delta_r} \\
L_{\delta_a} & L_{\delta_r} \\
N_{\delta_a} & N_{\delta_r} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_a \\ \delta_r
\end{bmatrix}
$$

SEE STENGEL BOOK PAGE 293 HAS THE FULL LINEAR EQUATIONS WILL ALL THE TERMS IN THEM!

18.4 Overview of Equations for Spherical, Rotating Earth

In this section the equations of motion describing the flight of a vehicle in the Earth's atmosphere are described. The Earth is assumed to be spherical, and rotating about the z-axis of an inertially fixed reference frame. The Earth's atmosphere is assumed to move with the relative rotation of the Earth.

Force Equations   In (Bilimoria, Schmidt)

$$
\left. \frac{dV_A}{dt} \right|_B + \omega_{B,I} \times V_A + \omega_{E,I} \times V_A + \omega_{E,I} \times (\omega_{E,I} \times \mathscr{R}) = g + (F_A + F_T)/m
\tag{18.18}
$$

Where $V_A = \begin{bmatrix} u & v & w \end{bmatrix}^\top$ and $\omega_{B,I} = \begin{bmatrix} p & q & r \end{bmatrix}^\top$. The force equations in F_B are given by: (Etkin pg 123- 143)

$$
X - mg\sin\theta = m[\dot{u} + (q_B^E + q)w - (r_B^E + r)v]
$$
$$
Y + mg\cos\theta\sin\phi = m[\dot{v} + (r_B^E + r)u - (p_B^E + p)w]
$$
$$
Z + mg\cos\theta\sin\phi = m[\dot{w} + (p_B^E + p)v - (q_B^E + q)u]
$$

Moment Equations

$$J\frac{d\omega_{B,I}}{dt}\bigg|_B + \omega_{B,I} \times J\omega_{B,I} = M_A + M_T \qquad (18.19)$$

The moment equations in F_B are given by the following, where as in the force equations, the moments L, M, and N are computed from look-up tables.

$$L = I_x\dot{p} - I_{yz}(q^2 - r^2) - I_{zx}(\dot{r} + pq) - I_{xy}(\dot{q} - rp) - (I_y - I_z)qr$$
$$M = I_y\dot{q} - I_{zx}(r^2 - p^2) - I_{xy}(\dot{p} + qr) - I_{yz}(\dot{r} - pq) - (I_z - I_x)rp$$
$$N = I_z\dot{r} - I_{xy}(p^2 - q^2) - I_{yz}(\dot{q} + rp) - I_{zx}(\dot{p} - qr) - (I_x - I_y)pq$$

Orientation Equations   The orientation kinematic equation is given by

$$\omega_{B,V}|^{Eu} = T\omega_{B,V}|^B$$

where the following is the transformation matrix T from body axes to Euler axes

$$T = \begin{bmatrix} 1 & \tan(\theta)\sin(\phi) & \tan(\theta)\cos(\phi) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)/\cos(\theta) & \cos(\phi)/\cos(\theta) \end{bmatrix}$$

Expressing this equation in terms of the scalar components

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p_v \\ q_v \\ r_v \end{bmatrix}$$

where

$$\begin{bmatrix} p_v \\ q_v \\ r_v \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} - R_{BV} \begin{bmatrix} (\omega_{earth} + \dot{\tau})\cos\lambda \\ -\dot{\lambda} \\ -(\omega_{earth} + \dot{\tau})\sin\lambda \end{bmatrix}$$

Etkin uses captial letters to denote angular velocity of the aircraft with respect to the vehicle carrying frame, whereas Bilimoria and Schmidt use subscript v.

$$\begin{bmatrix} p_v \\ q_v \\ r_v \end{bmatrix} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

The following rotation matrix is used to rotate a vector A expressed in the the vehicle carrying frame f_V to be expressed in body frame f_B. That is: $A|^B = R_{BV}A|^V$. This rotation matrix R_{BV} is given by (Bilimoria and Schmidt use the notation $[T]$):

$$R_{BV} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

The rotation matrix from the body frame f_B to the vehicle carrying frame f_V is given by:

$$R_{VB} = R_{BV}^{-1} = R_{BV}^{\top}$$

Kinematics:

$$V_B^E = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix} \tag{18.20}$$

Finally, the absolute angular velocity are shown, as well as the angular velocity components due to the Earth's rotation in F_B:

$$\omega_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \qquad \omega_B^E = \begin{bmatrix} p_B^E \\ q_B^E \\ r_B^E \end{bmatrix} = L_{BV} \begin{bmatrix} \cos \lambda \\ 0 \\ -\sin \lambda \end{bmatrix} \omega_{\text{earth}}$$

Navigation Equations   The trajectory kinematics are given by rotating the absolute velocity components of F_B into F_{EC}. Bilimoria and Schmidt use $[T]^{\top}$ instead of L_{VB} like Etkin. The navigation kinematic equation is given by

$$\boxed{V_A|^V = R_{BV}^{\top} V_A|^B}$$

$$\begin{bmatrix} \dot{\lambda}\mathscr{R} \\ \dot{\tau}\mathscr{R}\cos\lambda \\ -\dot{\mathscr{R}} \end{bmatrix} = L_{VB} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

18.5   Representation of Uncertainties in Aircraft Model

18.5.1   Center of Gravity Shift

Start with the nonlinear 6-DOF equations of motion for flat earth. We could show the effect of the CG shift as well using the equations of motion for a spherical, rotating earth, but essentially its the same thing. Looking at the force and moment equations from Stevens and Lewis page 110, and shown as equations 6 and 7 above, these are just Newtons second law. In these equations, there is no specification to where the origin of the body-fixed coordinate system is. So, when it comes to the moment equations with moment M and moment of inertia terms J, we haven't yet specified about which point this moment is to be taken. This moment M is the entire moment on the vehicle? Aerodynamic, thrust, and gravity? Need to look what the convention is for moment in body axes. Should be CG. Then, after I figure out how the moment changes when the CG shifts in the body axis frame, need to convert to stability axes and show where the uncertain terms are in stability axis representation. See Yechout page 153 for transformation matrix from body axes to stability axes. Look at pitch equation

$$\dot{Q} = \frac{M}{J_{yy}}$$

Consider taking moments about a fixed point on the aircraft given by the aircrafts nominal CG location. This moment has terms due to aerodynamics, thrust, and gravity. When the CG shifts to a new location, if we take moments about the nominal CG location, the new moment M will change, as well as Jyy. The new

Jyy can be found using parallel axis theorem. The only change in this moment is that due to gravity. So we can write this as

$$\dot{Q} = \frac{M_{\text{new}}}{J_{yy,\text{new}}}$$

where

$$J_{yy,\text{new}} = J_{yy} + m(\Delta x^2 + \Delta z^2)$$

where Δx and Δz are the CG shift, and

$$M_{\text{new}} = M - mg(\Delta x \cos\theta + \Delta z \sin\theta)$$

giving

$$\dot{Q} = \frac{M - mg(\Delta x \cos\Theta + \Delta z \sin\Theta)}{J_{yy} + m(\Delta x^2 + \Delta z^2)}$$

Linearizing this equation where $Q = Q_{\text{eq}} + q$ and $\Theta = \Theta_{\text{eq}} + \theta$ and taylor series expansion for sin and cosine

$$\cos\Theta \approx \cos\Theta_{\text{eq}} - \sin\Theta_{\text{eq}}\theta$$
$$\sin\Theta \approx \sin\Theta_{\text{eq}} + \cos\Theta_{\text{eq}}\theta$$

$$[J_{yy} + m(\Delta x^2 + \Delta z^2)]\dot{q} = M - M_{mg}$$

where

$$M_{mg} = mg[\Delta x(\cos\theta_{\text{eq}} - \sin\theta_{\text{eq}}\Delta\theta) + \Delta z(\sin\theta_{\text{eq}} + \cos\theta_{\text{eq}}\Delta\theta)]$$
$$= mg[\Delta z \cos\theta_{\text{eq}} - \Delta x \sin\theta_{\text{eq}}]\Delta\theta + mg[\Delta x \cos\theta_{\text{eq}} + \Delta z \sin\theta_{\text{eq}}]$$

$$[J_{yy} + m(\Delta x^2 + \Delta z^2)]\dot{q} = M - M_{mg}$$

when we linearize, will do Taylor Series expansion, drop higher order terms. Note that the moment M at equilibrium is zero. Taylor Series expansion is given by

$$f(X) \approx f(X_{\text{eq}}) + \left.\frac{\partial f}{\partial X_1}\right|_{\text{eq}} x_1 + \left.\frac{\partial f}{\partial X_2}\right|_{\text{eq}} x_2 + \ldots$$

So using the Taylor Series expansion to get the new linearized M we have

$$M = M_{\text{eq}} + \frac{\partial M}{\partial u}u + \frac{\partial M}{\partial w}w + \frac{\partial M}{\partial \dot{w}}\dot{w} + \frac{\partial M}{\partial q}q + \frac{\partial M}{\partial \dot{q}}\dot{q} + \frac{\partial M}{\partial \delta_e}\delta_e$$

so

$$J_{yy,\text{new}}\dot{q} = \left(\frac{\partial M}{\partial u}u + \frac{\partial M}{\partial w}w + \frac{\partial M}{\partial \dot{w}}\dot{w} + \frac{\partial M}{\partial q}q + \frac{\partial M}{\partial \dot{q}}\dot{q} + \frac{\partial M}{\partial \delta_e}\delta_e\right) - M_{mg}$$

Throwing out small terms

$$J_{yy,\text{new}}\dot{q} = \left(\frac{\partial M}{\partial w}w + \frac{\partial M}{\partial \dot{w}}\dot{w} + \frac{\partial M}{\partial q}q + \frac{\partial M}{\partial \delta_e}\delta_e\right) - M_{mg}$$

18.5.2 Thinking About CG Shift and Doing Moment About New CG Location

model aircraft as a lift and drag force acting at the center of pressure. Consider moments about the CG. At equilibrium flight condition, the moment about the CG is zero. If the CG shifts to a new location and we take moments about that new location, the pitching moment effect of the wing is still the same, but there is additional moment due to the lift acting along a nonzero lever arm. Assume that the moment of inertia doesn't change due to CG shift. Then the moment about the new CG is the following, where M is moment about the original CG location.

$$\dot{Q} = \frac{M - L\Delta x \cos\alpha - D\Delta x \sin\alpha + D\Delta z \cos\alpha - L\Delta z \sin\alpha}{J_{yy}}$$

18.5.3 CG Shift

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{V_0} & 1 + \frac{Z_q}{V_0} \\ M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} Z_{\delta_e} \\ M_{\delta_e} \end{bmatrix} \delta_e \tag{18.21}$$

We wish to use the method described above to design a pitch-rate tracking adaptive controller, when only the pitch-rate measurement is available.

$$M = M(\alpha, q, \delta_e)$$

In the conventional aircraft model, the total moment about aircraft center of gravity, in the longitudinal direction, is due to lift and drag contributions from the wing and horizontal tail. Looking at the equation for $\dot{\alpha}$, and with the total lift for Z independent of the CG location, no uncertainty will result in this equation due to CG shift. In the $\dot{q}$ equation the moment about the y-axis M is dependent on the CG location. The moment change due to a longitudinal CG shift Δx is

$$\Delta M = \big\{ (L_w + L_t)\cos\alpha + (D_w + D_t)\sin\alpha \big\}\Delta x$$

The lift and drag terms are linear in α, and the tail contributions linear in δ_e. Taking the partial derivatives of M, to determine the stability derivatives, we see perturbation in M_α and M_{δ_e}. We then see with $Z_{\delta_e} \approx 0$

$$\begin{bmatrix} \frac{Z_\alpha}{V_0} & 1 + \frac{Z_q}{V_0} \\ M_\alpha & M_q \end{bmatrix} + \begin{bmatrix} Z_{\delta_e} \\ M_{\delta_e} \end{bmatrix} \Lambda \begin{bmatrix} w_{p_1} & w_{p_2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{Z_\alpha}{V_0} & 1 + \frac{Z_q}{V_0} \\ M_\alpha & M_q \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ M_{\delta_e}\Lambda w_{p_1} & M_{\delta_e}\Lambda w_{p_2} \end{bmatrix}$$

18.5.4 Stability Derivative Uncertainties

C_{M_α} and C_{N_β} have already shown this for pitching moment coefficient, can show same thing on lateral-directional equations of motion for yawing moment coefficient and several others as well

18.5.5 Control Surface Effectiveness

Λ

18.5.6 Actuator Saturation
18.5.7 Guidance Control Research

Consider the following linear model describing the longitudinal flight dynamics of an aircraft, where α is the angle-of-attack, q the pitch rate, θ is pitch angle, and h is the altitude. It is assumed in this problem that α is not measurable, but acceleration sensors aligned with the vehicle body axes are, where a_z is the vertical acceleration. The follow equations show the dynamics of this system with the available system outputs.

$$
\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{V_{\text{eq}}} & 1 + \frac{Z_q}{V_{\text{eq}}} & 0 & 0 \\ M_\alpha + \frac{M_{\dot{\alpha}} Z_\alpha}{V_{\text{eq}}} & M_q + M_{\dot{\alpha}} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -V_{\text{eq}} & 0 & V_{\text{eq}} & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \\ h \end{bmatrix} + \begin{bmatrix} \frac{Z_{\delta_e}}{V_{\text{eq}}} \\ M_{\delta_e} \\ 0 \\ 0 \end{bmatrix} \delta_e
$$

$$
\begin{bmatrix} a_z \\ q \\ \theta \\ h \end{bmatrix} = \begin{bmatrix} Z_\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \\ h \end{bmatrix} + \begin{bmatrix} Z_{\delta_e} \\ 0 \\ 0 \\ 0 \end{bmatrix} \delta_e
$$

The control goal is to design an adaptive controller which will allow the vehicle to accurately track a reference altitude trajectory in the presence of uncertainties.

$$
\begin{aligned}
\ddot{h} &= -V_{\text{eq}}(\dot{\theta} - \dot{\alpha}) \\
&= V_{\text{eq}}(\dot{\alpha} - q) \\
&= Z_V V_T + Z_\alpha \alpha + (V_{\text{eq}} + Z_q)q - g\sin(\gamma_{\text{eq}})\theta - V_{\text{eq}} X_{\delta_{\text{th}}} \sin(\alpha_{\text{eq}})\delta_{\text{th}} + Z_{\delta_e}\delta_e - V_{\text{eq}}q \\
&= Z_V V_T + Z_\alpha \alpha + Z_q q - g\sin(\gamma_{\text{eq}})\theta - V_{\text{eq}} X_{\delta_{\text{th}}} \sin(\alpha_{\text{eq}})\delta_{\text{th}} + Z_{\delta_e}\delta_e
\end{aligned}
$$

Analysis of Open-Loop Dynamics

Transfer function from elevator to acceleration is relative degree zero, non-minimum phase. The altitude is essentially the acceleration twice integrated.

$$
\frac{a_z}{\delta_e} = \frac{-18.6(s + 4.894)(s - 4.354)}{s^2 + 0.9414s + 1.816}
$$

18.6 Linear Aircraft Models

In the previous chapter, simple linear models describing the longitudinal and lateral-directional dynamics of a flight vehical were derived from general equations of motion by linearization, decoupling, and order reduction.

$$
\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{V_0} & 1 + \frac{Z_q}{V_0} \\ M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} \frac{Z_{\delta_e}}{V_0} \\ M_{\delta_e} \end{bmatrix} \delta_e
$$

$$
A_z = Z_\alpha \alpha + Z_{\delta_e}\delta_e
$$

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$$
\begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} Y_\beta & 0 & -1 & \frac{g}{U_{\text{eq}}} \\ L_\beta & L_p & L_r & 0 \\ N_\beta & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} 0 & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}
$$

18.7   Actuator Models

This note shows the block diagram representation for a second order actuator with rate and limit saturations. The second order transfer function describing the deflection of the actuator due to an actuator command is:

$$
\frac{\delta}{\delta_{\text{cmd}}} = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}
$$

Cross multiplying:

$$
\delta s^2 + \delta 2\zeta\omega_n s + \delta\omega_n{}^2 = \delta_{\text{cmd}}\omega_n{}^2
$$

$$
\ddot{\delta} + 2\zeta\omega_n\dot{\delta} + \delta\omega_n{}^2 = \delta_{\text{cmd}}\omega_n{}^2
$$

Solving for $\ddot{\delta}$

$$
\ddot{\delta} = \omega_n{}^2(\delta_{\text{cmd}} - \delta) - 2\zeta\omega_n\dot{\delta}
$$

This expression can be used to begin the block diagram representation. Once $\ddot{\delta}$ is had, it is integrated to get $\dot{\delta}$ and fed back to complete the block diagram.
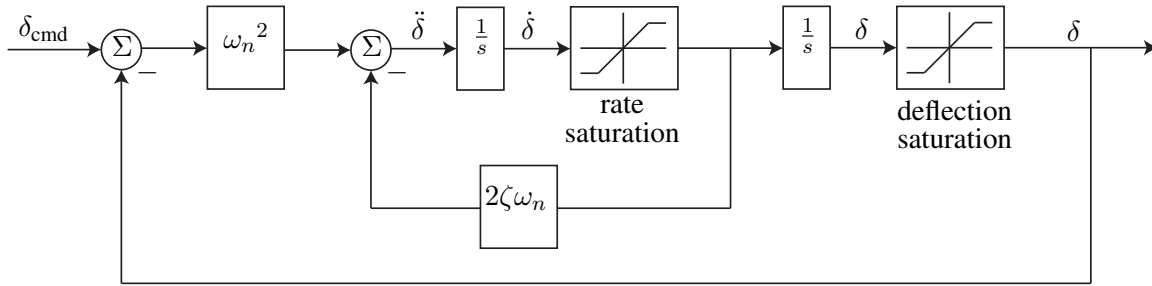
Figure 18.1: Second order actuator block diagram

Part III

Linear Systems

Chapter 19

Real Analysis

19.1 Notation and Preliminaries
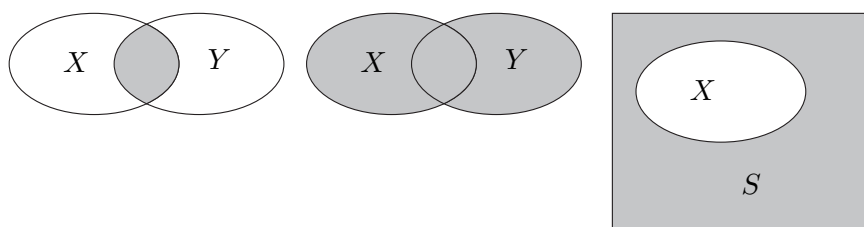
Figure 19.1: From left to right: $X \cap Y$, $X \cup Y$, and X^{c}

19.1.1 Random Stuff

- Zero (0) is even

- Rational numbers are countable

19.2 Chapter 1

- associativity: $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$

- commutativity: $x + y = y + x$ and $xy = yx$

- distributivity: $x(y + z) = xy + xz$

Definition 5 Fundamental theorem of arthimetic *Every positive integer ≥ 2 can be written as a product of primes in exactly one way up to rearrangements. See Hully and Wright Introduction to the Theory of Numbers sec 2.11.*

Remark 1 A set is completely characterized by the elements in it. That is to say if X and Y are sets then $X = Y$ if and only if, for all x, $x \in X$ if and only if $x \in Y$.

Definition 6 Proper subset *A set X is a proper subset of Y if $X \subset Y$ but $X \neq Y$.*

Example 10 Pythagorean Proposition There exists no such rational number r such that $r^2 = 2$.

PROOF By contradiction. Suppose $r = m/n$ is a rational number, that $r^2 = 2$, and that m and n have no common factor.

$$r = \frac{m}{n}$$

$$r^2 = \frac{m^2}{n^2} = 2$$

Rearranging the fraction

$$m^2 = 2n^2$$

Any integer n when squared remains an integer. When this integer n^2 is multiplied by two, the resulting quantity $2n^2$ is an even number. Thus m^2 is an even number, and thus m must be even as well. Since m is an even number, it can be expressed as the following, for some integer k.

$$m = 2k$$

Substituting this expression into the above gives

$$2k^2 = n^2$$

By the same argument as before, n is also an even number. With m and n both even, they have a common factor, 2, and the beginning supposition is contradicted. □

Definition 7 1.5 — Order *If S is a set, a relation $<$ is an order on S if:*

(i) for any $x, y \in S$ exactly one of the following holds

$$x < y \qquad x = y \qquad x > y$$

(ii) For any $x, y, z \in S$ $x < y$ and $y < z$ implies $x < z$

The relation $<$ acts on the elements of the set S. Some examples of sets on which the order is defined are $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$, where $<$ is the usual relation, that is $<$ means for $r, s \in \mathbb{Q}$ that $r < s$ means $s - r$ is a positive rational number.

Another example is the set $\mathbb{L} = \{strings\ of\ letters\}$ where the relation $<$ is lexicographic order. That is, the order operation is defined on $\mathbb{L}$ that puts strings of letters (words) in order as they are found in the dictionary: alphabetical order.

A non-example is subsets of $\mathbb{N}$ under C? That $\{1, 2\}$ not comparable to $\{3, 4\}$? It is important to think beyond the traditional greater than or less than. We will see later a way in which the complex numbers can be an ordered set, although the complex numbers cannot be an ordered field.

Definition 8 1.6 — Ordered set *An ordered set is a set S in which an order relation is defined.*

Definition 9 1.7 — Bounded above / upper bound *if S is an ordered set and $E \subset S$ then E is bounded above if there is a $\beta \in S$ such that $\beta \geq \gamma$ $\forall \gamma \in E$. Such a β is called an upper bound for E.*

Definition 10 1.8 — Least upper bound / supremum *If $E \subset S$ is bounded above and if $\alpha \in S$ obeys both of the following*

  (i) α is an upper bound for E

  (ii) If $\gamma < \alpha$ then γ is not an upper bound of E

then α is the least upper bound of E, and we write $\alpha = \sup E$. The supremum α need not be a member of E.

    Similarly define: bounded below/lower bound, and greatest lower bound/infimum.

Definition 11 1.10 — Least upper bound property *An ordered set S has the least upper bound property if any set $E \subset S$ which is bounded above has a least upper bound. That is that $\sup E$ exists in S.*

Example 11 $\mathbb{N}$ or $\mathbb{Z}$ under $<$.

Example 12 The following is a counterexample. $\mathbb{Q}$ under $<$. The set $E = \{p \in \mathbb{Q} | p^2 < 2\}$ is bounded above but has no least upper bound.

Theorem 1 1.11 *If S is an ordered set with the least upper bound property, and $B \subset S$ is nonempty and bounded below, then $L = \{\beta \in S : \beta$ is a lower bound for $B\}$ is bounded above and $\sup L = \inf B$*

PROOF Recall the definition for least-upper-bound property which states that the $\sup B$ exists in S.

$$\underline{ S }$$
$$\underline{ L }\; \underline{ B }$$
$$\alpha = \sup L = \inf B$$

1. Since B is bounded below, L is not empty.

2. By 1.7 Definition for lower bound, $L \subset S$ consists of all $y \in S$ such that $y \le x$ for every $x \in B$.

$$L = \{y \in S \mid y \le x \; \forall x \in B\}$$

3. Every $x \in B$ is an upper bound of L by 1.7 Definition.

4. Thus L is bounded above

5. Because S is an ordered set with the least-upper-bound property, $L \subset S$, and L is bounded above, by definition $\alpha = \sup L$ exists in S.

6. If we take a value γ with $\gamma < \alpha$, then γ is not an upper bound of L, and $\gamma \notin B$.

7. Every $x \in B$ is an upper bound for L, but we don't know yet that the least upper bound of L lives in B. All we know is that the least upper bound α is equal to or less than the smallest upper bound that lives in B. That is $\alpha \leq x$ for every $x \in B$.

8. From step (2) where the definition for a bound was used to define L, we see that $\alpha \in L$.

9. If we pick a $\beta > \alpha$, then $\beta \notin L$ since α is an upper bound for L, so picking something larger cannot be in L.

10. Since L is the set of lower bounds of B, and that if we take a value $\beta > \alpha$ that $\beta \notin L$ then β is not a lower bound. But we already showed that $\alpha \in L$ was a lower bound for B. Therefore α is the greatest lower bound.

11. $\alpha = \inf B$

$\square$

19.2.1 Fields

Remark 2 The neutral element 0 defined in the A4 the field axiom for addition is unique.

PROOF Proof strategy: assume there exists another neutral element $\hat{0}$ and then show that the two neutral elements must be equal. The definition of neutral element is a 0 such that $0 + x = x \; \forall x \in \mathbb{F}$. Following this definition:

$$0 + x = x \quad \forall x \in \mathbb{F}$$
$$\hat{0} + x = x \quad \forall x \in \mathbb{F}$$

Using these definitions for our two neutral elements, both of the following must hold

$$0 + \hat{0} = \hat{0}$$

$$\hat{0} + 0 = 0$$

giving

$$\hat{0} = \hat{0} + 0 = 0$$

and so the two neutral elements must be in fact the same

$$\boxed{\hat{0} = 0}$$

$\square$

Remark 3 The additive inverse element defined in the A5 the field axiom for addition is unique.

PROOF Proof strategy: assume there exists another inverse element of a and show that the two inverse elements of a must be the same. $\square$

Some notes on fields. Given the field axioms, the neutral elements for addition and multiplication are unique. In addition, the additive and multiplicative inverse elements are also unique.

Theorem 2 1.20 — Archimedean Principle *For every x, $y \in \mathbb{R}$ and $x > 0$ there exists $n \in \mathbb{N}$ such that*

$$nx > y$$

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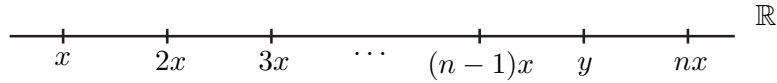
PROOF *(By contradiction)* Let $A = \{nx \mid n \in \mathbb{N}\}$. To prove that $nx > y$ we want to assume the opposite, and show how this leads to a contradiction. So, the opposite of $nx > y$ is to assume that

$$nx \leq y$$

for all $n \in \mathbb{N}$. If this was true, then y would be an upper bound of A. By 1.19 Theorem, $\mathbb{R}$ has the least upper bound property, giving, with 1.10 Definition, that the least upper bound of A must be in $\mathbb{R}$. This is stated

$$\alpha = \sup A \quad \alpha \in \mathbb{R}$$

So, since $x > 0$ and $\alpha = \sup A$, we can say

$$\alpha - x < \alpha$$

Since we defined α to be the least upper bound, and $x > 0$, then taking a value $\alpha - x$ will not be an upper bound of A. Since A is the set of mx where $m \in \mathbb{N}$, with an upper bound α, if we take a value $\alpha - x$ which is less than α, we know there will be some $m \in \mathbb{N}$ such that $mx \in A$. This means then that for some $m \in \mathbb{N}$

$$\alpha - x < mx$$

Rearranging this expression we get

$$\alpha < x(m + 1)$$

We see that $x(m + 1)$ lives in A since $m + 1$ is a natural number. However, we said that α was the least upper bound of A. That is, no value which lives in A can possibly be larger than α. This is a contradiction to our original assumption that $nx \leq y \ \forall n \in \mathbb{N}$, so we know then that

$$\boxed{nx > y}$$

□

Corollary *That is, for every $x \in \mathbb{R}$, $x > 0$ there is $n \in \mathbb{N}$ such that*

$$0 < 1/n < x$$

PROOF Assume there is an $x \in \mathbb{R}$ with $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$. If this were true, we would get $0 < x < 1/n$, and cross-multiplying would give $1/x > n$ for all $n \in \mathbb{N}$. □

Corollary *For every real number x there is an integer n such that $x < n \leq x + 1$.*

PROOF □

Corollary *This corollary to the Archimedean Principle states that between any two real numbers there is a rational number.*

PROOF Start with two real numbers $x, y \in \mathbb{R}$ on the number line, with $y > x$. We want to show that somewhere between x and y there is a rational number, no matter how close x and y are together. If $y = x + 1$, than we know there will be an integer on this interval. The thought process is: as x and y get

closer together, we can scale the interval up by an integer and stretch it out until x and y are a distance of 1 apart, find the integer that is between x and y, and then divide this integer by how much we had to scale up the interval, giving us a natural number.

The quantity $z = y - x$ is a real number. We know from the corollary that states there are no infinitely small numbers that $1/n < z$. Or, $1/n < y - x$. We then "stretch" this interval by n, multiplying both sides of the inequality.

$$1 < ny - nx$$

rearranging

$$1 + nx < ny$$

Now it is also clear that $nx < nx + 1$ and we also know there must be an integer m in between nx and $nx + 1$.

$$nx < m < 1 + nx < ny$$

Now divide back through by n

$$x < m/n < (1 + nx)/n < y$$

and the proof is complete

$$\boxed{x < \frac{m}{n} < \frac{x + 1}{n} < y}$$

$\square$

Theorem 3 1.21 Theorem — Extraction of nth roots / uniqueness of nth roots

Definition 12 Axiom of Completeness *Every set of real numbers that is bounded above has a least upper bound.*

Definition 13 Well-Ordering Principle *Every non-empty set of natural numbers has a least element.*

Theorem 4 1.35 — Cauchy-Schwartz Inequality *For complex numbers $a_1, \ldots, a_n$; $b_1, \ldots, b_n \in \mathbb{C}$, or $a_j, b_j \in \mathbb{C}$ then*

$$\left| \sum_{j=1}^{n} a_j \overline{b_j} \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

For real vectors in $\mathbb{R}^n$ this is expressed as

$$|x^\mathsf{T} y| \leq \|x\| \|y\|$$

PROOF First define the following quantities. Let

$$A = \sum_{j=1}^{n} |a_j|^2 \in \mathbb{R} \qquad B = \sum_{j=1}^{n} |b_j|^2 \in \mathbb{R} \qquad C = \sum_{j=1}^{n} a_j \overline{b_j} \in \mathbb{C}$$

and let both $A > 0$ and $B > 0$, or the Cauchy-Schwartz inequality is trivial. Start by evaluating the following expression

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = ?$$

Letting $z = Ba_j - Cb_j$ this can be written

$$\sum_{j=1}^{n} |z|^2 = ?$$

Using Rudin Theorem 1.32 the absolute value of a complex number z is given by $|z| = (z\overline{z})^{1/2}$, allowing the above expression to be written

$$\sum_{j=1}^{n} |z|^2 = \sum_{j=1}^{n} [(z\overline{z})^{1/2}]^2 = \sum_{j=1}^{n} z\overline{z}$$

Substituting back in $z = Ba_j - Cb_j$

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = \sum_{j=1}^{n} (Ba_j - Cb_j)(\overline{Ba_j - Cb_j})$$

Using the property from Rudin Theorem 1.31 for complex numbers z and w that $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\ \overline{w}$, and Definition 1.30 that the conjugate of a real number is just itself gives

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = \sum_{j=1}^{n} (Ba_j - Cb_j)(\overline{Ba_j} - \overline{Cb_j})$$

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = \sum_{j=1}^{n} (Ba_j - Cb_j)(\overline{B}\ \overline{a_j} - \overline{C}\ \overline{b_j})$$

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = \sum_{j=1}^{n} (Ba_j - Cb_j)(B\overline{a_j} - \overline{C}\ \overline{b_j})$$

Multiplying out

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = \sum_{j=1}^{n} B^2 a_j \overline{a_j} - Ba_j \overline{C}\ \overline{b_j} - Cb_j B\overline{a_j} + Cb_j \overline{C}\ \overline{b_j}$$

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = \sum_{j=1}^{n} B^2 a_j \overline{a_j} - \sum_{j=1}^{n} Ba_j \overline{C}\ \overline{b_j} - \sum_{j=1}^{n} Cb_j B\overline{a_j} + \sum_{j=1}^{n} Cb_j \overline{C}\ \overline{b_j}$$

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = B^2 \sum_{j=1}^{n} a_j \overline{a_j} - B\overline{C} \sum_{j=1}^{n} a_j \overline{b_j} - CB \sum_{j=1}^{n} b_j \overline{a_j} + C\overline{C} \sum_{j=1}^{n} b_j \overline{b_j}$$

Using Rudin Definition 1.32 for the absolute value of a complex number

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = B^2 \sum_{j=1}^{n} |a_j|^2 - B\overline{C} \sum_{j=1}^{n} a_j \overline{b_j} - CB \sum_{j=1}^{n} b_j \overline{a_j} + C\overline{C} \sum_{j=1}^{n} |b_j|^2$$

Substituting in the expressions for A, B, and C

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = B^2 A - B\overline{C}C - CB \sum_{j=1}^{n} b_j \overline{a_j} + C\overline{C}B$$

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = B^2 A - CB \sum_{j=1}^{n} b_j \overline{a_j}$$

Because $C = \sum_{j=1}^{n} a_j \overline{b_j}$ then $\overline{C} = \sum_{j=1}^{n} \overline{a_j} b_j$ allowing the above to be written

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = B^2 A - CB \overline{C}$$

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = B^2 A - B|C|^2$$

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = B(AB - |C|^2)$$

Because (by Rudin Theorem 1.33) the absolute value of a complex quantity is always non-negative, the left hand side of the above equation is non-negative. Therefore

$$B(AB - |C|^2) \geq 0$$

Since $B > 0$ then

$$AB - |C|^2 \geq 0$$

Finally, substituting back in the expressions for A, B, and C

$$\sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2 - \left| \sum_{j=1}^{n} a_j \overline{b_j} \right|^2 \geq 0$$

rearranging

$$\boxed{\left| \sum_{j=1}^{n} a_j \overline{b_j} \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2}$$

which is the desired inequality. □

Example 13   Prove the following inequality for vectors x and y

$$\big| \|x\| - \|y\| \big| \leq \|x + y\|$$

PROOF   First start by expressing x as

$$x = x + y - y$$

with norm

$$\|x\| = \|x + y - y\|$$

using $a = x + y$ and $b = -y$

$$\|x\| = \|a + b\|$$

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By the triangle inequality

$$\|x\| = \|a + b\| \leq \|a\| + \|b\| = \|x + y\| + \| - y\|$$

$$\|x\| \leq \|x + y\| + \|y\|$$

giving

$$\boxed{\|x\| - \|y\| \leq \|x + y\|}$$

Next express y as

$$y = y + x - x$$

with norm

$$\|y\| = \|y + x - x\|$$

now using $a = x + y$ and $c = -x$

$$\|y\| = \|a + c\|$$

By the triangle inequality

$$\|y\| = \|a + c\| \leq \|a\| + \|c\| = \|x + y\| + \| - x\|$$

$$\|y\| \leq \|x + y\| + \|x\|$$

giving

$$\boxed{\|y\| - \|x\| \leq \|x + y\|}$$

If $\|x\| > \|y\|$ then $\big|\|x\| - \|y\|\big| = \|x\| - \|y\|$ and the inequality is satisfied by the first boxed equation. If $\|y\| > \|x\|$ then $\big|\|x\| - \|y\|\big| = \|y\| - \|x\|$ and the inequality is then satisfied by the second equation. That is, for real numbers a and b

$$|a - b| = \begin{cases} a - b & \text{if } a > b \\ 0 & \text{if } a = b \\ b - a, & \text{if } a < b \end{cases}$$

so saying $|a - b| < x$ is the same as saying $a - b < x$ and $b - a < x$. So another way to think about the proof is to prove two things: Prove both of the following inequality for vectors x and y

$$\|x\| - \|y\| \leq \|x + y\|$$

and

$$\|y\| - \|x\| \leq \|x + y\|$$

Which is what was done. $\qquad\square$

19.2.2 Problems

Example 14 Rudin Ch1 Prob 8 No order can be defined in the complex field that turns it into an ordered field.

PROOF In order to attempt a solution to this problem, review 1.17 Definition on page 7 of what an ordered field is. An ordered field, which is also an ordered set, must therefore satisfy 1.5 Definition of an order relation. Note that the neutral element is $0 \in \mathbb{C} = (0,0)$ and begin with a complex number $z \in \mathbb{C}$, with $z \neq 0$. By trichotomy, one and only one of the following statements can hold

$$z > 0 \quad \text{or} \quad z < 0$$

Then state that the complex number $i = (0,1)$ is not equal to the neutral element, and therefore must either be greater than or less than the neutral element. Examine both of these cases. Suppose first that $i > 0$

$$i > 0$$
$$ii > 0i$$
$$i^2 > 0$$
$$-1 > 0$$

Now examine the case when $i < 0$. Then

$$-i > 0$$
$$(-i)(-i) > 0$$
$$i^2 > 0(-i)$$
$$-1 > 0$$

$\square$

19.3 Chapter 2

- Injective = one-to-one

- Surjective = onto

- Bijective = both injective and surjective

Proposition 1 A set is infinite if and only if it may be put into one-to-one correspondence with a proper subset of itself.

PROOF Assume the set X is finite, and has n elements where $n \in \mathbb{N}$, and assume the set Y is a proper subset of X. Then, because X is finite and Y is a proper subset of X, Y must have fewer elements in it than X. That is, Y will have m elements, where $m \in \mathbb{N}$ with $m < n$. The only way to put two finite sets

in one-to-one correspondence is if they have the same number of elements, which X and Y do not. This proves the "if" part of the statement, concluding that... $\qquad\square$

Theorem 5 2.12 *Let $\{E_n\}$, $n = 1, 2, 3, \ldots$, be a sequence of countable sets, and put*

$$S = \bigcup_{n=1}^{\infty} E_n$$

Then S is countable.

PROOF Basically take each set E_i and write its elements in a row. This will arrange all of the elements of each set E_i in an infinite array. Then form a sequence by ordering the elements along the diagonals. If any two of the sets E_i have elements in common, these will appear more than once in the sequence. Hence there is a subset T of the set of all the positive integeres such that $S \sim T$, which shows that S is at most countable by 2.8 Theorem. Since $E_1 \subset S$ and E_1 is infinite, S is infinite, and thus countable. $\qquad\square$

Theorem 6 2.14 *Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable.*

PROOF Here is the set A, whose members themselves are infinite sequences of 0 and 1.

$$A = \{\underbrace{\{1, 0, 0, 1, \ldots\}}_{s_1}, \underbrace{\{1, 0, 1, 0, \ldots\}}_{s_2}, \underbrace{\{0, 1, 1, 0, \ldots\}}_{s_3}, \underbrace{\{1, 1, 1, 0, \ldots\}}_{s_4}, \ldots\}$$

Let E be a countable subset of A.

$$E = \{s_1, s_2, s_3, s_4, \ldots\}$$

Arrange the elements of E by stacking the rows formed by the sequences s_i on top of each other, making an array. Then define a sequence s by taking the diagonal values from this array, but switching 0 to 1 and vice versa.

	1	2	3	4
s_1	1	0	0	1
s_2	1	0	1	0
s_3	0	1	1	0
s_4	1	1	1	0

So the sequence s is

$$s = \{0, 1, 0, 1, \ldots\}$$

Because at least one element in each sequence s_i was changed, this new sequence s is different from all of the $s_i \in E$. The existence of this sequence which is in A but not in E shows that E is a proper subset of A. Because we took an arbitrary countable subset of A and showed that it is a proper subset of A, then A is uncountable (for otherwise A would be a proper subset of A which is absurd). $\qquad\square$

Definition 14 2.18a — Neighborhood *A neighborhood around a point p...*
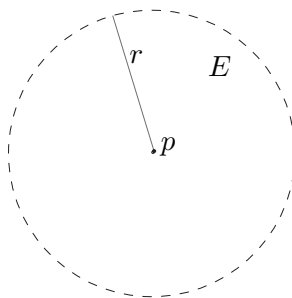
Figure 19.2: Rudin 2.18 Definition neighborhood of point p

Theorem 7 2.19 *Every neighborhood is an open set.*

PROOF First reviewing some definitions:

- By 2.18 Definition (a) a neighborhood about the point p is the set of all points within some radius r of p

- A set E is open if all of the points of E satisfy the definition of interior point

- A point p is an interior point of the set E if there is a neighborhood N of point p that is entirely contained in E: $N \subset E$.

Now continue with the proof.
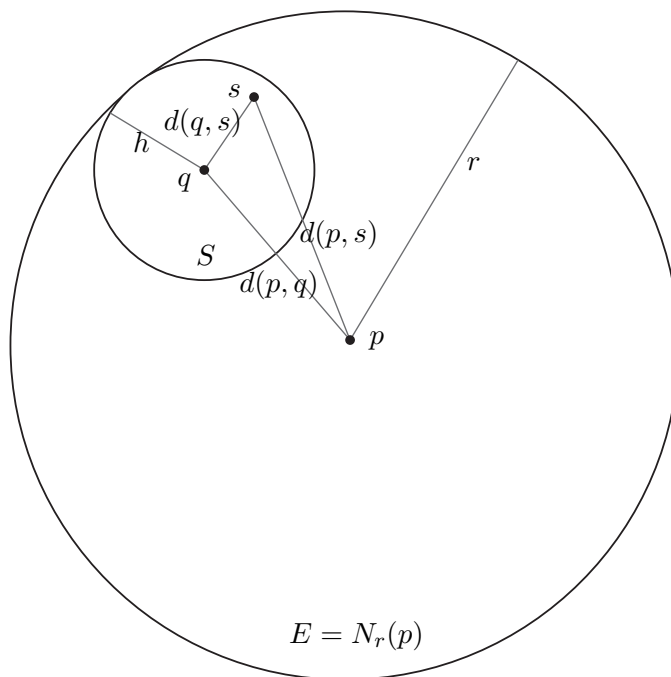
Figure 19.3: Rudin 2.19 theorem

1. Start with a neighborhood $E = N_r(p)$ around the point p.

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2. Pick a point q which is inside this neighborhood E.

3. Then define a radius h around the point q, where the radius h exists such that the following is true

$$d(p, q) = r - h$$

4. Then, we have by the triangle inequality that there is a point s such that

$$d(p, s) \leq d(p, q) + d(q, s)$$

5. But if we consider the points $s \in S$ such that $d(q, s) < h$, we can write

$$d(p, s) \leq d(p, q) + d(q, s) < d(p, q) + h$$

6. But earlier we defined $d(p, q) = r - h$ so we can substitute this in to get

$$d(p, s) \leq d(p, q) + d(q, s) < r - h + h$$

finally giving

$$d(p, s) < r$$

and the point s is in E.

7. This proof shows that if we start with a set E, any point q inside E has a neighborhood around it which is entirely contained in E. This neighborhood we called S, and all of the points $s \in S$ are in E, so the neighborhood $S \subset E$, and the neighborhood E is an open set.

$\square$

Theorem 8 2.23   *A set E is open if and only if its complement is closed.*

PROOF Insert figure.

1. Suppose E^c is closed

2. Pick $x \in E$

3. By definition $x \notin E^c$

   • Also, x is not a limit point of E^c. This is because we supposed E^c was closed, which means E^c contains all its limit points. And since $x \notin E^c$ it is not a limit point of E^c.

4. Because x is not a limit point, that means we can put any size neighborhood $N_r(x)$ of radius r around the point x that will not intersect E^c anywhere. That means this neighborhood is entirely in E. By the definition of an interior point, because we found a neighborhood $N_r(x)$ around the point x that was entirely in E, the point x is an interior point.

Because this point x was picked arbitrarily and found to be an interior point of E, that means that every point of E is an interior point of E, and by definition, E is open.

1. Now suppose E is open

2. Let x be a limit point of E^c (without saying whether or not x is in E^c)

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3. By definition of limit point, every neighborhood $N_r(x)$ around x contains a point $y \in E^c$ with $y \neq x$

4. Therefore x is not an interior point of E

    - Because for x to be an interior point of E there has to exist some neighborhood around x that is entirely contained in E.
    - However, no matter how small we pick this neighborhood, it will always contain a $y \in E^c$ with $y \neq x$
    - This means that neighborhood is not a subset of E

5. By definition, for E to be open, every point of E must be an interior point of E

    - Since x is not an interior point of E $x \notin E$
    - Thus $x \in E^c$

Thus for an arbitrarily selected limit point in E^c, it is contained in E^c. This shows that E^c contains all of its limit points, so E^c is closed. $\qquad\square$

Theorem 9 2.24 *Page 34.*

(a) *For any collection $\{G_\alpha\}$ of open sets, $\cup_\alpha G_\alpha$ is open*

(b) *For any collection $\{F_\alpha\}$ of closed sets, $\cup_\alpha F_\alpha$ is closed*

(c) *For any finite collection $G_1, \ldots, G_n$ of open sets, $\cup_{i=1}^n G_i$ is open*

(d) *For any finite collection $F_1, \ldots, F_n$ of closed sets, $\cup_{i=1}^n F_i$ is closed*

PROOF proof $\qquad\square$

19.3.1 Convex Sets and Functions

Definition 15 Line segment *A line segment connecting points y and x which belong to the set $\Omega \subset \mathbb{R}^n$ is shown in the figure below. The point z can be described as $z = y + \lambda m$ where $\lambda \in [0,1]$. The set of all points z is a line segment L. That is $L = \{y + \lambda m \mid \lambda \in [0,1]\}$.*
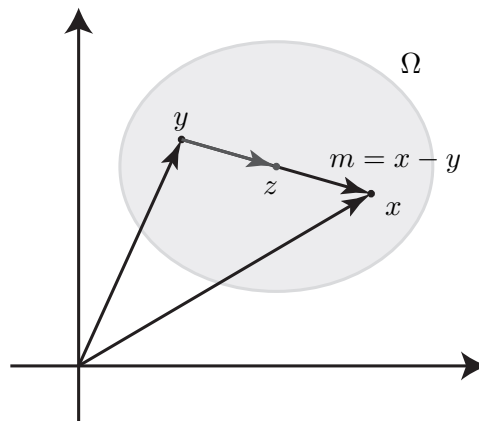
Figure 19.4: Line segment in subset $\Omega \subset \mathbb{R}^n$

Every point between y and x can be written $z = y + \lambda m \mid \lambda \in [0, 1]$ where $y + m = x \rightarrow m = x - y$.

$$z = y + \lambda m$$
$$z = y + \lambda(x - y)$$
$$z = y + \lambda x - \lambda y$$

$$\boxed{z = \lambda x + (1 - \lambda)y}$$

Definition 16 2.17 — Segment, interval, k-cell, ball, convex set *See Rudin page 31.*
    Ball *If $x \in \mathbb{R}^k$ and $r > 0$ the open (or closed) ball B with center at x and radius r is defined to be the set of all $y \in \mathbb{R}^k$ such that $|y - x| < r$ (or $|y - x| \leq r$).*
    Convex set *A set $\Omega \subset \mathbb{R}^n$ is convex if for two points x and y that belong to Ω, all points on the line segment L also belong to Ω. That is:*

$$\boxed{[\forall x, y \in \Omega \subset \mathbb{R}^n] \rightarrow [z = \lambda x + (1 - \lambda)y \in \Omega] \quad \forall\, 0 \leq \lambda \leq 1}$$
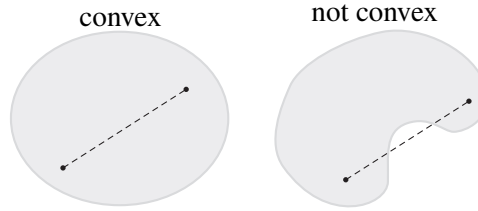
Figure 19.5: $\mathbb{R}^2$ examples

PROOF *(Balls are convex)* A ball with center at x and radius r is given by $|y - x| < r$ where the equality is possible for a closed ball. In order to prove that a ball is convex, pick two arbitrary points within the ball, and show that the line between these points is in the ball, by showing that it satisfies the definition for convex set. The points we will pick are a and b, and in order to be in the ball must satisfy

$$|a - x| < r$$
$$|b - x| < r$$

any point z between these two points must then satisfy

$$|z - x| < r$$

Since z is given by the line segment between a and b, this can be written using $z = \lambda a + (1 - \lambda)b$ as the following, where we want the inequality to hold, but have not yet proved that it does.

$$|\lambda a + (1 - \lambda)b - x| < r$$

Now we start manipulating this expression

$$|\lambda a - \lambda x + (1 - \lambda)b - x + \lambda x| < r$$

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$$|\lambda(a - x) + (1 - \lambda)b - (1 - \lambda)x| < r$$

$$|\lambda(a - x) + (1 - \lambda)(b - x)| < r$$

By the triangle inequality we have that

$$|\lambda(a - x) + (1 - \lambda)(b - x)| \leq |\lambda(a - x)| + |(1 - \lambda)(b - x)|$$

$$|\lambda(a - x) + (1 - \lambda)(b - x)| \leq \lambda|a - x| + (1 - \lambda)|b - x|$$

And because the points a and b were in our open ball to begin with $|a - x| < r$ and $|b - x| < r$ hold.

$$|\lambda(a - x) + (1 - \lambda)(b - x)| \leq \lambda|a - x| + (1 - \lambda)|b - x| < \lambda r + (1 - \lambda)r$$

$$|\lambda(a - x) + (1 - \lambda)(b - x)| < r$$

$$|\lambda a + (1 - \lambda)b - x| < r$$

which is the inequality we were trying to prove. □

19.3.2 Convex Function

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if the graph of the function lies below the line segment joining any two points of the graph. That is:

$$\boxed{f(z) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \forall 0 \leq \lambda \leq 1}$$
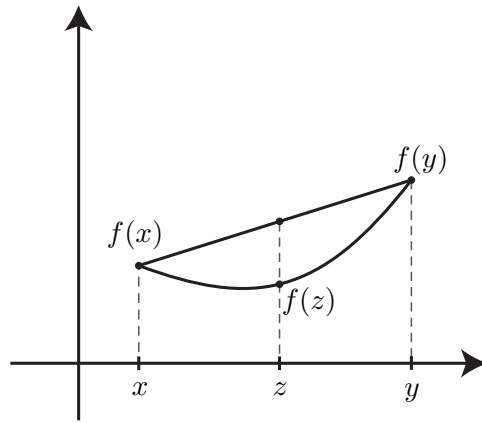
Figure 19.6: Convex function

Equivalently, a function is convex if its epigraph (the set of points on or above the graph of the function) is a convex set.

PROOF *(Every norm is a convex function)* For a function $f(z)$ to be a convex function, it must satisfy the following:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \forall 0 \leq \lambda \leq 1$$

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For $f(z) = \|z\|$ with $z = \lambda x + (1-\lambda)y$ we have $f(z) = \|\lambda x + (1-\lambda)y\|$.

$$\|\lambda x + (1-\lambda)y\| \leq \|\lambda x\| + \|(1-\lambda)y\|$$

by the triangle inequality.

$$\|\lambda x\| + \|(1-\lambda)y\| = \lambda\|x\| + (1-\lambda)\|y\|$$

by homogeneity, giving

$$\|\lambda x + (1-\lambda)y\| \leq \lambda\|x\| + (1-\lambda)\|y\|$$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

which is the condition we needed to satisfy for the function $f(z) = \|z\|$ to be convex. $\qquad\square$

PROOF *(Norm squared is a convex function)* For a function $f(z)$ to be a convex function, it must satisfy the following:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall 0 \leq \lambda \leq 1$$

For $f(z) = \|z\|^2$ with $z = \lambda x + (1-\lambda)y$ the following must be satisfied.

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2$$

Rearranging this expression

$$\|\lambda x + (1-\lambda)y\|^2 - \lambda\|x\|^2 - (1-\lambda)\|y\|^2 = ?$$

writing out the norms using the definition for Euclidean norm from Rudin page 16 that $\|a\| = (a\bullet a)^{1/2}$ and then $\|a+b\| = [(a+b)\bullet(a+b)]^{1/2}$. So $\|a+b\|^2 = (a+b)\bullet(a+b)$

$$\|\lambda x + (1-\lambda)y\|^2 - \lambda\|x\|^2 - (1-\lambda)\|y\|^2 = [\lambda x + (1-\lambda)y]\bullet[\lambda x + (1-\lambda)y] - \lambda\|x\|^2 - (1-\lambda)\|y\|^2$$

$$= \lambda^2 x\bullet x + 2\lambda(1-\lambda)x\bullet y + (1-\lambda)^2 y\bullet y - \lambda\|x\|^2 - (1-\lambda)\|y\|^2$$

$$= \lambda^2\|x\|^2 + 2\lambda(1-\lambda)x\bullet y + (1-\lambda)^2\|y\|^2 - \lambda\|x\|^2 - (1-\lambda)\|y\|^2$$

$$= \lambda(\lambda-1)\|x\|^2 + 2\lambda(1-\lambda)x\bullet y + (1-\lambda)[(1-\lambda)-1]\|y\|^2$$

$$= -\lambda(1-\lambda)\|x\|^2 + 2\lambda(1-\lambda)x\bullet y - \lambda(1-\lambda)\|y\|^2$$

$$= -\lambda(1-\lambda)\left(\|x\|^2 - 2x\bullet y + \|y\|^2\right)$$

Using the inner product definition in Rudin $\|a-b\|^2 = (a-b)\bullet(a-b) = a\bullet a - 2a\bullet b + b\bullet b = \|a\|^2 - 2a\bullet b + \|b\|^2$ giving

$$\|\lambda x + (1-\lambda)y\|^2 - \lambda\|x\|^2 - (1-\lambda)\|y\|^2 = -\lambda(1-\lambda)\|x-y\|^2$$

The quantity $\lambda(1-\lambda)$ is non-negative for $0 \leq \lambda \leq 1$, and $\|x-y\|^2$ is also always non-negative, so

$$\|\lambda x + (1-\lambda)y\|^2 - \lambda\|x\|^2 - (1-\lambda)\|y\|^2 = -\lambda(1-\lambda)\|x-y\|^2 \leq 0$$

$$\|\lambda x + (1-\lambda)y\|^2 - \lambda\|x\|^2 - (1-\lambda)\|y\|^2 \leq 0$$

which is rearranged to yield what we were originally trying to prove

$$\boxed{\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2}$$

$\qquad\square$

19.4   Chapter 3 — Numerical Sequences and Series

Definition 17 3.12 — Complete   *A metric space in which every Cauchy sequence converges.*

> **Example 15 Complete metric spaces** All Euclidean metric spaces and all compact metric spaces are complete.

19.5   Chapter 4 — Continuity

The following definition of continuity of a function is known as a $\delta - \varepsilon$ definition. Other definitions can be given in terms of limits of functions or limits of sequences. This definition is like that found in Slotine pg 123.

19.5.1   Continuous Function

Roughly speaking, a continuous function is one in which "small" changes in the input to the function result in "small" changes in the output of the function. In other words, when the input to the function is changed an infinitesimal amount, there should be no "jumps" in the output of the function. The definition is given below.

Definition 18 Continuous function   *(Rudin Page 85) A function $f(x) : E \subset X \to Y$ is continuous at the point p if:*

$$\forall \varepsilon > 0, \ \exists \delta(\varepsilon, p) > 0 \ such \ that \ \forall x, \ d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$$

What this says is that for all p in the domain of the function, and any value of ε that we would like to pick, for the function to be continuous there must exists a δ such that for any x value we pick that is within δ of p, this implies that the function value $f(x)$ is within ε of $f(p)$. The following figure shows a continuous function.
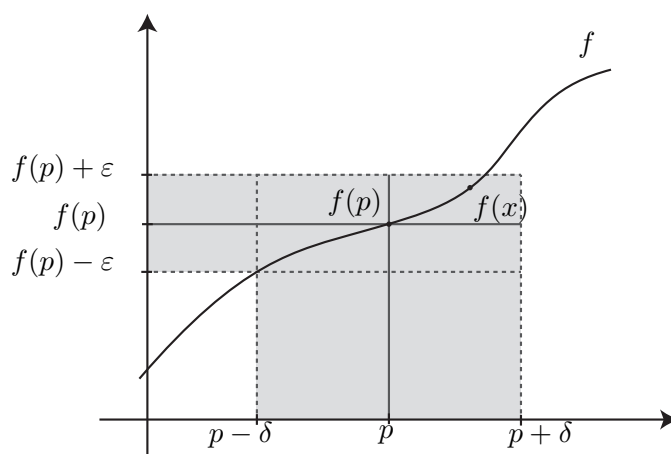
Figure 19.7: Continuous function

When the function $f(x) : \mathbb{R} \to \mathbb{R}$ is continuous at the point p if:

$$\forall \varepsilon > 0, \ \exists \delta(\varepsilon, p) > 0 \text{ such that } \forall x, \ |x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$$

We pick an arbitrary point p in the domain of the function. We then sweep across all possible values of ε. Regardless of how we pick ε, there is always a δ such that if x is within this δ, that $f(x)$ will be inside ε. No matter how p, ε, and x are picked, we can always find a δ that satisfies the definition. This may be more clear with an example of a discontinuous function.
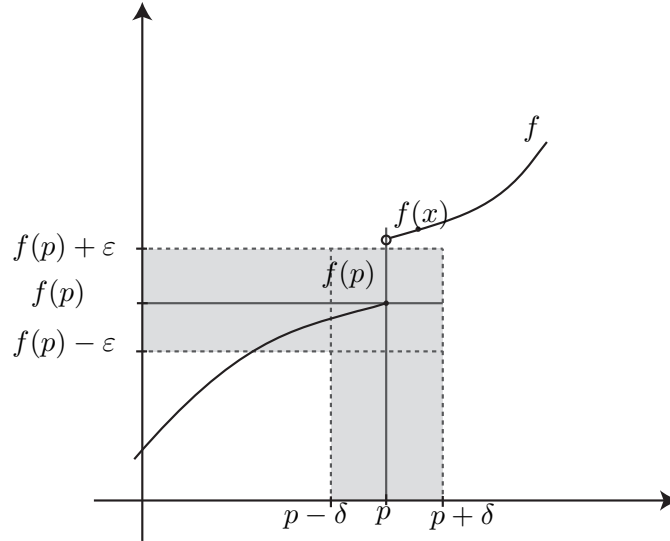
Figure 19.8: Discontinuous function

In this example, a value of p is selected. For all values of ε we want to pick (such as the one shown here) and all x (such as the one shown here), we must be able to find a δ such that $f(x)$ is within ε on $f(c)$. From this figure, regardless of how big or small of a δ that is picked, $f(x)$ will never be inside the ε region shown. Therefore this function is not continuous.

19.6 Uniform Continuity

For a continuous function, we said that once we selected an ε, we simply needed to be able to find a δ such that if x was inside the δ region, that $f(x)$ would be inside the ε region. Nothing was ever stated how large or small the value of δ could or needed to be, only that one existed.

For a function to be uniformly continuous, the value of δ does not depend on p. In particular, δ does not shrink as $p \to \infty$.

Definition 19 Uniformly Continuous function   *A function $f(x) : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if:*

$$\forall \varepsilon > 0, \ \exists \delta(\varepsilon) > 0 \ \forall p, \ \forall x : \ |x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$$

Example 16 The following example shows a continuous function which is not uniformly continuous. Take the function $f(x) : \mathbb{R} \to \mathbb{R}$ below:

$$f(x) = x^2$$

Example 17 The following example shows a continuous function which is not uniformly continuous. Take the function $f(x) : \mathbb{R} \to \mathbb{R}$ below:

$$f(x) = e^x$$

Chapter 20

Preliminaries

20.1 Introduction

20.1.1 Complex Numbers

Any complex number can be represented in exponential form, recalling Euler's formula $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. Considering this representation of a complex number $re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$ we can see that in the complex plane r represents the magnitude of the complex number, and θ is the angle. Additionally, the angle of a product of two complex numbers is the sum of the angles of each complex number. That is

$$\angle c_1 c_2 = \angle c_1 \angle c_2$$

20.1.2 Inverting a 2×2 Matrix

For the following 2×2 matrix A

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the inverse A^{-1} is:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

20.2 Norms

20.2.1 Introduction

In this note several types of norms will be explained. Given two real numbers, the notion of the "size" of these numbers is apparent. However, given quantity such as a vector or a matrix, we may by interested in how "big" they are when compared to another vector or matrix, respectively. A norm is a strictly positive measure of the "magnitude" of such a quantity. In particular, a norm is an operation on a vector that returns a non-negative quantity.

In the following section, the capital letter A will be reserved to denote matrices, while the lower case letter x will be used for vectors, and time dependency will be added to imply time-varying vector-values signals. $H(t)$, $g(t)$ and $G(s)$ for systems described by transfer functions? Vectors are assumed to be columns. The four types of norms described in this section are:

1. Vector

2. Matrix

3. Signal

4. System

Properties of Norms

Norms satisfy three basic properties below, where $x, y, z \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$.

1. Norms are always non-negative: $\|x\| \geq 0$, and the norm of the zero element is zero: $\|x\| = 0$ if and only if $x = 0$
2. Norms are scalable: $\|\alpha x\| = |\alpha| \|x\|$
3. Norms satisfy the triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$

There are some more properties of norms which can be constructed from these four.

20.2.2   Vector Norms

There are several common norms which are used to determine the "length" of a given vector, described below. Often vector norms are denoted by the lowercase letter l_p.

p-Norm   Denoted $\|x\|_p$, and defined as follows.

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

2-Norm   The 2-norm, or Euclidean-norm is denoted $\|x\|_2 = \|x\|$. This is basically an extension of Euclidean length to n-dimensional spaces. Taken by squaring each entry of the vector, adding up all the squares, and then taking the square root.

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

$$\|x\|_2 = \sqrt{x^{\mathrm{H}} x}$$

1-Norm   The 1-norm, or Taxicab norm is denoted $\|x\|_1$. This norm corresponds to distance in a similar way as the Euclidean norm, but is given by the length of the vector that a taxicab driver would have to drive along a rectangular street grid. It is taken by adding up the absolute value of each entry of the vector and adding them all up.

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

∞-Norm   Denoted $\|x\|_\infty$. The infinity norm of a vector x is the largest magnitude of an entry of the vector.

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

Examples

<div style="border:1px solid">

Example 18 Vector norms Let $v = \begin{bmatrix} 3 & 0 & -4 \end{bmatrix}^\top$. The following norms are

$$\|v\|_2 = \sqrt{v^\top v} = \sqrt{9 + 0 + 16} = 5$$
$$\|v\|_1 = 3 + 4 = 7$$
$$\|v\|_\infty = 4$$

</div>

20.2.3 Matrix Norms

Matrix norms are an extension of vector norms. After having been introduced to vector norms, it is not immediately intuitive how to extend this concept to a meaningful measure of the "size" of a given matrix. Two matrix norms described below are induced norms and entry wise norms.

Induced Norms

Induced norms are norms on a matrix A which require multiplication of the A by a vector x, and then evaluate the norm of the resulting vector, normalized by $\|x\|_p$.

Induced p-Norm   The general definition for an induced norm is given below.

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max \left\{ \|Ax\|_p \ : \ x \in K^n \text{ with } \|x\|_p = 1 \right\}$$

where $p \in \mathbb{N}^+$. Depending on what particular value of p is used for evaluating this norm, there are different interpretations, some of the common ones which are described below.

Induced 1-Norm   For the induced taxicab, or 1-norm, this corresponds to taking the maximum absolute column sum of the matrix.

Induced ∞-Norm   For the induced infinity norm, it is calculated by taking the maximum absolute row sum of the matrix.

Induced 2-Norm   The induced Euclidean, or 2-norm for a square matrix A is given by the largest singular value of A. The singular values of A are given by taking the square root of the eigenvalues of $A^H A$. That is:

$$\|A\|_1 = \sqrt{\lambda_{\max}(A^H A)} = \sigma_{\max}(A)$$

Entrywise Norms

Entrywise norms are matrix norms which are taken directly on a matrix A itself, without first requiring multiplication by a vector x.

Frobenius Norm The Frobenius norm is calculated by taking the absolute value of each entry of the matrix A, squaring it, adding them all together, and then taking the square root.

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|A_{i,j}|^2} = \sqrt{\text{tr}(A^H A)} = \sqrt{\text{tr}(A A^H)}$$

Examples

> **Example 19 Matrix norms** Let $A = \begin{bmatrix} 3 & 0 \\ 2 & -4 \end{bmatrix}^\top$. The following norms are
>
> $$\|A\|_2 = 4.76$$
> $$\|A\|_1 = 5$$
> $$\|A\|_\infty = 6$$
> $$\|A\|_F = 5.39$$

20.2.4 Signal Norms

The next quantity which we apply the concept of a norm to is that of a signal. Given a signal (either vector-valued or scalar), we wish to determine some measure of its magnitude. Perhaps the signal is very large at one instance of time, and small everywhere else, or maybe it is moderately large for all time. Whatever the case may be, we wish to have some way to quantify these varying degrees of the "largeness" of a signal.

Often signal norms are denoted using the capital letter L_p. The notation L in L_p refers to the fact that the integrand in (2.7) should be Lebesgue-integrable for the integral to exist. This is a generalization of the standard (Riemann) integral to a more general class of functions. See DDV Chapter 15. Signals norms are generalizations of vector norms. The signal norms we consider in this subsection are the following:

- 1-norm: action

- 2-norm squared: signal energy

- ∞-norm: peak signal magnitude

- signal power (average energy)

For the vector valued signal $x(t)$ the following norms are given.

L_p Norm

$$\|x(t)\|_{L_p} = \left(\int_{-\infty}^{\infty} \|x(t)\|^p dt \right)^{\frac{1}{p}}$$

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L_1 Norm

$$\|x(t)\|_{L_1} = \int_{-\infty}^{\infty} \|x(t)\| dt$$

L_2 Norm

$$\|x(t)\|_{L_2} = \sqrt{\int_{-\infty}^{\infty} \|x(t)\|^2 dt}$$

L_∞ Norm

$$\|x(t)\|_{L_\infty} = \sup_t \|x(t)\|$$

Examples

Example 20 Signal norms Let $x(t) =$. The following norms are

$$\|x(t)\|_{L_2} =$$
$$\|x(t)\|_{L_1} =$$
$$\|x(t)\|_{L_\infty} =$$

Existence of Signals in Normed Spaces?

Explain here what notation $e(t) \in \mathcal{L}^\infty$ and so on.

20.2.5 System Norms

In addition to the H_2 norm, which we have seen gives a characterization of the average gain of a system, a perhaps more fundamental norm for systems is the H_∞ norm, which provides a measure of a worst-case system gain. The H_∞ norm is simply a measure of the largest factor by which any sinusoid is magnified by the system.

L_1 Induced Norm

L_2 Induced Norm

$$\|G(s)\|_2 = \sqrt{\int_0^{\infty} \text{tr}[g(t)^{\text{T}} g(t)] dt}$$

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}[G(j\omega)^{\text{H}} G(j\omega)] d\omega}$$

$$\|G(s)\|_2 = \sup_\omega (\sigma_{\max} G(j\omega))$$

$$\|G(s)\|_2 = \sqrt{\text{tr}(B^{\text{T}} Q B)} \quad \text{where} \quad QA + A^{\text{T}} Q + C^{\text{T}} C = 0$$

$$\|G(s)\|_2 = \sqrt{\text{tr}(C P C^{\text{T}})} \quad \text{where} \quad AP + PA^{\text{T}} + BB^{\text{T}} = 0$$

20.3 Canonical State-Space Forms

There are four standard, or canonical state space forms into which a transfer function can be written. Two of these forms, namely the observable canonical form and the controllable canonical form, are often called companion forms. The other two canonical forms are modal and Jordan canonical forms. Each of these four different state space realizations has its own uses and benefits, which will be outlined below. The form of controllable and observable canonical forms will be shown below, where the former will contain a row whose coefficients are that of the transfer function denominator polynomial, and the latter a column. If the row of coefficients is on the bottom of the system A matrix, this is sometimes referred to as controllability, and on the top referred to as controller. Likewise the column of coefficients on the left is called observer, and on the right observability. See Kailath (1980).

Often the easiest way to demonstrate the process of writing a transfer function in state space form is by example. The following examples will illustrate the conversion process for small systems, but this procedure will extend to systems with arbitrarily large numerators and denominators. For these first examples, assume every transfer function is strictly proper, i.e. the order of the numerator is less than the order of the denominator. See the following section for proper transfer functions.

Controller Canonical Form

Given the following transfer function, convert to controller canonical form. Converting this transfer function to a state space representation in controller canonical form will result in a system that is guaranteed to be controllable.

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

A transfer function is an expression of an output/input. If y is used to indicate an output, u an input, and x a state, the above transfer function can be written:

$$G(s) = \frac{y}{u} = \frac{y}{x}\frac{x}{u}$$

Using this representation, the transfer function can be split into two products:

$$\frac{y}{x} = b_2 s^2 + b_1 s + b_0$$

and:

$$\frac{x}{u} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Rearranging each of these equations and taking the inverse Laplace transform:

$$y = b_2 \ddot{x} + b_1 \dot{x} + b_0 x$$

and:

$$\dddot{x} + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = u$$

The order of the system, i.e. the number of state variables needed is equal to the order of the transfer function denominator. For this example, 3 state variables are needed. Define x_1 to be the first state variable, and build

each subsequent state variable as a derivative of the first one.

$$x_1 = x$$
$$x_2 = \dot{x}_1 = \dot{x}$$
$$x_3 = \dot{x}_2 = \ddot{x}$$

Plugging in these state variable definitions:

$$y = b_2 x_3 + b_1 x_2 + b_0 x_1$$

and:

$$\dot{x}_3 + a_2 x_3 + a_1 x_2 + a_0 x_1 = u$$

Rearranging the above:

$$\dot{x}_3 = -a_2 x_3 - a_1 x_2 - a_0 x_1 + u$$

Rearranging the state variable definitions and assembling the above into matrix form gives:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -a_0 x_1 - a_1 x_2 - a_2 x_3 + u$$

and:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Observer Canonical Form

$$G(s) = \frac{y}{u} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

$$y(s^3 + a_2 s^2 + a_1 s + a_0) = u(b_2 s^2 + b_1 s + b_0)$$

$$\dddot{y} + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_2 \ddot{u} + b_1 \dot{u} + b_0 u$$

$$\dddot{y} = b_2 \ddot{u} - a_2 \ddot{y} + b_1 \dot{u} - a_1 \dot{y} + b_0 u - a_0 y$$

$$\dot{y} = b_2 u - a_2 y + \int (b_1 u - a_1 y) + \iint (b_0 u - a_0 y)$$

$$y = x_1$$

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$$\dot{x}_1 = b_2 u - a_2 x_1 + \int (b_1 u - a_1 x_1) + \iint (b_0 u - a_0 x_1)$$

$$x_2 = \int (b_1 u - a_1 x_1) + \iint (b_0 u - a_0 x_1)$$

$$\dot{x}_2 = (b_1 u - a_1 x_1) + \int (b_0 u - a_0 x_1)$$

$$x_3 = \int b_0 u - a_0 x_1$$

$$\dot{x}_3 = b_0 u - a_0 x_1$$

$$\dot{x}_1 = -a_2 x_1 + x_2 + b_2 u$$
$$\dot{x}_2 = -a_1 x_1 + x_3 + b_1 u$$
$$\dot{x}_3 = -a_0 x_1 + b_0 u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Modal Canonical Form

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{(s + a_1)(s + a_2)(s + a_3)}$$

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{(s + a_1)(s + a_2)(s + a_3)} = \frac{r_1}{(s + a_1)} + \frac{r_2}{(s + a_2)} + \frac{r_3}{(s + a_3)}$$

$$b_2 s^2 + b_1 s + b_0 = (s + a_2)(s + a_3) r_1 + (s + a_1)(s + a_3) r_2 + (s + a_1)(s + a_2) r_3$$

$$b_2 s^2 + b_1 s + b_0 = [s^2 + (a_2 + a_3) + a_2 a_3] r_1 + [s^2 + (a_1 + a_3) + a_2 a_3] r_2 + [s^2 + (a_2 + a_3) + a_1 a_2] r_3$$

Equating polynomial coefficients:

$$b_2 = r_1 + r_2 + r_3$$
$$b_1 = (a_2 + a_3) r_1 + (a_1 + a_3) r_2 + (a_1 + a_2) r_3$$
$$b_0 = a_2 a_3 r_1 + a_1 a_3 r_2 + a_1 a_2 r_3$$

For the arbitrarily large transfer function, this will give n equations in n unknowns, where n is the order of the denominator polynomial, allowing the partial fraction expansion to be completed.

$$G(s) = \frac{y}{u} = \frac{r_1}{(s + a_1)} + \frac{r_2}{(s + a_2)} + \frac{r_3}{(s + a_3)}$$

$$\frac{y}{u} = \frac{y}{x}\frac{x}{u} = \frac{r_1}{(s + a_1)} + \frac{r_2}{(s + a_2)} + \frac{r_3}{(s + a_3)}$$

$$\frac{x}{u} = \frac{x_1}{u} + \frac{x_2}{u} + \frac{x_3}{u} = \frac{r_1}{(s + a_1)} + \frac{r_2}{(s + a_2)} + \frac{r_3}{(s + a_3)}$$

$$\frac{x_1}{u} = \frac{r_1}{(s + a_1)}$$

$$\frac{x_2}{u} = \frac{r_2}{(s + a_2)}$$

$$\frac{x_3}{u} = \frac{r_3}{(s + a_3)}$$

$$\dot{x}_1 = -a_1 x_1 + r_1 u$$

$$\dot{x}_2 = -a_2 x_2 + r_2 u$$

$$\dot{x}_3 = -a_3 x_3 + r_3 u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Jordan Canonical Form

Lec. 7—6

20.3.1 Proper Transfer Functions

20.4 Singular Values

A measure of the "smallness" of the matrix is needed: the singular values. The singular value decomposition of a matrix M is the following factorization, where $(\cdot)^*$ denotes the Hermitian, or conjugate transpose.

$$M = U\Sigma V^*$$

where U and V^* are real or complex unitary matrices, i.e. $U^*U = UU^* = I$, and Σ is diagonal matrix with nonnegative real numbers on the diagonal. The diagonal entries of Σ are the singular values of M. Taking the Hermitian transpose of M gives $M^* = (U\Sigma V^*)^* = V\Sigma^*U^*$. Both pre- and post- multiplying M with its Hermitian transpose:

$$MM^* = U\Sigma V^*V\Sigma^*U^* = U\Sigma\Sigma^*U^*$$
$$M^*M = V\Sigma^*U^*U\Sigma V^* = V\Sigma^*\Sigma V^*$$

Since Σ is diagonal, this can be rearranged into the following eigenvalue problems, allowing the singular values to be found as the non-zero eigenvalues of M^*M or MM^*.

$$(MM^*)U = U(\Sigma\Sigma^*)$$
$$(M^*M)V = V(\Sigma^*\Sigma)$$

When the matrix M is a transfer function matrix $H(j\omega)$, the magnitude at a any frequency ω will depend on the direction of the input. Different singular values will be excited depending on the input. However, for any input the magnitude of $H(j\omega)$ is bounded above by its maximum singular value $\overline{\sigma}(H(j\omega))$ and below by its minimum singular value $\underline{\sigma}(H(j\omega))$. For performance and stability robustness $\underline{\sigma}(H(j\omega))$ should be large at low frequencies and $\overline{\sigma}(H(j\omega))$ should be small at high frequencies.

To generate the singular value plot for the linear system $\dot{x} = Ax + Bu$ with output given by $y = Cx$, the transfer function matrix H must first be calculated:

$$H(s) = C(sI - A)^{-1}B$$

When the linear system $\dot{x} = Ax + Bu$ is augmented with integral error states, and full state feedback control law $u = -Kx$ is used, the linear system becomes $\dot{x} = (A - B_1K)x + B_2x_{i,r}$, $y = Ix$, with transfer function matrix

$$H(s) = I(sI - (A - B_1K))^{-1}B_2$$

The singular values for each transfer function matrix H above can then be plotted versus frequency.

20.5 Positive Definiteness

The matrix $BB^{\mathrm{T}} = (BB^{\mathrm{T}})^{\mathrm{T}}$ is symmetric for all B. In addition, $BB^{\mathrm{T}} > 0$ is positive definite. Definition of positive definiteness: $x^{\mathrm{T}}BB^{\mathrm{T}}x > 0$. $x \in \mathbb{R}^{n \times 1}$ and $B \in \mathbb{R}^{n \times m}$. Define $C = B^{\mathrm{T}}x$, where $C \in \mathbb{R}^{m \times 1}$ giving $C^{\mathrm{T}}C = \|C\|_2^2 > 0 \,\forall C \neq 0$

20.6 Sensitivity Analysis

20.6.1 Introduction

The system $\dot{x} = f(x, u)$ describing the dynamics of the GHV was previously linearized and written

$$\dot{x} = Ax + Bu$$

where the state vector x for the model is in body axes as shown in previous work. It is desired to transform this linear system to use a new state which describes the GHV dynamics in stability axes. The subscript $(\cdot)_s$ indicates stability axes, and the state vector is shown below.

$$x_s = [\ V_T \quad \alpha \quad \beta \quad p \quad q \quad r \quad \phi \quad \theta \quad \psi \quad \lambda \quad \tau \quad \mathscr{R}\ ]^T$$

After completing the transformation from body to stability axes, it was verified that the system poles had not changed between A and A_s. Next, a linear transformation was needed in order to rearrange the states in x_s so that the longitudinal and lateral/directional states would be grouped together, and any states which did not influence the flight dynamics could be truncated. This transformation was accomplished using the transformation matrix $\mathbb{O}$. The entries of $\mathbb{O}$ are zero everywhere, with the exception of ones placed to rearrange the states as described above.

$$x_s' = [\ V_T \quad \alpha \quad q \quad \theta \quad \mathscr{R}\ |\ \beta \quad p \quad r \quad \phi\ |\ \psi \quad \lambda \quad \tau\ ]^T$$

Defining the transformation

$$x_s' = \mathbb{O}x_s$$

$$x_s = \mathbb{O}^{-1}x_s'$$

$$\dot{x}_s = \mathbb{O}^{-1}\dot{x}_s'$$

$$\dot{x}_s = A_s x_s + B_s u$$

$$\mathbb{O}^{-1}\dot{x}_s' = A_s \mathbb{O}^{-1}x_s' + B_s u$$

$$\dot{x}_s' = \mathbb{O}A\mathbb{O}^{-1}x_s' + \mathbb{O}B_s u$$

$$A_s' = \mathbb{O}A_s\mathbb{O}^{-1}$$

$$\dot{x}_s' = A_s' x_s' + B_s' u$$

20.6.2 Examining Modal Decomposition

After rearranging the original state-space system with state vector x into system with state vector x_s', the system was examined to check the validity of decoupling the lateral and longitudinal dynamics, and the truncation of the navigation states. Looking at the following transformed state-space system shown above, and considering only the initial condition response, the following autonomous system results

$$\dot{x}_s' = A_s' x_s'$$

The following transformation is introduced

$$x_s' = \mathbb{V}q$$

where $\mathbb{V}$ is the matrix of eigenvectors $\mathbb{V} \triangleq [\ \mathbf{v}_1\ |\ \cdots\ |\ \mathbf{v}_n\ ]$ giving

$$\dot{x}_s' = \mathbb{V}\dot{q}$$

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$$\dot{q} = \mathbb{V}^{-1} A_s' \mathbb{V} q$$

The matrix $\mathbb{V} A_s' \mathbb{V}^{-1}$ must be examined.

$$A_s' \mathbb{V} = A_s' [\ \mathbf{v}_1 \ \vdots \ \cdots \ \vdots \ \mathbf{v}_n \ ] = [\ A_s' \mathbf{v}_1 \ \vdots \ \cdots \ \vdots \ A_s' \mathbf{v}_n \ ] = [\ \mathbf{v}_1 \lambda_1 \ \vdots \ \cdots \ \vdots \ \mathbf{v}_n \lambda_n \ ] = \mathbb{V} \Lambda$$

giving

$$\dot{q} = \Lambda q$$

where Λ is the diagonal matrix of eigenvalues. The solution is given by

$$q(t) = e^{\Lambda t} q(0)$$

Selecting an initial condition as a scalar multiple of an eigenvector $\mathbf{v}_i$, i.e. $x(0) = \alpha_i \mathbf{v}_i$ then $q(0) = \mathbb{V}^{-1} x(0) = \alpha_i \mathbb{V}^{-1} \mathbf{v}_i$. But since $\mathbb{V}^{-1} \mathbf{v}_i = \mathbb{I}$ where $\mathbb{I}$ is the identity matrix, $\mathbb{V}^{-1} \mathbf{v}_i$ is just the ith column of $\mathbb{I}$. In other words, the initial condition $q(0)$ will have zeros everywhere, and α in the ith row.

$$q(t) = \alpha e^{\lambda_i t}$$

This shows that only the mode corresponding to λ_i will be present in the response from an initial condition along the ith eigenvector. The response in terms of x_s' corresponding to λ_i is then given by

$$x_s'(t) = \alpha_i e^{\Lambda_i t} \mathbf{v}_i$$

Based on this unforced modal response, if any entries in $\mathbf{v}_i$ are "small" relative to the others, the corresponding states are thus not influential in determining the initial condition response.

20.6.3 The Sensitivity Matrix

Chapter 21

State Feedback Control

21.1 Introduction
21.2 The Regulation Problem

The plants for which we will seek to design controllers will be represented in the following form.

$$\dot{x}_p = A_p x_p + B_p u$$

The LQR controller will be explained in this section. The LQR control is a full state feedback, static gain controller. That is, the control input is calculated using the entire plant state x_p, and multiplied by a constant gain matrix K_p to get the control input u. The control law for the LQR controller is

$$u = K_p^\top x_p$$

The method of obtaining the gain matrix will not be discussed here. There is a lot of information available about the process of obtaining K_p, but for the purposes of this section, what needs to be known is only that given a controllable plant, we can use MATLAB to generate K_p that will stabilize a given plant, and place the closed-loop poles in a "good" way. This command is: `Kp = -lqr(Ap,Cp,Qlqr,Rlqr)'`

The LQR controller as described above is of little use for the control applications we are interested in, as it eliminates the ability to provide external reference commands, and serves only to regulate the system to the origin.

21.3 The Tracking Problem

This control architecture is known as the LQR-servomechanism, LQR-PI control, or LQR with integral action. This structure uses LQR control where the plant is augmented with an integrator to ensure tracking of a reference command. This is done by adding states to the state-space representation, and adding an additional input matrix through which the reference command enters. Given the following open-loop plant

$$\begin{aligned}
\dot{x}_p &= A_p x_p + B_p u \\
z &= C_{pz} x_p + D_{pz} u
\end{aligned} \tag{21.1}$$

where $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times m}$, and $C_p \in \mathbb{R}^{\ell \times n_p}$ and $z \in \mathbb{R}^{n_e}$ is the regulated output, where the number of regulated outputs is not to exceed the number of inputs, that is $n_e \leq m$. Given that the state is available for measurement, the control goal is to design a control input u so that the closed-loop system has bounded

solutions and z tends to the reference command z_{cmd} asymptotically. In order to ensure command tracking, we introduce integral action, and for this purpose an additional state x_e is defined as

$$\dot{x}_e = z_{cmd} - z$$

and the plant in (21.1) is augmented to lead to the following extended open-loop dynamics

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_e \end{bmatrix} + \begin{bmatrix} B_p \\ -D_{pz} \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} z_{cmd} \tag{21.2}$$

The system in (21.2) can be written more compactly as

$$\dot{x} = Ax + Bu + B_{cmd} z_{cmd} \tag{21.3}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $B_{cmd} \in \mathbb{R}^{n \times n_e}$, and $C \in \mathbb{R}^{p \times n}$ are the known matrices given by

$$A = \begin{bmatrix} A_p & 0_{n_p \times n_e} \\ -C_{pz} & 0_{n_e \times n_e} \end{bmatrix} \quad B = \begin{bmatrix} B_p \\ -D_{pz} \end{bmatrix} \quad B_{cmd} = \begin{bmatrix} 0_{n_p \times m} \\ I_{n_e \times n_e} \end{bmatrix}$$

Thus, when the augmented system in Equation (21.2) reaches steady state, $\dot{x}_e$ will be zero, and $z = z_{cmd}$. The following control law will be used:

$$u = K_x^\top x \tag{21.4}$$

Substituting the control law (21.4) into (21.3)

$$\dot{x} = Ax + BK_x^\top x + B_{cmd} z_{cmd}$$
$$= \left(A + BK_x^\top \right) x + B_{cmd} z_{cmd}$$

The gain $K_x^\top$ is selected to ensure the closed loop matrix $A + BK_x^\top$ is a Hurwitz.

Example 21 Integral Augmented LQR Control Synthesis: Longitudinal Aircraft Dynamics Consider the following state-space model describing the longitudinal short-period dynamics of an aircraft. Given this system, the goal is to design a state feedback controller to track pitch rate commands.

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{V} & 1 \\ M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} \frac{Z_{\delta_e}}{V} \\ M_{\delta_e} \end{bmatrix} \delta_e$$

where α is the angle of attack, q the pitch rate, and δ_e the elevator deflection angle. The state vector is $x_p = \begin{bmatrix} \alpha & q \end{bmatrix}^\top$. With pitch rate as an output his system can be expressed more compactly as

$$\dot{x}_p = A_p x_p + B_p u$$
$$z = C_{pz} x_p$$

where

$$C_{pz} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

To enforce reference tracking we augment the system with an integrator as described above, and get the following extended open-loop dynamics

$$
\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{q}_e \end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{V} & 1 & 0 \\ M_\alpha & M_q & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ q_e \end{bmatrix} + \begin{bmatrix} \frac{Z_{\delta_e}}{V} \\ M_{\delta_e} \\ 0 \end{bmatrix} \delta_e + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q_{\mathrm{cmd}}
$$

As shown above, these extended open-loop dynamics can be written more compactly as

$$
\dot{x} = Ax + Bu + B_{\mathrm{cmd}} z_{\mathrm{cmd}}
$$

```matlab
1   %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
2   % makeLQRPI.m
3   % Flight Control Tools: Make LQR-PI Controller
4   % Written by: Daniel Wiese, Wednesday 15-October-2014
5   % Updated by: Daniel Wiese, Sunday 18-October-2015
6   %------------------------------------------------------------------------
7   %
8   % This script makes an LQR-PI controller given a system A and B matrices,
9   % and Q and R weights. Additionally the user must specify which (up to two)
10  % state variables should be augmented with an integral error state. Specify
11  % which state variable(s) are to be integrated using a scalar parameter
12  % corresponding to the ith state variable. If only one integrator is to be
13  % used, specify only the first 'e', and it doesn't matter what the second
14  % 'e' is. Pass the weighting matrices Q and R to the function as vectors,
15  % which will act as the diagonal entries of the weights.
16  %
17  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
18
19  function [Kx, P, A, Am, B, Bcmd, C, D, n] = makeLQRPI(Ap, Bp, Cp, Cpz, Dpz, ...
        Qlqr, Rlqr)
20
21  % Find number of states and inputs of \dot{x} = Ax + Bu
22  [np, m] = size(Bp);
23  [l, ¬] = size(Cp);
24  [ne, ¬] = size(Cpz);
25
26  % The total size of the augmented system will be original size plus error
27  n = np + ne;
28  p = l + ne;
29
30  if rank(ctrb(Ap,Bp)) ≠ np
31      error('The plant is not controllable!')
32  end
33
34  if rank([Ap, Bp; Cpz, Dpz]) ≠ n
35      error('Selection of regulated output destroys controllability!')
36  end
37
38  % Augment the system with the integral error state
```

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```
39   A = [Ap, zeros(np,ne); -Cpz, zeros(ne,ne)];
40   B = [Bp; -Dpz];
41   Bcmd = [zeros(np,ne); eye(ne,ne)];
42   C = [Cp, zeros(l,ne); zeros(ne,np), eye(ne,ne)];
43   D = zeros(p,m);
44
45   % Solve for the LKQ feedback gain
46   [Kx, P, ¬] = lqr(A, B, diag(Qlqr,0), diag(Rlqr,0));
47   Kx = -Kx';
48   Am = (A + B*Kx');
49
50   end
51
52   %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

21.3.1   State-Space Controller Representation

Now that the fundamental loop transfer functions are found, we now want to be able to actually apply this when using LQR-PI full state feedback controllers. Thus, it is important to show an example of how to represent an LQR-PI controller in state space form, and from there be able to calculate the loop transfer functions.
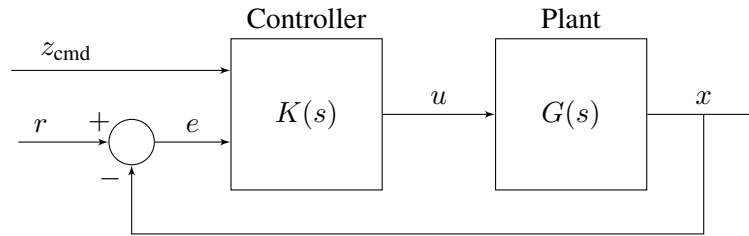
Figure 21.1: System block diagram

The LQR-PI controller is represented as the following, where the subscript $(\cdot)_c$ is used to denote "controller", and the only state in the controller is the integral error state x_e.

$$\dot{x}_e = A_c x_e + B_c \begin{bmatrix} z_{\text{cmd}} \\ -x_p \end{bmatrix}$$

$$u = C_c x_e + D_c \begin{bmatrix} z_{\text{cmd}} \\ -x_p \end{bmatrix}$$

The control law is a gain matrix multiplied by the state $x = \begin{bmatrix} x_p^\top & x_e^\top \end{bmatrix}^\top$ which is an augmentation of the plant state x_p with error state x_e.

$$\dot{x}_e = z_{\text{cmd}} - z$$

$$u = K_x^\top x$$

$$= \begin{bmatrix} K_p^\top & K_e^\top \end{bmatrix} \begin{bmatrix} x_p \\ x_e \end{bmatrix}$$

$$\dot{x}_e = z_{\text{cmd}} - C_{pz} x_p - D_{pz} u$$
$$u = K_e^\top x_e + K_p^\top x_p$$

$$\dot{x}_e = z_{\text{cmd}} - C_{pz} x_p - D_{pz}(K_e^\top x_e + K_p^\top x_p)$$
$$u = K_e^\top x_e + K_p^\top x_p$$

$$\dot{x}_e = -D_{pz} K_e^\top x_e + z_{\text{cmd}} - (C_{pz} + D_{pz} K_p^\top) x_p$$
$$u = K_e^\top x_e + K_p^\top x_p$$

$$\dot{x}_e = -D_{pz} K_e^\top x_e + \begin{bmatrix} 1 & C_{pz} + D_{pz} K_p^\top \end{bmatrix} \begin{bmatrix} z_{\text{cmd}} \\ -x_p \end{bmatrix}$$

$$u = K_e^\top x_e + \begin{bmatrix} 0 & -K_p^\top \end{bmatrix} \begin{bmatrix} z_{\text{cmd}} \\ -x_p \end{bmatrix}$$

Looking at this we can see that

LQR Controller State-Space Model

$$A_c = -D_{pz} K_e^\top$$
$$B_c = \begin{bmatrix} 1 & C_{pz} + D_{pz} K_p^\top \end{bmatrix}$$
$$C_c = K_e^\top$$
$$D_c = \begin{bmatrix} 0 & -K_p^\top \end{bmatrix}$$

(21.5)

Example 22 Integral Augmented LQR Controller State-Space Representation: Longitudinal Aircraft Dynamics The general form of the plant as shown above, but repeated here for the longitudinal subsystem is

$$\dot{x}_p = A_p x_p + B_p u$$
$$y_p = C_p x_p$$
$$z = C_{pz} x_p + D_{pz} u$$

For this longitudinal subsystem the plant states are angle of attack α, and pitch rate q, with the plant input being the elevator deflection angle δ_e. That is, $x_p = \begin{bmatrix} \alpha & q \end{bmatrix}^\top$.

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{V} & 1 \\ M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} \frac{Z_{\delta_e}}{V} \\ M_{\delta_e} \end{bmatrix} \delta_e$$

Plugging in some numbers

$$A_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C_{pz} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D_{pz} = \begin{bmatrix} 0 \end{bmatrix}$$

The general form of the controller is

$$\dot{x}_e = A_c x_e + B_c \begin{bmatrix} z_{\mathrm{cmd}} \\ -x_p \end{bmatrix}$$

$$u = C_c x_e + D_c \begin{bmatrix} z_{\mathrm{cmd}} \\ -x_p \end{bmatrix}$$

Inputs to controller are $z_{\mathrm{cmd}} = \alpha_{\mathrm{cmd}}$ and $e = -x_p = \begin{bmatrix} -\alpha & -q \end{bmatrix}^\top$. Combining these two inputs into one input $\begin{bmatrix} \alpha_{\mathrm{cmd}} & -\alpha & -q \end{bmatrix}^\top$ we get

$$\dot{x}_e = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{\mathrm{cmd}} \\ -\alpha \\ -q \end{bmatrix}$$

$$u = k_e x_e + \begin{bmatrix} 0 & -k_\alpha & -k_q \end{bmatrix} \begin{bmatrix} \alpha_{\mathrm{cmd}} \\ -\alpha \\ -q \end{bmatrix}$$

$$\boxed{A_c = 0 \quad B_c = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \quad C_c = \begin{bmatrix} k_e \end{bmatrix} \quad D_c = \begin{bmatrix} 0 & -k_\alpha & -k_q \end{bmatrix}}$$

This state space representation was then converted to a transfer matrix representation for frequency domain analysis. This transfer matrix was 1×3: controller input α_{cmd}, $-\alpha$, and $-q$ and the output was the elevator deflection angle δ_{e}.

21.3.2 Properties of Extended Open-Loop Dynamics
Controllability

Hautus controllability test Hespanha book page 113.

Theorem 10 Popov-Belevitch-Hautus Test for Controllability *The pair (A, B) is controllable if and only if*

$$\mathrm{rank}\left(\begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) = n, \qquad \forall \lambda \in \mathbb{C}$$

If we apply this test to the extended open-loop dynamics, we get

$$\mathrm{rank}\left(\begin{bmatrix} A_p - \lambda I & 0 & B_p \\ -C_{pz} & -\lambda I & -D_{pz} \end{bmatrix}\right) = n_p + n_e$$

for all $\lambda \neq 0$. So we just need to check the rank when $\lambda = 0$ giving the following:

$$\mathrm{rank}\left(\begin{bmatrix} A_p & B_p \\ -C_{pz} & -D_{pz} \end{bmatrix}\right) = n_p + n_e$$

Observability

Consider the observability matrix for Σ_p. Since this system is observable, it has full rank.

$$\text{rank} \begin{bmatrix} C_p \\ C_p A_p \\ C_p A_p^2 \\ \vdots \\ C_p A_p^{n_p-1} \end{bmatrix} = n_p$$

Now look at the controllability matrix for the augmented system

$$\begin{bmatrix} \begin{bmatrix} C_p & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} C_p & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix} \\ \begin{bmatrix} C_p & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix}^2 \\ \vdots \\ \begin{bmatrix} C_p & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix}^{n-1} \end{bmatrix}$$

Note that

$$\begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix}^2 = \begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix} \begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix} = \begin{bmatrix} A_p^2 & 0 \\ -C_{pz} A_p & 0 \end{bmatrix}$$

Note that

$$\begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix}^n = \begin{bmatrix} A_p^n & 0 \\ -C_{pz} A_p^{n-1} & 0 \end{bmatrix}$$

So the observability matrix for the extended open-loop dynamics becomes

$$\begin{bmatrix} \begin{bmatrix} C_p & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} C_p A_p & 0 \\ -C_{pz} & 0 \end{bmatrix} \\ \begin{bmatrix} C_p A_p^2 & 0 \\ -C_{pz} A_p & 0 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} C_p A_p^{n-1} & 0 \\ C_{pz} A_p^{n-2} & 0 \end{bmatrix} \end{bmatrix}$$

And we can see that the n_e columns that were added to the observability matrix for the extended open-loop dynamics are linearly independent. So the augmented controllability matrix is full rank.

Rank of B, C, and CB

The proof is trivial.

Transmission Zeros

Given system $\Sigma_p = (A_p, B_p, C_p)$ show that integral augmentation does not add any transmission zeros to the system. The Rosenbrock system matrix for Σ_p is

$$R_p(s) = \begin{bmatrix} sI - A_p & -B_p \\ C_p & D_{pz} \end{bmatrix}$$

The transmission zeros are the values of s which make $R_p(s)$ lose rank. For the augmented system the Rosenbrock matrix is

$$R(s) = \begin{bmatrix} sI - A_p & 0 & -B_p \\ C_{pz} & sI & D_{pz} \\ C_p & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

The augmented Rosenbrock matrix has $2n_e$ rows added and n_e columns. When the Σ_p is tall, that is $m \le \ell$, then the maximum rank of $R(s)$ is $n_p + n_e + m$. The rank of $R(s)$ only drops for values of s which are transmission zeros of Σ_p due to the extra n_e columns being added are linearly independent for all s, due to the identity at the bottom.

21.3.3   Using Feed Forward

Finish this section, see 16.31 notes.
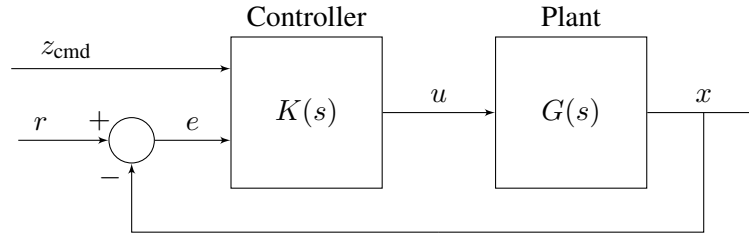
Figure 21.2: System block diagram

Chapter 22

Output Feedback Control

22.1 Introduction

When doing full state feedback pole placement problems, it is often the case that the desired pole locations can be achieved without using all of the gains. When this happens, the question may arise regarding what values to assign to these "extra" gains. In regards to stability (that is, the closed-loop eigenvalue locations) they can be assigned arbitrarily. However, one may postulate that these gains may be assigned in such a way that will improve the closed-loop performance of the system. Additionally, the idea of designing a control system with only some of the state variables being fed back to the controller should seem like a reasonable idea.

22.2 Projective Output Feedback

It can be shown that assignment of these "extra" gains allows the eigenvectors to be selected, in addition to only the eigenvalues. These performance benefits may manifest themselves as increased robustness???

Example 23 Pole placement Put an example here indicating a full-state feedback pole placement problem with extra degrees of freedom.

22.2.1 Static Projective Output Feedback

The full-state feedback LQR controller is a design which selects feedback gains optimally, based on some user specified performance weights. Such a design results in good closed-loop performance, but relies on the entire state vector being accessible. In cases where the entire state is not accessible, projective output feedback will allow an LQR like full state feedback design to be achieved, without requiring full state accessibility. The following notes are based on Eugene's book Chapter 6, page 165.

Given a state-space plant of the following form

$$\dot{x}_p = A_p x_p + B_p u$$
$$y_p = C_p x_p$$

A full state feedback LQR controller can be designed, with control law

$$u = K_p^\top x_p$$

giving the following full-state feedback closed loop system

$$\dot{x}_p = (A_p + B_p K_p^\top) x_p$$
$$y_p = C_p x_p$$

$$A_{\text{cl,fs}} = A_p + B_p K_p^\top$$

$$\dot{x}_p = A_{\text{cl,fs}} x_p$$

Now it is desired to design an output feedback controller, using gain feedback on only the output y_p instead of the entire state x_p. That is

$$u = K_y^\top y_p$$

giving the following output feedback closed loop system

$$\dot{x} = (A_p + B_p K_y^\top C_p) x_p$$

$$A_{\text{cl,op}} = A_p + B_p K_y^\top C_p$$

$$\dot{x}_p = A_{\text{cl,op}} x_p$$

The idea is to make the closed loop "A" matrix when using output feedback as close to the closed-loop "A" matrix when using full state feedback. This can be done by requiring that n_y eigenvalues of $A_{\text{cl,op}}$ are equal to those of $A_{\text{cl,fs}}$, where n_y is the number of outputs.

The eigenvalue problem can be written

$$A_{\text{cl,fs}} V = V \Lambda$$

We can require that n_y eigenvalues of $A_{\text{cl,fs}}$ are maintained in $A_{\text{cl,op}}$. That is, taking these n_y values of Λ gives Λ_{n_y}. In other words, this requirement gives n_y eigenvectors V_{n_y} that satisfy

$$A_{\text{cl,fs}} V_{n_y} = A_{\text{cl,op}} V_{n_y}$$

Solving this equation we can find K_y

$$(A_p + B_p K_x^\top) V_{n_y} = (A_p + B_p K_y^\top C_p) V_{n_y}$$

$$K_x^\top V_{n_y} = K_y^\top C_p V_{n_y}$$

$$\boxed{K_y^\top = K_x^\top V_{n_y} (C_p V_{n_y})^{-1}}$$

To implement this procedure, first design the full-state feedback LQR controller. Then pick the eigenvalues and corresponding eigenvectors of this full-state feedback design "A" matrix $A_{\text{cl,fs}}$ which should be kept in the output feedback design. This gives you V_{n_y} and Λ_{n_y}. Using V_{n_y} and K_x, solve for the output feedback gain K_y.

22.2.2 Dynamic Projective Output Feedback

Finish this section using the example problem in Eugene's book.

22.2.3 Comparing Full State and Output Projective Feedback

gain margins? phase margins? what happens to these margins if the output feedback gain is increased? what about an output feedback design which simply takes the full-state feedback gain matrix and deletes the feedback gains corresponding to states which are not accessible? If output feedback is done by just zeroing out gains corresponding to state variables which are not available for feedback, the system may not be stable.

22.3 State Observer/Estimator

Another way of dealing with control problems in which the full state is not available for feedback is to design a state observer or estimator. The general block diagram for a closed-loop estimator is shown in the figure below.
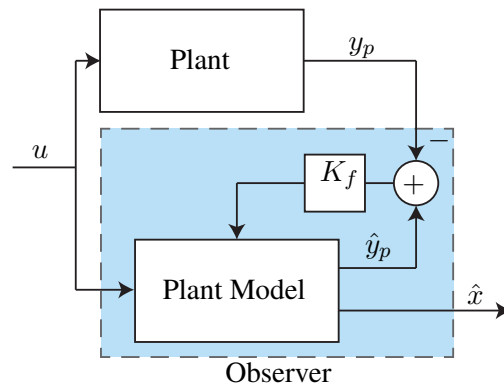
Figure 22.1: Closed loop estimator

The idea is that an estimator contains a linear model of what the plant is expected to be. In reality, the parameters assigned within the observer's plant model may not be perfect, but hopefully they are close. The feedback gain K_f is then used to correct the plant model based on differences between the actual output y and estimated output $\hat{y}$. Then, as this output error is reduced, the state estimate $\hat{x}_p$ is improved, and this estimate can then be used for feedback control.

22.3.1 LQE Estimator Design: Kalman Filter

Plant given by the LTI state space system. State-space form of plant

$$\dot{x}_p = A_p x_p + B_p u + B_{p2} w$$
$$y_p = C_p x_p + D_p u$$

Explain the input disturbance term B_{p2} that is up there later, but use this one for now.

$$\boxed{\begin{aligned} \dot{x}_p &= A_p x_p + B_p u \\ y_p &= C_p x_p + D_p u \end{aligned}}$$

The following figure shows a more detailed block diagram of the closed loop estimator, and how one might construct the block diagram in SIMULINK using basic gains, integrators, and summing blocks.
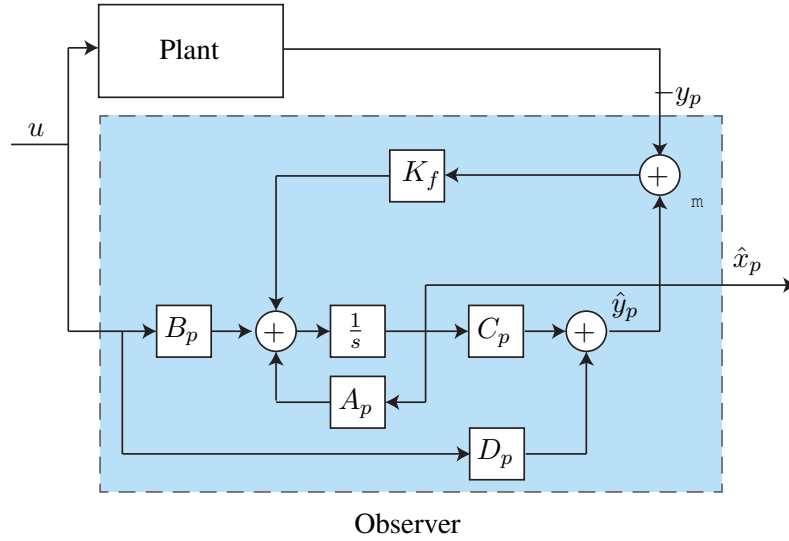
Figure 22.2: Closed loop estimator

A closed-loop estimator will then be used to reconstruct the plant state x_p. We will call this full state estimate $\hat{x}_p$. The state-space representation for the closed-loop estimator as shown in the block diagram above is given by

$$\dot{\hat{x}}_p = A_p\hat{x}_p + B_pu + K_f(\hat{y}_p - y_p)$$
$$\hat{y}_p = C_p\hat{x}_p + D_pu$$

Inserting the known expression for the plant output

$$\dot{\hat{x}}_p = A_p\hat{x}_p + B_pu + K_f(C_p\hat{x}_p + D_pu - y_p)$$
$$\hat{y}_p = C_p\hat{x}_p + D_pu$$

Combining terms gives the following closed-loop estimator state-space equations below

$$\boxed{\begin{aligned} \dot{\hat{x}}_p &= (A_p + K_fC_p)\hat{x}_p + (B_p + K_fD_p)u - K_fy_p \\ \hat{y}_p &= C_p\hat{x}_p + D_pu \end{aligned}}$$

Where the inclusion of the state estimate as an output in the above equations is to emphasize that these are the equations which would be used to implement an estimator to use, as possibly part of a controller, in simulation. The input to the estimator block would be the control u, measured plant output y_p, and the output of the estimator would be the estimated state $\hat{x}_p$. This estimated state would then be used in a feedback control law where ordinarily the actual state x_p would have been used, if it were accessible. Such a control architecture is known as a dynamic output feedback compensator, and is described in more detail in following sections.

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How to obtain the Kalman filter gain is not covered here, but K_f is given by

$$K_f = P_f C^\top R_0^{-1}$$
$$0 = AP_f + P_f A^\top + Q_0 - P_f C^\top R_0^{-1} C P_f$$

The Kalman filter gain K_f can be found by using the MATLAB function `lqr`. In order to implement the Kalman filter as given in these equations, the command would be: `Kf=-lqr(Ap',Cp',Qf,Rf)'`.

22.4 The Regulation Problem

Now it is desired to add the integral error state of the LQR-PI controller to the DOFB architecture. This will allow external reference commands to be given and followed with zero steady-state error, as well as using only the output, and not the full state, for feedback.

$$\dot{x}_p = A_p x_p + B_p u$$
$$y_p = C_p x_p + D_p u \tag{22.1}$$

where $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times m}$, and $C_p \in \mathbb{R}^{\ell \times n_p}$. Given that the state is available for measurement, the control goal is to design a control input u so that the closed-loop system has bounded solutions and the system is regulated to the origin. Now we would like to improve on this idea of using the estimated state $\hat{x}_p$ in feedback, just like in the case of full-state LQR feedback control. We would now like to use this same procedure, but instead of feeding back on the actual state x_p, we will use a state estimator to generate an estimate of the state $\hat{x}_p$. The combining of an optimal state estimator and optimal controller is known as LQG control. "For LTI systems with Gaussian models for disturbances and measurement noise, the Kalman filter is the optimal state estimator. When optimal control (LQR) is combined with optimal state estimation (Kalman filter) the control design is called the Linear Quadratic Gaussian (LQG) problem." [2] The control law is given by

$$u = K_p^\top \hat{x}_p$$

We design a state estimator as described above, and with the proposed control law the combined system, called the compensator, is described by the following equations

$$\dot{\hat{x}}_p = A_p \hat{x}_p + B_p u + K_f(\hat{y}_p - y_p)$$
$$\hat{y}_p = C_p \hat{x}_p + D_p u$$
$$u = K_p^\top \hat{x}_p$$

Substituting the control law and output equation in

$$\dot{\hat{x}}_p = A_p \hat{x}_p + B_p K_p^\top \hat{x}_p + K_f[C_p \hat{x}_p + D_p K_p^\top \hat{x}_p - y_p]$$
$$u = K_p^\top \hat{x}_p$$

Expanding

$$\dot{\hat{x}}_p = A_p \hat{x}_p + B_p K_p^\top \hat{x}_p + K_f C_p \hat{x}_p + K_f D_p K_p^\top \hat{x}_p + K_f(-y_p)$$
$$u = K_p^\top \hat{x}_p$$

Grouping terms

<div style="border:1px solid red; padding:1em;">

Output Feedback Regulator

$$\dot{\hat{x}}_p = (A_p + B_p K_p^\top + K_f C_p + K_f D_p K_p^\top)\hat{x}_p + K_f(-y_p)$$
$$u = K_p^\top \hat{x}_p$$

</div>

The basic block diagram which describes this combination of an estimator with controller looks just like the LQR controller, and is shown below. However, now the output y_i is being fed back to the controller, instead of the full state x_i, and the compensator $K(s)$ is of increased complexity.
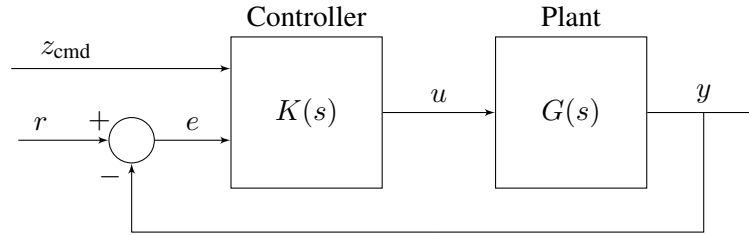
Figure 22.3: System block diagram

The following diagram shows a little bit more clearly the internal workings of the DOFB-regulator.
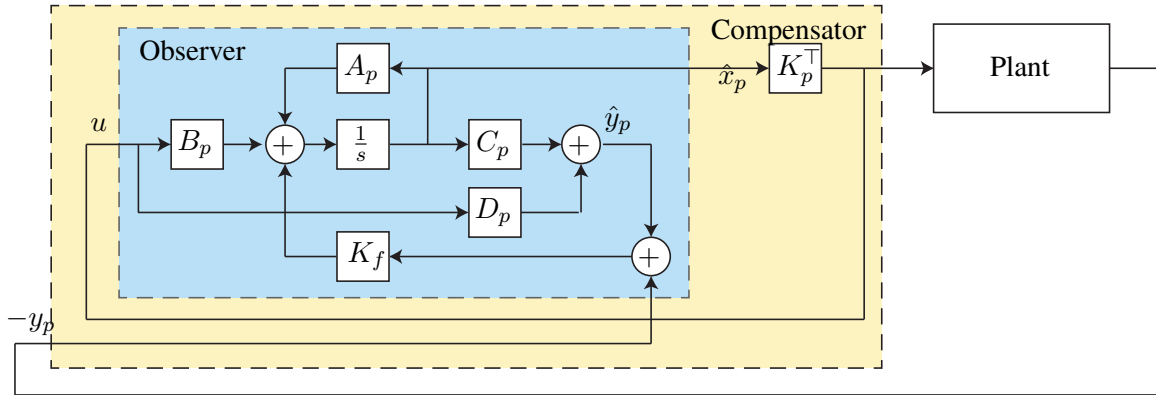
It doesn't matter if the estimator feedback term is $\hat{y}_p - y_p$ or $y_p - \hat{y}_p$. All this does is change the sign of K_f. We will use the representation shown above to facilitate representation of these equations in the block diagram form with negative feedback at the summing junction for input r_2. In addition, the feedback control law can be written with a positive or negative sign. All of the terms can be combined and substituted and rearranged within these above equations revealing that this dynamic output feedback compensator takes as its input the plant output y, and uses it with an LQR full state feedback control law operating on the estimated state $\hat{x}$ to generate the output u. This is called a dynamic output feedback compensator. This can be implemented as shown in the following block diagram, where the compensator $K(s)$ will contain these additional dynamics, instead of just the integrator and proportional gains as in the LQR-PI controller case.

This representation is also good because the only input to the compensator is the output feedback $e = -y_p$, as is represented in the equations.

22.5 The Tracking Problem

Now it is desired to add the integral error state of the LQR-PI controller to the DOFB architecture. This will allow external reference commands to be given and followed with zero steady-state error, as well as using only the output, and not the full state, for feedback.

$$
\begin{aligned}
\dot{x}_p &= A_p x_p + B_p u \\
y_p &= C_p x_p + D_p u \\
z &= C_{pz} x_p + D_{pz} u
\end{aligned}
\tag{22.2}
$$

where $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times m}$, and $C_p \in \mathbb{R}^{\ell \times n_p}$ and $z \in \mathbb{R}^{n_e}$ is the regulated output, where the number of regulated outputs is not to exceed the number of inputs, that is $n_e \leq m$. Given that the state is available for measurement, the control goal is to design a control input u so that the closed-loop system has bounded solutions and z tends to the reference command z_{cmd} asymptotically. In order to ensure command tracking, we introduce integral action, and for this purpose an additional state x_e is defined as

$$
\dot{x}_e = z_{\text{cmd}} - z
$$

and the plant in (22.2) is augmented to lead to the following extended open-loop dynamics

$$
\begin{bmatrix} \dot{x}_p \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_e \end{bmatrix} + \begin{bmatrix} B_p \\ -D_{pz} \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} z_{\text{cmd}}
\tag{22.3}
$$

The system in (22.3) can be written more compactly as

$$
\begin{aligned}
\dot{x} &= Ax + Bu + B_{\text{cmd}} z_{\text{cmd}} \\
y &= Cx + Du
\end{aligned}
\tag{22.4}
$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $B_{\text{cmd}} \in \mathbb{R}^{n \times n_e}$, and $C \in \mathbb{R}^{p \times n}$ are the known matrices given by

$$
A = \begin{bmatrix} A_p & 0_{n_p \times n_e} \\ -C_{pz} & 0_{n_e \times n_e} \end{bmatrix} \quad B = \begin{bmatrix} B_p \\ -D_{pz} \end{bmatrix} \quad B_{\text{cmd}} = \begin{bmatrix} 0_{n_p \times m} \\ I_{n_e \times n_e} \end{bmatrix} \quad C = \begin{bmatrix} C_p & 0_{\ell \times n_e} \\ 0_{n_e \times n} & I_{n_e \times n_e} \end{bmatrix}
$$

In the tracking problem for state feedback, the control law was $u = K_x^\top x$ where $x = \begin{bmatrix} x_p^\top & x_e^\top \end{bmatrix}^\top$. In the case of output feedback x_p must be replaced by the estimate as

$$
\begin{aligned}
u &= \begin{bmatrix} K_p^\top & K_e^\top \end{bmatrix} \begin{bmatrix} \hat{x}_p \\ x_e \end{bmatrix} \\
&= K_p^\top \hat{x}_p + K_e^\top x_e
\end{aligned}
$$

We now need to design an observer to generate the state estimate.

22.5.1 Non-Augmented Observer

We would now like to use this same procedure, but instead of feeding back on the actual state x_p, we will use the estimated state $\hat{x}_p$. The integral error term will be the difference between the commanded output value y_cmd and the measured value y_p. It can also be defined as the difference between commanded output and estimated output, but that will give different results. I don't really know what the advantages/disadvantages of doing it either way are yet. The basic closed-loop estimator equation, with integral error, and feedback control law The more detailed block diagram is shown in the following figure.
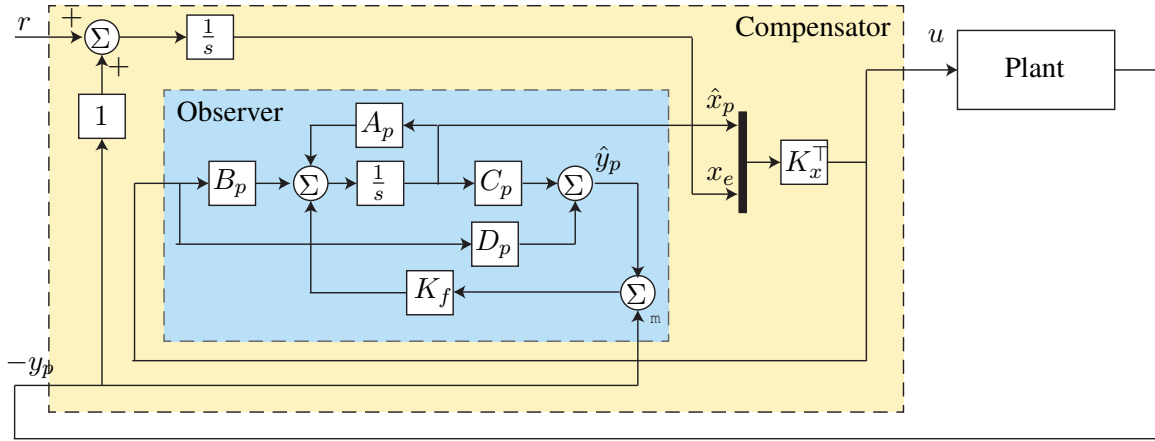
Figure 22.4: Observer controller and plant

Propose the following observer

$$\dot{\hat{x}}_p = A_p\hat{x}_p + B_p u + K_f(\hat{y}_p - y_p)$$
$$\hat{y}_p = C_p\hat{x}_p + D_p u$$

Combining this observer with the integral error equation and the control law

$$\dot{\hat{x}}_p = A_p\hat{x}_p + B_p u + K_f(\hat{y}_p - y_p)$$
$$\hat{y}_p = C_p\hat{x}_p + D_p u$$
$$\dot{x}_e = z_\text{cmd} - z$$
$$u = K_p^\top\hat{x}_p + K_e^\top x_e$$

$$\dot{\hat{x}}_p = A_p\hat{x}_p + B_p(K_p^\top\hat{x}_p + K_e^\top x_e) + K_f[C_p\hat{x}_p + D_p(K_p^\top\hat{x}_p + K_e^\top x_e) - y_p]$$
$$\dot{x}_e = z_\text{cmd} - z$$
$$u = K_p^\top\hat{x}_p + K_e^\top x_e$$

$$\dot{\hat{x}}_p = A_p\hat{x}_p + B_pK_p^\top\hat{x}_p + B_pK_e^\top x_e + K_fC_p\hat{x}_p + K_fD_pK_p^\top\hat{x}_p + K_fD_pK_e^\top x_e - K_fy_p$$
$$\dot{x}_e = z_\text{cmd} - z$$
$$u = K_p^\top\hat{x}_p + K_e^\top x_e$$

$$\dot{\hat{x}}_p = (A_p + B_p K_p^\top + K_f C_p + K_f D_p K_p^\top)\hat{x}_p + (B_p K_e^\top + K_f D_p K_e^\top)x_e - K_f y_p$$

$$\dot{x}_e = z_{\mathrm{cmd}} - z$$

$$u = K_p^\top \hat{x}_p + K_e^\top x_e$$

Combining into a state space representation gives the dynamic output feedback compensator with servomechanism: DOFB-Servo.

$$\begin{bmatrix} \dot{\hat{x}}_p \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A_p + B_p K_p^\top + K_f C_p + K_f D_p K_p^\top & B_p K_e^\top + K_f D_p K_e^\top \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_p \\ x_e \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} z_{\mathrm{cmd}} + \begin{bmatrix} K_f \\ 0 \end{bmatrix}(-y_p) + \begin{bmatrix} 0 \\ I \end{bmatrix}(-z)$$

$$u = \begin{bmatrix} K_p^\top & K_e^\top \end{bmatrix} \begin{bmatrix} \hat{x}_p \\ x_e \end{bmatrix}$$

Combining both inputs into one the final state-space representation for the DOFB-Servo compensator is had, and will now be used for evaluation of the loop transfer functions.

Non-Augmented Observer (Full Compensator)

$$\begin{bmatrix} \dot{\hat{x}}_p \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A_p + B_p K_p^\top + K_f C_p + K_f D_p K_p^\top & B_p K_e^\top + K_f D_p K_e^\top \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_p \\ x_e \end{bmatrix} + \begin{bmatrix} 0 & K_f & 0 \\ I & 0 & I \end{bmatrix} \begin{bmatrix} z_{\mathrm{cmd}} \\ -y_p \\ -z \end{bmatrix}$$

$$u = \begin{bmatrix} K_p^\top & K_e^\top \end{bmatrix} \begin{bmatrix} \hat{x}_p \\ x_e \end{bmatrix}$$

22.5.2 Augmented Observer

The plant for which we will design a dynamic output feedback compensator for is given by

$$\dot{x}_p = A_p x_p + B_p u$$
$$y_p = C_p x_p + D_p u$$
$$z = C_{pz} x_p + D_{pz} u$$

where $x_p \in \mathbb{R}^{n_p}$, $u \in \mathbb{R}^m$, $y_p \in \mathbb{R}^\ell$ and $z \in \mathbb{R}^{n_e}$ is the regulated output. The integral error state is defined by

$$\dot{x}_e = z_{\mathrm{cmd}} - z \tag{22.5}$$

Using the error description in (22.5), the state vector x_p is augmented to include this error by including x_e as a state variable

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A_p & 0_{n_p \times n_e} \\ -C_{pz} & 0_{n_e \times n_e} \end{bmatrix} \begin{bmatrix} x_p \\ x_e \end{bmatrix} + \begin{bmatrix} B_p \\ -D_{pz} \end{bmatrix} u + \begin{bmatrix} 0_{n_p \times n_e} \\ I_{n_e \times n_e} \end{bmatrix} z_{\mathrm{cmd}} \tag{22.6}$$

Writing the linear state-space representation in (22.6) more compactly using $x = \begin{bmatrix} x_p^\top & x_e^\top \end{bmatrix}^\top$ as

$$\dot{x} = Ax + Bu + B_{\mathrm{cmd}} z_{\mathrm{cmd}}$$
$$y = Cx + Du$$

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where $x \in \mathbb{R}^n$, $z_{\text{cmd}} \in \mathbb{R}^{n_e}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $B_{\text{cmd}} \in \mathbb{R}^{n \times n_e}$ and where

$$A = \begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix} \quad B = \begin{bmatrix} B_p \\ -D_{pz} \end{bmatrix} \quad B_{\text{cmd}} = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad C = \begin{bmatrix} C_p & 0_{\ell \times n_e} \\ 0_{n_e \times n_p} & I_{n_e \times n_e} \end{bmatrix} \quad D = \begin{bmatrix} D_p \\ 0_{n_e \times m} \end{bmatrix}$$

and where $y \in \mathbb{R}^p$ where $p = \ell + n_e$. The difference in this case versus the non-augmented observer approach is that now we will generate an estimator for the augmented system, and use that entire estimate for feedback. This includes an estimate of the error state. Consider the following observer with control law

$$\dot{\hat{x}} = A\hat{x} + Bu + B_{\text{cmd}}z_{\text{cmd}} + L(\hat{y} - y)$$
$$\hat{y} = C\hat{x} + Du$$
$$u = K_x^\top \hat{x}$$

$$\dot{\hat{x}} = A\hat{x} + BK_x^\top \hat{x} + B_{\text{cmd}}z_{\text{cmd}} + LC\hat{x} + LDK_x^\top \hat{x} - Ly$$
$$u = K_x^\top \hat{x}$$

$$\dot{\hat{x}} = (A + BK_x^\top + LC + LDK_x^\top)\hat{x} + B_{\text{cmd}}z_{\text{cmd}} - Ly$$
$$u = K_x^\top \hat{x}$$

Augmented Observer (Observer Only)

$$\dot{\hat{x}} = (A + BK_x^\top + LC + LDK_x^\top)\hat{x} + \begin{bmatrix} B_{\text{cmd}} & L \end{bmatrix} \begin{bmatrix} z_{\text{cmd}} \\ -y \end{bmatrix}$$

$$u = K_x^\top \hat{x}$$

(22.7)

The above is a representation only of the observer, and not the whole compensator, as the integral error state is also a part of the compensator and needs to be included in the dynamics

$$L = \begin{bmatrix} L_p & L_e \end{bmatrix}$$

$$y = \begin{bmatrix} y_p \\ z \end{bmatrix}$$

$$\begin{bmatrix} z_{\text{cmd}} & -y_p & -z \end{bmatrix}^\top$$

$$\dot{\hat{x}} = (A + BK_x^\top + LC + LDK_x^\top)\hat{x} + B_{\text{cmd}}z_{\text{cmd}} - L_p y_p - L_e z$$
$$\dot{x}_e = z_{\text{cmd}} - z$$
$$u = K_x^\top \hat{x}$$

Augmented Observer (Full Compensator)

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A + BK_x^\top + LC + LDK_x^\top & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ x_e \end{bmatrix} + \begin{bmatrix} B_{\text{cmd}} & L_p & L_e \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_{\text{cmd}} \\ -y_p \\ -z \end{bmatrix}$$

$$u = K_x^\top \hat{x}$$

(22.8)
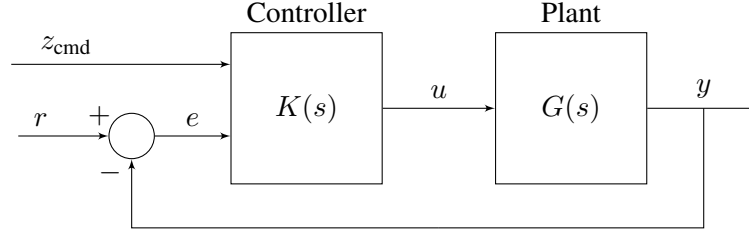
Figure 22.5: System block diagram

22.5.3 Loop Transfer Recovery

See Lavretsky, Wise page (184) 193. In analyzing the compensator and understanding loop transfer recovery, we consider two control systems: one state feedback and one output feedback. If the state feedback system is designed using LQR, then it will have excellent margins. If the output feedback system is designed using LQG, its margins may be arbitrarily bad. The process of loop transfer recover attempts to make the LQG controller like the LQR controller. While this statement is rather vague, the two systems will never be the same, as the LQG controller has many states, whereas the LQR controller without integral action has no states. However, what we mean by making the LQG controller look like the LQR is that the loop shapes in the frequency domain will be the same over a frequency range of interest, thus recovering the LQR margins.

To do this comparison, we break the loop at the input of both the LQR and LQG controllers. We look at the transfer function from u_i to u_o and see how we can change the observer gain so that these two transfer functions start to look the same. The plant has state-space representation

$$\dot{x}_p = A_p x_p + B_p u_i$$

The transfer function from the plant input to its output is given by

$$\frac{x_p}{u_i} = (sI - A_p)^{-1} B_p$$

The LQR compensator is described by

$$u_o = -K_p^\top x_p$$

Multiplying the two transfer functions together gives the transfer function through the control loop when it is broken at the plant input for the LQR controller

$$\frac{u_o}{u_i} = -K_p^\top (sI - A_p)^{-1} B_p$$

Now let's analyze the output feedback compensator. In this case the plant equation must include the output. If we dont use a D term the plant is described by

$$\dot{x}_p = A_p x_p + B_p u_i$$
$$y_p = C_p x_p$$

The transfer function from the plant input to its output is given by

$$\frac{y_p}{u_i} = C_p(sI - A_p)^{-1} B_p$$

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The LQG compensator is described by

$$\dot{\hat{x}}_p = (A_p - K_f C_p - B_p K_x)\hat{x} + K_f y_p$$
$$u_o = -K_p^\top \hat{x}_p$$

This gives the following transfer function from the input to the compensator (which is the plant output) to the compensator output (the control)

$$\frac{u_o}{y_p} = -K_p^\top (sI - A_p + K_f C_p + B_p K_p^\top)^{-1} K_f$$

Multiplying the two transfer functions together gives the transfer function through the control loop when it is broken at the plant input

$$\frac{u_o}{u_i} = -K_p^\top (sI - A_p + K_f C_p + B_p K_p^\top)^{-1} K_f C_p (sI - A_p)^{-1} B_p$$

22.5.4   An LTR Alternative to Output Feedback

Consider the following MIMO uncertain open-loop system

$$\begin{aligned}\dot{x}_p &= A_p x_p + B_p u \\ y_p &= C_p x_p \\ z &= C_{pz} x_p + D_{pz} u\end{aligned} \tag{22.9}$$

where $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times m}$, $C_p \in \mathbb{R}^{\ell \times n_p}$, $C_{pz} \in \mathbb{R}^{n_e \times n_p}$ are constant *known* matrices. z is the regulated output, and the number of regulated outputs cannot exceed the number of inputs, that is $n_e \leq m$. The goal is to design a control input u which will make z tend to the reference command z_{cmd} asymptotically.

In order to ensure command tracking, we introduce integral action, and for this purpose an additional state x_e is defined as

$$\dot{x}_e = z_{\text{cmd}} - z$$

and the plant in (22.9) is augmented to lead to the following extended open-loop dynamics

$$\begin{aligned}\begin{bmatrix} \dot{x}_p \\ \dot{x}_e \end{bmatrix} &= \begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_e \end{bmatrix} + \begin{bmatrix} B_p \\ -D_{pz} \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} z_{\text{cmd}} \\ \begin{bmatrix} y_p \\ x_e \end{bmatrix} &= \begin{bmatrix} C_p & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_p \\ x_e \end{bmatrix}\end{aligned} \tag{22.10}$$

The system in (22.10) can be written more compactly as

$$\begin{aligned}\dot{x} &= Ax + Bu + B_{\text{cmd}} z_{\text{cmd}} \\ y &= Cx\end{aligned} \tag{22.11}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $B_{\text{cmd}} \in \mathbb{R}^{n \times n_e}$, and $C \in \mathbb{R}^{p \times n}$ are the known matrices given by

$$A = \begin{bmatrix} A_p & 0_{n_p \times n_e} \\ -C_{pz} & 0_{n_e \times n_e} \end{bmatrix} \quad B = \begin{bmatrix} B_p \\ -D_{pz} \end{bmatrix} \quad B_{\text{cmd}} = \begin{bmatrix} 0_{n_p \times m} \\ I_{n_e \times n_e} \end{bmatrix} \quad C = \begin{bmatrix} C_p & 0_{\ell \times n_e} \\ 0_{n_e \times n} & I_{n_e \times n_e} \end{bmatrix}$$

Note that $p = \ell + n_e$. We make the following assumptions about the system $\Sigma = (A, B, C, 0)$ in (22.11).

Assumption 1

    a) (A, B) is controllable.

    b) (A, C) is observable.

    c) B, C, and CB are full rank.

    d) Any transmission zeros of Σ are strictly stable.

    e) Σ is tall: $p \geq m$.

Remark 4 Assumptions 1a and 1b are standard. Assumption 1c implies that inputs and outputs are not redundant, as well as a MIMO equivalent of relative degree unity. Assumption 1d is a standard requirement for adaptive control. Assumption 1e can be considered without loss of generality as the case of wide systems $p < m$ holds by duality.

Remark 5 Given a system $\Sigma_p = (A_p, B_p, C_p, 0)$ which satisfies

- (A_p, B_p) is controllable.

- (A_p, C_p) is observable.

- B_p, C_p, and $C_p B_p$ are full rank.

- Any transmission zeros of Σ_p are strictly stable.

- The rank of the following matrix is full

$$\text{rank}\left(\begin{bmatrix} A_p & B_p \\ -C_{pz} & -D_{pz} \end{bmatrix}\right) = n_p + n_e$$

when augmented with the integral error state as shown in (22.10) also satisfies Assumption 1a-d. In other words, under these assumptions, integral error augmentation does not destroy controllability or observability, the rank conditions, nor does it add any transmission zeros [3].

22.6 PI-Observer and Controller

Consider again the same plant as before, but this time with a constant sensor bias d_{out} on the output as represented in the figure below.
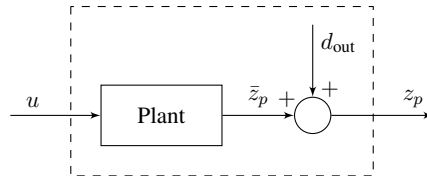
The equations describing this system are

$$\dot{x}_p = A_p x_p + B_p u$$
$$\bar{z}_p = C_p x_p$$
$$z_p = \bar{y}_p + d_{\text{out}}$$

First augment this state-space representation with an integral error state $\dot{x}_e = z_{\text{cmd}} - \bar{z}_p$ for the controller

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ -C_p & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_e \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} z_{\text{cmd}}$$

This system can be represented more compactly as follows

$$\dot{x} = Ax + Bu + B_{\text{cmd}} z_{\text{cmd}}$$

From which we write the regulated output $\bar{z}_p$ and the corresponding biased regulated output z_p as follows, and also we can define the measured outputs in the same way as $\bar{y}_p$ and y_p where the measured outputs contain the regulated outputs, but also include as additional outputs the integral error state.

$$\begin{aligned} \bar{z}_p &= \begin{bmatrix} C_p & 0 \end{bmatrix} x \\ &= C_z x \\ z_p &= \begin{bmatrix} C_p & 0 \end{bmatrix} x + d_{\text{out}} \\ &= C_z x + d_{\text{out}} \end{aligned}$$

and the measured outputs are

$$\begin{aligned} \bar{y}_p &= \begin{bmatrix} C_p & 0 \\ 0 & I \end{bmatrix} x \\ &= Cx \\ y_p &= \begin{bmatrix} C_p & 0 \\ 0 & I \end{bmatrix} x + \begin{bmatrix} I \\ 0 \end{bmatrix} d_{\text{out}} \\ &= Cx + B_D d_{\text{out}} \end{aligned}$$

Together these equations are

$$\begin{aligned} \dot{x} &= Ax + Bu + B_{\text{cmd}} z_{\text{cmd}} \\ \bar{y}_p &= Cx \\ y_p &= Cx + B_D d_{\text{out}} \\ \bar{z}_p &= C_z x \\ z_p &= C_z x + d_{\text{out}} \end{aligned}$$

Using the control law

$$u = K_x^\top x$$

And substituting in, the compact form of the plant augmented with the integral error state, and closed loop with control law is

$$\boxed{\begin{aligned} \dot{x} &= A_{\text{ref}} x + B_{\text{cmd}} z_{\text{cmd}} \\ \bar{y}_p &= Cx \\ y_p &= Cx + B_D d_{\text{out}} \end{aligned}}$$

So, at this point we have basically assumed the state x_p was available for measurement and the plant was completely known in determining the above system. We can now write these dynamics in a different form to include the disturbance as a state.

$$
\begin{bmatrix} \dot{x} \\ \dot{d}_{\text{out}} \end{bmatrix} = \begin{bmatrix} A_{\text{ref}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ d_{\text{out}} \end{bmatrix} + \begin{bmatrix} B_{\text{cmd}} \\ 0 \end{bmatrix} z_{\text{cmd}}
$$

This system can be represented more compactly as follows

$$
\dot{x}_f = A_f x_f + B_f z_{\text{cmd}}
$$

Outputs

$$
\begin{aligned}
\bar{z}_p &= \begin{bmatrix} C_z & 0 \end{bmatrix} x_f \\
&= \begin{bmatrix} C_p & 0 & 0 \end{bmatrix} x_f \\
&= \bar{C}_{zf} x_f \\
z_p &= \begin{bmatrix} C_z & 1 \end{bmatrix} x_f \\
&= \begin{bmatrix} C_p & 0 & 1 \end{bmatrix} x_f \\
&= C_{zf} x_f
\end{aligned}
$$

and the measured outputs are

$$
\begin{aligned}
\bar{y}_p &= \begin{bmatrix} C_p & 0 & 0 \\ 0 & I & 0 \end{bmatrix} x_f \\
&= \bar{C}_f x_f \\
y_p &= \begin{bmatrix} C_p & 0 & I \\ 0 & I & 0 \end{bmatrix} x_f \\
&= C_f x_f
\end{aligned}
$$

So everything is

$$
\begin{aligned}
\dot{x}_f &= A_f x_f + B_f z_{\text{cmd}} \\
\bar{y}_p &= \bar{C}_f x_f \\
y_p &= C_f x_f \\
\bar{z}_p &= C_{zf} x_f \\
z_p &= C_{zf} x_f
\end{aligned}
$$

where

$$
x_f = \begin{bmatrix} x_p^\top & x_e^\top & d_{\text{out}}^\top \end{bmatrix}^\top
$$

So now we have the plant with a closed-loop LQR-PI controller and disturbance with integrator all expressed in a single system. The problem is that we cannot implement the above controller as the state x_p which was used in the control law is unavailable. We attempt to estimate not only this state, but also the error state and disturbance. That is

$$
\begin{aligned}
\dot{\hat{x}}_f &= A_f \hat{x}_f + B_f z_{\text{cmd}} + K_f (\hat{y}_p - y_p) \\
\hat{\bar{y}}_p &= \bar{C}_f \hat{x}_f \\
\hat{y}_p &= C_f \hat{x}_f
\end{aligned}
$$

where

$$\hat{x}_f = \begin{bmatrix} \hat{x}_p^\top & \hat{x}_e^\top & \hat{d}_{\text{out}}^\top \end{bmatrix}^\top$$

Expanding this estimator representation we get the following

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{d}}_{\text{out}} \end{bmatrix} = \begin{bmatrix} A_{\text{ref}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d}_{\text{out}} \end{bmatrix} + \begin{bmatrix} B_{\text{cmd}} \\ 0 \end{bmatrix} z_{\text{cmd}} + \begin{bmatrix} L_v \\ L_I \end{bmatrix} (\hat{y}_p - y_p)$$

giving

$$\begin{aligned}
\dot{\hat{x}}_f &= A_{\text{ref}}\hat{x}_f + B_{\text{cmd}}z_{\text{cmd}} + L_v(\hat{y}_p - y_p) \\
\dot{\hat{d}}_{\text{out}} &= L_I(\hat{y}_p - y_p) \\
\hat{\bar{y}}_p &= \bar{C}_f \hat{x}_f
\end{aligned}$$

Design PI-Observer First, Then Control Later Dealing with an output bias is a dual problem to that of dealing with an input bias using an input integrator like in LQR-PI control. The first step is to represent the system equations to include the disturbance as an augmented state as shown below. Mathematically, these equations are exactly the same as above, just represented in a different way.

$$\begin{bmatrix} \dot{x}_p \\ \dot{d}_{\text{out}} \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ d_{\text{out}} \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u$$

$$z_p = \begin{bmatrix} C_p & 1 \end{bmatrix} \begin{bmatrix} x_p \\ d_{\text{out}} \end{bmatrix} + D_p u$$

More compactly these equations can be written

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
z_p &= Cx + D_p u
\end{aligned}$$

Now we design an estimator for this new system. The state z and corresponding estimated state $\hat{z}$ are

$$z = \begin{bmatrix} x_p \\ d_{\text{out}} \end{bmatrix} \qquad \hat{z} = \begin{bmatrix} \hat{x}_p \\ \hat{d}_{\text{out}} \end{bmatrix}$$

This allows the biased output y_p and the non-biased output $\bar{y}_p$ to be estimated, and thus the estimate of the bias to be determined.

$$\begin{aligned}
\dot{\hat{z}} &= A\hat{z} + Bu + K_f(\hat{y}_p - y_p) \\
\hat{y}_p &= C\hat{z} + D_p u \\
\hat{\bar{y}}_p &= \bar{C}\hat{z} + D_p u
\end{aligned}$$

where the output matrix for the estimate of the unbiased output is

$$\bar{C} = \begin{bmatrix} C_p & 0 \end{bmatrix}$$

Can also have output matrix for disturbance estimate

$$C_{\hat{d}} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

To make the proportional and integral components of this observer more clear, represent the observer above by splitting up K_f and writing

$$\dot{\hat{x}}_p = A_p\hat{x}_p + B_pu + K_P(\hat{y}_p - y_p)$$
$$\dot{\hat{d}}_{\text{out}} = K_I(\hat{y}_p - y_p)$$
$$\hat{y}_p = C_p\hat{x}_p + \hat{d}_{\text{out}}$$
$$\hat{\bar{y}}_p = C_p\hat{x}_p$$

where

$$K_f = \begin{bmatrix} K_P \\ K_I \end{bmatrix}$$

The thinking here is a little bit different than when creating the integral error state in LQR-PI controller. In LQR-PI control, this error state is constructed by considering measurements which are available, and asking ourselves what signals should be differenced to make an error which should then be driven to zero. That is, if we have an unbiased output measurement which we want to drive to some reference value, we difference these, then call this difference the derivative of the error state. Then, by augmenting the state-space description with this error state, we convince ourselves that at steady state this error derivative must become zero, requiring the measurement to be equal to the command.

With the PI observer it is a little bit different. That $\hat{y}_p$ will track y_p at steady state in the presence of an output disturbance is not what we are interested in, but rather that this process gives convergence of $\hat{x}_p$ to x_p. It is from this estimated state which we can then construct an estimate of the unbiased output $\hat{\bar{y}}_p$. In thinking of this problem it is convenient to think always of the plant as the system which includes the output bias, and concern ourselves only with estimating $\hat{x}_p$.

From here we can design a controller. Here we will use an LQR-PI controller, which has control law

$$u = K_p^\top \hat{x}_p + K_e^\top x_e$$

where the integral error state is defined as

$$\dot{x}_e = r - H\hat{x}_p$$

Chapter 23

Frequency Domain Analysis

23.1 Introduction

Frequency response method: give a system a sinusoidal input. The output (will always be?) a sinusoid, with a magnitude and phase which may be different than the input, but the frequency will be the same. Then, sweep the input frequency across a wide range, and observe how the gain and phase shift of the measured output change with frequency.

23.2 Compensator Analysis Using Loop Transfer Functions

References: see [2] (Chapter 5) and Astrom and [4] (Chapter 11).

Relative Stability

This block diagram below encompasses the types of controller designs we want to consider. If $z_{\text{cmd}} = 0$, this is the classical block diagram, where the error signal e is the only input to the controller. When $r = 0$, the input is given directly through z_{cmd}, as is the case with the LQR-PI controller.
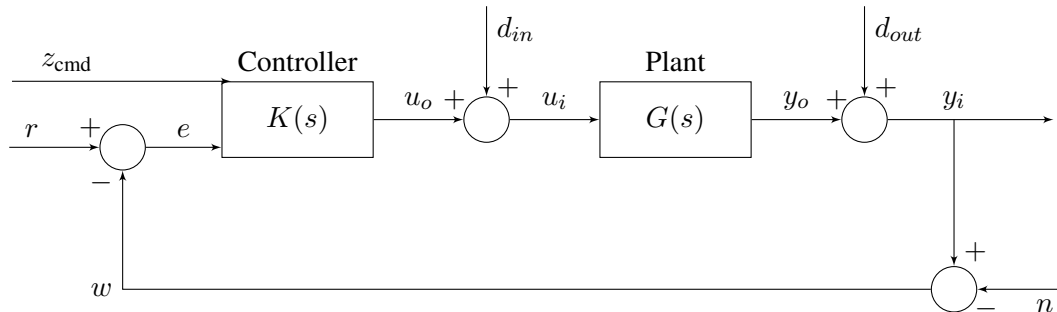
Figure 23.1: General MIMO feedback control block diagram

The following section will explain how to determine the various transfer functions of interest which describe a plant and compensator in the block diagram form shown in Figure 23.1. In order to find the different loop transfer functions, the control loop will be broken at different points, and the transfer function at this broken point evaluated. This is used to analyze what happens if noise or disturbances are injected

at these different loop break points. The following show how to find the loop transfer functions for MIMO plants and controller. This is important because unlike SISO systems, multiplication of matrices is not commutative.

Note also:

$$L(I + L)^{-1} = (I + L)^{-1}L$$

23.2.1 Input

Loop Transfer Function Breaking the loop at the plant input u means to evaluate the transfer function from "input" u_i to "output" u_o while all other signals are zero.

$$y = G(s)u_i$$

$$e = -y$$

$$u_o = K(s)e = -K(s)y$$

$$u_o = -K(s)G(s)u_i = -L_u(s)u_i$$

$$\boxed{L_u(s) = K(s)G(s)}$$

Return Difference Transfer Function Differencing the "input" u_i and "output" u_o using the expression for output from above:

$$u_i - u_o = u_i - K(s)e$$

$$u_i - u_o = u_i + K(s)G(s)u_i$$

$$u_i - u_o = (I + K(s)G(s))u_i$$

$$u_i - u_o = (I + L_u(s))u_i$$

$$\boxed{I + L_u(s)}$$

When plotting this loop shape, it will be large at low frequencies and tend to unity at high frequencies, since the loop transfer function $L_u \to 0$ at high frequencies. This loop will dip below 0 dB, before leveling out, and the more it dips below 0 dB the worse the gain margin is.

Input Sensitivity Transfer Function $S_u(s)$ The input sensitivity transfer function is from d_{in} to u_i while all other signals are zero.

$$u_i = u_o + d_{\text{in}}$$

$$u_o = K(s)e = -K(s)y$$

$$y = G(s)u_i$$

$$u_i = -K(s)G(s)u_i + d_{\text{in}}$$

$$(I + K(s)G(s))u_i = d_{\text{in}}$$

$$u_i = (I + K(s)G(s))^{-1}d_{\text{in}}$$

$$u_i = (I + L_u(s))^{-1}d_{\text{in}}$$

$$\boxed{S_u(s) = (I + L_u(s))^{-1}}$$

Complementary Input Sensitivity Transfer Function $T_u(s)$ Unlike the $T_y(s)$, the input complementary sensitivity transfer function does not represent an relationship between any of the system inputs and outputs. This transfer function is given by using the identity $S_u(s) + T_u(s) = I$, hence the name complementary sensitivity transfer function, but otherwise there is little intuition regarding what this transfer function represents.

$$S_u(s) + T_u(s) = I$$
$$(I + L_u(s))^{-1} + T_u = I$$
$$I + (I + L_u(s))T_u = I + L_u(s)$$
$$T_u = (I + L_u(s))^{-1}L_u(s)$$
$$T_u = S_u(s)L_u(s)$$

With $L(I + L)^{-1} = (I + L)^{-1}L$, which can be shown by pre- and post- multiplying both sides by $(I + L)$, we can also write $T_u(s)$ as

$$T_u(s) = L_u(I + L_u(s))^{-1}$$

Stability Robustness

$$1 + L_u^{-1}$$

At low frequencies $L_u^{-1} \to 0$, so this loop shape will be unity for low frequencies and large at high frequencies, since at high frequencies $L_u^{-1} \to \infty$. It will dip below 0 dB before going up, and the more it dips below 0 dB, the worse the gain margin will be.

23.2.2   Output

Loop Transfer Function $L_y(s)$ Breaking the loop at the plant output y means to evaluate the transfer function from "input" y_i to "output" y_o while all other signals are zero.

$$e = -y_i$$

$$u_i = u_o = K(s)e = -K(s)y_i$$

$$y_o = G(s)u_i$$

$$y_o = -G(s)K(s)y_i = -L_y(s)y_i$$

$$\boxed{L_y(s) = G(s)K(s)}$$

Return Difference Transfer Function   Differencing the "input" y_i and "output" y_o using the expression for output from above:

$$y_i - y_o = y_i + G(s)K(s)y_i$$

$$y_i - y_o = (I + G(s)K(s))y_i$$

$$y_i - y_o = (I + L_y(s))y_i$$

$$\boxed{I + L_y(s)}$$

Output Sensitivity Transfer Function $S_y(s)$ The output sensitivity is from d_{out} to y_i with all other signals are zero. "The sensitivity function tells how the variations in the output are influenced by feedback" [4]

$$y_i = y_o + d_{\text{out}}$$

$$y_o = G(s)u_i = G(s)u_o$$

$$u_o = K(s)e = -K(s)y_i$$

$$y_i = -G(s)K(s)y_i + d_{\text{out}}$$

$$(I + G(s)K(s))y_i = d_{\text{out}}$$

$$y_i = (I + G(s)K(s))^{-1}d_{\text{out}}$$

$$y_i = (I + L_y(s))^{-1}d_{\text{out}}$$

$$\boxed{S_y(s) = (I + L_y(s))^{-1}}$$

At frequencies where plant disturbances are to be rejected, we want $S_y(s) \to 0$.

Complementary Output Sensitivity Transfer Function $T_y(s)$ Unlike $T_u(s)$, this transfer function represents the relationship between the noise n and the output $y_i = y_o$. This transfer function also satisfies the identity $S_y(s) + T_y(s) = I$, hence the name complementary sensitivity transfer function.

$$e = r - w$$

$$w = y_i - n$$

$$y_i = G(s)K(s)e$$

$$e = r - y_i + n$$

$$e = r - G(s)K(s)e + n$$

$$(I + G(s)K(s))e = r + n$$

$$e = (I + G(s)K(s))^{-1}n$$

$$y_i = G(s)K(s)e$$

$$y_i = G(s)K(s)(I + G(s)K(s))^{-1}n$$

$$\boxed{T_y(s) = L_y(s)(I + L_y(s))^{-1}}$$

This transfer function is the same as that between r and the output $y_i = y_o$.

23.2.3   The Gang of Six

Now that the basic transfer functions have been found for a generic plant and controller feedback system, and an example of how to find the transfer function for the LQR-PI has been presented, we want to explain how to use the transfer functions to create some plots that will aid in control system analysis and design. The following figure shows singular value plots of the "gang of six".

23.2.4 Margins for MIMO System
23.3 Stuff
23.3.1 Gain and Phase of Poles and Zeros

These two examples that follow connect the transfer function representation of the integrator and differentiator to a differential equation and show how, when the system is given a sinusoidal input, that the zero causes the output to lead the input, and the pole causes the output to lag behind the input.

$$\frac{x}{u} = \frac{1}{s}$$

The differential equation is

$$\dot{x} = u$$

This whole thing of gain and phase is for a given sinusoidal input, so $u(t) = A\sin(\omega t)$ and then we want to see how that will affect the output $x(t)$ for this system.

$$\frac{dx}{dt} = A\sin(\omega t)$$

separate and integrate back

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} A\sin(\omega t)dt$$

giving

$$x - x_0 = -\frac{A}{\omega}\cos(\omega t) + \frac{A}{\omega}\cos(\omega t_0)$$

$$x(t) = -\frac{A}{\omega}\cos(\omega t) + \frac{A}{\omega}\cos(\omega t_0) + x_0$$

$$x(t) = -\frac{A}{\omega}\cos(\omega t)$$

but $-\cos(x) = \sin(x - 90°)$ so the input and corresponding output can be written

$$u(t) = A\sin(\omega t)$$
$$x(t) = \frac{A}{\omega}\sin(\omega t - 90°)$$

So we can see that the magnitude of the output depends inversely on ω, getting smaller as ω gets larger. More importantly we can see clearly that there is a $90°$ phase lag from the input to the output. Consider now a differentiator.

$$\frac{x}{u} = s$$

gives

$$x = \dot{u}$$

and with $u(t) = A \sin(\omega t)$ we have

$$x(t) = A\omega \cos(\omega t)$$

together the input and output are

$$u(t) = A \sin(\omega t)$$
$$x(t) = A\omega \cos(\omega t)$$

But we have $\cos(x) = \sin(x + 90°)$ giving

$$u(t) = A \sin(\omega t)$$
$$x(t) = A\omega \sin(x + 90°)$$

and here we can see that there is a positive phase shift, or say that the output leads the input by 90 degrees.

23.3.2 How to Make Bode Plots

first normalize the transfer function, so all the factors look like $(1 + sa_i)$

23.3.3 Nyquist Plots
23.3.4 Output Filters
Types of Filters

Talk about the different types of filters here, including elliptical filter. Talk about why I want to use any particular one, and which ones work best.

First Order Low-Pass Filter

Filters can be used to condition an output signal which is to be used in feedback. A low pass filter is given by the following transfer function, where the input to the filter is u and the output is y.

$$\frac{y}{u} = \frac{10}{s + 10}$$

Such a filter is best implemented in SIMULINK by representing it in state-space form, thus allowing the initial conditions to be set, which cannot be done using a transfer function representation.

$$\frac{y}{u} = \frac{y}{x}\frac{x}{u} = \frac{10}{s + 10}$$
$$\frac{x}{u} = \frac{1}{s + 10}$$
$$\frac{y}{x} = 10$$
$$\dot{x} + 10x = u$$

$$\boxed{\begin{array}{l} \dot{x} = -10x + u \\ y = 10x \end{array}}$$

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$$A = -10 \quad B = 1 \quad C = 10 \quad D = 0$$

This state-space representation of a low-pass filter can now be implemented in SIMULINK, but the initial conditions must be specified. The filter takes as its input the unfiltered state x and outputs the filtered state x_f. When the simulation begins, if the initial conditions are not set, the filter will cause an abrupt spike in the state before settling down. The initial conditions must be set such that when the simulation begins, at the first time step the output signal is the same as the input signal, and the filter state derivative should be zero. That is, $\dot{x}(0) = 0$ and $x_f(0) = x(0)$. Using the state as the input $u = x$, the filtered state as the output $y = x_f$, and the actual internal filter state $x = x_s$...

$$0 = -10x_s(0) + x(0)$$
$$x_f(0) = 10x_s(0)$$

We see $x_s(0) = x(0)/10 = (0)$ will make the filter state derivative $\dot{x}_s = 0$ and the filtered output the same as the input.

Chapter 24

Linear System Stuff

24.1 The Matrix Exponential

24.1.1 General System Stability

Given a linear ODE of the following form

$$a_n y^{(n)} + \cdots + a_2 \ddot{y} + a_1 \dot{y} = b_m u^{(m)} + \cdots + b_2 \ddot{u} + b_1 \dot{u}$$

with a specified input function $u(t)$, the solution is the output function $y(t)$. We say the system is BIBO stable if, given a bounded input function $u(t)$, the output function $y(t)$ is bounded for all time. That is $|y(t)| < \alpha < \infty$. Considering the Laplace transform of this system

$$Y(s)(a_n s^n + \cdots + a_2 s^2 + a_1 s) = U(s)(b_m s^m + \cdots + b_2 s^2 + b_1 s)$$

$$\frac{Y(s)}{U(s)} = \frac{(b_m s^m + \cdots + b_2 s^2 + b_1 s)}{(a_n s^n + \cdots + a_2 s^2 + a_1 s)} = \frac{N(s)}{D(s)} = G(s)$$

And for BIBO stability this corresponds to the denominator $D(s)$ having all of its roots have real part that is negative. When taking the inverse Laplace transform of $G(s)$ if any of the roots were positive, this would result in the output taking the input and multiplying it by some increasing exponential. So, regardless of which bounded input was fed to the system, the output would grow unbounded. That is why stability requires the roods of $D(s)$ be negative. Considering the transfer function representation of a stable system such as

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = G(s)$$

Now, although the system is stable, we may want to know how stable it is. We do this by looking at the roots of the denominator (which must all be all in the LHP for stability) and see how far they have to go before they become in the right half plane.

24.1.2 Stability Margins

Stability margins for a system have no meaning when considered outside the context of feedback. Take a stable system $G(s)$. If an additional gain or time delay is introduced between the input and the system, the output will be delayed, and scaled by this gain. The same goes for delays or gains at the output. But these will not effect the stability of the system. Likewise if the system $G(s)$ is unstable, regardless of how the input is magnified or time-shifted, the resulting output will always "blow up". Stability margins for a system

$G(s)$ have to do with placing the system in feedback, and then determining how delays or gains added to the system will affect stability for the closed-loop system. Consider the following system where the plant $G(s)$ is stable. (It doesn't have to be) and we are designing a feedback controller to achieve some performance objectives out of the closed-loop system. The input to the system is $U(s)$ and the output $Y(s)$.

$$\frac{Y(s)}{U(s)} = H(s) = \frac{KG}{1 + KGH}$$

Call the quantity

$$L = KGH$$

giving

$$\frac{Y(s)}{U(s)} = H(s) = \frac{KG}{1 + L}$$

Now the loop transfer function L simply takes as an input a complex frequency s, and the result is a complex number with some magnitude and phase. It is not obvious what values of s would result in what complex numbers from L, or even what a "good" or "bad" value of L is. However, we can see from this equation that if $L = -1$ that the closed-loop system will "blow up", which is bad.

Intuitively, we know that given an arbitrary input, if the system is amplifying the signal through the feedback loop, the output will become larger and larger, indicating system instability. One way to think about this is: after some initial amount of time with an arbitrary input, turn the input off. If the signal within the loop eventually decays, the closed-loop system is stable. If the signal in the loop grows, the system is unstable.

We consider the case when the input to the system is an arbitrary sinusoidal input.

Consider the block diagram of the system H above, with the loop broken at the output. Inject a signal δ_{in} into H and see what happens when it goes through the blocks and comes out of the output y and call this signal δ_{out}. Notice that the output signal is

$$\delta_{\text{out}} = -L\delta_{\text{in}}$$

Again keeping in mind that the input and output are both sinusoidal signals, and if we consider sweeping across all frequencies with our input signal δ_{in}, the output will be a sine wave of the same frequency that has been phase shifted and scaled by some magnitude. When is L bad? What must L do to the input to be bad?

Consider the case where we find the frequency such that L has a phase shift of $180°$. This is equivalent to flipping the sine wave, or multiplying it by a negative. When this happens the magnitude of L then must be less than unity, otherwise the output will be the same phase as the input, but larger? Is this true?

What is the explanation for why $|L(s)|$ can be larger than unity when its phase is $-180°$? Finish this explanation.

In any case, looking at the denominator, we know that $L(s) = -1$ is bad. So, we sweep across all input frequencies and look at when $|L(\omega)| = 1$, we then want to know, at this value of ω, how far is the phase from being $-180°$? This is called the phase margin.

For gain margin, sweep across all the frequencies ω and find the one such that $\angle L(\omega) = -180°$. Then, for this value of ω, determine how far the magnitude of L is from being unity. This is gain margin.

24.1.3 Time Delay Margin

$s = \sigma + j\omega$ Considering sinusoidal inputs only, that is $s = j\omega$ Considering a Bode plot of a transfer function $G(s)$, the phase margin is Consider an input $u(t)$ and output $y(t)$. The input and output of the system can be represented as an exponential. That is consider the input

$$u(t) = e^{st}$$

The output will be of a similar form, of which we are trying to find the specifics when there is a delay between the input and output

$$y(t) = u(t - \tau)$$

and the input with this delay is

$$u(t - \tau) = e^{s(t-\tau)} = e^{st}e^{-s\tau}$$

and we can write

$$y(t) = e^{st}e^{-s\tau}$$

which is

$$y(t) = u(t)e^{-s\tau}$$

and so we can see given an output that is delayed by τ from the input is represented by $e^{-s\tau}$. Taking a transfer function with delay

$$\frac{Y(s)}{U(s)} = G(s)e^{-s\tau}$$

Now we want to know when this system is placed in feedback, how large would the delay have to get before the closed-loop system becomes unstable? A delay does not affect the magnitude of the system, only the phase. Look at the transfer function $G(s)$ when its magnitude is 1. Then we want to know what the relationship of the delay to phase is, where we are looking for how much phase shift can occur before $G(s)$ with the delay hits -1. Evaluating G with the delay at the frequency where its magnitude is 1.

$$G(j\omega_{cg})e^{-j\omega_{cg}\tau} = -1$$
$$\angle(G(j\omega_{cg})e^{-j\omega_{cg}\tau}) = -180°$$

The magnitude of G is 1 and the magnitude of the delay is 1, so now we just need to check when the combined phase of the two is $-180°$.

$$\angle(G(j\omega_{cg})e^{-j\omega_{cg}\tau}) = \angle G(j\omega_{cg}) + \angle e^{-j\omega_{cg}\tau} = -180°$$

and the angle of the delay is

$$\angle e^{-j\omega_{cg}\tau} = -\omega_{cg}\tau$$
$$\angle G(j\omega_{cg}) - \omega_{cg}\tau = -180°$$

Recalling the definition of phase margin

$$\text{PM} = 180° + \angle G(j\omega_{cg})$$

we get

$$PM = \omega_{cg}\tau$$

and solving for the delay τ we get

$$\boxed{\tau = \frac{PM}{\omega_{cg}}}$$

Magnitude more than 1 OK only when phase is $\pm 180°$ The maximum tolerable time delay is one which causes the transfer function $\frac{Y(s)}{U(s)}$ to become -1,

24.1.4 Introduction

Consider the following linear, time varying, autonomous state space equation where $A(t) \in \mathbb{R}^{n \times n}$ and $x(t) \in \mathbb{R}^{n \times 1}$.

$$\dot{x}(t) = A(t)x(t)$$

We want to gain insight into solution of this matrix equation by first considering the time invariant scalar equation

$$\dot{x}(t) = x(t)$$

Proposing a solution of the following form:

$$x(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

differentiating

$$\dot{x}(t) = 0 + 1 + t + \frac{t^2}{2!} + \dots$$

and notice that the function $x(t)$ is in fact the derivative of itself, and satisfies the given equation. Define this function $x(t) = e^t$, which is described by the following sum, to be the exponential function, noting that $0! = 1$.

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

(See Rudin pg 63 for definition of e) Extending this function definition to the scalar first order differential equation

$$\dot{x}(t) = ax(t)$$

where $a \in \mathbb{R}$. We are looking for a function $x(t)$ which, when differentiated, results in $x(t)$ being multiplied my a. We propose a solution $x(t) = e^{at}$ by replacing each t in the above definition by at. That is

$$x(t) = e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots$$

which, when differentiated gives

$$\dot{x}(t) = 0 + a + a^2 t + \frac{a^3 t^2}{2!} \dots$$
$$= a\left(1 + at + \frac{a^2 t^2}{2!} \dots\right)$$
$$= ae^{at}$$
$$= ax(t)$$

which solves the given differential equation. In general

$$x(t) = e^{at} c$$

is a solution, where $c = x(0)$

$$x(t) = e^{at} x(0)$$

Or more generally for $t_0 \neq 0$ the solution to the differential equation $\dot{x}(t) = ax(t)$ is given by

$$x(t) = e^{a(t - t_0)} x(t_0)$$

24.1.5 Matrix Exponential

Similarly, if the following solution is proposed for the matrix differential equation

$$x(t) = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

it can be differentiated in the same way as the scalar case, thus defining the function known as the *matrix exponential*.

$$e^{At} \triangleq I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

giving the following solution to the matrix differential equation $\dot{x}(t) = Ax(t)$

$$x(t - t_0) = e^{A(t - t_0)} x(t_0)$$

The quantity given by evaluating the matrix exponential is known as the *state transition matrix*:

$$x(t - t_0) = \Phi(t, t_0) x(t_0)$$

which is sometimes written $\Phi(t - t_0)$ as well. The state transition matrix may be time varying, or time invariant, depending on the system from which it resulted. The following examples will help show different way in which the state transition matrix can be found by evaluating the matrix exponential.

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Diagonal A Matrix

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

Finding the State Transition Matrix

Example 24 Using the definition — Time Invariant For the system $\dot{x}(t) = Ax(t)$ with A given below, the matrix exponential can be evaluated using the series definition.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^{A(t-t_0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(t - t_0) + \frac{1}{2!}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 (t - t_0)^2 + \dots$$

but

$$A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ for } n \in \mathbb{N} \geq 2$$

giving

$$e^{A(t-t_0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(t - t_0) = \begin{bmatrix} 1 & (t - t_0) \\ 0 & 1 \end{bmatrix}$$

So the state transition matrix is given by:

$$\Phi(t, t_0) = \begin{bmatrix} 1 & (t - t_0) \\ 0 & 1 \end{bmatrix}$$

and the solution to the state space equation is:

$$x(t) = \begin{bmatrix} 1 & (t - t_0) \\ 0 & 1 \end{bmatrix} x(t_0)$$

Example 25 Using Inverse Laplace Transform — Time Invariant For the system $\dot{x}(t) = Ax(t)$ with A given below, the matrix exponential can be evaluated using inverse Laplace transforms.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x(t) = \mathscr{L}^{-1}((sI - A)^{-1})x(t_0)$$

where

$$\Phi(t, t_0) = \mathscr{L}^{-1}((sI - A)^{-1})$$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$$

Using the formula to invert a 2×2 matrix:

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

Taking the inverse Laplace transform:

$$\mathscr{L}^{-1}\left(\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}\right) = \begin{bmatrix} 1 & (t - t_0) \\ 0 & 1 \end{bmatrix}$$

So the state transition matrix is given by:

$$\Phi(t, t_0) = \begin{bmatrix} 1 & (t - t_0) \\ 0 & 1 \end{bmatrix}$$

which matches the answer we got in the first example by using the series definition.

Example 26 Using the definition — Time Invariant The last example showed how the series definition could be used to find the state transition matrix for a simple linear time invariant system. This example was made particularly easy because the series terminated after the first two terms. In this example, the entire series will have to be considered, and recognized as a series representation of a commonly known trigonometric function. For the system $\dot{x}(t) = Ax(t)$ A is given by:

$$A = \begin{bmatrix} 0 & 1 \\ -a^2 & 0 \end{bmatrix}$$

Evaluating the first terms in the series definition of the matrix exponential:

$$A^2 = \begin{bmatrix} 0 & 1 \\ -a^2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -a^2 & 0 \end{bmatrix} = \begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -a^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a^2 \\ a^4 & 0 \end{bmatrix}$$

$$A^4 = A^3 A = \begin{bmatrix} 0 & -a^2 \\ a^4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -a^2 & 0 \end{bmatrix} = \begin{bmatrix} a^4 & 0 \\ 0 & a^4 \end{bmatrix}$$

$$A^5 = A^4 A = \begin{bmatrix} a^4 & 0 \\ 0 & a^4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -a^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a^4 \\ -a^6 & 0 \end{bmatrix}$$

Assembling these components using the series definition of the matrix exponential, the state transition matrix becomes

$$\Phi(t, t_0) = \left[\begin{array}{cc} 1 - a^2 \frac{t^2}{2!} + a^4 \frac{t^4}{4!} - \cdots & t - \cdots \\ -a^2 t + \cdots & 1 - \cdots \end{array} \right]$$

Recognizing the series in the entries of the state transition matrix, it can be simplified to:

$$\Phi(t, t_0) = \left[\begin{array}{cc} xxx & xxx \\ xxx & xxx \end{array} \right]$$

and the solution to the state space equation is:

$$x(t) = \left[\begin{array}{cc} xxx & xxx \\ xxx & xxx \end{array} \right] x(t_0)$$

See Hogan's notes *Unforced_LTI_Response.pdf*

Example 27 Directly from A Matrix — Time Varying 6.241 notes Wed. 2/29

$$A = \left[\begin{array}{cc} 0 & t \\ 0 & 0 \end{array} \right]$$

$$\dot{x}_1(t) = t x_2(t)$$
$$\dot{x}_2(t) = 0$$

$$\dot{x}_1(t) = t x_2(t_0)$$
$$\dot{x}_2(t) = 0$$

Example 28 Using Inverse Laplace Transform — Time Varying 6.241 HW 3.4. Find an expression for $\Phi(t_2, t_0)$ for the system $\dot{x}(t) = A(t)x(t)$ given

$$A = \left[\begin{array}{cc} 0 & 1 \\ -k & 0 \end{array} \right]$$

when

$$\begin{cases} k = 0 & \text{for } t < 0 \\ k = 1 & \text{for } t \geq 0 \end{cases}$$

for $t_0 < 0$ to $t_2 > 0$. Since the matrix $A(t)$ is time varying and described for two different cases of k, only on state transition matrix Φ will not be able to be used. Two state transition matrices are needed.

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0)$$

The first one will go from $t = t_0 < 0$ to $t = t_1 = 0$, and is given by:

$$\Phi(t_1, t_0) = \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(t_1 - t_0)\right)$$

$$= \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(-t_0)\right)$$

Using the series definition of the matrix exponential, the state transition matrix is calculated. The series definition can be used because the series terminates after the first two terms. That is all powers of A greater than 1 are zero.

$$\exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(-t_0)\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(-t_0) + \frac{1}{2!}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2(-t_0)^2 + \ldots$$

$$= \begin{bmatrix} 1 & -t_0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \ldots$$

$$\Phi(t_1, t_0) = \begin{bmatrix} 1 & -t_0 \\ 0 & 1 \end{bmatrix}$$

Now, the second state transition matrix $\Phi(t_2, t_1)$ must be calculated from $t = t_1 = 0$ to $t = t_2 > 0$:

$$\Phi(t_2, t_1) = \exp\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(t_2 - t_1)\right)$$

$$= \exp\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(t_2)\right)$$

The series definition can not be used to evaluate the state transition matrix in this case, since it does not terminate after two terms like it did before. Another way to evaluate the state transition matrix is the following:

$$x(t_2) = \mathscr{L}^{-1}((sI - A)^{-1})x(t_1)$$

where

$$\Phi(t_2, t_1) = \mathcal{L}^{-1}((sI - A)^{-1})$$

$$= \mathcal{L}^{-1}\left(\left(sI - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)^{-1}\right)$$

$$= \mathcal{L}^{-1}\left(\left(\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}\right)^{-1}\right)$$

Using the formula for the inverse of a 2×2 matrix:

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} = \frac{1}{\det(sI - A)}\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} = \frac{1}{s^2 + 1}\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$$

Now, taking the inverse Laplace transform of $(sI - A)^{-1}$ (which only depends on t_2 and not t_1) using a Laplace transform table, by hand:

$$\mathcal{L}^{-1}\begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} = \begin{bmatrix} \cos(t_2) & \sin(t_2) \\ -\sin(t_2) & \cos(t_2) \end{bmatrix}$$

Combining the two state transition matrices:

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0) = \begin{bmatrix} \cos(t_2) & \sin(t_2) \\ -\sin(t_2) & \cos(t_2) \end{bmatrix}\begin{bmatrix} 1 & -t_0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t_2) & \sin(t_2) - t_0\cos(t) \\ -\sin(t_2) & \cos(t_2) + t_0\sin(t_2) \end{bmatrix}$$

Several other state transition matrices $\Phi(t_2, t_0)$ would exist for different conditions of t_2 and t_0. For instance, if t_2 and t_0 were both greater than zero, the state transition matrix would only be calculated for $k = 1$ giving:

$$\Phi(t_2, t_0) = \begin{bmatrix} 1 & t_2 - t_0 \\ 0 & 1 \end{bmatrix}$$

24.1.6 Non-Autonomous Systems

In order to find the state of a state space model at time t_1 based on initial conditions at t_0 and in input $u(t)$, the methods of the state transition matrix can still be used, although they must be extended to include the input.

$$x(t_1) = \Phi(t_1, t_0)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

Example 29 Non-Autonomous System — Time Varying QR10B from 6.241 midterm review Given the following differential equation, find $y(2\pi)$. The initial conditions are $y(0) = 1$ and $\dot{y}(0) = 1$. The input $u(t) = 1$ for all time.

$$\ddot{y}(t) + k(t)y(t) = u(t) \quad \text{where} k(t) = \begin{cases} 1 & \text{for } \sin(t) > 0 \\ 0 & \text{for } \sin(t) \leq 0 \end{cases}$$

Putting this information into the differential equation, and based on the final time at which we are asked to calculate y, this problem can be broken into two parts, and solved separately using the formula above. That is, with $t_0 = 0$ and $t_1 = \pi$, we will first find the value of $y(t_1)$ and $\dot{y}(t_1)$. Over this time interval $k(t) = 1$ is a constant, and the problem can be solved as a time invariant one. Once the values of y and $\dot{y}$ are had at t_1, the problem will be repeated with $t_2 = 2\pi$.

The given differential equation will be rewritten for the first interval over which it will be solved, and the corresponding state space model for this system must be found.

$$\ddot{y} + y = 1$$

Define the following state vector $x(t)$:

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

with the state vector derivative

$$\dot{x}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix}$$

Using this state vector, the system equation can be expressed using the following state space model

$$\begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

From this state space model, the state transition matrix $\Phi(t_1, t_0)$ must be found.

$$\Phi(t_1, t_0) = \exp\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (t_1 - t_0) \right)$$

This state transition matrix may be more difficult to evaluate using the series definition of the matrix exponential, so the inverse Laplace transform method will be used:

$$\Phi(t_1, t_0) = \mathscr{L}^{-1}((sI - A)^{-1})$$

$$= \mathscr{L}^{-1}\left(\left(sI - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)^{-1} \right)$$

$$= \mathscr{L}^{-1}\left(\left(\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} \right)^{-1} \right)$$

$$= \mathscr{L}^{-1}\left(\frac{1}{s^2 + 1} \left(\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \right) \right)$$

$$= \mathscr{L}^{-1}\left(\begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \right)$$

Using a Laplace transform table, or by evaluating the inverse Laplace transforms by hand:

$$\Phi(t_1, t_0) = \begin{bmatrix} \cos(t_1 - t_0) & \sin(t_1 - t_0) \\ -\sin(t_1 - t_0) & \cos(t_1 - t_0) \end{bmatrix}$$

With the initial conditions plugged into the state vector:

$$x(t_0) = x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

All of the components to use the equation are had. Plugging them in:

$$
\begin{aligned}
x(\pi) &= \begin{bmatrix} \cos(\pi) & \sin(\pi) \\ -\sin(\pi) & \cos(\pi) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^\pi \begin{bmatrix} \cos(\pi - \tau) & \sin(\pi - \tau) \\ -\sin(\pi - \tau) & \cos(\pi - \tau) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^\pi \begin{bmatrix} \sin(\pi - \tau) \\ \cos(\pi - \tau) \end{bmatrix} d\tau \\
&= \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} \cos(\pi - \tau) \\ \sin(\pi - \tau) \end{bmatrix} \Bigg|_0^\pi \\
&= \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} \cos(0) - \cos(\pi) \\ \sin(0) - \sin(\pi) \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

This procedure must now be repeated for the second time interval, from $t_1 = \pi$ to $t_2 = 2\pi$, using the following state space model

$$\begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with the "initial" conditions:

$$x(t_1) = x(\pi) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Again, using the inverse Laplace transform method to find the state transition matrix $\Phi(t_2, t_1)$:

$$
\begin{aligned}
\Phi(t_2, t_1) &= \mathscr{L}^{-1}((sI - A)^{-1}) \\
&= \mathscr{L}^{-1}\left(\left(sI - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)^{-1} \right) \\
&= \mathscr{L}^{-1}\left(\left(\begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix} \right)^{-1} \right) \\
&= \mathscr{L}^{-1}\left(\frac{1}{s^2} \left(\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \right) \right) \\
&= \mathscr{L}^{-1}\left(\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \right)
\end{aligned}
$$

Using a Laplace transform table, or by evaluating the inverse Laplace transforms by hand:

$$\Phi(t_2, t_1) = \begin{bmatrix} 1 & t_2 - t_1 \\ 0 & 1 \end{bmatrix}$$

$$x(t_1) = x(\pi) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

All of the components to use the equation are had. Plugging them in:

$$
\begin{aligned}
x(2\pi) &= \begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \int_0^\pi \begin{bmatrix} 1 & 2\pi - \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \\
&= \begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \int_0^\pi \begin{bmatrix} 2\pi - \tau \\ 1 \end{bmatrix} d\tau \\
&= \begin{bmatrix} 1 - \pi \\ -1 \end{bmatrix} + \begin{bmatrix} 2\pi\tau - \frac{1}{2}\tau^2 \\ \tau \end{bmatrix} \Bigg|_0^\pi \\
&= \begin{bmatrix} 1 - \pi \\ -1 \end{bmatrix} + \begin{bmatrix} \frac{3}{2}\pi^2 \\ \pi \end{bmatrix} \\
&= \begin{bmatrix} 1 - \pi \\ -1 \end{bmatrix} + \begin{bmatrix} \frac{3}{2}\pi^2 \\ \pi \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{2}\pi^2 - \pi + 1 \\ \pi - 1 \end{bmatrix}
\end{aligned}
$$

24.2 Controllability, Reachability, and Observability

24.2.1 Introduction

The concepts of controllability, reachability, and observability will be explained, as well as the weaker conditions of stabilizability and detectability. The differences between these terms will be explained for continuous and discrete time. In a broad sense, the concept of controllability is the ability to command a system to do what we want it to do through the action of a control input. The concept of observability is the ability to see what is going on inside a dynamical system given the sensor outputs. To better illustrate these concepts, eigenvalue decomposition will be used, and is explained next.

24.2.2 Eigenvalue Decomposition

Eigenvalue decomposition is the representation of a matrix A in terms of its eigenvalues and eigenvectors. Only diagonalizable matrices can be factored this way, and the result is a diagonal matrix with the eigenvalues of A along the diagonal.

The matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. To check whether A is diagonalizable, first find its characteristic equation $\delta_A = \det(\lambda I - A) = |\lambda I - A| = 0$. From the characteristic equation, all of the eigenvalues λ_i can be found. Then find the eigenvectors by plugging in each of the eigenvalues into $(\lambda_i I - A)v_i = 0$ and solving for all of the eigenvectors v_i. Once all of the eigenvectors are obtained, their linear independence can be checked by combining them into an

eigenvector matrix V. If the determinant of this matrix is nonzero, the eigenvectors are linearly independent and the matrix A can be diagonalized. The diagonalizability of A can also be verified by attempting to do so through row operations, although if it is verified this way that A is diagonalizable, the eigenvalues and eigenvectors will have to be found anyway.

The eigenvalue problem, or eigenvalue equation can be stated as follows, and is the basis for finding the eigenvalues and eigenvectors of A.

$$Av_i = v_i\lambda_i$$

Assembling these n linearly independent eigenvectors in a matrix V, and the eigenvalues along the diagonals of a matrix Λ, the eigenvalue equation can be written in matrix form.

$$AV = \begin{bmatrix} Av_1 & Av_2 & \cdots & Av_n \end{bmatrix}$$

$$V\Lambda = \begin{bmatrix} v_1\lambda_1 & v_2\lambda_2 & \cdots & v_n\lambda_n \end{bmatrix}$$

$$AV = V\Lambda$$

Since V is composed of the n linearly independent eigenvectors, it has full rank, and is thus invertible, allowing the following to be written:

$$A = V\Lambda V^{-1}$$

$$\Lambda = V^{-1}AV$$

This is the process of diagonalizing a matrix A using its eigenvalues and eigenvectors.

Given the system $\dot{x} = Ax + Bu$ with state vector x, a linear transformation $z = Mx$ can be proposed, so long as M is full rank. The requirement of M to have full rank has to do with the necessity of all information in x to be preserved under the transformation. Since M has full rank, it is invertible, and the state vector x can be written

$$x = M^{-1}z$$

with derivative

$$\dot{x} = M^{-1}\dot{z}$$

substituting the transformed state into the system equation

$$M^{-1}\dot{z} = AM^{-1}z + Bu$$

$$\dot{z} = MAM^{-1}z + MBu$$

looking at this representation, and recalling that we can select M to be any matrix with full rank, we select $M = V$. The system can then be simplified and represented as follows

$$\dot{z} = \Lambda z + VBu$$

This shows that given a state space model $\dot{x} = Ax + Bu$, the system can be rewritten using a new state z where the matrix A has been transformed into a diagonal matrix Λ with the eigenvalues of A along the diagonal.

The following examples will help show the process of finding the eigenvalues and eigenvectors, checking the diagonalizability of A, and then finding the representation $\dot{z} = \Lambda z + VBu$ when possible.

Example 30 Eigenvalue decomposition with distinct eigenvalues Given the matrix A below, we would like to determine if it can be decomposed using its eigenvalues and eigenvectors.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Find the characteristic equation $\delta_A = \det(\lambda I - A) = 0$ of A. First, setting up the matrix $\lambda I - A$:

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 4 \end{bmatrix}$$

taking the determinant

$$(\lambda - 1)(\lambda - 4) - 4 = 0$$
$$\lambda^2 - 5\lambda = 0$$
$$\lambda(\lambda - 5) = 0$$

giving the following eigenvalues

$$\lambda_1 = 0$$
$$\lambda_2 = 5$$

both eigenvalues are distinct, so we expect that the corresponding eigenvectors be linearly independent. Finding the corresponding eigenvectors

$$(\lambda_i I - A)v_i = 0$$

$$\begin{bmatrix} \lambda_i - 1 & -2 \\ -2 & \lambda_i - 4 \end{bmatrix} \begin{bmatrix} v_{i,1} \\ v_{i,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 0$

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} -2a \\ a \end{bmatrix}$$

For $\lambda_2 = 5$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} a \\ 2a \end{bmatrix}$$

It is verified that the two eigenvectors are in fact linearly independent, as we expected. Eigenvectors are not true vectors. . . they are like a set of vectors. . . So any choice for the eigenvectors will work. Choosing $a = 1$ gives the following eigenvectors, which are arranged in the eigenvector matrix V.

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad V = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

Since the matrix A is diagonalizable, and we all of the eigenvalues and eigenvectors of A, we can write the following:

$$\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

The significance of this will become clearer with further examples.

Example 31 Eigenvalue decomposition with repeated eigenvalues Given the matrix A below, we would like to determine if it can be decomposed using its eigenvalues and eigenvectors.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This matrix has repeated eigenvalues, but is diagonalizable.

Example 32 Eigenvalue decomposition with repeated eigenvalues Given the matrix A below, we would like to determine if it can be decomposed using its eigenvalues and eigenvectors.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Find the characteristic equation $\delta_A = \det(\lambda I - A) = 0$ of A. First, setting up the matrix $\lambda I - A$:

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{bmatrix}$$

taking the determinant

$$(1 - \lambda)(1 - \lambda) = 0$$
$$(\lambda - 1)^2 = 0$$

giving the following eigenvalues

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$

With repeated eigenvalues there is no guarantee that the eigenvectors will be linearly independent. Finding the corresponding eigenvectors

$$(\lambda_i I - A)v_i = 0$$

$$\begin{bmatrix} \lambda_i - 1 & -1 \\ 0 & \lambda_i - 1 \end{bmatrix} \begin{bmatrix} v_{i,1} \\ v_{i,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = \lambda_2 = 1$

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

There is only one linearly independent eigenvector of A, so A is not diagonalizable.

24.2.3 Controllability, Reachability, and Stabilizability

Controllability and reachability are two concepts with relate to the ability to command a system to do what we want it to do through the action of a control input. In continuous time (CT) systems, these terms have the same meaning, given by the following definition:

Definition 20 Controllability (CT) *A continuous time system is controllable if for all initial states x_0 all states x, and some time t_1 which is greater than t_0, there exists a control input which will take the state x from $x(t_0) = x_0$ to $x(t_1) = x_1$.*

$$\forall x_0, x \in \mathbb{R}^n \forall T > t_0 \exists u(t)\big|_{t_0 \leq t \leq T} : x(t) : x(t_0) = x_0, x(t_1) = x_1$$

For discrete time systems, controllability and reachability are different, as given by the following definitions (Kalman 1961).

Definition 21 Controllability (DT) *A discrete time system is controllable if there exists a control input which will take the state to the origin in finite time.*

Definition 22 Reachability (DT) *A discrete time system is reachable if there exists a control input which will take the state from any initial state to any final state in finite time.*

24.2.4 Controllability versus Reachability

The need for the two different, but similar definitions for the discrete time case is explained here... Examples:

$$x(t+1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Include example of DT system which is controllable but not reachable

24.2.5 Applying the Concept of Controllability

Now that the definitions for controllability and reachability have been presented and explained, we will focus our attention only on CT systems. The process of determining if a system is controllable will be explained, and some examples using eigenvalue decomposition will be used to make the concept of controllability clear.To investigate the controllability of a system, the controllability matrix must be found. This matrix will serve as the basis for controllability calculations

- Explain how to derive the controllability matrix

$$M_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

In order for a system to be controllable, the controllability matrix must have full rank. For a system system with a single input this corresponds to the column vectors B, AB, A^2B ... being linearly independent. That is to say the square controllability matrix M_c must be non-singular; its determinant must be non-zero. The following examples show the process of determining the controllability for CT systems.

Example 33 A controllable system $\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Calculate the controllability matrix M_c:

$$M_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$$

The controllability matrix has full rank of 2, so the system is controllable.

Example 34 An uncontrollable system $\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Calculate the controllability matrix M_c:

$$M_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

The controllability matrix has rank 1, so the system is not controllable.

Determining the controllability of a system by checking the rank of the controllability matrix is procedural operation which results in only a *yes* or *no* answer to the question: *is this system controllable?* However, if a system is uncontrollable, this does not mean we must pack up and go home. We would like to gain some more insight into systems which are not controllable to better understand what is going on.

When we say that a CT system is uncontrollable, this does not mean that we have zero influence over the output of the system, but rather that not every state vector in the state space can be achieved. The range of the controllability matrix M_c (the span of the columns of M_c) gives the possible state vectors that can be achieved by the system by application of control. For a square matrix M_c, if its columns are linearly independent, it will span $\mathbb{R}^n$, and any state vector in the state space $\mathbb{R}^n$ can be achieved. The state vectors that are achievable by application of control are called controllable **primal states**. We call a state vector a primal state to differentiate it from individual state elements or components of the state vector.

If M_c does not have full rank, then there are primal states in the state space which are not spanned by the columns of M_c.

So, we know that for a system that is not controllable, there may be some primal states which are controllable, and some that are not. The uncontrollable states are best described in terms of uncontrollable **dual states**. The concept of dual states may be best explained by first providing some examples.

Example 35 Revisited: A controllable system Consider again the system $\dot{x} = Ax + Bu$ with the following A and B matrices.

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The controllability matrix was found before to have full rank, meaning the system is controllable. Based on the definition of controllability for CT systems, we know it is possible to to command the state vector x to any desired value. To better see this we will represent this system using the transformed state $z = Vx$. This results in the system

$$\dot{z} = \Lambda z + VBu$$

where Λ is given by

$$\Lambda = V^{-1}AV$$

Finding V and Λ: Find the characteristic equation $\delta_A = \det(\lambda I - A) = 0$ of A. First, setting up the matrix $\lambda I - A$:

$$\lambda I - A = \begin{bmatrix} \lambda + 1 & 0 \\ -1 & \lambda - 1 \end{bmatrix}$$

taking the determinant

$$(\lambda - 1)(\lambda + 1) = 0$$

giving the following eigenvalues

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

both eigenvalues are distinct, so we expect that the corresponding eigenvectors be linearly independent. Finding the corresponding eigenvectors

$$(\lambda_i I - A)v_i = 0$$

$$\begin{bmatrix} \lambda_i + 1 & 0 \\ -1 & \lambda_i - 1 \end{bmatrix} \begin{bmatrix} v_{i,1} \\ v_{i,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 1$

$$\begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v_1 = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

For $\lambda_2 = -1$

$$\begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2a \\ a \end{bmatrix}$$

using $a = 1$

$$V = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

allowing the system to be written

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} z + \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

simplifying

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} z + \begin{bmatrix} -2 \\ 3 \end{bmatrix} u$$

From this representation using the new state vector z, it may now be more clear that since z_1 and z_2 are decoupled, and the control effort can effect both state variables, thus allowing any state to be reached through application of a specific control input.

- How to find unreachable primal states? Is that even a thing?

- Explain what dual states are

- Do controllable dual states mean anything?

- Explain controllability of a system (fully controllable system) versus controllability of a mode

Example 36 Revisited: An uncontrollable system $\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The controllability matrix M_c was calculated and found to be rank deficient (it did not have full rank) and thus the system is not controllable. What else can we find out about this uncontrollable system?

24.2.6 How Controllable?

condition number of controllability matrix

24.2.7 Observability and Detectability

$$M_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

24.2.8 Reachable Primal States, Uncontrollable Dual States

Given a state-space model, be able to find the set of all reachable primal states, and uncontrollable dual states.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

- Reachable primal states are given by the span of the columns of the controllability matrix M_c

Computing the controllability matrix:

$$M_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & a+2 \end{bmatrix}$$

Reachable Primal States For this square 3×3 matrix, if it has full rank, its columns would span $\mathbb{R}^3$ and the entire states-space, i.e. $\mathbb{R}^3$ would be a reachable primal state. Because it is square, we can take the determinant. When the determinant is zero, the controllability matrix loses rank, and then its columns would not span $\mathbb{R}^3$.

$$\det(M_c) = -a + 1$$

So the controllability matrix is full rank for $a \neq 1$, and it loses rank when $a = 1$. With full rank, the reachable primal states are anything in $\mathbb{R}^3$. This can be written:

> for $a \neq 1$ the reachable primal state is: $x = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for $c_1, c_2, c_3 \in \mathbb{R}$

When $a = 1$ and the controllability matrix loses rank, we need to still find the set of reachable primal states. That is, with $a = 1$, what is the span of the columns of M_c? In the case when $a = 1$, M_c becomes:

$$M_c = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

By inspection, or through a series of row reductions, it is seen that only two of the columns are linearly independent. Thus, the reachable primal states when $a = 1$ are given by a linear combination of any two of the columns of M_c. That is:

> for $a = 1$ the reachable primal state is: $x = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ for $c_1, c_2 \in \mathbb{R}$

Uncontrollable Dual States Uncontrollable states are a linear combination of each of the entries of the state vector x. That is, an uncontrollable state z_i is given by:

$$z_i = Px \quad \text{where } P \text{ is a row vector}$$

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The row vector P is such that:

$$PM_c = 0 \quad \text{or, alternatively} \quad M_c^{\mathrm{T}} P^{\mathrm{T}} = 0$$

Using the second expression this gives:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & a+2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When $a \neq 1$, $P = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. The answer would be given as:

"when $a \neq 1$ the uncontrollable dual state is given by: $z = Px$, with P defined by: $P = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$."

When $a = 1$, the following system would have to be solved for p_1, p_2, and p_3.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$p_2 = -p_3$$

$$p_1 + p_2 + 2p_3 = 0 \quad \rightarrow \quad p_1 + p_3 = 0 \quad \rightarrow \quad p_1 = -p_3$$

$$p_1 + 2p_2 + 3p_3 = 0 \quad \rightarrow \quad -p_3 - 2p_3 + 3p_3 = 0 \quad \rightarrow \quad 0p_3 = 0$$

So, when $a = 1$, p_3 can be selected arbitrarily, and then $p_1 = -p_3$ and $p_2 = -p_3$. The answer would be given as:

"when $a = 1$ the uncontrollable dual state is given by: $z = Px$, with P defined by: $P = k \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}$ for $k \in \mathbb{R}$."

Reachability of System A system is reachable if the entire state-space is reachable. That is, the only uncontrollable dual state is $z = 0$.

24.2.9 Observable Dual States, Unobservable Primal States

Unobservable Primal States Unobservable primal states given by the nullspace of M_o, which is: $M_o x = 0$.

$$M_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ a & 2 & 1 \\ a & a+2 & 2 \end{bmatrix}$$

Take the determinant of M_o:

$$\det(M_o) = a(a-1)$$

The observability matrix loses rank when $a \in \{0, 1\}$.

- When the observability matrix M_o has full rank, the system has no unobservable primal states.

- When M_o loses rank, there will be some unobservable primal states

To find the unobservable primal states, plug in each value of a which make M_o lose rank, and solve $M_o x = 0$. For $a = 0$:

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By inspection, x_1 can be made anything, and $x_2 = x_3 = 0$.

> for $a = 0$ the unobservable primal state is: $x = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for $k \in \mathbb{R}$

For $a = 1$:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving:

$$x_2 = -x_3$$

$$x_1 + 2x_2 + x_3 = 0 \quad \rightarrow \quad x_1 - x_3 = 0 \quad \rightarrow \quad x_1 = x_3$$

$$x_1 + 3x_2 + 2x_3 = 0 \quad \rightarrow \quad x_3 - 3x_3 + 2x_3 = 0 \quad \rightarrow \quad 0 x_3 = 0$$

x_3 can be selected arbitrarily, with $x_1 = x_3$, and $x_2 = -x_3$. This gives the following unobservable primal state:

> for $a = 1$ the unobservable primal state is: $x = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ for $k \in \mathbb{R}$

Observable Dual States

- Observable dual states are given by linear combinations of Cx, CAx, $CA^2 x$ and so forth.

To find the observable dual states expressed as $z = Px$ where P is a row, transpose the observability matrix and take the linearly independent columns. In the example above, when $a \notin \{0, 1\}$ the observability matrix has full rank, and all of its rows and columns are linearly independent. Thus, when it is transposed, the span of these linearly independent columns can be expressed as:

> Observable dual state for $a \notin \{0, 1\}$: $x = c_1 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

For $a \in \{0, 1\}$ the observability loses rank, and the observable dual states are found by taking the linearly independent columns of $M_o{}^\mathsf{T}$. For $a = 0$:

$$M_o{}^\mathsf{T} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

and only two of the columns are linearly independent, giving:

> Observable dual state for $a = 0$: $x = c_1 \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}$

For $a = 1$:

$$M_o{}^\mathsf{T} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

and only two of the columns are linearly independent, giving:

$$\boxed{\text{Observable dual state for } a = 1: \quad x = c_1 \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}}$$

24.3 More Linear System Stuff

24.3.1 Zeros

Rosenbrock matrix

$$R(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}$$

Let's take $D = 0$ for now

$$R(s_0) = \begin{bmatrix} s_0 I - A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

s_0 is a transmission zero if R loses rank and so there is a nonzero vector $v_0 = [\begin{array}{cc} x_0^\top & u_0^\top \end{array}]^\top$ such that $Rv_0 = 0$.

Zeros Introduced by Postcompensator

The zeros are s_0 and v_0 such that $(s_0)v_0 = 0$. So if we post compensate the output as

$$R_s(s_0) = \begin{bmatrix} s_0 I - A & B \\ -S_1 C & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then all of the zeros of R are also zeros of R_s, and there might be additional zeros. We can see by looking at

$$C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

and we know that $C_1 x_0 = C_2 x_0 = C_3 x_0 = 0$. But if we square down using

$$S_1 = \begin{bmatrix} S_{11} & S_{12} & S_{13} \end{bmatrix}$$

$$S_1 C = [S_{11} C_1 + S_{12} C_2 + S_{13} C_3]$$

and

$$[S_{11} C_1 + S_{12} C_2 + S_{13} C_3] x_0 = 0$$

$$S_{11} C_1 x_0 + S_{12} C_2 x_0 + S_{13} C_3 x_0 = 0$$

so there can be additional zeros.

Chapter 25

Robust Control

25.1 H_2 Optimization Using Completion of Squares

General state-space form of plant for which the optimal controller $K(s)$ will be designed:

$$
\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u \\
e &= C_1 x + D_{11} w + D_{12} u \\
y &= C_2 x + D_{21} w
\end{aligned}
$$

The optimization setup has control singularity when either of the following matrices are not left invertible for all $\omega \in [0, \infty]$ and for D_{12} at $\omega = \infty$.

$$
E_c(s) = \begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} \quad \text{and} \quad D_{12}
$$

The optimization setup has sensor singularity when either of the following matrices are not right invertible all $\omega \in [0, \infty]$ and for D_{21} at $\omega = \infty$.

$$
E_m(s) = \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} \quad \text{and} \quad D_{21}
$$

A matrix A is right invertible if there is a matrix B such that $AB = I$. A matrix A is left invertible if there is a matrix B such that $BA = I$. For a square matrix A, left and right invertibility are the same, and is just regular invertibility.

$$
A_{\text{right}}{}^{-1} = A^{\mathsf{T}}(AA^{\mathsf{T}})^{-1}
$$

$$
A_{\text{left}}{}^{-1} = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}
$$

Finding the optimal controller $K(s)$ with the general form:

$$
\begin{aligned}
\dot{x}_e &= (A + LC_2 + B_2 F)x_e - Ly \\
u &= Fx_e
\end{aligned}
$$

General form of completion of squares used to find the optimal control and observer gains F and L, respectively, where the vertical bar brackets mean 2-norm:

$$
|C_1 x + D_{12} u|^2 + 2x' P_c (Ax + B_2 u) = |D_{12}(u - Fx)|^2
$$

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$$|B_1'\psi + D_{21}'\xi|^2 + 2\psi P_e(A'\psi + C_2'\xi) = |D_{21}'(\xi - L'\psi)|^2$$

where $A + B_2 F$ and $A + LC_2$ are Hurwitz matrices.

Once the stabilizing solutions for P_c and P_e and thus F and L are found, the corresponding minimal square of the closed loop H_2 norm is given by:

$$J_{\min} = \text{tr}(B_1'P_cB_1) + \text{tr}(D_{12}FP_eF'D_{12}')$$

Also, once the matrices F and L are found, a transfer function for the controller can be written by using...

$$K(s) = C(sI - A)^{-1}B + D$$

Example 37 A state space plant of the general form has matrices:

$$A = a \quad B_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad B_2 = 1$$
$$C_1 = 0 \quad D_{11} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad D_{12} = 1$$
$$C_2 = 1 \quad D_{21} = \begin{bmatrix} b & 0 \end{bmatrix}$$

Substituting these matrices into the first completion of squares equation:

$$|u|^2 + 2x'P_c(ax + u) = |u - Fx|^2$$

Since the system is scalar, $x' = x$, and also $|n|^2 = n^2$ allows this expression to be simplified to:

$$u^2 + 2xP_c(ax + u) = (u - Fx)^2$$
$$u^2 + 2xP_c(ax + u) = u^2 - 2Fxu + F^2x^2$$
$$2P_cax^2 + 2xP_cu = -2Fxu + F^2x^2$$

Equating coefficients on both sides:

$$2P_ca = F^2 \quad \text{and} \quad P_c = -F$$

These equations need to be combined into the Riccati equation for P_c in terms of the plant parameters. This equation will have multiple solutions for P_c, and the stabilizing one that makes $A + B_2 F$ a Hurwitz matrix should be selected.

$$2P_ca = P_c^2$$

$$P_c(P_c - 2a) = 0$$

$$P_c = 0, \ 2a$$

If $a < 0$, the plant is stable, and no control is even needed to stabilize it. That is, the stabilizing solution is $F = 0$, giving $P_c = 0$. However, when $a > 0$ the plant is not stable,

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and the stabilizing solution $F = -2a$, and $P_c = 2a$ is used. In general, the larger value of P_c should be selected as the stabilizing solution.

$$
\begin{array}{lll}
\text{for } a < 0 & F = 0 & P_c = 0 \\
\text{for } a > 0 & F = -2a & P_c = 2a
\end{array}
$$

Substituting the plant matrices into the second completion of squares equation does not simplify as easily as the first equation.

$$|B_1'\psi + D_{21}'\xi|^2 + 2\psi P_e(A'\psi + C_2'\xi) = |D_{21}'(\xi - L'\psi)|^2$$

This equation is scalar, with $\psi \in \mathbb{R}$ and $\xi \in \mathbb{R}$? Comparing the two completion of square equations and looking at dimensions, ψ should have same dimensions as x, and ξ same dimensions as y. So, for this problem ψ and ξ are scalars, and so are L and P_e.

I am pretty sure the absolute value sign that Magretski uses is actually the vector 2-norm, making the completion of squares actually written:

$$\|B_1'\psi + D_{21}'\xi\|_2^2 + 2\psi P_e(A'\psi + C_2'\xi) = \|D_{21}'(\xi - L\psi)\|_2^2$$

substituting values from problem:

$$\left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \psi + \begin{bmatrix} b \\ 0 \end{bmatrix} \xi \right\|_2^2 + 2\psi P_e(a\psi + \xi) = \left\| \begin{bmatrix} b \\ 0 \end{bmatrix} (\xi - L\psi) \right\|_2^2$$

$$\left\| \begin{bmatrix} b\xi \\ \psi \end{bmatrix} \right\|_2^2 + 2\psi P_e(a\psi + \xi) = b^2(\xi - L\psi)^2$$

$$\left\| \begin{bmatrix} b\xi \\ \psi \end{bmatrix} \right\|_2 = \sqrt{(b\xi)^2 + \psi^2}$$

$$b^2\xi^2 + \psi^2 + 2\psi P_e(a\psi + \xi) = b^2(\xi - L\psi)^2$$

$$b^2\xi^2 + \psi^2 + 2aP_e\psi^2 + 2P_e\psi\xi = b^2(\xi - L\psi)^2$$

$$b^2\xi^2 + \psi^2 + 2aP_e\psi^2 + 2P_e\psi\xi = b^2(\xi^2 - 2L\psi\xi + L^2\psi^2)$$

$$b^2\xi^2 + \psi^2 + 2aP_e\psi^2 + 2P_e\psi\xi = b^2\xi^2 - 2b^2L\psi\xi + b^2L^2\psi^2$$

$$\psi^2 + 2aP_e\psi^2 + 2P_e\psi\xi = -2b^2L\psi\xi + b^2L^2\psi^2$$

$$\psi^2(1 + 2aP_e) + 2P_e\psi\xi = -2b^2L\psi\xi + b^2L^2\psi^2$$

Equating coefficients on both sides:

$$1 + 2aP_e = b^2L^2 \quad \text{and} \quad 2P_e = -2b^2L$$

Again combining these equations to find an equation for P_e with multiple solutions by first solving for L:

$$L = -\frac{P_e}{b^2}$$

$$L^2 = \frac{P_e{}^2}{b^4}$$

$$1 + 2aP_e = b^2\frac{P_e{}^2}{b^4}$$

$$1 + 2aP_e = \frac{P_e{}^2}{b^2}$$

$$P_e{}^2 - 2ab^2P_e - b^2 = 0$$

Solving this quadratic equation for P_e

$$P_e = \frac{2ab^2 \pm \sqrt{4a^2b^4 + 4b^2}}{2}$$

$$P_e = \frac{2ab^2 \pm \sqrt{4b^2(a^2b^2 + 1)}}{2}$$

$$P_e = \frac{2ab^2 \pm 2b\sqrt{a^2b^2 + 1}}{2}$$

$$P_e = ab^2 \pm b\sqrt{a^2b^2 + 1}$$

In general, select the larger value of P_e to be the stabilizing solution.

$$\boxed{P_e = ab^2 + b\sqrt{a^2b^2 + 1}}$$

Solve for L:

$$\boxed{L = -a - \frac{\sqrt{a^2b^2 + 1}}{b}}$$

Small Gain Theorem

For the feedback interconnection of LTI system P and Δ a memoryless system, the small gain theorem states that if the product of L_2 gains for P and Δ are less than one, that the closed loop L_2 gain will satisfy the following, where the L_2 gains are denoted by γ, and the closed loop system is G

$$\text{if} \quad \gamma_P\gamma_\Delta < 1 \quad \text{then:} \quad \gamma_G \leq \frac{\gamma_P}{1 - \gamma_P\gamma_\Delta}$$

25.2 Q-Parameterization

In this note Q-parameterization is used to express a given feedback control structure in terms of a different structure with feed-forward only. The closed-loop system G is to be expressed using Q-parameterization, where e can be any general output.
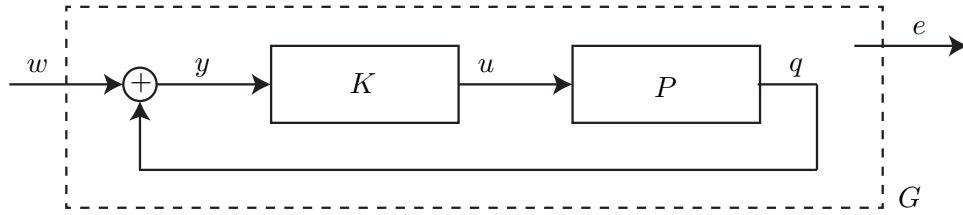
Figure 25.1: General feedback control block diagram for closed-loop system G

25.2.1 General Form of Plant

The plant P in the block diagram is given in its general form by:

Continuous Time

$$\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u \\
e &= C_1 x + D_{11} w + D_{12} u \\
y &= C_2 x + D_{21} w
\end{aligned}$$

Discrete Time

$$\begin{aligned}
x(t+1) &= Ax(t) + B_1 w(t) + B_2 u(t) \\
e(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
y(t) &= C_2 x(t) + D_{21} w(t)
\end{aligned}$$

Q-parameterization basically lets the closed loop plant from above be expressed as the following system. Then, from the equations for S_2 and S_1, the blocks G_0, G_1, and G_2 can be found.
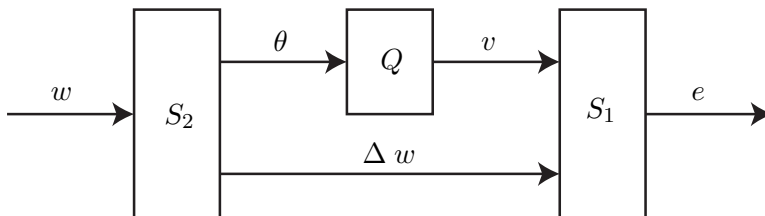
Figure 25.2: Feedback control block diagram
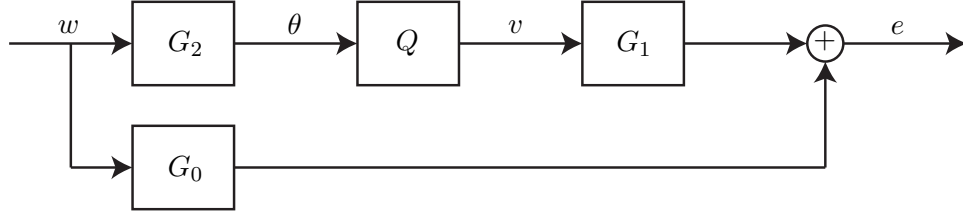
Figure 25.3: Feedback control block diagram

$$S_2: \begin{cases} \dot{\Delta} = (A + LC_2)\Delta + (B_1 + LD_{21})w \\ \theta = C_2\Delta + D_{21}w \end{cases}$$

$$S_1: \begin{cases} \dot{x} = (A + B_2F)x + B_2(v - F\Delta) + B_1w \\ e = (C_1 + D_{12}F)x + D_{12}(v - F\Delta) + D_{11}w \end{cases}$$

25.3   LQR-PI With Anti-windup
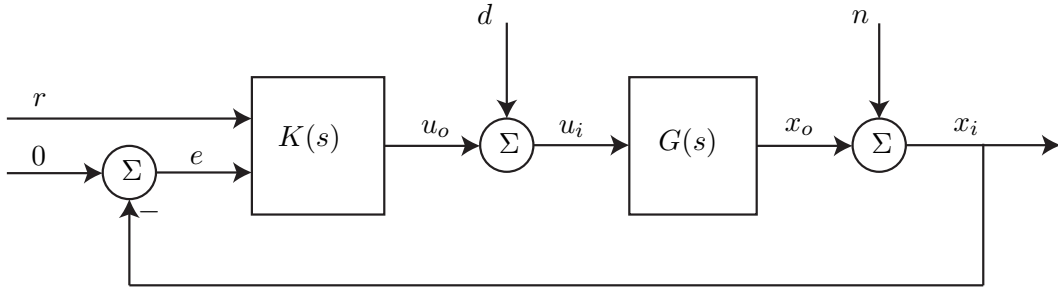
Figure 25.4: MIMO Control Block Diagram for LQR-PI

The plant which we would like to control is given in state-space form as

$$\dot{x}_p = A_p x_p + B_p u$$
$$y = C_p x_p + D_p u$$

where everything is scalar for the velocity plant subsystem. An LQR-PI controller can be represented in state-space form as

$$\dot{x}_e = A_c x_e + B_c e + B_r r$$
$$u = K_V e + K_e x_e$$

where $e = -V_T$ is the full state feedback. Since

$$\dot{x}_e = V_{T,\text{cmd}} - V_T$$

and $r = V_{T,\text{cmd}}$ the controller can be rewritten

$$\dot{x}_e = 0x_e - B_c V_T + B_r V_{T,\text{cmd}}$$
$$u = K_V V_T + K_e x_e$$

In order to implement anti-windup, logic needs to be written which will reset the integrator. This logic will require that the actual velocity is greater than the commanded velocity, and the throttle input is saturating for the integrator to be reset. This will keep integration error from accumulating, causing the throttle to continue to saturate, even while the commanded velocity has been exceeded. The question is what to reset the integrator to.

In a classical control feedback setup, where the integration error $e = r - y$ needs only to be reset to zero, with LQR-PI this is not the case. For LQR-PI the reset value is found by first determining the throttle input required to maintain the commanded velocity in equilibrium, based on the plant parameters. That is

$$\dot{V}_T = A_p V_T + B_p u_{th}$$

$$0 = A_p V_T + B_p u_{th}$$

want $V_T = V_{T,\text{cmd}}$

$$u_{th} = \frac{-A_p V_{T,\text{cmd}}}{B_p}$$

Then, knowing the required throttle to maintain equilibrium, the value of x_e that must exist at equilibrium to maintain this condition (since V_T, K_V, and K_e are known) can be found

$$u_{th} = K_V V_T + K_e x_e$$

again ith $V_T = V_{T,\text{cmd}}$

$$x_e = \frac{u_{th} - K_V V_{T,\text{cmd}}}{K_e}$$

which is the value the integrator must be reset to.

25.4   MIMO Zeros: Introduction

MIMO zeros for non square systems, MIMO zeros for square systems, how many zeros there will be, how squaring up by augmenting B moves existing zeros. Zeros of MIMO system not zeros of each TF in the transfer matrix. See 16.31 Lecture 8.

Definition 23 MIMO zeros   *Zeros of a MIMO system are ζ_0 such that $\lim_{s \to \zeta_0} (H(s)u(s)) = 0$.*

Example 38 The following system has a zero at $s = 3$.

$$H(s) = \begin{bmatrix} 1 & \frac{1}{s-3} \\ 0 & 1 \end{bmatrix} \qquad u(s) = \begin{bmatrix} -1 \\ s-3 \end{bmatrix}$$

u is finite.

$$\lim_{s \to 3} \left(\begin{bmatrix} 1 & \frac{1}{s-3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ s-3 \end{bmatrix} \right) = \lim_{s \to 3} \begin{bmatrix} 0 \\ s-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$H(s)$ has a pole at a frequency p_0 if some entry of $H(s)$ has a pole at $s = p_0$.

From DDV book: $H(s)$ has a zero at ζ_0 if it *drops rank* at $s = \zeta_0$. This particular defnition corresponds to what is termed a *transmission zero*.

$H(s)$ has full column rank if there is no rational vector $u(s) \neq 0$ such that $H(s)u(s) = 0$. At a transmission zero of $H(s)$, it will drop rank, and then there is a $u_0 \neq 0$ such that $H(\zeta_0)u_0 = 0$. The problem with this is if the MIMO system has a pole at the same frequency as a zero, and so this zero may not be detected.

MIMO transfer functions can have poles and zeros at the same frequency. So the refined zero definition is the limit one above. The above example is from DDV. We can see there is a pole at 3, but if we look in the limit as $s \to 3$ we see that the second column looks like the first one:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \infty \\ 1 \end{bmatrix}$$

(the ratio of the terms in each column are ∞) So we use the updated definition text and find the zero.

The following matrix is the Rosenbrock matrix.

$$R(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$

- $sI - A$ is full rank, except at system poles

- **Input decoupling zeros** are where the following matrix loses rank, which can only happen when $sI - A$ loses rank.
$$R_I(s) = \begin{bmatrix} sI - A & -B \end{bmatrix}$$
  Input decoupling zeros are a subset of system poles

- **Output decoupling zeros** are s where the following matrix loses rank
$$R_O(s) = \begin{bmatrix} sI - A \\ C \end{bmatrix}$$
  Output decoupling zeros are a subset of system poles

- **Invariant zeros** are values of s where $\text{rank}(S(s)) < \min\{n + \text{rank}(B), n + \text{rank}(C)\}$

- For a controllable and observable system, the invariant zeros are the same as the transmission zeros

Difference between invariant zeros and transmission zeros?
Difference between a system's transfer function $G(s) = C(sI - A)^{-1}B$ and its realization $\Sigma_p = \{A, B, C\}$?

Chapter 26

Passivity

The concept of passivity was first introduced by Popov. The essential feature of a passive system is its inability to increase its own energy [5]. For example, a network of passive components, e.g. inductors, resistors, and capacitors does not generate any energy and is therefore stable. Another example given in Reference [6] is a gravity tank into which water flows in to the top of the tank and out through the bottom. It is straightforward to show that the rate of change of stored energy in the tank is less than that supplied to it by the inlet flow rate. Passivity relates a system's input and output to the storage function and thus defines a set of useful input-output properties [6]. The following document repeats some definitions from the literature and presents an example of a passive and non-passive system using a mass-spring-damper.

Consider the following square system Σ:

$$
\begin{aligned}
\dot{x} &= f(x, u) \\
y &= h(x, u)
\end{aligned}
\tag{26.1}
$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $y \in \mathcal{Y} \subset \mathbb{R}^m$ and $u \in \mathcal{U} \subset \mathbb{R}^m$. The *Instantaneous Power Supply Rate* is denoted [7]

$$
w = w(u(t), y(t))
\tag{26.2}
$$

Assumption 2 [7], pp. 327 Given the system in (26.1) the instantaneous power supply rate $w(t)$ satisfies

$$
\int_{t_0}^{t_1} |w(\tau)| d\tau < \infty \qquad \forall t \geq 0
$$

Definition 24 Dissipative System and Storage Function *[7], pp. 327. The system in* (26.1) *with supply rate w is dissipative if there exists a positive semi-definite function $S(x) : \mathcal{X} \to \mathbb{R}^+$, called the Storage Function such that*

$$
S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} w(\tau) d\tau
\tag{26.3}
$$

for all $x \in \mathcal{X}$, $u \in \mathcal{U}$ and $t \geq 0$.

The inequality in (26.3) is called the *dissipation inequality*, and says that the stored energy $S(x(t_1))$ at time t_1 can never be greater than the sum of the initial energy in the system at time t_0 plus the energy into the system between t_0 and t_1. In other words, the system in (26.1) is dissipative if it is not generating any internal power: the increase in energy (storage function) during the interval (t_0, t_1) is no greater than the energy supplied (via the supply rate) to it.

Definition 25 Passive System *[8] pp. 1229 The system Σ in (26.1) in passive if it is dissipative with respect to the following supply rate*

$$w(u(t), y(t)) = y(t)^\top u(t) \qquad (26.4)$$

and the storage function S satisfies $S(0) = 0$.

Example 39 Mass-spring-damper Take a mass-spring-damper system, and let $m = 1$, $b = 1$, and $k = 1$. The transfer function from force to *velocity* is given by the following:

$$\frac{V(s)}{F(s)} = \frac{s}{s^2 + s + 1}$$

Given a sinusoidal input

$$u(t) = f(t) = \sin(\omega t) \qquad (26.5)$$

we can show the particular solution has the following form

$$y(t) = v(t) = A\sin(\omega t + \phi) \qquad (26.6)$$

where the phase shift $\phi \in (0, \pi/2)$. With this input and choice of output we use Definition 24 and show that the system is dissipative with respect to the following supply rate, as given in Definition 25.

$$w(u(t), y(t)) = A\sin(\omega t + \phi)\sin(\omega t) \qquad (26.7)$$

We evaluate the following integral from Definition 24 from t_0 to t_1

$$\int_{t_0}^{t_1} A\sin(\omega\tau + \phi)\sin(\omega\tau)d\tau$$

and using the identity

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a - b) - \cos(a + b)\right]$$

we can rewrite and then evaluate the integral as follows

$$\int_{t_0}^{t_1} A\sin(\omega\tau + \phi)\sin(\omega\tau)d\tau = \frac{1}{2}\int_{t_0}^{t_1}\cos(\phi) - \cos(2\omega\tau + \phi)d\tau$$

$$= \frac{1}{2}\cos(\phi)\tau\Big|_{t_0}^{t_1} - \frac{1}{4\omega}\sin(2\omega\tau + \phi)\Big|_{t_0}^{t_1}$$

$$= \frac{1}{2}\cos(\phi)(t_1 - t_0) - \left[\frac{1}{4\omega}\sin(2\omega t_1 + \phi) - \frac{1}{4\omega}\sin(2\omega t_0 + \phi)\right]$$

$$(26.8)$$

From this we can see the bracketed term is bounded and the first term is always positive because the phase shift between force and velocity is $\phi \in (0, \pi/2)$. This example is not sufficient to show that the mass-spring-damper with velocity as an output is or is not passive, but is useful to see.

However, if *position* is selected as the output, the transfer function is given by

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + s + 1}$$

If now this system is driven by a sinusoidal input as in (26.5) we can again show that the output is given by (26.6) where now $\phi \in (0, \pi)$. As before, the supply rate in Definition 25 is given by (26.7). We evaluate the integral from Definition 24 with this supply rate and get (26.8). The difference now is that $\phi \in (0, \pi)$ and so the first term in (26.8) for the case where *position* is the output is negative for all $\phi > \pi/2$, while the bracketed term remains bounded. So we can see that when driving the system with a sufficiently large input frequency, we can pick t_0 and t_1 such that no positive semi-definite storage function exists satisfying the inequality (26.3). Thus the spring-mass-damper system with force as an input and *position* as an output is not passive.

26.1   Difference Between Passive and Positive Real

For linear time invariant systems they are the same, as long as the system is detectable.[6] Passive systems are positive real. "The notion of Positive Real system may be seen as a generalization of the positive definiteness of a matrix to the case of a dynamical system with inputs and outputs" Reference [9] Positive-realness is a property of a function of a complex variable s. From Reference [6], pp. 14—15. The input-output property of passive systems is called positive realness. Passivity implies positive realness. For a detectable LTI system passivity is equivalent to positive realness. For LTI systems passive and positive real are the same.

Definition 26 Available Storage   *[7], pp. 327 The available storage, S_a of a dynamical system Σ with supply rate w is the function from X into R^e, where R^e is the extended real number system $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$, is defined by*

$$S_a(x) = \sup_{x \to t_1 \geq 0} - \int_0^{t_1} w(t)dt$$

where the notation $x \to$ denotes the supremum over all motions starting in state x at time 0 and where the supremum is taken over all $u \in \mathcal{U}$.

Theorem 11 *[7] pp. 328 The available storage, S_a, is finite for all $x \in X$ if and only if Σ is dissipative.*

Example 40 Mass with velocity output Consider a mass m with force f as the input and velocity v as the output. This system is represented as

$$f = m\dot{v}$$

or in the form of (26.9) with $x = v$, $u = f$, $y = v$, $f(x) = 0$, $g(x) = \frac{1}{m}$ and $h(x) = x$. This system is passive if it is dissipative with respect to $w = fv$. That is the inequality in

Definition 24 must be satisfied for all x, u, and $t \geq 0$. The mass starts from rest at time $t_0 = 0$ and is driven by a sinusoidal force input

$$f = \sin(t)$$

The velocity of the system is given by

$$v = \frac{1}{m} \int_0^t \sin(\tau)d\tau$$
$$= \frac{1}{m}(1 - \cos(t))$$

Look at the inequality in (26.10) with this particular choice of f and the corresponding v, the system is passive if it satisfies the following inequality

$$0 \leq \frac{1}{m} \int_0^t \sin(\tau)(1 - \cos(\tau))d\tau$$

This function is periodic and we can look at the integral over the first period and argue that this inequality is always satisfied.

Example 41 Mass with position output If we go through the same process as in Example 40 only now taking the position output, it is easy to show that our choice of f does not satisfy Definition 24 for all t, indicating that in this case the system is *not* passive.

$$f = m\ddot{x}$$

integrating, with zero initial conditions

$$\dot{x} = \frac{1}{m} \int_0^t f d\tau$$

This system is in the form of (26.1) with $u = f$ and $y = \dot{x}$. we choose $u = \sin(t)$,

$$\dot{x} = \frac{1}{m} \int_0^t \sin(\tau)d\tau$$
$$= -\frac{1}{m} \cos(\tau)\big|_0^t$$
$$= -\frac{1}{m}\big(\cos(t) - 1\big)$$

Integrating again

$$x = \frac{1}{m} \int_0^t 1 - \cos(\tau)d\tau$$
$$= \frac{1}{m}\big(t - \sin(t)\big)$$

Now we can let $m = 1$ and calculate the supply rate w as

$$w = ux$$
$$= \sin(t)\big(t - \sin(t)\big)$$

Using Definition 26 to determine the available storage

$$S_a(x) = \sup_{x \to t_1 \geq 0} - \int_0^{t_1} \sin(t)\big(t - \sin(t)\big) dt$$

Integrating this we get

$$S_a(x) = \sup_{x \to t_1 \geq 0} -\frac{1}{4}\big(- 2t + 4\sin(t) + \sin(2t) - 4t\cos(t)\big)\Big|_0^{t_1}$$

which is not finite as $t_1 \to \infty$. Thus, by Theorem 11 this system is not dissipative. Since it is not dissipative with respect to the supply rate $w = uy$, by Definition 25 this system is not passive.

26.2 More Stuff

Reference [10] gives some definitions of passive and SPR. If the storage function in (26.3) is differentiable, we can write (26.3) as

$$\frac{dS}{dt} \leq w(t)$$

which says that the rate of increase of system energy is no greater than the input power [6]. Consider now a special case of (26.1)

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x) \tag{26.9}$$

If we look at Definition 24 and Definition 25, the system in (26.9) with $x_0 = 0$ is passive if

$$0 \leq S(x(t)) \leq \int_0^t y^\top(\tau)u(\tau)d\tau \tag{26.10}$$

for all $x \in \mathcal{X}$, $u \in \mathcal{U}$ and $t \geq 0$.

Definition 27 Positive-Real System   *[8] pp. 1230 The system in* (26.9) *is positive-real if for all $t \geq 0$*

$$0 \leq \int_0^t y^\top(\tau)u(\tau)d\tau$$

whenever $x(0) = 0$.

Definition 28 Positive-Real System   *[11] pp. 63 A rational function $H(s)$ of the complex variable $s = \sigma + j\omega$ is PR if*

(i) $H(s)$ is real for real s

(ii) $Re[H(s)] \geq 0$ for all $Re[s] > 0$

The essential feature of a passive system is its inability to increase its own energy.[5] Think about a system that is just a mass. E is energy, P is power *into* the system, and power is the time rate of change of energy

$$\frac{d}{dt}[\text{stored energy}] = \text{external power input} + \text{internal power generation}$$

$$P = \frac{d}{dt}E$$

Integrating both sides

$$E(t) = E(0) + \int_0^t P(\tau)d\tau$$

A passive element is one for which $E(t) \geq 0$ for all t. For if not, this would mean that the integral of power is negative, even more negative than $E(0)$, meaning the system increased its own energy. If we think about our system that is a mass, power is given by force times velocity

$$E(t) = E(0) + \int_0^t f(\tau)v(\tau)d\tau$$

And we know this is true. However, the definition for passive systems is more complicated than this, as it is an input output property of a system, and thus depends on which inputs and outputs are used.

26.2.1 The Dissipation Inequality

We said before that a passive system is one that cannot deliver more energy than it has received. But when we say "energy", what we mean is the integral of an *instantaneous power supply rate*, which does not have to correspond to physical energy.

The definition of passivity relates the change in storage along solutions to the total supply, which is given by the integral of the supply rate, and the supply rate $y^\top u$ involves only the input and output. The *supply*, or *instantaneous power supply rate* $w(u, y)$ is given by

$$w(u, y) = y^\top u$$

$$\dot{S} = y^\top u - g$$

where g is used in Slotine's book Reference [12] as the negative of the internal power generation term, i.e. $g \geq 0$ for passive system. S is a *storage function*, and is positive-semidefinite. Since S need only be positive semi-definite, it does not make it necessarily a Lyapunov function. The storage function for a passive linear system must be quadratic.

26.2.2 How to Determine if a System is Passive

Definition 29 *a strictly stable linear SISO is passive if and only if*

$$\forall \omega \geq 0, \ Re[h(j\omega)] \geq 0$$

Definition 30 *a strictly stable linear SISO is passive if and only if its phase shift in response to a sinusoidal input is always less than or equal to $90°$.*

Properties

1. Necessary conditions for passivity: system be minimum phase and relative degree 0 or 1.

2. Passive system are not necessarily stable. This is only the case if a positive-definite storage function is used, but since the storage function only needs to be positive-semi-definite, stability is not always ensured by passivity [6].

The poles of LTI passive systems have negative real parts.

26.2.3 Interconnections of Passive Systems

Parallel and feedback interconnections of passive systems are passive.[9]

"If the system $Z(s)$ is SPR, this implies that the system remains stable for any positive static gain, even arbitrarily large, i.e. $Z(s)/(1 + kZ(s))$ is stable for any $k \geq 0$."

26.2.4 Positive Real Systems in Adaptive Control

"The choice of the Lyapunov function is simplified substantially when the transfer function of the relevant linear time-invariant system is strictly positive real." [13]

$$\dot{e}_x = (A + LC + B\Psi^\top)e_x + B\Lambda \widetilde{\Theta}^\top x_m$$
$$e_y = Ce_x$$

26.3 Householder Transformation

A Householder transformation is a linear transformation that can be used to transform a given matrix into one which is upper diagonal. This transformation is Hermitian and unitary. For operations on the matrices describing a linear state space system, all entries will be real valued, and so in the rest of this document we assume the Householder transformation matrix H is real valued, and give its properties with this assumption.

$$H^\top = H^{-1}$$

When applying the Householder transformation we essentially want to find the matrix $H \in \mathbb{R}^{n \times n}$ such that given a vector $x \in \mathbb{R}^n$, when multiplied by H gives the following

$$Hx = y$$

where

$$y = \begin{bmatrix} \hat{x} & 0 \ldots & 0 \end{bmatrix}^\top$$
$$= \hat{x}\mathbf{e}_1^\top$$

where $\mathbf{e}_1$ is the standard basis vector

$$\mathbf{e}_1 = \begin{bmatrix} 1 & 0 \ldots & 0 \end{bmatrix}^\top$$

26.3.1 Properties

The Householder matrix is a matrix of the following form

$$H = I - 2ww^\top$$

where $w \in \mathbb{R}^{n \times 1}$ is a unit vector. Evaluate the following

$$
\begin{aligned}
H^\top H &= (I - 2ww^\top)^\top (I - 2ww^\top) \\
&= (I - 2ww^\top)(I - 2ww^\top) \\
&= I - 2ww^\top - 2ww^\top + 4ww^\top ww^\top \\
&= I - 4ww^\top + 4w(w^\top w)w^\top \\
&= I
\end{aligned}
$$

From this we see that the following property is in fact satisfied by H

$$H^\top = H^{-1}$$

Suppose we pick the unit vector w as follows

$$w = \frac{u}{\|u\|}$$

where

$$u = x - \|x\|\mathbf{e}_1$$

giving

$$w = \frac{x - \|x\|\mathbf{e}_1}{\|x - \|x\|\mathbf{e}_1\|}$$

This gives

$$
\begin{aligned}
Hx &= (I - 2ww^\top)x \\
&= x - 2\left(\frac{x - \|x\|\mathbf{e}_1}{\|x - \|x\|\mathbf{e}_1\|}\right)\left(\frac{x - \|x\|\mathbf{e}_1}{\|x - \|x\|\mathbf{e}_1\|}\right)^\top \\
&= \\
&= \|x\|\mathbf{e}_1
\end{aligned}
$$

So we have defined H, showed that it is an orthogonal transformation, and showed how to pick w so that $Hx = \hat{x}\mathbf{e}_1$. The following section will clearly lay out the steps to code this algorithm for taking a matrix and making it upper diagonal.

26.3.2 Applying Householder Transformation

Given a matrix $B \in \mathbb{R}^{N \times M}$

$$B = \begin{bmatrix} B_1 & B_2 & \dots & B_m \end{bmatrix}$$

where each $B_i \in \mathbb{R}^{N \times 1}$. We apply the Householder transformation to the first column first, and the proceed with the remaining columns. Define

$$\mathbf{e}_1 \in \mathbb{R}^{n \times 1}$$

Let

$$x = B_i(i : N, i)$$

Note each $x \in \mathbb{R}^{n \times 1}$, where $n = N - i$. Calculate

$$\hat{x} = \|x\|$$

Calculate

$$w = \frac{x - \|x\|\mathbf{e}_1}{\|x - \|x\|\mathbf{e}_1\|}$$

Basically the idea is that at transforming the columns of B in each step the calculated Householder matrix decreases in size, but we need to keep the matrix $N \times N$, so that when we multiply $H_2 H_1 B$, that H_2 will not alter the first column of $H_1 B$.

$$H_1 B = \begin{bmatrix} \hat{x}_1 \mathbf{e}_1 & B_2 & \dots & B_m \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 1 & 0 \\ 0 & H_{\text{temp},2} \end{bmatrix}$$

Then, to find the matrix T, which will transform B to an upper diagonal form, multiply the H matrices together as

$$T = H_m \cdots H_2 H_1$$

And we can show that T is also an orthogonal transformation. Because each $H_i^\top = H_i^{-1}$ and so

$$\begin{aligned}
T^\top &= (H_m \cdots H_2 H_1)^\top \\
&= H_1^\top H_2^\top \cdots H_m^\top \\
&= H_1^{-1} H_2^{-1} \cdots H_m^{-1} \\
&= (H_m \cdots H_2 H_1)^{-1} \\
&= T^{-1}
\end{aligned}$$

26.4   Controls Quals Notes
26.4.1   Frequency Domain

Frequency response method: give a system a sinusoidal input. The output (will always be?) a sinusoid, with a magnitude and phase which may be different than the input, but the frequency will be the same. Then, sweep the input frequency across a wide range, and observe how the gain and phase shift of the measured output change with frequency.

26.4.2   Gain and Phase of Poles and Zeros

Why do zeros add phase and poles reduce phase? These two examples that follow connect the transfer function representation of the integrator and differentiator to a differential equation and show how, when the system is given a sinusoidal input, that the zero causes the output to lead the input, and the pole causes the output to lag behind the input.

$$\frac{x}{u} = \frac{1}{s}$$

The differential equation is

$$\dot{x} = u$$

This whole thing of gain and phase is for a given sinusoidal input, so $u(t) = A\sin(\omega t)$ and then we want to see how that will affect the output $x(t)$ for this system.

$$\frac{dx}{dt} = A\sin(\omega t)$$

separate and integrate back

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} A\sin(\omega t)dt$$

giving

$$x - x_0 = -\frac{A}{\omega}\cos(\omega t) + \frac{A}{\omega}\cos(\omega t_0)$$

$$x(t) = -\frac{A}{\omega}\cos(\omega t) + \frac{A}{\omega}\cos(\omega t_0) + x_0$$

$$x(t) = -\frac{A}{\omega}\cos(\omega t)$$

but $-\cos(x) = \sin(x - 90°)$ so the input and corresponding output can be written

$$u(t) = A\sin(\omega t)$$

$$x(t) = \frac{A}{\omega}\sin(\omega t - 90°)$$

So we can see that the magnitude of the output depends inversely on ω, getting smaller as ω gets larger. More importantly we can see clearly that there is a $90°$ phase lag from the input to the output.

Consider now a differentiator.

$$\frac{x}{u} = s$$

gives

$$x = \dot{u}$$

and with $u(t) = A\sin(\omega t)$ we have

$$x(t) = A\omega\cos(\omega t)$$

together the input and output are

$$u(t) = A\sin(\omega t)$$
$$x(t) = A\omega\cos(\omega t)$$

But we have $\cos(x) = \sin(x + 90°)$ giving

$$u(t) = A\sin(\omega t)$$
$$x(t) = A\omega\sin(x + 90°)$$

and here we can see that there is a positive phase shift, or say that the output leads the input by 90 degrees.

26.4.3 How to Make Bode Plots

first normalize the transfer function, so all the factors look like $(1 + sa_i)$

Part IV

Adaptive Control

Chapter 27

Introduction to Adaptive Control

What is adaptive control? Why adaptive control? Explain in simple terms some of the benefits of adaptive control, and given an overview of some applications that are well suited to adaptive control. Provide some neat and simple motivating examples.

Adaptive Control — the control of plants with unknown parameters. These notes will only cover continuous-time adaptive control.

Explain tracking error, and parameter error.

Also explain how our stability proofs are centered around driving the tracking error to zero (and the parameter error too, if we can) and so our stability proofs look at stability of the origin.

27.1   Preliminaries

Include basic stuff here on set notation and more. Many students may not have seen this before.

$\mathbb{R}$ set of real numbers $\mathbb{R}^n$ set of real-valued vectors with length n. Taken by convention to mean column vector? $\mathbb{R}^{n \times m}$ set of real-valued matrices of size $n \times m$.

27.1.1   Classes of Systems

In the following systems, x, y, and u are the state, output, and input, respectively. θ is an unknown parameter. The following is a nonlinear time varying system

$$\dot{x} = f(x, u, \theta, t)$$
$$y = h(x, u, \theta, t)$$

The following is a linear time varying (LTV) system

$$\dot{x} = A(\theta, t)x + B(\theta, t)u$$
$$y = H(\theta, t)x$$

The following is a linear time invariant (LTI) system.

$$\dot{x} = A(\theta)x + B(\theta)u$$
$$y = H(\theta)x$$

In this class, we will mostly look at LTI systems

27.1.2 Four/Five Classes of Adaptation

1. Passive

2. Input-Signal (Characteristic)

3. System-Variable

4. Extremum
    (a) Parameter-perturbation method
    (b) Sensitivity method
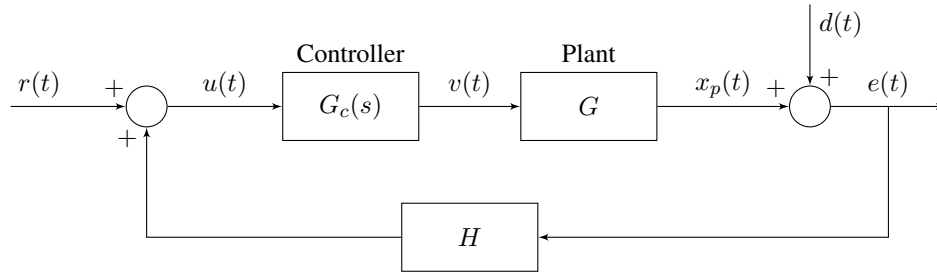
Extremum Adaptation

Idea in extremum adaptation: J cost function, θ is an adjustable parameter. Details regarding the determination of the gradient... The changing of theta changes the cost. θ like a knob, J like a gauge... dial the knob to minimize the reading on the gauge... Adaptive law: (MIT Rule)

$$\dot{\theta} \propto \frac{\partial J}{\partial \theta} \quad \text{(gradient)}$$

27.1.3 Control Goal

Given unknown plant P with parameter θ_p, design controller C with parameter θ_c. C is kind of like the inverse of P.

27.1.4 Direct and Indirect Adaptive Control

Why parameter estimation? In a control system, if a plant parameter is known, a suitable stabilizing controller can be selected based on the known plant parameter. When the plant parameter is unknown, can estimate the plant parameter, and then use this estimate to find the control gain. Essentially, direct and indirect adaptive control are described by the following.

Indirect Adaptive Control: θ_p is unknown: estimate θ_p as $\hat{\theta}_p$, then compute $\theta_c = f(\hat{\theta}_p)$

Direct Adaptive Control: θ_c unknown. Estimate $\hat{\theta}_c$ — direct adaptive control

27.1.5 Motivation for Reference Model

Consider the following transfer function representation first order plant with input u and output x_p.

$$\frac{x_p}{u} = \frac{k_p}{s - a_p}$$

Our goal is to make the plant output x_p follow some desired output x_d. We define the *tracking error*, the difference between the actual output and desired output as

$$e = x_p - x_d$$

This is represented in the following block diagram.
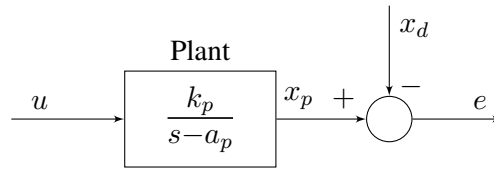
In order to achieve the goal, we must choose the control input u so that $e \to 0$. However, x_p can not follow any arbitrary command we would like, for example a series of steps, or some other non-smooth commands. Instead, we must ask the system to do something which it can actually do, which in this case a first order system can at best follow first order responses. This is where the need for the reference model comes in. We transform the problem of shaping the input to shaping the reference model so that x_m is as close to x_d as possible. The reference model is

$$\frac{x_m}{x_d} = \frac{k_m}{s - a_m}$$

Now we modify our goal slightly based on the statement above, where we don't try to track x_d, instead we try to track x_m. That is find u so that e goes to zero, where now the tracking error is

$$e = x_p - x_m$$

In the case of indirect adaptive control, we may not use the reference model output or the error signal above in the process of controlling the plant, but we keep the reference model in mind as basically the model describing where we want to put the poles of the closed-loop plant.

27.2   Direct and Indirect Adaptive Control

The following example talks a little bit about the difference between direct and indirect adaptive control.

Example 42 Direct and indirect adaptive control Consider the following transfer function representations of a plant and reference model below

$$\frac{x_p}{u} = \frac{k_p}{s - a_p}$$

$$\frac{x_m}{x_d} = \frac{k_m}{s - a_m}$$

in state space these are represented as

$$\dot{x}_p(t) = a_p x_p(t) + k_p u(t)$$
$$\dot{x}_m(t) = a_m x_m(t) + k_m r(t)$$

Assume for this example that $k_p = k_m = 1$, $a_p = 1$, $a_m = -1$ giving

$$\frac{x_p}{u} = \frac{1}{s-1}$$

$$\frac{x_m}{x_d} = \frac{1}{s+1}$$

With this knowledge, we can see that we need to shift the plant pole from 1 to -1 to match the reference model. Fortunately we can do this through feedback. The goal is to find the feedback gain θ that will place the pole of the plant so it matches the pole of the reference model. This will ensure $e \to 0$.
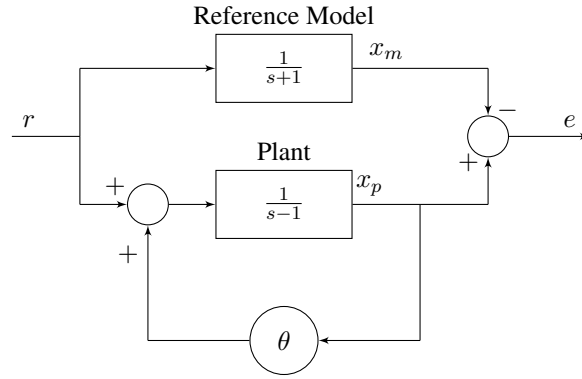
The closed loop transfer function is

$$\frac{x_p}{r} = \left.\frac{k_p}{s - a_p - k_p\theta}\right|_{k^*\theta^*}$$

substituting numerical values we get

$$\frac{x_p}{r} = \frac{1}{s - 1 - \theta}$$

From this expression, we can see that using a feedback gain of $\theta = -2$ will place the closed-loop plant pole to match the reference model pole, and will result in (after initial conditions) perfect tracking. Basically, if we knew what a_p was, we could then back-calculate the required value of θ to place the pole where we wanted. This problem of trying to identify a_p and then using this estimate to determine θ is called **indirect control**.

In direct control, we need not know, or attempt to determine what the value of a_p is, but we know there exists a feedback gain θ that will put the pole in the right place. Once the pole is in the right place, the tracking error $e \to 0$. The problem of **direct control** is to adjust θ based on the tracking error output e alone.

If we wait and identify a_p, we can then back-calculate what θ needs to be. This is indirect adaptive

control. $\theta = -\hat{a}_p + a_m$. In Direct adaptive control, all we need to know is that the ideal parameter that results in perfect tracking *exists*, and we seek instead to identify θ. Identify θ as $\hat{\theta}$. to adjust DC gain. Define θ^* and k^* as the values of θ, k so that plant + controller = reference model.

The closed loop transfer function of the following plant

$$\frac{x_p}{u} = \frac{k_p}{s + a_p}$$

with control law $u = \theta x_p + kr$ is given as follows, where the ideal values k^* and θ^* are used for the adaptive gains

$$\frac{x_p}{r} = \left.\frac{kk_p}{s - a_p - k_p\theta}\right|_{k^*\theta^*}$$

matching condition

$$k^* = \frac{k_m}{k_p}$$

$$\theta^* = \frac{a_p - a_m}{k_p}$$

so if a_p and k_p are unknown, then θ^* and k^* unknown. In the direct control approach, seek to identify control parameters θ^* and k^* as θ and k. but we need to adjust $\theta(t)$ and $k(t)$ so that $\theta(t) \to \theta^*$ and $k(t) \to k^*$. But his is hard to realize since θ^* and k^* are unknown. So now modify the goal so that instead we adjust θ and k so that $e(t) \to 0$. Now that controller is time varying, no more transfer functions, no Laplace operator... Laplace transforms don't exist.

27.2.1 Direct Adaptive Control

This section will introduce direct adaptive control using the following example. In this example, there is no attempt to identify the unknown plant parameter, and the feedback gain is adjusted using only the tracking error between the reference model and actual plant responses.

Example 43 Direct adaptive control of a scalar system with one unknown The plant and reference model are represented below

$$\dot{x}_p(t) = a_p x_p(t) + k_p u(t)$$
$$\dot{x}_m(t) = a_m x_m(t) + k_m r(t)$$

where

k_p known

a_p unknown

We would like to design a control input u such that the plant state x_p follows the reference model state x_m in the presence of the unknown parameter a_p. That is, we want $\lim_{t\to\infty} e(t) = 0$, where the error is defined as $e(t) = x_p - x_m$. The block diagram for this problem is shown below.
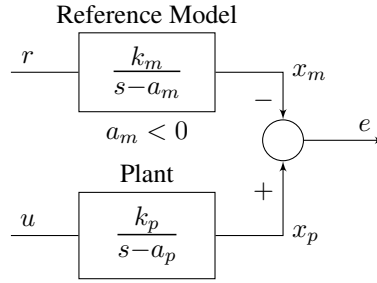
We propose the following controller

$$u(t) = \theta(t)x_p(t) + k^*r(t)$$

which can be represented in the following block diagram
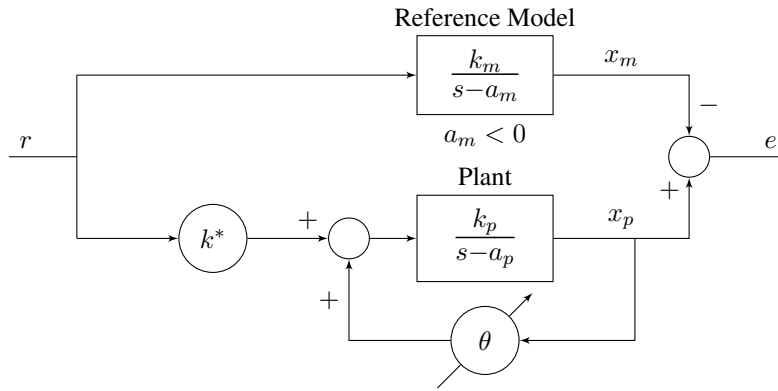
Substitute the proposed control law into the plant equation

$$\dot{x}_p(t) = \big(a_p + k_p\theta(t)\big)x_p(t) + k_pk^*r(t)$$

Recall that we define θ^* as the fixed values of $\theta(t)$ such that the closed-loop plant matches the reference model. This is known as the matching condition. That is, we compare the plant to the reference model and solve the following expressions for θ^*. Since k_p is known, we know k^* and can use that directly in the control law.

$$a_p + k_p\theta^* = a_m$$
$$k_pk^* = k_m$$

Solving, we obtain

$$k^* = \frac{k_m}{k_p}$$
$$\theta^* = \frac{a_m - a_p}{k_p}$$

We define the parameter error as the difference between the actual parameter value, and the "ideal" parameter value

$$\tilde{\theta}(t) = \theta(t) - \theta^*$$

Using $\theta(t) = \theta^* + \tilde{\theta}(t)$ and substituting we get

$$\dot{x}_p(t) = (a_p + k_p\theta^* + k_p\tilde{\theta}(t))x_p + k_m r(t)$$

Using $a_p + k_p\theta^* = a_m$ we get

$$\dot{x}_p(t) = a_m x_p(t) + k_p\tilde{\theta}(t)x_p(t) + k_m r(t)$$

Defining tracking error as

$$e(t) = x_p(t) - x_m(t)$$

and differentiating

$$\dot{e}(t) = \dot{x}_p(t) - \dot{x}_m(t)$$
$$= a_m x_p(t) + k_p\tilde{\theta}(t)x_p(t) + k_m r(t) - a_m x_m(t) - k_m r(t)$$
$$= a_m e(t) + k_p\tilde{\theta}(t)x_p(t)$$

And we can recognize this as error model 3, as shown in the following block diagram.
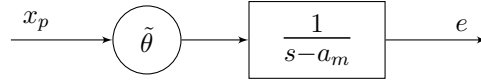
We now propose the following candidate Lyapunov function in order to prove stability of this adaptive system

$$V(e(t), \tilde{\theta}(t)) = \frac{1}{2}e^2(t) + \frac{1}{2}\tilde{\theta}^2(t)$$

Differentiating along system trajectories we get

$$\dot{V}(e(t), \tilde{\theta}(t)) = e(t)\dot{e}(t) + \tilde{\theta}(t)\dot{\tilde{\theta}}(t)$$
$$= e(t)\left(a_m e(t) + k_p\tilde{\theta}^\top(t)x_p(t)\right) + \tilde{\theta}(t)\dot{\tilde{\theta}}(t)$$
$$= a_m e^2(t) + e(t)k_p\tilde{\theta}^\top(t)x_p(t) + \tilde{\theta}(t)\dot{\tilde{\theta}}(t)$$

We propose now the following adaptive parameter update law

$$\boxed{\dot{\tilde{\theta}}(t) = -e(t)k_p x_p(t)}$$

Substituting this into $\dot{V}$ we get

$$\dot{V}(e(t), \tilde{\theta}(t)) = a_m e^2(t) + e(t)k_p\tilde{\theta}(t)x_p(t) - e(t)k_p\tilde{\theta}(t)x_p(t)$$
$$= a_m e^2(t)$$

And since $a_m < 0$, $\dot{V} \leq 0$, i.e. negative semi-definite. While it may appear that $\dot{V}$ is negative definite, as it is a negative quadratic, we must remember that V, and thus its derivative are functions of both e and $\tilde{\theta}$, and so $\dot{V}$ may be zero when its inputs are nonzero.

The previous example illustrated a basic adaptive law on a scalar plant with one unknown parameter. In the next example, we will consider the case where the plant has two unknown parameters, but the sign of the input parameter is known. This will require a modified Lyapunov function and parameter update law in order to prove stability.

In addition, this introductory example to direct adaptive control showed only stability in the sense of Lyapunov. That is, we showed only that the errors would remain bounded. In direct adaptive control examples to follow, we will include more thorough stability analysis, and show that the tracking error does tend to zero.

We return now to direct adaptive control to show more general adaptive controllers that can handle additional unknown plant parameters, as well as introduce additional tuning tools. From this point onward, explicit time dependency of different quantities will be dropped, only being used in order to emphasize this dependency.

Example 44 Direct adaptive control of a scalar system with two unknowns The plant and reference model are represented below, where a_p and k_p are unknown, but the sign of k_p is known.

$$\dot{x}_p = a_p x_p + k_p u$$
$$\dot{x}_m = a_m x_m + k_m r$$

where

<div align="center">

k_p unknown (but with known sign)

a_p unknown

</div>

We would like to design a control input u such that the plant state x_p follows the reference model state x_m in the presence of the unknown parameters a_p and k_p. That is, we want $\lim_{t \to \infty} e(t) = 0$, where the error is defined as $e(t) = x_p - x_m$. The block diagram for this problem is shown below.
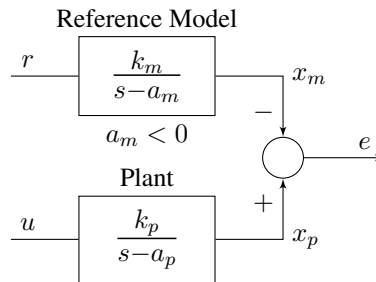
We propose the following controller

$$u = \theta x_p + kr$$

which can be represented in the following block diagram
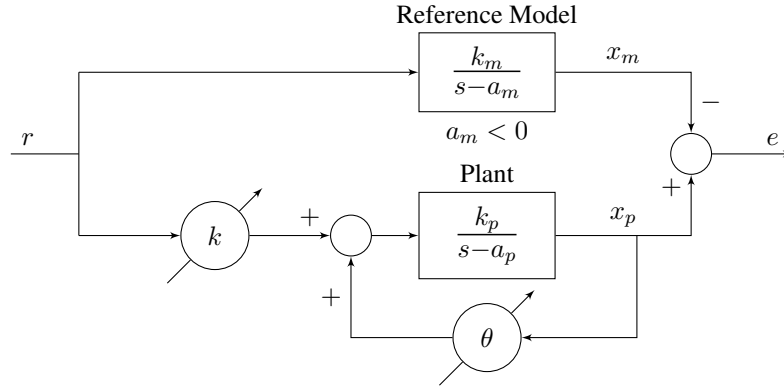
Substitute the proposed control law into the plant equation

$$\dot{x}_p(t) = (a_p + k_p\theta)x_p + k_pkr$$

Recall that we define θ^* and k^* as the fixed values of $\theta(t)$ and $k(t)$, respectively, such that the closed-loop plant matches the reference model. This is known as the matching condition. That is, we compare the plant to the reference model and solve the following expressions for θ^* and k^*.

$$a_p + k_p\theta^* = a_m$$
$$k_pk^* = k_m$$

Solving, we obtain

$$k^* = \frac{k_m}{k_p}$$
$$\theta^* = \frac{a_m - a_p}{k_p}$$

We define the parameter error as the difference between the actual parameter value, and the "ideal" parameter value

$$\tilde{\theta} = \theta - \theta^*$$
$$\tilde{k} = k - k^*$$

Using $\theta = \theta^* + \tilde{\theta}$ and $k = k^* + \tilde{k}$ and substituting we get

$$\dot{x}_p = (a_p + k_p\theta^* + k_p\tilde{\theta})x_p + k_p(k^* + \tilde{k})r$$

Using $a_p + k_p\theta^* = a_m$ and $k_pk^* = k_m$ we get

$$\dot{x}_p = a_mx_p + k_p\tilde{\theta}x_p + k_mr + k_p\tilde{k}r$$

Define the parameter $\bar{\theta}$ as the column vector which contains both parameters θ and k. From this point forward we may drop the explicit time dependence on some terms for ease of exposition.

$$\bar{\theta}^\top = \begin{bmatrix} \theta & k \end{bmatrix}$$

and the error

$$\tilde{\bar{\theta}}^\top = \begin{bmatrix} \tilde{\theta} & \tilde{k} \end{bmatrix} = \begin{bmatrix} \theta & k \end{bmatrix} - \begin{bmatrix} \theta^* & k^* \end{bmatrix}$$

or $\tilde{\bar{\theta}} = \bar{\theta} - \bar{\theta}^*$. The regressor

$$\phi^\top = \begin{bmatrix} x_p & r \end{bmatrix}$$

The plant equation can then be expressed as

$$\dot{x}_p = a_m x_p + k_m r + k_p \tilde{\bar{\theta}}^\top \phi$$

Defining tracking error as

$$e = x_p - x_m$$

and differentiating

$$\dot{e} = \dot{x}_p - \dot{x}_m$$
$$= a_m x_p + k_m r + k_p \tilde{\bar{\theta}}^\top \phi - a_m x_m - k_m r$$
$$= a_m e + k_p \tilde{\bar{\theta}}^\top \phi$$

which we again recognize as error model 3 as shown in the block diagram below.
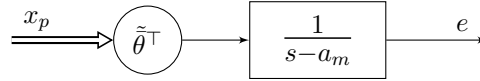
We propose the following candidate Lyapunov function in order to prove stability of this adaptive system

$$V(e, \tilde{\bar{\theta}}) = \frac{1}{2}e^2 + \frac{1}{2}|k_p|\tilde{\bar{\theta}}^\top \tilde{\bar{\theta}}$$

Differentiating along system trajectories we get

$$\dot{V}(e, \tilde{\bar{\theta}}) = e\dot{e} + |k_p|\tilde{\bar{\theta}}^\top \dot{\tilde{\bar{\theta}}}$$
$$= e(a_m e + k_p \tilde{\bar{\theta}}^\top \phi) + |k_p|\tilde{\bar{\theta}}^\top \dot{\tilde{\bar{\theta}}}$$
$$= a_m e^2(t) + e k_p \tilde{\bar{\theta}}^\top \phi + |k_p|\tilde{\bar{\theta}}^\top \dot{\tilde{\bar{\theta}}}$$

We propose now the following adaptive parameter update law

$$\boxed{\dot{\tilde{\bar{\theta}}} = -\text{sgn}(k_p)e\phi}$$

Substituting this into $\dot{V}$ we get

$$\dot{V}(e, \tilde{\bar{\theta}}) = a_m e^2 + e k_p \tilde{\bar{\theta}}^\top \phi - |k_p|\text{sgn}(k_p)e\tilde{\bar{\theta}}^\top \phi$$
$$= a_m e^2 + e k_p \tilde{\bar{\theta}}^\top \phi - e k_p \tilde{\bar{\theta}}^\top \phi$$
$$= a_m e^2$$

And since $a_m < 0$, $\dot{V} \leq 0$, i.e. is negative semi-definite.

Error Convergence

In the direct adaptive control example in this section and the previous section we have proposed stable update laws, and used a Lyapunov function with $\dot{V} \leq 0$ to show stability of the adaptive system. However, in these two examples, we have said nothing about whether our control goal of $e(t) \to 0$ as $t \to \infty$ was actually achieved. In this section, some additional tools are provided to allow us to show the convergence of the error to zero.

Considering Example 44, we left off with $\dot{V} \leq 0$. We now need to use Barbalat's lemma to show that $\lim_{t \to \infty} e(t) = 0$, and to do that, we need to find some various signal norms, described in what follows. Since $V > 0$ and $\dot{V} \leq 0$, we have $V(t) \leq V(0) < \infty$. Thus $V(t)$ is bounded, and so its arguments e and $\tilde{\theta}$ must be bounded also. Since r is bounded and the reference model is stable, x_m is bounded, and so we get that x_p is bounded. This can be compactly stated as $e,\ x_p,\ \tilde{\theta} \in \mathcal{L}_\infty$.

To apply Barbalat's lemma, all we have left to do is show $e \in \mathcal{L}_2$. To do this, note that

$$\int_0^t \dot{V}(\tau)d\tau = V(t) - V(0)$$

Since V is non increasing and positive definite, $V(0) - V(t) \leq V(0)$. This gives

$$-\int_0^t \dot{V}(\tau)d\tau \leq V(0)$$

Substituting in our expression for $\dot{V} = a_m e^2$, remembering that $a_m < 0$

$$|a_m| \int_0^t e^2(\tau)d\tau \leq V(0)$$

which is equivalent to

$$|a_m| \int_0^t \|e(\tau)\|^2 d\tau \leq V(0) < \infty$$

which simplifies to

$$\sqrt{\int_0^t \|e(\tau)\|^2 d\tau} < \infty$$

Recognize that this is just $\|e(t)\|_{L_2} < \infty$ we write $e \in \mathcal{L}_2$. Finally, we need to show the boundedness of $\dot{e}$ so we can apply Barbalat's lemma. In addition to the boundedness of the signals shown above, we also assume the reference input is bounded: $r \in \mathcal{L}_\infty$. So, looking at the error dynamics with $\phi,\ \tilde{\theta} \in \mathcal{L}_\infty$ we see that $\dot{e} \in \mathcal{L}_\infty$, and so the conditions to apply Barbalat's lemma are met, so $\lim_{t \to \infty} e(t) = 0$.

27.2.2   Indirect Adaptive Control

In indirect adaptive control, we will now create an identifier which we use to estimate the plant parameters. From these estimates of the plant parameters we determine the desired control gain. The structure of the identifier is motivated by the standard linear observer below. Assuming the plant parameters were known, the observer equation would be written as

$$\dot{\hat{x}}_p = a_p \hat{x}_p + k_p u + k_f(x_p - \hat{x}_p)$$

where k_f is the observer gain that is selected to give stable observer dynamics. However, if we knew the plant parameters we could pick the appropriate control gains to place the closed-loop poles as desired. The observer equation is modified to use instead the plant parameter estimates, and we call the following the identifier.

$$
\begin{aligned}
\dot{\hat{x}}_p &= \hat{a}_p \hat{x}_p + \hat{k}_p u + k_f (x_p - \hat{x}_p) \\
&= (\hat{a}_p - k_f)\hat{x}_p + k_f x_p + \hat{k}_p u
\end{aligned}
$$

We then choose k_f so the identifier is stable. Choose $k_f = \hat{a}_p - a_m$ giving

$$
\boxed{\dot{\hat{x}}_p = a_m \hat{x}_p + (\hat{a}_p - a_m)x_p + \hat{k}_p u}
$$

So now the plant, identifier, and reference model are the following. Note that we won't actually use the reference model for control, but maintain it as a model as to where we would ultimately like to place the closed-loop poles of the plant.

$$
\begin{aligned}
\dot{x}_p(t) &= a_p x_p(t) + k_p u(t) \\
\dot{\hat{x}}_p(t) &= a_m \hat{x}_p(t) + (\hat{a}_p(t) - a_m)x_p(t) + \hat{k}_p(t)u(t) \\
\dot{x}_m(t) &= a_m x_m(t) + k_m r(t)
\end{aligned}
$$

We now define the estimation error

$$
e_i(t) = x_p(t) - \hat{x}_p(t)
$$

and use the control law

$$
u(t) = \theta(t)x_p(t) + k(t)r(t)
$$

So now the identifier is used to determine parameter estimates $\hat{a}_p$ and $\hat{k}_p$, and the problem becomes how to use these parameter estimates to use in for finding $\theta(t)$ and $k(t)$ to use in the control law.

Algebraic Parameter Adjustment   The first method to determine the control parameters is algebraic. Substituting the control law into the plant equation above, we recall that the matching condition is the existence of the "ideal" adaptive gains such that the closed-loop plant matches the reference model. If we knew what the plant parameters were, we could calculate θ and k in the control law using the matching condition. In indirect adaptive control, we take the same approach, but use instead the parameter estimates.

$$
k = \frac{k_m}{\hat{k}_p}
$$

$$
\theta = \frac{a_m - \hat{a}_p}{\hat{k}_p}
$$

Dynamic Parameter Adjustment   Talk about dynamic adjustment of the control parameters here, and refer to page 121. Use the following matching-like errors.

$$
\begin{aligned}
\epsilon_\theta &= \hat{a}_p + \hat{k}_p \theta - a_m \\
\epsilon_k &= \hat{k}_p k - k_m
\end{aligned}
$$

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Identifier errors

$$e_i = x_p - \hat{x}_p$$
$$\dot{\hat{a}}_p = e_i x_p - \epsilon_\theta$$
$$\dot{\hat{k}}_p = e_i u - \theta \epsilon_\theta - k \epsilon_k$$

The plant and identifier equation are

$$\dot{x}_p = a_p x_p + k_p \theta x_p + k_p k r$$
$$\dot{\hat{x}}_p = a_m \hat{x}_p + (\hat{a}_p - a_m) x_p + \hat{k}_p u$$

Parameter errors

$$\tilde{a}_p = \hat{a}_p - a_p$$
$$\tilde{k}_p = \hat{k}_p - k_p$$

Propose the following candidate Lyapunov function

$$V(e_i, \tilde{\theta}, \tilde{k}, \tilde{a}_p, \tilde{k}_p) = \frac{1}{2}\left[e_i^2 + |k_p|(\tilde{\theta}^2 + \tilde{k}^2) + \tilde{a}_p^2 + \tilde{k}_p^2 \right]$$

time differentiating

$$\dot{V} = e_i \dot{e}_i +$$

27.2.3 Appendix

Previously, we have discussed only methods to identify unknown parameters in certain systems. As control engineers, our primary goal is to stabilize and control a given system in some desirable way, even if we are never able to determine exactly the values of the unknown parameters within the plant. To be more specific about our control goal, and to guide stability analysis, we separate the goal of stability into two portions.

1. **Stability** refers to the boundedness of tracking and parameter errors. That is, $e, \tilde{\theta} \in \mathcal{L}_\infty$

2. **Asymptotic stability** refers to the tracking and parameter errors approaching zero asymptotically. That is: $e, \tilde{\theta} \to 0$ as $t \to \infty$

However, again, as control engineers, we may not be interested in the entirety of (2). That is, we are primarily interested in driving $e \to 0$. Convergence of the parameter error to zero is also good, and the conditions required for $\tilde{\theta} \to 0$ will be discussed more later.

Chapter 28

Stability Theory

28.1 Lyapunov Stability

There are several ways in which the stability of equilibria can be defined which are outlined in these notes. Only autonomous systems are covered, looking at both continuous and discrete time cases.

Lyapunov Stability Analysis gives two approaches can be taken to analyze a system and see what type stability an equilibrium point satisfies. Lyapunov's first, or indirect method can be used to prove whether a system is stable, unstable, or draw no conclusion about stability. Lyapunov's second, or direct method can only prove system stability.

28.1.1 Stability of Autonomous Systems

When talking about the stability of autonomous systems, it is always done relative to an equilibrium point. Equilibrium points must first be found, and it is the stability of these points which must be studied. For linear systems there exists only one equilibrium, so the stability of this equilibrium point can be equivalently described by saying the stability of the system.

28.1.2 Equilibrium Points

Given the following autonomous system, the system's equilibrium points must first be found. (DDV 13.2)

$$\dot{x}(t) = f(x(t), t)$$

The point x_{eq} is an equilibrium point of the continuous system if $f(x_{\text{eq}}(t), t) = 0, \forall t \geq 0$. If the system is started in the state x_{eq} at time t_0, it will remain there for all time. Nonlinear systems can have multiple equilibrium points (or equilibria). For a linear time-invariant system

$$\dot{x}(t) = Ax$$

there is one equilibrium point (the origin) if A is nonsingular, otherwise there are an infinity of equilibrium points, which are contained in the nullspace of A.

The key here is that for linear systems with multiple equilibrium points, these points are not isolated. But for non-linear systems with multiple equilibrium points, there can be infinitely many isolated equilibrium points, for example consider a pendulum.

28.1.3 Stability Definitions

Consider the following dynamical system

$$\dot{x}(t) = f(x(t), t)$$
$$x(t_0) = x_0$$

Denote the equilibrium point as x_{eq}.

Definition 31 Stability *The equilibrium is stable if for all $\epsilon > 0$ there exists a $\delta(\epsilon, t_0) > 0$ such that $\|x_0\| \leq \delta$ implies $\|x(t)\| \leq \epsilon$ for all $t \geq t_0$.*

Stability is often referred to as stable in the sense of Lyapunov (ISL). A system that is stable is one which the system trajectory can be kept close to an equilibrium point by starting sufficiently close to the equilibrium. This is the weakest form of stability, and is also known as *marginally stable*. It is important to make the point that this must hold for *any* ϵ that can be picked, not just one particular and carefully selected special case. An equilibrium point that is not stable ISL is termed unstable. (DDV 13.2)

Definition 32 Attractive

Attractivity implies that all trajectories starting in a neighborhood of the equilibrium point eventually approach the equilibrium point.

Remark 6 Attractivity does not imply stability.

1. **Local asymptotic stability** A system which is stable ISL, and satisfies the *additional* constraint below is called locally asymptotically stable.

   - $\exists r$ such that if $\|x(t_0)\| < r$, then $x(t) \to \bar{x}$ as $t \to \infty$

   This statement says that if the starting point $x(t_0)$ is inside the circle centered about $\bar{x}$ with radius r, that the system trajectories will actually converge to $\bar{x}$. It is important to note that there exist systems which satisfy only this additional constraint *without* satisfying the first constraint of being stable ISL. Such systems are *not* asymptotically stable.

2. **Global asymptotic stability** A system which is globally asymptotically stable extends the definition of local asymptotic stability from a circle of radius r to the entire state space. In other words, beginning from *any* initial conditions $x(t_0)$ then $x(t) \to \bar{x}$ as $t \to \infty$. This is discussed in further detail using Lyapunov's second method.

28.2 Lyapunov Stability Analysis

Using these three definitions of stability, tools are now needed which will allow a system to be analyzed to determine if an equilibrium is stable, and if so, which type of stability the equilibrium point satisfies.

28.2.1 Lyapunov's First (Indirect) Method

This method involves linearizing the nonlinear system about an equilibrium point $\bar{x}$ in order to develop a local conclusion about the stability of the *nonlinear* system. If the linearized system has poles that are all strictly in the left-half complex plane, the equilibrium point is locally asymptotically stable. If the linearized system has any poles that are strictly in the right-half complex plane, equilibrium point is unstable. If the linearized system as any eigenvalues which are zero, no conclusion can be drawn about the stability of the equilibrium point. In this case, essentially the higher order terms that were lost in linearization will determine whether or not the equilibrium is stable or not. (DDV 14.3)

28.2.2 Lyapunov's Second (Direct) Method

Lyapunov's second method requires the construction of a scalar, energy like Lyapunov function of the state which satisfies the properties which follow. This function $V(x(t))$ is proposed as a "candidate Lyapunov function", and if the properties are satisfied, it becomes a Lyapunov function.

- V is locally positive definite

  - $V(0) = 0$
  - $V(x(t)) > 0, 0 < \|x(t)\| < r$ for some r

- $\dot{V}(x(t)) = \frac{d}{dt}V(x(t)) = \frac{d}{dx}V(x(t))\frac{dx}{dt}$ is locally negative semidefinite

  - $\dot{V}(0) = 0$
  - $\dot{V}(x(t)) \leq 0, 0 < \|x(t)\| < r$ for some r

Chain rule:

$$\dot{V}(x(t)) = \frac{dV(x(t))}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt}$$

$$\frac{dz(x(t), y(t))}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

The Lyapunov function which satisfies these three conditions proves the equilibrium point is locally stable ISL.

The condition of stability can be further improved if $\dot{V}(x(t))$ is negative definite, i.e. $\dot{V}(x(t)) < 0, 0 < \|x(t)\| < r$ for some r. Satisfying this condition results in asymptotic stability (DDV 13.4). Lyapunov's second method can be extended to prove global stability if the function $|V(x)| \rightarrow \infty$ as $\|x\| \rightarrow \infty$ (i.e. $V(x(t))$ is radially unbounded) *and* $\dot{V}(x(t))$ is negative definite on the entire state space. If a Lyapunov function cannot be found, this does not necessarily mean that the system is unstable, but only that a suitable Lyapunov function could not be found. Therefore, Lyapunov's direct method cannot be used to prove a system is unstable. In the stability proof for the adaptive controllers presented in these notes, we get $\dot{V} \leq 0$, so from this alone all we have is stability ISL. However, we will show other tools we have to show that the system is asymptotically stable without requiring $\dot{V} < 0$.

28.2.3 The Lyapunov Equation

To prove stability of the following continuous time, linear, autonomous system, a quadratic Lyapunov function will suffice.

$$\dot{x}(t) = Ax(t)$$

Propose the following quadratic Lyapunov function, where P must be chosen such that it is positive definite (i.e. $x^\top P x > 0 \ \forall x \neq 0$).

$$V(x(t)) = x^\top P x, \quad x \in \mathbb{R}^n$$

As long as P is positive definite $V(x(t))$ will be a suitable Lyapunov function. Taking the time derivative of the Lyapunov function, and substituting $\dot{x} = Ax$ gives:

$$\dot{V}(x) = \dot{x}^\top P x + x^\top P \dot{x}$$
$$= (Ax)^\top P x + x^\top P A x$$
$$= x^\top A^\top P x + x^\top P A x$$
$$= x^\top (A^\top P + P A) x$$
$$= -x^\top Q x$$

The resulting matrix $Q = -(A^\top P + PA)$ is symmetric as well. By picking P such that it is not only symmetric and positive definite, but such that Q is negative definite, the quadratic Lyapunov function will prove the linear system is globally asymptotically stable. However, we cannot guarantee for any given positive definite matrix P that when we solve for Q it will be positive definite. However, if we specify $Q > 0$ and A is stable, there always exists $P = P^\top > 0$.

28.3 Barbalat's Lemma

1. V is positive definite

2. $\dot{V}$ is negative semi-definite

   - With V positive definite and $\dot{V}$ negative semi-definite, V is bounded. That is, at the initial time $t = 0$ we have $V(x(t = 0), \theta(t = 0))$ and from here (since $\dot{V}$ is negative semidefinite) the value of V can only decrease. V is bounded below by zero since it is positive definite. Finally, we say that since V is bounded from above by $V(t = 0)$ and bounded from below by 0, that it is bounded. And because V is bounded and positive definite (actually probably some other condition, but it is true for a quadratic function) then the arguments of V are bounded

3. $\dot{V}$ is uniformly continuous, which follows from $\ddot{V}$ being bounded

   - We evaluate $\ddot{V}$ and since now we know the arguments of V are bounded, we use this to bound $\ddot{V}$, thus showing uniform continuity of $\dot{V}$

With these three conditions met Barbalat's Lemma states that $\dot{V} \to 0$ as $t \to \infty$. We then look at $\dot{V}$ and since it is tending to zero, its arguments must go to zero. Since $\dot{V}$ at this point is probably a function of the state error only, we say the state error tends to zero, although we can't necessarily say anything about the parameter error (unless the parameters are in $\dot{V}$ also... it is easiest to show this with examples).

Thus far, we have shown that $e \in \mathcal{L}_\infty$ is bounded. Now the goal is to use Barbalat's Lemma to prove $e \to 0$. That is, show the system is asymptotically stable?

Lemma 1 Slotine pg 122 *If the differentiable function $g(t) = \int_0^t f(\tau)d\tau$ has a finite limit as $t \to \infty$, and if $\dot{g}(t) = f(t)$ is uniformly continuous, then $\dot{g}(t) = f(t) \to 0$ as $t \to \infty$.*

Lemma 2 Lemma 2.12 Annaswamy pg 85 *If $f : \mathbb{R}^+ \to \mathbb{R}$ is uniformly continuous for $t \geq 0$, and if the limit of the integral*

$$\lim_{t \to \infty} \int_0^t |f(\tau)| d\tau$$

exists and is finite, then

$$\lim_{t \to \infty} f(t) = 0$$

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Comparing these two lemmas, $f(t) = \dot{g}(t)$. We want to use Barbalat's lemma to show that $e \to 0$. The steps are given in the next section.

Using Barbalat's Lemma

- Show that e is uniformly continuous for $t \geq 0$.

    - To do this need to show that $\dot{e}$ is bounded, which we did.

- Then show $\lim_{t \to \infty} \int_0^t |e(\tau)| d\tau$ exists and is finite

    - To do this... ???

- Then we will have shown that $\lim_{t \to \infty} e(t) = 0$

Chapter 29

Adaptive Parameter Identification

Section 3.2.1 from book.

29.1   Parameter Identification: Scalar Algebraic Systems

29.1.1   Non-Recursive Schemes

Method 1

Consider the scalar algebraic system with input $u(t)$, unknown scalar parameter θ, and output $y(t)$. We want to identify θ.
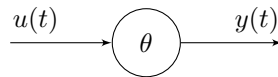
Want to identify θ using measurements $\{u(t), y(t)\}$. One method

$$\hat{\theta}(t) = \frac{y(t)}{u(t)}$$

provided θ is not zero, $\hat{\theta} \to \theta$.

Method 2

Another method

$$\hat{\theta}(t) = \frac{y(t)u(t)}{u(t)^2}$$

Denote the estimated value as $\hat{\theta}$, and compute the estimated parameter as

$$\hat{\theta}(t) = \frac{y(t)}{u(t)}$$

which can be written as follows. But why?

$$\hat{\theta}(t) = \frac{y(t)u(t)}{u(t)^2}$$

provided $u(t) \neq 0$.

Method 3

Now want to set it up using the cost function approach described above. Put u into θ and $\hat{\theta}$, take outputs y and $\hat{y}$ and difference them. Define this difference as the error $e = y - \hat{y}$.
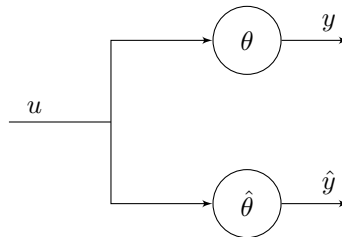
Becomes
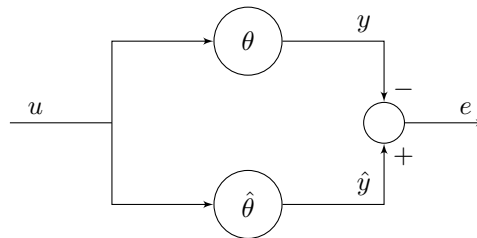
Then

$$J = e^2$$

And the goal is to find $\hat{\theta}$ by minimizing J

$$\hat{\theta} = \arg\min_{\theta} J$$

Expanding J

$$J = [y - \hat{y}]^2$$
$$= [y - \hat{\theta}u]^2$$

Evaluating the gradient

$$\frac{\partial J}{\partial \hat{\theta}} = 2[\hat{\theta}u - y]u$$
$$= 2\hat{\theta}u^2 - 2yu$$

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Setting this equal to zero and solving for the parameter estimate

$$\hat{\theta}(t) = \frac{y(t)u(t)}{u(t)^2}$$

which is the same estimate achieved above.

$$
\begin{aligned}
e &= \hat{y} - y \\
&= \hat{\theta}u - y \\
&= \hat{\theta}u - \theta u \\
&= \tilde{\theta}u
\end{aligned}
$$

Method 4

To accommodate noise, define a cost function as an integral of the error squared over some interval of time. Then find the estimate $\hat{\theta}$ as the value where the gradient of the cost function is zero. That is, the gradient will be zero when the cost function is at a minimum. The error is given by

$$e = y - \hat{y}$$

this is shown in the following figure The cost function

$$
\begin{aligned}
J &= \int_t^{t+T} e(\tau)^2 d\tau \\
&= \int_t^{t+T} \big(y(\tau) - \hat{y}(\tau)\big)^2 d\tau \\
&= \int_t^{t+T} \big(y(\tau) - \hat{\theta}(\tau)u(\tau)\big)^2 d\tau
\end{aligned}
$$

The goal is to minimize the cost J, and so at this minimum we have

$$\frac{\partial J}{\partial \hat{\theta}} = 0$$

and with

$$
\begin{aligned}
\frac{\partial J}{\partial \hat{\theta}} &= \int_t^{t+T} \frac{\partial}{\partial \hat{\theta}}\big(y(\tau) - \hat{\theta}(\tau)u(\tau)\big)^2 d\tau \\
&= \int_t^{t+T} 2\big(y(\tau) - \hat{\theta}(\tau)u(\tau)\big)u(\tau) d\tau
\end{aligned}
$$

and then setting $\partial J / \partial \hat{\theta} = 0$ we have

$$\int_t^{t+T} 2\big(y(\tau) - \hat{\theta}(\tau)u(\tau)\big)u(\tau) d\tau = 0$$

$$\int_t^{t+T} y(\tau)u(\tau) - \hat{\theta}(\tau)u(\tau)^2 d\tau = 0$$

$$\int_t^{t+T} y(\tau)u(\tau) d\tau = \int_t^{t+T} \hat{\theta}(\tau)u(\tau)^2 d\tau$$

$$\hat{\theta}(t) = \frac{\int_t^{t+T} y(\tau)u(\tau)d\tau}{\int_t^{t+T} u^2(\tau)d\tau}$$

Finish this explanation. Why pull out $\hat{\theta}$ like that?

Method 5

The estimate $\hat{\theta}(t)$ can be obtained by solving the differential equation obtained by setting $\dot{\hat{\theta}}$ equal to the negative gradient of the cost function. Using the following cost function

$$J = \frac{1}{2}\left(y(t) - \hat{\theta}(t)u(t)\right)^2$$

and evaluating the gradient

$$\frac{\partial J}{\partial \hat{\theta}} = -(y(t) - \hat{\theta}(t)u(t))u(t)$$
$$= -u(t)(\theta u(t) - \hat{\theta}(t)u(t))$$
$$= -u(t)^2(\theta - \hat{\theta}(t))$$

Defining the following parameter estimation error

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta$$

The gradient can be expressed

$$\frac{\partial J}{\partial \hat{\theta}} = u(t)^2\tilde{\theta}(t)$$

and make the estimate update

$$\dot{\hat{\theta}}(t) = -\frac{\partial J}{\partial \hat{\theta}}$$

giving

$$\dot{\hat{\theta}}(t) = \dot{\tilde{\theta}}(t) = -u^2(t)\tilde{\theta}$$

29.1.2   Recursive Schemes
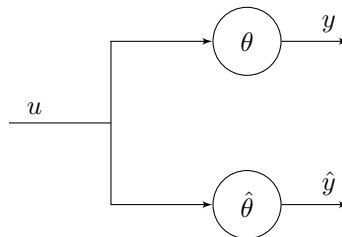
Becomes
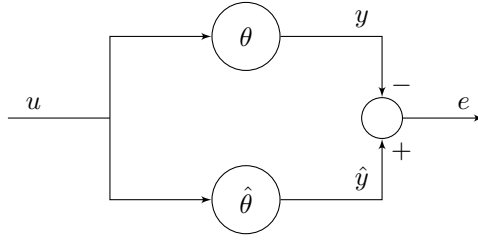
Use quadratic cost function in error $J = e^2 = [\hat{\theta}u - y]^2$. Recursive scheme: Identify $\hat{\theta}(t)$ at every instant. Set $\dot{\hat{\theta}} \propto \frac{\partial J}{\partial \theta}$. $\dot{\hat{\theta}} = -[\hat{\theta}u - y]u = -u^2\hat{\theta} + uy$
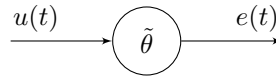
29.2 Parameter Identification: Vector Algebraic Systems

29.2.1 Non-Recursive Schemes
Method 1
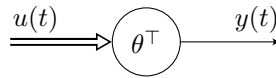
So identify as

$$\hat{\theta} = u^{-1}(t)Y(t)$$

If u^{-1} exists, $\hat{\theta} = \theta$.

Method 2 and 3

Same as before with scalar case

$$\theta = [U^\top U]^{-1}$$

alternatively

$$J = \|U\hat{\theta} - Y\|^2$$

and $\frac{\partial J}{\partial \hat{\theta}} = 0 \Rightarrow \hat{\theta} = [U^\top U]^{-1}U^\top Y$ $\min_{\hat{\theta}} J$ that implies

$$\hat{\theta} = [U^\top U]^{-1}U^\top Y$$

29.2.2 Recursive Schemes
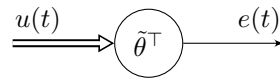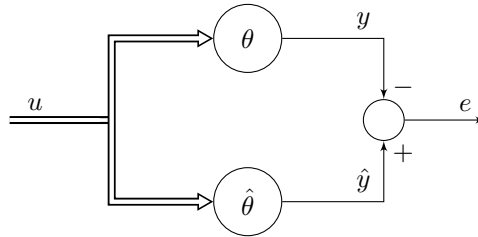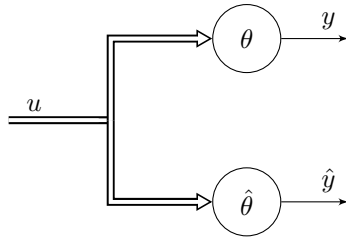
$$\dot{\hat{\theta}} = -u^2\hat{\theta} + u(u\theta)$$

$$\dot{\tilde{\theta}} = -u^2\tilde{\theta}$$

$$J = e^2 = (u^\top\hat{\theta} - y)^2$$

$$\dot{\hat{\theta}} \propto \frac{\partial J}{\partial \theta} = -u[u^\top\hat{\theta} - y]$$

Set

$$\dot{\tilde{\theta}} = -u(t)u^\top(t)\tilde{\theta}$$

$\dot{\tilde{\theta}} = A(t)\tilde{\theta}$ where $A(t) = u(t)u^\top(t)$, where $A(t)$ is a symmetric $n \times n$ matrix of rank 1.

$$[uu^\top]x = \lambda x$$

$x = u$

$$uu^\top u = (u^\top u)u$$

$x = u^\perp\ uu^\top u^\perp = (O)u^\perp\ u^\perp\ n - 1$ such vectors u is said to be **persistently exciting** if it has the full rank property.

$$\frac{1}{T}\int_t^{t+T} u(\tau)u^\top(\tau)d\tau \geq \alpha I_{n\times n}$$

For some T and $\alpha > 0$. This property is sometimes called **uniform observability**. Sinusoids are best vector for convergence.

Example 45 Identification of a parameter in a vector algebraic system Vector $\theta \in \mathbb{R}^n$. Again input u output y.

$$U = \begin{bmatrix} u^\top(t_1) \\ u^\top(t_2) \\ \dots \\ u^\top(t_n) \end{bmatrix}_{n\times n} \qquad \theta_{n\times 1} = \begin{bmatrix} y(t_1) \\ y(t_2) \\ \dots \\ y(t_n) \end{bmatrix}_{n\times 1}$$

where the output vector is Y

$$U\theta = Y$$

giving

$$\theta = U^{-1}Y$$

and then

$$U^\top U\theta = U^\top Y$$

giving

$$\theta = [U^\top U]^{-1}$$

alternatively

$$J = \|U\hat\theta - Y\|^2$$

and $\frac{\partial J}{\partial \theta} = 0 \Rightarrow \hat\theta = [U^\top U]^{-1}U^\top Y$ Recursive scheme: Identify $\hat\theta(t)$ at every instant. Set $\dot{\hat\theta} \propto \frac{\partial J}{\partial \theta}$

29.2.3 Overview of Persistent Excitation

sufficient condition to determine the parameter is

$$J = \int_t^{t+T} u^2 d\tau > \varepsilon$$

this is persistent excitation!

29.3 Introduction to Error Models

Error models are covered in detail in Chapter 7 of the book, and we will introduce them a little bit here. In the adaptive control problems we will consider, we are adjusting some parameter, either as part of an

estimator or controller. As this parameter is adjusted, we monitor an error signal. The goal is then to adjust our parameter in a way that drives the error to zero. Error models allow us to determine a set up update laws for the parameter which are applicable to many different problems by reducing the problem to a known error model form. These error models are introduced here and will be covered in detail later.

29.3.1   Error Model 1

Error model 1 is an algebraic error model. That is, the relationship between the input and error is an algebraic relationship with parameter error. Consider the following estimator block diagram
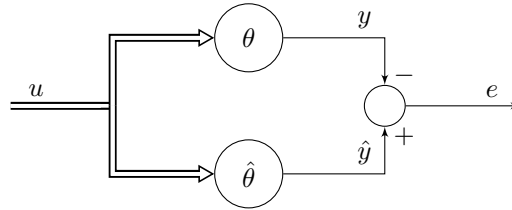
Defining the parameter error as $\tilde{\theta} = \hat{\theta} - \theta$ the block diagram can be expressed
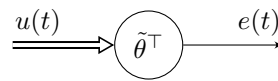
The relationship between the output error e and parameter error $\widetilde{\theta}$ is written as

$$e(t) = \tilde{\theta}^\top u$$

Now, given this error model, we want to adjust $\tilde{\theta}$ in some way to make $e \to 0$.

29.3.2   Error Model 3
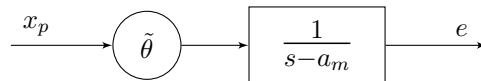
29.4   Parameter Identification: Dynamic Systems

Cannot use MIT rule for dynamic systems.

### 29.4.1   Scalar Systems: Single Parameter Identification

The following examples show how to identify a *single* parameter in a first-order systems. To identify multiple parameters in a scalar systems is also possible, and done in a following example.

Error Model 3

Example 46 DC motor: Identification of a single parameter in a scalar system (Error Model 3) Consider the following transfer function representation of a DC motor, where the input is terminal voltage V and the output is the angular velocity of the motor shaft ω.

$$\frac{\omega}{V} = \frac{K}{Js + B} = \frac{\frac{K}{J}}{s + \frac{B}{J}}$$

where B is the friction J the inertia, and K is the DC gain of the system. This transfer function can be represented by the following block diagram
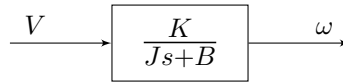
We can simplify the representation by parameterizing the transfer function as

$$\frac{\omega}{V} = \frac{a_1}{s + \theta_1}$$

where $a_1 = \frac{K}{J}$ and $\theta_1 = \frac{B}{J}$. For this problem we assume that the sign of the parameters B, J, and K are all known and furthermore that

$$a_1 \textbf{ known}$$
$$\theta_1 \geq 0 \textbf{ unknown}$$

where θ_1 is the unknown constant quantity which we would like to identify, where its positivity in this example is due to the physical parameters which it represents. Note that it is hard to identify unstable systems, and to do so we would need a stabilizing controller and perform closed-loop identification. More on this later. Using this parameterization, the block diagram can be written
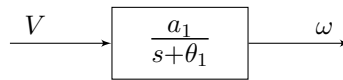
We can pull the known constant a_1 out of the transfer function as shown in the following block diagram
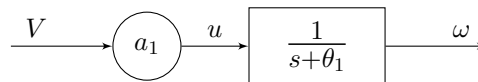
We can then look at the transfer function from u to ω

$$\frac{\omega}{u} = \frac{1}{s + \theta_1}$$

In state space, this plant can be represented as follows. The second equation is the estimation equation to identify θ_1.

$$\dot{\omega} = -\theta_1\omega + u$$
$$\dot{\hat{\omega}} = -\hat{\theta}_1\hat{\omega} + u$$

Define the tracking error as

$$e = \hat{\omega} - \omega$$

Differentiate the tracking error

$$\dot{e} = \dot{\hat{\omega}} - \dot{\omega}$$
$$= -\hat{\theta}_1\hat{\omega} + u + \theta_1\omega - u$$
$$= -\hat{\theta}_1\hat{\omega} + \theta_1\omega$$

Add and subtract $\theta_1\hat{\omega}$ from both sides

$$\dot{e} = \theta_1\omega - \hat{\theta}_1\hat{\omega} + \theta_1\hat{\omega} - \theta_1\hat{\omega}$$
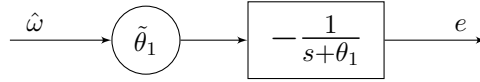$$= -\theta_1(\hat{\omega} - \omega) - (\hat{\theta}_1 - \theta_1)\hat{\omega}$$

and with $\tilde{\theta}_1 = \hat{\theta}_1 - \theta_1$ this gives the following error dynamics

$$\boxed{\dot{e} = -\theta_1 e - \tilde{\theta}_1\hat{\omega}}$$

Which we call **error model 3**. The transfer function for these error dynamics is

$$\frac{e}{\hat{\omega}} = -\frac{\tilde{\theta}_1}{s + \theta_1}$$

with the following block diagram representation

In order to determine a stable identification scheme, we propose the following Lyapunov function.

$$V(\tilde{\theta}, e) = \frac{1}{2}(\tilde{\theta}_1^2 + e^2)$$

Time differentiating

$$\dot{V} = \tilde{\theta}_1\dot{\tilde{\theta}}_1 + e\dot{e}$$
$$= \tilde{\theta}_1\dot{\tilde{\theta}}_1 + e(-\theta_1 e - \tilde{\theta}_1\hat{\omega})$$
$$= -\theta_1 e^2 + \tilde{\theta}_1\dot{\tilde{\theta}}_1 - e\tilde{\theta}_1\hat{\omega}$$

Noting that $\dot{\tilde{\theta}}_1 = \dot{\hat{\theta}}_1$ select the following parameter estimation law

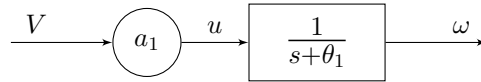$$\boxed{\dot{\hat{\theta}}_1 = e\hat{\omega}}$$

This gives

$$\dot{V} = -\theta_1 e^2$$

with $\dot{V} \leq 0 \Rightarrow e$ and $\tilde{\theta}$ are bounded. More on this later.

Error Model 1

Example 47 DC motor: Identification of a single parameter in a scalar system (Error Model 1) Consider again the DC motor example of example 46. In that example, we parameterized the first order plant as

$$V \longrightarrow \boxed{a_1} \xrightarrow{u} \boxed{\dfrac{1}{s+\theta_1}} \xrightarrow{\omega}$$

with transfer function from u to ω given by

$$\frac{\omega}{u} = \frac{1}{s + \theta_1}$$

We again assume

$$a_1 \textbf{ known}$$
$$\theta_1 \geq 0 \textbf{ unknown}$$

Write the transfer function as

$$\begin{aligned}
\frac{\omega}{u} &= \frac{1}{s + \theta_1} \\
&= \frac{1}{s + \theta_m} \cdot \frac{s + \theta_m}{s + \theta_1} \\
&= \frac{1}{s + \theta_m} \cdot \frac{1}{\frac{s+\theta_1}{s+\theta_m}} \\
&= \frac{1}{s + \theta_m} \cdot \frac{1}{\frac{\theta_m - \theta_m}{s+\theta_m} + \frac{s+\theta_1}{s+\theta_m}} \\
&= \frac{1}{s + \theta_m} \cdot \frac{1}{1 - \frac{\theta_m - \theta_1}{s+\theta_m}}
\end{aligned}$$

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where $\theta_m > 0$ is a known, positive parameter that the control designer picks, which should be a reasonable time constant for the DC motor in this case. Define $\theta \triangleq \theta_1 - \theta_m$ and simplify this transfer function as

$$\frac{\omega}{u} = \frac{1}{s + \theta_m} \cdot \frac{1}{1 + \frac{\theta}{s + \theta_m}}$$

and realize this in the following block diagram representation, where we are using two states to represent a first order system.
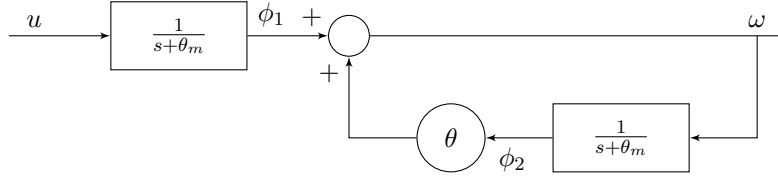
The only unknown entity is the parameter θ. Note that ϕ_1 and ϕ_2 are synthesizable signals that can be had online. So, we can express ω and its estimate $\hat{\omega}$ as

$$\omega = \phi_1 + \theta\phi_2$$
$$\hat{\omega} = \phi_1 + \hat{\theta}\phi_2$$

Define the tracking error as follows, and simplify

$$e = \hat{\omega} - \omega$$
$$= (\hat{\theta} - \theta)\phi_2$$

This gives the following relationship between input ϕ_2 and error, **error model 1**.

$$\boxed{e = \tilde{\theta}\phi_2}$$

where the parameter estimation error is defined as

$$\tilde{\theta} = \hat{\theta} - \theta$$

Define the following Lyapunov function candidate

$$V(\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^2$$

Differentiating

$$\dot{V} = \tilde{\theta}\dot{\tilde{\theta}}$$

Select the following parameter estimation law, noting that $\dot{\hat{\theta}} = \dot{\tilde{\theta}}$

$$\dot{\hat{\theta}} = -e\phi_2$$

Which simplifies to

$$\dot{\tilde{\theta}} = -\tilde{\theta}\phi_2^2$$

Substituting this into the time derivative of V

$$\dot{V} = -\tilde{\theta}^2\phi_2^2$$

Essentially converted the problem of identifying a single parameter in a first order dynamical system from error model 3 as shown in example 46 to error model 1.

## 29.4.2 Vector Systems: Multiple Parameter Identification

The following examples are for vector systems, that is systems with more than one unknown parameter. These examples are of a scalar system with multiple parameters, using two approaches: error model 1 and error model 3.

Error Model 3

Example 48 DC motor: identification of multiple parameters in a scalar system (Error model 3) Same system as in the first DC motor example, only this time two unknown parameters.

$$\frac{\omega}{V} = \frac{a_1}{s + \theta_1}$$

Block diagram

$$u \longrightarrow \boxed{\frac{a_1}{s+\theta_1}} \longrightarrow \omega$$
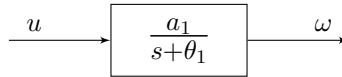
This time we assume

$$a_1 \text{ unknown}$$

$$\theta_1 \geq 0 \text{ unknown}$$

The differential equation describing the plant is given by

$$\dot{\omega} = -\theta_1\omega + a_1V$$

Generate an estimate of the plant output as follows, using parameter estimates in place of the unknown parameters

$$\dot{\hat{\omega}} = -\hat{\theta}_1\hat{\omega} + \hat{a}_1V$$

Define the following output error and parameter errors

$$e = \hat{\omega} - \omega$$

$$\tilde{\theta}_1 = \hat{\theta}_1 - \theta_1$$

$$\tilde{a}_1 = \hat{a}_1 - a_1$$
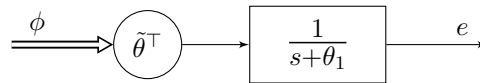
The output error dynamics are given by

$$\dot{e} = -\hat{\theta}_1\hat{\omega} + \hat{a}_1 V + \theta_1\omega - a_1 V$$

Add and subtract $\theta_1\hat{\omega}$

$$\begin{aligned}
\dot{e} &= -\hat{\theta}_1\hat{\omega} + \theta_1\hat{\omega} - \theta_1\hat{\omega} + \hat{a}_1 V + \theta_1\omega - a_1 V \\
&= (\theta_1 - \hat{\theta}_1)\hat{\omega} - \theta_1(\hat{\omega} - \omega) + \tilde{a}_1 V \\
&= -\theta_1 e - \tilde{\theta}_1\hat{\omega} + \tilde{a}_1 V \\
&= -\theta_1 e + \tilde{\theta}^\top \phi
\end{aligned}$$

where

$$\bar{\theta} = \begin{bmatrix} \theta_1 \\ a_1 \end{bmatrix} \qquad \text{and} \qquad \phi = \begin{bmatrix} -\hat{\omega} \\ V \end{bmatrix}$$
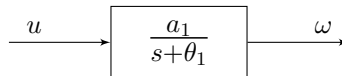

Error Model 1

Example 49 DC motor: identification of multiple parameters in a scalar system (Error model 1) Same system as in the first DC motor example, only this time two unknown parameters.

$$\frac{\omega}{u} = \frac{a_1}{s + \theta_1}$$

Block diagram

This time we assume
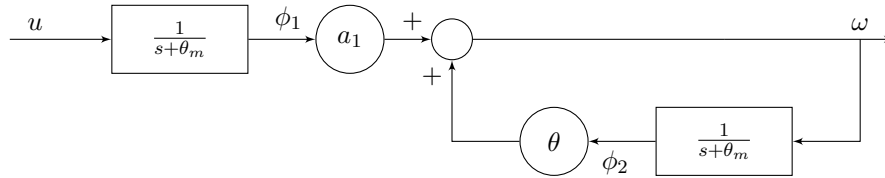
$$a_1 \text{ unknown}$$
$$\theta_1 \geq 0 \text{ unknown}$$

As last time in example 47 Write the transfer function as

$$\frac{\omega}{u} = \frac{a_1}{s + \theta_1}$$

$$= \frac{a_1}{s + \theta_m} \cdot \frac{s + \theta_m}{s + \theta_1}$$

$$= \frac{a_1}{s + \theta_m} \cdot \frac{1}{\frac{s+\theta_1}{s+\theta_m}}$$

$$= \frac{a_1}{s + \theta_m} \cdot \frac{1}{\frac{\theta_m - \theta_m}{s+\theta_m} + \frac{s+\theta_1}{s+\theta_m}}$$

$$= \frac{a_1}{s + \theta_m} \cdot \frac{1}{1 - \frac{\theta_m - \theta_1}{s+\theta_m}}$$

where $\theta_m > 0$ is a known, positive parameter that the control designer picks, which should be a reasonable time constant for the DC motor in this case. Define $\theta \triangleq \theta_1 - \theta_m$ and simplify this transfer function as

$$\frac{\omega}{u} = \frac{a_1}{s + \theta_m} \cdot \frac{1}{1 + \frac{\theta}{s+\theta_m}}$$

and realize this in the following block diagram representation, where we are using two states to represent a first order system.

$$\omega = a_1 \phi_1 + \theta \phi_2$$
$$= \bar{\theta}^\top \phi$$

where
$$\bar{\theta} = \begin{bmatrix} a_1 & \theta \end{bmatrix}^\top \qquad \phi = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix}^\top$$

Define the estimated parameter
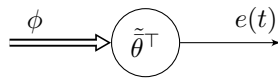$$\hat{\omega} = \hat{\bar{\theta}}^\top \phi$$

with error $e = \hat{\omega} - \omega$ given by

$$\boxed{e = \tilde{\bar{\theta}}^\top \phi}$$

Where the parameter error is
$$\tilde{\bar{\theta}} = \hat{\bar{\theta}} - \bar{\theta}$$

This is again error model 1, but now the unknown parameter is a vector. This error model has the following block diagram representation

$$\phi \Longrightarrow \left(\tilde{\theta}^{\top}\right) \xrightarrow{\;e(t)\;}$$

Problem cast as output... For more on this, see page 275 of text.

Chapter 30

Adaptive PI, PID, Phase-Lead Control

In this chapter, the velocity control of a DC motor is considered. Applying Newton's second law with angular motor displacement θ gives the following description of the motor, where the input is the motor torque τ directly (we assume motor torque is directly proportional to voltage, and neglect fast electrical dynamics).

$$J\ddot{\theta} + \dot{\theta}x = \tau$$

Written in terms of the measured output, angular velocity $\omega = \dot{\theta}$, this gives

$$J\dot{\omega} + B\omega = \tau$$

Using the state $x = \omega$ to represent angular velocity, this can finally be written

$$J\dot{x} + Bx = \tau \tag{30.1}$$

Thus the transfer function between the motor torque τ and angular velocity x is given by the following transfer function

> **DC motor transfer function:** $\qquad G_p(s) = \dfrac{x}{\tau} = \dfrac{1}{Js + B}$

Note that because this is a motor, the rotational inertia $J > 0$ and viscous damping $B > 0$, giving the plant a stable pole at $-B/J$, and no zeros. This is a very simple system to consider for this control problem: first order and stable. **The control goal is: have $x \to x_d$ where x_d is the desired angular velocity.** The controller will be connected with the plant as shown by the following block diagram
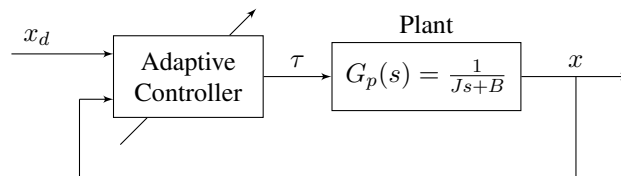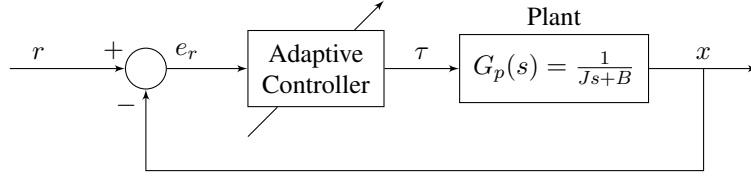
The input to the controller is the *reference error* $e_r = r - x$. Note that $x_d \neq r$. This implies that the final control architecture will use another element in addition to the controller shown in the block diagram above.

30.1 Adaptive PI Control

In this section we consider the design of an adaptive PI controller to achieve the proposed control goal. In fact, two adaptive PI controllers will be presented. First, the "original design", followed by a "better design" which makes a small modification moving a damping term from feedforward to feedback.

30.1.1 Adaptive PI Controller: Original Design
Algebraic Part

The first step in the design of the adaptive PI controller is to first design the nominal PI controller: the PI controller we would pick if J and B were known. This step is called the algebraic part. Then we consider J and B are unknown and develop the adaptive version. This step is called the analytic part. This procedure is called the *certainty equivalence principle*: develop solution when parameters are known, then replace parameters by estimates. As the proposed controller is a PI, the control block will be represented by $G_c(s)$ as in the following block diagram.
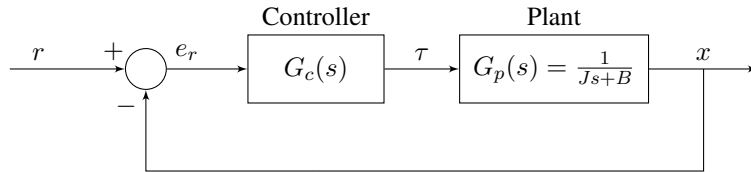
A standard PI controller is written in transfer function form as

$$G_c(s) = \frac{\tau}{e_r} = k_p + \frac{k_i}{s}$$
$$= \frac{sk_p + k_i}{s}$$
$$= \frac{k_p\left(s + \frac{k_i}{k_p}\right)}{s}$$

where the control gains $k_p > 0$ and $k_i > 0$. From this transfer function representation, we can see that this controller has a pole at the origin and a zero in the LHP at $-k_i/k_p$. This control law can also be expressed as

$$\tau(t) = k_p(t)e_r(t) + k_i(t)\left(\int_0^t e_r(\tau)d\tau\right)$$

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In order to design the nominal PI controller, we find the closed-loop transfer function $W_{cl}(s)$, as shown in the block diagram below. This is so the precompsenator, which generates r from x_d, can be defined as the inverse $W_{cl}(s)^{-1}$ giving a final transfer function from x_d to x of unity, accomplishing the control goal.
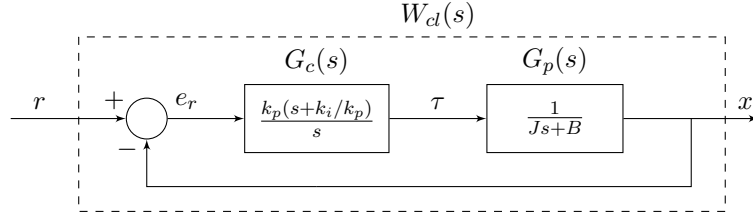
This closed-loop transfer function is

$$W_{cl}(s) = \frac{x}{r} = \frac{G_c G_p}{1 + G_c G_p}$$

$$= \frac{k_p s + k_i}{s(Js + B) + k_p s + k_i}$$

Next, reparameterize the control law as

$$\boxed{\begin{array}{l} k_p = K \\ k_i = K\lambda \end{array}}$$

where, since $k_p > 0$ and $k_i > 0$ then $K > 0$, $\lambda > 0$. The reason behind this reparameterization is explained later. Writing the controller transfer function using this reparameteriaztion gives

$$G_c(s) = \frac{K(s + \lambda)}{s}$$

The closed-loop transfer function when using the controller reparameterization is

$$W_{cl}(s) = \frac{x}{r} = \frac{K(s + \lambda)}{s(Js + B) + K(s + \lambda)}$$

Using this reparameterization the block diagram is represented as below
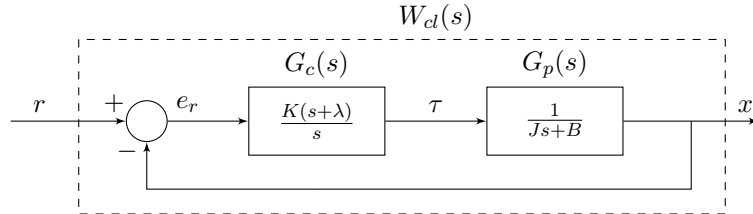
The characteristic polynomial is given by

$$s(Js + B) + K(s + \lambda) = 0$$
$$Js^2 + (B + K)s + K\lambda = 0$$

Using the Routh-Hurwitz criterion for this system, the second order system is stable if all of the polynomial coefficients are positive, which is satisfied given $J > 0$, $B > 0$, and $\forall\ K, \lambda > 0$.

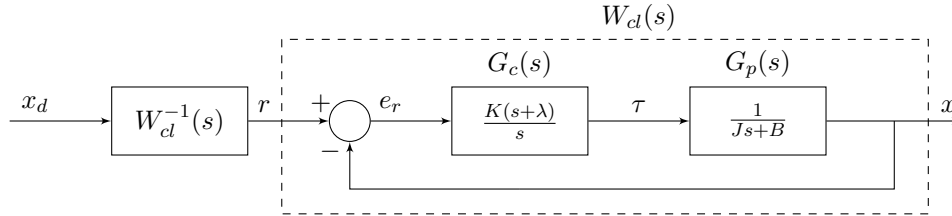The idea now is to, assuming the plant parameters are known exactly, figure out what reference command r we would have to give to ensure perfect tracking. If the closed loop system is given by $W_{cl}(s)$, we can determine r by inverting the closed-loop system, so the transfer function from x_d to x would be unity, giving perfect tracking. See block diagram below.

Inverting the closed-loop system we get

$$W_{cl}^{-1}(s) = \frac{r}{x_d} = \frac{1 + G_c G_p}{G_c G_p}$$
$$= \frac{1}{G_c G_p} + 1$$
$$= (Js + B)G_c(s)^{-1} + 1$$

The reference r can also be expressed in terms of x_d as

$$r = \left[W_c^{-1}(s)\right] x_d$$
$$= [(Js + B)G_c^{-1}(s) + 1]x_d$$
$$= (Js + B)G_c^{-1}(s)x_d + x_d$$
$$= JG_c^{-1}(s)\dot{x}_d + BG_c^{-1}(s)x_d + x_d$$

defining ω_d as

$$\omega_d = [G_c^{-1}(s)]x_d$$
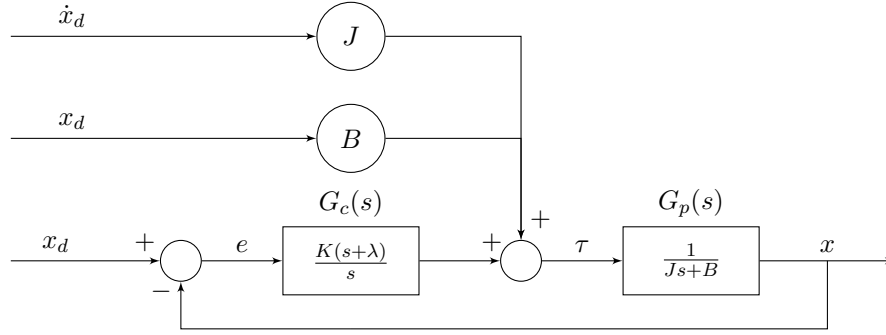
gives

$$r = J\dot{\omega}_d + B\omega_d + x_d$$

where $\dot{\omega}_d$, ω_d, and x_d are known signals. So assuming we knew the plant parameters exactly, and given the desired trajectory and its derivative, the above equation defines the reference command r such that perfect tracking of x_d by x would be achieved. This can be represented by the following block diagram.
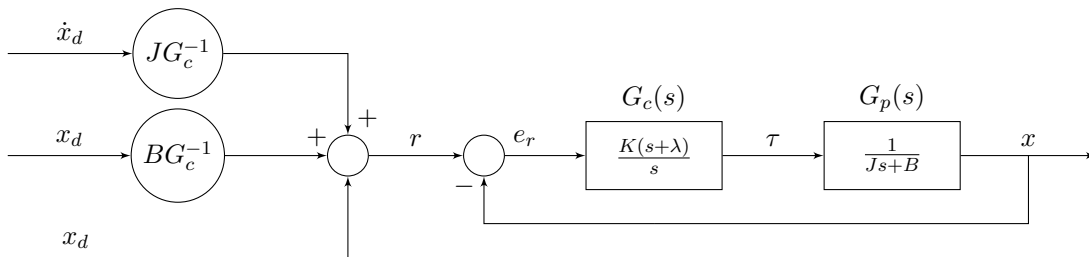
This can be rearranged as

where the *tracking error* e is given by

$$e = x_d - x$$

So the total control input is

$$\tau = J\dot{x}_d + Bx_d + G_c(s)e$$

Writing the PI controller transfer function $G_c(s)$ as

$$G_c(s) = K + \frac{K\lambda}{s}$$

then $G_c(s)e$ can be written

$$G_c(s)e = Ke + K\lambda \int e dt$$

finally allowing the total control input to be written as

$$\tau = J\dot{x}_d + Bx_d + Ke + K\lambda \int e(\tau)d\tau$$

$$= J\dot{x}_d + Bx_d + K\left(e + \lambda \int e(\tau)d\tau\right)$$

Define the following error

$$e_2 = e + \lambda \int e(\tau)d\tau$$

This allows the control law to finally be written as

$$\tau = J\dot{x}_d + Bx_d + Ke_2$$

Analytic Part

In the analytic part the control law that was defined above when it was assumed the plant parameters where completely known is taken and the unknown values of the plant parameters are replaced with estimates of their values, denoted with the "hat" symbol.

Adaptive PI control law, original design:	$\tau = \hat{J}\dot{x}_d + \hat{B}x_d + Ke_2$

(30.2)

Substituting the adaptive control law (30.2) into the plant equation (30.1) gives

$$J\dot{x} = -Bx + \hat{J}\dot{x}_d + \hat{B}x_d + Ke_2$$

$$\dot{x} = \frac{1}{J}(\hat{J}\dot{x}_d + \hat{B}x_d - Bx + Ke_2)$$

The tracking error dynamics are expressed

$$\dot{e} = \dot{x}_d - \dot{x}$$

$$= \dot{x}_d - \frac{1}{J}(\hat{J}\dot{x}_d + \hat{B}x_d - Bx + Ke_2)$$

$$= \frac{J}{J}\dot{x}_d - \frac{\hat{J}}{J}\dot{x}_d - \frac{1}{J}(\hat{B}x_d - Bx + Ke_2)$$

$$= -\frac{\tilde{J}}{J}\dot{x}_d - \frac{1}{J}(\hat{B}x_d - Bx + Ke_2)$$

where the rotational inertia parameter estimation error is defined as

$$\tilde{J} = \hat{J} - J$$

Using the error definition for e_2 above, the dynamics are described by

$$\dot{e}_2 = \dot{e} + \lambda e$$

$$= -\frac{\tilde{J}}{J}\dot{x}_d - \frac{1}{J}(\hat{B}x_d - Bx + Ke_2) + \lambda e$$

But these error dynamics do not conveniently include the viscous damping parameter estimation error.

Stability

Summary

<div style="border:1px solid red;">

Adaptive PI: Original Design

Plant:	$J\dot{x} + Bx = \tau$
Control:	$\tau = \hat{J}\dot{x}_d + \hat{B}x_d + Ke_2$
Error:	$e = x_d - x$
	$e_2 = e + \lambda \int e(\tau)d\tau$
Parameterization:	$k_p = K$
	$k_i = K\lambda$

</div>

30.1.2 Adaptive PI Controller: Better Design

Say some stuff here about exactly what the idea is behind using positive damping feedback, and how this will help with the adaptive part of the design to follow, as compared to the adaptive design when the damping feedback is not used. This can be represented using the following block diagram
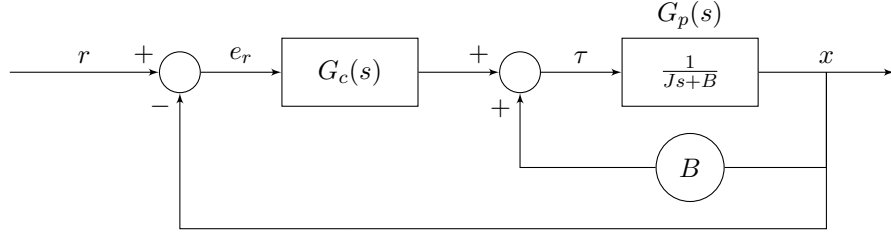
Comparing this to the last block diagram of the original adaptive PI design, it is clear that this feedback modification is equivalent to changing the signal x_d which is fed forward through the B to a feedback from x. It should already be somewhat obvious from the previous e_2 error dynamics how this is helpful, as it introduces the viscous damping parameter estimation error.

Algebraic Part

The block diagram above is the same as the following block diagram. Again, we complete the algebraic part first. Closing the inner loop with B feedback, the equivalent plant can be expressed

$$\frac{\frac{1}{Js+B}}{1 - \frac{B}{Js+B}} = \frac{\frac{1}{Js+B}}{\frac{Js}{Js+B}} = \frac{1}{Js}$$

This is represented by the following block diagram
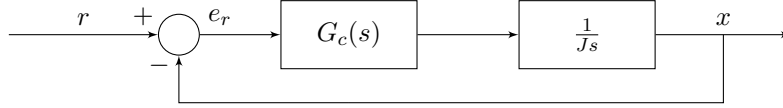
Again the same PI control law is used for $G_c(s)$.

$$G_c(s) = \frac{k_p s + k_i}{s}$$

Using this new structure with the proposed positive rate feedback, we must check the stability of the closed loop system W_{cl}.

$$W_{cl}(s) = \frac{x}{r} = \frac{G_c(s)\frac{1}{Js}}{1 + G_c(s)\frac{1}{Js}}$$
$$= \frac{G_c(s)}{Js + G_c(s)}$$
$$= \frac{\frac{k_p s + k_i}{s}}{\frac{Js^2}{s} + \frac{k_p s + k_i}{s}}$$
$$= \frac{k_p s + k_i}{Js^2 + k_p s + k_i}$$

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The stability of the closed-loop system can then be checked using the Routh-Hurwitz criterion. For a second order system this states that the closed-loop system will be stable if the characteristic polynomial has no sign changes. Thus, the closed-loop system is stable for all $J > 0$, $k_p > 0$, $k_i > 0$.

We now proceed in the same way as before: figure out what r should be based on x_d to give us perfect tracking. This is how we find the feed-forward part shown in the block diagram below.
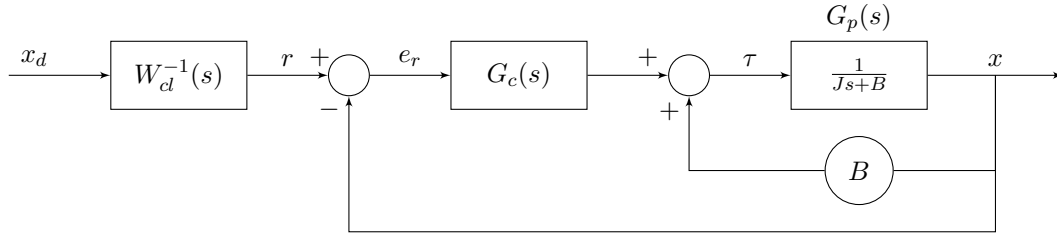
Using the expression for W_{cl} above and inverting it, the reference signal r can be found in terms of x_d. First invert W_{cl} as follows

$$
\begin{aligned}
W_{cl}^{-1}(s) &= \frac{Js + G_c(s)}{G_c(s)} \\
&= (Js + G_c(s))G_c^{-1}(s) \\
&= JsG_c^{-1}(s) + 1
\end{aligned}
$$

Now using $W_{cl}^{-1}(s)$ to determine x_d

$$
\begin{aligned}
r &= [W_{cl}^{-1}(s)]x_d \\
&= [1 + JsG_c^{-1}(s)]x_d \\
&= x_d + [JG_c^{-1}(s)]\dot{x}_d
\end{aligned}
$$

The block diagram above can then be represented as
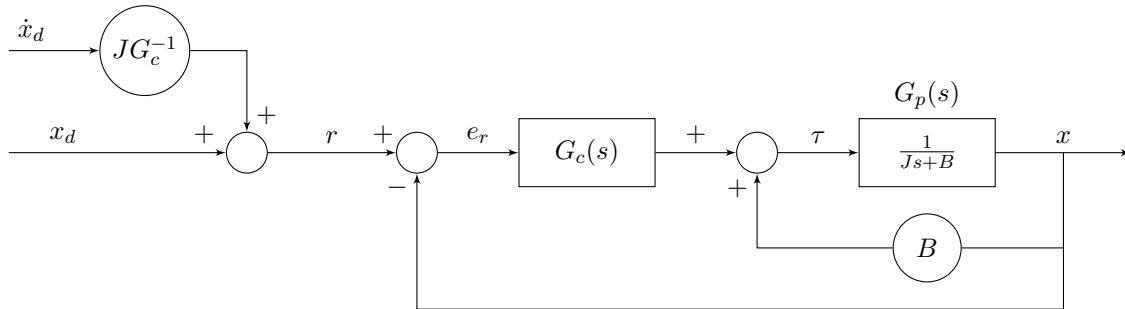
which can be again redrawn as
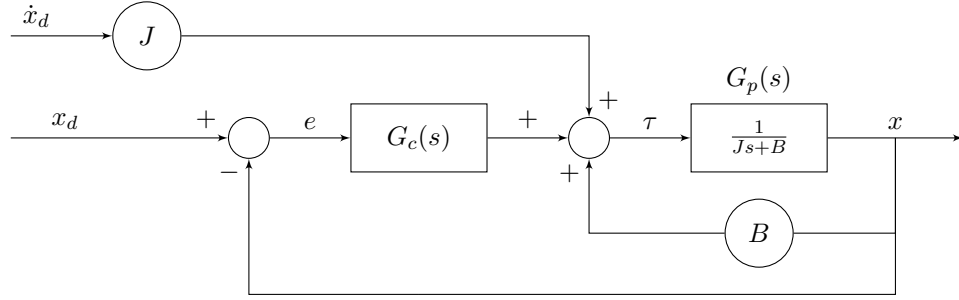
where the tracking error e is defined the same as before. Again we reparameterize the control law, with a different reparameterization this time. Let

$$k_p = K + J\lambda$$
$$k_i = K\lambda$$

The original PI control law, with this reparameterization, can be written

$$G_c(s) = \frac{s(K + J\lambda) + K\lambda}{s}$$

Whether by inspectiong the above block diagram or working through the algebra, the control input τ is given by

$$\tau = J\dot{x}_d + Bx + G_c(s)e$$

With the quantity $G_c(s)$ written as

$$G_c(s)e = (K + J\lambda)e + K\lambda \int edt$$

the control input can be expressed

$$\tau = J\dot{x}_d + Bx + (K + J\lambda)e + K\lambda \int edt$$
$$= J\dot{x}_d + Bx + Ke + J\lambda e + K\lambda \int edt$$
$$= J(\dot{x}_d + \lambda e) + Bx + K\left(e + \lambda \int edt\right)$$

Defining the following errors

$$e_1 = \dot{x}_d + \lambda e$$
$$e_2 = e + \lambda \int edt$$

finally the control input of the "better" adaptive PI can be written

$$\tau = Je_1 + Bx + Ke_2$$

This can be represented by the following block diagram
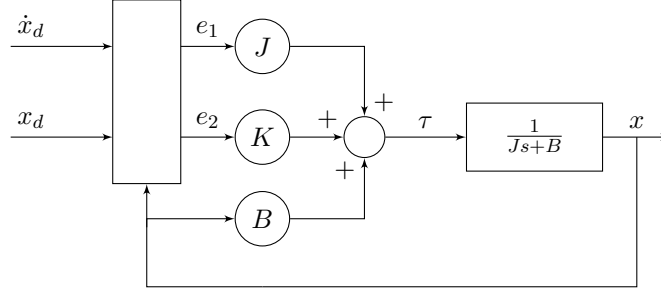
The control input can be expressed more compactly as

$$\tau = \theta_0^\top \omega_0(t)$$

where

$$\theta_0 = \begin{bmatrix} J & B & K \end{bmatrix}^\top$$

and

$$\omega_0(t) = \begin{bmatrix} e_1(t) & x(t) & e_2(t) \end{bmatrix}^\top$$

Again looking at the closed-loop transfer function using the PI controller with positive damping feedback

$$W_{cl}(s) = \frac{k_p s + k_i}{J s^2 + k_p s + k_i}$$

Using the parameterization from before, the characteristic polynomial is

$$J s^2 + (K + J\lambda)s + K\lambda$$

which, using Routh-Hurwitz criterion is stable if all of the coefficients have the same sign. So stable for

$$J > 0 \quad K > 0 \quad \lambda > 0$$

and then

$$r = W_c^{-1}(s)x_d \quad \Rightarrow \quad x \to x_d$$

Completes algebraic part. Now go to analytic part.

Analytic Part

In the analytic part we take the control law that we had before when we assumed the plant parameters where completely known, and replace the values of the plant parameters in the control law with estimates of their values.

Adaptive PI control law, better design: $\qquad \tau = \hat{J}(t)e_1(t) + \hat{B}(t)x(t) + Ke_2(t)$ $\qquad$ (30.3)

This is shown in the following block diagram
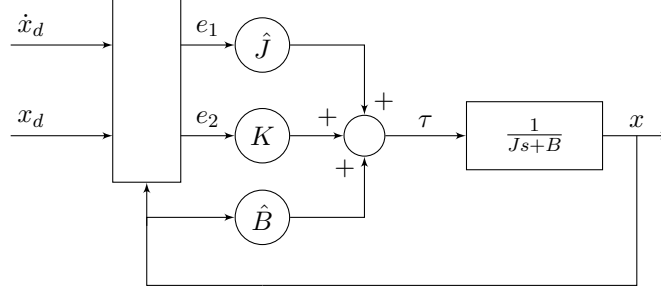
Rearranging the plant dynamics from (30.1)

$$J\dot{x} = -Bx + \tau$$

and substituting in the adaptive control law (30.3) gives

$$J\dot{x} = -Bx + \hat{B}x + \hat{J}e_1 + Ke_2$$
$$= \tilde{B}x + \hat{J}e_1 + Ke_2$$

where damping coefficient parameter error is

$$\tilde{B} = \hat{B} - B$$

Finally this gives the following expression for $\dot{x}$

$$\dot{x} = \frac{1}{J}\left(\tilde{B}x + \hat{J}e_1 + Ke_2\right)$$

Time differentiating the error e we get

$$\dot{e} = \dot{x}_d - \dot{x}$$
$$= \dot{x}_d - \frac{1}{J}\left(\tilde{B}x + \hat{J}e_1 + Ke_2\right)$$

Time differentiating e_2

$$\dot{e}_2 = \dot{e} + \lambda e$$
$$= \dot{x}_d - \frac{1}{J}\left(\tilde{B}x + \hat{J}e_1 + Ke_2\right) + \lambda e$$
$$= (\dot{x}_d + \lambda e) - \frac{1}{J}\left(\tilde{B}x + \hat{J}e_1 + Ke_2\right)$$
$$= e_1 - \frac{1}{J}\left(\tilde{B}x + \hat{J}e_1 + Ke_2\right)$$
$$= \frac{J}{J}e_1 - \frac{\tilde{B}x}{J} - \frac{\hat{J}}{J}e_1 - \frac{Ke_2}{J}$$
$$= -\left(\frac{\hat{J}}{J} - \frac{J}{J}\right)e_1 - \frac{\tilde{B}x}{J} - \frac{Ke_2}{J}$$
$$= -\frac{\tilde{J}}{J}e_1 - \frac{\tilde{B}x}{J} - \frac{Ke_2}{J}$$
$$= -\frac{K}{J}e_2 - \frac{1}{J}\left(\tilde{B}x + \tilde{J}e_1\right)$$

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which is error model 3. Using the following regressor

$$\omega(t) = \begin{bmatrix} -x \\ -e_1 \end{bmatrix}$$

and parameter vector

$$\tilde{\theta} = \begin{bmatrix} \tilde{B} \\ \tilde{J} \end{bmatrix}$$

The error dynamics for $\dot{e}_2$ can be expressed in the following block diagram.
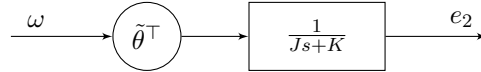
Stability

Since the plant is a DC motor we know $\text{sgn}(J) > 0$. Goal now is to drive $e_2 \to 0$. We attempt to find stable update laws using the following proposed Lyapunov function

$$V = \frac{1}{2}\left(Je_2^2 + \frac{1}{\gamma_1}\tilde{J}^2 + \frac{1}{\gamma_2}\tilde{B}^2 \right)$$

Time differentiating

$$\dot{V} = Je_2\dot{e}_2 + \frac{1}{\gamma_1}\tilde{J}\dot{\tilde{J}} + \frac{1}{\gamma_2}\tilde{B}\dot{\tilde{B}}$$

$$\dot{V} = Je_2\left(-\frac{K}{J}e_2 - \frac{1}{J}\left(\tilde{B}x + \tilde{J}e_1 \right) \right) + \frac{1}{\gamma_1}\tilde{J}\dot{\tilde{J}} + \frac{1}{\gamma_2}\tilde{B}\dot{\tilde{B}}$$

$$= -Ke_2^2 - \tilde{B}xe_2 - \tilde{J}e_1e_2 + \frac{1}{\gamma_1}\tilde{J}\dot{\tilde{J}} + \frac{1}{\gamma_2}\tilde{B}\dot{\tilde{B}}$$

And we can see that if we choose the following adaptive laws

$$\dot{\tilde{J}} = \gamma_1 e_2 e_1$$

$$\dot{\tilde{B}} = \gamma_2 e_2 x$$

and substitute them into the $\dot{V}$ equation

$$\dot{V} = -Ke_2^2$$

Thus $\dot{V} \le 0$. With $e_2, \tilde{J}, \tilde{B} \in \mathcal{L}_\infty \Rightarrow \dot{e}_2 \in \mathcal{L}_\infty$. And $e_2 \in \mathcal{L}_\infty \Rightarrow e \in \mathcal{L}_\infty$ and $e \in \mathcal{L}_\infty \Rightarrow e_1 \in \mathcal{L}_\infty$. Therefore $\dot{e}_2 \in \mathcal{L}_\infty$ and $e_2 \in \mathcal{L}_2 \Rightarrow \lim_{t\to\infty} e_2(t) = 0 \Rightarrow \lim_{t\to\infty} e(t) = 0$.

Summary

┌─────────────────── Adaptive PI: Better Design ───────────────────┐

Plant: $J\dot{x} + Bx = \tau$

Control: $\tau = \hat{J}e_1 + \hat{B}x + Ke_2$

Error: $e = x_d - x$

$e_1 = \dot{x}_d + \lambda e$

$e_2 = e + \lambda \int e(\tau)d\tau$

Parameterization: $k_p = K + J\lambda$

$k_i = K\lambda$

Update laws: $\dot{\hat{J}} = \gamma_1 e_2 e_1$

$\dot{\hat{B}} = \gamma_2 e_2 x$

└──┘

Example 50 Simulation: Adaptive PI Control of a DC Motor This example applies the better adaptive controller to a DC motor with unknown moment of inertia $J = 2$ and damping $B = 0.5$. The reparameterized controller gains are $K = 1$ and $\lambda = 1$. For the nonadaptive controller, where constant estimates of the motor parameters are used, the values are $\hat{J} = 0.5$ and $\hat{B} = 1$.
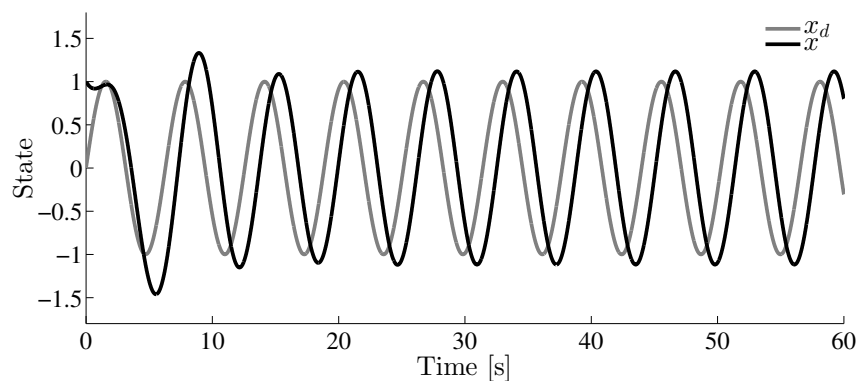
Figure 30.1: Nonadaptive PI control of DC motor when $\hat{J}$ and $\hat{B}$ are fixed.
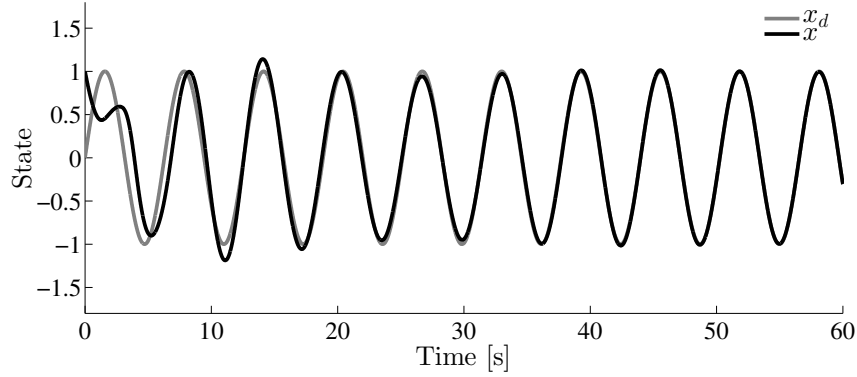
Figure 30.2: Adaptive PI control of DC motor.

30.1.3 More on the Parameterization of $G_c(s)$

$$G_c(s) = \frac{K(s+\lambda)}{s} = \frac{(K+J\lambda)s + k\lambda}{s}$$

reparameterization gives damping ratio which is less dependent on J.

$$\zeta = \frac{\sqrt{k}}{2\sqrt{J\lambda}}$$

versus

$$\zeta = \frac{k + J\lambda}{2\sqrt{kJ\lambda}}$$

30.2 Adaptive PID Control

The goal of this section is to design an adaptive PID controller for the following second order plant.

$$G_p(s) = \frac{1}{s(Js + B)}$$

where J, B, λ, and k are all positive. This is the same motor control problem that we used for the adaptive PI controller, but we note now that the output x is the motor position instead of velocity. This control architecture is represented by the following block diagram. The tracking error is

$$e = x_d - x$$

In addition, we again assume both x and $\dot{x}$ are available for measurement.
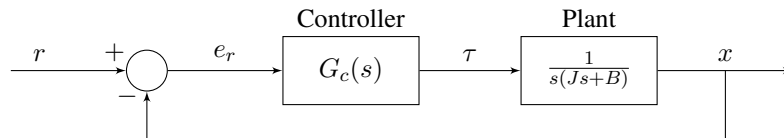
Again we begin with the algebraic part: the solution for J, B known. This time we use a PID controller, given in general by

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s$$
$$= \frac{K_p s + K_i + K_d s^2}{s}$$

30.2.1 Adaptive PID Controller: Original Design

We can reduce number of control parameters from three to two by requiring both of the controller zeros lie at the same place

$$G_c(s) = \frac{K(s + \lambda)^2}{s}$$

We find the closed-loop transfer function W_{cl} from r to x to see under what conditions the system will be stable.

$$W_{cl}(s) = \frac{x}{r} = \frac{G_c G_p}{1 + G_c G_p}$$
$$= \frac{\frac{K(s+\lambda)^2}{s} \frac{1}{s(Js+B)}}{1 + \frac{K(s+\lambda)^2}{s} \frac{1}{s(Js+B)}}$$
$$= \frac{K(s + \lambda)^2}{s^2(Js + B) + K(s + \lambda)^2}$$

Looking at the closed-loop characteristic equation, and using the Routh-Hurwitz stability criterion, the system will be stable for

$$K < \frac{J\lambda}{2} - B$$
$$\lambda > 0$$

Now that stability can be met, we want to figure out what r should be based on x_d so that $x \to x_d$. This is the goal of tracking. In the same way as with the PI controller, we consider a feed-forward controller which will, assuming all of the plant parameters are known exactly, cancel out the plant dynamics, allowing x to perfectly track x_d after transients.
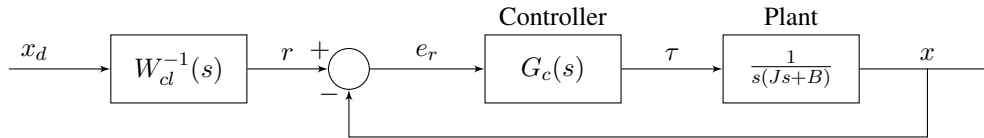
where

$$W_{cl}^{-1}(s) = \frac{1 + G_c G_p}{G_c G_p}$$
$$= \frac{1}{G_c G_p} + 1$$
$$= 1 + s(Js + B)G_c^{-1}$$

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and r is given by

$$
\begin{aligned}
r &= W_{cl}^{-1}(s)x_d \\
&= x_d + [s(Js + B)G_c(s)^{-1}]x_d \\
&= x_d + BG_c(s)^{-1}sx_d + JG_c(s)^{-1}s^2x_d
\end{aligned}
$$

using

$$
\omega_d = G_c(s)^{-1}x_d
$$

the reference can be written as

$$
r = x_d + B\dot{\omega}_d + K\ddot{\omega}_d
$$

30.2.2 Adaptive PID Controller: Better Design

Before we proceed with the PID control design, we again want to improve our design as we did in the case of the PI controller by using positive rate feedback, as shown in the following block diagram.
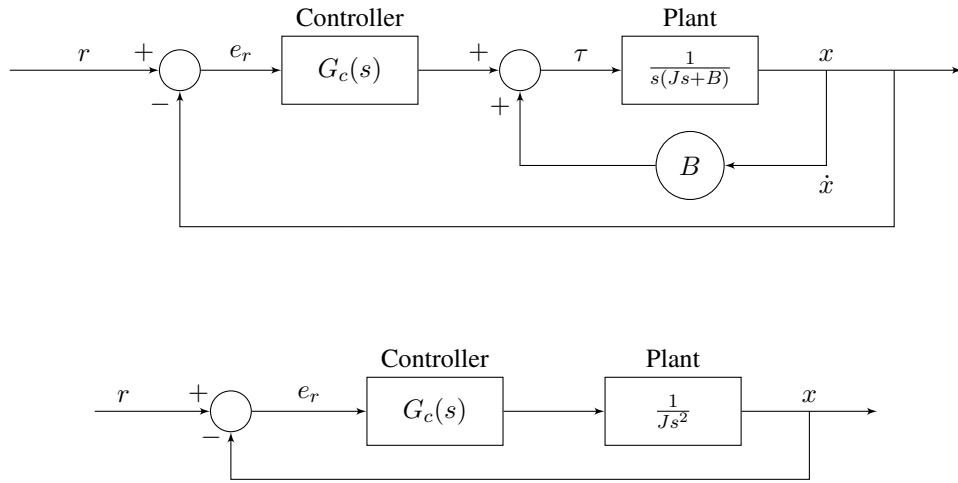
Algebraic Part

Now if we call the transfer function from r to x as $W_{cl}(s)$, we have

$$
\begin{aligned}
W_{cl}(s) &= \frac{G_c(s)\frac{1}{Js^2}}{1 + G_c(s)\frac{1}{Js^2}} \\
&= \frac{G_c}{Js^2 + G_c}
\end{aligned}
$$

So now we put the feed-forward control on the block diagram above as
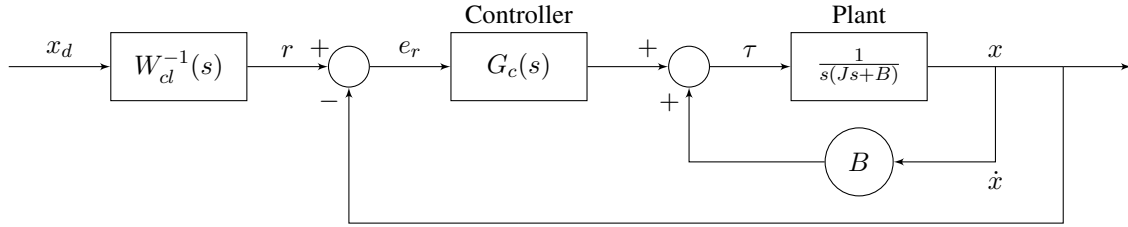
where

$$W_{cl}^{-1}(s) = \frac{Js^2 + G_c}{G_c}$$
$$= 1 + Js^2 G_c^{-1}$$

characteristic polynomial stable $\forall B, J$ **SHOW THIS HERE WITH ROUTH-HURWITZ**

$$r = W_{cl}^{-1}(s)x_d$$
$$= x_d + JG_c^{-1}(s)\ddot{x}_d$$

which is represented in the block diagram below
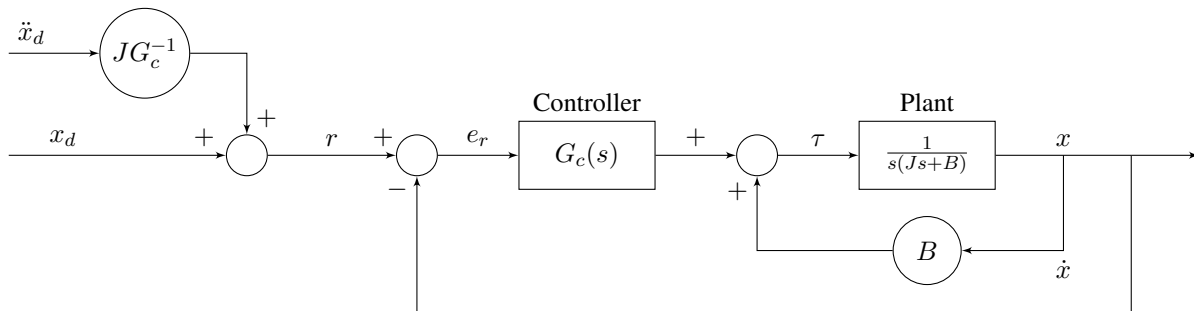
which can be again redrawn as
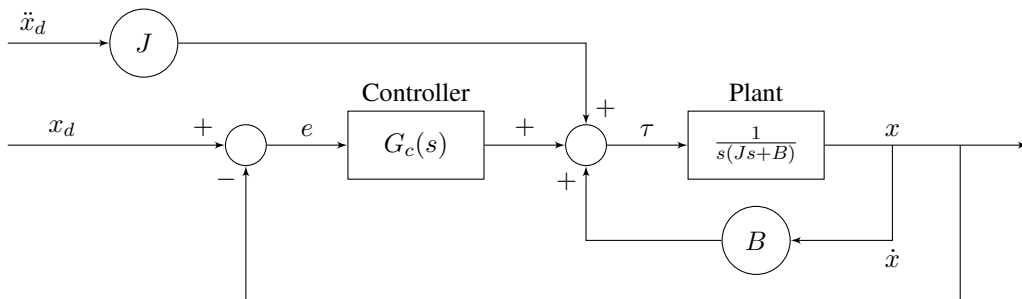
Reparameterize the controller as

$$G_c(s) = \frac{(K + 2\lambda J)s^2 + (2\lambda K + \lambda^2 J)s + K\lambda^2}{s}$$

$$= (K + 2\lambda J)s + (2\lambda K + \lambda^2 J) + K\lambda^2 \frac{1}{s}$$

and then

$$G_c e = (K + 2\lambda J)\dot{e} + (2\lambda K + \lambda^2 J)e + K\lambda^2 \int e\,dt$$

With structure of $G_c(s)$

$$\tau = B\dot{x} + J\ddot{x}_d + G_c e$$

$$= B\dot{x} + J\ddot{x}_d + (K + 2\lambda J)\dot{e} + (2\lambda K + \lambda^2 J)e + K\lambda^2 \int e\,dt$$

$$= B\dot{x} + J\ddot{x}_d + K\dot{e} + 2\lambda J\dot{e} + 2\lambda K e + \lambda^2 J e + K\lambda^2 \int e\,dt$$

$$= J(\ddot{x}_d + 2\lambda\dot{e} + \lambda^2 e) + B\dot{x} + K\left(\dot{e} + 2\lambda e + \lambda^2 \int e\,dt\right)$$

using the following errors

$$e_1 = \ddot{x}_d + 2\lambda\dot{e} + \lambda^2 e$$

$$e_2 = \dot{e} + 2\lambda e + \lambda^2 \int e\,dt$$

the total control input is

$$\tau = Je_1 + B\dot{x} + Ke_2$$

This completes the nominal PID design, or algebraic part.

Analytic Part

We now proceed with the analytic part by first replacing the plant parameter values in the control law with their estimates.

$$\tau = \hat{J}e_1 + \hat{B}\dot{x} + Ke_2$$

closed-loop

$$\ddot{x} = \frac{1}{J}(-B\dot{x} + \tau)$$

$$= \frac{1}{J}(-B\dot{x} + \hat{J}e_1 + \hat{B}\dot{x} + Ke_2)$$

$$= \frac{1}{J}((\hat{B} - B)\dot{x} + \hat{J}e_1 + Ke_2)$$

$$= \frac{1}{J}(\tilde{B}\dot{x} + \hat{J}e_1 + Ke_2)$$

$$= \frac{K}{J}e_2 + \frac{1}{J}\left(\hat{J}e_1 + \tilde{B}\dot{x}\right)$$

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where
$$\tilde{B} = \hat{B} - B$$

and

$$\ddot{e} = \ddot{x}_d - \ddot{x}$$
$$= \ddot{x}_d - \frac{K}{J}e_2 - \frac{1}{J}\left(\hat{J}e_1 + \tilde{B}\dot{x}\right)$$

use definition of e_2

$$\dot{e}_2 = \ddot{e} + 2\lambda\dot{e} + \lambda^2 e$$
$$= \ddot{x}_d - \frac{K}{J}e_2 - \frac{1}{J}\left(\hat{J}e_1 + \tilde{B}\dot{x}\right) + 2\lambda\dot{e} + \lambda^2 e$$
$$= e_1 - \frac{K}{J}e_2 - \frac{1}{J}\left(\hat{J}e_1 + \tilde{B}\dot{x}\right)$$
$$= -\frac{K}{J}e_2 + \left(\frac{J}{J} - \frac{\hat{J}}{J}\right)e_1 - \frac{\tilde{B}}{J}\dot{x}$$
$$= -\frac{K}{J}e_2 - \frac{\tilde{J}}{J}e_1 - \frac{\tilde{B}}{J}\dot{x}$$

where

$$\tilde{J} = \hat{J} - J$$

and then some stuff don't really know why this is here

$$\omega = \begin{bmatrix} -e_1 \\ \dot{x} \end{bmatrix}$$

and

$$\tilde{\theta} = \begin{bmatrix} \tilde{J} \\ \tilde{B} \end{bmatrix}$$

Propose the following

$$V = \frac{1}{2}\left(Je_2^2 + \tilde{J}^2 + \tilde{B}^2\right)$$

Time differentiating

$$\dot{V} = Je_2\dot{e}_2 + \tilde{J}\dot{\tilde{J}} + \tilde{B}\dot{\tilde{B}}$$
$$\dot{V} = Je_2\left(-\frac{K}{J}e_2 - \frac{\tilde{J}}{J}e_1 - \frac{\tilde{B}}{J}\dot{x}\right) + \tilde{J}\dot{\tilde{J}} + \tilde{B}\dot{\tilde{B}}$$
$$= -Ke_2^2 - \tilde{J}e_1e_2 - \tilde{B}\dot{x}e_2 + \tilde{J}\dot{\tilde{J}} + \tilde{B}\dot{\tilde{B}}$$

And we can see that if we choose the following adaptive laws

$$\dot{\tilde{J}} = e_2e_1$$
$$\dot{\tilde{B}} = e_2\dot{x}$$

and substitute them into the $\dot{V}$ equation

$$\dot{V} = -Ke_2^2$$

3rd order error model

Summary

PI was velocity control, PID was position control.

<table>
<tr><td colspan="2" align="center">Adaptive PID: Better Design</td></tr>
<tr><td>**Plant:**</td><td>$J\ddot{x} + B\dot{x} = \tau$</td></tr>
<tr><td>**Control:**</td><td>$\tau = \hat{J}e_1 + \hat{B}\dot{x} + Ke_2$</td></tr>
<tr><td>**Error:**</td><td>$e = x_d - x$</td></tr>
<tr><td></td><td>$e_1 = \ddot{x}_d + 2\lambda e + \lambda^2 e$</td></tr>
<tr><td></td><td>$e_2 = \dot{e} + 2\lambda e + \lambda^2 \int e\, d\tau$</td></tr>
<tr><td>**Update laws:**</td><td>$\hat{J} = \gamma_1 \int_0^t e_2(\tau)e_1(\tau)d\tau$</td></tr>
<tr><td></td><td>$\hat{B} = \gamma_2 \int_0^t e_2(\tau)\dot{x}(\tau)d\tau$</td></tr>
</table>

30.3 Adaptive Phase-Lead Control

The goal of this section is to design an adaptive phase-lead controller for the following second order plant.

$$G_m(s) = \frac{1}{s(Js + B)}$$

where J, B, are the motor inertia and damping, respectively, so we know they are positive. This is the same motor control problem that we used for the adaptive PID controller, where the output x is the motor position. This control architecture is represented by the following block diagram, where the controller is given by

$$G_c(s) = \frac{k(s + z_c)}{s + p_c}$$

where $0 < z_c < p_c$. In addition, we again assume both x and $\dot{x}$ are available for measurement.
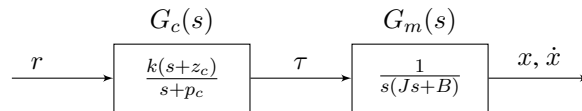
Because we assumed that x and $\dot{x}$ are measurable, we can take as the output

$$y = \dot{x} + ax$$

where $a > 0$ and express the system block diagram as follows, where we define $G_p(s)$ as

$$G_p(s) = \frac{s + a}{s(Js + B)}$$

giving the following block diagram
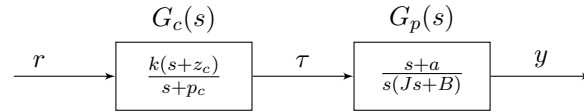
Now put a gain in the feedback path, to allow us to have additional control over the closed-loop system.
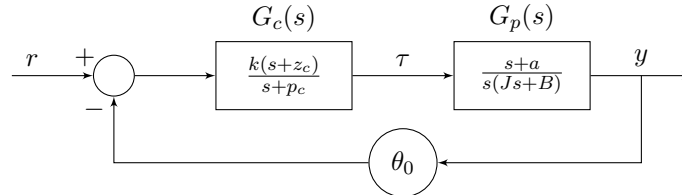
ROOT LOCUS

CLP locations depend on B and J

stable $\forall\ J > 0\ B > 0\ k > 0\ 0 < z_c < p_c\ \theta_0 > 0$ phase margin depend on J, B.

For desired pole-locations θ_0 depends on J, B

$k > 0\ 0 < z_c < p_c\ a > 0$

Tracking: Goal: $y \to y_d$. The closed-loop transfer function from r to y is

$$W_{cl}(s) = \frac{G_c(s)G_p(s)}{1 + \theta_0 G_c(s)G_p(s)}$$

Assuming all the plant parameters are known, we want to design a feed-forward controller $W_{cl}^{-1}(s)$ to give the appropriate reference signal r so that the transfer function from y_d to y is unity. In block diagram this is
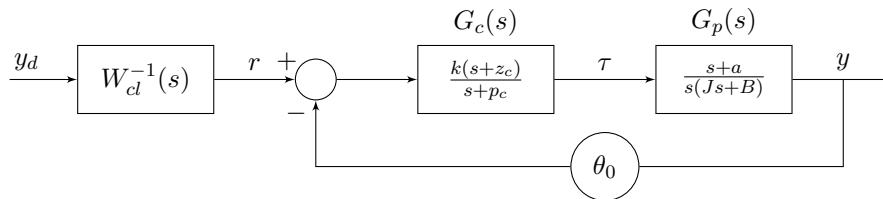
we want

$$\frac{y}{y_d} = \frac{r}{y_d}\frac{y}{r} = 1$$

so

$$W_{cl}^{-1}(s) = \frac{1 + \theta_0 G_c(s)G_p(s)}{G_c(s)G_p(s)}$$

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and then

$$\frac{r}{y_d} = \frac{1 + \theta_0 G_c(s) G_p(s)}{G_c(s) G_p(s)}$$

$$= \frac{1}{G_c(s) G_p(s)} + \theta_0$$

$$= \left[\left(\frac{s(Js + B)}{s + a} \right) G_c^{-1}(s) \right] + \theta_0$$

giving

$$r = \left[\left(\frac{s(Js + B)}{s + a} \right) G_c^{-1}(s) \right] y_d + \theta_0 y_d$$

defining

$$G_c^{-1}(s) y_d = \bar{\omega}_d$$

and

$$\frac{1}{s + a} \bar{\omega}_d = \omega_d$$

the reference signal r becomes

$$r = \theta_0 y_d + [s(Js + B)] \omega_d$$

$$= \theta_0 y_d + J\ddot{\omega}_d + B\dot{\omega}_d$$

Fixed phase-lead design will be:
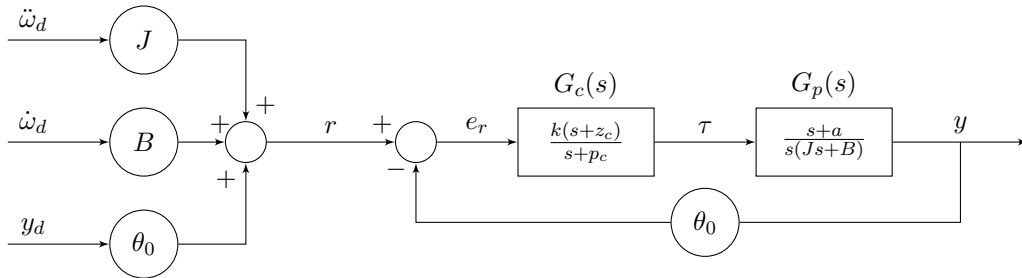
To get the reference model, we assume the plant parameters are known, and replace θ_0 with a nominal value θ_0^* such that the resulting closed-loop system as shown below is SPR, and when $\theta_0 = \theta_0^*$ then we have $y = y_d$.

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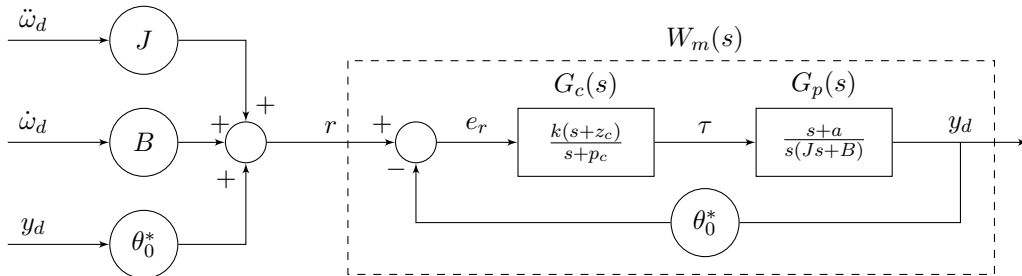
θ_0^* desired control parameter which produces desired closed-loop pole locations. If B and J are unknown this implies θ_0^* is unknown. So the output y_d from the reference model is given as

$$y_d = W_m(s)\theta^{*\top}\omega_D$$

where

$$\theta^* = \begin{bmatrix} J & B & \theta_0^* \end{bmatrix}^\top$$

and

$$\omega_D = \begin{bmatrix} \ddot{\omega}_d & \dot{\omega}_d & y_d \end{bmatrix}^\top$$

WHAT TO DO WHEN θ_0^* is unknown??????

So the transfer function $W_m(s)$ from r to y is given by

$$
\begin{aligned}
W_m(s) &= \frac{G_c(s)G_p(s)}{1 + \theta_0^* G_c(s)G_p(s)} \\[2mm]
&= \frac{\frac{k(s+z_c)}{s+p_c}\frac{s+a}{s(Js+B)}}{1 + \theta_0^*\frac{k(s+z_c)}{s+p_c}\frac{s+a}{s(Js+B)}} \\[2mm]
&= \frac{\frac{k(s+z_c)(s+a)}{(s+p_c)s(Js+B)}}{\frac{(s+p_c)s(Js+B)}{(s+p_c)s(Js+B)} + \frac{\theta_0^* k(s+z_c)(s+a)}{(s+p_c)s(Js+B)}} \\[2mm]
&= \frac{\frac{k(s+z_c)(s+a)}{(s+p_c)s(Js+B)}}{\frac{(s+p_c)s(Js+B)+\theta_0^* k(s+z_c)(s+a)}{(s+p_c)s(Js+B)}} \\[2mm]
&= \frac{k(s+z_c)(s+a)}{(s+p_c)s(Js+B) + \theta_0^* k(s+z_c)(s+a)}
\end{aligned}
$$

To get the adaptive phase-lead block diagram, instead of using known and nominal values, we have to use estimates of the plant parameters (since they are not known) and a time varying parameter θ_0.

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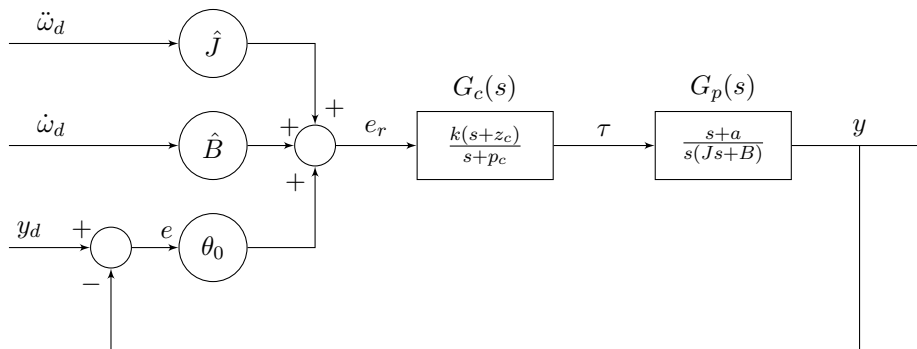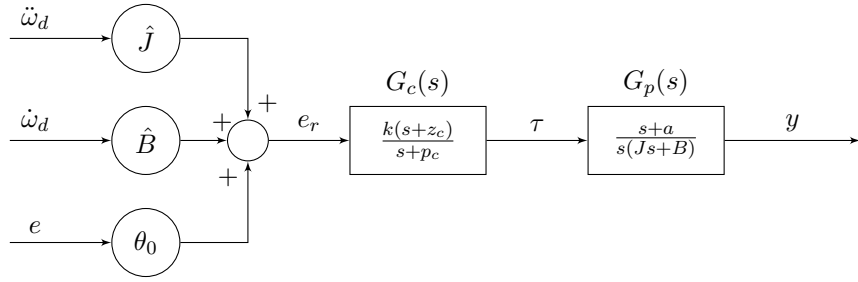
Using the following parameter error definitions, we split the parameter estimates into the actual and error

$$\tilde{\theta}_0 = \theta_0 - \theta_0^*$$
$$\tilde{J} = \hat{J} - J$$
$$\tilde{B} = \hat{B} - B$$

This gives the following block diagram representation
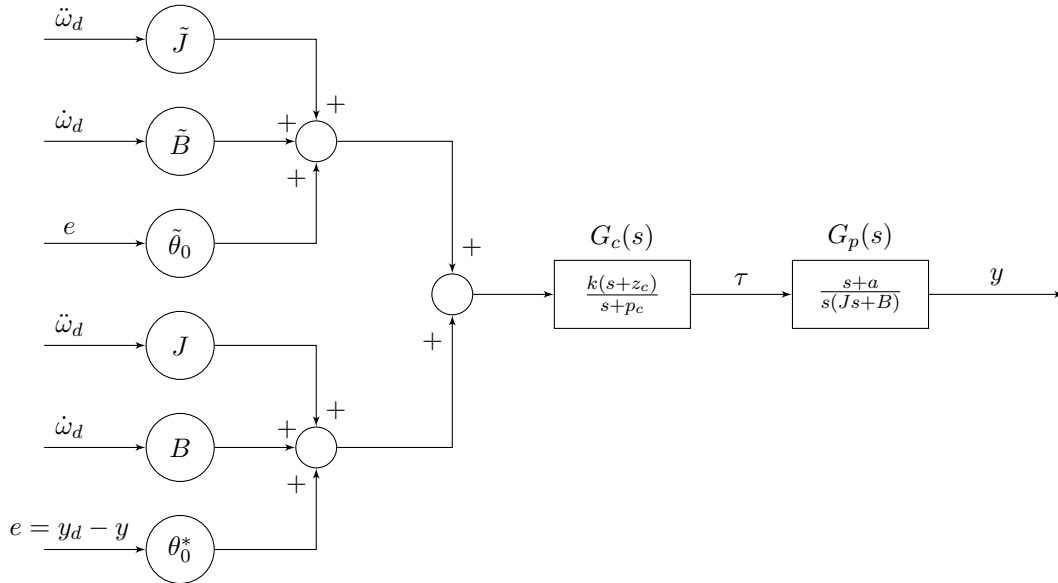
Also, recognizing that $e = y_d - y$, we can re-absorb a portion of the signal e which goes through θ_0^*, giving the following block diagram
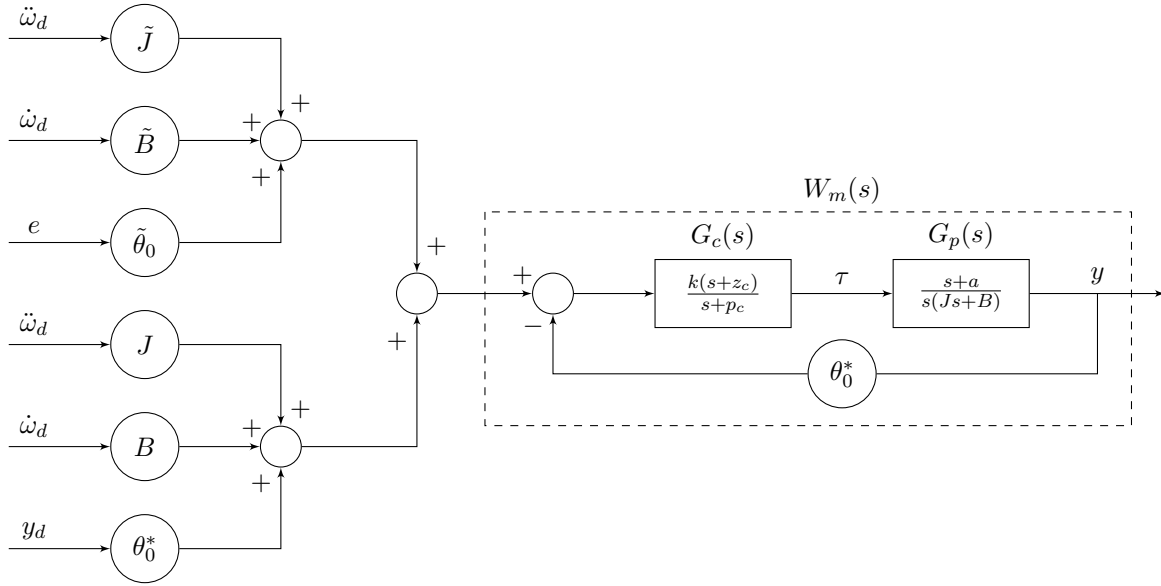
So at this point, we can simplify the above plant representation as the following.

$$y = W_m(s)\big(\tilde{\theta}^\top \omega + \theta^{*\top} \omega_D\big)$$

where

$$\tilde{\theta} = \begin{bmatrix} \tilde{J} & \tilde{B} & \tilde{\theta}_0 \end{bmatrix}^\top$$

and

$$\omega = \begin{bmatrix} \ddot{\omega}_d & \dot{\omega}_d & e \end{bmatrix}^\top$$

Defining the error as

$$e_y = y - y_d$$

and evaluating this error we get

$$\begin{aligned} e_y &= W_m(s)(\tilde{\theta}^\top \omega + \theta^{*\top} \omega_D) - W_m(s)\theta^{*\top} \omega_D \\ &= W_m(s)\tilde{\theta}^\top \omega \end{aligned}$$

use nominal phase-lead controlled nominal plant model to use as reference model.
$|phase W_m(s)| < 90 degrees$
positive correlation of u and y where u is input to y... if you were to take an integral (parsevals theorem) integrate uy from 0 to t will be lower bounded by some number...

Chapter 31

States Accessible Identification and Control

31.1 Preliminaries

In the adaptive control examples shown so far, a Lyapunov function has been used to prove stability when an appropriate update law has been proposed. The cancellation of terms in the stability proof relied on the fact that the terms were all scalar, and scalar multiplication is commutative, allowing the terms to be rearranged and cancelled. In the case where the terms are vectors and matrices, the commutativity of multiplication does not hold. Because of this, the trace operator must be used to allow us a way of still being able to rearrange terms.

31.1.1 The Trace Operator

Trace is linear operator which operates on square matrices. The trace operator sums the diagonal entries of its argument. Trace has the following properties for square matrices A, B, and scalars c.

- $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
- $\text{tr}(cA) = c \cdot \text{tr}(A)$
- $\text{tr}(AB) = \text{tr}(BA)$
- $\text{tr}(A^\top) = \text{tr}(A)$

For an $n \times m$ matrix C and an $m \times n$ matrix D, the following also holds.

- $\text{tr}(CD) = \text{tr}(DC)$

This allows more generally for the following to hold, where none of the matrices need be square, as long as their product is square.

- $\text{tr}(WXYZ) = \text{tr}(XYZW)$

These are called cyclic permutations, and follows exactly from the fifth property above. A final "property" that is a consequence of this, is that the inner product of two $n \times 1$ vectors a and b is the trace of the outer product of a and b. That is

- $\text{tr}(ab^\top) = b^\top a$

This is because we use the rule that allows the two quantities to be switched, after which the $n \times n$ quantity $ab^\top$ becomes the scalar quantity $b^\top a$, and the trace of a scalar is just itself. This is essential for use in the adaptive control of multi-input systems, as it allows vectors to be rearranged to facilitate coming up with update laws. We will see examples of this in the following sections.

31.2 Identification

Sec 3.4

$$\dot{x}_p = A_p x_p + B_p u$$

where $x_p \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A_p \in \mathbb{R}^{n \times n}$, $B_p \in \mathbb{R}^{n \times m}$. For identification assume:

1. u bounded $u \in \mathcal{L}_\infty$
2. A_p Hurwitz

The identifier equation is written

$$\dot{\hat{x}}_p = A_m \hat{x}_p + (\hat{A}_p - A_m)x_p + \hat{B}_p u$$

where $\hat{x}_p \in \mathbb{R}^n$. The estimation and parameter errors are

$$e = \hat{x}_p - x_p$$
$$\tilde{A}_p = \hat{A}_p - A_p$$
$$\tilde{B}_p = \hat{B}_p - B_p$$

Note that the order of the terms in the error e is different when doing estimation versus control. That is, for identification the error is $e = \hat{x}_p - x_p$ and for control the error is $e = x_p - x_m$. In general, on the right hand side is the "target". In the case of estimating we say target to mean that we want the estimation to follow plant itself, versus in control x_m is on the right because we want the plant x_p to follow the reference model. We then evaluate the error dynamics

$$
\begin{aligned}
\dot{e} &= \dot{\hat{x}}_p - \dot{x}_p \\
&= A_m \hat{x}_p + (\hat{A}_p - A_m)x_p + \hat{B}_p u - A_p x_p - B_p u \\
&= A_m \hat{x}_p + \hat{A}_p x_p - A_m x_p + \hat{B}_p u - A_p x_p - B_p u \\
&= A_m (\hat{x}_p - x_p) + (\hat{A}_p - A_p)x_p + (\hat{B}_p - B_p)u \\
&= A_m e + \tilde{A}_p x_p + \tilde{B}_p u
\end{aligned}
$$

Choose the following update laws

$$\dot{\tilde{A}}_p = \dot{\hat{A}}_p = -P e x_p^\top$$
$$\dot{\tilde{B}}_p = \dot{\hat{B}}_p = -P e u^\top$$

with transpose

$$\dot{\tilde{A}}_p^\top = -x_p e^\top P$$
$$\dot{\tilde{B}}_p^\top = -u e^\top P$$

and propose the following Lyapunov function

$$V(e, \tilde{A}_p, \tilde{B}_p) = e^\top P e + \operatorname{tr}\left(\tilde{A}_p^\top \tilde{A}_p + \tilde{B}_p^\top \tilde{B}_p\right)$$

Time differentiating the Lyapunov function we get the following, where the trace operator allows cyclic permutations of the operands and then we can remove the trace operator

$$\dot{V} = \dot{e}^\top Pe + e^\top P\dot{e} + \text{tr}\left(\dot{\tilde{A}}_p^\top \tilde{A}_p + \tilde{A}_p^\top \dot{\tilde{A}}_p + \dot{\tilde{B}}_p^\top \tilde{B}_p + \tilde{B}_p^\top \dot{\tilde{B}}_p\right)$$

$$= \dot{e}^\top Pe + e^\top P\dot{e} + 2\text{tr}\left(\dot{\tilde{A}}_p^\top \tilde{A}_p + \dot{\tilde{B}}_p^\top \tilde{B}_p\right)$$

$$= \dot{e}^\top Pe + e^\top P\dot{e} + 2\text{tr}\left(\dot{\tilde{A}}_p^\top \tilde{A}_p\right) + \text{tr}\left(\dot{\tilde{B}}_p^\top \tilde{B}_p\right)$$

$$= (A_m e + \tilde{A}_p x_p + \tilde{B}_p u)^\top Pe + e^\top P(A_m e + \tilde{A}_p x_p + \tilde{B}_p u) + 2\text{tr}\left(\dot{\tilde{A}}_p^\top \tilde{A}_p\right) + \text{tr}\left(\dot{\tilde{B}}_p^\top \tilde{B}_p\right)$$

$$= e^\top(A_m^\top P + PA_m)e + 2e^\top P\tilde{A}_p x_p + 2e^\top P\tilde{B}_p u + 2\text{tr}\left(-x_p e^\top P\tilde{A}_p\right) + 2\text{tr}\left(-u e^\top P\tilde{B}_p\right)$$

$$= e^\top(A_m^\top P + PA_m)e + 2e^\top P\tilde{A}_p x_p + 2e^\top P\tilde{B}_p u + 2\text{tr}\left(-e^\top P\tilde{A}_p x_p\right) + 2\text{tr}\left(-e^\top P\tilde{B}_p u\right)$$

$$= e^\top(A_m^\top P + PA_m)e + 2e^\top P\tilde{A}_p x_p + 2e^\top P\tilde{B}_p u - 2e^\top P\tilde{A}_p x_p - 2e^\top P\tilde{B}_p u$$

$$= -e^\top Qe$$

31.3 Direct Control
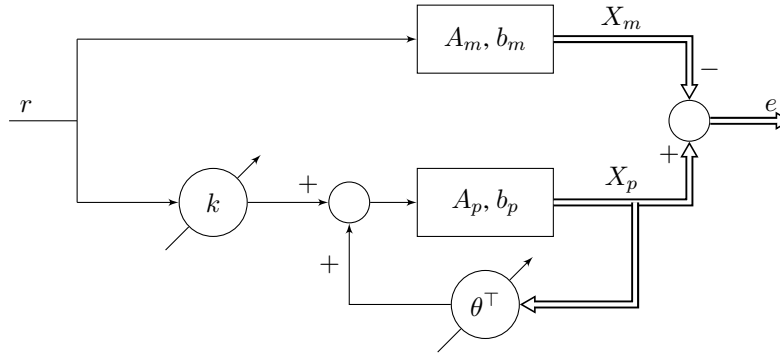
31.3.1 Single Input States Accessible
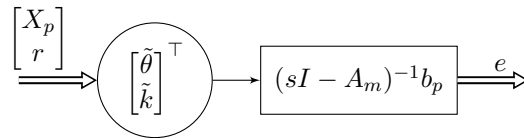
Lyapunov function

$$V(e, \tilde{\theta}) = e^\top Pe + \tilde{\theta}^\top \tilde{\theta}$$

Time differentiating

$$\dot{V} = \dot{e}^\top Pe + e^\top P\dot{e} + \dot{\tilde{\theta}}^\top \tilde{\theta} + \tilde{\theta}^\top \dot{\tilde{\theta}}$$

31.3.2   States Accessible
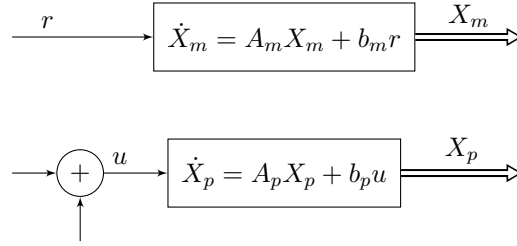
Case (a): A_p is unknown B_p is known

$$\dot{x}_p = A_p x_p + B_p u$$
$$\dot{x}_m = A_m x_m + B_m r$$

Control law

$$u = \theta(t) x_p + k^* r$$

$$\dot{x}_p = A_p x_p + B_p \left(\theta x_p + k^* r \right)$$
$$= (A_p + B_p \theta) x_p + B_p k^* r$$

Matching conditions

$$A_p + B_p \theta^* = A_m$$
$$B_p k^* = B_m$$

A_p and A_m have n^2 degrees of freedom. θ^* has nm degrees of freedom. Matching condition harder than controllability... more than just matching the eigenvalues... have to match eigenvectors too... The tracking and parameter errors are

$$e = x_p - x_m$$
$$\tilde{\theta} = \theta - \theta^*$$

plant equation becomes

$$\dot{x}_p = A_p x_p + B_p \theta x_p + B_p k^* r$$
$$= A_p x_p + B_p \theta^* x_p + B_p \tilde{\theta} x_p + B_p k^* r$$
$$= (A_p + B_p \theta^*) x_p + B_p \tilde{\theta} x_p + B_p k^* r$$
$$= A_m x_p + B_p \tilde{\theta} x_p + B_m r$$

Error dynamics

$$\dot{e} = A_m x_p + B_p \tilde{\theta} x_p + B_m r - A_m x_m - B_m r$$
$$= A_m e + B_p \tilde{\theta} x_p$$

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Error Model 2
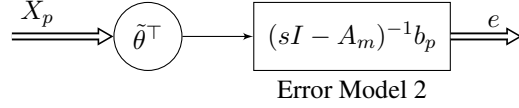
Lyapunov function

$$V(e, \tilde{\theta}) = e^\top P e + \mathrm{tr}\left(\tilde{\theta}^\top \tilde{\theta}\right)$$

Time differentiating

$$\begin{aligned}
\dot{V} &= \dot{e}^\top P e + e^\top P \dot{e} + \mathrm{tr}\left(\dot{\tilde{\theta}}^\top \tilde{\theta} + \tilde{\theta}^\top \dot{\tilde{\theta}}\right) \\
&= \dot{e}^\top P e + e^\top P \dot{e} + \mathrm{tr}\left(\dot{\tilde{\theta}}^\top \tilde{\theta}\right) + \mathrm{tr}\left(\tilde{\theta}^\top \dot{\tilde{\theta}}\right) \\
&= \dot{e}^\top P e + e^\top P \dot{e} + 2\mathrm{tr}\left(\dot{\tilde{\theta}}^\top \tilde{\theta}\right) \\
&= (A_m e + B_p \tilde{\theta} x_p)^\top P e + e^\top P(A_m e + B_p \tilde{\theta} x_p) + 2\mathrm{tr}\left(\dot{\tilde{\theta}}^\top \tilde{\theta}\right) \\
&= e^\top (A_m^\top P + P A_m) e + (B_p \tilde{\theta} x_p)^\top P e + e^\top P(B_p \tilde{\theta} x_p) + 2\mathrm{tr}\left(\dot{\tilde{\theta}}^\top \tilde{\theta}\right) \\
&= -e^\top Q e + 2 e^\top P B_p \tilde{\theta} x_p + 2\mathrm{tr}\left(\dot{\tilde{\theta}}^\top \tilde{\theta}\right)
\end{aligned}$$

Propose the following update law

$$\dot{\tilde{\theta}} = \dot{\theta} = -B_p^\top P e x_p^\top$$

with transpose

$$\dot{\tilde{\theta}}^\top = -x_p e^\top P B_p$$

Plugging into $\dot{V}$

$$\begin{aligned}
\dot{V} &= -e^\top Q e + 2 e^\top P B_p \tilde{\theta} x_p + 2\mathrm{tr}\left(-x_p e^\top P B_p \tilde{\theta}\right) \\
&= -e^\top Q e + 2 e^\top P B_p \tilde{\theta} x_p + 2\mathrm{tr}\left(-e^\top P B_p \tilde{\theta} x_p\right) \\
&= -e^\top Q e
\end{aligned}$$

FINISH ME: All signals bounded

$$-\int_0^\infty \dot{V} = -\int -e^\top Q e = V(0) - V(\infty) = \int e^\top Q e \leq V(0)$$

and then

$$\int e^\top \lambda_{\min}(Q) e = \int e^\top Q e \leq V(0)$$

note $e^\top Q e \geq e^\top \lambda_{\min}(Q) e$

$$\|e\|_{L_2}^2 \leq \frac{V(0)}{\lambda_{\min}(Q)}$$

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ORM: Case (b) A_p unknown $B_p = B\Lambda$ where B is known   Plant and reference model

$$\dot{x}_p = A_p x_p + B_p u$$
$$\dot{x}_m = A_m x_m + B_m r$$

Control law

$$u = \theta(t) x_p + k(t) r$$

the plant equation becomes

$$\dot{x}_p = A_p x_p + B_p \left(\theta x_p + kr \right)$$
$$= \left(A_p + B_p \theta \right) x_p + B_p kr$$

Matching conditions

$$A_p + B\Lambda \theta^* = A_m$$
$$B\Lambda k^* = B_m$$

Update laws

$$\dot{\theta} = -B^\top P e x_p^\top$$
$$\dot{k} = -B^\top P e r^\top$$

errors

$$e = x_p - x_m$$
$$\tilde{\theta} = \theta - \theta^*$$
$$\tilde{k} = k - k^*$$

plant equation becomes

$$\dot{x}_p = A_p x_p + B_p \theta x_p + B_p kr$$
$$= A_p x_p + B_p \theta^* x_p + B_p \tilde{\theta} x_p + B_p k^* r + B_p \tilde{k} r$$
$$= (A_p + B\Lambda \theta^*) x_p + B\Lambda \tilde{\theta} x_p + B\Lambda k^* r + B\Lambda \tilde{k} r$$
$$= A_m x_p + B\Lambda \tilde{\theta} x_p + B_m r + B\Lambda \tilde{k} r$$

error

$$\dot{e} = A_m x_p + B\Lambda \tilde{\theta} x_p + B_m r + B\Lambda \tilde{k} r - A_m x_m - B_m r$$
$$= A_m e + B\Lambda \tilde{\theta} x_p + B\Lambda \tilde{k} r$$

Lyapunov

$$V(e, \tilde{\theta}, \tilde{k}) = e^\top P e + \text{tr} \left(\tilde{\theta}^\top \Lambda \tilde{\theta} \right) + \text{tr} \left(\tilde{k}^\top \Lambda \tilde{k} \right)$$

Time differentiating

$$\dot{V} = \dot{e}^\top Pe + e^\top P\dot{e} + \text{tr}\left(\dot{\tilde{\theta}}^\top \Lambda \tilde{\theta}\right) + \text{tr}\left(\tilde{\theta}^\top \Lambda \dot{\tilde{\theta}}\right) + \text{tr}\left(\dot{\tilde{k}}^\top \Lambda \tilde{k}\right) + \text{tr}\left(\tilde{k}^\top \Lambda \dot{\tilde{k}}\right)$$

$$= \dot{e}^\top Pe + e^\top P\dot{e} + 2\text{tr}\left(\dot{\tilde{\theta}}^\top \Lambda \tilde{\theta}\right) + 2\text{tr}\left(\dot{\tilde{k}}^\top \Lambda \tilde{k}\right)$$

$$= (A_m e + B\Lambda \tilde{\theta} x_p + B\Lambda \tilde{k} r)^\top Pe + e^\top P(A_m e + B\Lambda \tilde{\theta} x_p + B\Lambda \tilde{k} r) + 2\text{tr}\left(\dot{\tilde{\theta}}^\top \Lambda \tilde{\theta}\right) + 2\text{tr}\left(\dot{\tilde{k}}^\top \Lambda \tilde{k}\right)$$

$$= -e^\top Qe + 2e^\top P(B\Lambda \tilde{\theta} x_p + B\Lambda \tilde{k} r) + 2\text{tr}\left(\dot{\tilde{\theta}}^\top \Lambda \tilde{\theta}\right) + 2\text{tr}\left(\dot{\tilde{k}}^\top \Lambda \tilde{k}\right)$$

$$= -e^\top Qe + 2e^\top PB\Lambda \tilde{\theta} x_p + 2e^\top PB\Lambda \tilde{k} r + 2\text{tr}\left(\dot{\tilde{\theta}}^\top \Lambda \tilde{\theta}\right) + 2\text{tr}\left(\dot{\tilde{k}}^\top \Lambda \tilde{k}\right)$$

Use the following update laws

$$\dot{\tilde{\theta}} = -B^\top Pe x_p^\top$$
$$\dot{\tilde{k}} = -B^\top Pe r^\top$$

with transpose

$$\dot{\tilde{\theta}}^\top = -x_p e^\top PB$$
$$\dot{\tilde{k}}^\top = -r e^\top PB$$

Plugging these into the $\dot{V}$ we get

$$\dot{V} = -e^\top Qe + 2e^\top PB\Lambda \tilde{\theta} x_p + 2e^\top PB\Lambda \tilde{k} r - 2\text{tr}\left(x_p e^\top PB\Lambda \tilde{\theta}\right) - 2\text{tr}\left(r e^\top PB\Lambda \tilde{k}\right)$$

$$= -e^\top Qe + 2e^\top PB\Lambda \tilde{\theta} x_p + 2e^\top PB\Lambda \tilde{k} r - 2\text{tr}\left(e^\top PB\Lambda \tilde{\theta} x_p\right) - 2\text{tr}\left(e^\top PB\Lambda \tilde{k} r\right)$$

$$= -e^\top Qe + 2e^\top PB\Lambda \tilde{\theta} x_p + 2e^\top PB\Lambda \tilde{k} r - 2e^\top PB\Lambda \tilde{\theta} x_p - 2e^\top PB\Lambda \tilde{k} r$$

$$= -e^\top Qe$$

31.4   Tuning Gains and Closed-Loop Reference Model

From this point forward, all control will be assumed to be direct, unless stated otherwise.

31.4.1   Addition of Tuning Gain to Update Law

Briefly revisit the two adaptive control examples from last lecture: control of a scalar plant with one, and two unknowns. It is worth mention here that in both cases the parameter updated laws that were selected could have been modified to include a tuning gain, usually denoted by γ. The inclusion of this term requires the Lyapunov function to be modified, but provides the control engineer a degree of freedom which can be used to vary the learning rate of the system, allowing the time response of the adaptive system to be tuned. The following example shows the changes that must be made to the update law and Lyapunov function for the control of the scalar plant with two unknowns.

Example 51 Addition of adaptive tuning gain to update law Consider the example of adaptive control of the scalar plant with two unknowns. We now propose the following candidate Lyapunov function, which contains the γ^{-1} term, where $\gamma > 0$

$$V(e, \tilde{\bar{\theta}}) = \frac{1}{2}e^2 + \frac{1}{2}\gamma^{-1}|k_p|\tilde{\bar{\theta}}^\top \tilde{\bar{\theta}}$$

Differentiating along system trajectories we get

$$\dot{V}(e, \tilde{\bar{\theta}}) = e\dot{e} + \gamma^{-1}|k_p|\tilde{\bar{\theta}}^\top \dot{\tilde{\bar{\theta}}}$$
$$= e(a_m e + k_p \tilde{\bar{\theta}}^\top \phi) + \gamma^{-1}|k_p|\tilde{\bar{\theta}}^\top \dot{\tilde{\bar{\theta}}}$$
$$= a_m e^2 + e k_p \tilde{\bar{\theta}}^\top \phi + \gamma^{-1}|k_p|\tilde{\bar{\theta}}^\top \dot{\tilde{\bar{\theta}}}$$

We then propose the following parameter update law

$$\dot{\tilde{\bar{\theta}}} = -\gamma \operatorname{sgn}(k_p)e\phi$$

Substituting this into $\dot{V}$ we get

$$\dot{V}(e, \tilde{\bar{\theta}}) = a_m e^2 + e k_p \tilde{\bar{\theta}}^\top \phi - |k_p|\operatorname{sgn}(k_p)e\tilde{\bar{\theta}}^\top \phi$$
$$= a_m e^2 + e k_p \tilde{\bar{\theta}}^\top \phi - e k_p \tilde{\bar{\theta}}^\top \phi$$
$$= a_m e^2$$

As in the previous examples $\dot{V} \leq 0$, and the stability proof follows previous direct adaptive control examples, but now we have added the additional flexibility of a tuning parameter.

This tuning gain changes the learning rate and thus the time response of the system, as shown in the following simulation examples.

Example 52 Adaptive control of a scalar plant when changing γ Consider the following transfer function representations of a plant and reference model below where $a_p = 1$, $k_p = 2$, $a_m = -1$, $k_m = 1$.

$$\frac{x_p}{u} = \frac{k_p}{s - a_p} \qquad \frac{x_m}{x_d} = \frac{k_m}{s - a_m}$$

The following plots show the time response of this adaptive system for three different values of γ, showing the rapid oscillations associated with a large learning rate.
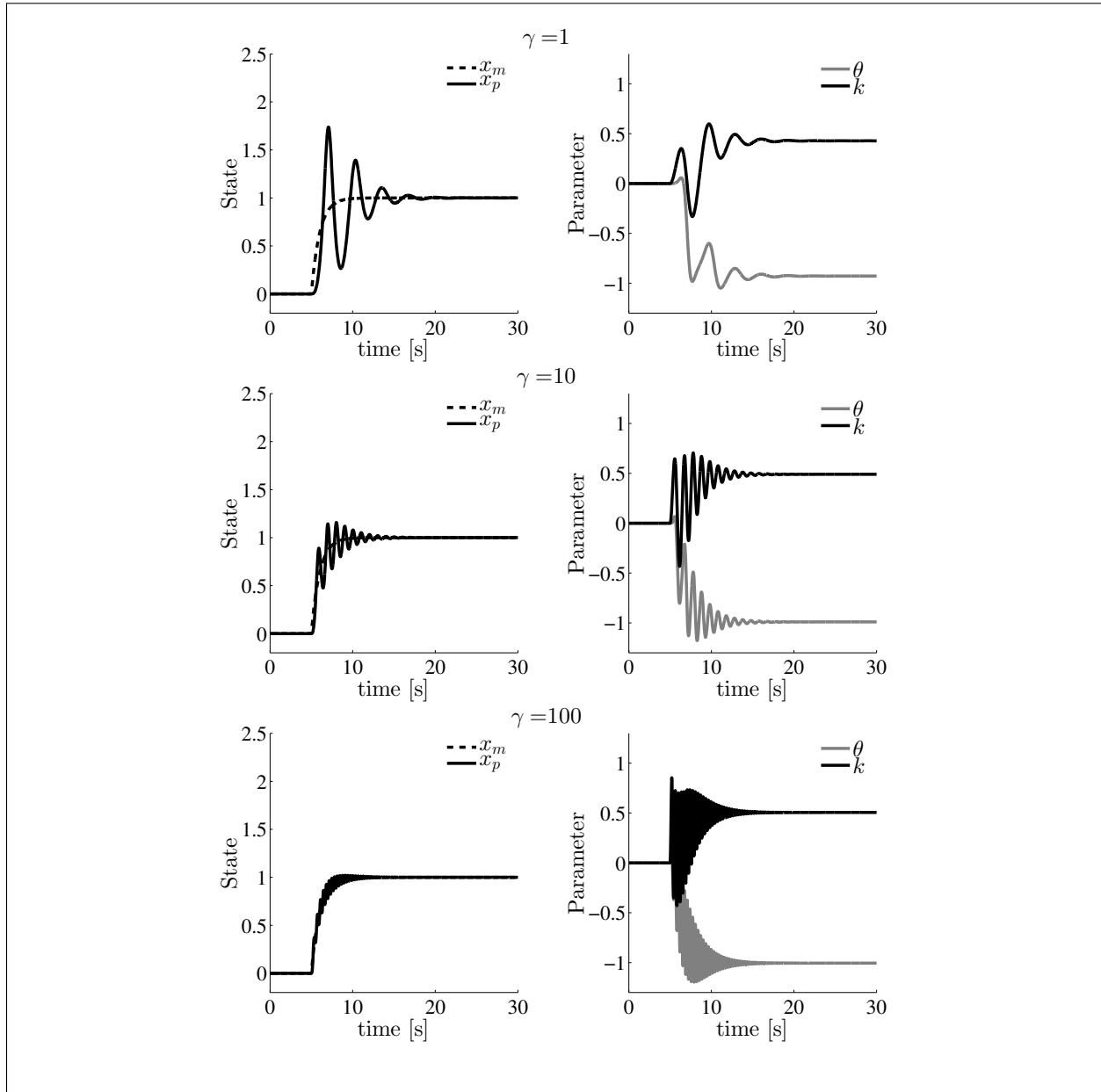
Transient Performance

In finding a stable adaptive controller, we have only just introduced the first method by which the control engineer has been able to exercise any design freedom. In all of our adaptive controllers thus far, we have only proved boundedness and convergence of some signals to zero. In other words, we have shown only that the error will go to zero in the limit as $t \to \infty$, but we have said nothing about what the system response will do between now and infinity. This section will introduce a method which provides additional flexibility to tune the adaptive controller to provide a desirable transient response.

31.4.2 Closed-Loop Reference Model Adaptive Control

The reference model used before is modified with the addition of an observer-like gain, $\ell < 0$, and is called closed-loop reference model, as opposed to the reference model we have seen before, which we call open-loop reference models. The addition of this term allows the reference model response to deviate from what was originally determined to be the desired response. Examples to follow will help make this more clear. Note, however, that the tracking error is now denoted e^c, where $e^c = x_p - x_m^c$. This is to differentiate it from the "real", or open-loop reference model error $e^o = x_p - x_m^o$.

$$\dot{x}_m^c = a_m x_m^c + b_m r - \ell e^c$$

Additionally, note that if $\ell = 0$, the open-loop reference model is recovered.

Example 53 Direct adaptive control of a scalar system with two unknowns: Closed-Loop Reference Model We again consider the scalar plant and reference model below, where a_p and b_p are unknown, but the sign of b_p is known.

$$\dot{x}_p = a_p x_p + k_p u$$
$$\dot{x}_m^c = a_m x_m^c + k_m r - \ell e^c$$

Propose the following controller
$$u = \theta x_p + k r$$

Block
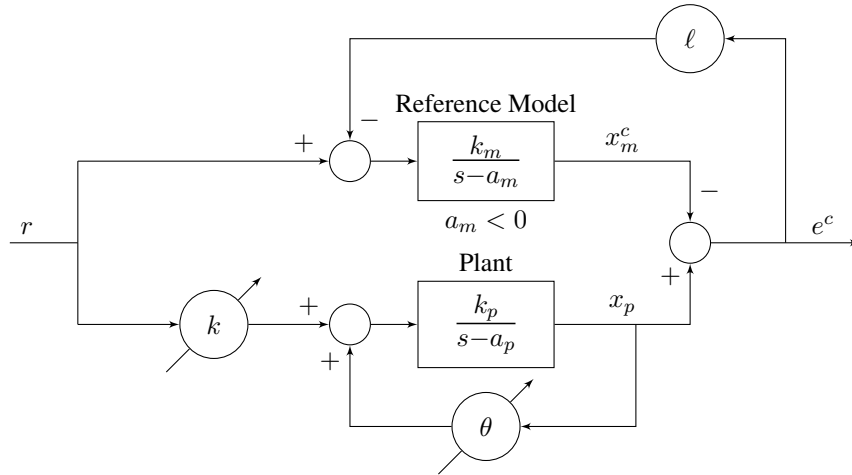
Substituting in the control law, the plant equation can then be expressed as

$$\dot{x}_p = a_m x_p + k_p \theta x_p + k_p k r$$

or

$$\dot{x}_p = a_m x_p + k_p \bar{\theta}^\top \phi$$

---

where the parameter and regressor are

$$\bar{\theta}^\top = \begin{bmatrix} \theta & k \end{bmatrix} \qquad \text{and} \qquad \phi^\top = \begin{bmatrix} x_p & r \end{bmatrix}$$

The tracking error is

$$
\begin{aligned}
\dot{e}^c &= \dot{x}_p - \dot{x}_m^c \\
&= a_m x_p + b_m r + b_p \tilde{\bar{\theta}}^\top \phi - a_m x_m^c - b_m r + \ell e^c \\
&= (a_m + \ell)e^c + b_p \tilde{\bar{\theta}}^\top \phi
\end{aligned}
$$

We now propose the following candidate Lyapunov function in order to prove stability of this adaptive system

$$V(e^c, \tilde{\bar{\theta}}) = \frac{1}{2}e^{c2} + \frac{1}{2}\gamma^{-1}|b_p|\tilde{\bar{\theta}}^\top \tilde{\bar{\theta}}$$

Differentiating along system trajectories we get

$$
\begin{aligned}
\dot{V}(e^c, \tilde{\bar{\theta}}) &= e^c \dot{e}^c + \gamma^{-1}|b_p|\tilde{\bar{\theta}}^\top \dot{\tilde{\bar{\theta}}} \\
&= e^c \left((a_m + \ell)e^c + b_p \tilde{\bar{\theta}}^\top \phi \right) + \gamma^{-1}|b_p|\tilde{\bar{\theta}}^\top \dot{\tilde{\bar{\theta}}} \\
&= (a_m + \ell)e^{c2} + e^c b_p \tilde{\bar{\theta}}^\top \phi + \gamma^{-1}|b_p|\tilde{\bar{\theta}}^\top \dot{\tilde{\bar{\theta}}}
\end{aligned}
$$

We propose now the following adaptive parameter update law

$$\boxed{\dot{\tilde{\bar{\theta}}} = -\gamma \,\mathrm{sgn}(b_p)e^c \phi}$$

Substituting this into $\dot{V}$ we get

$$\dot{V}(e^c, \tilde{\bar{\theta}}) = (a_m + \ell)e^{c2}$$

And so we have $\dot{V} \leq 0$.

Example 54 CRM control of a scalar plant Consider the following transfer function representations of a plant and reference model below where $a_p = 1$, $k_p = 2$, $a_m = -1$, $k_m = 1$.

$$\frac{x_p}{u} = \frac{k_p}{s - a_p}$$

$$\frac{x_m}{x_d} = \frac{k_m}{s - a_m}$$
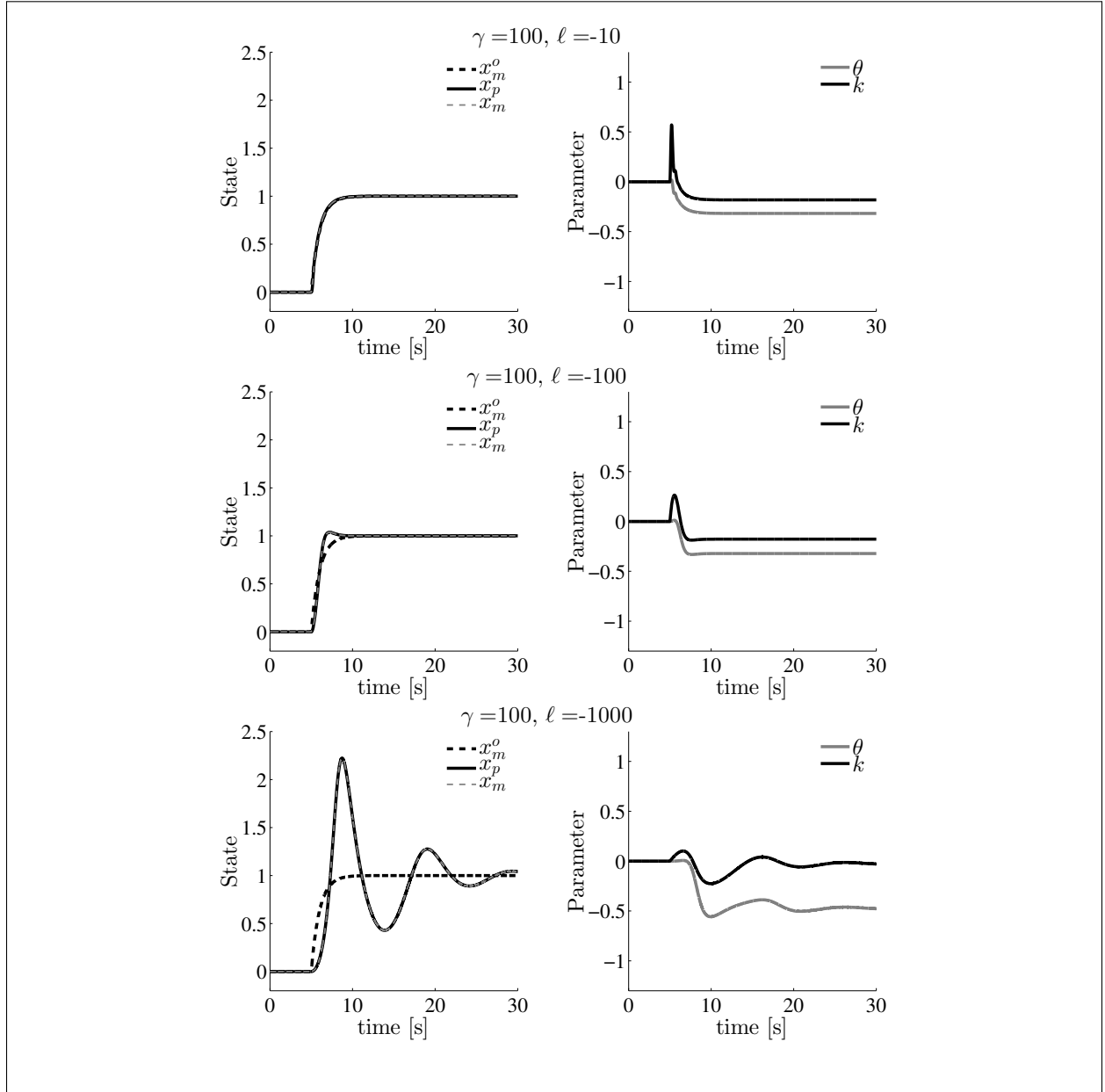
Analyzing the Transient Performance of CRM Adaptive Control

In this section we will compute some analytical bounds on several quantities when using CRM adaptive control on a scalar plant as shown in the example above.

$\mathcal{L}_\infty$ norm of e^c

Since V is positive definite and $\dot{V}$ negative semi-definite, along system trajectories we have

$$V(e^c(t), \tilde{\bar{\theta}}(t)) \leq V(e^c(0), \tilde{\bar{\theta}}(0)) < \infty$$

Expanding the left hand side we have

$$\frac{1}{2}e^{c2} + \frac{1}{2}\gamma^{-1}|b_p|\tilde{\bar{\theta}}^\top\tilde{\bar{\theta}} \le V(e^c(0), \tilde{\bar{\theta}}(0))$$

and since $\frac{1}{2}\gamma^{-1}|b_p|\tilde{\bar{\theta}}^\top\tilde{\bar{\theta}} \ge 0$ we can simplify this inequality to

$$\frac{1}{2}e^{c2} \le V(e^c(0), \tilde{\bar{\theta}}(0))$$

which shows that $e^c(t)$ is bounded, with bound given by

$$(e^c(t))^2 \le 2V(e^c(0), \tilde{\bar{\theta}}(0))$$

which is

$$\boxed{\|e^c(t)\|^2_{L_\infty} \le 2V(0)}$$

$\mathcal{L}_2$ norm of e^c

Integrating the Lyapunov time derivative

$$\int_0^\infty \dot{V}(e^c(\tau), \tilde{\bar{\theta}}(\tau))d\tau = V(\infty) - V(0)$$

Substituting in the expression for $\dot{V}$ we get

$$\int_0^\infty (a_m + \ell)(e^c(\tau))^2 d\tau = V(\infty) - V(0)$$

Pulling out a negative, and with $V(\infty) \ge 0$ we get

$$-\int_0^\infty (a_m + \ell)(e^c(t))^2 = V(0) - V(\infty) \le V(0)$$

Pulling the constant terms out of the integral

$$-(a_m + \ell)\int_0^\infty (e^c(t))^2 \le V(0)$$

Recalling that $a_m < 0$ and $\ell < 0$ this can be written

$$-(a_m + \ell)\int_0^\infty (e^c(t))^2 = |a_m + \ell|\int_0^\infty (e^c(t))^2 \le V(0)$$

so we can write

$$\int_0^\infty e^c(t)^2 \le \frac{V(0)}{|a_m + \ell|}$$

Substituting in the expression for $V(0)$ from the Lyapunov function proposed above

$$\int_0^\infty e^c(t)^2 \le \frac{\frac{1}{2}e(0)^2 + \frac{|b_p|}{2\gamma}\tilde{\bar{\theta}}^\top(0)\tilde{\bar{\theta}}(0)}{|a_m + \ell|}$$

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which is the $\mathcal{L}_2$ norm of $e^c(t)$.

$$\boxed{\|e^c(t)\|_{L_2} \leq \frac{V(0)}{|a_m + \ell|}}$$

and we can evaluate the initial condition of the Lyapunov function $V(0)$ as

$$V(0) = \frac{1}{2}e(0)^2 + \frac{|b_p|}{2\gamma}\tilde{\tilde{\theta}}^\top(0)\tilde{\tilde{\theta}}(0)$$

$\mathcal{L}_2$ norm of $\dot{k}$

From the update law, taking $\hat{\theta}$ apart into the θ and k entries, the update law for k is

$$\dot{k} = -\gamma\mathrm{sgn}(b_p)e^c r$$

squaring both sides this becomes

$$|\dot{k}|^2 = \gamma^2 e^{c2} r^2$$

integrating both sides

$$\int_0^\infty |\dot{k}|^2 d\tau = \int_0^\infty \gamma^2 e^c(t)^2 r^2 d\tau$$

Next, we can write an inequality by replacing the value of $r(t)$ at every instant in time with its supremum over all time. Thus the r term is no longer time dependent, and can be pulled out of the integral with γ giving

$$\int_0^\infty |\dot{k}|^2 d\tau \leq \int_0^\infty \gamma^2 \sup_t |r(t)|^2 e^c(t)^2 d\tau = \gamma^2 \|r(t)\|_{L_\infty}^2 \int_0^\infty e^c(\tau)^2 d\tau$$

$$\int_0^\infty |\dot{k}|^2 d\tau \leq \gamma^2 \|r(t)\|_{L_\infty}^2 \|e^c(t)\|_{L_2}$$

We then recognize several terms as $\mathcal{L}_p$ norms, and the inequality becomes

$$\|\dot{k}(t)\|_{L_2}^2 \leq \gamma^2 \|r(t)\|_{L_\infty}^2 \|e^c(t)\|_{L_2}^2$$

$$\boxed{\|\dot{k}(t)\|_{L_2}^2 \leq \frac{\gamma^2 \|r(t)\|_{L_\infty}^2 V(0)}{|a_m + \ell|}}$$

$\mathcal{L}_2$ norm of $\dot{\theta}$

L_∞ norm of $x_m(t)$ The solution to the reference model ODE $\dot{x}_m^c = a_m x_m^c + b_m r - \ell e^c$ is

$$x_m(t) = \exp(a_m t)x_m(0) + \int_0^t k_m\exp(a_m(t - \tau))r(\tau)d\tau + (-\ell)\int_0^t \exp(a_m(t - \tau))e(\tau)d\tau$$

where the first two terms are

$$\leq \|x_m^o\|_{L_\infty}$$

Use Cauchy-Schwarz inequality for second term

$$\int_0^\infty |fg|d\tau \le \sqrt{\int_0^\infty |f|^2}\sqrt{\int_0^\infty |g|^2}$$

which can also be written

$$\|fg\|_{L_1} \le \|f\|_{L_2}\|g\|_{L_2}$$

so second term is

$$(-\ell)\infty_0^t\exp(a_m(t-\tau))e(\tau)d\tau \le \sqrt{\int_0^t \exp(2a_m(t-\tau))d\tau}\sqrt{\int_0^\infty e^2 d\tau}$$

$$\|x_m(t)\|_{L_\infty} \le \|x_m^o\|_{L_\infty} + \frac{|\ell|}{\sqrt{|2a_m|}\|e\|_{L_2}}$$

$$\|x_m(t)\|_{L_\infty}^2 \le 2\|x_m^o\|_{L_\infty}^2 + \frac{\ell^2}{a_m}\|e\|_{L_2}^2$$

$$\|x_m(t)\|_{L_\infty}^2 \le 2\|x_m^o\|_{L_\infty}^2 + \frac{\ell^2}{a_m}\frac{V(0)}{|a_m+\ell|}$$

$\mathcal{L}_2$ norm of $\dot\theta$ From the update law, taking $\hat\theta$ apart into the θ and k entries, the update law for θ is

$$\dot\theta = -\gamma\text{sgn}(k_p)ex_p$$
$$= -\gamma\text{sgn}(k_p)e(e+x_m)$$

squaring both sides

$$|\dot\theta|^2 = \gamma^2 e^2(e+x_m)^2$$

We have the following inequality

$$(a+b)^2 \le 2a^2 + 2b^2$$

which we can see by expanding the following non-negative expression

$$0 \le (a-b)^2 = a^2 - 2ab + b^2$$

adding $a^2 + 2ab + b^2$ to both sides we get

$$a^2 + 2ab + b^2 \le 2a^2 + 2b^2$$
$$(a+b)^2 \le 2a^2 + 2b^2$$

Using this inequality

$$(e+x_m)^2 \le 2e^2 + 2x_m^2$$

we can write

$$|\dot\theta|^2 \le \gamma^2 e^2(2e^2 + 2x_m^2)$$
$$\le 2\gamma^2 e^2 e^2 + 2\gamma^2 e^2 x_m^2$$

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look at first part of expression on right hand side

$$\int_0^\infty |\dot\theta|^2 d\tau \le 2\gamma^2 \int_0^\infty e^2 e^2 d\tau$$

$$\le 2\gamma^2 \int_0^\infty |\sup_t e(t)|^2 e^2 d\tau$$

$$\le 2\gamma^2 \|e(t)^2\|_{L_\infty} \int_0^\infty e^2 d\tau$$

$$\le 2\gamma^2 \|e(t)^2\|_{L_\infty} \|e(t)^2\|_{L_2}$$

can say

$$V(t) \le V(0)$$

and so we also have

$$\frac{1}{2} e(t)^2 \le V(0)$$

so

$$e(t)^2 \le 2V(0)$$

$$\|e\|_{L_2}^2 \le \frac{V(0)}{|a_m + \ell|}$$

$$\le 2\gamma^2 2V(0) \frac{V(0)}{|a_m + \ell|}$$

$$\|\dot\theta\|_{L_2} \le \frac{4\gamma^2 V(0)^2}{|a_m + \ell|}$$

PART ONE ONLY. Now part 2:

$$\|\dot\theta\|_{L_2} \le 2\gamma^2 \int_0^\infty e^2 x_m^2 dt$$

$$\le 2\gamma^2 \|x_m\|_{L_\infty} \int_0^\infty e^2 dt$$

combining parts 1 and 2

$$\le 2\gamma^2 \left[2\|x_m^o\|_{L_\infty}^2 + \frac{\ell^2}{a_m} \|e\|_{L_2}^2 \right] \|e\|_{L_2}^2$$

$$\le \frac{4\gamma^2 \|x_m^o\|_{L_\infty}^2 V(0)^2}{|a_m + \ell|} + \frac{\ell^2}{|a_m|} \frac{V(0)^2}{|a_m + \ell|^2}$$

The following example shows the effect that ℓ has on the time response of a CRM-adaptive controlled scalar system.

Chapter 32

Output Feedback Adaptive Control

- rational
- analytic
- PR/SPR transfer functions
- KY Lemma
- Control of $n^* = 1$ systems
    - Unknown high frequency gain only
    - Unknown high frequency gain and zeros

32.1 Definitions

rational function H(s)

$$H(s) = \frac{P(s)}{Q(s)}$$

where $P(s)$ and $Q(s)$ are polynomials, where a polynomial is

$$P(s) = p_0 + p_1 s + \cdots + p_n s^n$$

Let $H(s) : \mathbb{C} \to \mathbb{C}$ is analytic $\forall\, s \in \Omega$ means that $H(s)$ has a derivative $\forall\, s \in \Omega$.

Example 55 Analytic Transfer Function Let $H(s) = \frac{P(s)}{Q(s)}$, $H(s)$ is analytic $\forall\, s \notin \beta$ means all poles are in β. Analytic basically means where the poles are. Because differentiating in Laplace domain is multiplication by s.

Definition 33 2.6.1 *A rational function $H(s)$ of the complex variable $s = \sigma + j\omega$ is positive real (PR) if*

1. *$H(s)$ is real $\forall$ real s*

2. *$Re[H(s)] \geq 0 \,\forall\, Re[s] > 0$*

Put some examples here

Definition 34 2.7 *A rational function is SPR (strictly positive real) if $H(s-\varepsilon)$ is PR for some $\varepsilon \in \mathbb{R} > 0$.*

Put some examples here

Definition 35 2.8 *A rational function is SPR if and only if*

1. $H(s)$ is analytic in $RE[s] \geq 0$. ie. $H(s)$ is a stable transfer function.. poles in LHP.

2. $Re[H(j\omega)] > 0 \ \forall \ \omega \in (-\infty, infty)$

3. If $n^* = 1 \ \lim_{\omega^2 \to \infty} \omega^2 Re[H(j\omega)] > 0$

relative degree:

$$\frac{P(s)}{Q(s)} = \frac{\cdots + p_m s^m}{\cdots + q_n s^n}$$

$n^* = n - m$

Lemma 3 2.4 — MKY *Given $\gamma \geq 0$, vectors b, c, A stable, $L = L^\top$*

$$\dot{x} = Ax + Bu$$
$$Y = c^\top x + d$$

$Re[H(j\omega)] = Re[\frac{\gamma}{2} + c^\top(j\omega I - A)^{-1}b] > 0 \ \forall \ \omega \in (-\infty, \infty)$ then $\exists \varepsilon > 0$ a vector q and $P = P^\top > 0$
$s+$

1. $A^\top P + PA = -qq^\top - \varepsilon L$
2. $Pb - c = \sqrt{\gamma}q$

Lemma 4 KY

$$\dot{x} = Ax + bu$$
$$y = c^\top x$$

If $c^\top(sI - A)^{-1}b$ is SPR, then $\exists P = P^\top > 0, Q = Q^\top > 0$

1. $A^\top P + PA = -Q$
2. $Pb = c$

32.2 Output Feedback Control of Plants with Relative Degree 1

Relative degree of the plant $W_p(s)$ is $n^* = 1$

$$W_p(s) = k_p \frac{Z_p(s)}{R_p(s)}$$

where k_p is the HFG (high frequency gain) and $Z_p(s)$ and $R_p(s)$ are monic polynomials. A monic polynomial is one that is represented $z_m s^m + z_{m-1} s^{m-1} + \ldots$ where $z_m = 1$.

32.2.1 Case 1: k_p Unknown Only

(page 186) We assume the sign of k_p is known, but the value is unknown.
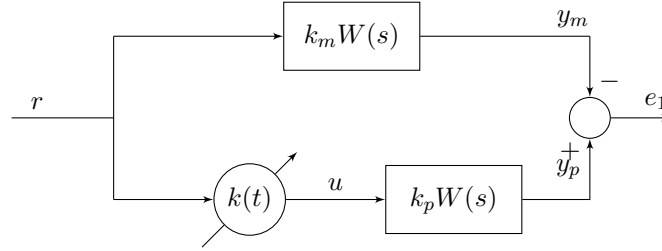
k_p unknown, known sign. Assume $W_p(s)$ is stable, and $W'_p(s)$ is SPR, where

$$W_p(s) = k_p W'_p(s)$$
$$= k_p \frac{Z_p(s)}{R_p(s)}$$

The reference model is

$$W_m(s) = k_m \frac{Z_m(s)}{R_m(s)}$$
$$= k_m \frac{Z_p(s)}{R_p(s)}$$

The plant and reference model outputs are

$$y_p = W'_p(s) k_p k(t) r$$
$$= W'_m(s) k_p k(t) r$$

and

$$y_m = W'_m(s) k_m r$$

And the matching condition is the value of $k(t)$ called k^* such that

$$k^* = \frac{k_m}{k_p}$$

The output tracking error is

$$
\begin{aligned}
e_1 &= y_p - y_m \\
&= W'_m(s) k_p k r - W'_m(s) k_m r \\
&= W'_m(s) k_p \left[k - \frac{k_m}{k_p} \right] r \\
&= W'_m(s) k_p (k - k^*) r \\
&= W'_m(s) k_p \tilde{k} r
\end{aligned}
$$

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where the parameter error is

$$\tilde{k} = k - k^*$$

where $W'_m(s)$ is SPR.
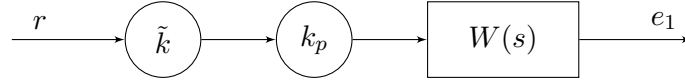
Error Model 3

$$\implies \quad \dot{\tilde{k}} = -sign(k_p)e_1 r$$

$\underline{W(s) \text{ must be strictly positive real.}} \implies$

- $Z_m(s)$ must have roots in C^-.

- Poles and Zeros of $W(s)$ must be interlaced.

Time domain equivalent

$$\dot{e} = A_{mn}e + b_{mn}\tilde{k}r(t)$$
$$e_y = c_{mn}^\top e$$
$$e_1 = k_p e_y$$

Which matches Lemma 5.1 on page 185, giving an update law of

$$\dot{\tilde{k}} = -\text{sgn}(k_p)e_1 r$$

$$k_p[c_{mn}^\top[sI - A_{mn}]^{-1}b_{mn}]\tilde{k}r$$

where this is equal to

$$k_p W'_m(s)\tilde{k}r(t)$$

Show stability using Lyapunov function

$$V = e^\top P e + \frac{1}{|k_p|}\tilde{k}^2$$

Time differentiating

$$\dot{V} = e^\top A_{mn}^\top P e + e^\top P A_{mn} e + \tilde{k}r b_{mn}^\top P e + e^\top P b_{mn}\tilde{k}r(t) + \frac{2}{|k_p|}\tilde{k}\dot{\tilde{k}}$$

$$= -e^\top Q e + 2e_y\tilde{k}r(t) + \frac{2}{|k_p|}\tilde{k}\dot{\tilde{k}}$$

$$= -e^\top Q e + 2\frac{e_1\text{sgn}(k_p)}{|k_p|}\tilde{k}r(t) - \frac{2}{|k_p|}\tilde{k}\text{sgn}(k_p)e_1 r$$

$$= -e^\top Q e$$

1. $A^\top P + PA = -Q$

2. $Pb_{mn} = c_{mn}$

32.2.2 Case 2: k_p Unknown, $Z_p(s)$ Unknown

(page 187)

plant of TF

$$y_p(t) = k_p \frac{Z_p(s)}{R_p(s)} u(t)$$

where $\text{sgn}(k_p)$ was known, $Z_p(s)$ unknown, but with stable roots, and $R_p(s)$ known, stable roots. We choose $R_m(s) = R_p(s)$ to make our lives easier.
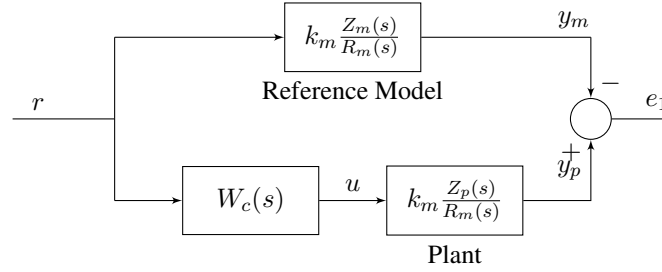
This is an adaptive controller in the forward loop. Don't need feedback to change zero locations. (check this).

k_p unknown, but with known sign. $Z_p(s)$ unknown, minimum phase (stable zeros). $R_p(s)$ is known and stable.

Goal: $k_p \frac{Z_p(s)}{R_p(s)} \to k_m \frac{Z_m(s)}{R_m(s)}$

Pick $R_m(s) = R_p(s)$

Realization of $W_c(s)$

$$w_c(s) = \frac{k_m}{k_p} \frac{Z_m(s)}{Z_p(s)}$$

realized as dynamics

$$u = \theta_{1c}^\top w_1 + k_c r$$
$$\dot{w}_1 = \Lambda w_1 + \ell u$$

where (Λ, ℓ) controllable

$$\lambda(s) = \det(sI - \Lambda) = Z_m(s)$$

$$Z_m(s) - Z_p(s) = t_1^*(s)$$

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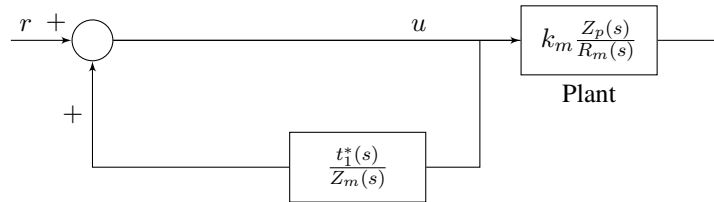
$$W_c(s) = \frac{Z_m(s)}{Z_p(s)} = \frac{Z_m(s)}{Z_m(s) - t_1^*(s)} \tag{32.1}$$

$$= \frac{1}{1 - \frac{t_1^*(s)}{Z_m(s)}} \tag{32.2}$$

$$k_c = \frac{K_m}{K_p}$$

$$\theta_{1c}^\top (sI - \Lambda)^{-1} \ell = \frac{Z_m(s) - Z_p(s)}{Z_m(s)}$$

matching conditions

$$K^* = K_c$$

$$\theta_1^* = \theta_{1c}$$

Realization of $t_1(s)/Z_m(s)$
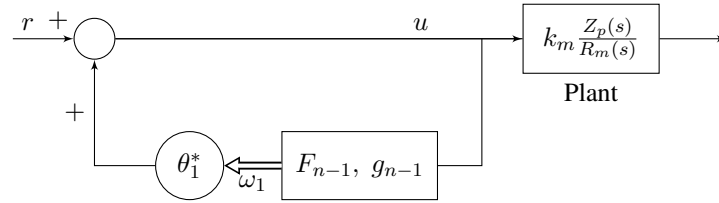
Algebraic part: $\dfrac{t_1(s)}{Z_m(s)} = \qquad\qquad \theta_{1c}^T (sI - F_{n-1})^{-1} g_{n-1}$ $\tag{32.3}$

$$t_1(s) = t_1^*(s) \implies \qquad\qquad \theta_{1c} = \theta_{1c}^* \tag{32.4}$$

$$\frac{W_m(s)}{k^*}$$

$$\frac{k_m \frac{Z_m}{R_m}}{k^*}$$

$$k_p \frac{z_m}{R_m}$$

$$y_p(t) = k_p W_m'(s)[k^* + \tilde{\theta}_1^\top \bar{w}_1]$$

$$\tilde{\theta}_1 = \begin{bmatrix} k - k^* \\ \theta_1 - \theta_1^* \end{bmatrix}$$

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$$\bar{w}_1 = \begin{bmatrix} r \\ w_1 \end{bmatrix}$$

Reference

$$y_m = k_m w'_m(s)r = k_p w'_m(s)k^*r$$

$$e_1 = y_p - y_m$$

$$e_1 = k_p w'_m(s)[\tilde{\theta}_1^\top \bar{w}_1]$$

$$\dot{\tilde{\theta}}_1 = \dot{\theta}_1 = -\text{sgn}(k_p)e_1\bar{w}_1$$

Everything is in the book. Remark, book doesn't use prime notation, so in the error equation they have

$$e_1 = \frac{k_p}{k_m} w_m(s)[\tilde{\theta}_1^\top \bar{w}_1]$$

Writing out in state-space form

$$\dot{x}_p = A_p x_p + b_p u y_p \qquad\qquad = k_p c_p^\top x$$

where the book calls $h_p^\top = k_p c_p^\top$

$$\begin{bmatrix} \dot{x}_p \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A_p & b_p\theta_1^{*\top} \\ 0 & \Lambda + \ell\theta_1^{*\top} \end{bmatrix} \begin{bmatrix} x_p \\ w_1 \end{bmatrix} + \begin{bmatrix} b_p \\ \ell \end{bmatrix} k^*r + \begin{bmatrix} b_p \\ \ell \end{bmatrix} \tilde{\theta}_1^\top w_1$$

$$y_p = K_p[c_p^\top 0]$$

$$x = \begin{bmatrix} x_p \\ w_1 \end{bmatrix}$$

Reference Model

$$\begin{bmatrix} \dot{x}_p^* \\ \dot{w}^* \end{bmatrix} = \begin{bmatrix} A_p & b_p\theta_1^{*\top} \\ 0 & \Lambda + \ell\theta_1^{*\top} \end{bmatrix} \begin{bmatrix} x_p^* \\ w_1^* \end{bmatrix} + \begin{bmatrix} b_p \\ \ell \end{bmatrix} k^*r + \begin{bmatrix} b_p \\ \ell \end{bmatrix} \tilde{\theta}_1^\top w_1$$

$$y_m = K_p[c_{p1}^\top 0]^\top$$

$$x_{mn} = \begin{bmatrix} x_p^* \\ w_1^* \end{bmatrix}$$

where the m means reference model and n means non-minimal

$$\dot{x}_{mn} = A_{mn}x_{mn} + b_{mn}k^*r$$
$$y - C_{mn}^\top x_{mn}$$

and in book $h_{mn}^\top = K_p[C_{p1}^\top 0]^\top$

$$y_m = K_p C_{mn}^\top (sI - A_{mn})^{-1} b_{mn} k^* r$$
$$= K_p w_m' - (s) k^* r$$

where w_m' is SPR by design

ERRORS

$$e = x - x_m$$

$$\dot{e} = A_{mn} e + b_{mn} \tilde{\theta}_1^\top w_1$$
$$e_1 = K_p C_{mn}^\top e$$

Stability

$$V = e^\top P e + \frac{1}{|k_p|} \tilde{\theta}^\top \tilde{\theta}$$

From KY Lemma we have

$$A_{mn}^\top P + P A_{mn} = -Q$$

and

$$P b_{mn} = c_{mn}$$

$$\dot{V} = -e^\top Q e + 2 e^\top P b_{mn} \tilde{\theta}^\top \bar{w}_1 + \frac{2 \tilde{\theta} \dot{\tilde{\theta}}}{|k_p|}$$

$$\dot{V} = -e^\top Q e$$

extra work was to show internal states of system stable.

Output Feedback Control of Plants with Relative Degree 1 (Continued)

Last time plant of TF

$$y_p(t) = k_p \frac{Z_p(s)}{R_p(s)} u(t)$$

where $\text{sgn}(k_p)$ was known, $Z_p(s)$ unknown, but with stable roots, and $R_p(s)$ known, stable roots. We chose last time $R_m(s) = R_p(s)$ to make our lives easier.

32.2.3 Case 3: k_p Unknown, $R_p(s)$ Unknown

(page 190) As before, we assume the sign of k_p is known.
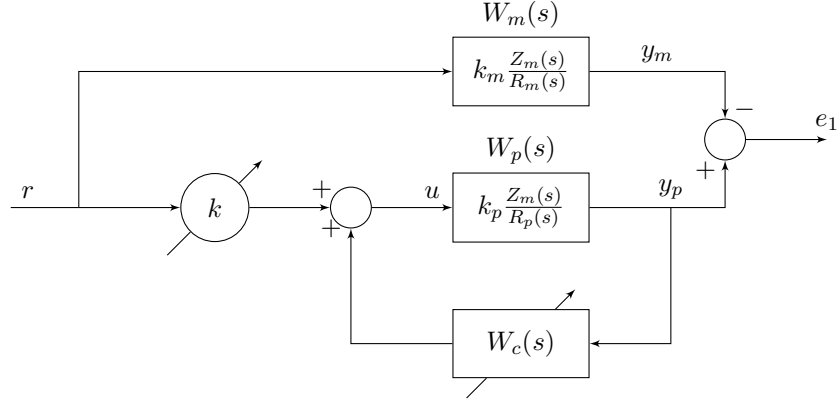
The controller is covered in detail in Narendra, Annaswamy on page 190, and the important equations are repeated here. The goal is to find a controller transfer function $W_c(s)$, which has constant parameters θ_0 and θ_2, and feedforward gain k such that the closed-loop transfer function $W_o(s)$ from r to y_p is $W_m(s)$. If these parameters take the constant values $k = k^*$, $\theta_0 = \theta_{0c}$, and $\theta_2 = \theta_{2c}$ then $W_o(s)$ is given by

$$W_o(s) = \frac{k^* k_p Z_m(s)}{R_p(s)\left[1 - k_p \frac{Z_m(s)}{R_p(s)} W_c(s)\right]}$$

The controller is defined by

$$\dot{\omega}_2 = \Lambda \omega_2 + \ell y_p$$
$$u = kr + \theta_0 y_p + \theta_2^\top \omega_2$$

When θ_0 and θ_2 are the constants θ_{0c} and θ_{2c} respectively, the controller can be written as

$$u = kr + W_c(s) y_p$$

where

$$W_c(s) = \theta_{0c} + \theta_{2c}^\top (sI - \Lambda)^{-1} \ell$$
$$= \frac{D(s)}{\lambda(s)}$$
$$= \frac{D(s)}{Z_m(s)}$$

giving

$$W_o(s) = \frac{k^* k_p Z_m(s)}{R_p(s)\left[1 - k_p \frac{Z_m(s)}{R_p(s)} \frac{D(s)}{Z_m(s)}\right]}$$

If the following matching conditions are satisfied

$$k^* k_p = k_m$$

and

$$R_p(s) - k_p D^*(s) = R_m(s)$$

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then $W_o(s) = W_m(s)$.

The block diagram below is another representation which shows the architecture for which the controller and update laws will be found.
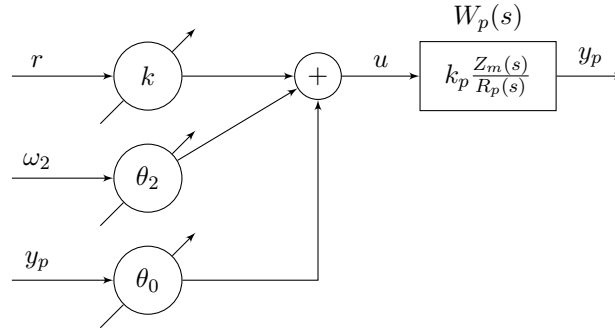
Using the following parameter errors, this block diagram is further simplified.

$$k = \psi + k^*$$
$$\theta_2 = \phi_2 + \theta_2^*$$
$$\theta_0 = \phi_0 + \theta_0^*$$

By manipulating the inputs, and using the matching conditions, the following block diagram is obtained.
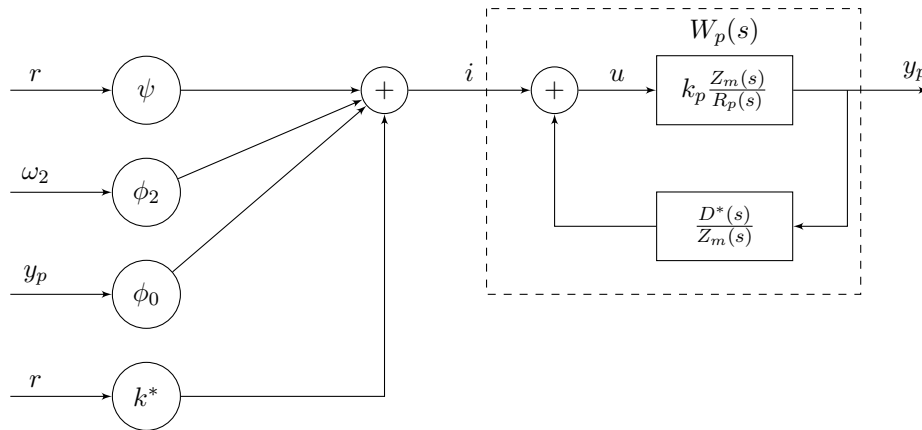
32.2.4   Case 4: General $n^* = 1$ Case

(page 192)
Assumptions

1. sign k_p known

2. plant zeros of $Z_p(s)$ are stable (because we are doing pole-zero cancellation)

3. $n^* = 1$

GOAL: As before, want system to look like (follow this:)

$$k_m \frac{Z_m(s)}{R_m(s)} = k_m W'_m(s) = W_m(s)$$

where $W'_m(s)$ is SPR. The differential equations that completely describe the controller are given on page 194. If we express the control input to the plant using these equations, we get the following block diagram.
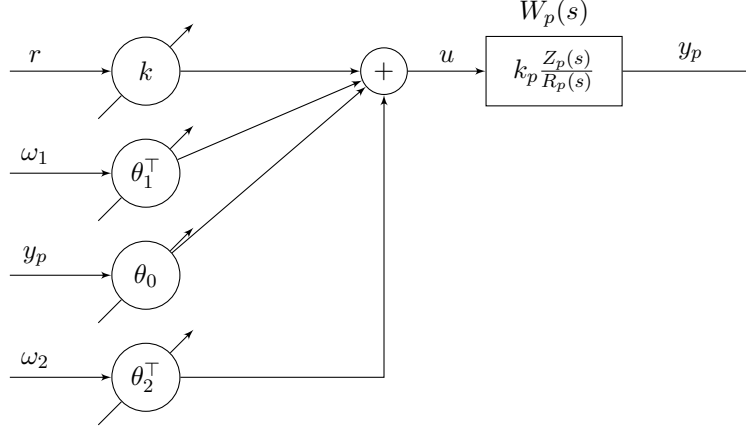
The signals ω_1 and ω_2 are given by the following differential equations, where Λ is a stable $(n-1) \times (n-1)$ matrix.

$$\dot{w}_1 = \Lambda w_1 + \ell u$$
$$\dot{w}_2 = \Lambda w_2 + \ell y_p$$

We would like to show that if we replace time-varying adaptive values with fixed, nominal values, that such values exist such that the closed-loop plant equation from r to y_p can be made to be the same as the reference model. The transfer functions for the blocks which represent the differential equations for ω_1 and ω_2 are

$$\omega_1 = (sI - \Lambda)^{-1} \ell u$$
$$\omega_2 = (sI - \Lambda)^{-1} \ell y_p$$

And we can represent the inverse of the matrix $(sI - \Lambda)$ as

$$(sI - \Lambda)^{-1} = \frac{C'(s)}{\det(sI - \Lambda)}$$

where $\text{com}(sI - \Lambda) = C'(s)$ is the matrix of cofactors. We not also that each element of the matrix $C'(s)$ is a polynomial of degree $n - 2$. If we define $\lambda(s) = \det(sI - \Lambda)$ we can write

$$(sI - \Lambda)^{-1} = \frac{C'(s)}{\lambda(s)}$$

and so

$$\omega_1 = \frac{C'(s)}{\lambda(s)}\ell u$$

$$\omega_2 = (sI - \Lambda)^{-1}\ell y_p$$

Assume for now that the parameters in the block diagram above are fixed. Using these transfer functions the block diagram can be represented as
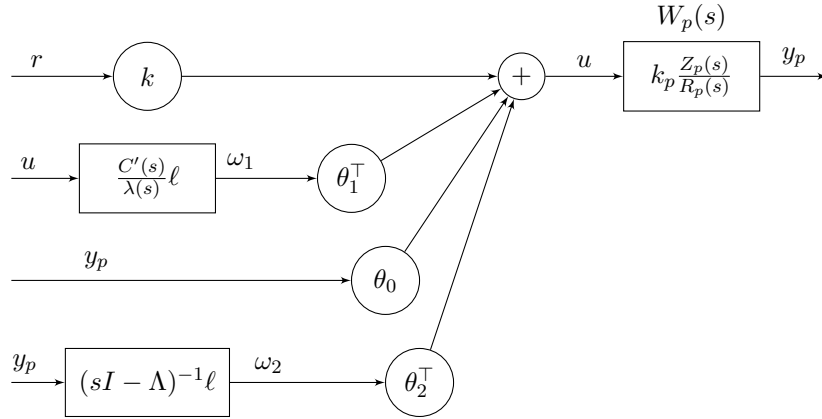
Wrapping some of the signals into the block diagram
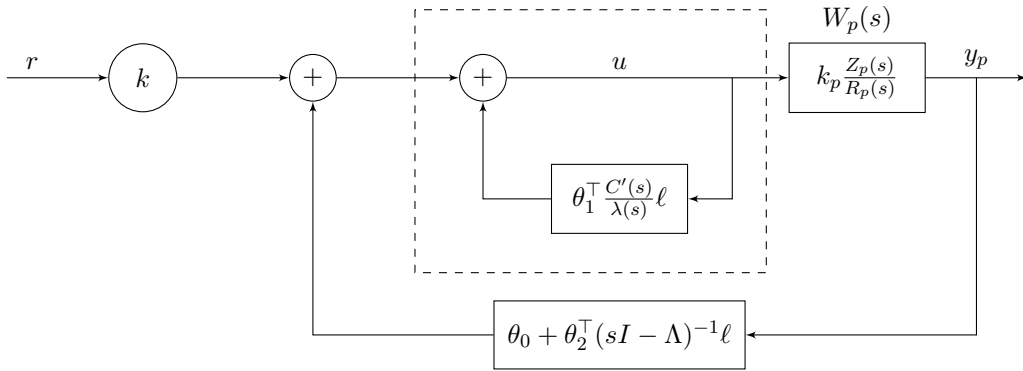
Simplifying the feed-forward part, and expressing the feedback controller as

$$\theta_0 + \theta_2^\top (sI - \Lambda)^{-1}\ell = \frac{D(s)}{\lambda(s)}$$

we get
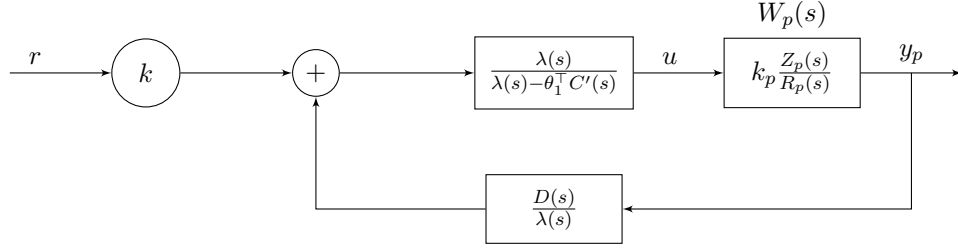
Writing the closed-loop transfer function from r to y we have

$$\frac{y_p}{r} = \frac{\frac{\lambda(s)}{\lambda(s) - \theta_1^\top C'(s)} \frac{k_p Z_p(s)}{R_p(s)}}{1 - \frac{\lambda(s)}{\lambda(s) - \theta_1^\top C'(s)} \frac{k_p Z_p(s)}{R_p(s)} \frac{D(s)}{\lambda(s)}}$$

$$= \frac{\frac{\lambda(s) k_p Z_p(s)}{(\lambda(s) - \theta_1^\top C'(s)) R_p(s)}}{1 - \frac{\lambda(s) k_p Z_p(s) D(s)}{(\lambda(s) - \theta_1^\top C'(s)) R_p(s) \lambda(s)}}$$

$$= \frac{\lambda(s) k_p Z_p(s)}{(\lambda(s) - \theta_1^\top C'(s)) R_p(s) - k_p Z_p(s) D(s)}$$

Recall that we can pick Λ to be any stable matrix we want.

$C(s)$ has $n - 1$ degrees of freedom, but its a polynomial of order $n - 2$ but a polynomial of order 5 for example, has 6 coefficients. $D(s)$ has n degrees of freedom, and its polynomial order is $n - 1$. $Z_m(s)$ is monic, so its highest power of s has a coefficient of unity, and it has $n - 1$ DOF, and its polynomial order is $n - 1$. Because in a monic polynomial one of its coefficients is fixed.

Using the assumption that $Z_m(s)$ is a stable polynomial, we pick Λ such that $\lambda(s) = Z_m(s)$. We also know by looking at the degrees of freedom in the polynomials that there is a particular fixed value of θ_1, which we call θ_1^* such that

$$\lambda(s) - \theta_1^{*\top} C'(s) = Z_p(s)$$

and we call

$$\theta_1^{*\top} C'(s) = C^*(s)$$

and we know $C^*(s)$ is order $n - 2$, since it is taking a linear combination of elements of the cofactor matrix, each of which has order $n - 2$ (or less?). So the output is expressed

$$\frac{y_p}{r} = \frac{Z_m(s) k_p Z_p(s)}{Z_p(s) R_p(s) - k_p Z_p(s) D(s)}$$

$$= \frac{Z_m(s) k_p}{R_p(s) - k_p D(s)}$$

And finally, the degrees of freedom are available in θ_0 and θ_2 such that ideal values exist such that $R_p(s) - k_p D^*(s) = R_m(s)$.

$$W_{ff}(s) = K_c \frac{\lambda(s)}{\lambda(s) - C(s)}$$

and

$$\lambda(s) = \det(sI - \Lambda)$$
$$C(s) = \theta_{1c}^\top (sI - \Lambda)^{-1}\ell$$

$$W_{fb}(s) = \frac{D(s)}{\lambda(s)} = \theta_{0c} + \theta_{2c}^\top (sI - \Lambda)^{-1}\ell$$

$$y_p = \frac{K_c K_p Z_p(s)\lambda(s)}{(\lambda(s) - C(s))R_p(s) - K_p Z_p(s)D(s)}$$

Choose $\lambda(s) = Z_m(s)$ which comes through choosing Λ

$$y_p = \frac{K_c K_p Z_p(s)Z_m(s)}{(Z_m(s) - C(s))R_p(s) - K_p Z_p(s)D(s)}$$

$C(s)$ has $n-1$ degrees of freedom, but its a polynomial of order $n-2$ but a polynomial of order 5 for example, has 6 coefficients. $D(s)$ has n degrees of freedom, and its polynomial order is $n-1$. $Z_m(s)$ is monic, so its highest power of s has a coefficient of unity, and it has $n-1$ DOF, and its polynomial order is $n-1$. Because in a monic polynomial one of its coefficients is fixed.
$\exists C^*(s), \ D^*(s)$
Matching conditions:
so we know the follow is true because of polynomial orders and stuff

$$Z_m(s) - C^*(s) = Z_p(s)$$

similarly for $R_p(s)$

$$R_p(s) - K_p D^*(s) = R_m(s)$$

third matching condition is

$$K^* = \frac{K_m}{K_p}$$

substitute $C^*(s)$, $D^*(s)$, and $K^*(s)$

$$
\begin{aligned}
y_p &= \frac{K_c K_p Z_p(s)Z_m(s)}{Z_p(s)R_p(s) - K_p Z_p(s)D(s)} \\
&= \frac{K_c K_p(s)Z_m(s)}{R_p(s) - K_p(s)D(s)} \\
&= \frac{K_c K_p(s)Z_m(s)}{R_m(s)} \\
&= K_m \frac{Z_m(s)}{R_m(s)}
\end{aligned}
$$

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FOR GENERAL $n^* = 1$ CASE:
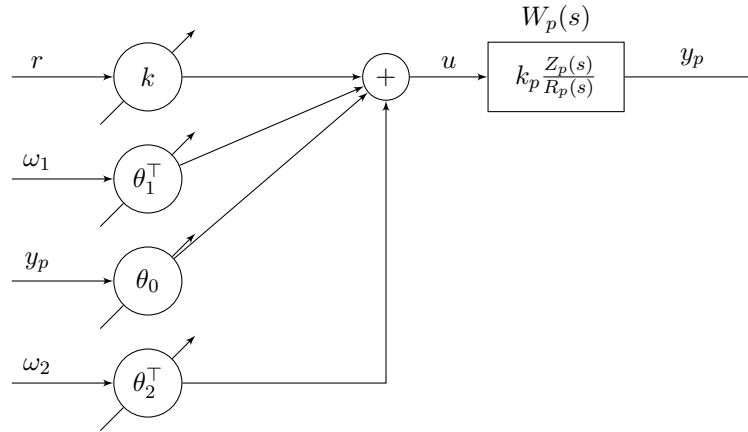
Using the following parameter errors, this block diagram can be expressed as

$$k = \psi + k^*$$
$$\theta_1 = \phi_1 + \theta_1^*$$
$$\theta_0 = \phi_0 + \theta_0^*$$
$$\theta_2 = \phi_2 + \theta_2^*$$
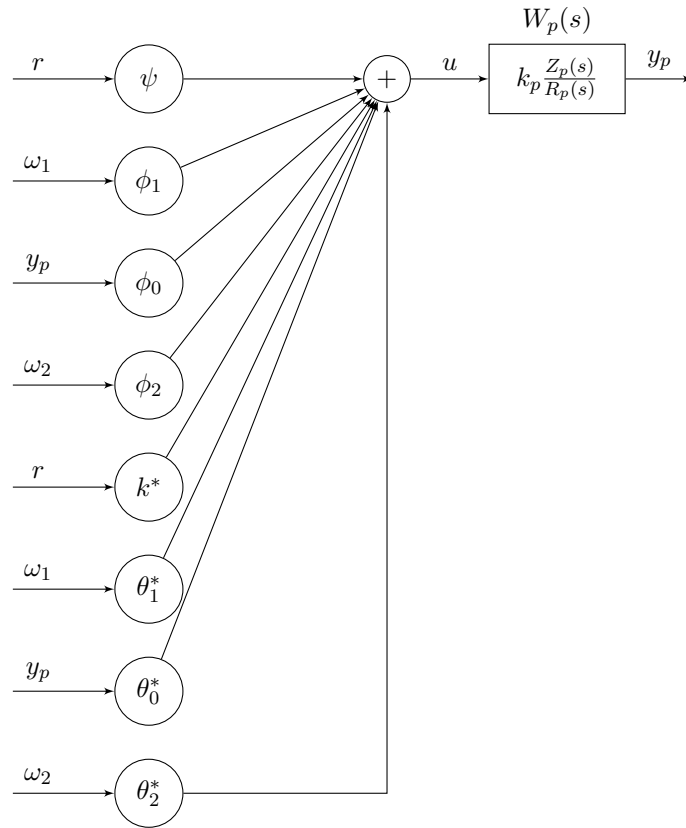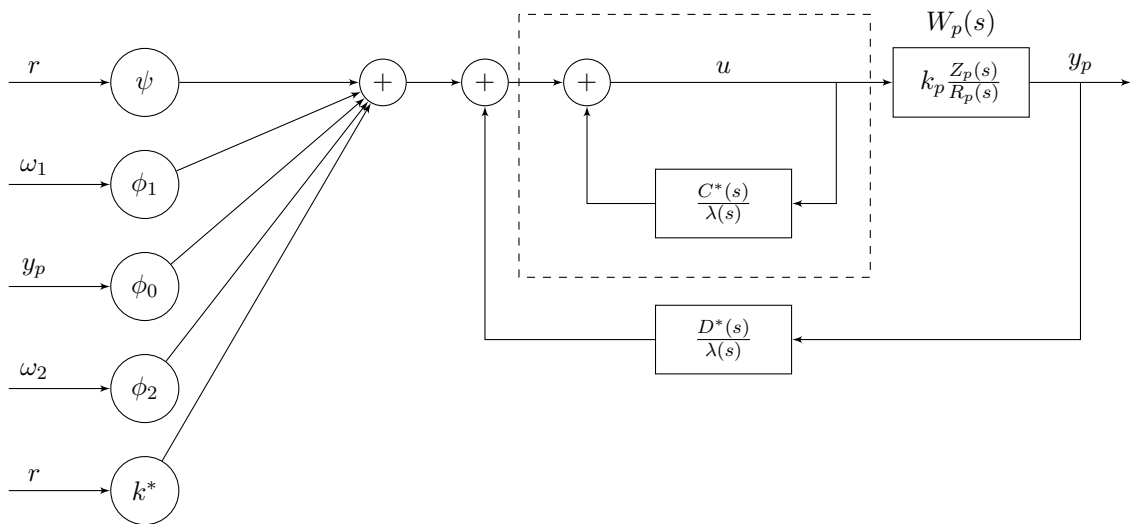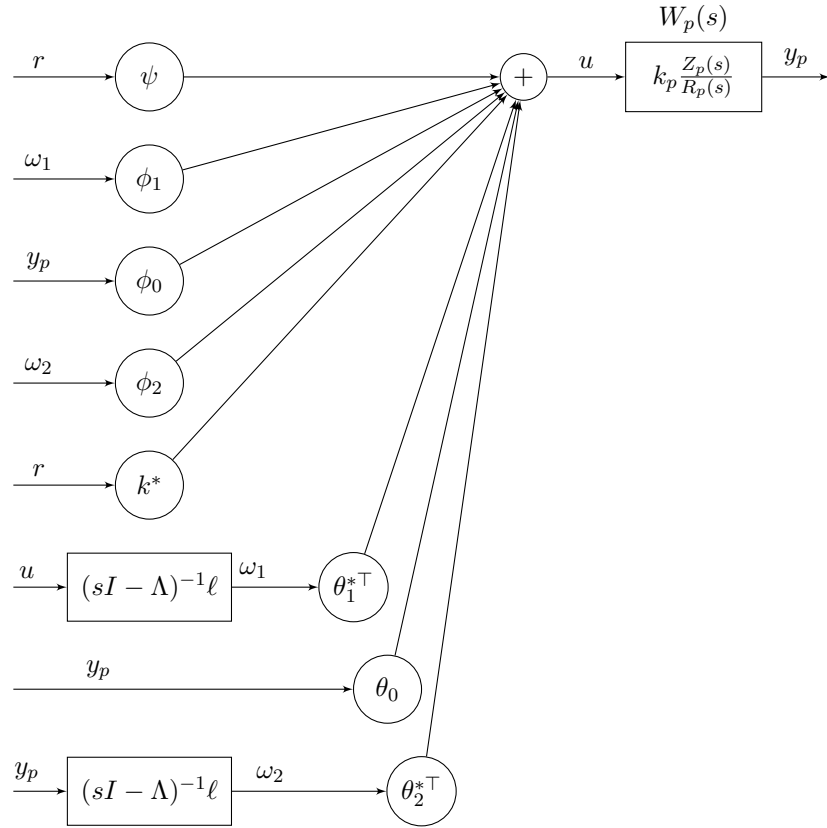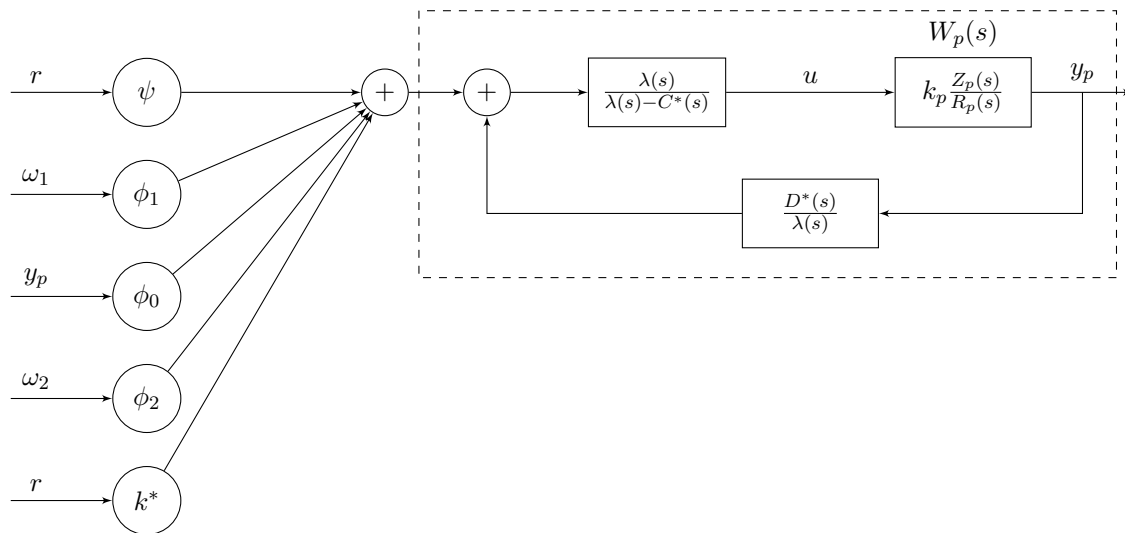
32.3   Output Feedback of Plants with $n^* \geq 2$

Finish this section

Chapter 33

Robustness Modifications

33.1 Introduction

This chapter: robust adaptive control - with disturbances - first-order

uncertainties: non-parametric such as disturbances (constant, time-varying, input- disturbances, output-disturbances) time-varying parameters unmodeled dynamics

This material is covered in Chapter 8 of the text.

33.1.1 Adaptive Control Without Disturbance

Consider the following plant, reference model, and control law as we have seen before at the beginning of the class.

Plant:	$\dot{x}_p = a_p x_p + u$		(33.1)
Reference model:	$\dot{x}_m = a_m x_m + r$		(33.2)
Control input:	$u = \theta x_p + r$		(33.3)

where $a_m < 0$. With tracking error $e = x_p - x_m$, matching condition $a_p + \theta^* = a_m$ and parameter error $\tilde{\theta} = \theta - \theta^*$ we get the error dynamics $\dot{e} = a_m e + \tilde{\theta} x_p$ and use the Lyapunov function $V = \frac{1}{2}(e^2 + \tilde{\theta}^2)$ to show that choosing an update law $\dot{\theta} = -e x_p$ results in $\dot{V} = -a_m e^2$ and ultimately $\lim_{t \to \infty} e(t) = 0$. Together the differential equations that describe the system are

Tracking error:	$\dot{e} = -a_m e + \tilde{\theta}(e + x_m)$	(33.4)
Update law:	$\dot{\tilde{\theta}} = -e(e + x_m)$	(33.5)

The origin of these error dynamics is a stable equilibrium. The following example gives a numerical example of the above stable error dynamics.

Example 56 No Disturbance: Numerical Simulation Consider the dynamics with dis-

turbance in equations (33.4) and (33.5) above, with the following values.

$$a_m = 1$$
$$x_m = 2$$
$$e(0) = 1$$
$$\tilde{\theta}(0) = 5$$
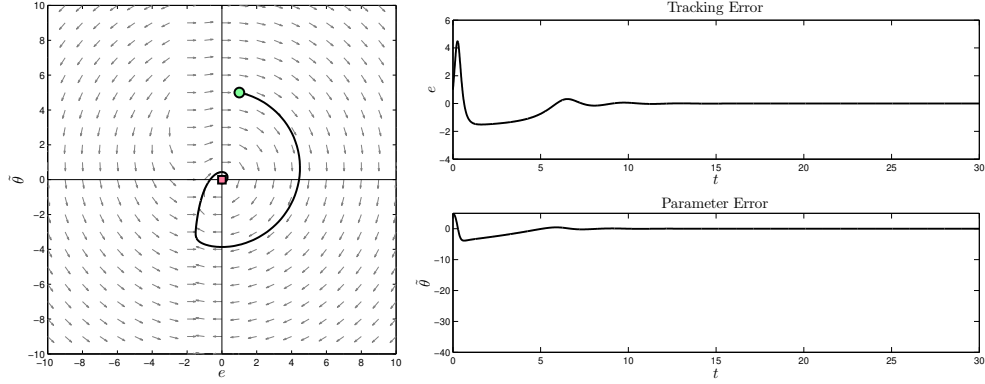
33.1.2 Adaptive Control With Disturbance

Consider again the same plant with bounded time-varying disturbance $\nu(t) \leq \nu_{\max}$, and same reference model and control law

Plant with disturbance:	$\dot{x}_p = a_p x_p + u + \nu$	(33.6)
Reference model:	$\dot{x}_m = a_m x_m + r$	(33.7)
Control input:	$u = \theta x_p + r$	(33.8)

where $a_m < 0$, tracking error $e = x_p - x_m$, matching condition $a_p + \theta^* = a_m$ and parameter error $\tilde{\theta} = \theta - \theta^*$. Together the differential equations that describe this system with the disturbance d are

Tracking error:	$\dot{e} = a_m e + \tilde{\theta}(e + x_m) + \nu$	(33.9)
Update Law	$\dot{\tilde{\theta}} = -e(e + x_m)$	(33.10)

And proposing the same candidate Lyapunov function we attempt to prove stability

$$V = \frac{1}{2}(e^2 + \tilde{\theta}^2)$$

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Time differentiating we get

$$
\begin{aligned}
\dot{V} &= e\dot{e} + \tilde{\theta}\dot{\tilde{\theta}} \\
&= a_m e^2 + e\tilde{\theta}x_p + e\nu - \tilde{\theta}ex_p \\
&= a_m e^2 + e\nu \\
&\leq a_m|e|(-|e| + \frac{|\nu|}{a_m})
\end{aligned}
$$

So $\dot{V}$ is sign indefinite, and $\dot{V} < 0$ in a set which is not compact. $\Rightarrow e, \tilde{\theta}$ need not be bounded. We can see the unboundedness of the parameters using the following numerical example.

Example 57 Constant Disturbance: Numerical Simulation Consider the dynamics with disturbance in equations (33.9) above, with the following values. The plot shows unbounded behavior of θ.

$$
\begin{aligned}
a_m &= -1 \\
x_m &= 2 \\
\nu &= -5 \\
e(0) &= 1 \\
\tilde{\theta}(0) &= 5
\end{aligned}
$$
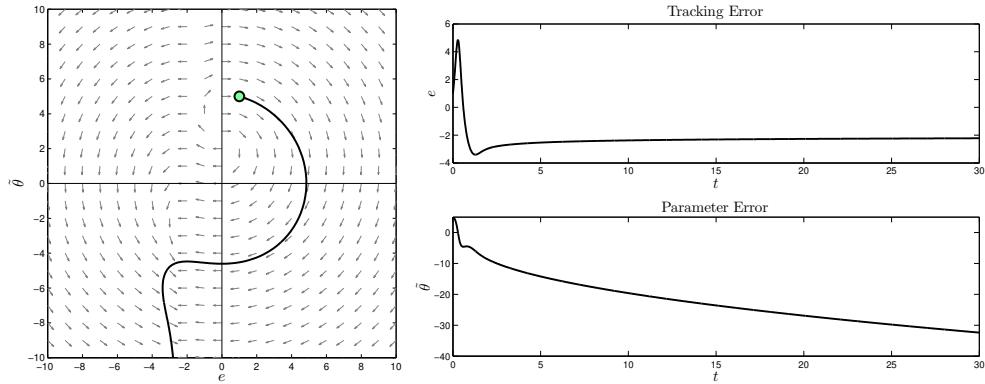
Clearly this is undesirable, but it is not unreasonable to have input disturbances present in real system. So, what do we do?

33.2   Exact Cancellation of Disturbances

Before we talk about how to cancel disturbances in adaptive control, let us first recall how input disturbances are dealt with in classical control.

33.2.1 Disturbances in Classical Control

Consider the following plant and controller block diagram with disturbance ν.
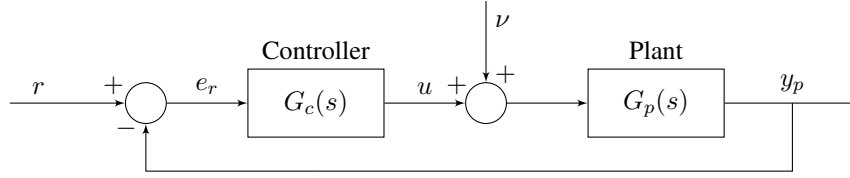
Recall: system type with respect to reference is the number of free integrators in the loop. System type with respect to disturbance is the number of free integrators in the controller. Depending on what the type of the system is will tell us what kind of input disturbances we can reject. To reject a constant disturbance, need to have system type 1 with respect to disturbance.

Goal design G so that closed-loop $|y - r| \to 0$. G must have an integrator of some kind. Closed-loop TF:

33.2.2 Exact Cancellation of a Constant Disturbance

We will use the idea of system type with respect to disturbance to design an adaptive controller for a plant with an unknown constant disturbance as shown in the following block diagram. Take ν to be a constant disturbance.
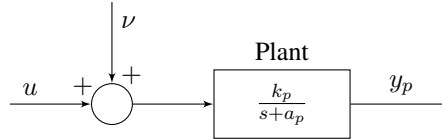
The objective is to have the output of this plant track the output of a strictly positive real reference model with transfer function

$$W_m(s) = \frac{k_m}{s + a_m}$$

In order to design a control system that can reject this constant disturbance ν, we propose putting a proper transfer function in the forward loop that contains an integrator. We can use a PI controller as shown in the following block diagram, where the zero location is arbitrary.
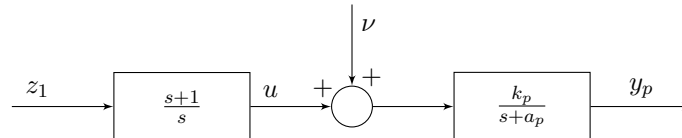
Note that this PI controller can be realized as shown in the following block diagram
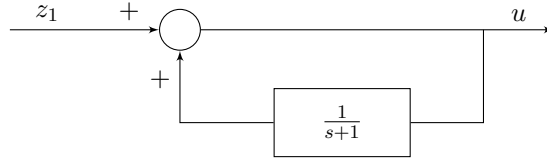
With this integrator in the controller, we could use classical control methods to close the loop and have a controller which would reject the constant disturbance ν. However, we need to consider that the plant parameters are unknown, and come up with a way to match the closed loop system to the reference model. Consider closing the loop as shown in the following block diagram
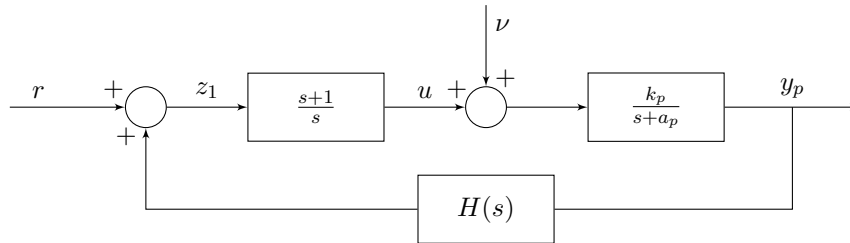
Looking at the closed-loop transfer function $T(s)$ we have

$$T(s) = \frac{\frac{k_p(s+1)}{s(s+a_p)}}{1 - \frac{k_p(s+1)N_H}{s(s+a_p)D_H}}$$

$$= \frac{D_H k_p(s+1)}{D_H(s^2 + a_p s) - k_p(s+1)N_H}$$

and we need to make this look like the reference model, with the exception of the DC gain, which we can handle by putting a feed-forward gain in front of r at the end. So, to make this look like our reference model, $H(s)$ must allow us the degrees of freedom to do a pole-zero cancellation leaving just k_p in the numerator. If $H(s)$ was a constant, we wouldn't have enough degrees of freedom. So $H(s)$ should have two parameters in the numerator since we need to match two coefficients, and to keep the transfer function proper, the denominator of $H(s)$ should have a pole. Use

$$H(s) = \frac{\theta(s+1) + d}{s+1}$$

This can be realized as shown in the following block diagram
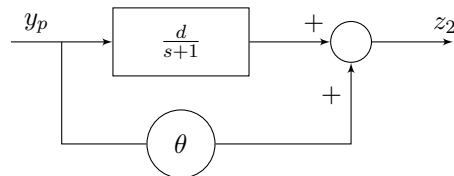
Putting everything together we get the following block diagram

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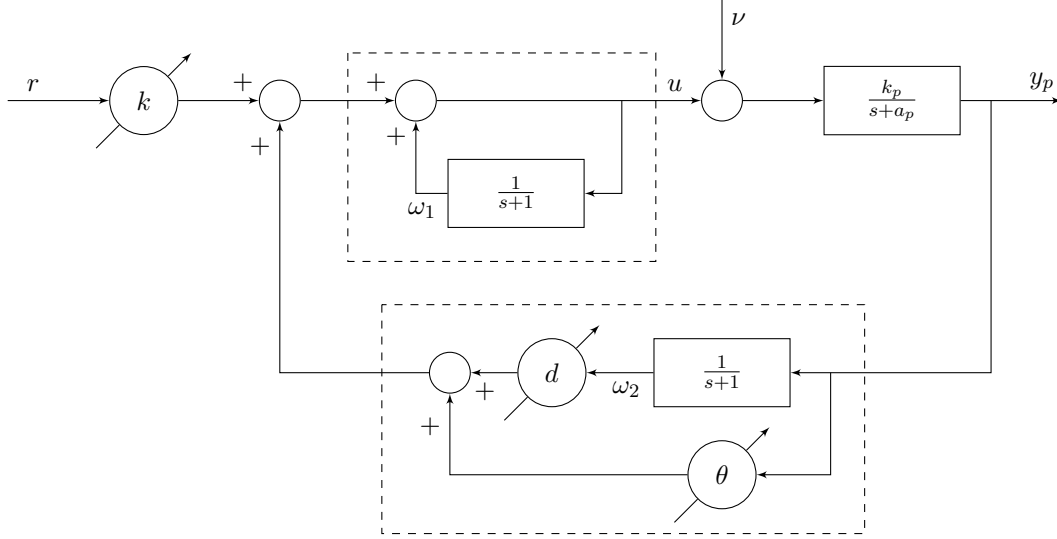
The total control input is

$$\boxed{\textbf{Control:} \qquad u = kr + \omega_1 + d\omega_2 + \theta y_p} \qquad (33.11)$$

Using this control, by writing down the closed-loop transfer function $T(s)$ we can show that ideal parameters k^*, d^*, and θ^* exist which will match this system to the SPR reference model. The problem is that this approach only works for constant disturbances (recall system type with respect to disturbance from classical control). This idea can be extended to other types of disturbances (see page 299 in the text) but the drawback of this approach is that the precise nature of the disturbance must be known a priori. In the following sections we will look at alternatives to direct cancellation in order to ensure boundedness of errors while maintaining a desired level of performance.

What to make G? Assume k_p known. Closed-loop TF is See Chapter 8. (8.1 or 8.2)

Example 58 Constant Disturbance Case (i) page 303. Assume $r = 0$ and $x_m(0) = 0$, and so we have $x_m = 0$, giving $e = x_p$ and let $\nu = -\nu_{\max}$. With these values equations (33.9) become

$$\dot{e} = a_m e + \tilde{\theta} e - \nu_{\max}$$
$$\dot{\tilde{\theta}} = -x_p^2$$

If $x_p \notin \mathcal{L}^2$ then $\tilde{\theta}$ will be unbounded.

Example 59 Constant Disturbance Case (ii) page 303. Consider $r = r_0$, $x_m = x_0 > 0$ and $\nu(t) = -\nu_{\max}$ where $\nu_{\max} > 0$.

$$\dot{e} = a_m e + \tilde{\theta}(e + x_0) - \nu_{\max}$$
$$\dot{\tilde{\theta}} = -e(e + x_0)$$

two cases:

(i) $\frac{\nu_0}{a_m} > 0 \; \tilde{\theta} \to \infty$

(ii) $\frac{\nu_0}{a_m} < 0 \; \tilde{\theta}, \; e$ bounded

Want to prevent wind-up/drift in $\tilde{\theta}$

$$\tilde{\theta} = -\int_0^t e(e + x_m)dt$$

$$\dot{\tilde{\theta}} = -ex_p$$

source of wind-up

Equilibrium Points Consider equations (33.9) with the derivatives set to zero, and consider the case when $x_m = 0$ and $\nu = -\nu_{\max}$. This gives

$$0 = a_m e + \tilde{\theta}e - \nu_{\max}$$
$$0 = -e^2$$

Which we can see there are no equilibrium points.

33.3 Modifications for Bounded Disturbances

The following subsection showed what can happen when there is a (INPUT) disturbance present in the plant. In this section we outline some modifications to the adaptive law that will be better. . .

1. σ-modification

2. Deadzone

3. e_1-modification

4. Projection

make sure $\dot{V}$ is not sign indefinite in an infinite channel. We will consider bounded disturbances $|\nu(t)| \leq \nu_{\max}$.

33.3.1 σ-Modification

See page 310. (of textbook?) Consider again the same plant with disturbance, reference model, and control law

Plant with disturbance:	$\dot{x}_p = a_p x_p + u + \nu$	(33.12)
Reference model:	$\dot{x}_m = a_m x_m + r$	(33.13)
Control input:	$u = \theta x_p + r$	(33.14)
Error dynamics:	$\dot{e} = a_m e + \tilde{\theta} x_p + \nu$	(33.15)

where again $a_m < 0$, $e = x_p - x_m$, $a_p + \theta^* = a_m$ and $\tilde{\theta} = \theta - \theta^*$.
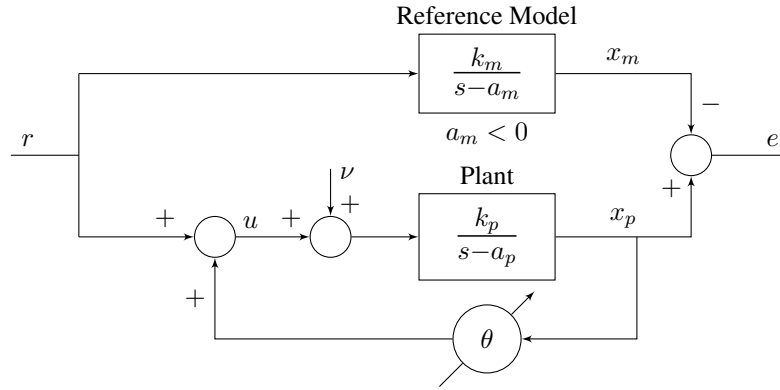
We now propose the following modified update law where $\sigma > 0$, which is known as σ-modification.

$$\dot{\tilde{\theta}} = -e x_p - \sigma\tilde{\theta}$$

Proposing the candidate Lyapunov function $V = \frac{1}{2}(e^2 + \tilde{\theta}^2)$ and time differentiating we get the following

$$\dot{V} = a_m e^2 + e\nu - \sigma\tilde{\theta}^2$$

However, we can't implement this proposed scheme, as we do not know what $\tilde{\theta}$ is, so we have to modify this update law as

$$\dot{\tilde{\theta}} = -e x_p - \sigma\theta$$
$$= -e x_p - \sigma(\tilde{\theta} + \theta^*)$$

σ-**Modification update law:**	$\dot{\theta} = -e x_p - \sigma\theta$

The differential equations the describe the system now are

σ-Modification Error Dynamics

Tracking error:	$\dot{e} = a_m e + \tilde{\theta}(e + x_m) + \nu$	(33.16)
Update law:	$\dot{\tilde{\theta}} = -e(e + x_m) - \sigma(\tilde{\theta} + \theta^*)$	(33.17)

Plugging the error dynamics and update law into $\dot{V}$ we get

$$\boxed{\dot{V} = a_m e^2 + e\nu - \sigma\tilde{\theta}^2 - \sigma\theta^*\tilde{\theta}}$$

From this expression, since $a_m < 0$, and $\sigma > 0$, if $|e|$ and $|\tilde{\theta}|$ are large enough, $\dot{V}$ will be negative, even if the sign of e, $\tilde{\theta}$ are unknown. We wish to find the region where $|e|$ and $|\tilde{\theta}|$ are sufficiently large so that $\dot{V}$ is guaranteed to be negative. We create the inequality below by assuming some known upper bound $|\nu| \leq \nu_{\max}$.

$$\dot{V} = a_m e^2 + e\nu - \sigma\tilde{\theta}^2 - \sigma\theta^*\tilde{\theta}$$
$$\leq a_m e^2 + |e|\nu_{\max} - \sigma\tilde{\theta}^2 - \sigma\theta^*\tilde{\theta}$$
$$\leq a_m e^2 + |e|\nu_{\max} - \sigma\tilde{\theta}^2 + \sigma|\theta^*||\tilde{\theta}|$$

Bound Method 1: Box   Since $a_m < 0$ we have $a_m = -|a_m|$ and from the inequality above can write

$$\dot{V} \leq -|a_m|\left(e^2 - \frac{|e|d_{\max}}{|a_m|}\right) - \sigma(\tilde{\theta}^2 - \theta^*\tilde{\theta})$$

Looking at the two terms of $\dot{V}$ separately we have the following inequality for the first term

$$-|a_m|\left(e^2 - \frac{|e|d_{\max}}{|a_m|}\right) = -|a_m|\left(\frac{1}{2}e^2 + \frac{1}{2}e^2 - \frac{|e|d_{\max}}{|a_m|} + \frac{1}{2}\frac{d_{\max}^2}{|a_m|^2} - \frac{1}{2}\frac{d_{\max}^2}{|a_m|^2}\right)$$
$$= -|a_m|\left(\frac{1}{2}e^2 + \frac{1}{2}\left(|e|^2 - 2\frac{|e|d_{\max}}{|a_m|} + \left(\frac{d_{\max}}{|a_m|}\right)^2\right) - \frac{1}{2}\frac{d_{\max}^2}{|a_m|^2}\right)$$
$$= -|a_m|\left(\frac{1}{2}|e|^2 + \frac{1}{2}\left(|e| - \frac{d_{\max}}{|a_m|}\right)^2 - \frac{1}{2}\left(\frac{d_{\max}}{|a_m|}\right)^2\right)$$
$$\leq -|a_m|\left(\frac{1}{2}|e|^2 - \frac{1}{2}\left(\frac{d_{\max}}{|a_m|}\right)^2\right)$$
$$= -|a_m|\frac{1}{2}\left(|e|^2 - \left(\frac{d_{\max}}{|a_m|}\right)^2\right)$$

and the following inequality for the second term

$$-\sigma\left(\tilde{\theta}^2 - |\tilde{\theta}||\theta^*|\right) = -\sigma\left(\frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\tilde{\theta}^2 - |\tilde{\theta}||\theta^*| + \frac{1}{2}|\theta^*|^2 - \frac{1}{2}|\theta^*|^2\right)$$
$$= -\sigma\left(\frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\left(|\tilde{\theta}|^2 - 2|\tilde{\theta}||\theta^*| + |\theta^*|^2\right) - \frac{1}{2}|\theta^*|^2\right)$$
$$= -\sigma\left(\frac{1}{2}|\tilde{\theta}|^2 + \frac{1}{2}\left(|\tilde{\theta}| - |\theta^*|\right)^2 - \frac{1}{2}|\theta^*|^2\right)$$
$$\leq -\sigma\left(\frac{1}{2}|\tilde{\theta}|^2 - \frac{1}{2}|\theta^*|^2\right)$$
$$= -\sigma\frac{1}{2}\left(|\tilde{\theta}|^2 - |\theta^*|^2\right)$$

From these two inequalities, we have

$$\dot{V} \leq -|a_m|\frac{1}{2}\left(|e|^2 - \left(\frac{d_{\max}}{|a_m|}\right)^2\right) - \sigma\frac{1}{2}\left(|\tilde{\theta}|^2 - |\theta^*|^2\right)$$

We can see that if we have both $|e| > \frac{d_{\max}}{|a_m|}$ and $|\tilde{\theta}| > |\theta^*|$ then $\dot{V} < 0$. This gives the following region D inside of which $\dot{V}$ is sign indefinite.

$$D = \left\{(e, \tilde{\theta}) \,\middle|\, |e| \leq \frac{d_{\max}}{|a_m|}, \ |\tilde{\theta}| \leq |\theta^*|\right\}$$

Bound Method 2: Ellipse We again consider the inequality for $\dot{V}$ above, which is repeated here

$$\dot{V} \leq -a_m e^2 + |e|d_{\max} - \sigma\tilde{\theta}^2 + \sigma|\theta^*||\tilde{\theta}|$$

Noticing the presence of the quadratic and linear terms in both e and $\tilde{\theta}$, this inequality resembles an elliptical region. Motivated by this and using

$$-a_m\left(|e| - \frac{d_{\max}}{2a_m}\right)^2 = -a_m|e|^2 + |e|d_{\max} - \frac{d_{\max}^2}{4a_m}$$

and

$$-\sigma\left(|\tilde{\theta}| - \frac{|\theta^*|}{2}\right)^2 = -\sigma|\tilde{\theta}|^2 + \sigma|\theta^*||\tilde{\theta}| - \frac{\sigma|\theta^*|^2}{4}$$

the inequality for $\dot{V}$ becomes

$$\begin{aligned}
\dot{V} &\leq -a_m e^2 + |e|d_{\max} - \sigma\tilde{\theta}^2 + \sigma|\theta^*||\tilde{\theta}| \\
&= -a_m e^2 + |e|d_{\max} - \frac{d_{\max}^2}{4a_m} - \sigma\tilde{\theta}^2 + \sigma|\theta^*||\tilde{\theta}| - \frac{\sigma|\theta^*|^2}{4} + \frac{d_{\max}^2}{4a_m} + \frac{\sigma|\theta^*|^2}{4} \\
&= -a_m\left(|e| - \frac{d_{\max}}{2a_m}\right)^2 - \sigma\left(|\tilde{\theta}| - \frac{|\theta^*|}{2}\right)^2 + \frac{d_{\max}^2}{4a_m} + \frac{\sigma|\theta^*|^2}{4}
\end{aligned}$$

We can now see from this inequality for $\dot{V}$ that if the combined magnitude of the first two terms (which are negative) is greater than the combined magnitude of the second two terms (which are both positive) $\dot{V} < 0$. That is, if the following inequality holds

$$a_m(|e| - k_1)^2 + \sigma(|\tilde{\theta}| - k_2)^2 > k_3$$

where

$$k_1 = \frac{d_{\max}}{2a_m}$$
$$k_2 = \frac{|\theta^*|}{2}$$
$$k_3 = \frac{d_{\max}^2}{4a_m} + \frac{\sigma|\theta^*|^2}{4}$$

then $\dot{V} < 0$. We recognize the following set D

$$D = \{(e, \tilde{\theta}) \mid a_m(|e| - k_1)^2 + \sigma(|\tilde{\theta}| - k_2)^2 \leq k_3\}$$

as an ellipse in the $(e, \tilde{\theta})$ space.

Example 60 Constant Disturbance: Numerical Simulation Consider now the differential equations for tracking and parameter error in (33.16) with disturbance and σ-modification. The following plots show the system response for different numerical values.

$$a_m = 1$$
$$a_p = 4$$
$$d = 0$$
$$e(0) = 5$$
$$\tilde{\theta}(0) = 5$$
$$x_m = 0$$
$$\sigma = 1$$

$$x_m = 3$$
$$\sigma = 1$$

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$$x_m = 3$$
$$\sigma = 0.5$$
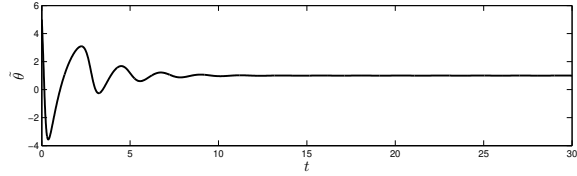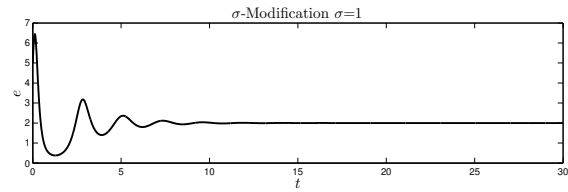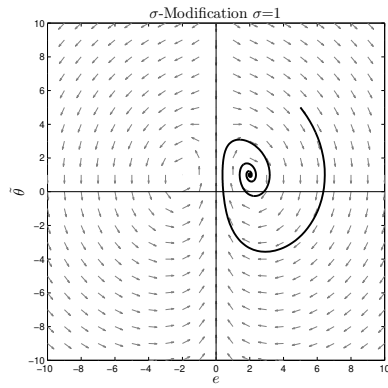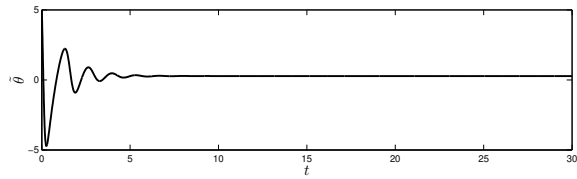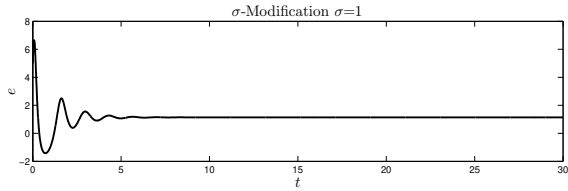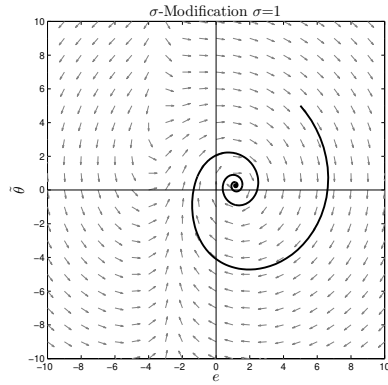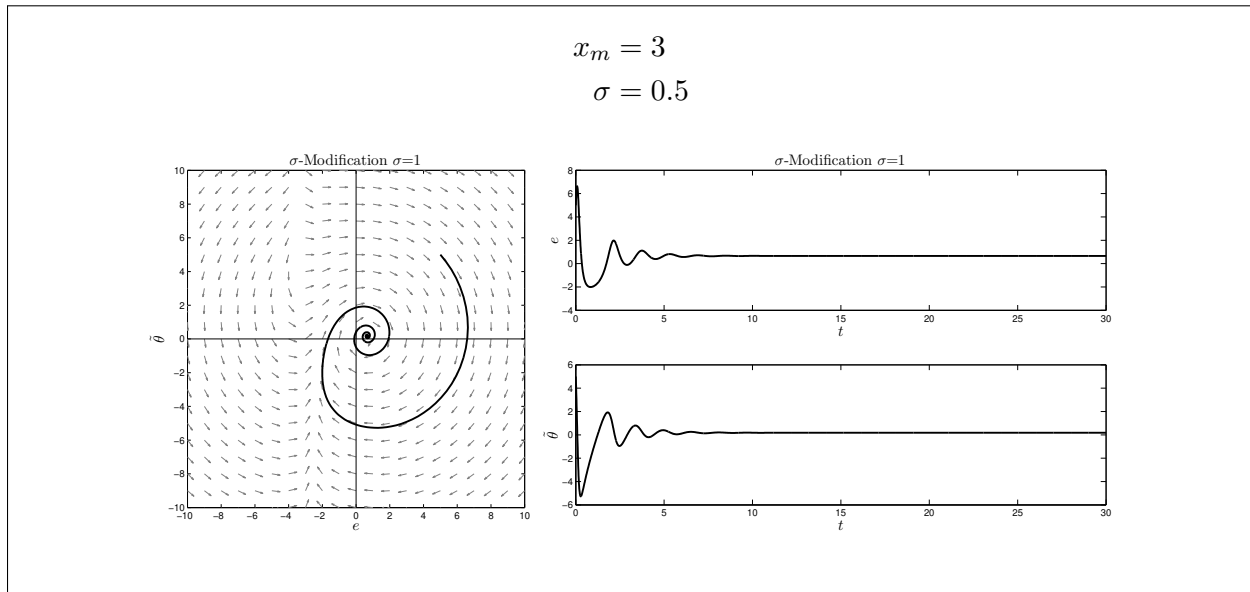
Equilibrium Points   Consider equations (33.16) with the derivatives set to zero, and consider $x_m = 0$ and $d = -d_{\max}$

$$0 = -a_m e + \tilde{\theta} e - d_{\max}$$
$$0 = -e^2 - \sigma \tilde{\theta} - \sigma \theta^*$$

On Vector Systems

A_p unknown, B_p known.

$$\dot{x}_p = A_p x_p + B_p u + d(t)$$
$$\dot{x}_m = A_m x_m + B_p r$$
$$u = \theta x_p + r$$
$$\dot{\theta} = -B_p^\top P e x_p^\top - \sigma I_{n \times n} \theta$$

```
1   %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
2   % Daniel Wiese
3   % 2.153 - Adaptive Control
4   % Monday 14-April-2014
5   % example_sigmamod_v1.m
6   %----------------------------------------------------------------------------
7   % This script simulates and plots the response of an adaptive system with input
8   % disturbance. We can plot the response of the nominal system without sigmamod or
9   % disturbance (stable), turn the disturbance on without sigmamod and show
10  % unboundedness of error (unstable), and then use sigmamod to make stable.
11  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
12  clear all;
13  clc;
14  close all;
```

```matlab
15  restoredefaultpath;
16  thismfile=dbstack('-completenames');
17  thisdir=fileparts(thismfile.file);
18  cd(thisdir);
19  addpath(fullfile(fileparts(pwd)));
20  plotdir='Z:\Dropbox\Dan\DOCUMENTS\School\MIT\Dan_Latex\FIGURES\Adaptive_Class\Lec12_Robust';
21  addpath('Z:\Dropbox\Dan\DOCUMENTS\School\MIT\Codes\tools');
22  s=tf('s');
23  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
24  %Set up simulation parameters
25  tsim=60;
26  am=-1;
27  nu=5;
28  sigma=1;
29  thetastar=-5; %=am-ap
30  %Initial conditions
31  ym=1;
32  e0=5;
33  thetatilde0=5;
34
35  %--------------------------------------------------------------------------------
36  %Plotting parameters
37  AxesLineWidth=2;
38  PlotLineWidth=2;
39  PlotFontSizeTitle=16;
40  PlotFontSizeLab=16;
41  PlotFontSizeLegend=16;
42  AxesLineColor=[0.3,0.3,0.3];
43  AxesLineStyle='--';
44  emin=-10;
45  emax=10;
46  eplotmin=-10;
47  eplotmax=10;
48  ThetaTildePlotmin=-10;
49  ThetaTildePlotmax=10;
50
51  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
52  %PLOT THE VECTOR FIELD
53  %Define the error differential equations
54  f = @(t,Y) [am*Y(1)+Y(2)*(Y(1)+ym)+nu;
55              -Y(1)*(Y(1)+ym)-sigma*(Y(2)+thetastar)];
56  %Set up a linspace grid over which to determine the direction of the vector Ydot
57  y1 = linspace(emin,emax,21);
58  y2 = linspace(emin,emax,21);
59  [x,y] = meshgrid(y1,y2);
60  %Preallocate the derivatives
61  edot=zeros(size(x));
62  phidot=zeros(size(x));
63  %Solve for the derivatives
64  t=0;
65  for ii = 1:length(y1)
66      for jj=1:length(y2)
67          Yprime = f(t,[y1(ii); y2(jj)]);
68          edot(ii,jj) = Yprime(1);
69          phidot(ii,jj) = Yprime(2);
70      end
```

```matlab
71  end
72  %Normalize length of vectors
73  L=2*sqrt(edot.^2+phidot.^2);
74
75  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
76  %SIMULATE THE ERROR DYNAMICS IN TIME
77  sim('model_sigmamod_v1')
78
79  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
80  %% PLOT THE RESULTS
81  cd(plotdir)
82  figure(1)
83  quiver(x,y,edot'./L',phidot'./L',0,'color',[0.5,0.5,0.5]);
84  hold on
85  plot([0,0],[ThetaTildePlotmin,ThetaTildePlotmax],'color',[0 0 0])
86  plot([eplotmin,eplotmax],[0,0],'color',[0 0 0])
87  plot(eout.signals.values,thetatildeout.signals.values,'linestyle','-','color', ...
88      [0,0,0],'linewidth',PlotLineWidth);
89  plot(eout.signals.values(1),thetatildeout.signals.values(1),'linestyle','o', ...
90      'LineWidth',2,'MarkerEdgeColor','k','MarkerFaceColor',[0.5 1 0.6], ...
91      'MarkerSize',10);
92  plot(eout.signals.values(end),thetatildeout.signals.values(end),'linestyle','s', ...
        ...
93      'LineWidth',2,'MarkerEdgeColor','k','MarkerFaceColor',[1 0.49 0.63], ...
94      'MarkerSize',10);
95  xlim([eplotmin,eplotmax])
96  ylim([ThetaTildePlotmin,ThetaTildePlotmax])
97  daspect([1 1 1])
98  plot_title1=strcat('$x_{m}=$',sprintf('%0.0f',ym),',  $\nu=$',sprintf('%0.0f',nu));
99  title(plot_title1,'interpreter','latex','FontSize',PlotFontSizeTitle)
100 xlabel('$e$','interpreter','latex','FontSize',PlotFontSizeLab,'interpreter','latex')
101 ylabel('$\tilde{\theta}$','interpreter','latex','FontSize',PlotFontSizeLab, ...
102     'interpreter','latex')
103 set(gcf,'Units','pixels');
104 set(gcf,'PaperUnits','inches','PaperPosition',[0 0 10 6]);
105 set(gcf,'PaperPositionMode','manual')
106 set(gcf,'InvertHardCopy','off');
107 set(gcf,'color',[1 1 1])
108 box on
109 print('-depsc','-r600','sim_r1nu5sigma1_phase.eps');
110
111 figure(2)
112 subplot(2,1,1)
113 plot(eout.time,eout.signals.values,'linewidth',PlotLineWidth,'color',[0 0 0])
114 % plot_title1=strcat('Tracking Error');
115 plot_title1=strcat('Tracking Error: ',' $x_{m}=$',sprintf('%0.0f',ym),',  ...
        $\nu=$',sprintf('%0.0f',nu));
116 title(plot_title1,'interpreter','latex','FontSize',PlotFontSizeTitle)
117 xlabel('$t$','interpreter','latex','FontSize',PlotFontSizeLab,'interpreter','latex')
118 ylabel('$e$','interpreter','latex','FontSize',PlotFontSizeLab,'interpreter','latex')
119 xlim([0 tsim]);
120 ylim([-4 6]);
121 box on
122 subplot(2,1,2)
123 plot(thetatildeout.time,thetatildeout.signals.values,'linewidth',PlotLineWidth, ...
124     'color',[0 0 0])
```

```
125  plot_title1=strcat('Parameter Error');
126  title(plot_title1,'interpreter','latex','FontSize',PlotFontSizeTitle)
127  xlabel('$t$','interpreter','latex','FontSize',PlotFontSizeLab,'interpreter','latex')
128  ylabel('$\tilde{\theta}$','interpreter','latex','FontSize',PlotFontSizeLab, ...
129      'interpreter','latex')
130  xlim([0 tsim]);
131  ylim([-40 5]);
132  box on
133  set(gcf,'Units','pixels');
134  set(gcf,'PaperUnits','inches','PaperPosition',[0 0 10 6]);
135  set(gcf,'PaperPositionMode','manual')
136  set(gcf,'InvertHardCopy','off');
137  set(gcf,'color',[1 1 1]);
138  print('-depsc','-r600','sim_r1nu5sigma1_tres.eps');
139
140  cd(thisdir)
141
142  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

33.3.2   Dead-zone Modification

Consider the following plant with time-varying disturbance $d(t)$ with $|d(t)| \leq d_{\max}$, reference model and control law

$$\dot{x}_p = a_p x_p + u + d(t)$$
$$\dot{x}_m = a_m x_m + r$$
$$u = \theta x_p$$

where $a_m < 0$. With tracking error $e = x_p - x_m$, matching condition $a_p + \theta^* = a_m$ and parameter error $\tilde{\theta} = \theta - \theta^*$ we get the error dynamics $\dot{e} = a_m e + \tilde{\theta} x_p + d$ and use the Lyapunov function $V = \frac{1}{2}(e^2 + \tilde{\theta}^2)$. Time differentiating V we get

$$\dot{V} = a_m e^2 + e d(t) + \tilde{\theta} \dot{\tilde{\theta}} + \tilde{\theta} e x_p$$

rewrite as

$$\dot{V} \leq -|a_m||e|\left(|e| - \frac{d_{\max}}{|a_m|}\right) + \tilde{\theta}\dot{\tilde{\theta}} + \tilde{\theta} e x_p$$

Propose the following update law with dead-zone modification

$$\dot{\tilde{\theta}} = \dot{\theta} = \begin{cases} -e x_p & \text{if } (e, \tilde{\theta}) \in D_1^c \\ 0 & \text{if } (e, \tilde{\theta}) \in D_1 \end{cases}$$

where the region D_1 is

$$D_1 = \left\{ (e, \tilde{\theta}) \,\middle|\, |e| \leq \frac{d_{\max}}{|a_m|} \right\}$$

There are two cases

(i) $\forall (e, \tilde{\theta}) \in D_1^c$ then $\dot{V} \leq 0$

(ii) $\forall (e, \tilde{\theta}) \in D_1$ then $\dot{\theta} = 0$

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33.3.3 e_1 Modification

Finish this section

33.3.4 Projection

See Eugene's book starting on page 348.

The Gradient Operator

$$\nabla(f(x)) = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \frac{\partial f}{\partial x_2}\mathbf{e}_2 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n$$

Define gradient of a function f to always be a column vector? Or define the orientation of gradient vector on the orientation of the input to function $f : \mathbb{R}^k \to \mathbb{R}$?

$$\nabla(f(x)) = \left[\begin{array}{cccc} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{array} \right]^{\mathsf{T}}$$

$$\nabla^{\mathsf{T}}(f(x)) = (\nabla(f(x)))^{\mathsf{T}} = \left[\begin{array}{cccc} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{array} \right]$$

Write $\nabla^{\mathsf{T}} f = \nabla^{\mathsf{T}}(f(x))$ and $\nabla f = \nabla(f(x))$

Lemma 5 11.1 *Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then for any constant $\delta > 0$ the following set is convex:*

$$\boxed{\Omega_\delta = \{\theta \in \mathbb{R}^n | f(\theta) \le \delta\}}$$

Basically what this says is that the set Ω_δ of all inputs θ to the convex function $f(x)$ such that $f(\theta) \le \delta$ is convex.

Proof of Lemma 11.1 First define $\Omega_\delta = \{\theta \in \mathbb{R}^n | f(\theta) \le \delta\}$ which we want to prove is convex. Let $\theta_1, \ \theta_2 \in \Omega_\delta$. Then $f(\theta_1) \le \delta$ and $f(\theta_2) \le \delta$ by definition in order to belong to Ω_δ.

Because the function $f(x)$ is convex, for $\theta_1 \le \theta \le \theta_2$, that is: $\theta = \lambda\theta_1 + (1 - \lambda)\theta_2 \ \forall\lambda \in [0,1] \Rightarrow f(\theta) \le \lambda f(\theta_1) + (1 - \lambda)f(\theta_2)$.

Because $f(\theta_1) \le \delta$ and $f(\theta_2) \le \delta$, then we can substitute into the inequality giving $f(\theta) \le \lambda\delta + (1 - \lambda)\delta$, giving $f(\theta) \le \delta$. Since $f(\theta) \le \delta$ then $\theta \in \Omega_\delta$.
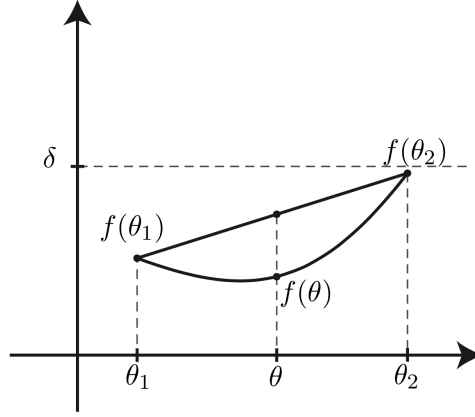
Figure 33.1: Lemma 11.1

Lemma 6 11.2 *Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Choose a constant $\delta > 0$ and consider the set $\Omega_\delta = \{\theta \in \mathbb{R}^n | f(\theta) \le \delta\} \subset \mathbb{R}^n$. Let θ^*, $\theta_b \in \Omega_\delta$. Assume θ^* is an interior point of Ω_δ, and θ_b is a boundary point of Ω_δ. That is*

$$f(\theta^*) < \delta$$
$$f(\theta_b) = \delta$$

Then the following inequality holds

$$\boxed{(\theta^* - \theta_b)^\mathsf{T} \nabla f(\theta_b) \le 0}$$

This basically says that the gradient vector of a function $f(x)$, evaluated at the boundary of a convex level set generated by this function, always points away from the set.

Proof of Lemma 11.2 From the figure below, $\theta_b + m = \theta^*$, giving $m = \theta^* - \theta_b$. Since $f(x)$ is a convex function, for $\theta = \lambda\theta^* + (1 - \lambda)\theta_b$, $f(\theta) \le \lambda f(\theta^*) + (1 - \lambda)f(\theta_b)$. That is

$$f(\lambda\theta^* + (1 - \lambda)\theta_b) \le \lambda f(\theta^*) + (1 - \lambda)f(\theta_b)$$

rearranging

$$f(\lambda\theta^* + \theta_b - \lambda\theta_b) \le \lambda f(\theta^*) + f(\theta_b) - \lambda f(\theta_b)$$
$$f(\theta_b + \lambda(\theta^* - \theta_b)) \le f(\theta_b) + \lambda(f(\theta^*) - f(\theta_b))$$
$$f(\theta_b + \lambda(\theta^* - \theta_b)) - f(\theta_b) \le \lambda(f(\theta^*) - f(\theta_b))$$

Dividing both sides by $\lambda \in (0, 1]$

$$\frac{f(\theta_b + \lambda(\theta^* - \theta_b)) - f(\theta_b)}{\lambda} \le f(\theta^*) - f(\theta_b)$$

Since $f(\theta^*) < \delta$ and $f(\theta_b) = \delta$, the inequality can be rewritten

$$\frac{f(\theta_b + \lambda(\theta^* - \theta_b)) - f(\theta_b)}{\lambda} \le f(\theta^*) - f(\theta_b) < 0$$

$$\frac{f(\theta_b + \lambda(\theta^* - \theta_b)) - f(\theta_b)}{\lambda} < 0$$

Following the definition in Rudin page 217, and taking the following limit as $\lambda \to 0$, usually called the directional derivative and denoted $D_m f$ gives

$$\lim_{\lambda \to 0} \frac{f(\theta_b + \lambda m) - f(\theta_b)}{\lambda} = \nabla f(\theta_b) \cdot m \leq 0$$

where the inequality is no longer strict after taking the limit and we have

$$m^{\mathsf{T}} \nabla f(\theta_b) \leq 0$$

and substituting back in for m

$$(\theta^* - \theta_b)^{\mathsf{T}} \nabla f(\theta_b) \leq 0$$
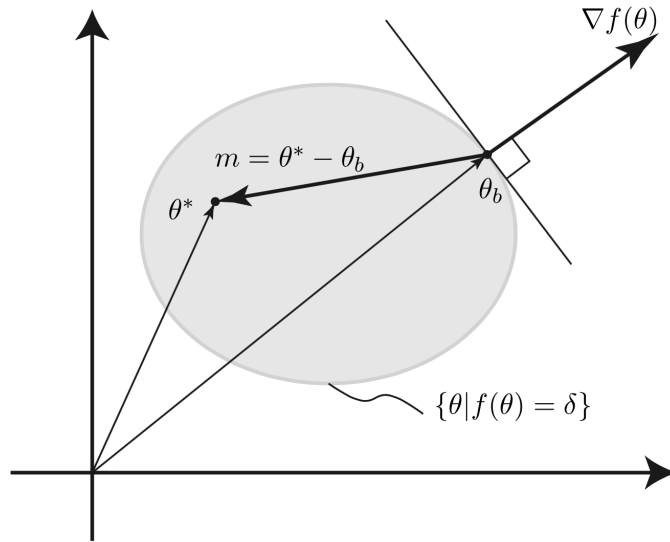
Figure 33.2: Lemma 11.2

Convex Sets

For a convex function $f(x) : \mathbb{R}^n \to \mathbb{R}$, by lemma 11.1 the following sets are convex. For $\theta \in \mathbb{R}^n$ belonging to the convex set Ω_0

$$\Omega_0 = \{\theta \in \mathbb{R}^n | f(\theta) \leq 0\}$$

For $\theta \in \mathbb{R}^n$ belonging to the convex set Ω_1

$$\Omega_1 = \{\theta \in \mathbb{R}^n | f(\theta) \leq 1\}$$

Then it is obvious that $\Omega_0 \subseteq \Omega_1$.

The Projection Operator

The idea behind the projection operator is to subtract off the component of the vector y which is perpendicular to the surface of the ball made by $f(\theta)$, and only keep the component which parallel to the surface, thus enforcing the vector y will never leave the ball. The perpendicular component of y is found by using the gradient at the ball's surface: $\nabla f(\theta)$. However, in order to ...

the vector y in the figure below by subtracting off the component of y which is normal to the surface of the ball made by $f(\theta)$. We know the direction of this normal vector, $\nabla f(\theta)$, but do not know the length. Therefore the component which must be subtracted off of y is a scalar multiple of $\nabla f(\theta)$. We denote this projection $\mathrm{Proj}(\theta, y)$.
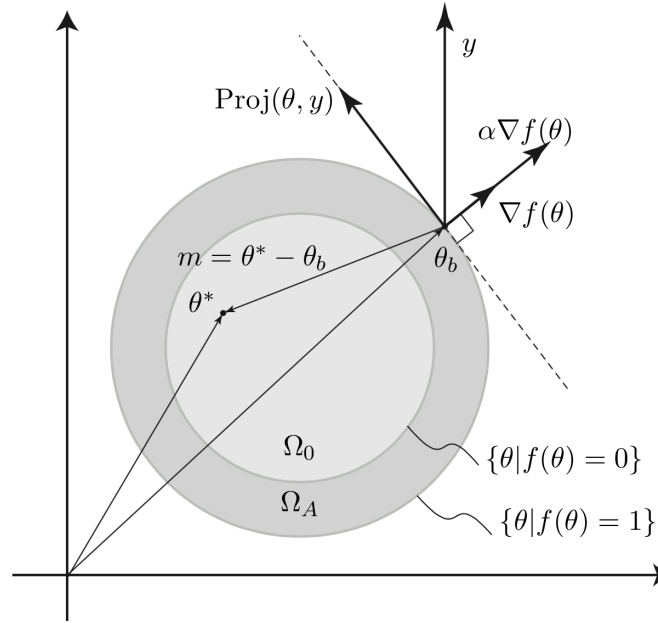
Figure 33.3: Projection

The projection operator is defined as

$$\mathrm{Proj}(\theta, y) = y - \alpha \nabla f(\theta)$$

where α is a scalar that multiplies the gradient vector so as to exactly subtract off the component of y which is perpendicular to the surface of the ball. Left multiplying both sides by $\nabla^{\mathsf{T}} f(\theta)$

$$\nabla^{\mathsf{T}} f(\theta)\mathrm{Proj}(\theta, y) = \nabla^{\mathsf{T}} f(\theta)[y - \alpha \nabla f(\theta)]$$

with $\nabla^{\mathsf{T}} f(\theta)\mathrm{Proj}(\theta, y) = 0$

$$0 = \nabla^{\mathsf{T}} f(\theta)y - \nabla^{\mathsf{T}} f(\theta)\alpha \nabla f(\theta)$$
$$\alpha \nabla^{\mathsf{T}} f(\theta)\nabla f(\theta) = \nabla^{\mathsf{T}} f(\theta)y$$

The quantity $\nabla^{\mathsf{T}} f(\theta)\nabla f(\theta) = \|\nabla f(\theta)\|^2$ is scalar, and so α can be solved for

$$\alpha = \frac{\nabla^{\mathsf{T}} f(\theta)y}{\|\nabla f(\theta)\|^2}$$

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Since α is scalar, we can write $\text{Proj}(\theta, y) = y - \nabla f(\theta)\alpha$ and substitute α in to get

$$\text{Proj}(\theta, y) = y - \frac{\nabla f(\theta)\nabla^\mathsf{T} f(\theta)}{\|\nabla f(\theta)\|^2} y$$

So the projection operator is given below in terms of θ and y, where the first condition requires that $\theta > 0$ is not within the inner ball, and that the vector y is pointing outwardly. Otherwise, the projection operator does nothing. Note the addition of the extra $f(\theta)$ term in the projection operator. This term is to dial the projection on gradually from when it first comes on until it is fully on. This extra term varies from 0 to 1, turning up the projection in the annulus of the convex function f.

$$\text{Proj}(\theta, y) = \begin{cases} y - \frac{\nabla f(\theta)\nabla^\mathsf{T} f(\theta)}{\|\nabla f(\theta)\|^2} yf(\theta), & \text{if } f(\theta) > 0 \wedge y^\mathsf{T}\nabla f(\theta) > 0 \\ y, & \text{otherwise} \end{cases}$$

Example

Assumption: have an a-priori upper-bound for $\|\theta^*\|$.
Projection in the scalar case.

$$\dot{e} = -a_m e + \tilde{\theta}(e + x_m) + d \tag{33.18}$$

$$\dot{\tilde{\theta}} = \text{Proj}(\theta, -e(e + x_m)) \tag{33.19}$$

Need a convex function to use projection. Use the following quadratic function

$$f(\theta) = a\theta^2 - h$$

where $h > 0$ and $a > 0$. The projection operator becomes the following for the scalar case

$$\text{Proj}(\theta, y) = \begin{cases} y(1 - f(\theta)), & \text{if } f(\theta) > 0 \wedge yf'(\theta) > 0 \\ y, & \text{otherwise} \end{cases}$$

where $f'(\theta) = \frac{df}{d\theta}$.

Example 61 Constant Disturbance: Numerical Simulation Consider the dynamics with disturbance in equations (33.18) above, with the following values. The plot shows the projection operator bounding θ.

$$a_m = 1$$
$$a_p = 4$$
$$x_m = 2$$
$$d = -5$$
$$e(0) = 1$$
$$\tilde{\theta}(0) = 5$$
$$\theta^* = -5$$
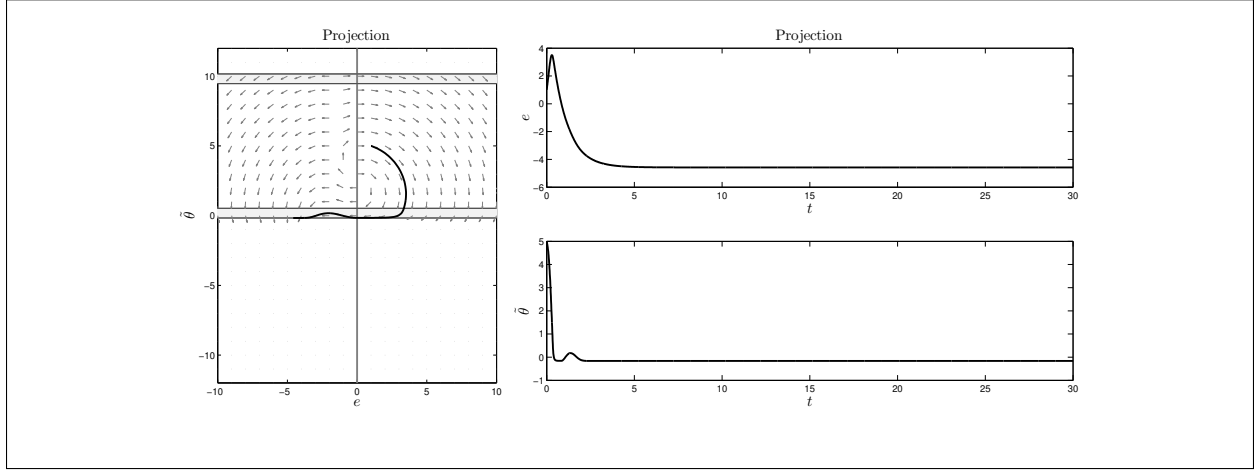
Equilibrium Points   Solve equation (33.18) with the time derivatives set to zero. Let $x_m = 0$ and let $d = -d_{\max}$

$$0 = -a_m e + \tilde{\theta} e - d_{\max}$$
$$0 = \text{Proj}(\theta, -e^2)$$

Considering first the case when projection is not active, there are no equilibrium solutions. Considering the case when projection is active, we have

$$0 = -a_m e + \tilde{\theta} e - d_{\max}$$
$$0 = -e^2(1 - a\theta^2 + h)$$

The only equilibrium solutions that exist are when

$$1 - a\theta^2 + h = 0$$

giving

$$\tilde{\theta}_{\text{eq}} = \pm\sqrt{\frac{1+h}{a}} - \theta^*$$

and

$$e_{\text{eq}} = \frac{d_{\max}}{-a_m + \tilde{\theta}_{\text{eq}}}$$

Γ-Projection

In adaptive control, the vector y which is projected onto the θ ball may sometimes need to scaled by a factor Γ. It may seem intuitive to implement this as $\Gamma\text{Proj}(\theta, y)$ or as follows

$$\text{Proj}_\Gamma(\theta, \Gamma y) = \Gamma y - \alpha \nabla f(\theta)$$

which does not work. The idea behind Γ-projection is the same as that for the standard projection, with the inclusion of the scaling gain Γ in the algorithm in order to facilitate its use in an adaptive control law.

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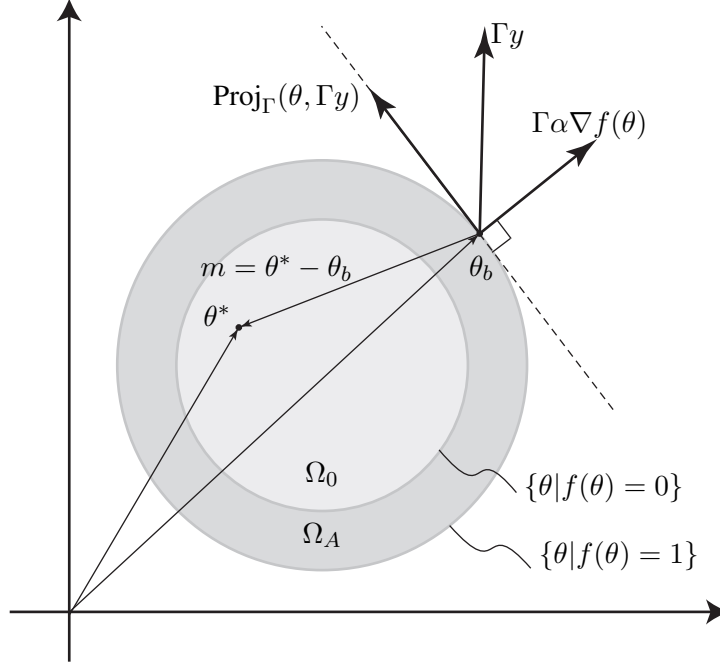
Figure 33.4: Γ-Projection

The $\Gamma-$projection operator is defined as

$$\mathrm{Proj}_\Gamma(\theta, \Gamma y) = \Gamma y - \Gamma \alpha \nabla f(\theta)$$

And we note the conditions under which we want to consider the projection to be "active". When the vector θ is inside the inner ball, that is θ is such that $f(\theta) \leq 0$ we want projection to be off. Then, when θ is in the annulus such that $0 < f(\theta) \leq 1$ we want projection to only be active when the vector Γy is pointing in an outwards sense. The outward pointing of Γy is given by when the following dot product is is positive

$$\Gamma y \cdot \nabla f(\theta) > 0$$

that is

$$(\Gamma y)^\mathsf{T} \nabla f(\theta) > 0$$

$$y^\mathsf{T} \Gamma \nabla f(\theta) > 0$$

Again left multiplying both sides by $\nabla^\mathsf{T} f(\theta)$

$$\nabla^\mathsf{T} f(\theta)\mathrm{Proj}_\Gamma(\theta, \Gamma y) = \nabla^\mathsf{T} f(\theta)(\Gamma y - \Gamma \alpha \nabla f(\theta))$$

$$\nabla^\mathsf{T} f(\theta)\mathrm{Proj}_\Gamma(\theta, \Gamma y) = \nabla^\mathsf{T} f(\theta)\Gamma y - \nabla^\mathsf{T} f(\theta)\Gamma \alpha \nabla f(\theta)$$

with $\nabla^\mathsf{T} f(\theta)\mathrm{Proj}_\Gamma(\theta, y) = 0$

$$0 = \nabla^\mathsf{T} f(\theta)\Gamma y - \nabla^\mathsf{T} f(\theta)\Gamma \alpha \nabla f(\theta)$$

$$\alpha \nabla^\mathsf{T} f(\theta)\Gamma \nabla f(\theta) = \nabla^\mathsf{T} f(\theta)\Gamma y$$

The quantity $\nabla^\mathsf{T} f(\theta)\Gamma\nabla f(\theta)$ is scalar, and so is invertible, and α can be expressed

$$\alpha = \frac{\nabla^\mathsf{T} f(\theta)\Gamma y}{\nabla^\mathsf{T} f(\theta)\Gamma\nabla f(\theta)}$$

Since α is scalar, we can write $\mathrm{Proj}_\Gamma(\theta,\Gamma y) = \Gamma y - \Gamma\nabla f(\theta)\alpha$ and substitute in α to get

$$\mathrm{Proj}_\Gamma(\theta,\Gamma y) = \Gamma y - \Gamma\frac{\nabla f(\theta)\nabla^\mathsf{T} f(\theta)}{\nabla^\mathsf{T} f(\theta)\Gamma\nabla f(\theta)}\Gamma y$$

And finally, the projection operator can expressed as follows, where we again note the addition of the extra $f(\theta)$ to "dial on" projection smoothly within the annulus of the $f(\theta)$ ball.

Γ−Projection

$$\mathrm{Proj}_\Gamma(\theta,\Gamma y) = \begin{cases} \Gamma y - \Gamma\frac{\nabla f(\theta)\nabla^\mathsf{T} f(\theta)}{\nabla^\mathsf{T} f(\theta)\Gamma\nabla f(\theta)}\Gamma y f(\theta), & \text{if } f(\theta) > 0 \wedge y^\mathsf{T}\Gamma\nabla f(\theta) > 0 \\ \Gamma y, & \text{otherwise} \end{cases}$$

Prove the following, where $\tilde{\theta} = \theta - \theta^*$ and where the convex function f is selected such that $\theta^* \in \{\theta \in \mathbb{R}^n | f(\theta) < 0\}$. In implementation, this can be easily enforced by selecting the inner radius to be an order of magnitude larger than the the largest expected uncertainty.

$$\tilde{\theta}^\mathsf{T}\left(\Gamma^{-1}\mathrm{Proj}_\Gamma(\theta,\Gamma y) - y\right) \le 0$$

There are two cases that θ can take: either in the inner ball, or in the annulus. If θ is in the inner ball, that is θ is such that $f(\theta) \le 0$ then $\mathrm{Proj}(\theta,\Gamma y) = \Gamma y$ and we have

$$\tilde{\theta}^\mathsf{T}\left(\Gamma^{-1}\Gamma y - y\right) = 0$$

If θ is in the annulus, that is θ is such that $0 < f(\theta) \le 1$ then we look at the quantity

$$\tilde{\theta}^\mathsf{T}\left(\Gamma^{-1}\mathrm{Proj}_\Gamma(\theta,\Gamma y) - y\right) = -\tilde{\theta}^\mathsf{T}\left(y - \Gamma^{-1}\mathrm{Proj}_\Gamma(\theta,\Gamma y)\right)$$

Substituting the definition of projection in

$$-\tilde{\theta}^\mathsf{T}\left[y - \Gamma^{-1}\left(\Gamma y - \Gamma\frac{\nabla f(\theta)\nabla^\mathsf{T} f(\theta)}{\nabla^\mathsf{T} f(\theta)\Gamma\nabla f(\theta)}\Gamma y f(\theta)\right)\right]$$

and simplifying

$$-\tilde{\theta}^\mathsf{T}\left[y - \Gamma^{-1}\Gamma y - \Gamma^{-1}\Gamma\frac{\nabla f(\theta)\nabla^\mathsf{T} f(\theta)}{\nabla^\mathsf{T} f(\theta)\Gamma\nabla f(\theta)}\Gamma y f(\theta)\right]$$

more

$$\tilde{\theta}^\mathsf{T}\left(\frac{\nabla f(\theta)\nabla^\mathsf{T} f(\theta)}{\nabla^\mathsf{T} f(\theta)\Gamma\nabla f(\theta)}\Gamma y f(\theta)\right)$$

more

$$\tilde{\theta}^\mathsf{T}\nabla f(\theta)\frac{\nabla^\mathsf{T} f(\theta)\Gamma y f(\theta)}{\nabla^\mathsf{T} f(\theta)\Gamma\nabla f(\theta)}$$

And using Lemma 11.2 we have that $\tilde{\theta}^\mathsf{T}\nabla f(\theta) \le 0$, the denominator is positive since Γ is positive definite, and the numerator is a positive scalar, as defined for projection to be "on" (the vector Γy must point outwardly)

$$\tilde{\theta}^\mathsf{T}\left(\Gamma^{-1}\mathrm{Proj}_\Gamma(\theta,\Gamma y) - y\right) = \tilde{\theta}^\mathsf{T}\nabla f(\theta)\frac{\nabla^\mathsf{T} f(\theta)\Gamma y f(\theta)}{\nabla^\mathsf{T} f(\theta)\Gamma\nabla f(\theta)} \le 0$$

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Implementation of Projection in Adaptive Control

The projection operator is used in the adaptive control gain update law shown below to bound the adaptive gain θ and prevent it from becoming too large.

$$\dot{\theta} = \text{Proj}(\theta, -\Gamma x e^\mathsf{T} P B_1 \text{sign}(\Lambda))$$

$$\dot{\theta} = \text{Proj}_\Gamma(\theta, -\Gamma x e^\mathsf{T} P B_1 \text{sign}(\Lambda))$$

$$\tilde{\theta}^\mathsf{T}[\Gamma^{-1}\text{Proj}_\Gamma(\theta, \Gamma y) - y] \leq 0$$

Convex Function in Adaptive Law

In order to use the projection operator, a convex function $f(x)$ is needed. Every norm is a convex function. The basic convex function that will be used with the projection operator is the norm-squared. Two representations of the convex function $f(\theta; \theta_{\max}, \varepsilon) = f(\theta)$ are shown below.

Function 1: Inner and Outer Radius

$$f(\theta; \theta_{\max}, \varepsilon) = f(\theta) = \frac{(1 + \varepsilon)\|\theta\|^2 - \theta_{\max}{}^2}{\varepsilon \theta_{\max}{}^2} \quad \textbf{Function 1}$$

The value of $\theta_{\max}$ is such that $f(\theta : \|\theta\| = \theta_{\max}) = 1$. This is the outer radius of Ω_A which θ can never exceed. ε sets the radius of Ω_0 as $\theta_{\max}/(1 + \varepsilon)$.

$$f(\theta) = \frac{(1 + \varepsilon)\|\theta\|^2 - \theta_{\max}{}^2}{\varepsilon \theta_{\max}{}^2} = 1$$

$$(1 + \varepsilon)\|\theta\|^2 - \theta_{\max}{}^2 = \varepsilon \theta_{\max}{}^2$$

$$(1 + \varepsilon)\|\theta\|^2 = (1 + \varepsilon)\theta_{\max}{}^2$$

$$\|\theta\| = \theta_{\max}$$

$$f(\theta) = \frac{(1 + \varepsilon)\|\theta\|^2 - \theta_{\max}{}^2}{\varepsilon \theta_{\max}{}^2} = 0$$

$$(1 + \varepsilon)\|\theta\|^2 - \theta_{\max}{}^2 = 0$$

$$\|\theta\|^2 = \frac{\theta_{\max}{}^2}{1 + \varepsilon}$$

$$\|\theta\| = \frac{\theta_{\max}}{\sqrt{1 + \varepsilon}}$$
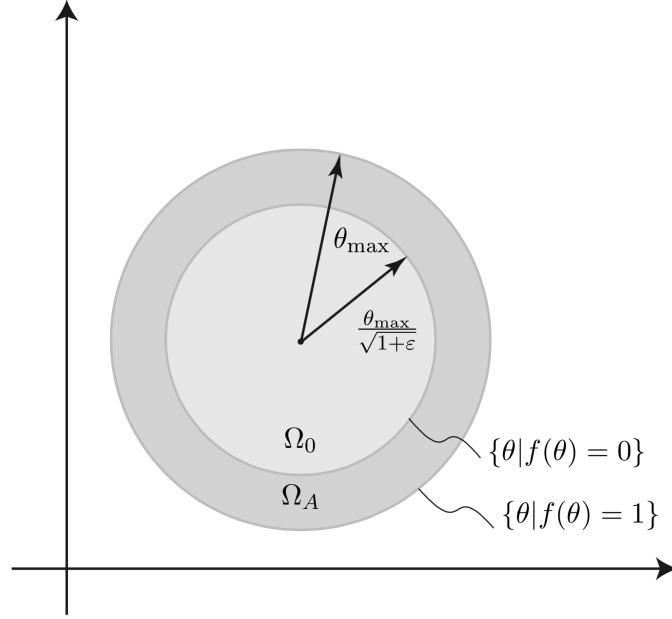
Figure 33.5: Eugene's convex function

Function 2: Inner Radius and Annulus

$$f(\theta; \theta_0, \Delta) = f(\theta) = \frac{\|\theta\|^2 - \theta_0{}^2}{2\Delta\theta_0 + \Delta^2} \quad \textbf{Function 2}$$

The value of θ_0 is the radius of the inner radius of Ω_0, and Δ is the width of the annulus Ω_A. The value of $\theta_0 + \Delta$ thus gives the outer radius of Ω_A such that $f(\theta : \|\theta\| = \theta_0 + \Delta) = 1$. This is value which θ can never exceed.

$$f(\theta) = \frac{\|\theta\|^2 - \theta_0{}^2}{2\Delta\theta_0 + \Delta^2} = 1$$

$$\|\theta\|^2 - \theta_0{}^2 = 2\Delta\theta_0 + \Delta^2$$

$$\|\theta\|^2 = \theta_0{}^2 + 2\Delta\theta_0 + \Delta^2$$

$$\|\theta\|^2 = (\theta_0 + \Delta)^2$$

$$\|\theta\| = \theta_0 + \Delta$$

$$f(\theta) = \frac{\|\theta\|^2 - \theta_0{}^2}{2\Delta\theta_0 + \Delta^2} = 0$$

$$\|\theta\|^2 - \theta_0{}^2 = 0$$

$$\|\theta\| = \theta_0$$

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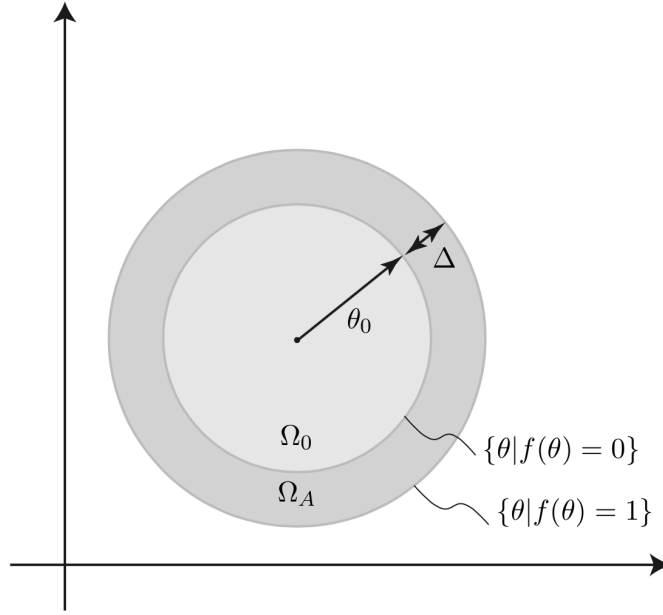
Figure 33.6: Convex function 2

Gradient of a Convex Function

For a convex function of the form $f(x) = a\|x\|^2 - b$ where $0 < a, b \in \mathbb{R}$ the gradient is computed as follows:

$$\nabla f(x) = a\nabla \left(\|x\|^2\right) = a\nabla(x^\mathsf{T}x)$$

$$x^\mathsf{T}x = x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2$$

$$\nabla(x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2) = \frac{\partial}{\partial x_1}x_1{}^2\hat{k}_1 + \frac{\partial}{\partial x_2}x_2{}^2\hat{k}_2 + \cdots + \frac{\partial}{\partial x_n}x_n{}^2\hat{k}_n$$

$$\nabla(x^\mathsf{T}x) = 2\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\mathsf{T}$$

$$\nabla f(x) = 2ax$$

Chapter 34

Uniform Asymptotic Stability and Parameter Convergence and Error Models

(Chapter 2): stability (asymptotic) not just error e but also parameter error to zero Stability Uniform asymptotic stability Parameter convergence Persistent excitation Linear systems Linear Time Invariant (LTI)

$$\dot{x} = Ax$$

where A is a matrix. Solution

$$x(t) = e^{A(t-t_0)}x(t_0)$$

$$A = V\Lambda V^{-1}$$

$$e^{At} = Ve^{\Lambda t}V^{-1}$$

stability determined by Λ solutions determined by Λ Linear Time Varying (LTV)

$$\dot{x} = A(t)x$$

Still a transition matrix $\Phi(t, t_0)$ where

$$x(t) = \Phi(t, t_0)x(t_0)$$

Remember: $\Phi(t, t_0)$ exists, and it is not always equal to $e^{\int_{t_0}^{t} A(\tau)d\tau}$ and so eigenvalues do not mean anything. Eigenvalues of $A(t)$ do not provide clues for stability. (example: time varying matrix with constant eigenvalues in LHP but LTV system is unstable. $x(t) \to \infty$) in scalar case $\dot{x} = a(t)x$ then $x(t) = \exp\{\int_{t_0}^{t} a(\tau)d\tau\}x(t_0)$ aside: Almost time-invariant systems

$$\dot{x} = \big[A + B(t)\big]x$$

basically $B(t)$ is small. Three cases

(i) $\|B(t)\| \to 0$ as $t \to \infty$

(ii) $B(t) \in \mathcal{L}^1$

(iii) $B(t) \in \mathcal{L}^2$

If A is Hurwitz $x(t) \to 0$ as $t \to \infty$.

34.1 Error Models

Return to adaptive systems had two error models we discussed

(i) Error model 1

(ii) Error model 3

34.1.1 Error model 1

In the case where the input u is related to the output e through algebra alone, the result is error model 1. In this case, there is no differential equation representation for e, and so the only error equation is for $\tilde{\theta}$. In error model 1, e is thus not a state, and so we don't need it in the Lyapunov function when proving stability.

Scalar Case

Consider the following scalar algebraic system. We wish to identify the parameter θ in the following diagram, where the input is u and the output y. An estimator is proposed, which, when given input u will generate the output $\hat{y}$.
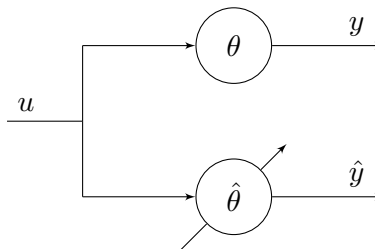
The better the estimate of the parameter is, the closer the output $\hat{y}$ will be to y. It is this difference that will be used to drive the parameter estimate, as shown in the following figure.
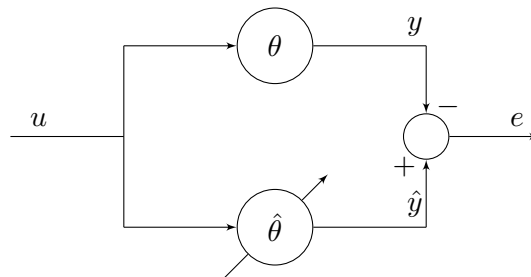
The output from the parameter and estimate can be represented

$$\theta(t)u(t) = y(t)$$
$$\hat{\theta}(t)u(t) = \hat{y}(t)$$

with tracking error and parameter error

Tracking Error:	$e = \hat{y} - y$	(34.1)
Estimation Error:	$\tilde{\theta} = \hat{\theta} - \theta$	(34.2)

the equations above are subtracted, giving

Error Model 1:	$e(t) = \tilde{\theta}(t)u(t)$	(34.3)

Which is represented with the following block diagram
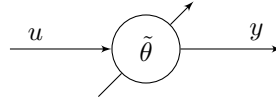
If we multiply both sides by $u(t)$ we get

$$e(t)u(t) = \tilde{\theta}(t)u(t)^2$$

which will indicate the sign of the error $\tilde{\theta}$ since the signs of $e(t)$ and $u(t)$ are known. So, a natural choice is to pick the adaptive law as follows. This gives an expression for the estimation error dynamics which depends explicitly on the input u. This expression is important, as it will allow us to see how any given input will influence the convergence of the estimation error, and thus our ability to correctly estimate the unknown parameter.

Update Law/Estimation Error Dynamics:	$\dot{\tilde{\theta}} = -u^2(t)\tilde{\theta}$	(34.4)

Using the expression for e from the error model 1 above we have

Parameter Update Law:	$\dot{\tilde{\theta}} = -e(t)u(t)$	(34.5)

This is a recursive law? Gradient

$$\frac{\partial J}{\partial \hat{\theta}} \propto [-u\hat{\theta} + y]u = [-u(\hat{\theta} - \theta)]u$$
$$= -(u\tilde{\theta})u$$
$$= -eu$$

Check stability with the proposed update law by using following candidate Lyapunov function

$$V(\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^2$$

We want $\tilde{\theta} \to 0$ as $t \to \infty$. Time differentiating V we have

$$\dot{V} = \tilde{\theta}\dot{\tilde{\theta}}$$
$$= -\tilde{\theta}eu$$
$$= -e(\tilde{\theta}u)$$
$$= -e^2 \leq 0$$

Thus V is non-increasing. Stable, $\tilde{\theta}$ bounded.

(i) u is constant, not equal to zero, then $\tilde{\theta}(t) \to 0$ as $t \to \infty$ convergence is uniform in $t \Rightarrow$ uniformly asymptotically stable.

(ii) $u(t) \to 0$ $\tilde{\theta}(t) = e^{-\int_{t_0}^{t} u^2(\tau)d\tau}\tilde{\theta}(t_0)$ so for this to go to zero then the integral needs to go to zero. If this integral tends to infinity as t tends to infinity then $\tilde{\theta}(t)$ goes to zero. If the integral is finite, then $\tilde{\theta}(t) = c\tilde{\theta}(t_0)$. If finite, don't have UAS, if infinite, have UAS.

(iii) The following property is defined as persistent excitation

$$\int_{t_0}^{t} u^2(\tau)d\tau \geq ct$$

$$\tilde{\theta}(t) = e^{-\int_{t_0}^{t} u^2(\tau)d\tau}\tilde{\theta}(t_0)$$

$$\|\tilde{\theta}(t)\| \leq e^{-ct}\|\tilde{\theta}(t_0)\| \to 0 \qquad \text{as} \qquad t \to \infty$$

Vector case
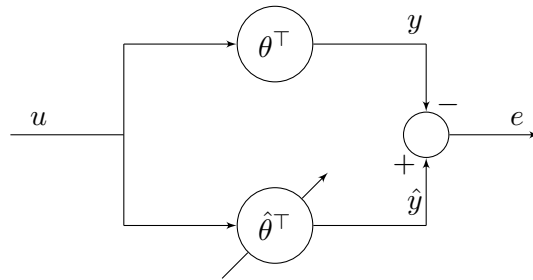
The output from the parameter and estimate can be represented

$$\theta^\top(t)u(t) = y(t)$$
$$\hat{\theta}^\top(t)u(t) = \hat{y}(t)$$

As in the scalar case, the tracking error and parameter error are given by

Tracking Error:	$e = \hat{y} - y$	(34.6)
Estimation Error:	$\tilde{\theta} = \hat{\theta} - \theta$	(34.7)

The equations above are subtracted, giving

Error Model 1:	$e(t) = \tilde{\theta}^{\top}(t)u(t)$	(34.8)

Which is represented by the following block diagram

Propose

$$\dot{\tilde{\theta}} = -(\hat{y} - y)u$$
$$= -(\hat{\theta}^{\top}(t)u(t) - \theta^{\top}u(t))u(t)$$
$$= -\tilde{\theta}^{\top}(t)u(t)u(t)$$
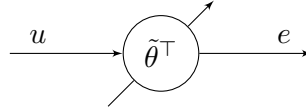
But

Estimation Error Dynamics:	$\dot{\tilde{\theta}} = -u(t)u^{\top}(t)\tilde{\theta}$	(34.9)

Check stability using following candidate Lyapunov function

$$V(\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^{\top}\tilde{\theta}$$

want $\tilde{\theta} \to 0$ as $t \to \infty$. Taking the time derivative of $\dot{V}$ we have

$$\dot{V} = \tilde{\theta}^{\top}\dot{\tilde{\theta}}$$
$$= -(\tilde{\theta}^{\top}u)(u^{\top}\tilde{\theta})$$
$$= -(\tilde{\theta}^{\top}u)^2$$
$$= -e^2$$

V is non-increasing. Look back at the estimation error dynamics. We can write this expression by using a matrix as

$$\dot{\tilde{\theta}} = -A(t)\tilde{\theta}$$

where $A(t) = u(t)u^\top(t)$. Like eigenvalues and eigenvectors... rank 1. Regressor is u that determines direction of adaptation. If $A(t)$ satisfies (persistent excitation of u) $\int_t^{t+T} A(\tau)d\tau \geq \alpha I$ for some α, I and $\forall t \geq t_0$ then $\tilde\theta(t) \to 0$ as $t \to == [U^\top U]$ being non-singular.

$$\dot{\hat\theta} \propto 2[y - \hat\theta u]u = -eu$$

$$\dot{\tilde\theta} = A(t)\tilde\theta$$

$$A(t) = -uu^\top$$

Goal $\tilde\theta(t) \to 0$ (Parameter convergence)

$\Rightarrow$ uniform asymptotic stability of $\dot{\tilde\theta} = A(t)\tilde\theta$ is needed

(i) $u(t)$ is constant. Assume $u(t) \in \mathbb{R}^2$ $\dot{\tilde\theta} = c(-c^\top\tilde\theta)$

$\tilde\theta(t)$ does not go to 0 as $t \to \infty$

We say that $u(t) \in \mathbb{R}^n$ is persistently exciting if...

34.1.2 Persistent Excitation

In this subsection, we will consider error model 1, with the estimation error dynamics given by (34.9). We will look at how the properties of u affect the convergence of this error.

The equilibrium of the estimation error dynamics (34.9) are uniformly asymptotically stable if and only if the input $u : \mathbb{R}^+ \to \mathbb{R}^n$ satisfies the following inequality for positive constants t_0, T_0, and α. This is equation (6.8) in the book on page 246.

$$\frac{1}{T_0} \int_t^{t+T_0} u(\tau)u^\top(\tau)d\tau \geq \alpha I \qquad \forall t \geq t_0$$

An equivalent definition is that of (6.9) on page 247 that states that the equilibrium of the estimation error dynamics (34.9) are uniformly asymptotically stable if and only if for every *unit vector* $w \in \mathbb{R}^n$ the following inequality is satisfied

$$\frac{1}{T_0} \int \|u^\top(\tau)w\|d\tau \geq \varepsilon_0 > 0 \qquad \forall t \geq t_0$$

for positive constants t_0, T_0, and ε_0.

$u(t) = c_1$ for $t_0 \leq t \leq t_0 + T_1$ $u(t) = c_2$ for $t_0 + T_1 \leq t \leq t_0 + T_0$

Recall that when using the update law in (xx) the proposed Lyapunov function had time derivative $\dot{V}(t) = -\left(u^\top(t)\tilde\theta(t)\right)^2$. If we integrate $\dot{V}$ from some initial time to some later time $+T_0$ we get

$$\int_{t_0}^{t_0+T_0} \dot{V}(\tau)d\tau = V(t_0 + T) - V(t)$$

$$= \int_{t_0}^{t_0+T_0} \left(u^\top(\tau)\tilde\theta(\tau)\right)^2 d\tau$$

$$= \int_{t_0}^{t_0+T_0} \|u^\top(\tau)\tilde\theta(\tau)\|^2 d\tau$$

If this integral is $\geq \varepsilon_1 \forall t \geq t_0$ then the error dynamics (34.9) are uniformly asymptotically stable. Next we need to use the Cauchy-Schwarz inequality, given below.

$$\boxed{\textbf{Cauchy-Schwarz Inequality:} \qquad \left(\int_a^b f(\tau)g(\tau)d\tau \right)^2 \leq \int_a^b \left(f(\tau) \right)^2 d\tau \int_a^b \left(g(\tau) \right)^2 d\tau} \quad (34.10)$$

Let $f(\tau) = \|u^\top(\tau)\tilde{\theta}(\tau)\|$, $g(\tau) = 1$, $a = t$, and $b = t + T_0$. Substituting this into this inequality we get

$$\left(\int_t^{t+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau \right)^2 \leq \int_t^{t+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|^2 d\tau \int_t^{t+T_0} d\tau$$

Integrating part of the right hand side and dividing since $T_0 > 0$ won't flip the direction of the inequality we have

$$\frac{1}{T_0} \left(\int_t^{t+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau \right)^2 \leq \int_t^{t+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|^2 d\tau$$

Putting the pieces so far together we have

$$V(t_0 + T) - V(t) \geq \frac{1}{T_0} \left(\int_t^{t+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau \right)^2$$

Now we can multiply through by T_0 and take the square root of both sides to get

$$\boxed{\int_t^{t+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau \leq \sqrt{T_0}\sqrt{V(t_0 + T_0) - V(t)}}$$

Now we will use a special form of the Minkowski inequality for $p = 1$. This inequality is

$$\boxed{\textbf{Minkoski Inequality:} \qquad \int \|f(\tau) + g(\tau)\|d\tau \leq \int \|f(\tau)\|d\tau + \int \|g(\tau)\|d\tau} \quad (34.11)$$

Now notice that the integrand of the above can be written as

$$\|u^\top(\tau)\tilde{\theta}(\tau)\| = \|u^\top(\tau)\big(\tilde{\theta}(t_0) - \tilde{\theta}(t_0) + \tilde{\theta}(\tau)\big)\|$$

Also note that $\| - h(\tau)\| = \|h(\tau)\|$ for any $h(\tau)$, and to use the Minkowski inequality define

$$f(\tau) = u^\top(\tau)\tilde{\theta}(\tau)$$
$$g(\tau) = u^\top(\tau)\big(\tilde{\theta}(t_0) - \tilde{\theta}(\tau)\big)$$

with

$$f(\tau) + g(\tau) = u^\top(\tau)\tilde{\theta}(t_0)$$

Plugging this into the Minkowski inequality we have

$$\int \|u^\top(\tau)\tilde{\theta}(t_0)\|d\tau \leq \int \|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau + \int \|u^\top(\tau)\big(\tilde{\theta}(t_0) - \tilde{\theta}(\tau)\big)\|d\tau$$

Rearranging

$$\boxed{\int \|u^\top(\tau)\tilde{\theta}(t_0)\|d\tau - \int \|u^\top(\tau)\big(\tilde{\theta}(t_0) - \tilde{\theta}(\tau)\big)\|d\tau \leq \int \|u^\top(\tau)\tilde{\theta}(t_0)\|d\tau}$$

Now look at the terms on the left hand side. Finish this reasoning later.

$$\|\tilde{\theta}(t_0)\|T_0\varepsilon_3 \leq \int \|u^\top(\tau)\tilde{\theta}(t_0)\|d\tau$$

and

$$u_{\max}T_0 \int_{t_1}^{t_1+T_0} \|\dot{\tilde{\theta}}\|d\tau \geq \int \|u^\top(\tau)\big(\tilde{\theta}(t_0) - \tilde{\theta}(\tau)\big)\|d\tau$$

With this we get

$$\|\tilde{\theta}(t_0)\|T_0\varepsilon_3 - u_{\max}T_0 \int_{t_1}^{t_1+T_0} \|\dot{\tilde{\theta}}\|d\tau \leq \int_t^{t+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau$$

If we substitute our expression for $\dot{\tilde{\theta}}(\tau) = -u(\tau)u^\top(\tau)\tilde{\theta}(\tau)$ into the second term on the left hand side

$$u_{\max}T_0 \int_{t_1}^{t_1+T_0} \|\dot{\tilde{\theta}}\|d\tau = u_{\max}T_0 \int_{t_1}^{t_1+T_0} \| - u(\tau)u^\top(\tau)\tilde{\theta}(\tau)\|d\tau$$

And since $u^\top(\tau)\tilde{\theta}(\tau)$ is a scalar, using the property of norms that for scalars c and vectors v that $\|c\|\|v\| = \|cv\|$ we can write the integrand as $\| - u(\tau)u^\top(\tau)\tilde{\theta}(\tau)\| = \|u(\tau)u^\top(\tau)\tilde{\theta}(\tau)\| = \|u(\tau)\|\|u^\top(\tau)\tilde{\theta}(\tau)\|$. Putting this into the integral

$$u_{\max}T_0 \int_{t_1}^{t_1+T_0} \|\dot{\tilde{\theta}}\|d\tau = u_{\max}T_0 \int_{t_1}^{t_1+T_0} \|u(\tau)\|\|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau$$

And we can write the inequality that $\|u(\tau)\|\|u^\top(\tau)\tilde{\theta}(\tau)\| \leq u_{\max}\|u^\top(\tau)\tilde{\theta}(\tau)\|$ Putting this into the integral we have

$$u_{\max}T_0 \int_{t_1}^{t_1+T_0} \|\dot{\tilde{\theta}}\|d\tau \leq u_{\max}^2 T_0 \int_{t_1}^{t_1+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau$$

Finally

$$\|\tilde{\theta}(t_0)\|T_0\varepsilon_3 - u_{\max}^2 T_0 \int_{t_1}^{t_1+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau \leq \int_t^{t+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau$$

which we can write

$$\|\tilde{\theta}(t_0)\|T_0\varepsilon_3 \leq \big(1 + u_{\max}^2 T_0\big) \int_t^{t+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau$$

rearranging

$$\boxed{\frac{\|\tilde{\theta}(t_0)\|T_0\varepsilon_3}{1 + u_{\max}^2 T_0} \leq \int_t^{t+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\|d\tau}$$

Combining this with what we had way above

$$\frac{\|\tilde{\theta}(t_0)\| T_0 \varepsilon_3}{1 + u_{\max}^2 T_0} \leq \sqrt{T_0}\sqrt{V(t_0 + T_0) - V(t)}$$

Squaring both sides

$$\geq \varepsilon_0 T_0 \|\tilde{\theta}(t_0)\| - u_{\max}^2 T_0 \int_{t_0}^{t_0+T_0} \|u^\top(\tau)\tilde{\theta}(\tau)\| d\tau$$

$$\geq \frac{\varepsilon_0 T_0}{1 + u_{\max}^2 T_0}\sqrt{2V(t_0)}$$

$$\Rightarrow V(t_0 + T) - V(t_0) \leq -\frac{2\varepsilon_0^2 T_0}{(1 + u_{\max}^2 T_0)^2} V(t_0)$$

$$V(t_0 + T) \leq \gamma V(t_0)$$

for $0 < \gamma < 1 \Rightarrow V(t) \to 0$ as $t \to \infty$.

34.1.3   Error Model 3
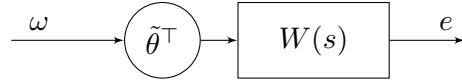
$$\dot{e} = Ae + b\omega^\top \tilde{\theta}$$

$$e_1 = h^\top e$$

$$\dot{\tilde{\theta}} = -e_1 \omega$$

$$h^\top (sI - A)^{-1} b = W(s)$$

$$\begin{bmatrix} \dot{e} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A & b\omega^\top \\ -\omega h^\top & 0 \end{bmatrix} \begin{bmatrix} e \\ \tilde{\theta} \end{bmatrix}$$

$$\dot{x} = A(t)x$$

$e = x_1 \quad \tilde{\theta} = x_2$

$$V = \frac{1}{2}(x_1^\top P x_1 + x_2^\top P x_2)$$

$$\dot{V} = -\frac{1}{2}x_1^\top Q x_1 \leq 0$$

$\dot{x} = A(t)x$ is UAS if T_0, ε_0, and δ_0 exist so that

$$\frac{1}{T_0} \int_{t_2}^{t_2+\delta_0} \omega^\top(\tau) w d\tau \| \geq \varepsilon_0$$

for all $t \geq t_0$ $[t_2, t_2 + \delta_0] \subset [t, t + T_0]$

Error model 3

$$\dot{\tilde{\theta}} = -e_1 \omega = -\omega W(s)[\omega^\top \tilde{\theta}]$$

Conditions on ω so that $\tilde{\theta}(t) \to 0$.

Error model 2 is the same as error model 3, but when all the states are accessible.

34.2 Persistent Excitation

34.2.1 Introduction

Summary of last class: looked at two error models.

Properties of persistently exciting signals (chapter 6)

three figures in R2

$$\omega(t) = \begin{bmatrix} \sin(\Omega_0 t) \\ \cos(\Omega_0 t) \\ \sin(\Omega_0 t + \phi) \end{bmatrix}$$

Not PE in $\mathbb{R}^3$ since the third component z-direction is a linear combination of the first two components, x and y-directions.

$$\omega(t) = \begin{bmatrix} \sin(\Omega_1 t) \\ \cos(\Omega_1 t) \\ \sin(\Omega_2 t) \\ \cos(\Omega_2 t) \\ \vdots \\ \sin(\Omega_n t) \\ \cos(\Omega_n t) \end{bmatrix}$$

$\Omega_i \neq \Omega_j$ then PE in $\mathbb{R}^{2n}$ need n distinct frequencies.

In $\omega \in \Omega_n$ then $T\omega \in \Omega_n$ where T nonsingular.

Suppose reference input $r(t) = \sum_{i=1}^n a_i \sin(\Omega_i t)$, $\Omega_i \neq \Omega_j$

Then

$$R = \begin{bmatrix} r \\ \dot{r} \\ \vdots \\ r^{(2n-1)} \end{bmatrix} \in \Omega_{2n} = T \begin{bmatrix} \sin(\Omega_1 t) \\ \cos(\Omega_1 t) \\ \sin(\Omega_2 t) \\ \vdots \\ \cos(\Omega_n t) \end{bmatrix}$$

where T nonsingular.

where $q(s)$ is a Hurwitz polynomial. If $u \in \Omega_1 \Rightarrow y \in \Omega_1$. Linear system same frequency comes through, so input PE means output PE.

another block diagram, same thing can be said about the output $p(s)$ Hurwitz. Problem if zero was on imaginary axis, that frequency component wouldn't show up in output. $u \in \Omega_1 \Rightarrow y \in \Omega_1$

another block diagram input r output x_m. $\dot{x}_m = Ax_m + br$. If $r = \sum_{i=1}^n a_i \sin \Omega_i t$ $x_m(t) \in \Omega_{2n}$. $A \in \mathbb{R}^{2n \times 2n}$ A Hurwitz, (A, b) controllable.

$x(t) = x_m(t) + e(t)$ and $x_m(t) \in \Omega_n$, $e(t) \to 0$ as $t \to \infty \Rightarrow x(t) \in \Omega_n$.

Return to Control Problem

1. Scalar
2. States accessible

34.2.2 Case (1) Scalar

Case 1: Scalar

Plant:	$\dot{x} = a_p x_p + bu$
Control:	$u = \theta x_p + kr$
Model:	$\dot{x}_m = a_m x_m + b_m r$
Tracking error:	$e = x_p - x_m$
Tracking error dynamics:	$\dot{e} = a_m e + b(\tilde{k}r + \tilde{\theta}x_p)$
Parameter error:	$\theta = \tilde{\theta} + \theta^*$
	$k = \tilde{k} + k^*$
Matching condition:	$a_p + b\theta^* = a_m$
	$bk^* = b_m$
Update laws:	$\dot{\tilde{\theta}} = -\text{sgn}(b)ex_p$
	$\dot{\tilde{k}} = -\text{sgn}(b)er$

Proposing the following candidate Lyapunov function

$$V = \frac{1}{2}(e^2 + |b|(\tilde{\theta}^2 + \tilde{k}^2))$$

Time differentiating

$$\dot{V} = a_m e^2$$

And so $e, \tilde{\theta}, \tilde{k} \in \mathcal{L}^\infty$ and $e(t) \to 0$ as $t \to \infty$ and $\text{sgn}(b) = +1$

so ω PE in $\mathbb{R}^2$ <=> $\tilde{\theta} \to 0!$ $\omega = \omega^* + Ce\,\omega^*$ PE in $\mathbb{R}^2 \Rightarrow \tilde{\theta} \to 0$.

34.2.3 Case (2) States Accessible

<div style="border:1px solid red; padding:1em;">

Case 2: States Accessible

Plant:	$\dot{x} = A_p x_p + bu$
Control:	$u = \theta^\top x_p + r$
Model:	$\dot{x}_m = A_m x_m + br$
Tracking error:	$e = x_p - x_m$
Tracking error dynamics:	$\dot{e} = A_m e + \tilde{\theta}^\top x_p$
Parameter error:	$\theta = \tilde{\theta} + \theta^*$
	$k = \tilde{k} + k^*$
Matching condition:	$a + b\theta^* = a_m$
	$bk^* = b_m$
Update laws:	$\dot{\tilde{\theta}} = -(e^\top P b) x_p$
	$\dot{\tilde{k}} = -\mathrm{sgn}(b) e r$

</div>

$Q > 0 \; A_m^\top P + P A_m = -Q$

x_p PE $<=> \tilde{\theta}(t) \to 0$ as $t \to \infty$ $x_p = x_m + e$ $x_m \in \Omega_n \Rightarrow \tilde{\theta}(t) \to 0$ as $t \to \infty$ r must have $\frac{n}{2}$ frequencies

34.3 Parameter Convergence in Adaptive Control

goal: want y to track y_m and choose y_m something that we can indeed track. Pick reference model that has enough degree of smoothness. y can at best follow a signal which has derivatives well defined up to the relative degree of plant. Relative degree of reference model must be at least that of plant or greater.

Goal: Choose u so e goes to zero.

First assume $n^* = 1$ and also assume $W_m(s)$ is SPR. Also assume k_m and k_p are known, with $k_m = k_p = 1$ for simplicity (for now).

Pick filters like in representation 2, except without dealing with the constant (for now) since $k_m = k_p = 1$.

$$v_1(s) = \frac{c(s)}{f(s)} u$$

where $f(s)$ Hurwitz, monic, $n - 1$th degree.

$C(s)$ is not monic, and $n - 2$th degree

$$v_2(s) = \frac{d(s)}{f(s)} u$$

$d(s)$ is not monic, and $n - 1$th degree

Transfer function from r to y

zeros can only be canceled out, and poles shifted.

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θ_1 should be $\theta_1 = f(s) - c(s) = Z_p'(s)$ so degrees of freedom are available to solve for $c(s)$ leaving us with polynomial $n - 2$th degree, which need not be monic. So $c^*(s)$ exists, and θ_1^* is the parameter that leads to $c^*(s)$. This allows the forward loop to be simplified as

This was all pretty much review of output feedback control for the general $n^* = 1$ case, with only the simplification that $k_p = k_m = 1$.

New part is if ω PE $\Rightarrow \tilde{\theta} \to 0$.

x_{mn} is non minimal state.

$$x_{mn} = \begin{bmatrix} x_m \\ \omega_1 \\ \omega_2 \end{bmatrix}_{3n-2}$$

$$\omega_{2n-1}^* = C x_{mn}$$

$$\dot{x}_{mn} = A_{mn} x_{mn} + b_{mn} r$$

Properties of PS

if (A, b) controllable and A is Hurwitz, then u being a sum of $n/2$ sinusoids, then x is persistently exciting in $\mathbb{R}^n$.

If (A, b) is not controllable, x will be persistently exciting in a lower dimension.

Say $x(t)$ PE in $\mathbb{R}^n$ denote that as $x \in \Omega_n$

If we have rectangular matrix $P_{r \times n}$ then $z = Px$, $z(t) \in \mathbb{R}^r$. Then if z is PE in $\mathbb{R}^r$, then $x \in \Omega_n^r$

Example 62

$$x = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

then $x \in \Omega_2$. If

$$x = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

then $x \in \Omega_2^1$. If

$$x = \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix}$$

and so x not PE. If

$$x = \begin{bmatrix} \sin(t) \\ \cos(t) \\ \sin(t + \phi) \end{bmatrix}$$

then $x \in \Omega_3^2$.

Nonminimal representation of the plant

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$$\dot{x}_{pn} = A_{mn}x_{pn} + b_{mn}(\tilde{\theta}^{\top}\omega + r)$$
$$\dot{\omega}_1 = F_{n-1}\omega_1 + g_{n-1}ue$$
$$\dot{\omega}_2 = F_{n-1}\omega_2 + g_{n-1}y$$

$$\omega = \omega^* + Ce$$

34.4 Summary

we introduced recursive schemes — adaptive laws for adjusting parameter. We introduced two error models. Error models relate something we can actually measure that tells how system behaves and responds to uncertainty e called tracking error and parameter error $\tilde{\theta}$. The analogy was turning a single knob without any markings on it. Hope is that $e \to 0$ means that $\tilde{\theta} \to 0$

Error model 1: was when the relationship between parameter and tracking error was algebraic. $\dot{\tilde{\theta}} = -eu$. One example of this is $y = \theta^{\top}u$ where θ unknown, and $\hat{\theta}$ is estimate. Error model 3 was when the relationship between parameter and tracking error has some dynamics in between it. For example a first order system. An example of this is is the motor from last lecture, which can be cast in terms of error model 1.

In both cases assume ω is smooth. That is $|\dot{\omega}| \leq M$ is finite.

Persistent excitation of $\omega \in \mathbb{R}^n$

$$\frac{1}{T_0} \int_t^{t+T_0} |\omega^{\top}(\tau)w| d\tau \geq \varepsilon_0$$

$\forall$ unit vectors $w \in \mathbb{R}^n$ $\forall t \geq t_0$

$$\int_t^{t+T_0} \omega(\tau)\omega^{\top}(\tau)d\tau \geq \alpha I > 0$$

$\forall t \geq t_0$
    ω PE $<=> \tilde{\theta}(t) \to 0$ as $t \to \infty$

Chapter 35

Adaptive Observers

35.1 Introduction

Last Class: Persistent Excitation
Today: Adaptive Observers and Parameter Convergence
Observers (Ch 4)
non-minimal representation characterized by $2n$ parameters.

$$
\dot{x}_1 = -\lambda x_1 + u
$$
$$
\dot{\omega}_1 = \Lambda \omega_1 + \ell u
$$
$$
\dot{\omega}_2 = \Lambda \omega_2 + \ell y_p
$$
$$
u = \theta_1^\top \omega_1 + \theta_0 y_p + \theta_2^\top \omega_2
$$

error models (1) and (2)

35.1.1 Nonadaptive Observers

State not available for measurement. Must estimate state as $\hat{x}$.
Before adaptive observer, first look at "regular" observer.
Plant where x is not measurable but y is

$$
\dot{x} = Ax + bu
$$
$$
y = h^\top x
$$

Standard solution is Luenberger observer

$$
\dot{\hat{x}} = A\hat{x} + bu + \ell(y - \hat{y})
$$
$$
\hat{y} = h^\top \hat{x}
$$

Proof is fairly simple. $e = \hat{x} - x$.

$$
\dot{e} = Ae + \ell h^\top(x - \hat{x})
$$
$$
= (A - \ell h^\top)e
$$

And if (A, h) is observable then $(A - \ell h^\top)$ is Hurwitz. So $e(t) \to 0$ as $t \to \infty$.

Parameterization of A, b, h.

$$A = \begin{bmatrix} a & I \\ a \dots & 0 \end{bmatrix}$$

$$h = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

35.1.2 Adaptive Observers

only n unknown values to estimate in A

$$\hat{A} = \begin{bmatrix} \hat{a} & I \\ \hat{a} \dots & 0 \end{bmatrix}$$

And same with b, where $\hat{b}$ is the estimate of b.

Also don't know what to pick as ℓ since A is unknown, so can't guarantee to pick the right ℓ so $A - \ell h^\top$ is Hurwitz.

$$A_m = \begin{bmatrix} a_m & I \\ a_m \dots & 0 \end{bmatrix}$$

$$Ax = A_m x + (a - a_m)y$$

Adaptive Observer (assume for now that $\ell = 0$)

$$\dot{\hat{x}} = A_m \hat{x} + (\hat{a} - a_m)y + \hat{b}u$$

rewrite plant

$$\dot{x} = A_m x + (a - a_m)y + bu$$

Error

$$\dot{e} = A_m e + (\hat{a} - a_m)y + (\hat{b} - b)u$$
$$\dot{e} = A_m e + \tilde{a}y + \tilde{b}u$$

Pick candidate Lyapunov function

$$V = e^\top P e + \tilde{a}^\top \tilde{a} + \tilde{b}^\top \tilde{b}$$

$$\dot{V} = -e^\top Q e + 2(e^\top P \tilde{a})y + 2(e^\top P \tilde{b})u + 2\dot{\tilde{a}}^\top \tilde{a} + 2\dot{\tilde{b}}^\top \tilde{b}$$
$$= -e^\top Q e + 2(ye^\top P + \dot{\tilde{a}}^\top)\tilde{a} + 2(ue^\top P + \dot{\tilde{b}}^\top)\tilde{b}$$

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The problem with using the following update laws (which would otherwise give $\dot{V} \leq 0$)

$$\dot{\hat{a}} = -Pey \qquad\qquad \dot{\hat{b}} = -Peu$$

is that we do not have the signal e.

Go back through and change A_m to $\bar{A}_m$ where

$$\bar{A}_m = A_m - \ell h^\top$$

$$\dot{e} = A_m e + (\hat{a} - a_m)y + (\hat{b} - b)u - \ell h^\top e$$
$$\dot{e} = \bar{A}_m e + \tilde{a}y + \tilde{b}u$$

What was the $\bar{A}_m$ stuff about???????

35.2 Minimal Representations

Minimal representations have more restrictive applications, harder to implement (synthesizing ν signal), are same order as plant, easier to look at (estimated states are states).

35.3 Nonminimal Representations

Nonminimal representations are easier to implement.

35.3.1 Nonminimal Representation 2

The problem is to design an adaptive observer for the following plant for $W_p(s)$ below, where $R_p(s)$ is monic and degree n, and $Z_p(s)$ is degree m, (need not be monic), where $m \leq n - 1$.

$$W_p(s) = \frac{Z_p(s)}{R_p(s)}$$

An example might help to show what is going on, so consider the given $W_p(s)$ below, where the goal is to determine a_0 in this transfer function.

$$W_p(s) = \frac{1}{s + a_0}$$

Given an $n-$th order plant, a minimal observer is one that is of order n; there are as many integrators in the observer as in the plant. The non-minimal observers have more integrations than the plant. Given the plant W_p above, we will try to see and understand representation 2, where we will use twice as many integrations in the observer as in the plant. Consider the following block diagram.
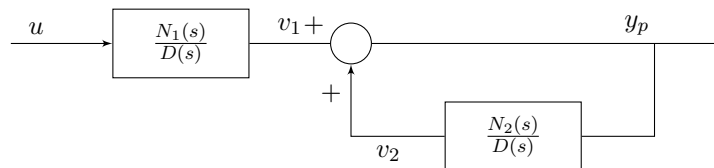
Look at the the transfer function from u to y_p and we get

$$W_p(s) = \frac{N_1(s)}{D(s) - N_2(s)}$$

Now, we can can pick $D(s)$ to be any stable monic polynomial which is the same order as the plant denominator polynomial, pick $N_1(s)$ to be the zero polynomial of $Wp(s)$, and then pick $N_2(s)$ so that $D(s) - N_2(s) = R_p(s)$. If we do that for our simple example above, we get

$$N_1(s) = 1 \qquad D(s) = s + \beta$$

where $\beta > 0$. Now evaluate $D(s) - N_2(s) = s + \beta - N_2(s) = s + a_0$ which gives $N_2(s) = \beta - a_0$. Putting these values into the block diagram
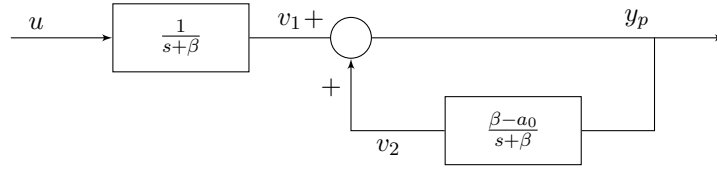
which is equivalent to the given $W_p(s)$. In this example $W_p(s)$ was first order, but we could have done exactly the same process for an n-th order system. Essentially, we can pick $D(s)$ to be any stable monic polynomial that we want, and then use the degrees of freedom in the two zero polynomials to place the poles and zeros of $W_p(s)$.

Let's now look at how this structure can be used to build an observer. To do this, we consider the transfer function blocks above represented in state space as shown in the block diagram below.
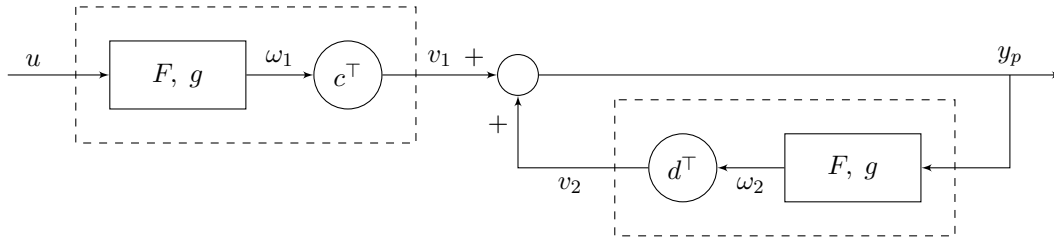
We have two filters shown using the dotted lines. The filters are defined in state-space form by the matrices $(F, g, c^\top)$ and $(F, g, d^\top)$. We showed above in our example how to pick these filters as transfer functions, but we need to see how to do this using the state-space representation. The representation of a transfer function in state-space is not unique. Control canonical form is a good choice.

$$\dot{x} = -\beta x + u$$
$$v_1 = 1$$

and

$$\dot{x} = -\beta x + y_p$$
$$v_2 = \beta - a_0$$

So we have $F = \beta$, $g = 1$, $c^\top = 1$ and $d^\top = \beta - a_0$. For the scalar plant examples, these parameters are also scalars, but they will be vectors when designing an observer for a higher order plant. Now we change the block diagram so we can try to use this idea to make our adaptive observer. Consider the equivalent block diagram

$$y_p = c^\top \omega_1 + d^\top \omega_2$$
$$= \theta^\top \omega$$

where

$$\theta = \begin{bmatrix} c^\top & d^\top \end{bmatrix}^\top \qquad \text{and} \qquad \omega = \begin{bmatrix} \omega_1^\top & \omega_2^\top \end{bmatrix}^\top$$
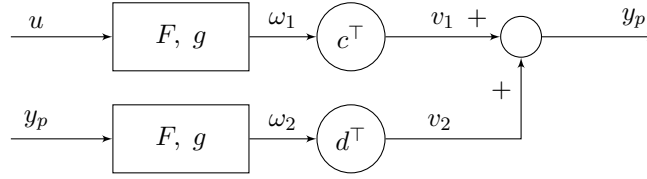
So our parameters are all contained in θ, and ω is a known signal. Now replace the parameters c and d with estimates, and note also that ω_1 and ω_2 need to be replaced with estimates as well. The ω's are estimated values and not the true values only because of errors in initial conditions. These errors will decay with a speed based on how F is picked. In block diagram this becomes
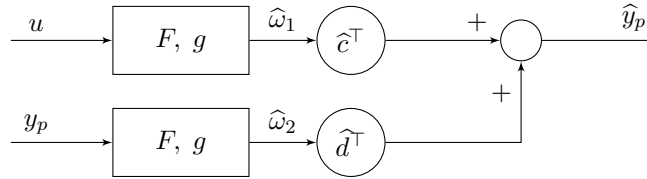
The output can be expressed

$$\widehat{y}_p = c^\top \widehat{\omega}_1 + d^\top \widehat{\omega}_2$$
$$= \widehat{\theta}^\top \widehat{\omega}$$

where

$$\widehat{\theta} = \begin{bmatrix} \widehat{c}^\top & \widehat{d}^\top \end{bmatrix}^\top \qquad \text{and} \qquad \widehat{\omega} = \begin{bmatrix} \widehat{\omega}_1^\top & \widehat{\omega}_2^\top \end{bmatrix}^\top$$

Now look at the error

$$e_1 = \widehat{y}_p - y_p$$
$$= \widehat{\theta}^\top - \theta^\top \omega$$

Define errors

$$\widetilde{\theta} = \widehat{\theta} - \theta \qquad \text{and} \qquad \widetilde{\omega} = \widehat{\omega} - \omega$$

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So $\theta = \widetilde{\theta} + \widehat{\theta}$ then we can rewrite error as

$$e_1 = \widehat{\theta}^\top \widehat{\omega} - \theta^\top (\widehat{\omega} - \widetilde{\omega})$$
$$= \widehat{\theta}^\top \widehat{\omega} - \theta^\top \widehat{\omega} + \theta^\top \widetilde{\omega}$$
$$= (\widehat{\theta}^\top - \theta^\top)\widehat{\omega} + \theta^\top \widetilde{\omega}$$
$$= \widetilde{\theta}^\top \widehat{\omega} + \theta^\top \widetilde{\omega}$$

And the term $\varepsilon(t) = \theta^\top \widetilde{\omega}$ tends to zero as $t \to \infty$. So we are essentially left with error model 1.

$$e_1 = \widetilde{\theta}^\top \widehat{\omega}$$

Start by representing the plant this way

$$W_p(s) = \left[\frac{Z_p(s)}{R_m(s)}\right]\left[\frac{R_m(s)}{R_p(s)}\right]$$

$$W_p(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}$$

Start by representing the plant this way

$$W_p(s) = \left[\frac{Z_p(s)}{R_m(s)}\right]\left[\frac{R_m(s)}{R_p(s)}\right]$$

Time Domain
Frequency Domain
$R_p(s)$ is monic, degree n $Z_p(s)$ is degree m, and need not be monic, where $m \le n - 1$

$$W_p(s) = \frac{\theta_{11} s^m + \theta_{12} s^{m-1} + \cdots + \theta_{1(m+1)}}{s^n + \theta_{21} s^{n-1} + \cdots + \theta_{2n}}$$

$$W_p(s) = \left[\frac{Z_p(s)}{R_m(s)}\right]\left[\frac{R_m(s)}{R_p(s)}\right]$$

So write each of the terms with bunch of unknown parameters. Pick (F_n, g_n) where $F_n : n \times n$ and $g_n : n \times 1$. Control canonical form

$$\frac{Z_p(s)}{R_m(s)}$$

$$\det(sI - F_n) = R_m(s)$$

$$\dot{\omega}_1 = F_n \omega_1 + g_n u$$
$$v_1 = \theta_1^\top \omega_1$$

Now second term

$$\frac{R_m(s)}{R_p(s)}$$

block diagram here

$$R'_p(s) = R_m(s) - R_p(s)$$

$n - 1$th degree polynomial, non-monic

$$\dot{\omega}_2 = F_n \omega_2 + g_n y$$
$$v_2 = \theta_2'^{\top} \omega_2$$

$$y = \theta_1^{\top} \omega_1 + \theta_2'^{\top} \omega_2$$

Adaptive Observer (Non-Minimal)

$$\dot{\hat{\omega}}_1 = F_n \hat{\omega}_1 + g_n u$$
$$\dot{\hat{\omega}}_2 = F_n \hat{\omega}_2 + g_n y$$
$$\hat{y} = \hat{\theta}_1^{\top} \hat{\omega}_1 + \hat{\theta}_2^{\top} \hat{\omega}_2$$

For parameter convergence: If ω is PE then $\tilde{\theta}(t) \to 0$ as $t \to \infty$.
note $\hat{\omega}$ is not the same as ω!

$$\hat{y} = \hat{\theta}_1^{\top} \hat{\omega}_1 + \hat{\theta}_2^{\top} \hat{\omega}_2$$
$$y = \theta_1^{\top} \omega_1 + \theta_2'^{\top} \omega_2$$
$$= \theta_1^{\top} \hat{\omega}_1 + \theta_2^{\top} \hat{\omega}_2 + \theta_1^{\top} (\omega_1 - \hat{\omega}_1) + \theta_2^{\top} (\omega_2 - \hat{\omega}_2)$$
$$= \theta^{\top} \hat{\omega} + \theta^{\top} \tilde{\omega}$$

$$y = \theta^{\top} \hat{\omega} + \varepsilon(t)$$

$$\dot{\omega}_1 =$$
$$\dot{\hat{\omega}}_1 =$$
$$\dot{\tilde{\omega}}_1 =$$
$$\dot{\tilde{\omega}}_2 =$$

$$\varepsilon(t) = \theta^{\top} \tilde{\omega}$$

$\to 0$ as $t \to \infty$
   If $\hat{\omega}$ is PE then $\tilde{\theta}(t) \to 0$ as $t \to \infty$

35.3.2 Nonminimal Representation 1

Here is another method of representing the plant for which an adaptive observer is designed (method 1) uses one less integrator $(2n - 1)$. The method above was method 2. (pretty sure, check book, might be other way around)

More Stuff on Adaptive Observers

started with luenberger and showed that the error model 2 type structure required $e^\top P$ and e is not available for measurement. Only a subset of e is available which is not enough to construct $e^\top P$.

Went to non minimal representation.

Consider only SISO system.

Representation (1)

P is nth order plant, observable and controllable. Thats what transfer function is. Do all pole zero cancellations and result is controllable observable representation of dynamic system. Since numerator and denominator polynomial do not share any factors, they are called co-prime.

Add extra poles and zeros to get a non minimal representation of plant.

The non-minimal state is

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

Name of game: build a way of getting state estimate and plant estimate.

Only reason ω and $\hat{\omega}$ are different is due to initial conditions.

$$\hat{y} = \hat{\theta}^\top \hat{\omega}$$
$$e_y = \hat{\theta}^\top \hat{\omega} - \theta^\top \omega + (\theta - \theta)^\top \hat{\omega}$$
$$= \tilde{\theta}^\top \hat{\omega} + \theta^\top (\hat{\omega} - \omega)$$
$$= \tilde{\theta}^\top \hat{\omega} \quad \text{after finite time}$$

$$\dot{\tilde{\theta}} = -e_y \hat{\omega}$$

$\tilde{\theta}$ is bounded, and $\tilde{\theta} \to 0$ iff $\hat{\omega}$ PE. And ω PE $\Rightarrow \hat{\omega}$ PE

Structure of ω

Assume (F_n, g_n) in control canonical form. $\det(sI - F_n) = f(s)$ and $f(s)$ is nth degree, Hurwitz, and monic.

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{f(s)} \\ \frac{s}{f(s)} \\ \vdots \\ \frac{s^{n-1}}{f(s)} \\ \frac{Z_p(s)}{R_p(s)f(s)} \\ \frac{sZ_p(s)}{R_p(s)f(s)} \\ \vdots \\ \frac{s^{n-1}Z_p(s)}{R_p(s)f(s)} \end{bmatrix} u = \begin{bmatrix} R_p(s) \\ sR_p(s) \\ \vdots \\ s^{n-1}R_p(s) \\ Z_p(s) \\ sZ_p(s) \\ \vdots \\ s^{n-1}Z_p(s) \end{bmatrix} \frac{1}{f(s)R_p(s)} u$$

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And we have in the denominator a $2n$th degree polynomial, Hurwitz, monic.

$$\omega = T \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{2n-1} \end{bmatrix} \frac{u}{f(s)R_p(s)}$$

ω is nonsingular transformation of this vector which is persistently exciting in dimension $2n - 1$.
T is nonsingular because Rp Zp coprime so each time you take a derivative ... ?

$$u' = \frac{1}{f(s)R_p(s)} u$$

$$U = \begin{bmatrix} u' \\ \dot{u}' \\ \vdots \\ u'^{(2n-1)} \end{bmatrix}$$

$$u = \sum_{i=1}^{n} a_i \sin \Omega_i t$$

Chapter 36

Closed-Loop Reference Model Based Output Feedback Adaptive Control

36.1 Introduction

nominal system use $T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$ to show separation principle.

36.1.1 CRM States Accessible

CRM: Case (b) A_p unknown $B_p = B\Lambda$ where B is known   Plant and reference model

$$\dot{x}_p = A_p x_p + B_p u$$
$$\dot{x}_m = A_m x_m + B_m r - L(x - x_m)$$

Control law

$$u = \theta(t)x_p + k(t)r$$

the plant equation becomes
    error dynamics

$$\dot{e} = (A_m + L)e + B\Lambda\tilde{\theta}x_p + B\Lambda\tilde{k}r$$

update laws

$$\dot{\theta} = -\Gamma_\theta B^\top P e x_p^\top$$
$$\dot{k} = -\Gamma_k B^\top P e r^\top$$

Lyapunov

$$(A_m + L)^\top P + P(A_m + L) = -Q$$
$$G = A_m + L$$
$$G^\top P + PG = -Q$$

Lyapunov

$$V = e^\top P e + \mathrm{tr}\left(\tilde{\theta}^\top \Gamma_\theta^{-1} \Lambda \tilde{\theta}\right) + \mathrm{tr}\left(\tilde{k}^\top \Gamma_k^{-1} \Lambda \tilde{k}\right)$$

Pick L such that $G = gI_{n \times n}$ where $g < 0$.

$$Q = |g| I_{n \times n}$$

$$P = \frac{1}{2} |g| I_{n \times n}$$

Tuning laws

$$\dot{\theta} = -\frac{1}{2} \Gamma_\theta B^\top e x_p^\top$$

$$\dot{k} = -\frac{1}{2} \Gamma_k B^\top e r^\top$$

Lyapunov

$$V = \frac{1}{2} e^\top e + \text{tr}\left(\tilde{\theta}^\top \Gamma_\theta^{-1} \Lambda \tilde{\theta}\right) + \text{tr}\left(\tilde{k}^\top \Gamma_k^{-1} \Lambda \tilde{k}\right)$$

$$\Rightarrow \int \|e\|^2 \leq \frac{V(0)}{|g|}$$

ORM: Case (c) A_p and B_p unknown   Only going to be local stability. Matching conditions will change because structure is different.

$$\dot{x}_p = A_p x_p + B_p u$$
$$\dot{x}_m = A_m x_m + B_m r$$

Matching condition has B_m inside instead of B_p

$$A_p + B_m \theta^* = A_m$$

$$B_p K^* = K_m$$

input

$$u = K\theta x_p + Kr$$

substitute control law into plant equation

$$
\begin{aligned}
\dot{x}_p &= A_p x_p + B_p(K\theta x_p + Kr) \\
&= A_p x_p + B_p K\theta x_p + B_p K r \\
&= A_p x_p + B_p K^* \theta x_p + B_p \tilde{K}\theta x_p + B_p K r \\
&= (A_p + B_p K^* \theta) x_p + B_p \tilde{K}\theta x_p + B_p K r \\
&= A_m x_p + B_p \tilde{K}\theta x_p + B_p K r \\
&= A_m x_p + B_p \tilde{K}\theta x_p + B_p K r + A_m x_p - A_m x_p \\
&= A_m x_p + A_p x_p + B_p(K^* + K - K^*)\theta x_p - A_m x_p + B_p K r \\
&= A_m x_p + \left(A_p + B_p(K^* + K - K^*)\theta - A_m\right) x_p + B_p K r
\end{aligned}
$$

Evaluating the error dynamics

$$
\begin{aligned}
\dot{e} &= A_m e + (A_p + B_p(K^* + K - K^*)\theta - A_m)\, x_p + B_p(K - K^*)r \\
&= A_m e + B_m(\theta - \theta^*)x_p + B_m(K^{*-1}K - I)\theta x_p + B_m(K^{*-1}K - I)r \\
&= A_m e + B_m \tilde{\theta} x_p + B_m(K^{*-1} - K^{-1})K\theta x_p + B_m(K^{*-1} - K^{-1})Kr \\
&= A_m e + B_m \tilde{\theta} x_p + B_m(K^{*-1} - K^{-1})K(\theta x_p + r) \\
&= A_m e + B_m \tilde{\theta} x_p + B_m(K^{*-1} - K^{-1})u \\
&= A_m e + B_m \tilde{\theta} x_p + B_m \tilde{\psi} u
\end{aligned}
$$

where

$$
\tilde{\theta} = \theta - \theta^*
$$
$$
\tilde{\psi} = K^{*-1} - K^{-1}
$$

Lyapunov

$$
V = e^\top P e + \operatorname{tr}\left(\tilde{\theta}^\top \tilde{\theta}\right) + \operatorname{tr}\left(\tilde{\psi}^\top \tilde{\psi}\right)
$$

update

$$
\frac{d}{dt}(K^{*-1} - K^{-1}) = K^{-1}\dot{K}K^{-1} = -B_m P e u^\top
$$

$$
\dot{K} = -K B_m P e u^\top K
$$

example 3.4. V not radially unbounded.

$$
V = e^2 + \tilde{\psi}^2
$$

$$
\tilde{\psi} = \frac{1}{k} - \frac{1}{k^*}
$$

$$
\tilde{k} = k - k^*
$$

$$
V = e^2 + \frac{\tilde{k}^2}{(\tilde{k} + k^*)^2 k^{*2}}
$$

36.2 Squaring up Output Feedback

Showing that if the squared up system satisfies $PB_{\text{aug}} = (SC)^\top$ that the non-augmented system satisfies an equivalent relationship: $PB = (S_1 C)^\top$.

$$
(A - LC)^\top P + P(A - LC) < 0
$$
$$
P[\, B \ B_1\,] = (SC)^\top
$$

$$
P[\, B \ B_1\,] = C^\top S^\top
$$

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

$$P[\, B \ B_1 \,] = C^\top [\, S_1^\top \ S_2^\top \,]$$

$$PB = C^\top S_1^\top$$

$$PB = (S_1 C)^\top$$

36.3 Eugene Visit Notes

Given

$$\dot{x}_p = A_p x_p + B_p \Lambda(u + \Theta^\top \Phi(x))$$
$$y_p = C_p x_p$$
$$z = C_{pz} x_p + D_{pz} u$$

where (A_p, B_p) is controllable, $\det \Lambda \neq 0$, (A_p, C_p) is observable.

$$\dot{e}_I = y_{\text{ref}} - y_{\text{cmd}}$$

$$y_{\text{reg}} = C_{p,\text{reg}} x_p + D_{p,\text{reg}} \Lambda(u + \Theta^\top \Phi(x))$$

Goal: given $e_{\max}$, T_0, find u such that $\|y_{\text{reg}} - y_{\text{cmd}}\| < e_{\max}$ for all $t > T_0 < \infty$. This is known as UUB, see Khalil. Define the following extended open-loop dynamics.

$$\dot{x} = Ax + B\Lambda(u + \Theta^\top \Phi(x)) + B_{\text{cmd}} y_{\text{cmd}}$$

$$\dot{x} = Ax + B\Lambda(u + \Theta^\top \Phi(x))$$

Khalil feedback linearization. Isidori (Alberto) Burns. Feedback linearizable systems can handle unmatched uncertainty reference model — command prefilted goal find u to asymptotically track x_m reference model x_m follows r with some error add and subtract trick Bu_{bl}

$$\dot{x} = Ax + B\Lambda(u + \Theta^\top \Phi(x)) + Bu_{\text{bl}} - Bu_{\text{bl}}$$
$$= Ax + Bu_{\text{bl}} + B\Lambda(u - \Lambda^{-1} u_{\text{bl}} + \Theta^\top \Phi(x))$$

$$u = u_{\text{bl}} + u_{\text{ad}}$$

$$\dot{x} = Ax + Bu_{\text{bl}} + B\Lambda(u_{\text{bl}} + u_{\text{ad}} - \Lambda^{-1} u_{\text{bl}} + \Theta^\top \Phi(x))$$
$$= Ax + Bu_{\text{bl}} + B\Lambda(u_{\text{ad}} + (I - \Lambda^{-1}) u_{\text{bl}} + \Theta^\top \Phi(x))$$
$$= Ax + Bu_{\text{bl}} + B\Lambda(u_{\text{ad}} + \bar{\Theta}^\top \bar{\Phi}(u_{\text{bl}}, x))$$

where

$$\bar{\Theta} = \begin{bmatrix} I - \Lambda^{-1} & \Theta^\top \end{bmatrix}^\top$$

$$\bar{\Phi}(u_{\text{bl}}, x) = \begin{bmatrix} u_{\text{bl}} \\ \Phi(x) \end{bmatrix}$$

so now we pick u_{ad} as

$$u_{\text{ad}} = -\hat{\bar{\Theta}}^\top \bar{\Phi}$$

The baseline part provides nominal system with desired performance and the adaptive part fights uncertainty.

$$\dot{\hat{x}} = A\hat{x} + Bu_{\text{bl}} + B\Lambda(u_{\text{ad}} + \hat{\bar{\Theta}}^\top \bar{\Phi}) + L(y - \hat{y})$$
$$= A\hat{x} + Bu_{\text{bl}} + L(y - \hat{y})$$

The output and state are

$$u_{\text{bl}} = -K_x \hat{x} + K_r r$$
$$\dot{\hat{x}} = (A - BK - LC)\hat{x} + BK_r r + Ly$$

these are stable:

$$A_{\text{ref}} = A - BK_x \qquad A - LC$$

But the reference model must be stable as well! *Challenge: Find conditions such that $A - BK - LC$ is Hurwitz.* This is for practical purposes. The idea being if an airplane is sitting on the ground and is powered on the estimator state should be zero, measurements zero, and reference zero. If we were to hit the plane we would be essentially resetting the initial conditions of the estimator through its input y. If it were not stable, the estimator state would blow up causing actuators to go crazy and we would not be a happy goose.

Rosenbrock matrix $y = 0$ no matter what u is

$$\dot{x} = Ax + Bu$$
$$0 = Cx + Du$$

Laplace transform (no IC)

$$sX = AX + BU$$
$$0 = CX + DU$$

$$(A - sI)X + BU = 0$$
$$CX + DU = 0$$

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When this matrix loses rank: transmission zeros

Example: Consider the system

$$\dot{\phi} = p$$
$$\dot{p} = u$$

So we have

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Then the Rosenbrock matrix is

$$\begin{bmatrix} -s & 1 & 0 \\ 0 & -s & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Eat a banana and sit in Hawaii.

Back to the tracking problem.

Note that our input choice $u = u_{\text{bl}} + u_{\text{ad}}$ is not unique.

Bellman — Dynamic programming

36.4 LMI Based Approach to Adaptive Output Feedback

36.5 Some Stuff

Corollary to KYP Lemma says that a $P = P^\top > 0$ satisfies $PB = C^\top$ if and only if $CB = (CB)^\top > 0$. So CB must be full rank. Proof?

Left multiply both sides of $PB = C^\top$ by $B^\top$

$$B^\top P B = (CB)^\top$$

36.6 Adaptive Control of a Pendulum

This document will explain the process of linearizing a simple pendulum, and using this linearized system to design a full state feedback LQR-PI controller. With the controller gain in the feedback loop to control the nonlinear system, the required reference torque input τ_r which gives a desired pendulum deflection angle θ_d can be found. This is known as the feed-forward gain. In this way, a reference torque can be commanded to the system resulting in a quick and accurate rotation of the pendulum to the desired angle, within some range around which the linearization is valid. Integrating the pendulum error removes the need to calculate necessary reference input and apply a feed-forward gain. Instead, a reference command can be given to the system, and the controller will apply control effort until the system reaches this reference value. At this point, the error will be zero.

Model reference adaptive control, or MRAC, is also explained and applied to the pendulum model. MRAC requires a reference model to be specified, against which the actual system will be compared. This is used if the actual plant had uncertain parameters. This controller would simply compare the response of the actual plant against the reference model, and adjust the gain to make the system behave as desired. With the adaptive controller complete, it can be added around the existing LQR-PI controller.

36.6.1   Pendulum Equations of Motion

The governing equations for a pendulum with a viscous damper and torsional restoring spring are shown below, where $J = ml^2$ is the moment of inertia of the pendulum, m is the mass of the pendulum, g is gravitational acceleration, l is the length of the pendulum, k_θ is a torsional spring constant, $k_{\dot\theta}$ is viscous damping coefficient, and τ is an external torque input provided by a motor or similar actuator. $\theta = 0$ when the pendulum is hanging vertically downward.
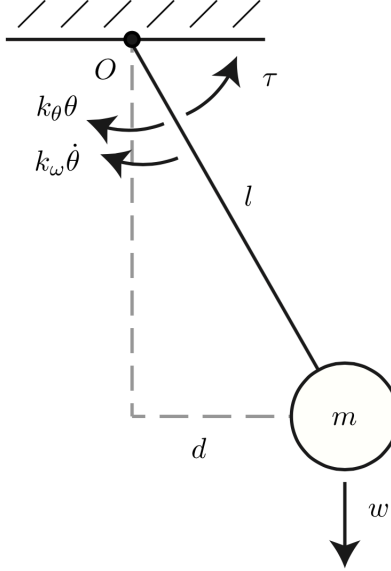
$$\sum M_O = J\ddot{\theta}$$

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) - \frac{k_\theta}{ml^2}\theta - \frac{k_{\dot\theta}}{ml^2}\dot{\theta} + \frac{1}{ml^2}\tau$$

Using the following state variables

$$X_1 = \theta$$
$$X_2 = \omega$$

and input $U = \tau$, the second order governing equation can be written using two first order equations

$$\dot{X}_1 = X_2$$
$$\dot{X}_2 = -\frac{g}{l}\sin(X_1) - \frac{k_\theta}{ml^2}X_1 - \frac{k_{\dot\theta}}{ml^2}X_2 + \frac{1}{ml^2}U$$

The state vector X is then:

$$X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}^{\mathrm{T}}$$

And the state equations take the form of:

$$\dot{X} = f(X, U)$$

36.6.2 Equilibrium Points and Linearization
Equilibrium Points

Equilibrium points for the pendulum must be found.

Equilibrium points are defined to be states where the state derivative is zero:

$$\dot{X}^* = f(X^*, U^*) = 0 \tag{36.1}$$

Linearization

Define perturbations about the equlibrium point:

$$X = X^* + x \tag{36.2}$$

$$U = U^* + u \tag{36.3}$$

Differentiating (36.2):

$$\dot{X} = \dot{X}^* + \dot{x}$$

but, using (36.1):

$$\dot{X}^* = 0$$

giving:

$$\dot{X} = \dot{x}$$

$$\dot{x} = f(X, U) = f(X^* + x, U^* + u)$$

Performing a Taylor series expansion, neglecting second order terms and higher:

$$\dot{x} = f(X^*, U^*) + \left. \frac{\partial f(X, U)}{\partial X} \right|_* x + \left. \frac{\partial f(X, U)}{\partial U} \right|_* u + \epsilon$$

with:

$$f(X^*, U^*) = 0$$

giving:

$$\dot{x} = \left. \frac{\partial f(X, U)}{\partial X} \right|_* x + \left. \frac{\partial f(X, U)}{\partial U} \right|_* u$$

where the subscript $(\cdot)_*$ indicates these matrices be evaluated at the equilibrium point found in the preceding section.

When the equations of motion are linearized this way, the resulting equations can be expressed:

$$\dot{x}_p = A_p x_p + B_p u$$

Linearizing the Pendulum

There are two cases of equilibrium points that will occur for this system: equilibrium points for which the input is zero, and equilibrium points for which there is a non-zero input.

Case I: $U^* \neq 0$

$$0 = \bar{X}_2$$

$$0 = -\frac{g}{l}\sin(\bar{X}_1) - \frac{k_\theta}{ml^2}\bar{X}_1 - \frac{k_{\dot\theta}}{ml^2}\bar{X}_2 + \frac{1}{ml^2}\bar{U}$$

For the above case, there will be infinitely many trim points. These trim points correspond to some non-zero torque input $\bar{U}$ that will maintain the pendulum stationary at some corresponding value ox $\bar{X}_1 = \bar\theta \neq 0$. However, it is usually desired to trim a system around a point which requires zero input, so this case will not be focused on any further. Instead, case II will be examined instead.

Case II: $U^* = 0$ Finding the equilibrium points which require zero input will result in the following equations:

$$0 = \bar{X}_2$$

$$0 = -\frac{g}{l}\sin(\bar{X}_1) - \frac{k_\theta}{ml^2}\bar{X}_1 - \frac{k_{\dot\theta}}{ml^2}\bar{X}_2$$

From these equations, there are again infinitely many trim points, although this time they are periodic, corresponding to the pendulum hanging vertically downward or balancing vertically upward.

$$\bar{X} = \begin{bmatrix} \{2n\pi\} \cup \{(2n+1)\pi\} \\ 0 \end{bmatrix}, n \in \mathbb{Z}$$

where the first set of $\bar{X}_1$ correspond to stable equilibrium points, and the second set of $\bar{X}_1$ corresponds to unstable equilibrium points.

Linearizing Pendulum

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} \end{bmatrix}_* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial U} \\ \frac{\partial f_2}{\partial U} \end{bmatrix}_* u$$

Using the pendulum example provided, and evaluating the partial derivatives results in:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(X_1) - \frac{k_\theta}{ml^2} & -\frac{k_{\dot\theta}}{ml^2} \end{bmatrix}_* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}_* u$$

and substituting the equilibrium values:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} - \frac{k_\theta}{ml^2} & -\frac{k_{\dot\theta}}{ml^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u \tag{36.4}$$

Or, more compactly:

$$\dot{x}_p = A_p x_p + B_p u$$

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36.6.3 LQR-PI Control Desgin

The following sections have described how to apply LQR control to a nonlinear system, and command the system to some desired equilibrium state. However, this required an analytical expression relating the closed loop input torque to deflection angle be found. Using this expression, the torque required to command the pendulum to θ_d had to be found for each desired rotation angle, and implemented using a feed-forward gain N. An LQR-PI controller will not require such a procedure. Instead, a reference will be given to the system, and the error between actual system output and reference input will be measured. This error will be integrated over time, and the controller will seek to reduce this, resulting in zero steady-state error.

The system is the same as above. Define error the difference between a particular single perturbation state variable of interest and the reference value $r = x_{p_i,\text{cmd}}$:

$$\dot{x}_e = r - x_{p_i} \tag{36.5}$$

Using the error description in (36.5), the state vector x_p can be augmented to include x_e, the integration of $\dot{x}_e$, as a state variable:

$$\frac{d}{dt} \begin{bmatrix} x_p \\ x_e \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ H & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_e \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \tag{36.6}$$

Or, more compactly:

$$\dot{x} = Ax + B_1 u + B_2 r \tag{36.7}$$

A and B are the matrices for the augmented linearized system, and, the selection matrix $H \in \mathbb{R}^{m \times n}$ is chosen according to the definition in (36.5). For the pendulum $n = 2$, and $m = 1$, so H is the 1×2 selection matrix below:

$$H = \begin{bmatrix} -1 & 0 \end{bmatrix}$$

Expanding the state vectors, and inserting some numerical values:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_e \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_e \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

Using this augmented state-space representation, the following control law will be used:

$$u = K^{\mathrm{T}} x \tag{36.8}$$

Note here that the control law does not depend on the commanded reference input r. This is because the reference input is coming into the system through the error term, $\dot{x}_e$, thus ensuring the system will go to the commanded value. The gain K will be found using LQR, minimizing the quadratic cost function using A and B_1. B_2 is not used since it is constant for a given reference r. Substituting the control law (36.8) into (36.7):

$$\dot{x} = Ax + B_1 K^{\mathrm{T}} x + B_2 r$$
$$\dot{x} = \left(A + B_1 K^{\mathrm{T}} \right) x + B_2 r$$

The gain K is selected using LQR, and ensures the closed loop matrix $\left(A + B_1 K^{\mathrm{T}} \right)$ is Hurwitz. This new resulting state-space form for the pendulum will have zero error at steady-state.

36.6.4 MRAC Control Design
Motivation for Adaptive Control

The linear full state feedback LQR-PI controller can be used to stabilize the pendulum around a commanded reference angle. The LQR-PI controller guarantees zero steady state error by inclusion of an error term in an augmented state-space representation.

The motivation for the adaptive control design lies behind the presence of uncertainties in system parameters. The linear controller that was designed for the nominal system should work well within some neighborhood of the equilibrium point, and without uncertainties.

In designing a MRAC, a linear reference model is created, which is stable and known. Essentially, its behavior is that which the original pendulum should mimic. The adaptive controller attempts to adjust the gain parameters so the actual system behaves like the reference model.

The Adaptive Controller

When the nominal equation $\dot{X} = f(X, U)$ is linearized about a trim point, it takes the form $\dot{x}_p = A_p x_p + B_p u$ as shown previously. With the presence of input uncertainties, the nonlinear equation can be represented as

$$\dot{X} = f(X, \Lambda U)$$

When this equation is linearized about the same flight condition, it takes the form

$$\dot{x}_p = A_{p\lambda} x_p + B_{p\lambda} u$$

where the uncertainty can be incorporated as $B_{p\lambda} = B_p \Lambda$ giving

$$\dot{x}_p = A_{p\lambda} x_p + B_p \Lambda u$$

When this uncertain linear plant is augmented with an integral error state, it can be represented

$$\frac{d}{dt} \begin{bmatrix} x_p \\ x_e \end{bmatrix} = \begin{bmatrix} A_{p\lambda} & 0 \\ H & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_e \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} \Lambda u + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

Or, more compactly, the overall plant to be controlled is given by:

$$\boxed{\dot{x} = A_\lambda x + B_1 \Lambda u + B_2 r} \tag{36.9}$$

Selection of the Reference Model The reference model is selected by designing a nominal full state feedback controller $u_{\text{nom}} = K^{\mathrm{T}} x$ where the gain was calculated to optimize control of the nominal plant. The reference model is of the form

$$\dot{x}_m = A_m x_m + B_m r \tag{36.10}$$

Using the augmented nominal state and input matrices

$$A = \begin{bmatrix} A_p & 0 \\ H & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} B_p \\ 0 \end{bmatrix} \quad B_m = B_2 \tag{36.11}$$

the reference model is given by

$$\dot{x}_m = (A + B_1 K^{\mathrm{T}}) x_m + B_2 r \tag{36.12}$$

In other words, the reference model is simply the nominal linear plant, closed using a controller which guarantees stability of the system.

Adaptive Law   With the form of the plant known, and a reference system available to compare it with, the adaptive law must be determined to stabilize the uncertain plant. First, the error between actual performance and desired performance must be quantified. Define the adaptive state error

$$e = x - x_m \tag{36.13}$$

to be the difference between actual state and desired state. Propose a control law of the following form, where $\theta(t) \in \mathbb{R}^{n \times m}$ is the adjustable control parameter

$$\boxed{u = (\theta^{\mathrm{T}} + K^{\mathrm{T}})x} \tag{36.14}$$

Substituting the control law (36.14) into (36.9):

$$\dot{x} = A_\lambda x + B_1 \Lambda (\theta^{\mathrm{T}} + K^{\mathrm{T}})x + B_2 r$$

$$\dot{x} = [A_\lambda + B_1 \Lambda (\theta^{\mathrm{T}} + K^{\mathrm{T}})]x + B_2 r$$

$$\dot{x} = [A_\lambda + B_1 \Lambda \theta^{\mathrm{T}} + B_1 \Lambda K^{\mathrm{T}}]x + B_2 r$$

Comparing this expression to the reference model, it is assumed that a constant, ideal feedback gain matrix θ^* exists that results in perfect reference model tracking such that

$$\boxed{A_\lambda + B_1 \Lambda (\theta^{*\mathrm{T}} + K^{\mathrm{T}}) = A_m} \tag{36.15}$$

which is known as the matching condition. Using the definition for A_m this can be written

$$A_\lambda + B_1 \Lambda (\theta^{*\mathrm{T}} + K^{\mathrm{T}}) = A + B_1 K^{\mathrm{T}}$$

Assuming there is no uncertainty in the A matrix, that is $A_\lambda = A$, this expression can be simplified to:

$$B_1 \Lambda (\theta^{*\mathrm{T}} + K^{\mathrm{T}}) = B_1 K^{\mathrm{T}}$$

$$B_1 \Lambda \theta^{*\mathrm{T}} + B_1 \Lambda K^{\mathrm{T}} = B_1 K^{\mathrm{T}}$$

$$B_1 \Lambda \theta^{*\mathrm{T}} = B_1 K^{\mathrm{T}} - B_1 \Lambda K^{\mathrm{T}}$$

$$B_1 \Lambda \theta^{*\mathrm{T}} = B_1 (I - \Lambda) K^{\mathrm{T}}$$

$$\theta^{*\mathrm{T}} = \Lambda^{-1}(I - \Lambda) K^{\mathrm{T}}$$

which is true since everything on the RHS ...

Adaptive control parameter gain error is defined as

$$\tilde{\theta} = \theta - \theta^*$$

Differentiating the error (36.13):

$$\dot{e} = \dot{x} - \dot{x}_m \tag{36.16}$$

Substituting (36.9) and (36.10) into (36.16):

$$\dot{e} = A_\lambda x + B_1 \Lambda u + B_2 r - (A_m x_m + B_m r)$$

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substituting the control law equation (36.14) and since $B_m = B_2$

$$\dot{e} = A_\lambda x + B_1\Lambda(\theta^{\mathrm{T}} + K^{\mathrm{T}})x + B_2 r - A_m x_m - B_2 r$$

$$\dot{e} = A_\lambda x + B_1\Lambda(\theta^{\mathrm{T}} + K^{\mathrm{T}})x - A_m x_m$$

Rearranging the matching condition equation (36.15)

$$A_\lambda = A_m - B_1\Lambda(\theta^{*\mathrm{T}} + K^{\mathrm{T}})$$

and substituting

$$\dot{e} = [A_m - B_1\Lambda(\theta^{*\mathrm{T}} + K^{\mathrm{T}})]x + B_1\Lambda(\theta^{\mathrm{T}} + K^{\mathrm{T}})x - A_m x_m$$

$$\dot{e} = A_m x - B_1\Lambda(\theta^{*\mathrm{T}} + K^{\mathrm{T}})x + B_1\Lambda(\theta^{\mathrm{T}} + K^{\mathrm{T}})x - A_m x_m$$

$$\dot{e} = A_m(x - x_m) - B_1\Lambda(\theta^{*\mathrm{T}} + K^{\mathrm{T}})x + B_1\Lambda(\theta^{\mathrm{T}} + K^{\mathrm{T}})x$$

$$\dot{e} = A_m(x - x_m) - B_1\Lambda\theta^{*\mathrm{T}}x - B_1\Lambda K^{\mathrm{T}}x + B_1\Lambda\theta^{\mathrm{T}}x + B_1\Lambda K^{\mathrm{T}}x$$

$$\dot{e} = A_m(x - x_m) - B_1\Lambda\theta^{*\mathrm{T}}x + B_1\Lambda\theta^{\mathrm{T}}x$$

$$\dot{e} = A_m e + B_1\Lambda(\theta^{\mathrm{T}} - \theta^{*\mathrm{T}})x$$

$$\dot{e} = A_m e + B_1\Lambda\tilde{\theta}^{\mathrm{T}}x \tag{36.17}$$

The goal of the adaptive controller is to drive the error $e(t)$ to zero: $\lim_{t\to\infty} e(t) = 0$. This will be accomplished by adjusting the parameter θ. The following candidate Lyapunov equation is proposed where P is symmetric positive definite, $\Gamma^{-1} \in \mathbb{R}^{n\times n}$ is a symmetric, invertible, positive definite user selected tuning gain matrix, and the operation $|\cdot|$ takes the absolute value of the entries of the argument.

$$\boxed{V = e^{\mathrm{T}}Pe + \mathrm{tr}\left(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}|\Lambda|\right)}$$

Differentiating

$$\dot{V} = \dot{e}^{\mathrm{T}}Pe + e^{\mathrm{T}}P\dot{e} + \mathrm{tr}(\dot{\tilde{\theta}}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}|\Lambda|) + \mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)$$

Substituting equation ((36.17)) and simplifying

$$\dot{V} = (A_m e + B_1\Lambda\tilde{\theta}^{\mathrm{T}}x)^{\mathrm{T}}Pe + e^{\mathrm{T}}P(A_m e + B_1\Lambda\tilde{\theta}^{\mathrm{T}}x) + \mathrm{tr}(\dot{\tilde{\theta}}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}|\Lambda|) + \mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)$$

$$\dot{V} = (A_m e)^{\mathrm{T}}Pe + (B_1\Lambda\tilde{\theta}^{\mathrm{T}}x)^{\mathrm{T}}Pe + e^{\mathrm{T}}PA_m e + e^{\mathrm{T}}PB_1\Lambda\tilde{\theta}^{\mathrm{T}}x + \mathrm{tr}(\dot{\tilde{\theta}}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}|\Lambda|) + \mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)$$

$$\dot{V} = e^{\mathrm{T}}A_m{}^{\mathrm{T}}Pe + x^{\mathrm{T}}\tilde{\theta}\Lambda B_1{}^{\mathrm{T}}Pe + e^{\mathrm{T}}PA_m e + e^{\mathrm{T}}PB_1\Lambda\tilde{\theta}^{\mathrm{T}}x + \mathrm{tr}(\dot{\tilde{\theta}}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}|\Lambda|) + \mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)$$

$$\dot{V} = e^{\mathrm{T}}(A_m{}^{\mathrm{T}}P + PA_m)e + x^{\mathrm{T}}\tilde{\theta}\Lambda B_1{}^{\mathrm{T}}Pe + e^{\mathrm{T}}PB_1\Lambda\tilde{\theta}^{\mathrm{T}}x + \mathrm{tr}(\dot{\tilde{\theta}}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}|\Lambda|) + \mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)$$

Because the quantity $x^{\mathrm{T}}\tilde{\theta}\Lambda B_1{}^{\mathrm{T}}Pe$ is a scalar, $x^{\mathrm{T}}\tilde{\theta}\Lambda B_1{}^{\mathrm{T}}Pe = (x^{\mathrm{T}}\tilde{\theta}\Lambda B_1{}^{\mathrm{T}}Pe)^{\mathrm{T}} = e^{\mathrm{T}}PB_1\Lambda\tilde{\theta}^{\mathrm{T}}x$ the above equation can be further simplified

$$\dot{V} = e^{\mathrm{T}}(A_m{}^{\mathrm{T}}P + PA_1)e + 2x^{\mathrm{T}}\tilde{\theta}\Lambda B_1{}^{\mathrm{T}}Pe + \mathrm{tr}(\dot{\tilde{\theta}}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}|\Lambda|) + \mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)$$

Substituting $-Q = A_m{}^\mathsf{T}P + PA_m$ into the equation above gives

$$\dot{V} = -e^\mathsf{T}Qe + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1{}^\mathsf{T}Pe + \mathrm{tr}(\dot{\tilde{\theta}}{}^\mathsf{T}\Gamma^{-1}\tilde{\theta}|\Lambda|) + \mathrm{tr}(\tilde{\theta}{}^\mathsf{T}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)$$

Proposing the following adaptive control gain update law, noting that $\dot{\tilde{\theta}} = \dot{\theta}$

$$\boxed{\dot{\theta} = -\Gamma x e^\mathsf{T}PB_1\mathrm{sign}(\Lambda)}$$

giving

$$\begin{aligned}\dot{\theta}^\mathsf{T} &= -\mathrm{sign}(\Lambda)(\Gamma x e^\mathsf{T}PB_1)^\mathsf{T} \\ &= -\mathrm{sign}(\Lambda)B_1{}^\mathsf{T}P^\mathsf{T}e x^\mathsf{T}\Gamma^\mathsf{T}\end{aligned}$$

Substituting this update law to continue evaluation of the candidate Lyapunov function's derivative

$$\dot{V} = -e^\mathsf{T}Qe + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1{}^\mathsf{T}Pe - \mathrm{tr}(\mathrm{sign}(\Lambda)B_1{}^\mathsf{T}P^\mathsf{T}e x^\mathsf{T}\Gamma^\mathsf{T}\Gamma^{-1}\tilde{\theta}|\Lambda|) - \mathrm{tr}(\tilde{\theta}{}^\mathsf{T}\Gamma^{-1}\Gamma x_p e^\mathsf{T}PB_1\mathrm{sign}(\Lambda)|\Lambda|)$$

Simplifying, recalling that $\Gamma = \Gamma^\mathsf{T}$ and $P = P^\mathsf{T}$ are symmetric

$$\dot{V} = -e^\mathsf{T}Qe + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1{}^\mathsf{T}Pe - \mathrm{tr}(B_1{}^\mathsf{T}Pe x^\mathsf{T}\tilde{\theta}\mathrm{sign}(\Lambda)|\Lambda|) - \mathrm{tr}(\tilde{\theta}{}^\mathsf{T}x e^\mathsf{T}PB_1\mathrm{sign}(\Lambda)|\Lambda|)$$

Because every entry in Λ has the same sign, $\mathrm{sign}(\Lambda)|\Lambda| = \Lambda$ giving

$$\dot{V} = -e^\mathsf{T}Qe + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1{}^\mathsf{T}Pe - \mathrm{tr}(B_1{}^\mathsf{T}Pe x^\mathsf{T}\tilde{\theta}\Lambda) - \mathrm{tr}(\tilde{\theta}{}^\mathsf{T}x e^\mathsf{T}PB_1\Lambda)$$

using property of trace $\mathrm{tr}(a) = \mathrm{tr}(a^\mathsf{T})$, and also that $\Lambda = \Lambda^\mathsf{T}$ is symmetric

$$\dot{V} = -e^\mathsf{T}Qe + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1{}^\mathsf{T}Pe - \mathrm{tr}(B_1{}^\mathsf{T}Pe x^\mathsf{T}\tilde{\theta}\Lambda) - \mathrm{tr}(\Lambda B_1{}^\mathsf{T}Pe x^\mathsf{T}\tilde{\theta})$$

rearranging using $\mathrm{tr}(ab) = \mathrm{tr}(ba)$

$$\dot{V} = -e^\mathsf{T}Qe + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1{}^\mathsf{T}Pe - \mathrm{tr}(B_1{}^\mathsf{T}Pe x^\mathsf{T}\tilde{\theta}\Lambda) - \mathrm{tr}(B_1{}^\mathsf{T}Pe x^\mathsf{T}\tilde{\theta}\Lambda)$$

combining terms

$$\dot{V} = -e^\mathsf{T}Qe + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1{}^\mathsf{T}Pe - 2\mathrm{tr}(B_1{}^\mathsf{T}Pe x^\mathsf{T}\tilde{\theta}\Lambda)$$

and finally using $\mathrm{tr}(ab^\mathsf{T}) = b^\mathsf{T}a$ with $a = B_1{}^\mathsf{T}Pe$ and $b^\mathsf{T} = x^\mathsf{T}\tilde{\theta}\Lambda$

$$\dot{V} = -e^\mathsf{T}Qe + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1{}^\mathsf{T}Pe - 2x^\mathsf{T}\tilde{\theta}\Lambda B_1{}^\mathsf{T}Pe$$

Finally simplifying to

$$\boxed{\dot{V} = -e^\mathsf{T}Qe}$$

So, if a symmetric, positive definite matrix Q exists, which solves the Lyapunov equation $A_m{}^\mathsf{T}P + PA_m + Q = 0$, the candidate Lyapunov function which was proposed will serve as a valid Lyapunov function for this system.

Implementing the Controller

Selecting the Tuning Gains

36.6.5 Improvements to MRAC Architecture
Modified Reference Model

Modifying the reference model is from before, where *G is a positive definite user selected matrix gain*, gives:

$$\dot{x}_m = A_m x_m + B_m r + G(x - x_m)$$

Matching condition still requires constant adaptive gain $\theta^{*\mathrm{T}}$ such that the following is satisfied:

$$A_\lambda + B_1 \Lambda(\theta^{*\mathrm{T}} + K^\mathrm{T}) = A_m = A + B_1 K^\mathrm{T}$$

Substituting the new reference model into the differentiated error equation $\dot{e} - x - x_m$:

$$\dot{e} = [A_\lambda + B_1 \Lambda \theta^\mathrm{T} + B_1 \Lambda K^\mathrm{T}]x + B_2 r - [A_m x_m + B_2 r + Ge]$$

rearranging and matching condition

$$A_\lambda = A_m - B_1 \Lambda(\theta^{*\mathrm{T}} + K^\mathrm{T})$$

and substituting

$$\dot{e} = [A_\lambda + B_1 \Lambda \theta^\mathrm{T} + B_1 \Lambda K^\mathrm{T}]x - [A_m x_m + Ge]$$

gives

$$\dot{e} = (A_m + G)e + B_1 \tilde{\theta}^\mathrm{T} x$$

Using the same candidate Lyapunov function as before

$$V = e^\mathrm{T} P e + \mathrm{tr}\left(\tilde{\theta}^\mathrm{T} \Gamma^{-1} \tilde{\theta}|\Lambda|\right)$$

and differentiating

$$\dot{V} = \dot{e}^\mathrm{T} P e + e^\mathrm{T} P \dot{e} + \mathrm{tr}(\dot{\tilde{\theta}}^\mathrm{T} \Gamma^{-1} \tilde{\theta}|\Lambda|) + \mathrm{tr}(\tilde{\theta}^\mathrm{T} \Gamma^{-1} \dot{\tilde{\theta}}|\Lambda|)$$

$$\dot{V} = [(A_m + G)e + B_1 \tilde{\theta}^\mathrm{T} x]^\mathrm{T} P e + e^\mathrm{T} P[(A_m + G)e + B_1 \tilde{\theta}^\mathrm{T} x] + \mathrm{tr}(\dot{\tilde{\theta}}^\mathrm{T} \Gamma^{-1} \tilde{\theta}|\Lambda|) + \mathrm{tr}(\tilde{\theta}^\mathrm{T} \Gamma^{-1} \dot{\tilde{\theta}}|\Lambda|)$$

Propose the same following adaptive control gain update law as before, noting that $\dot{\tilde{\theta}} = \dot{\theta}$.

$$\dot{\theta} = -\Gamma x e^\mathrm{T} P B_1 \mathrm{sign}(\Lambda)$$

Substituting the adaptive control gain update law gives

$$\dot{V} = -e^\mathrm{T} Q e - e^\mathrm{T} Q_G e$$

So, if symmetric, positive definite matrices Q and Q_G exist, which solves the Lyapunov equation $A_m^\mathrm{T} P + P A_m + Q = 0$, and $G^\mathrm{T} P + P G + Q_G = 0$ the candidate Lyapunov function which was proposed will serve as a valid Lyapunov function for this system.

Feed-Forward Gain

$$\begin{bmatrix} \dot{x} \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ H & 0 \end{bmatrix} \begin{bmatrix} x \\ x_e \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

at steady state:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ H & 0 \end{bmatrix} \begin{bmatrix} x^* \\ x_e^* \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u^* + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

$$\begin{bmatrix} 0 \\ -r \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ H & 0 \end{bmatrix} \begin{bmatrix} x^* \\ 0 \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u^*$$

$$\begin{bmatrix} 0 \\ -r \end{bmatrix} = \begin{bmatrix} A_p & B_p \\ H & 0 \end{bmatrix} \begin{bmatrix} x \\ u^* \end{bmatrix}$$

$$\begin{bmatrix} x^* \\ u^* \end{bmatrix} = \begin{bmatrix} A_p & B_p \\ H & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -r \end{bmatrix}$$

Chapter 37

Classical MIMO Adaptive Control

37.1   Introduction

This document provides a description of the analysis and synthesis of classical adaptive output feedback control as given in Reference [11], and aims to analyze the class of systems to which this method is applicable, and compare it to closed-loop reference model based output feedback, and the systems for which that method is applicable.

37.2   Preliminaries

$\mathbb{R}[s]$ is the ring of polynomials, $\mathbb{R}(s)$ is the field of rational functions, and $\mathbb{R}_p(s)$ is the ring of proper rational functions of a single variable with coefficients in $\mathbb{R}$, i.e. transfer matrices. We will typically use the letters $G(s)$ or $W_p(s)$ to represent rational (transfer) matrices, and $P(s)$ for polynomial matrices.

Definition 36 High frequency gain for SISO plant   *[11] p.183, p. 405. The high frequency gain of a transfer function $W_p(s)$ is the constant k_p that is pulled out leaving the numerator and denominator of $W_p(s)$ as monic polynomials. That is*

$$\lim_{s \to \infty} s^{n^*} W_p(s) \triangleq k_p$$

Definition 37 High frequency gain for multivariable plant   *[[11], p. 405] This definition depends on the structure of the right Hermite form $H_p(s)$ of $W_p(s)$*

$$\lim_{s \to \infty} H_p^{-1}(s) W_p(s) \triangleq K_p$$

Definition 38 Proper/strictly proper rational transfer matrix   *[14] p. 382. A rational TFM $G(s)$ is proper if $\lim_{s \to \infty} G(s) < \infty$ and strictly proper if $\lim_{s \to \infty} G(s) = 0$.*

Definition 39 Characteristic polynomial   *[15] p. 299. The characteristic polynomial of a transfer matrix $G(s)$ is defined as the least common multiple of the denominators of all minors of $G(s)$. The degree of this polynomial is defined as the McMillan degree of $G(s)$. We will typically denote the characteristic polynomial by $p(s)$.*

Example 63 Characteristic polynomial Given the following transfer function matrix, find the characteristic polynomial $p(s)$

$$G(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{2}{(s-2)^2} \\ \frac{1}{(s-1)^2} & \frac{1}{s+3} \end{bmatrix}$$

The 1×1 minors of $G(s)$ are:

$$\frac{1}{s-1} \qquad \frac{2}{(s-2)^2} \qquad \frac{1}{(s-1)^2} \qquad \frac{1}{s+3}$$

The 2×2 minor of $G(s)$ is:

$$\frac{1}{s-1}\frac{1}{s+3} - \frac{2}{(s-2)^2}\frac{1}{(s-1)^2} = \frac{(s-1)(s-2)^2}{(s-1)^2(s-2)^2}\frac{1}{s+3} - \frac{2}{(s-2)^2}\frac{1(s+3)}{(s-1)^2(s+3)}$$

From this we can see that the least common multiple of the minors of $G(s)$ is

$$p(s) = (s-1)^2(s-2)^2(s+3)$$

Definition 40 Transmission Zero *[16] Ch. 27. A rational matrix $H(s)$ of full column rank has a zero at $s = \zeta_0$ if there is a rational vector $u(s)$ such that $u(\zeta_0)$ is finite and nonzero, and $\lim_{s \to \zeta_0}[H(s)u(s)] = 0$.*

This definition seems hard to apply in practice, i.e. how to find the vector $u(s)$?

Example 64 Transmission zero This example was taken from Reference [16]. Let a transfer matrix $G(s)$ be given as follows

$$G(s) = \begin{bmatrix} 1 & \frac{1}{s-3} \\ 0 & 1 \end{bmatrix}$$

Using Definition 39, we can determine that $G(s)$ has a pole at $s = -3$. To determine the transmission zeros of $G(s)$ we apply Definition 40. We suspect there is a zero at $s = -3$ as well, and use $u(s) = \begin{bmatrix} -1 & s-3 \end{bmatrix}^\top$. Applying the definition

$$\lim_{s \to 3} \begin{bmatrix} 1 & \frac{1}{s-3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ s-3 \end{bmatrix} = \lim_{s \to 3} \begin{bmatrix} \frac{s-3}{s-3} - 1 \\ s-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we see that there is in fact a transmission zero at $s = -3$.

Definition 41 McMillan degree (rational matrix) *The McMillan degree of a system with transfer matrix $G(s)$ is given by $deg[p(s)]$, where $p(s)$ is the characteristic polynomial of $G(s)$. The McMillan degree of a system $G(s)$ gives the system order, the pole locations and their multiplicities.*

Definition 42 Minimum polynomial (rational matrix)   *[15] p.299. The monic least common denominator of all nonzero first-order minors of rational matrix $G(s)$ is called the minimal polynomial of $G(s)$ and is denoted by $d(s)$.*

Example 65 Minimum polynomial Given the following transfer matrix, find the minimum polynomial $d(s)$.

$$G(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-2)^2} \\ \frac{1}{(s-1)^2} & \frac{1}{s+3} \end{bmatrix}$$

Then $d(s)$ is given by

$$d(s) = (s-1)^2(s-2)^2(s+3)$$

Find an example for the minimum polynomial that is not the same as the characteristic polynomial

Definition 43 Unimodular (polynomial matrix)   *[[17], p.25] [[18], p.279] [[14], p.375] [Suter 148]. A unimodular polynomial matrix $Q(s)$ is a square polynomial matrix with inverse which is also a polynomial matrix. A necessary and sufficient condition is $\det Q(s)$ is a real number different from zero (thus not dependent on s).*

Definition 44 Unimodular (rational matrix)

Definition 45 Elementary operations   *[[17], p.25.]*

1. *Interchange of columns*

2. *Multiplications of columns by scalar*

3. *Replacement of column by itself plus a polynomial multiple of another column*

A unimodular matrix can always be obtained by elementary operations on an identity matrix.

Definition 46 Elementary matrices   *[[14], p.374.] The elementary operations can be represented by elementary matrices, postmultiplication by which correspond to elementary column operations, while premultiplication yields elementary row operations.*

Example 66 Elementary matrices For a 2×2 system, these unimodular (elementary) matrices will look like

$$R(s) = \begin{bmatrix} 1 & r(s) \\ 0 & 1 \end{bmatrix}$$

Definition 47 Hermite form (polynomial matrix)   *[[14], p.375] [[15], p. 532.] By elementary operations we can convert polynomial matrices to several "standard" forms. Here we shall describe the Hermite forms, which are obtained by using only row (or only column) operations.*

Definition 48 Determinantal devisor (polynomial matrix)   *[[15], p. 299.] The determinantal devisor D_i of a polynomial matrix $P(s)$ is the greatest common factor of all $i \times i$ minors of $P(s)$*

Definition 49 Smith form (polynomial matrix) *[[15], p. 533.] The Smith form of a polynomial matrix $P \in \mathbb{R}[s]$ is the decomposition*

$$P(s) = U_L(s)S_P(s)U_R(s)$$

where $U_L(s)$ and $U_R(s)$ are unimodular polynomial matrices. $S_P(s)$ is a diagonal matrix with entries that are monic polynomials. The entries of $S_P(s)$ are calculated using the determinantal devisors of $P(s)$. See Reference [15] for details.

Definition 50 Smith-McMillan form (rational matrix) *Smith-McMillan form is for a rational matrix $G(s)$. We identify the poles (and their multiplicities) and zeros of the transfer matrix using the Smith-McMillan form. This form is useful in obtaining the poles and zeros (with their multiplicities) from a given transfer matrix. Consider $G \in \mathbb{R}_p(s)$ and represent it as follows, where $P \in \mathbb{R}[s]$ is a polynomial matrix, and d is the minimum polynomial of G*

$$G(s) = \frac{P(s)}{d(s)}$$

The Smith-McMillan form of $G(s)$ then uses the Smith form of the polynomial matrix $P(s)$ as

$$G(s) = \frac{U_L(s)S_P(s)U_R(s)}{d(s)}$$

Example 67 Smith-McMillan form Consider the following transfer matrix given on page 414 of Narendra, Annaswamy.

$$W_p(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{2}{(s-2)^2} \\ \frac{1}{(s-1)^2} & \frac{1}{s+3} \end{bmatrix}$$

First find the minimum polynomial $d(s)$ of the rational matrix $W_p(s)$

$$d(s) = (s-1)^2(s-2)^2(s+3)$$

Express $W_p(s)$ using the minimum polynomial as follows, where $P(s)$ is a polynomial matrix

$$W_p(s) = \frac{P(s)}{d(s)}$$

where

$$P(s) = \begin{bmatrix} (s-1)(s-2)^2(s+3) & 2(s-1)^2(s+3) \\ (s-2)^2(s+3) & (s-1)^2(s-2)^2 \end{bmatrix}$$

We now put this polynomial matrix $P(s)$ into Smith form by finding the determinantal devisors of $P(s)$ as follows, with $D_0 \triangleq 1$. The 1×1 minors of $P(s)$ are:

$$(s-1)(s-2)^2(s+3) \qquad 2(s-1)^2(s+3) \qquad (s-2)^2(s+3) \qquad (s-1)^2(s-2)^2$$

The greatest common devisor of these matrices is 1, so we set

$$D_1(s) = 1$$

The 2×2 minor of $W_p(s)$ is:

$$(s-1)^3(s-2)^4(s+3) - 2(s-1)^2(s-2)^2(s+3)^2 = (s-1)^2(s-2)^2(s+3)[(s-1)(s-2)^2 - 2(s+3)]$$

The greatest common devisor of this is itself, so we set

$$D_2(s) = (s-1)^2(s-2)^2(s+3)[(s-1)(s-2)^2 - 2(s+3)]$$

We then calculate ϵ_i' as

$$\epsilon_i'(s) = \frac{D_i(s)}{D_{i-1}(s)}$$

giving

$$\epsilon_1'(s) = 1$$
$$\epsilon_2'(s) = (s-1)^2(s-2)^2(s+3)[(s-1)(s-2)^2 - 2(s+3)]$$

So the Smith form of $P(s)$ is

$$S_P(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s-1)^2(s-2)^2(s+3)[(s-1)(s-2)^2 - 2(s+3)] \end{bmatrix}$$

To get the Smith-McMillan form of $W_p(s)$ divide the Smith form $S_P(s)$ by the minimum polynomial $d(s)$.

$$SM_{W_p}(s) = \frac{1}{(s-1)^2(s-2)^2(s+3)} \begin{bmatrix} 1 & 0 \\ 0 & (s-1)^2(s-2)^2(s+3)[(s-1)(s-2)^2 - 2(s+3)] \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{(s-1)^2(s-2)^2(s+3)} & 0 \\ 0 & (s-1)(s-2)^2 - 2(s+3) \end{bmatrix}$$

The diagonal entries are in the form

$$\frac{\epsilon_i(s)}{\psi_i(s)}$$

with

$$\epsilon_1 = 1$$
$$\epsilon_2 = (s-1)(s-2)^2 - 2(s+3)$$

and

$$\psi_1 = (s-1)^2(s-2)^2(s+3)$$
$$\psi_2 = 1$$

And so the characteristic polynomial $p(s)$ can be calculated as

$$p(s) = \psi_1(s)\psi_2(s)$$
$$= (s-1)^2(s-2)^2(s+3)$$

The zero polynomial is given by

$$
\begin{aligned}
z(s) &= \epsilon_1(s)\epsilon_2(s) \\
&= (s-1)(s-2)^2 - 2(s+3) \\
&= s^3 - 5s^2 + 6s - 10
\end{aligned}
$$

We can see that this system is fifth order $n = 5$, with a pole with multiplicity two at $s = 1$, a pole with multiplicity two at $s = 2$, and a pole at $s = -3$. It also has three unstable transmission zeros.

37.2.1 Hermite Form

Hermite form is a "standard" form which is obtained by applying only elementary row (or only column) operations on a polynomial matrix $P \in \mathbb{R}[s]$. The definition of Hermite form is different for rational polynomial matrices. The elementary operations can be represented by multiplication by elementary matrices. When the Hermite form of $P(s)$ is obtained by right multiplication of $P(s)$ by a unimodular matrix $U_R(s)$, this is called the right Hermite form, denoted H_R. Similarly, the left Hermite form of P us obtained as $H_L(s) = U_L(s)P(s)$. We define the right Hermite form of $P(s)$ as the matrix $H_R(s)$ as having the following properties:

$$
H_R(s) = P(s)U_R(s)
$$

- It is lower triangular

- Each diagonal element is monic

- Each diagonal element has higher degree than any other element in the same row

We will try to find an extended definition to rational transfer matrices.

Polynomial Matrices

The procedure of applying elementary operations using elementary matrices to a polynomial matrix $P \in \mathbb{R}[s]$ is described in the following steps, and uses the fact that $a(s)$ can be written as $a(s) = q(s)b(s)+r(s)$ where the degrees of q and r are less than that of a. The problem is, given P we wish to find the right Hermite form of P. That is, find H_R and a unimodular matrix U_R such that $H_R = PU_R$.

1. Perform column operations so that in the first row the polynomial of the highest order is in the first column

2. Express the polynomials in the first row as

$$
\begin{bmatrix} q(s)b(s) + r(s) & b(s) \\ \times & \times \end{bmatrix}
$$

3. Right multiply P by the unimodular matrix U_1 given by

$$U_1 = \begin{bmatrix} 1 & 0 \\ -q(s) & 1 \end{bmatrix}$$

This results in a matrix with the following form

$$\begin{bmatrix} r(s) & b(s) \\ \times & \times \end{bmatrix}$$

4. Go back and repeat step 1

Example 68 Hermite form for polynomial matrix We provide an example to demonstrate the steps, which are given only for 2×2 matrices. Given

$$P(s) = \begin{bmatrix} -s^3 - 2s^2 + 1 & -(s+1)^2 \\ (s+2)^2(s+1) & 0 \end{bmatrix}$$

1. Perform column operations so that in the first row the polynomial of the highest order is in the first column. In the first row, the polynomial of the highest order is already in the first column so no column operations are necessary.

2. Express the polynomials in the first row as

$$b(s) = -s^2 - 2s - 1$$
$$-s^3 - 2s^2 + 1 = q(s)(-s^2 - 2s - 1) + r(s)$$

Now we try to make $q(s)$ have as high order as possible. Choose

$$q(s) = s$$

and then

$$-s^3 - 2s^2 + 1 = -s^3 - 2s^2 - s + r(s)$$

and so

$$r(s) = s + 1$$

So the first unimodular matrix is

$$U_1 = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}$$

3. Right multiply P by U_1 as

$$PU_1 = \begin{bmatrix} -s^3 - 2s^2 + 1 & -(s+1)^2 \\ (s+2)^2(s+1) & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} = \begin{bmatrix} s+1 & -(s+1)^2 \\ (s+2)^2(s+1) & 0 \end{bmatrix}$$

4. Repeat step 1

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From here onward we will simply be repeating steps 1—4. Swap the columns using the unimodular matrix

$$U_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

giving

$$PU_1U_2 = \begin{bmatrix} -(s+1)^2 & s+1 \\ 0 & (s+2)^2(s+1) \end{bmatrix}$$

By inspection we can see that

$$q(s) = -(s+1)$$
$$r(s) = 0$$

and so

$$U_3 = \begin{bmatrix} 1 & 0 \\ -(s+1) & 1 \end{bmatrix}$$

and so

$$PU_1U_2U_3 = \begin{bmatrix} -(s+1)^2 & s+1 \\ 0 & (s+2)^2(s+1) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (s+1) & 1 \end{bmatrix} = \begin{bmatrix} 0 & s+1 \\ (s+1)^2(s+2)^2 & (s+1)(s+2)^2 \end{bmatrix}$$

Swap columns

$$U_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

giving

$$H_R = PU_1U_2U_3U_4 = \begin{bmatrix} s+1 & 0 \\ (s+1)(s+2)^2 & (s+1)^2(s+2)^2 \end{bmatrix}$$

And this matrix satisfies the conditions of Hermite form, and the unimodular matrix U_R is given by

$$U_R = U_1U_2U_3U_4 = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(s+1) & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -s-1 \\ -s & s^2+s+1 \end{bmatrix}$$

Rational Matrices

Given a matrix $W_p(s)$, Hermite decomposition is expressing $W_p(s)$ as follows, where $Q_p(s)$ is unimodular and $H_p(s)$ is lower triangular.

$$W_p(s) = H_p(s)Q_p^{-1}(s)$$

and so $H_p(s)$ is given by

$$H_p(s) = W_p(s)Q_p(s)$$

Given a system with transfer matrix $W_p(s)$, before we try to find the Hermite form of this system, we first want to know whether the Hermite form will be diagonal or not. To do this we need to form the matrix

E as given in Reference [11], p.396. We calculate r_i as the minimum relative degree in the ith row of $W_p(s)$. We then calculate the rows of E as follows

$$E_i = \lim_{s \to \infty} s^{r_i} W_{p,i}(s) \tag{37.1}$$

where $W_{p,i}$ corresponds to the ith row of W_p.

Example 69 Checking structure of Hermite form Consider again the example from Reference [11], p.414.

$$W_p(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{2}{(s-2)^2} \\ \frac{1}{(s-1)^2} & \frac{1}{s+3} \end{bmatrix}$$

Evaluating the matrix E as described above with $r_1 = 1, r_2 = 1$ we obtain

$$E = \lim_{s \to \infty} \begin{bmatrix} \frac{s}{s-1} & \frac{2s}{(s-2)^2} \\ \frac{s}{(s-1)^2} & \frac{s}{s+3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And so E is nonsingular, and so the right Hermite form of $W_p(s)$ is diagonal.

Finding the Hermite Form Try to determine the definition for Hermite form of a rational matrix as described in Reference [11].

Example 70 Calculating the Hermite Form Back to our transfer matrix from Example 69. From that example we know the Hermite form $H_p(s)$ of $W_p(s)$ will be diagonal, and now we wish to find $H_p(s)$.

$$W_p(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{2}{(s-2)^2} \\ \frac{1}{(s-1)^2} & \frac{1}{s+3} \end{bmatrix}$$

want to find the Hermite form. Because $E(W_p(s))$ is nonsingular, by Corollary 10.1 in Reference [11] we know that $H_p(s)$ is diagonal.

$$H_p = W_p Q_p$$

37.2.2 Observability Index

Note that the observability index satisfies the following inequality, as found in Reference [19], p.157. Given $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times n}$ where C has full rank

$$\frac{n}{m} \leq \nu \leq \min(\bar{n}, n - q + 1)$$

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From Transfer Matrix

An upper bound on the observability index is required for classical MIMO adaptive output feedback. This upper bound can be computed given a transfer matrix $W_p(s)$ as described in Reference [11], p.406. It says an upper bound ν on the observability index can be obtained by knowing an upper bound n_{ij} on the order of the ij^{th} scalar entry in $W_p(s)$.

$$\nu = \left\lceil \frac{1}{m} \sum_{i,j} n_{ij} \right\rceil$$

State Space System

Given a state-space model, we can calculate the observability index using the procedure described in Reference [15], p.294. This process involves selecting the first n linearly independent rows of the observability matrix and then rearranging them.

37.3 Comparison of Classical and CRM Based Output Feedback Adaptive Control

Consider the following multivariable plant with m inputs, p outputs, and n state variables, where (A, B) is controllable and (A, C) is observable, and B and C are full rank

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{37.2}$$

with transfer matrix given by

$$W_p(s) \triangleq C(sI - A)^{-1}B \in \mathbb{R}_p^{p \times m}(s) \tag{37.3}$$

The aim of this section is to analyze and compare the assumptions necessary of the plant in (37.2) and (37.3) to determine conditions under which the classical and CRM based adaptive output feedback control solutions are applicable. The requirements of the former will depend on information contained in (37.3), while the latter on the state-space matrices of (37.2). These requirements are summarized below.

37.3.1 Requirements for Classical Adaptive Output Feedback

1. Plant must be square, that is $m = p$.

2. The plant has no unstable transmission zeros.

    • This is to ensure pole-zero cancellations do not occur in the right half plane.

3. The Hermite form $H_p(s)$ of $W_p(s)$ is diagonal.

    • The only information needed to determine whether the Hermite form $H_p(s)$ of $W_p(s)$ is diagonal is the relative degree of each entry of $W_p(s)$. See Corollary 10.1 in Reference [11], p.396.

4. The sign of the high frequency gain satisfies a sign definite condition.

5. An upper bound on the observability index is known.

37.3.2 Requirements for CRM Based Adaptive Output Feedback

1. Plant need not be square, that is $m \neq p$.

2. The plant has no unstable transmission zeros.

    - This is to ensure pole-zero cancellations do not occur in the right half plane.

3. CB is full rank

    - This is the multivariable equivalent of relative degree unity for a SISO system. We will show in the following slides how this relates to the relative degrees of the entries of $W_p(s)$.

4. The sign of the control input uncertainty is known.

    - That is, if uncertainty in control effectiveness is given by Λu, Λ is a diagonal matrix with entries of known sign

5. The system order is known

    - This information is contained in the state-space representation used for the CRM based method

37.3.3 Markov Parameters

The Markov parameters can be calculated directly from a system transfer matrix $W_p(s)$, or from the matrices of a state-space realization of W_p. The following definition and theorem relates information about these state space matrices to limits of the transfer matrix.

Definition 51 Markov Parameters *[[15], p.387.] Given a transfer matrix $W_p(s)$, calculate the Markov parameters as follows*

$$
\begin{aligned}
H_0 &= \lim_{s \to \infty} W_p(s) \\
H_1 &= \lim_{s \to \infty} s(W_p(s) - H_0) \\
H_2 &= \lim_{s \to \infty} s^2(W_p(s) - H_0 - H_1 s^{-1}) \\
&\vdots
\end{aligned}
$$

Theorem 12 Markov Parameters Realization *[[15], p.387.] The set (A, B, C, D) is a realization of $H(s)$ if and only if*

$$
\begin{aligned}
H_0 &= D \\
H_i &= CA^{i-1}B, \qquad i = 1, 2, \ldots
\end{aligned}
$$

PROOF See [15], p.387 for proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 7 Markov Parameters Realization Given a plant as in (37.2) with equal number of inputs and outputs, that is $m = p$, and with transfer matrix given by (37.3), when $CB = \lim_{s \to \infty} sW_p(s)$ is full rank, this essentially means that there is at least one relative degree one relationship between each input and one of the outputs of $W_p(s)$. This implies $r_i = 1$ for all $i = 1, \ldots, m$. Additionally, from the definition of E in (37.1), we can see that this implies that $E(W_p(s)) = CB$. Furthermore, since $E[W_p(s)] = CB$ is nonsingular, then $H_p(s)$ is diagonal and the elements of K_p are identical to the high frequency gains of the scalar transfer functions corresponding to the minimum relative degrees in each row. That is, $K_p = E[W_p(s)]$ [11], p. 405. We can see this also in the definition of the high frequency gain in Definition 37.

Theorem 13 *Given the plant in (37.2) where $m = p$, the Hermite form of $W_p(s)$ is diagonal if CB is full rank, and $E[W_p(s)] = CB$.*

PROOF Using Definition 51 of Markov Parameters and Theorem 12 we see that $CB = \lim_{s \to \infty} sW_p(s)$ is full rank implies that then $r_i = 1$ for all $i = 1, \ldots, m$, where r_i is the minimum relative degree of the entries in each row of $W_p(s)$. Writing out the series expansion for the resolvent of A in (37.3) we obtain

$$W_p(s) = C \left(\frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \ldots \right) B$$

From which, using the following definition for E as in (37.1) with $r_i = 1$ for all $i = 1, \ldots, m$ we obtain

$$
\begin{aligned}
E[W_p(s)] &= \lim_{s \to \infty} sW_p(s) \\
&= \lim_{s \to \infty} sC \left(\frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \ldots \right) B \\
&= CB
\end{aligned}
$$

And by corollary 10.1 in Reference [11], p.396 since CB is full rank thus E is full rank, the Hermite form of $G(s)$ is diagonal. $\qquad \square$

Corollary *Given the plant in (37.2) where $m = p$, if CB is full rank, the high frequency gain is given by*

$$K_p = CB$$

where the high frequency gain is given by Definition 37 as

$$\lim_{s \to \infty} H_p^{-1}(s)W_p(s) \triangleq K_p$$

PROOF When the Hermite form $H_p(s)$ of $G(s)$ is diagonal, $K_p = E[W_p(s)]$ [11], p.396. By Theorem 13 CB being full rank implies $H_p(s)$ is diagonal and $E[W_p(s)] = CB$ and so $K_p = CB$. $\qquad \square$

Theorem 14 *Given a relative degree one system $W_p(s) \in \mathbb{R}_p^{p \times m}$ with realization $(A, B, C, 0)$, both the classical and CRM based output feedback adaptive control methods are equally applicable if*

1. *The plant is square, that is $p = m$.*

2. *That plant has no transmission zeros in the closed right half plane.*

3. *The matrix B is of the form $B^*\Lambda$ where B^* is known and Λ is a diagonal nonsingular matrix with elements of known sign.*

37.3.4 Relating Classical and CRM Adaptive Control

From the preceding subsections and discussion, we can make the following comparison between the applicability of the classical and CRM based methods to systems in (37.2) and (37.3).

- For relative degree one systems, that is rank of CB is full, both methods are applicable if the system is square $m = p$.

- If the system is not square, the classical method will not work.

- If the system is not relative degree one, the CRM based method will not work

37.4 Adaptive Control Numerical Example

Consider the following LTI plant

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$ and the system matrices are given by

$$A = \begin{bmatrix} -2 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{37.4}$$

Evaluating the system transfer matrix $W_p(s)$ defined as follows

$$W_p(s) \triangleq C(sI - A)^{-1}B \in \mathbb{R}_p^{p \times m}(s)$$

for the system with matrices in (37.4) we get

$$W_p(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+2)^2} \\ \frac{1}{(s+1)^2} & \frac{1}{s-3} \end{bmatrix} \tag{37.5}$$

Note that this system is similar, but not identical to that in Example 67 taken from Reference [11]. In the subsections that follow we will design a classical adaptive output feedback controller for this system. We will check the transmission zeros and verify the requirement that all transmission zeros are stable, check that the plant has a diagonal Hermite form, then select the reference model, and build the adaptive controller.

37.4.1 Evaluate System Transmission Zeros

First calculate the minimum polynomial $d(s)$ of our rational matrix $W_p(s)$ as:

$$d(s) = (s + 1)^2(s + 2)^2(s - 3)$$

and express $W_p(s)$ as

$$W_p(s) = \frac{1}{d(s)}P(s)$$

where

$$P(s) = \begin{bmatrix} (s + 1)(s + 2)^2(s - 3) & 2(s + 1)^2(s - 3) \\ (s + 2)^2(s - 3) & (s + 1)^2(s + 2)^2 \end{bmatrix}$$

We now put this polynomial matrix $P(s)$ into Smith form by finding the determinantal devisors of $P(s)$ as follows, with $D_0 \triangleq 1$. The 1×1 minors of $P(s)$ are:

$$(s + 1)(s + 2)^2(s - 3) \qquad 2(s + 1)^2(s - 3) \qquad (s + 2)^2(s - 3) \qquad (s + 1)^2(s + 2)^2$$

The greatest common devisor of these matrices is 1, so we set

$$D_1(s) = 1$$

The 2×2 minor of $W_p(s)$ is:

$$(s+1)^3(s+2)^4(s-3) - 2(s+1)^2(s+2)^2(s-3)^2 = (s+1)^2(s+2)^2(s-3)[(s+1)(s+2)^2 - 2(s-3)]$$

The greatest common devisor of this is itself, so we set

$$D_2(s) = (s+1)^2(s+2)^2(s-3)[(s+1)(s+2)^2 - 2(s-3)]$$

We then calculate ϵ_i' as

$$\epsilon_i'(s) = \frac{D_i(s)}{D_{i-1}(s)}$$

giving

$$\epsilon_1'(s) = 1$$
$$\epsilon_2'(s) = (s+1)^2(s+2)^2(s-3)[(s+1)(s+2)^2 - 2(s-3)]$$

So the Smith form of $P(s)$ is

$$S_P(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s+1)^2(s+2)^2(s-3)[(s+1)(s+2)^2 - 2(s-3)] \end{bmatrix}$$

To get the Smith-McMillan form of $W_p(s)$ we divide the Smith form $S_P(s)$ by the minimum polynomial $d(s)$.

$$SM_{W_p}(s) = \frac{1}{(s+1)^2(s+2)^2(s-3)} \begin{bmatrix} 1 & 0 \\ 0 & (s+1)^2(s+2)^2(s-3)[(s+1)(s+2)^2 - 2(s-3)] \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(s+1)^2(s+2)^2(s-3)} & 0 \\ 0 & (s+1)(s+2)^2 - 2(s-3) \end{bmatrix}$$

The diagonal entries are in the form

$$\frac{\epsilon_i(s)}{\psi_i(s)}$$

with

$$\epsilon_1 = 1$$
$$\epsilon_2 = (s+1)(s+2)^2 - 2(s-3)$$

and

$$\psi_1 = (s+1)^2(s+2)^2(s-3)$$
$$\psi_2 = 1$$

And so the characteristic polynomial $p(s)$ can be calculated as

$$p(s) = \psi_1(s)\psi_2(s)$$
$$= (s+1)^2(s+2)^2(s-3)$$

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The zero polynomial is given by

$$z(s) = \epsilon_1(s)\epsilon_2(s)$$
$$= (s+1)(s+2)^2 - 2(s-3)$$
$$=$$

We can see that this system is fifth order $n = 5$, with a pole with multiplicity two at $s = -1$, a pole with multiplicity two at $s = -2$, and a pole at $s = 3$. It also has three stable transmission zeros.

37.4.2 Check Structure of Plant's Hermite Form

Check the matrix E to see if it is nonsingular. If so, the Hermite form $H_p(s)$ of $W_p(s)$ is diagonal, which will make control design easier. To do this, we first *find the minimum relative degree n_i of the elements in each row of $W_p(s)$*. For our plant with transfer function in (37.5) this is

$$n_1 = 1$$
$$n_2 = 1$$

(37.6)

Evaluate E as in Reference [11], p.396.

$$E_i = \lim_{s \to \infty} s^{r_i} G_i(s)$$

where $G_i(s)$ corresponds to the ith row of $G(s)$.

$$E_1 = \lim_{s \to \infty} s \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+2)^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$E_2 = \lim_{s \to \infty} s \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{s-3} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

So

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and since E is nonsingular, the plant's Hermite form $H_p(s)$ is diagonal.

37.4.3 Expressing the Plant's Hermite Form

We know the plant has a diagonal Hermite form. It is given by

$$H_p(s) = \begin{bmatrix} \frac{1}{\pi^{n_1}(s)} & 0 & \cdots & 0 \\ 0 & \frac{1}{\pi^{n_2}(s)} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\pi^{n_m}(s)} \end{bmatrix}$$

where $\pi(s)$ is any monic polynomial of degree 1 and n_i is the minimum relative degree of the elements of $W_p(s)$ in the ith row. Those values are given in (37.6).

Remark 8 Since $\pi(s)$ is any monic polynomial of degree 1, and since the class of reference models that we can use consists essentially of those asymptotically stable transfer matrices that are generated by the Hermite normal form of the plant, we will pick $\pi(s) = s + a$ where $a > 0$.

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Following this remark the plant in this example has Hermite form

$$H_p(s) = \begin{bmatrix} \frac{1}{s+a} & 0 \\ 0 & \frac{1}{s+a} \end{bmatrix} \tag{37.7}$$

37.4.4 Find the High Frequency Gain

To find K_p, use $K_p = \lim_{s \to \infty} H_p^{-1}(s) W_p(s)$, which for diagonal Hermite forms is the same as $K_p = E[W_p(s)]$.

$$K_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

37.4.5 Select the Reference Model

Pick the reference model transfer matrix $W_m(s)$ as

$$W_m(s) = H_p(s) Q_m(s)$$

where $Q_m(s)$ is an asymptotically stable unimodular matrix. For purposes of simplicity we can assume that $Q_m = \gamma I$, where γ is picked so that the DC gain of the components of the diagonal Hermite form, and thus reference model, have unity DC gain. With $H_p(s)$ in (37.7) this gives $\gamma = a$ and a reference model as

$$W_m(s) = \begin{bmatrix} \frac{a}{s+a} & 0 \\ 0 & \frac{a}{s+a} \end{bmatrix}$$

37.4.6 Calculate an Upper Bound on the Observability Index

Calculate an upper bound ν on the observability index using the following formula [11], p.406. Where n_{ij} is the relative degree of each transfer function in $W_p(s)$ in the ith row and j^{th} column.

$$\nu = \left\lceil \frac{1}{m} \sum_{i,j} n_{ij} \right\rceil$$

For our plant, this is

$$\boxed{\nu = 3}$$

37.4.7 Design Controller Filters

Using the upper bound ν on the observability index, we design $\nu - 1$ control input filters, and ν output filters as follows, where $r_q(s)$ is a Hurwitz, monic polynomial of degree $\nu - 1$.

$$\text{Control signal filter} \qquad \omega_i(t) = \frac{s^{i-1}}{r_q(s)} u(t) \qquad i = 1, \ldots \nu - 1$$

$$\text{Output filter} \qquad \omega_j(t) = \frac{s^{j-\nu}}{r_q(s)} y_p(t) \qquad j = \nu, \ldots 2\nu - 1$$

Each filter has a scalar denominator $r_q(s)$, and there are m components to each filter, and a total of $2\nu - 1$ filter, so the total number of integrations (i.e. the number of controller states) to generate the ω signals is $m(2\nu - 1)$. There is a parameter matrix corresponding to each ω signal, giving $m^2(2\nu - 1)$ parameters.

In the current example, with $\nu = 3$, this means there will be 5 total filters: 2 input filters and 3 output filters. And let $r_q(s)$ be defined as

$$r_q(s) = s^2 + s + 1$$

Gives

$$\omega_1(t) = \frac{1}{s^2 + s + 1}u(t) \qquad \omega_2(t) = \frac{s}{s^2 + s + 1}u(t)$$

$$\omega_3(t) = \frac{1}{s^2 + s + 1}y_p(t) \qquad \omega_4(t) = \frac{s}{s^2 + s + 1}y_p(t) \qquad \omega_5(t) = \frac{s^2}{s^2 + s + 1}y_p(t)$$

37.4.8 Control Input

$$u(t) = \Theta(t)\omega(t)$$

where

$$\omega(t) = \begin{bmatrix} r & \omega_1^\top & \cdots & \omega_{\nu-1}^\top & \omega_\nu^\top & \cdots & \omega_{2\nu-1}^\top \end{bmatrix}^\top$$

and

$$\Theta(t) = \begin{bmatrix} K_0 & C_1 & \cdots & C_{\nu-1} & D_0 & \cdots & D_{\nu-1} \end{bmatrix}$$

37.4.9 Update Laws

$$\dot{K}_0 = -e_1 r^\top \qquad \dot{C}_1 = -e_1 \omega_1^\top \qquad \dot{C}_2 = -e_1 \omega_2^\top$$

$$\dot{D}_0 = -e_1 \omega_3^\top \qquad \dot{D}_1 = -e_1 \omega_4^\top \qquad \dot{D}_2 = -e_1 \omega_5^\top$$

37.4.10 Summary

<div style="border:1px solid red; padding:10px;">

Example: Classical MIMO Adaptive Control

Plant: $\qquad\qquad\qquad\qquad W_p(s)$

Reference Model: $\qquad\quad W_m(s)$

Tracking Error: $\qquad\qquad e_1 = y_p - y_m$

Input Filter: $\qquad\qquad \omega_i(t) = \dfrac{s^{i-1}}{r_q(s)} u(t) \qquad i = 1, \ldots \nu - 1$

Output Filter: $\qquad\quad \omega_j(t) = \dfrac{s^{j-\nu}}{r_q(s)} y_p(t) \qquad j = \nu, \ldots 2\nu - 1$

Control Law: $\qquad\qquad u(t) = \Theta(t)\omega(t)$

Parameters: $\qquad\quad\; \Theta(t) = \begin{bmatrix} K_0 & C_1 & \ldots & C_{\nu-1} & D_0 & \ldots & D_{\nu-1} \end{bmatrix}$

Regressor: $\qquad\qquad\; \omega(t) = \begin{bmatrix} r & \omega_1^\top & \ldots & \omega_{\nu-1}^\top & \omega_\nu^\top & \ldots & \omega_{2\nu-1}^\top \end{bmatrix}^\top$

Update Law: $\qquad\qquad \dot{\Theta} = -e_1 \omega^\top$

</div>

37.5 Simulation Results
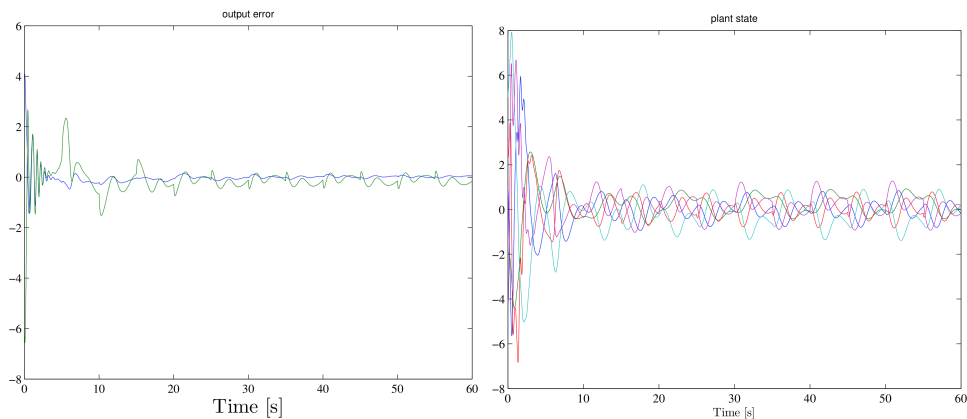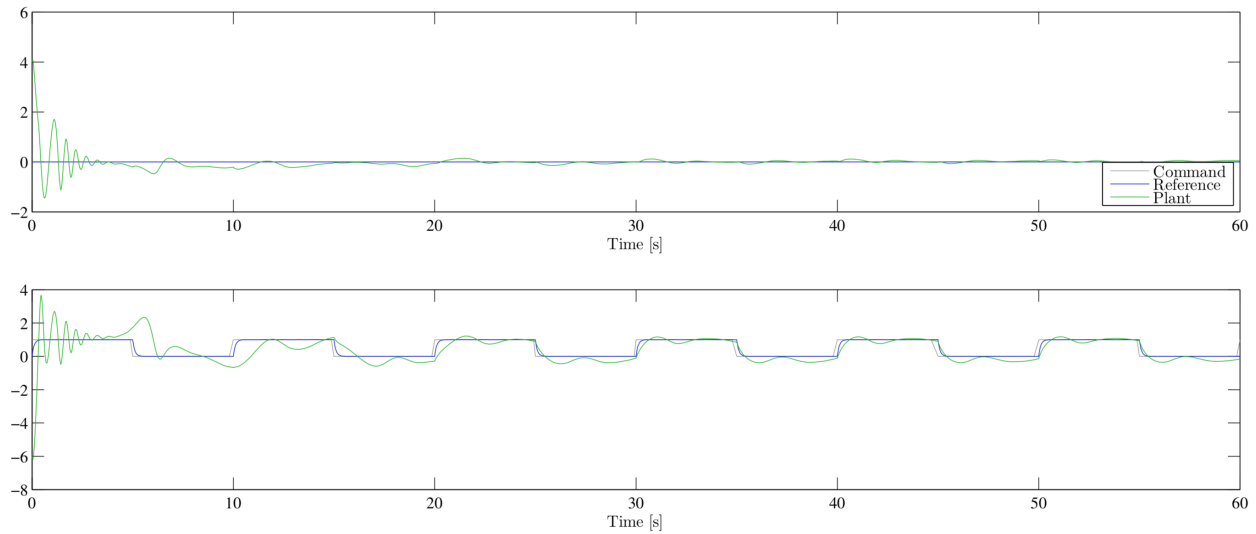
37.6   Appendix

Given a transfer matrix $W_p(s)$ we want to design a classical MIMO adaptive output feedback controller, and analyze and compare the structure of the controller to closed-loop reference model based output feedback controllers. To do this we would like to:

1. Given a transfer matrix, find its poles and zeros

    (a) To do this, we will use Smith-McMillan form

2. Convert a given transfer matrix to a minimal state-space realization. This can be accomplished in one of two ways:

    (a) Convert the transfer matrix directly to a minimal realization

    (b) Express the transfer matrix using a non-minimal realization, and then use Kalman decomposition to remove the uncontrollable/unobservable modes

3. Express a given transfer matrix using Hermite decomposition

37.6.1   Transmission Zeros

Of state-space and transfer matrix representations of systems. How to calculate them for a state-space representation? (use Smith-McMillan form for transfer matrix). Given a nominal system with only stable transmission zeros, what if the uncertainty introduces unstable transmission zeros?

Chapter 38

Advanced Adaptive Control

38.1 Adaptive Control of Plants with Arbitrary Relative Degree

Last class: parameter convergence summary: ω produces $\tilde{\theta}$ and we said that if ω is PE then $\tilde{\theta} \to 0$.

$\omega = \omega^* + Ce$ where ω^* comes from reference model, and so ω^* PE and $e \to 0 \Rightarrow \omega$ PE after some initial time $t \geq t_1$.

if r has n frequencies $\omega^* \in \mathbb{R}^{2n}$ is PE $\Rightarrow$ PE after $t \geq t_1 \Rightarrow \tilde{\theta} \to 0$. t_1 is dependent on $x(0)$ where x is the state of the adaptive system in adaptive control (1). Whereas t_1 is independent of the initial condition in adaptive observer (2).

(1) Adaptive control: uniform asymptotic stability (2) Adaptive observer: exponential stability

38.1.1 Adaptive Control of Plants with Arbitrary Relative Degree

Chapter 5 $Z_p(s)$: monic, degree $n - n^* = m$ $R_p(s)$: monic, degree n k_p: high frequency gain

need to know sign of k_p because the adaptive law needs the gradient information, that is which way the plant will move.

At a minimum y_m must be generated by a model with same relative degree as plant. It can be higher but not lower. Need an upper bound on relative degree. So pick $W_m(s)$ to have relative degree n^*. That is $W_m(s) = \frac{k_m}{D_m(s)}$ where $D_m(s)$ is monic, and degree n^*. Use same structure as before.

$\theta_{1c}^\top (sI - F_{n-1})^{-1} g_{n-1} = \frac{t_1(s)}{f(s)}$

$\theta_{20c} + \theta_{2c}^\top (sI - F_{n-1})^{-1} g_{n-1} = \frac{t_2(s)}{f(s)}$

TF from r to y_p:

$$
= \frac{k_c \frac{1}{1-\frac{t_1}{f}} \frac{k_p Z_p}{R_p}}{1 - \left[\frac{f(s)}{f(s)-t_1(s)}\right] \frac{k_p Z_p}{R_p} \frac{t_2}{f}}
$$

$$
= \frac{(k_c k_p) f(s) Z_p(s)}{R_p[f - t_1] - k_p Z_p t_2}
$$

$$
? = \frac{k_m}{D_m(s)}
$$

Need to check if we have the degrees of freedom $k_c = k^* = \frac{k_m}{k_p}$ Are there t_1^* and t_2^* such that

$$
R_p(s)[f(s) - t_1^*(s)] - k_p Z_p(s) t_2^*(s) =? = D_m(s) f(s) Z_p(s)
$$

where $R_p(s)$ is degree n, $f(s)$ is degree $n-1$, $t_1^*(s)$ is degree $n-2$, $Z_p(s)$ is degree m, $t_2(s)$ degree $n-1$ and $D_m(s)$ degree $n-m=n^*$, $f(s)$ is degree $n-1$, and $Z_p(s)$ is degree $m=n-n^*$. But t_1^* and $t_2^*(s)$ are not monic, so they have $n-1$ and n degrees of freedom, respectively.

So we know there is a solution to this equation. Nonsingularity comes from that $R_p(s)$ and $k_p Z_p(s)$ are coprime: that is they share no common factors.

Bezout-Identity guarantees existence and uniqueness of t_1^* and t_2^*.

(See book 5.2, 5.3)

Analytic part:

$$k(t) = k^* + \tilde{k}(t)$$
$$\theta_1(t) = \theta^* + \tilde{\theta}_1(t)$$
$$\theta_2(t) = \theta_2^* + \tilde{\theta}_2(t)$$
$$\theta_{20}(t) = \theta_{20}^* + \tilde{\theta}_{20}(t)$$

$\theta_1^{*\top}(sI - F_{n-1})^{-1} g_{n-1} = \frac{t_1^*(s)}{f(s)}$

$\theta_{20}^* + \theta_2^{*\top}(sI - F_{n-1})^{-1} g_{n-1} = \frac{t_2^*(s)}{f(s)}$

$$\tilde{\theta} = \begin{bmatrix} \tilde{k}(t) \\ \tilde{\theta}_1(t) \\ \tilde{\theta}_{20}(t) \\ \tilde{\theta}_2(t) \end{bmatrix}$$

$$\omega = \begin{bmatrix} r \\ \omega_1 \\ y_p \\ \omega_2 \end{bmatrix}$$

Error model

biggest problem is that the transfer function in this error model is not SPR. Want:

first assume k_p is known, then deal with the case where k_p is unknown later. e_a is fully realizable. ε_1 is augmented error.

Augmenting the system with the plus/minus part only contributes a $\tilde{\theta}$ part, since for θ^*, since its a constant, the transfer function operation commutes... That is doesn't matter whether you differentiate first and integrate second, or vice-versa.

Now we have error model 1 and immediately get boundedness of parameter errors. $\tilde{\theta}$ bounded.

$$V = \frac{1}{2}\tilde{\theta}^\top \tilde{\theta}$$
$$\dot{V} = -\varepsilon_1^2 \le 0$$
$$\dot{\tilde{\theta}} = -\varepsilon_1 \zeta$$

$\varepsilon_1 = e_1 + e_2$

next class: notion of slowly-varying parameters.

38.2 Adaptive Control of nth Order Systems with Output Feedback

Adaptive control of nth order systems with output feedback.

Block diagram

$e_1 + e_2 = \varepsilon_1$.

Error model is ζ through $\tilde{\theta}^\top$ to get ε_1.

The error model now allows us to use $\dot{\tilde{\theta}} = -\varepsilon_1 \zeta$ gives bounded solutions. To be able to attempt to get global...? make system almost time invariant by introducing a normalization.

$$\dot{\tilde{\theta}} = \frac{-\varepsilon_1 \zeta}{1 + \zeta^\top \zeta + \omega^\top \omega}$$

$$\dot{V} = \frac{-\varepsilon_1^2}{1 + \zeta^\top \zeta + \omega^\top \omega}$$

$\Rightarrow \tilde{\theta} \in \mathcal{L}^\infty$. So now system has been reduced to...?

Next property to establish

(ii) $\dot{\tilde{\theta}}$ is small.

$$\dot{V} = \frac{-\varepsilon_1^2}{1 + \zeta^\top \zeta + \omega^\top \omega} = -\left(\frac{\varepsilon_1}{\sqrt{1 + \zeta^\top \zeta + \omega^\top \omega}} \right)^2$$

$\int \dot{V} dt < \infty \Rightarrow \frac{\varepsilon_1}{\sqrt{1 + \zeta^\top \zeta + \omega^\top \omega}} \in \mathcal{L}^2$

so $\dot{\theta} = -\left(\frac{\varepsilon_1}{\sqrt{1 + \zeta^\top \zeta + \omega^\top \omega}} \right) \left(\frac{\zeta}{\sqrt{1 + \zeta^\top \zeta + \omega^\top \omega}} \right)$ and so $\dot{\theta} \in \mathcal{L}^2$.

Barbalat's lemma $\dot{\theta} \in \mathcal{L}^2, \dot{\theta} \in \mathcal{L}^\infty, \ddot{\theta} \in \mathcal{L}^\infty$ and so $\dot{\theta} \to 0$.

(iii) Boundedness of the state:

$$x = \begin{bmatrix} x_p \\ \omega \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} A_p & b_p \theta_1^\top & b_p \theta_{20} & b_p \theta_2^\top \\ & & & \end{bmatrix} x + \begin{bmatrix} b_p \\ g_{n-1} \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{x} = A_{cl} x + b(\tilde{\theta}^\top \omega + r)$$
$$y_p = h^\top x$$

$h^\top (sI - A_{cl})^{-1} b = W_m(s)$

BLOCK DIAGRAM

$\tilde{\theta} \in \mathcal{L}^\infty \ \dot{\tilde{\theta}} \to 0$

Properties of almost time invariant systems... growth rate of signals... $x(t)$ is bounded.

High order tuners.

Implicit in everything we have done so far in how we realize adaptation. Time varying parameter $\theta(t)$. Adjustment of θ was given by coming up with $\dot{\theta} =$ something. Stability procedure was Lyapunov based.

What about instead we came up with $\theta(t)$ was some function f through a transfer function $G(s)$ which is higher order tuner for $\theta(t)$. Pick $G(s)$ with a higher relative degree.

Problem:

BLOCK DIAGRAM

Relative degree 3, feedback through θ, where θ is unknown. Try to use adaptive observer.

$x_1 = \frac{1}{s+1}(u + \theta y)$

$\hat{x}_1 = \omega_{11} + \hat{\theta}\omega_{21}$

$x_2 = \frac{1}{(s+1)^2}(u + \theta y)$

$\hat{x}_2 = \omega_{12} + \hat{\theta}\omega_{22}$

$\hat{y} = \frac{1}{s+1}\hat{x}_2$

$e_y = \hat{y} - y = \frac{1}{s+1}(\hat{x}_2 - x_2) = \frac{1}{s+1}(\tilde{\theta}\omega_{22})$

So this is the error model and we are done.

$\dot{\tilde{\theta}} = -e_y\omega_{22} \Rightarrow$

$e, \tilde{\theta} \in \mathcal{L}^\infty$

if we assume $y \in \mathcal{L}^\infty$

then $\Rightarrow \hat{y} \in \mathcal{L}^\infty$

$\omega_{22} \in \mathcal{L}^\infty$.

$\Rightarrow \dot{e} \in \mathcal{L}^\infty$

$\Rightarrow e \to 0$

Adaptive Control

$y = \frac{1}{(s+1)^3}(u + \theta y) = \frac{1}{s+1}\left[\frac{1}{(s+1)^2}(u + \theta y)\right]$

$\hat{y} = \frac{1}{s+1}\left[\frac{1}{(s+1)^2}(u) + \hat{\theta}\omega_{22}\right]$

Choose u so that $\hat{y} \to 0$

ie $\frac{1}{(s+1)^2}u = -\hat{\theta}\omega_{22}$

$\Rightarrow u = (s+1)^2\left(-\hat{\theta}\omega_{22}\right) = -(s^2 + 2s + 1)(\hat{\theta}\omega_{22})$

$= -s\left(\hat{\theta}\dot{\omega}_{22} + \dot{\hat{\theta}}\omega_{22}\right) - 2\left(\hat{\theta}\dot{\omega}_{22} + \dot{\hat{\theta}}\omega_{22}\right) - \hat{\theta}\omega_{22}$

$= -\left(\hat{\theta}\ddot{\omega}_{22} + 2\dot{\hat{\theta}}\dot{\omega}_{22} + \ddot{\hat{\theta}}\omega_{22}\right) - 2\hat{\theta}\dot{\omega}_{22} - 2\dot{\hat{\theta}}\omega_{22} - \hat{\theta}\omega_{22}$

$= -\hat{\theta}(\ddot{\omega}_{22} + 2\dot{\omega}_{22} + 1) = -\hat{\theta}y - 2\dot{\hat{\theta}}(\dot{\omega}_{22} + \omega_{22}) - 2\dot{\hat{\theta}}\omega_{21} - \ddot{\hat{\theta}}\omega_{22}$

Suppose $\dot{\hat{\theta}} = -e_y\omega_{22}$

$\ddot{\hat{\theta}} = -\dot{e}_y\omega_{22} - e_y\dot{\omega}_{22}$

$\dot{e}_y = -e_y + \tilde{\theta}\omega_{22}$

Higher order tuning

Introduce $\bar{\theta}$ another estimate of θ need in general $n^* - 1$ estimates.

Error model hasn't changed

Add and subtract

$\dot{e}_y = -e_y + (\hat{\theta} - \bar{\theta})\omega_{22} + (\bar{\theta} - \theta)\omega_{22}$

$\tilde{\theta}_{\text{new}} = \bar{\theta} - \theta$

So if

$\dot{\tilde{\theta}}_{\text{new}} = -e\omega_{22}$

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and

$\hat{\theta}$ so that $\hat{\theta} \to \bar{\theta}$

i.e. $\hat{\theta} = f(x)$ where f is chosen so that $\hat{\theta} \to \bar{\theta}$ and $\ddot{\hat{\theta}}$ must be computable.

Use Lyapunov as the guideline for picking the derivative

$V = \frac{1}{2}(e_y^2 + \tilde{\theta}_{\text{new}}^2 + (\hat{\theta} - \bar{\theta})^2)$

plug and chug skipping algebra we get

$\dot{V} = -e_y^2 + 2e_y(\hat{\theta} - \bar{\theta})\omega_{22} + (\hat{\theta} - \bar{\theta})f(x)$

So choose $f(x)$ as

$f(x) = -(\hat{\theta} - \bar{\theta})\omega_{22}^2$

$\dot{V} = -(e_y - \omega_{22}(\hat{\theta} - \bar{\theta}))^2$

and

$e, \hat{\theta}, \bar{\theta} \in \mathcal{L}^\infty$

$\dot{\hat{\theta}} = f$

$\Rightarrow \ddot{\hat{\theta}} = \dot{f} = -(\dot{\hat{\theta}} - \dot{\bar{\theta}})\omega_{22}^2 - (\hat{\theta} - \bar{\theta})2\omega_{22}\dot{\omega}_{22}$

computable. Steven Morse. Notes of this stuff will be put online.

So 3 general methods that are available with output feedback for higher order plants

1. Augmented error approach in book

2. Higher order tuner (not in book)

3. Backstopping (also not in book, developed in 1990s)

38.3   Saturation Protection

38.3.1   Introduction and Motivation

Consider the following plant, reference model, and control law, which we have seen many times before, but note the difference between the control law which produces the desired command u_c and the input to the plant u. As before $a_m < 0$, and the sign of b_p is known.

Plant:	$\dot{x}_p = a_p x_p + b_p u$	(38.1)
Reference model:	$\dot{x}_m = a_m x_m + b_m r$	(38.2)
Control law:	$u_c = \theta x_p + kr$	(38.3)

Depending on the values of the adaptive parameters θ and k, the state x_p and the reference input r, the control input u might be larger than what the actuator is actually capable of. This is known as **actuator saturation**, which all actuators will have. This limit is a *known constraint*, for example how much torque a motor can apply, or how much a control surface on an aircraft can deflect. This limit $u_{\max} > 0$ is such that $|u| \leq u_{\max}$, after which the actuator experiences saturation. The control law with this saturation can be expressed as follows.

Control input:	$u = \begin{cases} u_c & \text{if } |u_c| \leq u_{\max} \\ u_{\max}\text{sgn}(u_c) & \text{if } |u_c| > u_{\max} \end{cases}$	(38.4)

The control law can also be written

$$u = u_{\max}\text{sat}\left(\frac{u_c}{u_{\max}}\right)$$

where the saturation function is given by

$$\text{sat}(x) = \begin{cases} x & \text{if } |x| \leq 1 \\ \text{sgn}(x) & \text{if } |x| > 1 \end{cases}$$

Before considering a control solution which accommodates the actuator saturation in (38.4), *the case where the actuator has no saturation limits will be considered first.* That is set $u = u_c$, recovering the standard adaptive system shown previously. Defining tracking error as

$$e = x_p - x_m \tag{38.5}$$

parameter error as

$$\tilde{\theta} = \theta - \theta^* \qquad \tilde{k} = k - k^* \tag{38.6}$$

and comparing this expression above to the reference model, we the following ideal constant parameters θ^* and k^* which satisfy the matching conditions

$$a_m = a_p + b_p \theta^* \qquad b_m = b_p k^* \tag{38.7}$$

where the ideal parameters can be solved as

$$\theta^* = \frac{a_m - a_p}{b_p} \qquad k^* = \frac{b_m}{b_p}$$

Differentiating (38.5) get tracking error dynamics

$$\dot{e} = a_m e + b_p \tilde{\theta} x_p + b_p \tilde{k} r$$

Proposing the following standard update update laws

> **Update laws:** $\dot{\tilde{k}} = -\text{sgn}(b_p) e r$
>
> $\dot{\tilde{\theta}} = -\text{sgn}(b_p) e x_p$

Stability can be verified by proposing the following candidate Lyapunov function.

$$V(e, \tilde{k}, \tilde{\theta}) = \frac{1}{2}\left(e^2 + |b_p|\tilde{k}^2 + |b_p|\tilde{\theta}^2\right)$$

Time differentiating gives

$$\dot{V} = a_m e^2 + e b_p \tilde{\theta} x_p + e b_p \tilde{k} r + |b_p|\tilde{k}\dot{\tilde{k}} + |b_p|\tilde{\theta}\dot{\tilde{\theta}}$$

from which the proposed update laws give $\dot{V} \leq 0$. Standard stability arguments follow. Simulating this system when the reference command is a sinusoid, and first if we look at the response without saturation, we can plot the plant state, reference model state, control input, and tracking error as shown below.
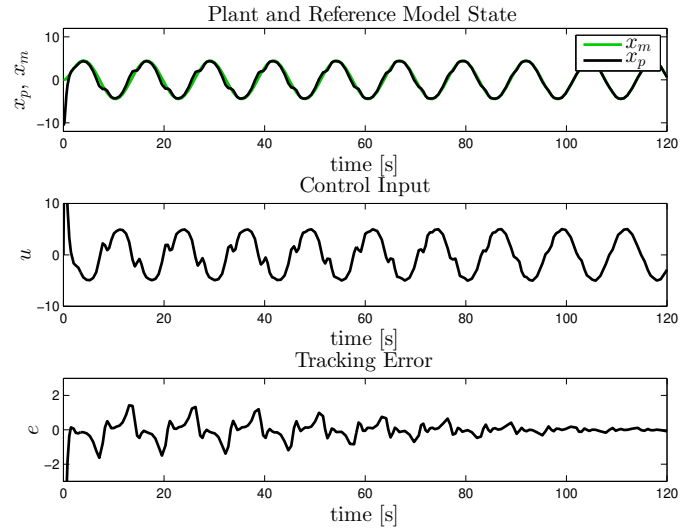
As expected, the tracking error tends to zero. If we introduce a saturation limit of $u_{\max} = 5$, say, even starting from zero initial tracking error, we obtain the following simulation response
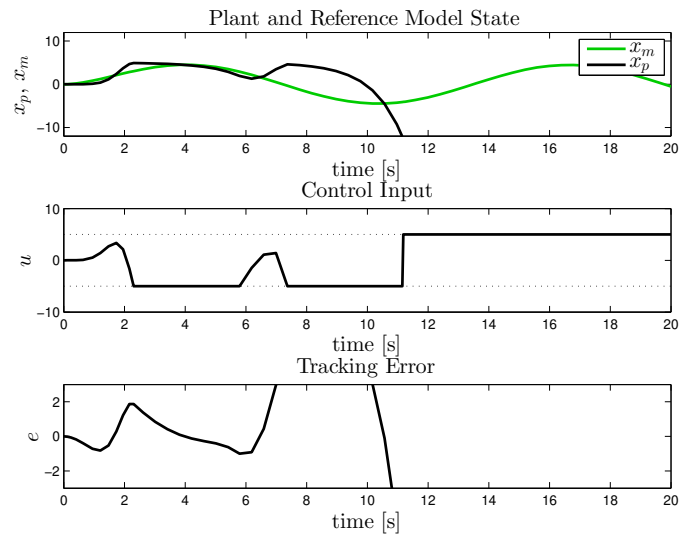
What is happening? Essentially is wind-up of the parameters. As shown before with time varying disturbances we need to modify our controller to accommodate for the effective input disturbance due to saturation. The difference in this case is that this disturbance is known.

38.3.2 Saturation Protection Control Design

In what follows, modifications to the adaptive system described above will be proposed so as to provide stability in the presence of actuator saturation. The control input to the actuator can be expressed in two parts: the total *desired* input signal as computed by the control law, and a component which subtracts off the portion of this control signal which the actuator is unable to produce. We call these components u and Δu,

respectively. Δu is the **control deficit**, and is known.

$$\boxed{u = u_c + \Delta u} \tag{38.8}$$

Substituting this expression for the control law into the plant equation we get

$$\dot{x}_p = (a_p + b_p\theta)x_p + b_pkr + b_p\Delta u$$

Differentiating the tracking error (38.5) the error dynamics are given by where, differentiating the state error

$$\dot{e} = a_px_p + b_pu - a_mx_m - b_mr$$

Substituting in the control input to the plant from the actuator (38.8) and the matching conditions (38.7) gives

$$\begin{aligned}
\dot{e} &= (a_m - b_p\theta^*)x_p + b_p(u_c + \Delta u) - a_mx_m - b_pk^*r \\
&= a_mx_p - b_p\theta^*x_p + b_pu_c + b_p\Delta u - a_mx_m - b_pk^*r \\
&= a_m(x_p - x_m) - b_p\theta^*x_p + b_pu_c + b_p\Delta u - b_pk^*r
\end{aligned}$$

and now substituting the control law (38.3) in

$$\begin{aligned}
\dot{e} &= a_me - b_p\theta^*x_p + b_p\theta x_p + b_pkr + b_p\Delta u - b_pk^*r \\
&= a_me + b_p(\theta - \theta^*)x_p + b_p(k - k^*)r + b_p\Delta u
\end{aligned}$$

and substituting the parameter errors (38.6) finally gives

$$\boxed{\dot{e} = a_me + b_p\tilde{\theta}x_p + b_p\tilde{k}r + b_p\Delta u} \tag{38.9}$$

The portion of the control signal which the actuator is unable to accommodate, Δu, can be looked at like a disturbance to the system which we have seen before. We would like to design the adaptive scheme to deal with the presence of this Δu. Because our actuator has a known saturation limit, essentially what we want to do is measure the error, but only try to reduce the portion of the error which we actually have the actuator authority to do so. That is, we define the **controllable error** e_u as

$$\boxed{e_u = e - e_\Delta}$$

We then obtain the controllable error e_u by subtracting the **deficit error** e_Δ from the measurable tracking error. The deficit error due to Δu is determined from the following differential equation, where the input β_Δ is unknown.

$$\boxed{\dot{e}_\Delta = a_me_\Delta + \beta_\Delta\Delta u} \tag{38.10}$$

When the control input is saturating, the controller cannot achieve any higher level of performance. That is, the controller should not seek to minimize an error signal which it is unable to do because of the limitation of the actuator. Instead we define the error e_u which takes the state error (the actual error we would like to minimize) and subtracts off the portion of the error due to the "disturbance" Δu which the controller input can do nothing about. This is the error we would like to use to drive adaptation. If the controller is

causing the input to saturate, there is no sense using an error signal which the controller cannot reduce to drive adaptation. The dynamics describing this error are given by

$$\dot{e}_u = \dot{e} - \dot{e}_\Delta$$

Plugging in (38.9) and (38.10)

$$\dot{e}_u = a_m e + b_p \tilde{\theta} x_p + b_p \tilde{k} r + b_p \Delta u - a_m e_\Delta - \beta_\Delta \Delta u$$
$$= a_m e_u + b_p \tilde{\theta} x_p + b_p \tilde{k} r + (b_p - \beta_\Delta) \Delta u$$

and define the last parameter

$$\boxed{\tilde{\beta} = b_p - \beta_\Delta} \tag{38.11}$$

finally gives

$$\boxed{\dot{e}_u = a_m e_u + b_p \tilde{\theta} x_p + b_p \tilde{k} r + \tilde{\beta} \Delta u} \tag{38.12}$$

This is the error we want to try to minimize. Now propose the following candidate Lyapunov function

$$\boxed{V(e_u, \tilde{\theta}, \tilde{k}, \tilde{\beta}) = \frac{1}{2} e_u{}^2 + \frac{1}{2} |b_p| \gamma_1^{-1} \tilde{\theta}^2 + \frac{1}{2} |b_p| \gamma_2^{-1} \tilde{k}^2 + \frac{1}{2} \gamma_3^{-1} \tilde{\beta}^2}$$

taking the time derivative

$$\dot{V} = e_u \dot{e}_u + |b_p| \gamma_1^{-1} \tilde{\theta} \dot{\tilde{\theta}} + |b_p| \gamma_2^{-1} \tilde{k} \dot{\tilde{k}} + \gamma_3^{-1} \tilde{\beta} \dot{\tilde{\beta}}$$

substituting in (38.12)

$$\dot{V} = e_u(a_m e_u + b_p \tilde{\theta} x_p + b_p \tilde{k} r + \tilde{\beta} \Delta u) + |b_p| \gamma_1^{-1} \tilde{\theta} \dot{\tilde{\theta}} + |b_p| \gamma_2^{-1} \tilde{k} \dot{\tilde{k}} + \gamma_3^{-1} \tilde{\beta} \dot{\tilde{\beta}}$$
$$= a_m e_u{}^2 + b_p \tilde{\theta} e_u x_p + b_p \tilde{k} e_u r + \tilde{\beta} e_u \Delta u + |b_p| \gamma_1^{-1} \tilde{\theta} \dot{\tilde{\theta}} + |b_p| \gamma_2^{-1} \tilde{k} \dot{\tilde{k}} + \gamma_3^{-1} \tilde{\beta} \dot{\tilde{\beta}}$$

Now the goal is to determine adaptive update laws for each of the parameters $\tilde{\theta}$, $\tilde{k}$, and $\tilde{\beta}$ such that all of the terms except the quadratic in e_u are left. That is

$$b_p \tilde{\theta} e_u x_p + |b_p| \gamma_1^{-1} \tilde{\theta} \dot{\tilde{\theta}} = 0$$
$$b_p \tilde{k} e_u r + |b_p| \gamma_2^{-1} \tilde{k} \dot{\tilde{k}} = 0$$
$$\tilde{\beta} e_u \Delta u + \gamma_3^{-1} \tilde{\beta} \dot{\tilde{\beta}} = 0$$

So this results in the following update laws

$$\dot{\tilde{\theta}} = -\gamma_1 \text{sgn}(b_p) e_u x_p$$
$$\dot{\tilde{k}} = -\gamma_2 \text{sgn}(b_p) e_u r$$
$$\dot{\tilde{\beta}} = -\gamma_3 e_u \Delta u$$

resulting in the following time derivative $\dot{V}$

$$\dot{V} = a_m e_u{}^2 \leq 0$$

And with $\dot{V}$ negative semi-definite, then V is bounded above by $V(t_0)$ and below by zero. Additionally, the arguments of V are also bounded for all $t \geq t_0$. Note that in the update laws, when implementing the sign on $\dot{\beta}_\Delta$ is flipped due to the definition of $\tilde{\beta}$ in (38.11).

38.3.3 Controller Summary

<div style="border:1px solid red; padding:1em;">

Saturation Protection

Plant:	$\dot{x}_p = a_p x_p + b_p u$
Reference Model:	$\dot{x}_m = a_m x_m + b_m r$
Control Law:	$u_c = \theta x_p + kr$
Control input:	$u = \begin{cases} u_c & \text{if } |u_c| \leq u_{\max} \\ u_{\max}\mathrm{sgn}(u_c) & \text{if } |u_c| > u_{\max} \end{cases}$
Control deficit:	$\Delta u = u - u_c$
Tracking Error:	$e = x_p - x_m$
Deficit Error:	$\dot{e}_\Delta = a_m e_\Delta + \beta_\Delta \Delta u$
Controllable Error:	$e_u = e - e_\Delta$
Update laws:	$\dot{k} = -\gamma_1 \mathrm{sgn}(b_p) e_u r$
	$\dot{\theta} = -\gamma_2 \mathrm{sgn}(b_p) e_u x_p$
	$\dot{\beta}_\Delta = \gamma_3 e_u \Delta u$

</div>

38.3.4 Conditions to Ensure Bounded Solutions

Because the parameters are bounded, there exists $\theta_{\max} < \infty$ and $k_{\max} < \infty$ such that for all $t \geq t_0$ the following hold

$$|\tilde{\theta}(t)| < \theta_{\max}$$

$$|\tilde{k}(t)| < k_{\max}$$

These maximum values can be found from the initial value of the Lyapunov equation at t_0. The maximum value of each of these parameters is found assuming all of the others are zero. That is

$$V(t_0) < V_0 = \frac{1}{2}|b_p|\gamma_1^{-1}\tilde{\theta}_{\max}{}^2$$

and

$$V(t_0) < V_0 = \frac{1}{2}|b_p|\gamma_2^{-1}\tilde{k}_{\max}{}^2$$

we can solve for $\tilde{k}_{\max}$ in terms of $\tilde{\theta}_{\max}$

$$\frac{1}{2}|b_p|\gamma_1^{-1}\tilde{\theta}_{\max}{}^2 = \frac{1}{2}|b_p|\gamma_2^{-1}\tilde{k}_{\max}{}^2$$

$$\gamma_1^{-1}\tilde{\theta}_{\max}{}^2 = \gamma_2^{-1}\tilde{k}_{\max}{}^2$$

$$\tilde{k}_{\max} = \sqrt{\frac{\gamma_2}{\gamma_1}}\tilde{\theta}_{\max}$$

$$\alpha = \sqrt{\frac{\gamma_2}{\gamma_1}}$$

$$\tilde{k}_{\max} = \alpha\tilde{\theta}_{\max}$$

and from above we have that

$$\sqrt{V(t_0)} < \sqrt{\frac{|b_p|}{2\gamma_1}}\tilde{\theta}_{\max}$$

We already showed the boundedness of the other parameters, now just have to show it for x_p. We do this by creating a Lyapunov-like function of x_p and show that its time derivative is decreasing. However, the boundedness of x_p is conditional, given that it is initially not too large, and Lyapunov function has a value which is initially not too large, thus ensuring the other parameters are initially not too large.

Propose the following Lyapunov-like function

$$W(x_p) = \frac{1}{2}x_p{}^2$$

taking its time derivative

$$\dot{W}(x_p) = x_p\dot{x}_p$$

Taking the equation above for x_p

$$\dot{x}_p = a_px_p + b_pu_{\text{sat}}$$

Case 1 Now there are two conditions to consider for this analysis. The first one is when the control input does not saturate. That is $|u| \le u_{\max}$ so $u = u_c$ and $\Delta u = 0$ giving

$$u = \theta x_p + kr$$

plugging this in to the plant equation

$$\dot{x}_p = a_px_p + b_p(\theta x_p + kr)$$

add subtract trick

$$\dot{x}_p = a_px_p + b_p\theta x_p + b_pkr - b_pk^*r + b_pk^*r - b_p\theta^*x_p + b_p\theta^*x_p$$

combining terms

$$\dot{x}_p = a_px_p + b_p\tilde{\theta}x_p + b_p\tilde{k}r + b_pk^*r + b_p\theta^*x_p$$
$$= (a_p + b_p\theta^*)x_p + b_p\tilde{\theta}x_p + b_p\tilde{k}r + b_pk^*r$$

Plugging this into $\dot{W}$ we get

$$\dot{W}(x_p) = x_p[(a_p + b_p\theta^*)x_p + b_p\tilde{\theta}x_p + b_p\tilde{k}r + b_pk^*r]$$
$$= (a_p + b_p\theta^*)x_p{}^2 + b_p\tilde{\theta}x_p{}^2 + b_p\tilde{k}rx_p + b_pk^*rx_p$$
$$= a_mx_p{}^2 + b_p\tilde{\theta}x_p{}^2 + b_p\tilde{k}rx_p + b_pk^*rx_p$$

From this equation we see that since a_m is negative, as long $|b_p\tilde{\theta}| < a_m$ then x_p can be made sufficiently large to ensure $\dot{W} < 0$.

$$a_mx_p{}^2 + b_p\tilde{\theta}_{\max}x_p{}^2 + b_p\tilde{k}rx_p + b_pk^*rx_p < 0$$

Additionally, to find the appropriate lower bound on x_p, we must assume "worst case" in the above expression. That is

$$a_m x_p{}^2 + b_p \tilde{\theta}_{\max} x_p{}^2 + b_p \tilde{k} r x_p + b_p k^* r x_p < (a_m + |b_p| \tilde{\theta}_{\max}) x_p{}^2 + |b_p| \tilde{k}_{\max} r_{\max} x_p + |b_p||k^*| r_{\max} x_p = 0$$

where

$$|\tilde{k}| < \tilde{k}_{\max} \quad \forall t \qquad |r| < r_{\max} \quad \forall t$$

Also, since a_m is negative we have that $a_m = -|a_m|$ and we are going to solve for $x_p = x_{\min}$

$$(-|a_m| + |b_p|\tilde{\theta}_{\max})x_p + (|b_p|\tilde{k}_{\max} r_{\max} x_p + |b_p||k^*| r_{\max}) = 0$$

and also with $\tilde{k}_{\max} = \tilde{\theta}_{\max}$

$$x_{\min} = \frac{|b_p| r_{\max}(\alpha \tilde{\theta}_{\max} + |k^*|)}{|a_m| - |b_p|\tilde{\theta}_{\max}}$$

This value $x_{\min}$ is such that $\dot{W} < 0$ when the control input is not saturating. As long as $x \geq x_{\min}$ then $\dot{W} < 0$. Another constraint on x_p comes from the plant dynamics.

$$\dot{x}_p = a_p x_p + b_p u \leq 0$$

$$|a_p| x_p - |b_p| u_{\max} \leq 0$$

$$x_{\max} = \frac{|b_p|}{|a_p|} u_{\max}$$

Case 2   Now the case where the control input does saturate must be considered. When this happens $\Delta u \neq 0$ and we have

$$u = u_{\max} \text{sgn}(\theta x_p + kr)$$

plugging this into the plant equation with $a_p = a_m - b_p \theta^*$

$$\begin{aligned}
\dot{x}_p &= a_p x_p + b_p u \\
&= (a_m - b_p \theta^*) x_p + b_p u_{\max} \text{sgn}(\theta x_p + kr) \\
&= a_m x_p + b_p u_{\max} \text{sgn}(\theta x_p + kr) - b_p \theta^* x_p
\end{aligned}$$

then

$$\dot{W} = a_m x_p^2 - b_p \theta^* x_p^2 + x_p b_p u_{\max} \text{sgn}(\theta x_p + kr)$$

and with $b_p x_p = |b_p x_p| \text{sgn}(b_p x_p)$ we get

$$\dot{W} = a_m x_p^2 - b_p \theta^* x_p^2 + u_{\max} |b_p x_p| \text{sgn}(b_p x_p) \text{sgn}(\theta x_p + kr)$$

Now there are two cases to consider.

(i) Case: $\text{sgn}(\theta x_p + kr) = -\text{sgn}(b_p x_p)$

(ii) Case: $\text{sgn}(\theta x_p + kr) = \text{sgn}(b_p x_p)$

Looking at case (i) first we get

$$\dot{W} = a_m x_p^2 - b_p \theta^* x_p^2 - u_{\max}|b_p x_p|$$

again want $\dot{W} < 0$ we we must satisfy

$$|a_p x_p{}^2| < |b_p x_p u_{\max}|$$

$$|a_p||x_p||x_p| < |b_p||x_p||u_{\max}|$$

$$|a_p||x_p| < |b_p||u_{\max}|$$

$$|x_p| < \frac{|b_p|u_{\max}}{|a_p|}$$

$$x_{\max} = \frac{|b_p|}{|a_p|}u_{\max}$$

Looking at case (ii) $\operatorname{sgn}(u) = \operatorname{sgn}(\theta x_p + kr) = \operatorname{sgn}(b_p x_p)$ and since $|u| = |\theta x_p + kr| > u_{\max}$ then

$$|u||b_p x_p| > u_{\max}|b_p x_p|$$

and the case we are considering gives

$$|b_p x_p| = b_p x_p \operatorname{sgn}(b_p x_p) = b_p x_p \operatorname{sgn}(\theta x_p + kr) = b_p x_p \operatorname{sgn}(u)$$

$$|u|\operatorname{sgn}(u)b_p x_p > u_{\max}|b_p x_p|$$

$$u b_p x_p > u_{\max}|b_p x_p|$$

$$u = \theta x_p + kr$$

$$\theta = \tilde{\theta} + \theta^*$$

$$u = (\tilde{\theta}x_p + \theta^* x_p + kr)$$

$$(\tilde{\theta}x_p + \theta^* x_p + kr)b_p x_p > u_{\max}|b_p x_p|$$

$$b_p \tilde{\theta}x_p^2 + \theta^* x_p^2 + b_p kr x_p > u_{\max}|b_p x_p|$$

Also with $\operatorname{sgn}(u) = \operatorname{sgn}(b_p x_p)$ then we get

$$\dot{W} = a_m x_p{}^2 - b_p \theta^* x_p{}^2 + u_{\max}|b_p x_p|$$

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38.3.5   Simulation Results
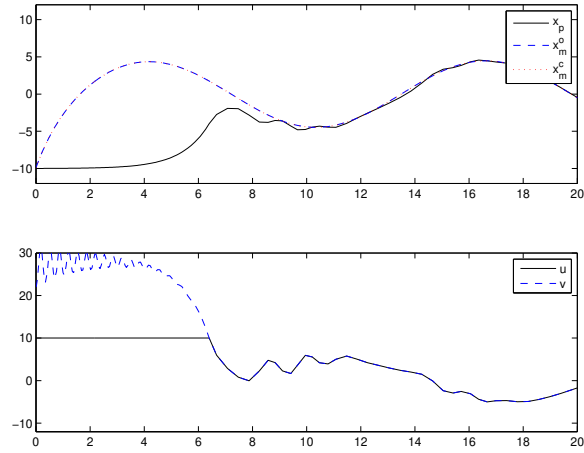
Figure 38.1: ORM: State and input
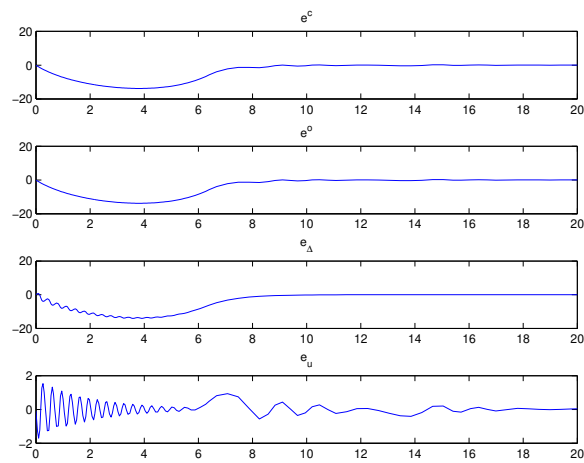
Figure 38.2: ORM: errors

450
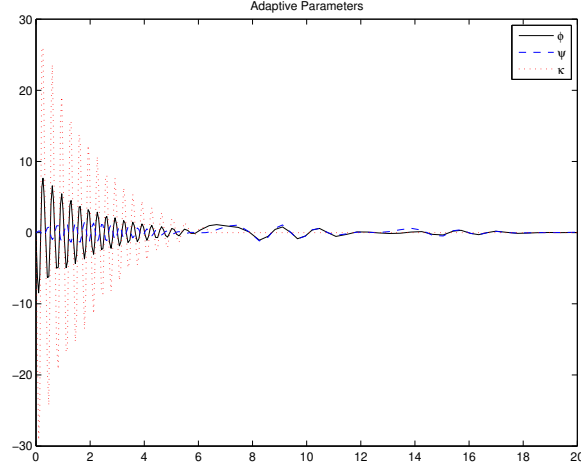
Figure 38.3: ORM: Parameters

38.3.6 Extension to Closed-Loop Reference Model

Setting $\ell = 0$ recovers the normal open-loop reference model case.

$$\dot{e}_\Delta = (a_m + \ell)e_\Delta + \beta_\Delta \Delta u$$

When the control input is saturating, the controller cannot achieve any higher level of performance. That is, the controller should not seek to minimize an error signal which it is unable to do because of the limitation of the actuator. Instead we define the error e_u which takes the state error (the actual error we would like to minimize) and subtracts off the portion of the error due to the "disturbance" Δu which the controller input can do nothing about. This is the error we would like to use to drive adaptation. If the controller is causing the input to saturate, there is no sense using an error signal which the controller cannot reduce to drive adaptation.

$$e_u = e^c - e_\Delta$$

The dynamics describing this error are given by

$$\dot{e}_u = \dot{e}^c - \dot{e}_\Delta$$

where, differentiating the state error

$$\dot{e}^c = \dot{x}_p - \dot{x}_m^c$$

plugging in

$$\dot{e}^c = a_p x_p + b_p u - a_m x_m^c - b_m r + \ell(x_p - x_m^c)$$

Substituting in the control law $u = u_c + \Delta u$ and the matching conditions $a_p = a_m - b_p \theta^*$ and $b_m = b_p k^*$ we get

$$\dot{e}^c = (a_m - b_p \theta^*)x_p + b_p(u_c + \Delta u) - a_m x_m^c - b_p k^* r + \ell(x_p - x_m^c)$$

and then

$$\dot{e}^c = a_m x_p - b_p \theta^* x_p + b_p u_c + b_p \Delta u - a_m x_m^c - b_p k^* r + \ell x_p - \ell x_m^c$$

and then

$$\dot{e}^c = (a_m + \ell)(x_p - x_m^c) - b_p \theta^* x_p + b_p u_c + b_p \Delta u - b_p k^* r$$

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and now substituting the control law $u_c = \theta x_p + kr$ in

$$\dot{e}^c = (a_m + \ell)e^c - b_p\theta^* x_p + b_p\theta x_p + b_p kr + b_p\Delta u - b_p k^* r$$

and then

$$\dot{e}^c = (a_m + \ell)e^c + b_p(\theta - \theta^*)x_p + b_p(k - k^*)r + b_p\Delta u$$

Now define the following parameter errors

$$\boxed{\tilde{\theta} = \theta - \theta^*}$$

$$\boxed{\tilde{k} = k - k^*}$$

and substituting these in

$$\dot{e}^c = (a_m + \ell)e^c + b_p\tilde{\theta}x_p + b_p\tilde{k}r + b_p\Delta u$$

Now returning to $\dot{e}_u$

$$\dot{e}_u = (a_m + \ell)e^c + b_p\tilde{\theta}x_p + b_p\tilde{k}r + b_p\Delta u - (a_m + \ell)e_\Delta - \beta_\Delta\Delta u$$

and then

$$\dot{e}_u = (a_m + \ell)e_u + b_p\tilde{\theta}x_p + b_p\tilde{k}r + (b_p - \beta_\Delta)\Delta u$$

and define the last parameter

$$\boxed{\tilde{\beta} = b_p - \beta_\Delta}$$

and then

$$\dot{e}_u = (a_m + \ell)e_u + b_p\tilde{\theta}x_p + b_p\tilde{k}r + \tilde{\beta}\Delta u$$

This is the error we want to try to minimize. Now propose the following candidate Lyapunov function

$$\boxed{V(e_u, \tilde{\theta}, \tilde{k}, \tilde{\beta}) = \frac{1}{2}e_u{}^2 + \frac{1}{2}|b_p|\gamma_1^{-1}\tilde{\theta}^2 + \frac{1}{2}|b_p|\gamma_2^{-1}\tilde{k}^2 + \frac{1}{2}\gamma_3^{-1}\tilde{\beta}^2}$$

taking the time derivative

$$\dot{V} = e_u\dot{e}_u + |b_p|\gamma_1^{-1}\tilde{\theta}\dot{\tilde{\theta}} + |b_p|\gamma_2^{-1}\tilde{k}\dot{\tilde{k}} + \gamma_3^{-1}\tilde{\beta}\dot{\tilde{\beta}}$$

substituting in $\dot{e}_u$

$$\dot{V} = e_u[(a_m + \ell)e_u + b_p\tilde{\theta}x_p + b_p\tilde{k}r + \tilde{\beta}\Delta u] + |b_p|\gamma_1^{-1}\tilde{\theta}\dot{\tilde{\theta}} + |b_p|\gamma_2^{-1}\tilde{k}\dot{\tilde{k}} + \gamma_3^{-1}\tilde{\beta}\dot{\tilde{\beta}}$$

and then

$$\dot{V} = (a_m + \ell)e_u{}^2 + b_p\tilde{\theta}e_u x_p + b_p\tilde{k}e_u r + \tilde{\beta}e_u\Delta u + |b_p|\gamma_1^{-1}\tilde{\theta}\dot{\tilde{\theta}} + |b_p|\gamma_2^{-1}\tilde{k}\dot{\tilde{k}} + \gamma_3^{-1}\tilde{\beta}\dot{\tilde{\beta}}$$

Now the goal is to determine adaptive update laws for each of the parameters $\tilde{\theta}$, $\tilde{k}$, and $\tilde{\beta}$ such that all of the terms except the quadratic in e_u are left. That is

$$b_p\tilde{\theta}e_u x_p + |b_p|\gamma_1^{-1}\tilde{\theta}\dot{\tilde{\theta}} = 0$$

$$b_p\tilde{k}e_u r + |b_p|\gamma_2^{-1}\tilde{k}\dot{\tilde{k}} = 0$$

$$\tilde{\beta}e_u\Delta u + \gamma_3^{-1}\tilde{\beta}\dot{\tilde{\beta}} = 0$$

So this results in the following update laws

$$\dot{\tilde{\theta}} = -\gamma_1\text{sgn}(b_p)e_u x_p$$

$$\dot{\tilde{k}} = -\gamma_2\text{sgn}(b_p)e_u r$$

$$\dot{\tilde{\beta}} = -\gamma_3 e_u\Delta u$$

resulting in the following time derivative $\dot{V}$

$$\dot{V} = (a_m + \ell)e_u{}^2 \leq 0$$

And with $\dot{V}$ negative semi-definite, then V is bounded above by $V(t_0)$ and below by zero. Additionally, the arguments of V are also bounded for all $t \geq t_0$.

Simulation Results
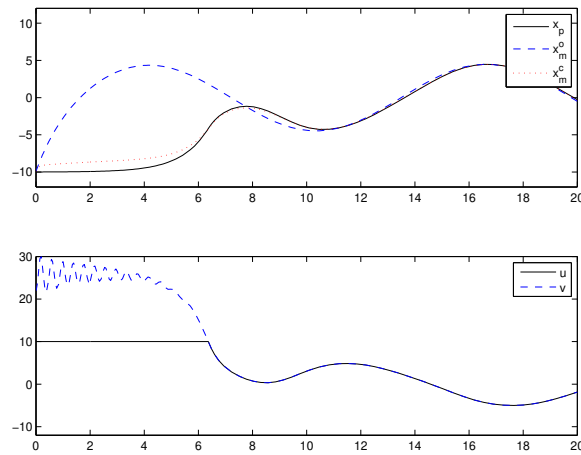
Figure 38.4: CRM: State and input
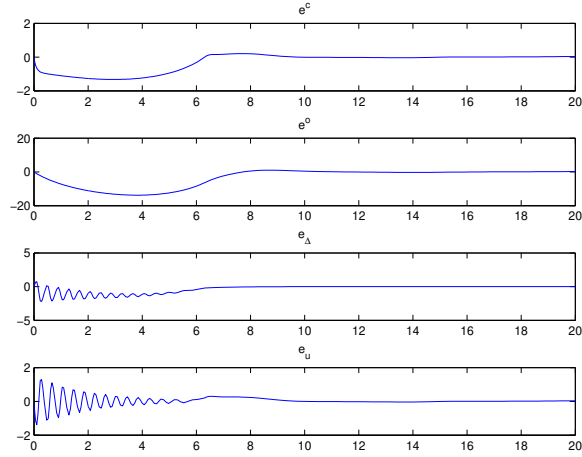
Figure 38.5: CRM: errors
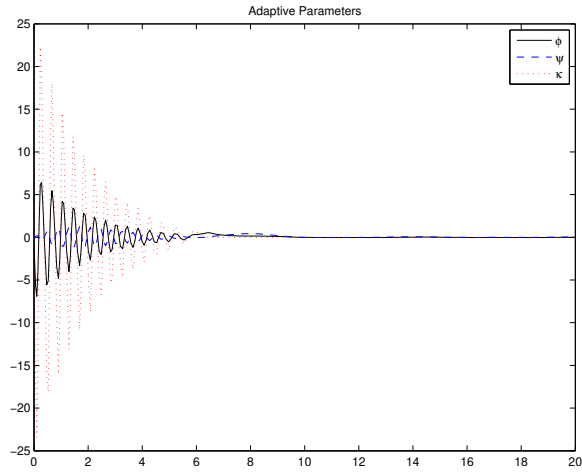
Figure 38.6: CRM: Parameters

38.3.7 Saturation Protection Higher Order with CRM

Plant to be controlled

$$\dot{x} = (A + B_1 \Lambda W^\mathsf{T})x + B_1 \Lambda u + B_2 x_\mathrm{cmd}$$

commanded control law

$$u_c = (\theta + K_x)^\mathsf{T} x$$

actual control input to the plant due to the position saturation of the actuators

$$u = \begin{cases} u_c & \text{if } |u_c| \leq u_\mathrm{max} \\ u_\mathrm{max}\mathrm{sgn}(u_c) & \text{if } |u_c| > u_\mathrm{max} \end{cases}$$

Δu is the control deficit.

$$\boxed{u = u_c + \Delta u}$$

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Plugging in the control into the plant equation and $r = x_{\mathrm{cmd}}$

$$\dot{x} = (A + B_1 \Lambda W^{\mathsf{T}})x + B_1 \Lambda (u_c + \Delta u) + B_2 r$$

and then

$$\dot{x} = (A + B_1 \Lambda W^{\mathsf{T}})x + B_1 \Lambda u_c + B_1 \Lambda \Delta u + B_2 r$$

The reference model is given by

$$\dot{x}_m^c = A_m x_m^c + B_m x_{\mathrm{cmd}} - L(x - x_m^c)$$

$$\dot{x}_m^c = A_m x_m^c + B_2 r - L(x - x_m^c)$$

where

$$A_m = A + B_1 K_x^{\mathsf{T}}$$

I am using a closed-loop reference model

$$\boxed{\dot{e}_\Delta = (A_m + L)e_\Delta + \beta_\Delta \Delta u}$$

and

$$\boxed{e_u = e^c - e_\Delta}$$

The dynamics describing this error are given by

$$\dot{e}_u = \dot{e}^c - \dot{e}_\Delta$$

where, differentiating the state error

$$\dot{e}^c = \dot{x} - \dot{x}_m^c$$

plugging in

$$\dot{e}^c = (A + B_1 \Lambda W^{\mathsf{T}})x + B_1 \Lambda u_c + B_1 \Lambda \Delta u + B_2 r - [A_m x_m^c + B_2 r - L(x - x_m^c)]$$

and then

$$\dot{e}^c = (A + B_1 \Lambda W^{\mathsf{T}})x + B_1 \Lambda u_c + B_1 \Lambda \Delta u - A_m x_m^c + L(x - x_m^c)$$

Using $A = A_m - B_1 K_x^{\mathsf{T}}$ and $u_c = (\theta + K_x)^{\mathsf{T}} x$

$$\dot{e}^c = (A_m - B_1 K_x^{\mathsf{T}} + B_1 \Lambda W^{\mathsf{T}})x + B_1 \Lambda (\theta + K_x)^{\mathsf{T}} x + B_1 \Lambda \Delta u - A_m x_m^c + L(x - x_m^c)$$

and then

$$\dot{e}^c = A_m x + (-B_1 K_x^{\mathsf{T}} + B_1 \Lambda W^{\mathsf{T}})x + B_1 \Lambda \theta^{\mathsf{T}} x + B_1 \Lambda K_x^{\mathsf{T}} x + B_1 \Lambda \Delta u - A_m x_m^c + L(x - x_m^c)$$

and then

$$\dot{e}^c = A_m(x - x_m^c) + (B_1 \Lambda \theta^{\mathsf{T}} + B_1 \Lambda K_x^{\mathsf{T}} - B_1 K_x^{\mathsf{T}} + B_1 \Lambda W^{\mathsf{T}})x + B_1 \Lambda \Delta u + L(x - x_m^c)$$

and then

$$\dot{e}^c = (A_m + L)(x - x_m^c) + (B_1 \Lambda \theta^{\mathsf{T}} + B_1 \Lambda K_x^{\mathsf{T}} - B_1 K_x^{\mathsf{T}} + B_1 \Lambda W^{\mathsf{T}})x + B_1 \Lambda \Delta u$$

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and then
$$\dot{e}^c = (A_m + L)(x - x_m^c) + B_1(\Lambda\theta^\mathsf{T} + \Lambda K_x^\mathsf{T} - K_x^\mathsf{T} + \Lambda W^\mathsf{T})x + B_1\Lambda\Delta u$$

and from the matching condition we have

$$\Lambda\left[W^\mathsf{T} + (\theta^* + K_x)^\mathsf{T}\right] = K_x^\mathsf{T}$$

which we can rearrange

$$\Lambda W^\mathsf{T} + \Lambda\theta^{*\mathsf{T}} + \Lambda K_x^\mathsf{T} = K_x^\mathsf{T}$$

and then

$$\Lambda K_x^\mathsf{T} - K_x^\mathsf{T} + \Lambda W^\mathsf{T} = -\Lambda\theta^{*\mathsf{T}}$$

substituting this in
$$\dot{e}^c = (A_m + L)(x - x_m^c) + B_1(\Lambda\theta^\mathsf{T} - \Lambda\theta^{*\mathsf{T}})x + B_1\Lambda\Delta u$$

and with $\tilde{\theta} = \theta - \theta^*$ we get

$$\dot{e}^c = (A_m + L)(x - x_m^c) + B_1\Lambda\tilde{\theta}^\mathsf{T}x + B_1\Lambda\Delta u$$

$$\dot{e}^c = (A_m + L)e^c + B_1\Lambda\tilde{\theta}^\mathsf{T}x + B_1\Lambda\Delta u$$

And using $\dot{e}_u = \dot{e}^c - \dot{e}_\Delta$

$$\dot{e}_u = (A_m + L)e^c + B_1\Lambda\tilde{\theta}^\mathsf{T}x + B_1\Lambda\Delta u - [(A_m + L)e_\Delta + \beta_\Delta\Delta u]$$

and then

$$\dot{e}_u = (A_m + L)e^c + B_1\Lambda\tilde{\theta}^\mathsf{T}x + B_1\Lambda\Delta u - (A_m + L)e_\Delta - \beta_\Delta\Delta u$$

and then

$$\dot{e}_u = (A_m + L)(e^c - e_\Delta) + B_1\Lambda\tilde{\theta}^\mathsf{T}x + B_1\Lambda\Delta u - \beta_\Delta\Delta u$$

and then

$$\dot{e}_u = (A_m + L)e_u + B_1\Lambda\tilde{\theta}^\mathsf{T}x + (B_1\Lambda - \beta_\Delta)\Delta u$$

defining $\tilde{\beta} = B_1\Lambda - \beta_\Delta$ we get the following augmented error dynamics

$$\dot{e}_u = (A_m + L)e_u + B_1\Lambda\tilde{\theta}^\mathsf{T}x + \tilde{\beta}\Delta u$$

$$\boxed{\dot{e}_u = \bar{A}_m e_u + B_1\Lambda\tilde{\theta}^\mathsf{T}x + \tilde{\beta}\Delta u}$$

Proposing the following candidate Lyapunov function

$$\boxed{V = e_u^\mathsf{T}Pe_u + \mathrm{tr}\left(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\tilde{\theta}|\Lambda|\right) + \tilde{\beta}^\mathsf{T}\Gamma_\beta^{-1}\tilde{\beta}}$$

Differentiating

$$\dot{V} = \dot{e}_u^\mathsf{T}Pe_u + e_u^\mathsf{T}P\dot{e}_u + \mathrm{tr}(\dot{\tilde{\theta}}^\mathsf{T}\Gamma^{-1}\tilde{\theta}|\Lambda|) + \mathrm{tr}(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|) + \dot{\tilde{\beta}}^\mathsf{T}\Gamma_\beta^{-1}\tilde{\beta} + \tilde{\beta}^\mathsf{T}\Gamma_\beta^{-1}\dot{\tilde{\beta}}$$

Substituting error dynamics equation from above, with the transpose

$$\dot{e}_u^\mathsf{T} = e_u^\mathsf{T}\bar{A}_m^\mathsf{T} + x^\mathsf{T}\tilde{\theta}\Lambda B_1^\mathsf{T} + \Delta u^\mathsf{T}\tilde{\beta}^\mathsf{T}$$

and simplifying

$$\dot{V} = (e_u^\mathsf{T}\bar{A}_m^\mathsf{T}+x^\mathsf{T}\tilde{\theta}\Lambda B_1^\mathsf{T}+\Delta u^\mathsf{T}\tilde{\beta}^\mathsf{T})Pe_u+e_u^\mathsf{T}P(\bar{A}_m e_u+B_1\Lambda\tilde{\theta}^\mathsf{T}x+\tilde{\beta}\Delta u)+\text{tr}(\dot{\tilde{\theta}}^\mathsf{T}\Gamma^{-1}\tilde{\theta}|\Lambda|)+\text{tr}(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)+\dot{\tilde{\beta}}^\mathsf{T}\Gamma_\beta^{-1}\tilde{\beta}+\tilde{\beta}^\mathsf{T}\Gamma$$

and then

$$\dot{V} = (e_u^\mathsf{T}\bar{A}_m^\mathsf{T}Pe_u)+(x^\mathsf{T}\tilde{\theta}\Lambda B_1^\mathsf{T}+\Delta u^\mathsf{T}\tilde{\beta}^\mathsf{T})Pe_u+(e_u^\mathsf{T}P\bar{A}_m e_u)+e_u^\mathsf{T}P(B_1\Lambda\tilde{\theta}^\mathsf{T}x+\tilde{\beta}\Delta u)+\text{tr}(\dot{\tilde{\theta}}^\mathsf{T}\Gamma^{-1}\tilde{\theta}|\Lambda|)+\text{tr}(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)+\dot{\tilde{\beta}}^\mathsf{T}$$

and then

$$\dot{V} = e_u^\mathsf{T}(\bar{A}_m^\mathsf{T}P+P\bar{A}_m)e_u+(x^\mathsf{T}\tilde{\theta}\Lambda B_1^\mathsf{T}+\Delta u^\mathsf{T}\tilde{\beta}^\mathsf{T})Pe_u+e_u^\mathsf{T}P(B_1\Lambda\tilde{\theta}^\mathsf{T}x+\tilde{\beta}\Delta u)+\text{tr}(\dot{\tilde{\theta}}^\mathsf{T}\Gamma^{-1}\tilde{\theta}|\Lambda|)+\text{tr}(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)+\dot{\tilde{\beta}}^\mathsf{T}\Gamma_\beta^{-1}\tilde{\beta}-$$

and with $\bar{A}_m^\mathsf{T}P + P\bar{A}_m = -Q$ we get

$$\dot{V} = -e_u^\mathsf{T}Qe_u+x^\mathsf{T}\tilde{\theta}\Lambda B_1^\mathsf{T}Pe_u+\Delta u^\mathsf{T}\tilde{\beta}^\mathsf{T}Pe_u+e_u^\mathsf{T}PB_1\Lambda\tilde{\theta}^\mathsf{T}x+e_u^\mathsf{T}P\tilde{\beta}\Delta u+\text{tr}(\dot{\tilde{\theta}}^\mathsf{T}\Gamma^{-1}\tilde{\theta}|\Lambda|)+\text{tr}(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)+\dot{\tilde{\beta}}^\mathsf{T}\Gamma_\beta^{-1}\tilde{\beta}+\tilde{\beta}^\mathsf{T}\Gamma_\beta$$

since the terms of $\dot{V}$ are all scalars we have

$$\dot{V} = -e_u^\mathsf{T}Qe_u + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1^\mathsf{T}Pe_u + 2\Delta u^\mathsf{T}\tilde{\beta}^\mathsf{T}Pe_u + \text{tr}(\dot{\tilde{\theta}}^\mathsf{T}\Gamma^{-1}\tilde{\theta}|\Lambda|) + \text{tr}(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|) + 2\tilde{\beta}^\mathsf{T}\Gamma_\beta^{-1}\dot{\tilde{\beta}}$$

The following adaptive control gain update laws are proposed

$$\boxed{\dot{\theta} = \text{Proj}_\Gamma(\theta, -\Gamma x e_u^\mathsf{T}PB_1\text{sgn}(\Lambda))}$$

$$\boxed{\dot{\tilde{\beta}} = -\Gamma_\beta Pe_u\Delta u}$$

Looking at only the $\tilde{\beta}$ terms

$$2\Delta u^\mathsf{T}\tilde{\beta}^\mathsf{T}Pe_u - 2\tilde{\beta}^\mathsf{T}\Gamma_\beta^{-1}\Gamma_\beta Pe_u\Delta u =$$

and since we are considering scalar input system

$$2\Delta u^\mathsf{T}\tilde{\beta}^\mathsf{T}Pe_u - 2\Delta u\tilde{\beta}^\mathsf{T}Pe_u = 0$$

and so $\dot{V}$ becomes

$$\dot{V} = -e_u^\mathsf{T}Qe_u + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1^\mathsf{T}Pe_u + \text{tr}(\dot{\tilde{\theta}}^\mathsf{T}\Gamma^{-1}\tilde{\theta}|\Lambda|) + \text{tr}(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)$$

using $\text{tr}(a) = \text{tr}(a^\mathsf{T})$, $\Lambda = \Lambda^\mathsf{T}$, $\Gamma = \Gamma^\mathsf{T}$, and $\text{tr}(ab) = \text{tr}(ba)$ we get

$$\dot{V} = -e_u^\mathsf{T}Qe + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1^\mathsf{T}Pe_u + 2\text{tr}(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\dot{\tilde{\theta}}|\Lambda|)$$

Substituting the adaptive control gain update law for θ

$$\dot{V} = -e_u^\mathsf{T}Qe + 2x^\mathsf{T}\tilde{\theta}\Lambda B_1^\mathsf{T}Pe_u + 2\text{tr}\left(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\text{Proj}_\Gamma(\theta, -\Gamma x e_u^\mathsf{T}PB_1\text{sgn}(\Lambda))|\Lambda|\right)$$

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Since the quantity $2x^\mathsf{T}\tilde{\theta}\Lambda B_1{}^\mathsf{T}Pe_u$ is scalar, it is equal to its trace, so we can write

$$\dot{V} = -e_u^\mathsf{T}Qe_u + 2\mathrm{tr}(x^\mathsf{T}\tilde{\theta}\Lambda B_1{}^\mathsf{T}Pe_u) + 2\mathrm{tr}\left(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\mathrm{Proj}_\Gamma(\theta, -\Gamma xe_u^\mathsf{T}PB_1\mathrm{sgn}(\Lambda))|\Lambda|\right)$$

The terms inside the first trace operator can be rearranged using $\mathrm{tr}(a) = \mathrm{tr}(a^\mathsf{T})$, $\Lambda = \Lambda^\mathsf{T}$, and $\mathrm{tr}(ab) = \mathrm{tr}(ba)$

$$\dot{V} = -e_u^\mathsf{T}Qe_u + 2\mathrm{tr}(\tilde{\theta}^\mathsf{T}xe_u^\mathsf{T}PB_1\Lambda) + 2\mathrm{tr}\left(\tilde{\theta}^\mathsf{T}\Gamma^{-1}\mathrm{Proj}_\Gamma(\theta, -\Gamma xe_u^\mathsf{T}PB_1\mathrm{sgn}(\Lambda))|\Lambda|\right)$$

Combining the trace operators

$$\dot{V} = -e_u^\mathsf{T}Qe_u + 2\mathrm{tr}\left(\tilde{\theta}^\mathsf{T}xe^\mathsf{T}PB_1\Lambda + \tilde{\theta}^\mathsf{T}\Gamma^{-1}\mathrm{Proj}_\Gamma(\theta, -\Gamma xe_u^\mathsf{T}PB_1\mathrm{sgn}(\Lambda))|\Lambda|\right)$$

$$\dot{V} = -e_u^\mathsf{T}Qe_u + 2\mathrm{tr}\left(\tilde{\theta}^\mathsf{T}\left(\Gamma^{-1}\mathrm{Proj}_\Gamma(\theta, -\Gamma xe_u^\mathsf{T}PB_1\mathrm{sign}(\Lambda))|\Lambda| + xe_u^\mathsf{T}PB_1\Lambda\right)\right)$$

Let $y = -xe_u^\mathsf{T}PB_1\mathrm{sign}(\Lambda)$ and with $\mathrm{sign}(\Lambda)|\Lambda| = \Lambda$

$$\dot{V} = -e_u^\mathsf{T}Qe_u + 2\mathrm{tr}\left(\tilde{\theta}^\mathsf{T}\left(\Gamma^{-1}\mathrm{Proj}_\Gamma(\theta, \Gamma y)|\Lambda| - y|\Lambda|\right)\right)$$

$$\dot{V} = -e_u^\mathsf{T}Qe_u + 2\mathrm{tr}\left(\tilde{\theta}^\mathsf{T}\left(\Gamma^{-1}\mathrm{Proj}_\Gamma(\theta, \Gamma y) - y\right)|\Lambda|\right)$$

And we have the following inequality (See my personal notes)

$$\tilde{\theta}^\mathsf{T}(\Gamma^{-1}\mathrm{Proj}_\Gamma(\theta, \Gamma y) - y) \le 0$$

So, if a symmetric, positive definite matrix P exists, which solves the Lyapunov equation $\bar{A}_m{}^\mathsf{T}P + P\bar{A}_m + Q = 0$, where Q is positive definite, then $\dot{V} = \dot{V}(e, \theta) \le 0$ is negative semidefinite, and $V(e, \theta)$ serves as a valid Lyapunov function for this system.

Chapter 39

Appendix

Example 71
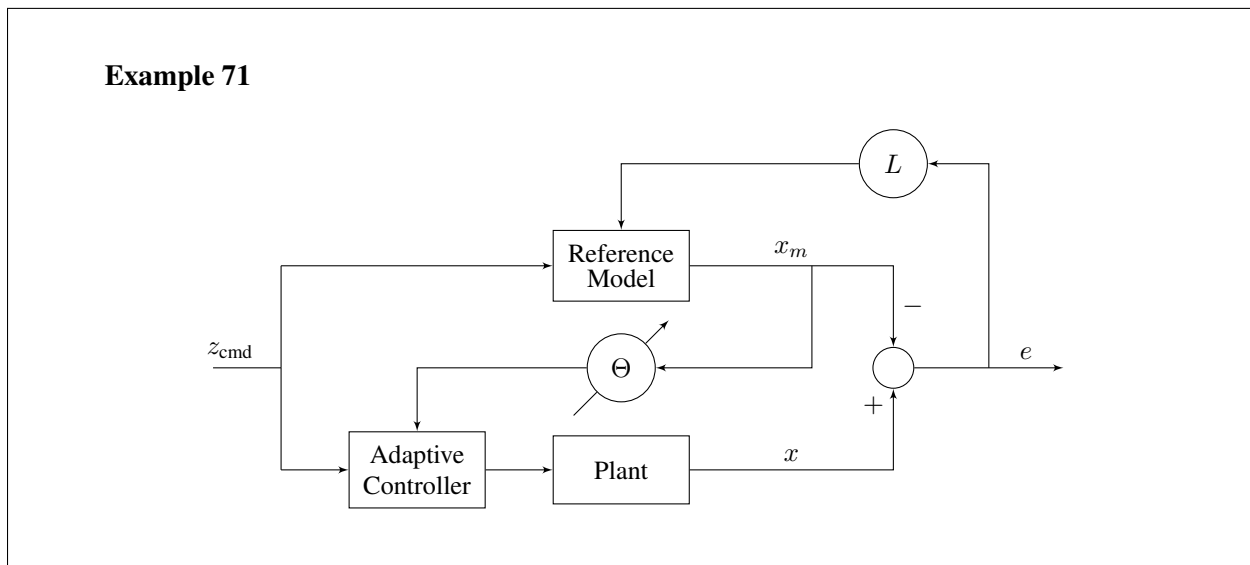
39.1 Eugene Lecture

Last class $n^* \geq 2$

EUGENE STUFF

Linear Control LQR (state feedback) PID (classical, output) H-inf (both) Pole placement (state feedback) LQG (LQG+kalman filter/estimator)

Nonlinear Control SMC (variable structure control) - good for DC motor control, bad for flight control Feedback linearization (dynamic inversion, works great in Matlab, not good in real life... no guarantees for robustness) Lyapunov redesign (mostly for systems with known dynamics, adaptive could kinda be considered to fall under here) Adaptive control

all the control strategies above are model based design/analysis

Plant –¿ model Models are all incorrect... Some more than others. Use model to cook up control, hoping that even though model is incorrect the control will work on plant Plant –¿ model –¿ controller

reason this works is robustness. Robustness is the key.

Dynamic inversion very sensitive. Dynamic inversion might work, but can take a lot of time to design analyze and tune.

H_∞ control on Harrier...

Example 72 X-45A super unstable. Time to departure 20ms. 3 gains in pitch 5 gains in roll/yaw used LQR control.

---

pretty much everything Boeing does (80—90 percent) is LQR

Adaptive control. Does it work? yes. Does it always work? no!

build robust control (e.g. LQR)

to fly all we need is 40* phase margin and 8dB gain margin.

LQR is really good for uncertainties in A and B but is highly sensitive when A and B are state dependent. For example, standing shock wave on top of delta wing as it moves with changes in flight condition.

LQR controller has no tolerance to "bubbles" in, for example, pitching moment versus elevator angle plot because of shock wave.

39.2 Dr. Annaswamy Lecturing

SISO system

u to y through plant P

P is of form $W_p(s) = \frac{k_p Z_p(s)}{R_p(s)}$

degrees of freedom... Filters of dimension $n - 1$

Error model

Introduce auxiliary error e_2, augmented error ε_1 where $\varepsilon_1 = e_1 + e_2$. Use ε_1 instead of e_1 in adaptive law.

Omega in through $W_m(s)$ and get ζ then use $\dot{\theta} = -\varepsilon_1 \zeta$

SHOW BLOCK DIAGRAM THAT GENERATES e_2.

BLOCK DIAGRAM FOR ANALYTIC PART, THAT ALGEBRAIC PART CANCELS

when $e_1 + e_2$: $W_m(s)[\tilde{\theta}^\top] + \tilde{\theta}^\top \zeta - W_m(s)[\tilde{\theta}^\top \omega]$ and assume for simplicity that $k^* = 1$ and so we get $\varepsilon_1 = \tilde{\theta}^\top \zeta$. So now use $\dot{\tilde{\theta}} = -\varepsilon_1 \zeta$.

Started out with adaptive system which was nonlinear and time-varying. And $\tilde{\theta}$ is bounded so this is reduced to a linear time-varying system. Can we then use this to prove stability? Need to take one more step.

Need to allow $\tilde{\theta}$ to grow slowly. Make it the normalized quantity.

39.3 Review

Class review

control of an uncertain dynamic system: adaptive control

two bins:

1. Input, states, output

2. parameters

in adaptive control measure as many of (1) as possible, and deal with as many in (2) as possible must assume there is a plant model, and since parameters are being attached to the model, must assume the structure of the model is known

one unifying theme of adaptive control/identificaation: error model

Two classes of error:

1. parameter

2. tracking

error model is "black box" between parameter error and tracking error. Want this black box to have a nice feature to . . .

1. error model 1: $e = \omega\tilde{\theta}$ e is linear regression of $\tilde{\theta}$

2. error model 2: dynamics between parameter error and e $w = \omega\tilde{\theta}$ then w goes into $\dot{e} = Ae + Bw$

3. error model 3: specific case of error model 2, and now only some part of the error is available e_1. And we needed dynamics to be SPR.

4. error model 4: uses augmented error approach with adding and subtracting. . . for arbitrary relative degree. Add e_1 and e_2 to get ε_1

Adaptive Control (1) First order plant $\frac{1}{s-a_p}$ so know model structure (its first order) put θ in feedback error model becomes x_p -¿ $\tilde{\theta}$ -¿ $\frac{1}{s-a_m}$ -¿ e $\dot{\tilde{\theta}} = -ex_p$

extend to $\frac{k_p}{s-a_p}$ where k_p has known sign, by adding a feedforward part error model becomes ω -¿ $\tilde{\theta}$ -¿ $\frac{k_p}{s-a_m}$ -¿ e $\omega = [r\,x_p]^\top$ $\dot{\tilde{\theta}} = -sgn(k_p)e\omega$

(2) States accessible $\dot{x}_p = A_p x_p + Bu$ $B\Lambda^* = B_m$ where Λ^* is diagonal with the sign of each of the elements on the diagonal is known. $\omega = [r^\top\,x_p^\top]^\top$ ω -¿ $\tilde{\theta}$ -¿ box -¿ e $\dot{\tilde{\theta}} = -sgn(\Lambda^*)\omega e^\top PB_m$

Adaptive Control: PI, PID, Phase-Lead popular for low order plant model simple example: $\frac{1}{s(Js+B)}$ parameterization shown in class shows the dependence of control parameters on plant parameters

Relative degree 2 add z into error model after $\tilde{\theta}$ and before dynamics where $z = \dot{\theta}^\top\bar{\omega}$ so $\bar{\omega}$ is a filtered version of ω, and now we get error model e in terms of this filtered ω

$$\dot{\tilde{\theta}} = -e\bar{\omega}$$
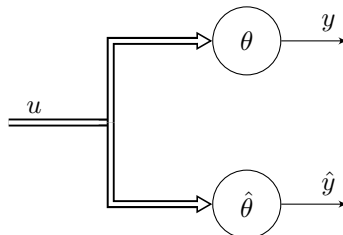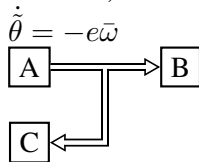
Extra block diagram
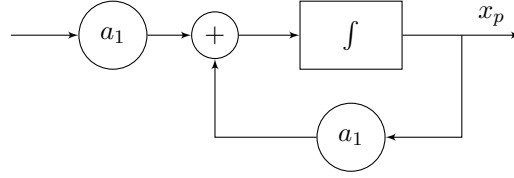
39.3.1 Time Varying Parameters

Consider the following time-varying plant, reference model, and control law

$$\dot{x}_p = a_p(t)x_p + u + d(t)$$
$$\dot{x}_m = a_m x_m + r$$
$$u = \theta x_p + r$$

Gives time-varying matching condition

Lyapunov

$$V = \frac{1}{2}(e^2 + \tilde{\theta}^2)$$

$$\dot{V} = a_m e^2 + \tilde{\theta}\dot{\tilde{\theta}} + \tilde{\theta}e x_p$$

$$\dot{V} = a_m e^2 + \tilde{\theta}\dot{\theta} + \tilde{\theta}e x_p + \tilde{\theta}\dot{\theta}^*$$

Propose

$$\dot{\theta} = -e x_p - \sigma\theta$$

rewrite $\dot{V}$

$$\dot{V} = a_m e^2 + \tilde{\theta}\dot{\theta}^* - \sigma\tilde{\theta}^2 - \sigma\tilde{\theta}\theta^*(t)$$

if we know $|\theta^*(t)| \le d_1$ and $|\dot{\theta}^*(t)| \le d_2$. Rewrite pull out $|\tilde{\theta}|$

$$\dot{V} \le -|a_m|e^2 - \sigma|\tilde{\theta}|\left(|\tilde{\theta}| - \frac{d_2}{\sigma} - d_1\right)$$

Part V

Nonlinear Control

Chapter 40

Nonlinear Control

40.1 Lyapunov Stability

40.1.1 Introduction

There are several ways in which the stability of equilibria can be defined which are outlined in these notes. Only autonomous systems are covered, looking at both continuous and discrete time cases.

Lyapunov Stability Analysis gives two approaches can be taken to analyze a system and see what type stability an equilibrium point satisfies. Lyapunov's first, or indirect method can be used to prove whether a system is stable, unstable, or draw no conclusion about stability. Lyapunov's second, or direct method can only prove system stability.

40.1.2 Stability of Autonomous Systems

When talking about the stability of autonomous systems, it is always done relative to an equilibrium point. Equilibrium points must first be found, and it is the stability of these points which must be studied. For linear systems there exists only one equilibrium, so the stability of this equilibrium point can be equivalently described by saying the stability of the system.

40.1.3 Equilibrium Points

Given the following autonomous systems (one continuous the other discrete), the system's equilibrium points must first be found. (DDV 13.2)

$$\dot{x}(t) = f(x(t)) \tag{40.1}$$

$$x(t+1) = f(x(t)) \tag{40.2}$$

The point x_{eq} is an equilibrium point of the continuous system if $f(x_{\text{eq}}(t)) = 0, \forall t \geq 0$, and an equilibrium point of the discrete system if $f(x_{\text{eq}}(t)) = x_{\text{eq}}, \forall t \geq 0$. If the system is started in the state x_{eq} at time t_0, it will remain there for all time. Nonlinear systems can have multiple equilibrium points (or equilibria), but linear systems can only have one.

40.1.4 Three Types of Stability

1. **Stability in the sense of Lyapunov (ISL)** A system that is stable ISL is one which the system trajectory can be kept close to an equilibrium point by starting sufficiently close to the equilibrium.

- Given any $\epsilon > 0$, $\exists \delta$ such that if $\|x(t_0)\| < \delta$, then $\|x(t_0)\| < \epsilon, \forall t > t_0$

This basically says that if the starting point $x(t_0)$ is inside the circle centered about $\bar{x}$ with radius δ, that the system trajectory can leave this region and into the larger circle with radius ϵ, but it can never leave that region. This is the weakest form of stability, and is also known as *marginally stable*. It is important to make the point that this must hold for *any* ϵ that can be picked, not just one particular and carefully selected special case. An equilibrium point that is not stable ISL is termed unstable. (DDV 13.2)

2. **Local asymptotic stability** A system which is stable ISL, and satisfies the *additional* constraint below is called locally asymptotically stable.

- $\exists r$ such that if $\|x(t_0)\| < r$, then $x(t) \to \bar{x}$ as $t \to \infty$

This statement says that if the starting point $x(t_0)$ is inside the circle centered about $\bar{x}$ with radius r, that the system trajectories will actually converge to $\bar{x}$. It is important to note that there exist systems which satisfy only this additional constraint *without* satisfying the first constraint of being stable ISL. Such systems are *not* asymptotically stable.

3. **Global asymptotic stability** A system which is globally asymptotically stable extends the definition of local asymptotic stability from a circle of radius r to the entire state space. In other words, beginning from *any* initial conditions $x(t_0)$ then $x(t) \to \bar{x}$ as $t \to \infty$. This is discussed in further detail using Lyapunov's second method.

40.2 Lyapunov Stability Analysis

Using these three definitions of stability, tools are now needed which will allow a system to be analyzed to determine if an equilibrium is stable, and if so, which type of stability the equilibrium point satisfies.

40.2.1 Lyapunov's First (Indirect) Method

This method involves linearizing the nonlinear system about an equilibrium point $\bar{x}$ in order to develop a local conclusion about the stability of the *nonlinear* system. If the linearized system has poles that are all strictly in the left-half complex plane, the equilibrium point is locally asymptotically stable. If the linearized system has any poles that are strictly in the right-half complex plane, equilibrium point is unstable. If the linearized system as any eigenvalues which are zero, no conclusion can be drawn about the stability of the equilibrium point. In this case, essentially the higher order terms that were lost in linearization will determine whether or not the equilibrium is stable or not. (DDV 14.3)

Stability of the linearized system The linearized system, with some eigenvalues on the imaginary axis $\lambda_i = 0$, can be unstable, while the nonlinear system is actually stable. This is why no conclusion about the nonlinear system can be drawn when any eigenvalues are zero. *Can the opposite be true? i.e. the linearized system with any $\lambda_i = 0$ is stable, but the nonlinear system is unstable? example?* The stability of the linear systems themselves when eigenvalues are on the imaginary axis will be analyzed in more detail later.

40.2.2 Lyapunov's Second (Direct) Method

Continuous Time Lyapunov's second method requires the construction of a scalar, energy like Lyapunov function of the state which satisfies the properties which follow. This function $V(x(t))$ is proposed as a "candidate Lyapunov function", and if the properties are satisfied, it becomes a Lyapunov function.

- V is locally positive definite

    - $V(0) = 0$
    - $V(x(t)) > 0, 0 < \|x(t)\| < r$ for some r

- $\dot{V}(x(t)) = \frac{d}{dt}V(x(t)) = \frac{d}{dx}V(x(t))\frac{dx}{dt}$ is locally negative semidefinite

    - $\dot{V}(0) = 0$
    - $\dot{V}(x(t)) \leq 0, 0 < \|x(t)\| < r$ for some r

The Lyapunov function which satisfies these three conditions proves the equilibrium point is locally stable ISL. The condition of stability can be further improved if $\dot{V}(x(t))$ is negative definite, i.e. $\dot{V}(x(t)) < 0, 0 < \|x(t)\| < r$ for some r. Satisfying this condition results in asymptotic stability. (DDV 13.4)

Lyapunov's second method can be extended to prove global stability if the function $|V(x)| \to \infty$ as $\|x\| \to \infty$ (i.e. $V(x(t))$ is radially unbounded) *and* $\dot{V}(x(t))$ is negative definite on the entire state space.

If a Lyapunov function cannot be found, this does not necessarily mean that the system is unstable, but only that a suitable Lyapunov function could not be found. Therefore, Lyapunov's direct method cannot be used to prove a system is unstable.

Discrete time $\dot{V}(x(t)) \triangleq V(x(t+1)) - V(x(t))$

40.2.3 The Lyapunov Equation

Continuous time To prove stability of the following continuous time, linear, autonomous system, a quadratic Lyapunov function will suffice.

$$\dot{x}(t) = Ax(t)$$

Propose the following quadratic Lyapunov function, where P must be chosen such that it is positive definite (i.e. $x^\mathsf{T} P x > 0 \; \forall x \neq 0$).

$$V(x(t)) = x^\mathsf{T} P x, \quad x \in \mathbb{R}^n$$

As long as P is positive definite $V(x(t))$ will be a suitable Lyapunov function. Taking the time derivative of the Lyapunov function, and substituting $\dot{x} = Ax$ gives:

$$\begin{aligned}
\dot{V}(x) &= \dot{x}^\mathsf{T} P x + x^\mathsf{T} P \dot{x} \\
&= (Ax)^\mathsf{T} P x + x^\mathsf{T} P A x \\
&= x^\mathsf{T} A^\mathsf{T} P x + x^\mathsf{T} P A x \\
&= x^\mathsf{T} (A^\mathsf{T} P + P A) x \\
&= x^\mathsf{T} Q x
\end{aligned}$$

The resulting matrix $Q = (A^\mathsf{T} P + P A)$ is symmetric as well. By picking P such that it is not only symmetric and positive definite, but such that Q is negative definite, the quadratic Lyapunov function will prove the linear system is globally asymptotically stable.

There are no other restrictions on the selection of P, so it desired to select a P which is symmetric?

Discrete time For the following linear, discrete time, autonomous system, a Lyapunov function is desired.

$$x(t+1) = Ax(t)$$

Propose the following quadratic Lyapunov function

$$V(x(t)) = x^{\mathsf{T}}Px, \quad x \in \mathbb{R}^n$$

$$\begin{aligned}
\dot{V}(x(t)) &\triangleq V(x(t+1)) - V(x(t)) \\
&= x(t+1)^{\mathsf{T}}Px(t+1) - x(t)^{\mathsf{T}}Px(t) \\
&= (Ax(t))^{\mathsf{T}}PAx(t) - x(t)^{\mathsf{T}}Px(t) \\
&= x(t)^{\mathsf{T}}A^{\mathsf{T}}PAx(t) - x(t)^{\mathsf{T}}Px(t) \\
&= x(t)^{\mathsf{T}}(A^{\mathsf{T}}PA - P)x(t) \\
&= x(t)^{\mathsf{T}}Qx(t)
\end{aligned}$$

The resulting matrix $Q = (A^{\mathsf{T}}PA - P)$ is symmetric as well. By picking P such that it is not only symmetric and positive definite, but such that Q is negative definite, the quadratic Lyapunov function will prove the linear system is globally asymptotically stable.

40.2.4 Stability of Linear Systems

As was mentioned previously in discussing Lyapunov's first method, the location of the eigenvalues in the complex plane determines the stability of a linear system. If the eigenvalues are in the open LHP the system is considered stable, and if the eigenvalues are in the open RHP the system is unstable. When all of the eigenvalues are non-positive, but some eigenvalues are on the imaginary axis, can any conclusions about stability be drawn then? (DDV 13.3)

The following system, while both of its eigenvalues are $\lambda = 0, 0$, is stable. This is because its Jordan blocks are 1×1.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The following system, while both of its eigenvalues are $\lambda = 0, 0$, is not stable. This is because it has one Jordan block of size 2×2.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Both of these examples are explained further in the Jordan normal form section.

40.2.5 Solving Lyapunov Problems

The following section will attempt to explain how to use Lyapunov stability analysis in solving problems.

- Is the system unstable?

  - if it is suspected that the system is unstable, Lyapunov's first method can be used to prove instability

- if Lyapunov's first method is used, and the system is locally stable, the second method can then be attempted to prove global stability

- Try to find a Lyapunov function V be which will prove stability of the system

  - try to make $\dot{V} = 0$ only when $x = x_0$ to prove asymptotic stability, otherwise as long as $\dot{V}$ is never positive the system is still stable

- For linear, autonomous, stable systems

  - use $V = x^{\mathsf{T}} P x$
  - find P by solving $A^{\mathsf{T}} P + P A = -Q$
  - where Q and P are both positive-definite
  - Q is a free design parameter: use a diagonal matrix

- For linear, non-autonomous, unstable systems

  - can use full state feedback $u = -Kx$ to stabilize the linear A matrix
  - the Lyapunov equation is now solved using the closed-loop A matrix, since A_{CL} is stable: $A_{CL}{}^{\mathsf{T}} P + P A_{CL} = -Q$

Sometimes having $\dot{V}$ negative semi-definite, and thus only being able to conclude stability in the sense of Lyapunov and not asymptotic stability is not enough. Must use invariance when this is the case.

40.2.6 Stuff

[20] The objective of the so-called "second method" of Lyapunov is this: *To answer questions of stability of differential equations, utilizing the given form of the equations but without explicit knowledge of the solutions.*

40.3 Lyapunov Stability

40.3.1 Lyapunov Functions

1. First find the equilibrium points of the system

- For first order system choose $V(x) = \frac{1}{2}(x - x_{\mathrm{eq}})^2$
- Evaluate $\dot{V}$ and if it is negative definite, the equilibrium is asymptotically stable.
- Then if V is radially unbounded, that is $V \to \infty$ as $\|x\| \to \infty$ then the equilibrium is globally asymptotically stable. *Is this true if there are more than one stable equilibrium point?*
- If $\dot{V}$ is only negative semi-definite, then the equilibrium is stable in the sense of Lyapunov.

  - At this point invariant set theorems can be used.

- For second order systems of the form $l(\dot{x})\ddot{x} + b(\dot{x}) + c(x) = 0$ with $b(\dot{x})$ and $c(x)$ in the first and third quadrants, then we can choose

$$V(x, \dot{x}) = \int_0^{\dot{x}} l(y)y\,dy + \int_0^x c(y)\,dy$$

  And if $l(\dot{x}) = 1$ then this reduces

$$V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \int_0^x c(y)\,dy$$

- If we have a second order linear system

$$A_1\ddot{x} + A_2\dot{x} + A_3 x = 0$$

  use

$$V = \frac{1}{2}\dot{x}^{\mathsf{T}} A_1 \dot{x} + \frac{1}{2}y^{\mathsf{T}} A_3 y^{\mathsf{T}}$$

- Using this Lyapunov function if/when we evaluate $\dot{V}$ and get negative semi-definite, we can only assert stability in the sense of Lyapunov

- To get local asymptotic stability, look at the condition that is required to make $\dot{V} = 0$. For example if $\dot{V} = -\dot{x}^2$ then we take $\dot{x} = 0$. Substitute this value $\dot{x} = 0$ into the system equations

- Then???

- Result is extended to global asymptotic stability if the integral $\int_0^y c(y)\,dy$ is radially unbounded

When analyzing a system with multiple equilibria, in order for a scalar function V to be considered a candidate Lyapunov function, it must be zero only at the equilibrium point, and positive everywhere else. Thus a single candidate Lyapunov function that can be used to determine stability about all of the equilibria cannot be proposed. Sticking to Lyapunov's second method would thus require a candidate Lyapunov function proposed for each of the equilibria, and then analyzed. However, in systems with multiple equilibria, we may wish to use only a single Lyapunov-like function in order to say something about the stability of the equilibria.

In order to do this we appeal to the global invariant set theorem, and necessarily relax the requirement for V to be positive definite. Therefore, we can only call the scalar function V a Lyapunov-like function.

We find $\dot{V}$ and evaluate the values of the state x that make $\dot{V} = 0$. We call this set of points R.

Then, using the system dynamics, we find a subset R that are equilibrium points and call this subset M.

Globally, we will converge to this set, although we don't know the particulars about which points within this set to which the system trajectories actually converge. Additional analysis is required. If the Lyapunov-like function is locally about an equilibrium "a bowl" this equilibrium point is a stable one. If the Lyapunov-like function about an equilibrium point is locally "a hill" we can't conclude the local instability of this point? Check this statement. But a linear analysis will indicate local instability.

40.4 Solving Lyapunov Problems

The following section will attempt to explain how to use Lyapunov stability analysis in solving problems.

- Is the system unstable?
    - if it is suspected that the system is unstable, Lyapunov's first method can be used to prove instability
    - if Lyapunov's first method is used, and the system is locally stable, the second method can then be attempted to prove global stability
- Try to find a Lyapunov function V be which will prove stability of the system
    - try to make $\dot{V} = 0$ only when $x = x_0$ to prove asymptotic stability, otherwise as long as $\dot{V}$ is never positive the system is still stable
- For linear, autonomous, stable systems
    - use $V = x^T P x$
    - find P by solving $A^T P + P A = -Q$
    - where Q and P are both positive-definite
    - Q is a free design parameter: use a diagonal matrix
- For linear, non-autonomous, unstable systems
    - can use full state feedback $u = -Kx$ to stabilize the linear A matrix
    - the Lyapunov equation is now solved using the closed-loop A matrix, since A_{CL} is stable: $A_{CL}{}^T P + P A_{CL} = -Q$

Sometimes having $\dot{V}$ negative semi-definite, and thus only being able to conclude stability in the sense of Lyapunov and not asymptotic stability is not enough. Must use invariance when this is the case.

40.5 Stuff

40.6 Sliding Mode Control

40.6.1 Introduction

Sliding, switching, "suction" control.

40.6.2 General Idea

Perfect tracking possible in presence of arbitrary parameter inaccuracies, but requires extremely high control activity. Provides good tradeoff between tracking performance and uncertainty. We consider systems of the form

$$x^{(n)} = f(x) + b(x)u$$

where $f(x)$ is not completely known, but imprecision on $f(x)$ upper bounded by a function of x. $b(x)$ not completely known, but known sign and also upper bounded. Goal: get the state x to track the desired state x_d. Define the state tracking error $\tilde{x}$ as

$$\tilde{x} = x - x_d$$

To make it easier to explain in these notes, we will consider when the system is of second order. (with some additional simplifications over the general form)

$$\ddot{x} = f(x) + b(x)u$$

$$\ddot{x} = f + u$$

The general form of the composite/switching variable s is

$$s = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}$$

which, when equal to zero, drives the state error $\tilde{x}$ to zero. For a the second order system this composite variable is

$$s = \left(\frac{d}{dt} + \lambda\right)^{1} \tilde{x} = \frac{d}{dt}\tilde{x} + \lambda\tilde{x}$$

giving

$$s = \dot{\tilde{x}} + \lambda\tilde{x}$$

for a third order

$$s = \left(\frac{d}{dt} + \lambda\right)^{2} \tilde{x} = \left(\frac{d^2}{dt^2} + 2\lambda\frac{d}{dt} + \lambda^2\right)\tilde{x}$$

giving

$$s = \ddot{\tilde{x}} + 2\lambda\dot{\tilde{x}} + \lambda^2\tilde{x}$$

and so on. Our goal is to enforce that this new composite variable s tends to zero. We do this by creating a candidate Lyapunov function V of this variable, take its time derivative, and choose the control input such that $\dot{V}$ is negative definite. That is, choose the following candidate Lyapunov equation

$$V(s) = \frac{1}{2}s^2$$

time differentiating

$$\dot{V} = s\dot{s}$$

From the definition of s we can evaluate $\dot{s}$

$$\dot{s} = \ddot{\tilde{x}} + \lambda\dot{\tilde{x}}$$

and substitute it in

$$\dot{V} = s(\ddot{\tilde{x}} + \lambda\dot{\tilde{x}})$$

$$\dot{V} = s(\ddot{x} - \ddot{x}_d + \lambda\dot{\tilde{x}})$$

to simplify things we can define a new variable x_r that satisfies the following

$$\ddot{x}_r = \ddot{x}_d - \lambda\dot{\tilde{x}}$$

substituting this into $\dot{V}$

$$\dot{V} = s(\ddot{x} - \ddot{x}_r)$$

and substituting in the system equation

$$\dot{V} = s(f + u - \ddot{x}_r)$$

At this point we are ready to design the control input u. However, instead of designing u simply to make $\dot{V}$ negative definite, we wish to impose some degree of "negativeness". That is, we want to design u to enforce

$$\dot{V} \leq \eta|s|$$

where $\eta > 0$ is a positive constant. With our choice of $V = \frac{1}{2}s^2$ this constraint can be expressed

$$\frac{1}{2}\frac{d}{dt}s^2 \leq -\eta|s|$$

This gives

$$s(f + u - \ddot{x}_r) \leq \eta|s|$$

We want to design an *equivalent control* $\hat{u}$ that would completely cancel out f if the parameters of f were completely known. That is

$$\hat{u} = -\hat{f} + \ddot{x}_r$$

The control law

$$u = \hat{u} - k\text{sgn}(s)$$

$$u = -\hat{f} + \ddot{x}_r - k\text{sgn}(s)$$

plugging in

$$s(f - \hat{f} - k\text{sgn}(s)) \leq \eta|s|$$

$$-s(\hat{f} - f) - sk\text{sgn}(s) \leq \eta|s|$$

and then

$$-\frac{s(\hat{f} - f)}{|s|} - k \leq \eta$$

and then

$$\pm(\hat{f} - f) - k \leq \eta$$

And we know a bound on the parameter error

$$|\hat{f} - f| \leq F$$

so we can pick k to satisfy

$$\pm(\hat{f} - f) - k \leq \eta \leq |\hat{f} - f| + \eta \leq k$$

and then

$$|\hat{f} - f| + \eta \leq F + \eta \leq k$$

and pick k such that it is the smallest value that will satisfy this inequality

$$k = F + \eta$$

40.6.3 Summary

The general form of the system to use switching control.

$$x^{(n)} = f(x) + b(x)u$$

So when the system is second order, use these (I don't know for sure what it would be for higher order systems.)

$$\tilde{x} = x - x_d$$
$$s = \dot{\tilde{x}} + \lambda\tilde{x}$$
$$s = \dot{x} - \dot{x}_r$$
$$\dot{x}_r = \dot{x}_d - \lambda\tilde{x}$$

40.6.4 Barbalat's Lemma

1. V is positive definite

2. $\dot{V}$ is negative semi-definite

   - With V positive definite and $\dot{V}$ negative semi-definite, V is bounded. That is, at the initial time $t = 0$ we have $V(x(t = 0), \theta(t = 0))$ and from here (since $\dot{V}$ is negative semidefinite) the value of V can only decrease. V is bounded below by zero since it is positive definite. Finally, we say that since V is bounded from above by $V(t = 0)$ and bounded from below by 0, that it is bounded. And because V is bounded and positive definite (actually probably some other condition, but it is true for a quadratic function) then the arguments of V are bounded

3. $\dot{V}$ is uniformly continuous, which follows from $\ddot{V}$ being bounded

   - We evaluate $\ddot{V}$ and since now we know the arguments of V are bounded, we use this to bound $\ddot{V}$, thus showing uniform continuity of $\dot{V}$

With these three conditions met Barbalat's Lemma states that $\dot{V} \to 0$ as $t \to \infty$. We then look at $\dot{V}$ and since it is tending to zero, its arguments must go to zero. Since $\dot{V}$ at this point is probably a function of the state error only, we say the state error tends to zero, although we can't necessarily say anything about the parameter error (unless the parameters are in $\dot{V}$ also...it is easiest to show this with examples).

Barbalat's Lemma

Thus far, we have shown that $e \in \mathcal{L}_\infty$ is bounded. Now the goal is to use Barbalat's Lemma to prove $e \to 0$. That is, show the system is asymptotically stable?

Stuff

- Continuity does not imply differentiability. e.g. absolute value

Example Slotine example page 122: $\dot{f}(t) \to 0 \not\Rightarrow f$ converges.

This example shows a function $f(t)$ which has a time derivative $\dot{f}(t)$ that goes to zero as $t \to \infty$, but the function $f(t)$ does not converge as $t \to \infty$.

$$f(t) = \sin(\ln(t))$$

$$\dot{f}(t) = \cos(\ln(t))\frac{d}{dt}(\ln(t)) = \frac{\cos(\ln(t))}{t}$$

$$\lim_{t\to\infty} \dot{f}(t) = \lim_{t\to\infty} \frac{\cos(\ln(t))}{t} = 0$$

However, just because $\dot{f}(t)$ approaches zero as $t \to \infty$ does not mean the function $f(t)$ will converge as $t \to \infty$, as is the case in this example.

Example Slotine example page 122: f converges $\not\Rightarrow \dot{f}(t) \to 0$.

This example shows a function $f(t)$ which converges as $t \to \infty$, with a time derivative $\dot{f}(t)$ that does not go to zero as $t \to \infty$.

$$f(t) = e^{-t}\sin(e^{2t})$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} e^{-t}\sin(e^{2t}) = 0$$

$$\dot{f}(t) = -e^{-t}\sin(e^{2t}) + 2\cos(e^{2t})$$

$$\lim_{t\to\infty} \dot{f}(t) = \lim_{t\to\infty} -e^{-t}\sin(e^{2t}) + 2\cos(e^{2t}) = \lim_{t\to\infty} 2\cos(e^{2t})$$

The function $f(t)$ has a time derivative $\dot{f}(t)$ which does not converge. It oscillates back and forth faster and faster.

40.7 Describing Functions and Limit Cycles

The purpose of a describing function is to represent a nonlinearity as an effective gain which varies based on the input signal. Describing functions are based on the assumption that the input to the nonlinearity is sinusoidal. The idea is that in response to a sinusoidal excitation, most nonlinearities will produce a periodic signal with frequencies being the harmonics of the input frequency. The assumption is made that the output may in be acceptably approximated by the first harmonic alone. Describing functions allow limit cycles to be found.

40.8 Contraction Analysis

plant

$$\dot{x} = f(x, u, t)$$

substitute control law

$$u(x, t)$$

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get

$$\dot{x} = f(x,t)$$

consider a virtual displacement

$$\delta\dot{x} = \frac{\partial f(x,t)}{\partial x}\delta x = \frac{\partial f}{\partial x}\delta x$$

When a trajectory is virtually displaced, we want the displacement to tend to zero with time. That is, we want the trajectory to eventually go back the the trajectory from which it was displaced. If all neighboring trajectories converge to each other (contraction behavior) global exponential convergence to a single trajectory can then be concluded. We define the distance of this virtual displacement as

$$\delta x^{\mathsf{T}}\delta x$$

For this distance to tend to zero, it is the same as its time derivative always being strictly negative. Evaluating the derivative

$$\frac{d}{dt}(\delta x^{\mathsf{T}}\delta x) = 2\delta x^{\mathsf{T}}\delta\dot{x} = 2\delta x^{\mathsf{T}}\frac{\partial f}{\partial x}\delta x$$

contraction analysis extends a number of desirable properties of linear system analysis to general nonlinear non-autonomous systems.

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