1 Question 1

a)

Yes. A poly-time constant factor approximation for the Independent Set (IS) problem does give a poly-time constant factor approximation for the Clique (CQ) problem.

First we assert that if U is a maximal independent set in G' (complement of the graph), then U is also the maximal clique in G. Hence $OPT_{IS}(G') = OPT_{CQ}(G)$. This can be seen by the question statement: "U is an independent set in $G \iff U$ is a clique in G'". This equivalence indicates the largest IS in G' is the largest CQ in G as there is a one-to-one mapping (largest clique corresponds to an IS and vice versa).

Suppose we had a poly-time approximation algorithm for IS. Now, let's see if we can use the approximation algorithm for IS to find a constant factor approximation algorithm for CQ given some graph G. Simply run the IS approximation algorithm on G' (poly-time operation) to get an IS U for G' with the bound $|U| \geq (1-\epsilon)|OPT_{IS}(G')|$. Since U would then be completely connected in G, it is also an approximation for the CQ problem. By the equality of the optimal solutions in the above paragraph, we would have this bound $|U| \geq (1-\epsilon)|OPT_{CQ}(G)|$. Thus, we get a poly-time constant factor approximation for the clique problem given the IS approximation algorithm.

b)

No. A poly-time constant factor approximation for the Independent Set (IS) problem does not give a poly-time constant factor approximation for the Vertex Cover (VC) problem.

First, we assert that if U is a maximal independent set then $V-U$ is a minimal vertex cover. Hence $|V| = |OPT_{IS}|+|OPT_{VC}|$. This can be seen by the question statement: "$U \subseteq V$ is an IS $\iff V-U$ is a VC". If U' is the largest IS with a corresponding VC $V-U'$, then because there are no IS's bigger than U', any smaller $V-U$ is not possible since U' is maximal. Thus, $V-U'$ is a minimal cover.

Suppose we had a poly-time approximation algorithm for IS. Then, we get the bound $|U'| \geq (1-\epsilon)|OPT_{IS}|$. Observe the following:

$$
\begin{aligned}
|V| &= |V| \\
|V| - |U'| &\leq |V| - (1-\epsilon)|OPT_{IS}| \\
|V| - |U'| &\leq |OPT_{IS}| + |OPT_{VC}| - (1-\epsilon)|OPT_{IS}| \\
|V| - |U'| &\leq |OPT_{IS}| + |OPT_{VC}| - |OPT_{IS}| + \epsilon|OPT_{IS}| \\
|V| - |U'| &\leq |OPT_{VC}| + \epsilon|OPT_{IS}| \\
|V| - |U'| &\leq \begin{cases} (1+\epsilon)|OPT_{VC}| & \text{if } |OPT_{VC}| \geq |OPT_{IS}| \\ |OPT_{VC}| + \epsilon|OPT_{IS}| & \text{else} \end{cases}
\end{aligned}
$$

From the above, it is clear that the approximation depends on the relative sizes of $|OPT_{VC}|$ and $|OPT_{IS}|$. Thus, for some ϵ, you would get a constant approx factor if the minimal VC is bigger than or equal to the maximal IS. However, if the maximal IS is bigger than the minimal VC, the approximation factor/bound can be violated as $|OPT_{VC}| + \epsilon|OPT_{IS}| > (1+\epsilon)|OPT_{VC}|$ (this is a reachable, tight bound for this case). Thus, a constant factor approximation for VC is not possible.