1 Question 2

a)

We will prove that this deterministic algorithm has competitive ratio $\frac{1}{\sqrt{B}}$.

Consider case 1, $M \geq T$. In this case, the algorithm will always accept a bid D such that $T = \sqrt{B} \leq D \leq M$ as such a bid will always appear in the sequence and the algorithm will accept the first one it sees. The optimal algorithm would pick M. Then, the ratio between this algorithm and the OPT is $\frac{ALG}{OPT} = \frac{D}{M}$. In the worst case, we minimize this ratio by considering the largest M and smallest D to get a lower bound of $\frac{\sqrt{B}}{B} = \frac{1}{\sqrt{B}}$.

Consider case 2, $M < T$. In this case, the algorithm will pick the last bid D such that $1 \leq D \leq M < T = \sqrt{B}$. The optimal algorithm would pick $M < \sqrt{B}$. Then, the ratio between this algorithm and OPT is $\frac{ALG}{OPT} = \frac{D}{M}$. In the worst case, we minimize the ratio by considering the largest M and the smallest D to get a lower bound of $\frac{1}{\sqrt{B}-1} > \frac{1}{\sqrt{B}}$. Take $\frac{1}{\sqrt{B}}$ as the lower bound. (Note that in the case where $B = 1$, then $\frac{1}{\sqrt{B}} = 1$ still satisfies as the lower bound).

Thus, in both possible cases, we see that the ratio between the algorithm's performance to that of the optimal algorithm is $\frac{ALG}{OPT} \geq \frac{1}{\sqrt{B}}$. Thus, $ALG \geq \frac{1}{\sqrt{B}}OPT$ and the algorithm is $\frac{1}{\sqrt{B}}$-competitive.

b)

We will prove that the expected competitive ratio is at least $\frac{1}{2\log B}$. Consider M such that it is $2^k \leq M < 2^{k+1}$ for any $k \in [1-1, 2-1, 3-1, \dots \lfloor \log B+1 \rfloor -1]$. Now, we can look at two cases based on the randomly chosen threshold. For a given k, either $T = 2^{i-1} \leq 2^k$ for some $i \in [1, 2, \dots, \lfloor \log B + 1 \rfloor]$ or $T > 2^k = 2^{i-1}$.

In case 1, the randomly chosen i can only take on the following values: $i \in [1, 2, k+1]$. Any greater value of i would surpass the bound 2^k. Since we are choosing i at random from $\lfloor +1 \rfloor$ values, the probability of each i would be $\frac{1}{\lfloor \log B + 1 \rfloor}$.