1 Question 2

a)

We will prove that this deterministic algorithm has competitive ratio $\frac{1}{\sqrt{B}}$.

Consider case 1, $M \geq T$. In this case, the algorithm will always accept a bid D such that $T = \sqrt{B} \leq D \leq M$ as such a bid will always appear in the sequence and the algorithm will accept the first one it sees. The optimal algorithm would pick M. Then, the ratio between this algorithm and the OPT is $\frac{ALG}{OPT} = \frac{D}{M}$. In the worst case, we minimize this ratio by considering the largest M and smallest D to get a lower bound of $\frac{\sqrt{B}}{B} = \frac{1}{\sqrt{B}}$.

Consider case 2, $M < T$. In this case, the algorithm will pick the last bid D such that $1 \leq D \leq M < T = \sqrt{B}$. The optimal algorithm would pick $M < \sqrt{B}$. Then, the ratio between this algorithm and OPT is $\frac{ALG}{OPT} = \frac{D}{M}$. In the worst case, we minimize the ratio by considering the largest M and the smallest D to get a lower bound of $\frac{1}{\sqrt{B}-1} > \frac{1}{\sqrt{B}}$. Take $\frac{1}{\sqrt{B}}$ as the lower bound. (Note that in the case where $B = 1$, then $\frac{1}{\sqrt{B}} = 1$ still satisfies as the lower bound).

Thus, in both possible cases, we see that the ratio between the algorithm's performance to that of the optimal algorithm is $\frac{ALG}{OPT} \geq \frac{1}{\sqrt{B}}$. Thus, $ALG \geq \frac{1}{\sqrt{B}}OPT$ and the algorithm is $\frac{1}{\sqrt{B}}$-competitive.

b)

We will prove that the expected competitive ratio is at least $\frac{1}{2\log B}$.

Consider any M such that $2^k \leq M < 2^{k+1}$ for any choice of $k \in [1-1, 2-1, 3-1, \ldots, \lfloor \log B + 1 \rfloor - 1]$. This means there exists a k in the stated range that satsifies any given M in this problem. To calculate the expected competitive ratio, we notice that i is a random variable that determines the threshold. The expectation is thus the sum over all possible i of the probability of choosing such i multiplied by the competitive ratio given such threshold $T = 2^{i-1}$.

Let's look at the part of the sum where the choice of i is such that $T = 2^{i-1} \leq 2^k$. To satisfy this bound, $i \in [1, 2, k+1]$ must be true. For any i in that given range, we know that the bid accepted is $D \geq T = 2^{i-1}$ and the optimal bid is $M < 2^{k+1}$. The competitive ratio is $\frac{ALG}{OPT} = \frac{D}{M}$. Let's establish a lower bound on the competitive ratio by setting $D = 2^{i-1}$ (worst case) and letting the denominator be a higher value than what is possible: $\frac{D}{M} \geq \frac{T}{M} > \frac{2^{i-1}}{2^{k+1}}$. This is a lower bound on the competitive ratio as a function of i. Now, the probability of choosing each i is $\frac{1}{\lfloor \log B + 1 \rfloor}$. The expected competitive ratio is now $\mathbf{E}(\frac{ALG}{OPT}) \geq \frac{1}{\lfloor \log B + 1 \rfloor} \sum_{i=1}^{k+1} \frac{2^{i-1}}{2^{k+1}} + \beta$, where β is the half of the expectation for when $i > k+1$.

Now, let's look at the part of the expectation where the choice of i is such that $T = 2^{i-1} > 2^k$. Since the choice of i must be an integer, we have $i \in [k+2, \ldots, \lfloor \log B + 1 \rfloor]$. However, for any choice of i in this range, we have $T = 2^{i-1} \geq 2^{k+1} > M$. Since the threshold is bigger than the maximum value in the sequence, the algorithm doesn't bid and gets a reward of 0. This means that the competitve ratio is also 0 for these choices of i. Thus, we see that $\beta = 0$ and the expected competitve ratio is $\mathbf{E}(\frac{ALG}{OPT}) \geq \frac{1}{\lfloor \log B + 1 \rfloor} \sum_{i=1}^{k+1} \frac{2^{i-1}}{2^{k+1}}$.

Let's simplify this expression:

$$
\begin{aligned}
\mathbf{E}(\frac{ALG}{OPT}) &\geq \frac{1}{\lfloor \log B + 1 \rfloor} \sum_{i=1}^{k+1} \frac{2^{i-1}}{2^{k+1}} \\
&\geq \frac{1}{\lfloor \log B + 1 \rfloor} \frac{1}{2^{k+2}} \sum_{i=1}^{k+1} 2^i \\
&\geq \frac{1}{\lfloor \log B + 1 \rfloor} \frac{1}{2^{k+2}} (2^{k+2} - 2) \\
&\geq \frac{1}{\lfloor \log B + 1 \rfloor} \frac{2^{k+1} - 1}{2^{k+1}} && 1 - \frac{1}{2^{k+1}} \geq \frac{1}{2}, \forall k \geq 0 \\
&\geq \frac{1}{2\lfloor \log B + 1 \rfloor}
\end{aligned}
$$

Thus, we have proven that this randomized algorithm has an expected competitve ratio at least $\frac{1}{2(\log B + 1)}$