# 1 Question 1

[10 marks] Suppose you have a splay tree with keys $1 \dots n$ and you access the keys sequentially, i.e. you search for $1, 2, \dots, n$, in that order, performing a splay on each one.

a) [2 marks] What is the structure of the final splay tree? Justify your answer.

**Answer:** We will define potential to be the number of elements in the array. Thus, when the array has $k$ elements, $\Phi = k$. This guarantees that $\Phi_i \geq 0$ for all $i$, confirming that it is a valid potential. Furthermore, we see that this allows the initial potential to be $\Phi_0 = 0$ as initially, we start off with an empty array.

b) Prove that the final potential is $\geq$ initial potential.

**Answer:** Initially, we start off with an empty array - there are 0 elements in the array. Thus, the initial potential is 0. The number of elements stored in the array can only ever be a non-negative number. Thus, the final array contains *at least* 0 elements, and so the final potential is at least 0, which means it's at least the initial potential.

c) Conclude that the amortized cost of each operation is $O(1)$.

**Answer:** In class, we were given the following theorem:

**Theorem 1** *If final potential $\geq$ initial potential, then amortized cost $\leq$ max charge.*

From part b), we know that final potential $\geq$ initial potential. Thus, we can apply this theorem to the two operations.

For the add operation, we are given the constant $charge = 2$. Thus, by the theorem, we know that amortized cost $\leq 2$. Since 2 is a constant, we know that amortized cost $\in O(1)$.

For the empty operation, we are given the constant $charge = 1$. Thus, by the theorem, we know that amortized cost $\leq 1$. Since 1 is a constant, we know that amortized cost $\in O(1)$.