1 Question 1

We will prove that the expected number of times m is updated is $O(logn)$.

Suppose that the random permutation of the array that we use to update m is given by $x_1, x_2, \ldots, x_n$. Let U represent the total number of updates to m. Let X_i represent the number of updates to m caused by x_i. It is trivial to see that $X_i \in \{0, 1\}$ as we will only compare x_i to m a single time, and an update occurs if and only if $x_i < m$. As m can only be updated by comparing it to some x_i, we see that $U = \sum_{i=1}^{n} X_i$. Thus, $\mathbf{E}(U) = \mathbf{E}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathbf{E}(X_i)$ by the linearity of expectation.

To calculate $\mathbf{E}(X_i)$, we make the observation that the integers are distinct and that we randomly permute the array. Thus, x_i is uniformly chosen (akin to uniform sampling without replacement) and that it has a $\frac{1}{n}$ probability to be the kth largest element in the array (since all integers are distinct and so the relative ordering in the sorted array is fixed). Suppose we are at the ith iteration (i.e we are comparing $x_i < m$). m is therefore the minimum of the $i-1$ integers that came before in the permutation: $m = \min(x_1, \ldots, x_{i-1})$. Since each x_i was placed in the permutation uniformly, we observe that the probability $\min(x_1, \ldots x_i) = x_i$ is $\frac{1}{i}$. Thus, since we only update m if x_i is the minimum seen so far by the ith iteration, we have $\mathbf{E}(X_i) = 1 \times \frac{1}{i} + 0 = \frac{1}{i}$. Note that $\mathbf{E}(X_1) = 1$ as m starts off at ∞.

Thus, we can calculate the expectation of the total number of updates like so $\mathbf{E}(U) = \sum_{i=1}^{n} \mathbf{E}(X_i) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, which are the harmonic numbers. Thus, the expected number of updates to m is $\mathbf{E}(U) \in O(\log n)$.