CHAPTER 15 CONIC SECTIONS

15.1 INTRODUCTION SECTIONS OF A CONE

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In order to understand the central ideas of Chapters 13 and 14, it was necessary to pay close attention to the precise wording of definitions and to the details of proofs, so the level of mathematical rigor in those chapters was rather high. However, we now turn to work that is mostly geometric in nature. We shall therefore rely much more heavily on reasoning based on spatial intuition and the kind of insight that can be obtained from carefully drawn figures.

Consider a circle C. Let A be the line through the center of C perpendicular to the plane of C, and let V be a point on A not in the plane of C, as shown in Fig. 15.1. Let P be a point on C, and draw the infinite straight line through P that also passes through V. As P moves around C, the line PV sweeps out a *right circular cone* with *axis* A and *vertex* V. Each of the lines PV is called a *generator* of the cone, and the angle α between the axis and any generator is called the *vertex angle*. The cone shown in Fig. 15.1 has a vertical axis, and the upper and lower portions of the cone that meet at the vertex are called the *nappes* of the cone.* In elementary geometry a cone is usually understood to be a solid figure occupying the bounded region of space that lies between V and the plane of C and is inside the surface we have just described. However, in the present context the cone is this surface itself, and is understood to consist of both nappes, extending to infinity in both directions.

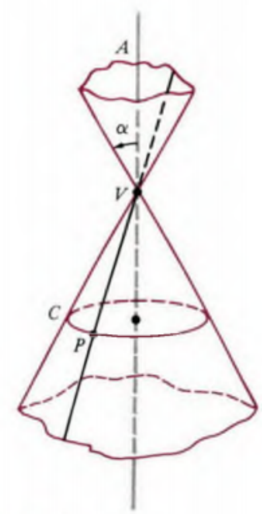

Figure 15.1

The curves obtained by slicing a cone with a plane that does not pass through the vertex are called *conic sections,* or simply *conics.* If the slicing plane is parallel to a generator, the conic is called a *parabola.* Otherwise, the conic is called an *ellipse* or a *hyperbola,* depending on whether the plane cuts just one or both nappes. The hyperbola is to be thought of as a single curve consisting of two "branches," one on each nappe. These three curves are illustrated in Fig. 15.2.

The three curves shown in Fig. 15.2 can be described in another way. Imagine a source of light placed at V and a circular ring placed at C. Then the shadow cast by the ring on a plane will be a parabola, an ellipse, or one branch of a hyperbola, depending on the steepness of the plane. If the plane is parallel to one of the lines joining V to C, we get a parabolic shadow; the shadow will be an ellipse if the plane is less steep than this, and part of a hyperbola if it is more steep.

*"Nappe" is from the French word *nappe,* meaning a sheet of something, perhaps cloth, as in "napkin" or "napery" (household linen).

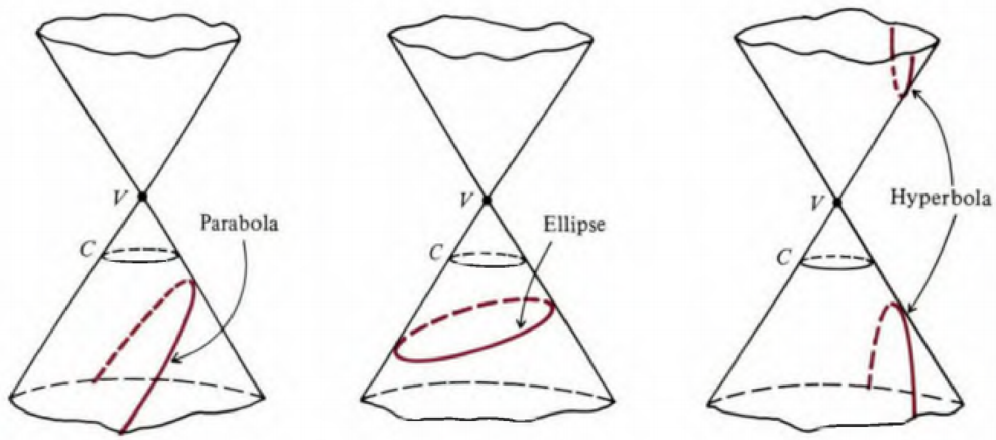

Figure 15.2

It should be noted that if we move each of the slicing planes in Fig. 15.2 parallel to itself until it passes through the vertex, then we get three so-called *degenerate* conic sections, namely, a single straight line, a point, and a pair of intersecting straight lines.

Many important discoveries in both pure mathematics and science have been linked to the conic sections. The classical Greeks—Archimedes, Apollonius and others—studied these beautiful curves for the sheer pleasure of it, as a form of play, without any thought of their possible uses. The first applications appeared almost 2000 years later, at the beginning of the seventeenth century. About the year 1604 Galileo discovered that if a projectile is fired horizontally from the top of a tower and is assumed to be acted on only by the force of gravity—that is, if air resistance and other complicating factors are ignored—then the path of the projectile will be a parabola. One of the great events in the history of astronomy occurred only a few years later, in 1609, when Kepler published his discovery that the orbit of Mars is an ellipse and then went on to suggest that all the planets move in elliptical orbits. And about 60 years after this, Newton was able to prove mathematically that an elliptical planetary orbit implies, and is implied by, an inverse square law of gravitational attraction. This led Newton to formulate and publish (in 1687) his famous theory of universal gravitation as the explanation of the mechanism of the solar system, which has been described as the greatest contribution to science ever made by one man. These developments took place hundreds of years ago, but the study of conic sections is far from outdated even today. Indeed, these curves are important tools for present-day explorations of outer space, and also for research into the behavior of atomic particles. Artificial satellites move around the earth in elliptical orbits, and the path of an alpha particle moving in the electric field of an atomic nucleus is a hyperbola. These examples and many others show that the importance of conic sections, both historically and in modern times, is difficult to exaggerate.

We shall be studying the conic sections as plane curves. For this purpose it is convenient to make use of equivalent definitions that refer only to the plane in which the curves lie and depend on special points in this plane called *foci* (*focus* is the singular). An ellipse can be defined as the set of all points in the plane the sum of whose distances d_1 and d_2 from two fixed points F and F' (the foci) is constant, as shown on the left in Fig. 15.3. A hyperbola is the set of all points for which the difference $|d_1 - d_2|$ is constant. And a parabola is the set of all

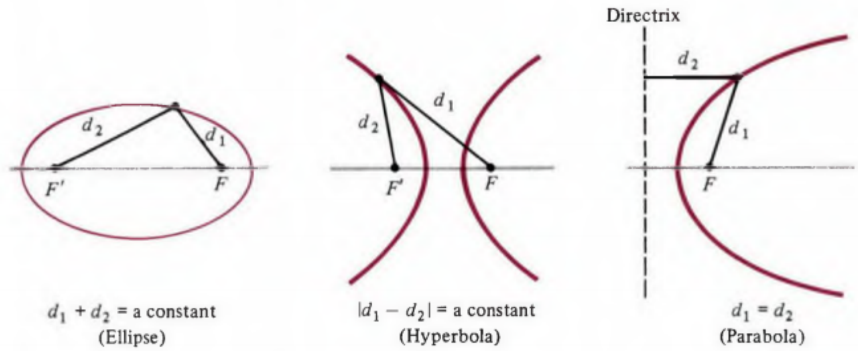

Figure 15.3

points for which the distance to a fixed point F (the focus) equals the distance to a fixed line (called the *directrix*).

There is a simple and elegant argument which shows that the focal property of an ellipse follows from its definition as a section of a cone. This proof uses the two spheres shown in Fig. 15.4, which are internally tangent to the cone along the horizontal circles C_1 and C_2, and are also tangent to the slicing plane at the points F and F'. If P is an arbitrary point on the ellipse, we must show that the sum of the distances $PF + PF'$ is constant in the sense that it does not depend on the particular position of P. To see this, we notice that if Q and R are the points on C_1 and C_2 that lie on the generator through P, then $PF = PQ$ and $PF' = PR$, because any two tangents to a sphere drawn from a common external point have the same length. It follows that $PF + PF' = PQ + PR = QR$; and the argument is completed by observing that QR, as the distance from C_1 to C_2 down a generator, has the same value for every position of P.

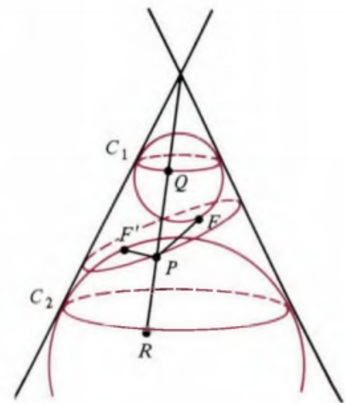

Figure 15.4 The focal property of an ellipse.

With slight modifications this proof also works for the hyperbola and the parabola. In the case of the hyperbola, we use one sphere in each portion of the cone, with both spheres tangent to the slicing plane. And for the parabola we use one sphere tangent to the slicing plane. The focus of the parabola is this point of tangency, and its directrix is the line of intersection of the slicing plane with the plane of the circle along which the sphere is internally tangent to the cone. Students should use these hints to draw suitable pictures and prove for themselves that the focal properties of the hyperbola and the parabola can be derived from their definitions as sections of a cone.

1. 标题：圆锥曲线（Conic Sections）的几何图像与核心性质

在这一部分课件中，我们围绕"圆锥曲线"这一重要主题展开，主要介绍了如何通过切割圆锥面（包含上下两个相对的圆锥部分，也称作nappes）来得到抛物线、椭圆和双曲线这三类曲线；并从历史和几何角度阐释了它们在数学与科学中的重要地位。随后，课件还对椭圆、双曲线及抛物线的焦点性质进行了初步解释，包含它们与定点（焦点）和定直线（准线）之间距离关系的定义，以及通过圆锥切割推导焦点性质的几何思路。

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2. 详细知识点与深入解析

下面我们按照图片中出现的顺序，逐点分析每一个知识点以及其细节内容，并联系上下文进行更深入的讲解。

（a）圆锥曲线的几何构造与"nappes"概念

- **圆锥与两张"nappes"**
  1. 在图 15.1 中，圆锥被视为一个"无厚度"的几何面，它实际上包含"上下两个对顶的圆锥面"。文中将这两个对顶的圆锥面部分分别称为两个"nappes"。
  2. 在一般几何课程中，我们常见的圆锥通常被理解为一个有厚度的实体，或仅看上半部分；但在更抽象的几何里，圆锥可以延伸到包含下半部分，使之对称地延伸到顶点两侧。这样，圆锥表面便含有两个"nappes"——上面的一张和下面的一张。

- **生成线（generator）与顶角（vertex angle）**
    1. 在图 15.1 中，令 C 表示圆的中心和底面所在平面，A 表示穿过这个圆心并垂直于 C 的直线。再取一个不在平面 C 上的点 V（它将成为圆锥的顶点）。
    2. 对于圆周上的每一个点 P，连线 PV 可以理解为生成线（generator）。当点 P 绕圆周一周时，PV 扫过整个圆锥面。
    3. 圆锥的轴是从 A 到顶点 V 这条线，生成线与轴形成的夹角被称作圆锥的顶角或生成线与轴线的夹角。
- **关于“nappes”一词**
  文中指出，“nappes”来自法语，意指一块“覆盖物”或“布料”的意思。之所以用这个词，是因为可以把上下的圆锥面理解为两片展开的“布面”，它们在顶点处“会合”，并向上下两侧延伸。

(b) 平面切圆锥面：椭圆、抛物线和双曲线

- **基本切割情形**
    1. 当一个平面去切割这个不经过顶点的圆锥面时（即与圆锥的顶点无交点），所得到的截线就被称为圆锥曲线（conic）。
    2. 如果这个切割平面与生成线平行，那么截线是**抛物线（parabola）**。
    3. 如果这个切割平面稍微“倾斜”一些，且只割到圆锥的一张 nappe，那么截线是**椭圆（ellipse）**。
    4. 如果切割平面截穿上下两张 nappe，那么得到的曲线则是**双曲线（hyperbola）**。
    5. 双曲线可以想象为“一个曲线分成两支”，各自分别位于圆锥的上半部分和下半部分。

- **平行切割与各种圆锥曲线**
  在图 15.2 中，通过想象我们把一个发光点放在圆锥顶点 V，把一个圆环放在底部 C，观察在不同倾斜角度下的切割平面所形成的"阴影"曲线，可以分别得到抛物线、椭圆和双曲线。
  - 如果切割平面平行于圆锥的一条生成线，则在"阴影"里会出现抛物线。
  - 若切割平面更倾斜一些，只与一张 nappe 相交，则出现封闭的椭圆。
  - 若切割平面更倾斜，穿过上下两张 nappe，则出现双曲线的一支（另一支来自另一侧的 nappe）。
- **退化（degenerate）圆锥曲线**
  当切割平面移动到最终经过顶点时，可能会出现退化情况：
  1. 如果平面刚好只"擦过"顶点，截线是**一条直线**。
  2. 如果平面仅与顶点重合且与生成线重合，可能出现**两条相交直线**或**一个点**。
     这三种情况（直线、一点、两条相交直线）被称为退化圆锥曲线。

（c）历史与重要应用：从伽利略、开普勒到牛顿

- **圆锥曲线在古代几何学中的地位**
  1. 古希腊数学家阿基米德、阿波罗尼乌斯等人，对圆锥曲线进行了系统研究，仅仅是出于对美丽曲线的兴趣，而尚未考虑其实用价值。
  2. 第一批应用直到 2000 年后才出现，进入十七世纪初期。
- **伽利略 (Galileo, 1604)**
  1. 伽利略发现，如果没有空气阻力，且仅在重力作用下，将一个抛射物从塔顶水平发射，那么它的轨道是**抛物线**。
  2. 这代表了圆锥曲线在物理学、力学中的首批重要应用之一。

- **开普勒 (Kepler, 1609) 与行星运动**
  1. 开普勒提出行星绕太阳运动的轨道是**椭圆**形。
  2. 随后又提出所有行星都按椭圆轨道绕太阳运动。
- **牛顿 (Newton, 1687) 与万有引力**
  1. 牛顿进一步用数学手段证明，如果天体在万有引力的反平方定律下运动，其轨道正是椭圆的一种形式，抛物线、双曲线都可以视为能量状态不同的轨道形式。
  2. 他在 1687 年发表的《自然哲学的数学原理》（Principia）中，系统阐述了引力理论，被誉为对人类科学影响最大的著作之一。
- **现代意义**
  1. 人造卫星绕地球运动是椭圆轨道；带电粒子在原子核电场中运动有时表现为双曲线或椭圆型轨道。
  2. 圆锥曲线在航天工程、原子物理以及数学研究中依旧扮演极其重要的角色。

（d）圆锥曲线的平面定义：焦点与准线

接下来课件介绍了当我们把圆锥曲线放到平面上时的“等效定义”或“焦点—准线”定义。也就是说，我们可以不再从圆锥的三维切割出发，而是用以下几何特性直接在平面上定义三类曲线。

1. **椭圆（Ellipse）**
   - 定义：在平面上，椭圆可以定义为到两个定点（称为焦点 F 和 F'）距离之和为常数的全部点的集合。
   - 记作：$d_1 + d_2 = \text{常数}$，其中 $d_1 = PF$，$d_2 = PF'$。
   - 几何意义：这一定义能够解释为什么椭圆是一个封闭曲线，也能说明焦点与椭圆上的点之间有着特殊的距离关系。

2. **双曲线（Hyperbola）**
   - 定义：在平面上，双曲线可以定义为到两个定点（焦点 F 和 F'）距离之差为常数的全部点的集合。
   - 记作：$|d_1 - d_2| = 常数$，其中 $d_1 = PF$，$d_2 = PF'$。
   - 几何意义：双曲线由两支分支构成，其焦点—准线结构和椭圆类似，但距离之差不变的几何条件使它呈"开放"状态。
3. **抛物线（Parabola）**
   - 定义：在平面上，抛物线可以定义为到一个定点（焦点 F）和一条定直线（准线）距离相等的全部点的集合。
   - 记作：$d_1 = d_2$，其中 $d_1 = PF$ 表示点 P 到焦点的距离，d_2 表示点 P 到准线的距离。
   - 几何意义：因为抛物线只有一个焦点与一条准线，相对于椭圆和双曲线来说，它"刚好"处于不封闭不双支的状态。

(e) 焦点性质的几何推导：利用圆锥切割与球面辅助

- **焦点性质：椭圆的"距离和"为常数**
  图 15.4 通过在圆锥内部安放两个"内切球"（分别与圆锥的某些水平圆 C_1, C_2 以及与切割平面相切）来说明：
  1. 当平面切割圆锥得到椭圆后，设 F, F' 为该椭圆在平面内的两个焦点。
  2. 取椭圆上一点 P，如果我们在圆锥中连接 P 到 F 和 F'，可以在球面切线的构造下发现 $PF + PF'$ 的和等于从球面外公切线长度之和，从而证明与点 P 的具体位置无关，始终保持某个定值。
  3. 这说明椭圆的"焦点—距离和"性质可以从三维圆锥切割来推导。

- **类比：抛物线和双曲线的“焦点性质”**
  1. 对抛物线而言，只需在圆锥中放置一个球面，并让它与切割平面正好相切于准线位置，焦点则是圆锥与球面接触对应的那一点。
  2. 对双曲线，需要在圆锥上下两部分各放一个球，使其分别与切割平面及圆锥相切，并与焦点位置相对应，从而可以得到“距离差”为常数的性质。
  3. 这些内容都需要学生进一步结合图形，理解球面、切割平面以及焦点之间的位置关系，从而“自发”地体会为何圆锥曲线自带这些焦点性质。

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3. 本次内容总结

1. **圆锥曲线的来源**：通过在三维空间切割一个上下对顶的圆锥面（包含两张 nappe），可以得到三类基本曲线：椭圆、抛物线和双曲线；当平面通过顶点时，还可能得到退化情况（直线、点、两条相交直线）。
2. **历史意义与应用**：从古希腊的纯数学研究，到伽利略分析抛体运动，开普勒发现行星椭圆轨道，牛顿提出万有引力定律，圆锥曲线成为人类认识自然界的重要工具。如今仍广泛应用于航天、原子物理以及各种力学和几何研究中。
3. **焦点-准线定义**：在平面上，我们可以用距离和/距离差恒定或与准线距离相等等几何条件来定义椭圆、双曲线、抛物线；这与圆锥切割的三维构造相一致。
4. **焦点性质证明思路**：可以借助圆锥中安放球面与切割平面相切的思路来理解为什么椭圆上任一点到焦点的距离和为常数（或双曲线的距离差为常数，抛物线的点到焦点和准线距离相等）。

15.2 ANOTHER LOOK AT CIRCLES AND PARABOLAS

15.2 ANOTHER LOOK AT CIRCLES AND PARABOLAS

Circles and parabolas were discussed fairly thoroughly in Chapter 1. However, that was a long time ago, and it may be helpful to give a very brief review of the main facts in order to assist students in fitting these topics into the context of the present chapter.

CIRCLES

Referring to Fig. 15.4, we see at once that a circle can be thought of as the special case of an ellipse obtained by taking the slicing plane perpendicular to the axis of the cone, so that the foci coincide. Nevertheless, for several reasons it is convenient to discuss circles separately, and to reserve the word "ellipse" for the case in which the foci are two distinct points.

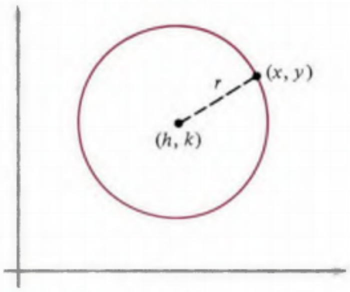

Figure 15.5 A circle.

A *circle*, therefore—as we very well know—can be defined as a plane curve consisting of the set of all points at a given fixed distance (called the *radius*) from a given fixed point (called the *center*). If $r > 0$ is the radius and (h, k) is the center, and if (x, y) is an arbitrary point on the circle (see Fig. 15.5), then by using the distance formula we can write the defining condition as

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

or

$$(x-h)^2 + (y-k)^2 = r^2, \tag{1}$$

which is the equation of the circle in standard form. By squaring the terms on the left of (1) and rearranging, this equation can be written in the form

$$x^2 + y^2 + Ax + By + C = 0. \tag{2}$$

Conversely, by completing the square on the x and y terms, any equation of the form (2) can be written in the form (1), and therefore represents a circle if $r^2 > 0$. As students will remember, there is a slight difficulty with this procedure as a result of the fact that the constant r^2 on the right of (1) may turn out to be zero or a negative number. In these cases, (1) can be thought of as the equation of a single point or the empty set.

PARABOLAS

As we saw in Section 15.1, a parabola can be defined as a plane curve consisting of the set of all points P that are equally distant from a given fixed point F and a given fixed line d, as shown on the left in Fig. 15.6. The fixed point is called the *focus*, and the fixed line is called the *directrix*. To find a simple equation for this curve, we introduce the coordinate system shown on the right in the figure, in which the focus is the point $F = (0, p)$, where p is a positive number, and the directrix is the line $y = -p$. If $P = (x, y)$ is an arbitrary point on the parabola, then by using the distance formula the defining condition can be written as

$$\sqrt{x^2 + (y-p)^2} = y + p. \tag{3}$$

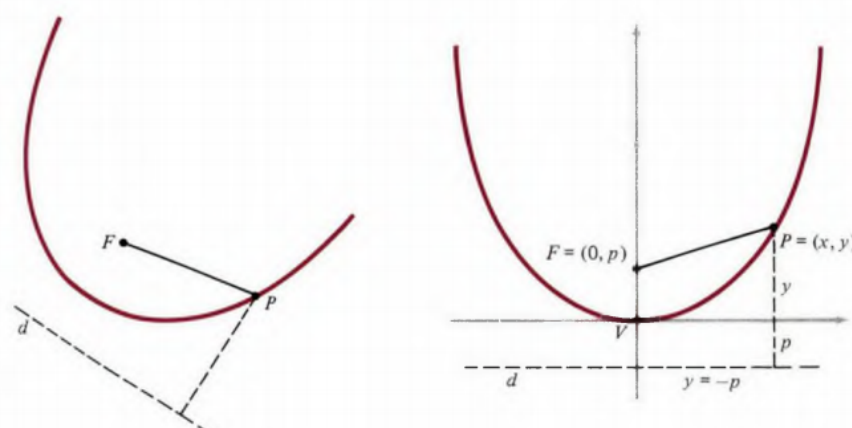

Figure 15.6 A parabola.

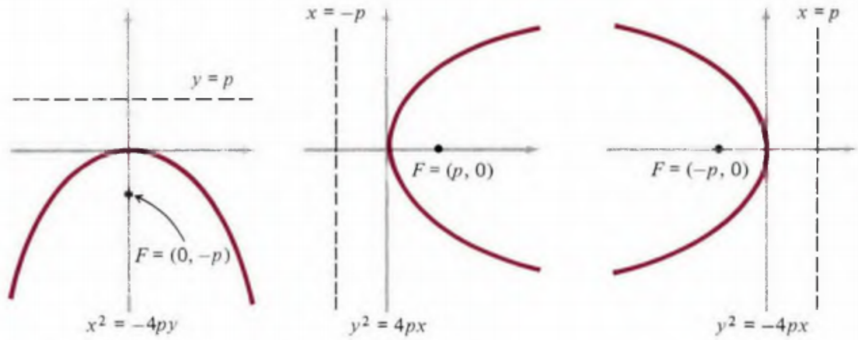

Figure 15.7

By squaring and simplifying we get

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

or

$$x^2 = 4py. \tag{4}$$

Conversely, by reversing the steps, it can be shown that (3) can be derived from (4). Equation (4) is therefore the equation of this particular parabola in standard form. The line through the focus perpendicular to the directrix is called the *axis* of the parabola, and the point V where the parabola intersects the axis is called the *vertex*. For the parabola (4), the axis is clearly the y-axis, and the vertex is the origin.

If we change the position of the parabola relative to the coordinate axes, we naturally change its equation. Three other simple positions, each with its corresponding equation, are shown in Fig. 15.7. Students should verify the correctness of all three equations. We emphasize that the constant p is always understood to be a positive number; geometrically, it is the distance from the vertex to the focus, and also from the vertex to the directrix.

We illustrate a further point about parabolas by considering the equation

$$x^2 - 8x - y + 19 = 0.$$

If we write this as $x^2 - 8x = y - 19$ and complete the square on x, then the result is

$$(x - 4)^2 = y - 3.$$

If we now introduce new variables x' and y' by writing

$$x' = x - 4,$$
$$y' = y - 3,$$

then our equation becomes

$$x'^2 = y'.$$

The graph of this equation is clearly a parabola with vertical axis whose vertex lies at the origin in the x', y' coordinate system, and this origin is located at the point (4, 3) in the x, y system, as shown in Fig. 15.8. In exactly the same way, any equation of the form

$$x^2 + Ax + By + C = 0, \qquad B \neq 0, \tag{5}$$

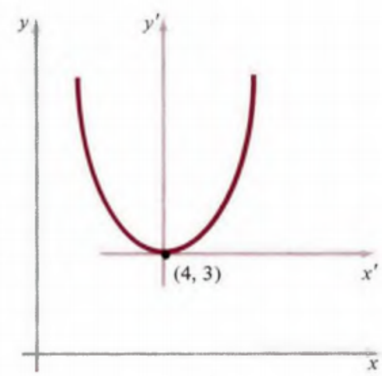

Figure 15.8

represents a parabola with vertical axis. The vertex of this parabola is easily located by completing the square on x, and in this way the equation can be written in the form

$$(x-h)^2 = 4p(y-k) \qquad \text{or} \qquad (x-h)^2 = -4p(y-k),$$

where the point (h, k) is the vertex.* Similarly, any equation of the form

$$y^2 + Ax + By + C = 0, \qquad A \neq 0,$$

represents a parabola with horizontal axis, and the geometric features of this parabola can be discovered by completing the square on y and writing the equation as

$$(y-k)^2 = 4p(x-h) \qquad \text{or} \qquad (y-k)^2 = -4p(x-h).$$

We conclude this section by describing the so-called *reflection property* of parabolas. Consider the tangent line at a point $P = (x, y)$ on the parabola $y^2 = 4px$ (Fig. 15.9), where $F = (p, 0)$ is the focus. As shown in the figure, let α be the angle between the tangent and the segment FP, and let β be the angle between the tangent and the horizontal line through P. In Problem 9, students are asked to prove that $\alpha = \beta$.

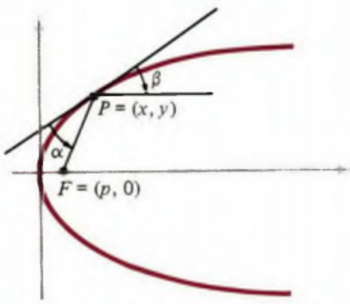

Figure 15.9 The reflection property.

This geometric property of parabolas has many applications. For example, it is used in the design of mirrors for searchlights. To construct such a mirror, revolve the parabola about its axis to form a surface of revolution, then coat the inside with silver paint to make a reflecting surface. If a source of light is placed at F, each ray will be reflected along a line parallel to the axis to form a beam of parallel rays. The same principle is used in a more important way in the design of mirrors for reflecting telescopes and solar furnaces, where rays of light that are parallel to the axis and come in toward the mirror are reflected in to the focus. This reflection property of parabolas also underlies the design of radar antennas and radio telescopes.

*We point out here that if $B = 0$ is allowed in (5), then the graph of the equation can be one straight line, or two parallel lines, or the empty set. For the particular equation $x^2 - 2x - k = 0$, or equivalently $(x-1)^2 = k + 1$, these cases correspond to $k = -1$, $k > -1$, and $k < -1$.

1. 标题：圆与抛物线的再探——特殊椭圆情形及标准方程与反射性质

本次课件（第 15.2 节）对“圆”和“抛物线”在圆锥曲线框架下进行了进一步讨论。尽管它们在前面的章节（特别是第 1 章）已有涉及，但这里将它们与圆锥切割及坐标方程更紧密地结合，并补充了反射性质等重要内容。以下是对课件中每一张图片及文字的系统整理和讲解。

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2. 详细知识点与深入解析

（A）圆（Circle）：椭圆的特殊情形

1. **圆是椭圆的特殊情况**
   - 在上一节我们了解到，椭圆是“到两个焦点距离之和保持常数”的点的集合。当两焦点重合时，这两个焦点就合为同一个点 (h,k)，也就是圆的圆心。
   - 在圆锥切割的角度上看：当切割平面与圆锥轴线垂直，就会使得原本分离的两个焦点重合，得到一个圆。
   - 虽然圆是椭圆的特例，但由于在应用和研究中经常单独使用“圆”这一概念，所以一般把椭圆一词留给“焦点不重合”的情形。
2. **圆的定义与方程**
   - **定义**：圆可以定义为到某个定点（圆心）距离相等的所有点的集合。
   - 设圆心为 (h,k)，半径为 r，则任意在圆上的点 (x,y) 满足
     $$\sqrt{(x-h)^2+(y-k)^2} \;=\; r.$$
   - 两边平方，得到圆的标准方程：
     $$(x-h)^2+(y-k)^2 \;=\; r^2.$$

- 若我们把这一方程展开并整理，即可写成二次形式：
  $$x^2 + y^2 + Ax + By + C \ = \ 0.$$
  通过“配平方”可以判断该方程是否真的代表一个圆，并求出圆心和半径。
- **退化情形**：如果整理后得到的常数 $r^2 \leq 0$，则方程所代表的曲线退化为一个点（当 $r^2 = 0$ ）或空集（当 $r^2 < 0$）。

3. **图示（Figure 15.5）**
   - 图中标明了圆心 (h, k) 和任意点 (x, y)，并用箭头表示 $\sqrt{(x-h)^2 + (y-k)^2} = r$。
   - 直观地看，这就是平面上所有与圆心距离固定为 r 的点。

（B）抛物线（Parabola）：与焦点—准线定义相结合

1. **抛物线的基本定义**
   - 与上一节的圆锥切割法一致，抛物线也可以从三维到二维来理解；但在平面几何中，我们有更直接的“焦点—准线”定义：
     > 抛物线是平面上所有到一个定点（焦点 F）和一条定直线（准线 d）距离相等的点的集合。
   - 图 15.6 左边给出了抛物线与其焦点 F 和准线 d 的示意关系。
2. **坐标系中的简化：焦点与准线的标准位置**
   - 为了推导简明方程，经常把抛物线放置在一个便于计算的坐标系中：
     - 设焦点 F 在 $(0, p)$，其中 $p > 0$。
     - 设准线为 $y = -p$。
     - 这样，抛物线的顶点（vertex）就在原点 $(0, 0)$，轴线是 y-轴。

- 任意点 (x, y) 在抛物线上，就满足

$$(\text{到焦点的距离}) = (\text{到准线的距离}).$$

  也即

$$\sqrt{(x-0)^2 + (y-p)^2} \;=\; \bigl|y - (-p)\bigr| \;=\; |y+p|.$$

- 为了具体得到方程，文中（见 (3) 式）将准线写成 $y = -p$，并设 (x, y) 满足

$$\sqrt{x^2 + (y-p)^2} \;=\; y + p,$$

  其中当 $y \geq -p$ 时，可省略绝对值写成 $y + p$。

3. **推导抛物线方程**

- **步骤**：对 $\sqrt{x^2 + (y-p)^2} = y + p$ 两边平方、再展开、整理，便可得到

$$x^2 = 4py.$$

- 这就是**标准形抛物线方程**，顶点在原点，轴与 y-轴重合，开口朝上（当 $p > 0$）。
- 如果把焦点放在其他位置，如 $(p, 0)$、(h, k) 等，就会相应得到其他形式的抛物线方程，比如：

$$y^2 = 4px, \quad (x-h)^2 = 4p(y-k), \quad \text{或 } (y-k)^2 = 4p(x-h).$$

- 图 15.6, 15.7, 15.8 展示了不同坐标轴定位时抛物线的形状及方程如何发生变化。

4. **配方法与抛物线的平移**
    - 课件中用到“配平方”的技巧来将通用二次方程 $x^2 - 8x - y + 19 = 0$ 转化为更熟悉的抛物线标准形。
    - 先对 x 项配方，即写成

      $$(x-4)^2 = y-3,$$

      然后再用平移坐标系 $\left(x' = x-4,\, y' = y-3\right)$ 来得到

      $$x'^2 = y',$$

      显示了在 (x', y')-系下，抛物线以 $(0,0)$ 为顶点并开口向上，明显是标准形式 $x'^2 = y'$ 的抛物线。

(C) 抛物线的反射性质 (Reflection Property)

1. **反射定理的叙述**
    - 在图 15.9 中，抛物线方程假设为 $y^2 = 4px$，其焦点 F 位于 $(p, 0)$。
    - 取抛物线上任意一点 $P = (x, y)$ 并作切线，记 α 为该切线与过 P 的水平线之间的夹角；β 为该切线与线段 FP（即由焦点到点 P 的连线）之间的夹角。
    - 反射性质说明：$\alpha = \beta$。也就是说，若一条光线从焦点射向抛物线上的点 P，则经抛物线反射后，它会平行于抛物线的轴线方向“弹出”（或反向，如果光线是由外部平行射向抛物线，再由抛物线反射聚集到焦点）。

2. **应用**
   - **搜索灯与抛物面反射镜**：把抛物线绕其轴旋转所形成的三维曲面称为"抛物面"，若在焦点放置光源，那么光线反射后会大致平行地射出，形成聚光效果。
   - **反射望远镜、太阳能炉**：利用平行光线经过抛物面镜反射后汇聚到焦点的原理进行高效聚热或观测。
   - **雷达天线**：雷达天线也常做成抛物面，以便把来自远方的电磁波聚焦到接收器（焦点）。

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3. 本次内容总结

1. **圆与椭圆的关系**：圆可以视为焦点合二为一的椭圆，也是当切割平面垂直于圆锥轴时得到的一种特殊圆锥曲线。
2. **圆的标准方程与退化情况**：$(x-h)^2+(y-k)^2=r^2$。若 $r=0$ 或虚数，便出现退化圆或空集。
3. **抛物线的焦点—准线定义**：到焦点与准线距离相等的一切点构成抛物线；在简化坐标系下，其标准方程可以呈现为 $x^2=4py$ 或 $y^2=4px$ 等多种形式。
4. **配方法与平移**：对于较一般的二次方程，可以通过配平方与坐标平移将其化为标准抛物线方程，进而辨别顶点位置、开口方向等信息。
5. **抛物线的反射性质**：光线从焦点出发，通过抛物线反射后变为平行于对称轴；或反之，平行光线反射后汇聚到焦点。这一几何特性在探照灯、反射望远镜、雷达天线等中具有重要应用价值。

15.3 ELIPSES

15.3 ELLIPSES

In Section 15.2 we gave "set of all points" definitions for both circles and parabolas. It is also possible to give "locus" definitions, in which each curve is defined —and thought of—as the path of a moving point that satisfies a certain condition as it moves. This language has the advantage of greater pictorial vividness. Thus, a parabola can be defined as the locus of a point that moves in such a way that it maintains equal distances from a given fixed point and a given fixed line.

Similarly, in accordance with Section 15.1, we can define an *ellipse* as the locus of a point P that moves in such a way that the sum of its distances from two fixed points F and F' is constant, as shown on the left in Fig. 15.13. To simplify later equations, we denote this constant by $2a$ and write the defining condition as

$$PF + PF' = 2a. \tag{1}$$

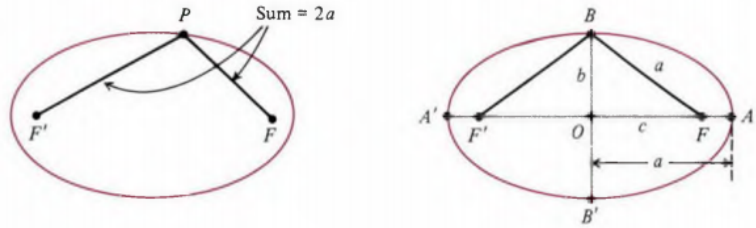

Figure 15.13

The two points F and F' are called the *foci* (plural of *focus*) of the ellipse because of the reflection property discussed in Remark 1 below. Since circles are not considered to be ellipses in this discussion, F and F' are understood to be two *distinct* points: $F \neq F'$.

The definition provides an easy way to draw an ellipse on a sheet of paper. We begin by fastening the paper to a drawing board with two tacks placed at F and F'. Next, we tie the ends of a piece of string to the tacks and pull the string taut with the point of a pencil. It is clear that if the pencil is moved around while the string is kept taut, then its point draws an ellipse. Because of this construction, the defining condition (1) is often called the *string property* of an ellipse.

We now introduce several standard notations for the dimensions of an ellipse. It is easy to see from the definition that the curve is symmetric with respect to the line through the foci, and also with respect to the perpendicular bisector of the segment FF'. On the right in Fig. 15.13 the segment AA' is called the *major axis* and the segment BB' is called the *minor axis* of the ellipse, and the point O where these axes intersect is called the *center.* The two points A and A' at the ends of the major axis are called the *vertices* of the ellipse. We denote the length of the minor axis by $2b$ and the distance between the foci by $2c$. It is clear that $BF = a$, because $BF + BF' = 2a$ and $BF = BF'$, so

$$a^2 = b^2 + c^2. \tag{2}$$

Since $AF + AF' = 2a$ and $AF' = FA'$, we see that $AA' = 2a$, so the length of the major axis is $2a$. The numbers a and b are called the *semimajor axis* and the *semiminor axis.*

It is easy to see from equation (2) that $b < a$. If b is very small compared with a, so that the ellipse is long and thin, then (2) shows that c is nearly as large as a, and the foci are near the ends of the major axis; and if b is nearly as large as a, so that the ellipse is nearly circular, then c is small, and the foci are close to the center. The ratio c/a is called the *eccentricity* of the ellipse and is denoted by e:

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}. \tag{3}$$

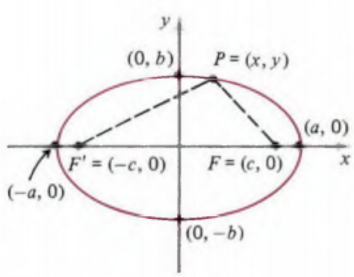

Figure 15.14

Notice that $0 < e < 1$. Nearly circular ellipses have eccentricity near 0, and long, thin ellipses have eccentricity near 1.

In order to find a simple equation for the ellipse, we take the x-axis along the segment FF' and the y-axis as the perpendicular bisector of this segment. Then the foci are $F = (c, 0)$ and $F' = (-c, 0)$, as shown in Fig. 15.14, and the defining condition (1) yields

$$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a \tag{4}$$

as the equation of the curve.

To simplify this equation, we follow the usual procedure for eliminating radicals, namely, solve for one of the radicals and square. If we move the first radical in (4) over to the right side, square both sides, and simplify, then we obtain

$$PF = \sqrt{(x - c)^2 + y^2} = a - \frac{c}{a}x \tag{5}$$

and

$$PF' = \sqrt{(x + c)^2 + y^2} = a + \frac{c}{a}x, \tag{6}$$

where (6) follows from (5) because $PF' = 2a - PF$. By squaring again and simplifying, either of these equations gives

$$\left(\frac{a^2 - c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{7}$$

By using (2) to simplify (7) still further, we now put the equation into its final form,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{8}$$

This argument shows that (8) is satisfied if (4) is. It can be shown, conversely, that (4) is satisfied if (8) is, but we omit the details. Equation (8) is therefore the standard form of the equation of the ellipse shown in Fig. 15.14.

We pause briefly to point out that equation (8) easily yields most of the simpler geometric features of the ellipse that are visible in Fig. 15.14. (i) If $y = 0$, then the equation tells us that $x = \pm a$, and if $x = 0$, then $y = \pm b$, so the curve crosses the x and y axes at the four points $(\pm a, 0)$ and $(0, \pm b)$. (ii) Since both terms x^2/a^2 and y^2/b^2 are nonnegative and their sum is 1, it follows that neither of them can be greater than 1, so $|x| \leq a$ and $|y| \leq b$. This means that the whole ellipse is contained in the rectangle whose sides are $x = \pm a$ and $y = \pm b$, and is therefore—unlike the parabola—a bounded curve. (iii) If (x, y) satisfies the equation, then so do $(x, -y)$ and $(-x, y)$, so the curve is symmetric with respect to both the x-axis and the y-axis. This tells us that to graph the complete curve it suffices to sketch the graph in the first quadrant and then extend it to the other quadrants by symmetry. The left-right symmetry of the ellipse that is so obvious from equation (8) is really rather remarkable, because most people contemplating Fig. 15.2 for the first time feel quite sure that an ellipse should be an egg-shaped oval which has a "small end" at the part of the ellipse nearest the vertex of the cone and a "big end" at the part farthest from this vertex—but of course this is not true.

We consider again formulas (5) and (6) for the right and left *focal radii* PF and PF', which can be written as

$$PF = a - \frac{c}{a}x = e\left[\frac{a}{e} - x\right] \tag{9}$$

and

$$PF' = a + \frac{c}{a}x = e\left(\frac{a}{e} + x\right) = e\left[x - \left(-\frac{a}{e}\right)\right], \tag{10}$$

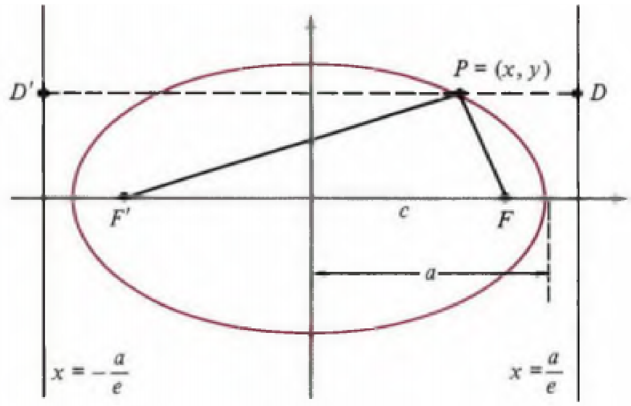

Figure 15.15

where $e = c/a$ is the eccentricity defined earlier. The quantities in brackets can be interpreted (see Fig. 15.15) as the distances PD and PD' from P to the lines $x = a/e$ and $x = -a/e$, respectively. Formulas (9) and (10) can therefore be written in the form

$$\frac{PF}{PD} = e \qquad \text{and} \qquad \frac{PF'}{PD'} = e. \tag{11}$$

Each of the lines $x = a/e$ and $x = -a/e$ is called a *directrix* of the ellipse. Equations (11) show that *an ellipse can be characterized as the locus of a point that moves in such a way that the ratio of its distance from a fixed point (a focus) to its distance from a fixed line (the corresponding directrix) equals a constant* $e < 1$. We shall see in Chapter 16 and elsewhere that this way of characterizing ellipses is often very useful.

1. 标题：椭圆的焦点定义、标准方程与准线性质

在这一部分（第 15.3 节）中，我们将系统介绍椭圆的"焦点—定长和"定义、几何结构参数（焦距、长短轴、离心率等）、标准方程的推导过程，以及与准线相关的另一种刻画方式。以下笔记将帮助我们全面理解椭圆的核心性质及其多种等价定义。

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2. 详细知识点与深入解析

（A）"焦点—定长和"定义与基本性质

1. **椭圆的焦点定义**
   - **定义**：椭圆可以看作是平面上一点 P 所构成的轨迹，满足其到两个固定点（焦点 F 与 F' ）的距离之和为某个常数。
   - 若我们记此常数为 $2a$，则可用数学式写作
   $$PF + PF' = 2a. \tag{1}$$
   - 在图 15.13 中，焦点分别是 F 和 F'，点 P 在椭圆上运行时，始终保持 $PF + PF' = 2a$ 。
   - **焦点不重合**：与上一节讲的圆不同，这里要求 $F \neq F'$；如果焦点重合，就会退化成一个圆。
2. **几何作图：拉线法（string property）**
   - 教材提到一种在纸上快速画出椭圆的方法：
     1. 将两枚图钉分别固定在纸面上作为焦点。
     2. 在这两个钉子上系一条长为 $2a$ 的线，然后用铅笔把线拉紧，让笔尖在纸上移动。
     3. 铅笔运动的轨迹即满足"到两个钉子距离之和为常数 $2a$"的条件，从而勾勒出一条椭圆。
   - 这形象地说明了公式 (1) 常被称为椭圆的"拉线性质"。

3. **椭圆的对称性与主要尺寸**
    - 椭圆关于"连结两焦点的直线"以及其垂直平分线具有对称性。
    - 文中引入了**主轴（major axis）与次轴（minor axis）**：
        - 通过焦点 F 与 F' 的那条线称为主轴（又称长轴），长轴长为 $2a$。
        - 与长轴中心垂直相交的那条轴称为次轴（又称短轴），短轴长为 $2b$。
        - 主轴和次轴的交点称为椭圆的**中心**（记为 O）。
    - 主轴两端分别是椭圆的**顶点（vertices）**，如图 15.13 右侧所示的 A 与 A'。
4. **焦点距、离心率**
    - 设椭圆中心为原点，焦点在 $(\pm c, 0)$，则两焦点间的距离为 $2c$。
    - 推导可得：
    $$a^2 \ = \ b^2 + c^2 \quad （其中 \ a > b）, \tag{2}$$
    说明椭圆的三个重要参数 a, b, c 之间存在勾股式关系。
    - **离心率（eccentricity）**：定义为
    $$e = \frac{c}{a}. \tag{3}$$
        - 当 $0 < e < 1$ 时，图形为椭圆；e 越小，椭圆越接近圆形；e 越大，则越"扁长"。
        - 若 $e = 0$，则 $c = 0$，说明焦点合一，椭圆退化为圆；若 e 逼近 1，则椭圆变得非常狭长。

（B）椭圆的标准方程与推导

1. **坐标系选取**
   - 为了得到椭圆的代数方程，通常把椭圆的中心放在坐标原点 O，焦点放在 $(\pm c, 0)$，主轴沿着 x-轴。这样椭圆上的点 (x, y) 应该满足

     $$\sqrt{(x-c)^2+y^2} \; + \; \sqrt{(x+c)^2+y^2} \; = \; 2a. \tag{4}$$

     这正是"到两焦点距离之和为 $2a$"的代数形式。见图 15.14。

2. **方程的消根与化简**
   - 教材里对式 (4) 进行了一系列标准操作：
     1. 先将一个根式移到右边，再平方；
     2. 整理后再移另一个根式，再继续平方；
     3. 最终将所有结果简化得到：

        $$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1. \tag{7}$$

     - 因为 $a^2=b^2+c^2$，可写 $a^2-c^2=b^2$。
     4. 最后即得到**标准形式**：

        $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{8}$$

   - 这便是以 x-轴为长轴、焦点在 $(\pm c, 0)$ 的椭圆方程。

3. **几何解读**
   - 从 (8) 可看出：
     - 当 $y=0$ 时，$x=\pm a$；当 $x=0$ 时，$y=\pm b$。故椭圆与坐标轴交点是 $(\pm a,0)$ 与 $(0,\pm b)$。
     - 整个椭圆处于以 $\pm a$ 和 $\pm b$ 为边界的矩形之内，是一个有界封闭曲线。
     - 由对称性可知，只需画出第一象限部分，再镜像到其他三象限即可获得完整椭圆。

(C) 准线与另一种椭圆刻画

1. **定义准线：Directrix**
   - 在图 15.15 中，教材还给出椭圆可用“焦点—准线”距离比为常数来定义：
     $$\frac{PF}{PD}=e,$$
     其中 P 为椭圆上任意一点，F 为焦点，D 为点 P 投影到准线上的垂足。
   - 对椭圆而言，对应每个焦点都有一条相应的准线：
     $$x=\frac{a}{e}\quad 和 \quad x=-\frac{a}{e}.$$
   - 由此可见：**椭圆同样可以定义为：平面上所有点的集合，使得它到某个定点（焦点）的距离与到某条定直线（准线）的距离之比为常数 $e<1$。**
2. **与抛物线、双曲线的类比**
   - 回想上一节，抛物线定义中是 $PF=PD$（比率为 1）; 双曲线定义则是 $PF/PD=e>1$。现在椭圆则是 $e<1$。
   - 因而，透过“焦点—准线比”可统一地看待圆锥曲线：$e=1$ 是抛物线，$e<1$ 是椭圆，$e>1$ 则是双曲线。

3. 本次内容总结

1. **焦点-距离和定义**：椭圆可定义为到两个焦点距离之和为常数（$2a$）的点的轨迹；与“拉线作图”方法和圆锥切割联系紧密。
2. **主轴、次轴与离心率**：主轴长 $2a$、次轴长 $2b$，焦点距中心 c，三者满足 $a^2=b^2+c^2$。椭圆的离心率 $e=c/a$。
3. **标准方程推导**：选取焦点在 $(\pm c,0)$ 的坐标系，可由“焦点—定长和”关系得到标准方程 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$。由此可分析椭圆顶点、对称性及有界性。
4. **准线与距离比定义**：椭圆也可通过“到焦点的距离 / 到准线的距离 = e”来刻画，其准线为 $x=\pm\frac{a}{e}$。该定义与抛物线、双曲线的准线定义一脉相承。

Example 1 Identify the graph of $16x^2 + 25y^2 = 400$ as an ellipse, and find its vertices, foci, eccentricity, and directrices. Sketch the graph.

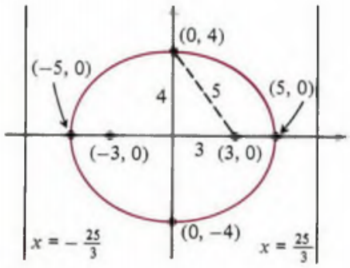

Figure 15.16

Solution First, we divide by 400 to convert the equation into the standard form

$$\frac{x^2}{25} + \frac{y^2}{16} = 1,$$

which on comparison with (8) tells us that the graph is an ellipse. Since $a^2 = 25$ and $b^2 = 16$, we have $a = 5$ and $b = 4$, so the vertices are $(\pm 5, 0)$ and the ends of the minor axis are $(0, \pm 4)$, as shown in Fig. 15.16. Next, $c^2 = a^2 - b^2 = 25 - 16 = 9$, so $c = 3$ and the foci are $(\pm 3, 0)$. Finally, the eccentricity is $e = c/a = \frac{3}{5}$, and the directrices are the vertical lines $x = \pm a/e = \pm \frac{25}{3}$.

In the above discussion it is assumed that the ellipse has its center at the origin and its foci on the x-axis. However, if its center is the origin and its foci lie on the y-axis, then its major axis is vertical and the roles of x and y are interchanged.

Example 2 Show that $9x^2 + 4y^2 = 36$ represents an ellipse, and find its vertices, foci, eccentricity, and directrices. Sketch the graph.

Solution As before, we divide by 36 to convert the given equation into the recognizable standard form

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Observe that here *the denominator of the y term is larger,* so we have an ellipse whose major axis is vertical. The semimajor and semiminor axes are clearly $a = 3$ and $b = 2$, so the vertices (see Fig. 15.17) are $(0, \pm 3)$ and the ends of the minor axis are $(\pm 2, 0)$. Since $c^2 = a^2 - b^2 = 9 - 4 = 5$, $c = \sqrt{5}$, and the foci are the points $(0, \sqrt{5})$ and $(0, -\sqrt{5})$ on the y-axis. The eccentricity is $e = c/a = \sqrt{5}/3$, and the directrices are the horizontal lines $y = \pm a/e = \pm 9/\sqrt{5} = \pm \frac{9}{5}\sqrt{5}$.

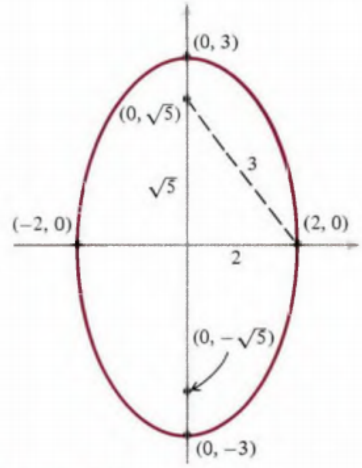

Figure 15.17

Examples 1 and 2 illustrate the fact that if we have an equation of the form

$$\frac{x^2}{(\ )^2} + \frac{y^2}{(\ )^2} = 1$$

with unequal denominators, then the equation represents an ellipse, and the question of whether the foci and major axis lie on the x-axis or the y-axis is determined by which denominator is larger.

In equation (8), x and y are the horizontal and vertical displacements from the axes of the ellipse to the point $P = (x, y)$. If the center is the point (h, k) instead of the origin, then these displacements are $x - h$ and $y - k$, and the equation of the ellipse becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \qquad (12)$$

Example 3 Show that $4x^2 + 16y^2 - 24x - 32y = 12$ is the equation of an ellipse, and find its vertices, foci, eccentricity, and directrices. Sketch the graph.

Solution The equation can be written as

$$4(x^2 - 6x) + 16(y^2 - 2y) = 12.$$

Completing the squares inside the parentheses, we obtain

$$4(x-3)^2 + 16(y-1)^2 = 64$$

or

$$\frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} = 1.$$

Comparison with (12) shows that this represents an ellipse with center $(3, 1)$, horizontal major axis, and semiaxes $a = 4$, $b = 2$, so the vertices (Fig. 15.18) are the points $(7, 1)$, $(-1, 1)$ and the ends of the minor axis are $(3, 3)$, $(3, -1)$. The foci are a distance $c = \sqrt{a^2 - b^2} = \sqrt{12} = 2\sqrt{3}$ to the right and left of the center, and are therefore the points $(3 \pm 2\sqrt{3}, 1)$. The eccentricity is $e = c/a = \frac{1}{2}\sqrt{3}$, and the directrices are vertical lines at a distance $a/e = 8/\sqrt{3} = \frac{8}{3}\sqrt{3}$ to the right and left of the center. Their equations are $x = 3 \pm \frac{8}{3}\sqrt{3}$.

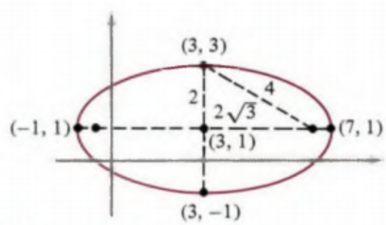

Figure 15.18

Remark 1 Like parabolas, ellipses also have a remarkable reflection property. Let P be a point on an ellipse with foci F and F', and let T be the tangent at P, as shown in Fig. 15.19. If T makes angles α and β with the two focal radii PF and PF', then $\alpha = \beta$. Students are asked to prove this in Problem 9.

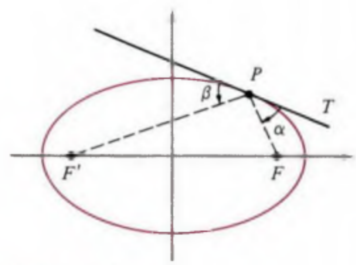

Figure 15.19

This reflection property has no important scientific applications like those we saw in the case of parabolas, but there is at least one mildly amusing consequence. Let the ellipse in the figure be revolved about its major axis to form a surface of revolution, and imagine that a room is built with its walls and ceiling having the shape of the upper part of this surface, with the two foci about shoulder height above the floor. Then a whisper uttered at one focus can be clearly heard a considerable distance away at the other focus even though it is inaudible at intermediate points, because the sound waves bounce off the walls and are reflected to the second focus, and furthermore arrive together because they all travel the same distance. There actually exist several rooms of this kind—known as *whispering galleries*—in certain American museums of science and in the castles of a few eccentric European monarchs.

A less frivolous application is in the new treatment for kidney stones called *lithotripsy* (from the Greek *lithos,* stone + *tripsis,* a rubbing or pounding). An ellipsoidal reflector is placed in such a position that the offending kidney stone is at one focus. High-intensity sound waves generated at the other focus are reflected harmlessly through the patient's body and are concentrated at the stone, which they pound into powder. The patient is spared having to go through surgery and recovers in a few days.

1. 标题：椭圆方程的应用示例与反射性质——顶点、焦点、离心率、准线及其在现实中的有趣应用

本次课件示例部分（见图 15.16 至图 15.19）主要演示了如何从给定的二次方程中辨识椭圆、化为标准形式，并进一步求出椭圆的顶点、焦点、离心率及准线等几何要素。同时，还延伸讨论了椭圆的反射性质及其在建筑声学和医疗中的应用。以下按示例顺序和图示内容作系统整理与讲解。

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2. 详细知识点与深入解析

（A）示例 1：识别并求解椭圆的几何要素

- **给定方程：**

$$16x^2 + 25y^2 = 400.$$

  化为标准形式：

  1. 两边同除以 400，得到
  $$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$
  2. 比较 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 可知：
  $$a^2 = 25 \quad \Rightarrow \quad a = 5, \quad b^2 = 16 \quad \Rightarrow \quad b = 4.$$

  - 因为 $a^2 > b^2$，说明主轴沿 x-轴，中心在原点。

- **顶点（Vertices）与短轴端点：**
  - 顶点在 $(\pm a, 0) = (\pm 5, 0)$。
  - 短轴端点在 $(0, \pm b) = (0, \pm 4)$。

- **焦点（Foci）**
  1. 记 c 为焦点到中心的距离，$c^2 = a^2 - b^2$。
  2. 这里 $c^2 = 25 - 16 = 9 \Rightarrow c = 3$。
  3. 焦点位于主轴上，即 $(\pm c, 0) = (\pm 3, 0)$。
- **离心率（Eccentricity）**

$$e = \frac{c}{a} = \frac{3}{5}.$$

- **准线（Directrices）**
  - 对应焦点在 $(\pm c, 0)$ 的椭圆，准线是与 y-轴平行的直线：

$$x = \pm\frac{a}{e} \; = \; \pm\frac{5}{\frac{3}{5}} \; = \; \pm\frac{25}{3}.$$

  - 即两条竖直直线 $x = \frac{25}{3}$ 与 $x = -\frac{25}{3}$。
- **图形与注解（Figure 15.16）**
  - 如图所示，椭圆以原点为中心，长轴为 10（从 -5 到 5），短轴为 8（从 -4 到 4）。

> **小结**：主轴沿 x-轴、$a = 5$、$b = 4$、$c = 3$、$e = \frac{3}{5}$，准线 $x = \pm\frac{25}{3}$。

（B）示例 2：纵向主轴的椭圆

- **给定方程：**

$$9x^2 + 4y^2 = 36.$$

  化为标准形式：

  1. 同除以 36 得
$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$
  2. 与 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 比较，这次

$$a^2 = 4,\ b^2 = 9.$$

  - 因为 $b^2 > a^2$，**主轴在 y-轴方向**，故此时可记 $a = 3, b = 2$ 也行，但往往我们仍将更大者视为 a。书中通常约定 $a \geq b$，因此这里

$$a = 3, \quad b = 2.$$

  - 这样意味着椭圆的"长轴"是上下方向，中心在 $(0, 0)$。

- **顶点与短轴端点：**
  - 顶点（长轴端点）在 $(0, \pm a) = (0, \pm 3)$。
  - 短轴端点在 $(\pm b, 0) = (\pm 2, 0)$。

- **焦点、离心率和准线：**
  1. $c^2 = a^2 - b^2 = 9 - 4 = 5 \Rightarrow c = \sqrt{5}$。
  2. 焦点在 $(0, \pm c)$，即 $(0, \pm\sqrt{5})$。
  3. 离心率 $e = c/a = \sqrt{5}/3$。
  4. 准线为水平直线：$y = \pm\frac{a}{e} = \pm\frac{3}{(\sqrt{5}/3)} = \pm\frac{9}{\sqrt{5}} = \pm\frac{9\sqrt{5}}{5}$.

- **图示（Figure 15.17）**
  - 椭圆在垂直方向上伸得更长，故其主轴是竖直的（从 -3 到 3 沿 y 轴）。

> **小结**：当 $\frac{y^2}{9}+\frac{x^2}{4}=1$ 时，中心在原点、主轴垂直、$a=3$、$b=2$、$c=\sqrt{5}$、$e=\frac{\sqrt{5}}{3}$，准线 $y=\pm\frac{9}{\sqrt{5}}$。

(C) 示例 3：一般二次方程的配平方求椭圆

- **给定方程**：

$$4x^2+16y^2\ -\ 24x\ -\ 32y\ =\ 12.$$

- **配平方步骤**
  1. 把 x 和 y 的项分组：

$$4(x^2-6x)+16(y^2-2y)=12.$$

  2. 给每组变量配平方：
     - x^2-6x 可写作 $(x-3)^2-9$。
     - y^2-2y 可写作 $(y-1)^2-1$。
  3. 代入得

$$4\big[(x-3)^2-9\big]\ +\ 16\big[(y-1)^2-1\big]=12.$$

$$4(x-3)^2-36\ +\ 16(y-1)^2-16=12.$$

$$4(x-3)^2+16(y-1)^2=12+36+16=64.$$

$$\frac{(x-3)^2}{16}+\frac{(y-1)^2}{4}=1.$$

- **结果分析**
  - 这是标准形 $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$，其中中心在 $(h,k)=(3,1)$。
  - $a^2=16\Rightarrow a=4$，$b^2=4\Rightarrow b=2$。因为 $a>b$，主轴水平。
  - 焦点距：$c^2=a^2-b^2=16-4=12\Rightarrow c=\sqrt{12}=2\sqrt{3}$。
  - 焦点：$(3\pm c,1)=\big(3\pm 2\sqrt{3},\,1\big)$。
  - 离心率：$e=\frac{c}{a}=\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$。
  - 准线：沿着主轴方向（水平）放置，即
  $$x=h\pm\frac{a}{e}\;=\;3\pm\frac{4}{\sqrt{3}/2}\;=\;3\pm\frac{4\times 2}{\sqrt{3}}\;=\;3\pm\frac{8}{\sqrt{3}}\;=\;3\pm\frac{8\sqrt{3}}{3}.$$
- **图示（Figure 15.18）**
  - 椭圆中心在 $(3,1)$，主轴长 $2a=8$，水平定向；顶点分别是 $(3\pm 4,\,1)$。

> **小结**：此例演示了"配平方"在判断二次方程是否椭圆并求其几何要素时的重要性。最终得出中心 $(3,1)$、$a=4$、$b=2$、$c=2\sqrt{3}$、$e=\frac{\sqrt{3}}{2}$、准线 $x=3\pm\frac{8\sqrt{3}}{3}$。

（D）椭圆的反射性质与应用（Figure 15.19）

1. **反射定理简述**
   - 类似抛物线具有"焦点—平行光线"反射特性，椭圆也有一个**反射性质**：
     > 若 P 为椭圆上一点，F、F' 分别是其焦点，作过 P 的切线 T，记 α 为切线与 PF 的夹角，β 为切线与 PF' 的夹角，则

$$\alpha = \beta.$$

- 这意味着：从一个焦点 F 发出的光线碰到椭圆后，会**反射通过另一个焦点** F'。

2. **“耳语长廊”(whispering gallery)**
   - 如果把椭圆绕长轴旋转，可得一个**旋转椭圆体**（三维）；将其上半部分做成房间的屋顶，并把两个焦点位置留在房间内合适的高度，则在一个焦点处轻声耳语，便能在另一个焦点处清晰听到，尽管中间位置也许很微弱。
   - 这种现象在某些科学博物馆或欧洲城堡里的“回音室”中可以体验到。
3. **医疗应用：体外碎石术 (lithotripsy)**
   - 医生将椭圆反射器的一焦点放置在产生高能超声波的位置，另一焦点对准病人肾结石所在位置。
   - 超声波从第一个焦点发出，经椭圆面反射后聚焦到另一个焦点处，对结石精准“击碎”，并减少对周围组织的损伤。

> **小结**：椭圆虽无抛物线那样广泛的聚光、聚热应用，但其反射特性仍在建筑声学和医疗领域有巧妙运用。

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3. 本次内容总结

1. **标准方程判断与配方技巧**
   - 透过示例 1、2、3，我们系统演示了从二次方程识别椭圆、化简至 $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ 的过程。
   - 通过判断 a^2 与 b^2 大小可辨主轴方向，并据此求出顶点、焦点、离心率、准线等核心信息。
2. **反射定理及现实应用**
   - 椭圆的“从一个焦点到另一焦点”的反射特性，带来一些有趣的“耳语长廊”现象，以及在现代医学中运用椭圆反射器进行肾结石碎石。
   - 由此可见，圆锥曲线中的椭圆并非纯理论产物，亦具备实用价值。
3. **与前面知识的呼应**
   - 本节继续深化了圆锥曲线（尤其是椭圆）的坐标方程、几何参数与重点性质。连同前面对抛物线、双曲线的分析，可以见到所有圆锥曲线都可用统一的焦点—准线概念来描述，也都蕴藏着各自独特的反射或聚焦特性。

Mercury	.21	Saturn	.06
Venus	.01	Uranus	.05
Earth	.02	Neptune	.01
Mars	.09	Pluto	.25
Jupiter	.05		

Figure 15.20 Eccentricities of planetary orbits.

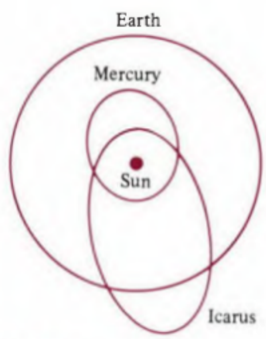

Figure 15.21

Remark 2 Except for small perturbations resulting from the influence of the other planets, each planet in the solar system revolves around the sun in an elliptical orbit with the sun at one focus. As we pointed out in Section 15.1, this phenomenon was discovered empirically by Kepler in the early seventeenth century, and was explained mathematically by Newton in the later decades of the same century. We shall give a detailed treatment of Newton's ideas at the end of Chapter 17.

Most of the planets, including the earth, have orbits that are nearly circular. This can be seen from the eccentricities given in the table in Fig. 15.20. Mercury, however, has a rather eccentric orbit, with $e = 0.21$, as does Pluto, with $e = 0.25$. Other bodies in the solar system have even more eccentric orbits, for instance, the flying mountains known as asteroids. Thus the asteroid Icarus, which was discovered at Mount Palomar in 1949 and is about 1 mi in diameter, has an orbit so eccentric, with $e = 0.83$, that at its closest approach to the sun (Fig. 15.21) it is halfway between the sun and the orbit of Mercury, and at its farthest it is out beyond the orbit of the earth.*

One of the most interesting objects in the solar system is Halley's Comet, which has eccentricity $e = 0.98$ and an orbit (Fig. 15.22) about 7 astronomical units wide by 35 astronomical units long. [One astronomical unit (AU) is the semimajor axis of the earth's orbit, approximately 93 million miles or 150 million kilometers.] The period of revolution of this comet around the sun is about 76 years. It appeared in 1910, and again in 1985–1986. It was observed in 1682, and the astronomer Edmund Halley (Newton's friend) successfully predicted its return in 1758, many years after his own death in 1742. This was one of the most convincing successes of Newton's theory of gravitation. At its closest approach,

*The surface temperature of Icarus has been estimated at about 900°F at its closest approach to the sun. Arthur C. Clarke has used this fact as the basis for a fine story, "Summertime on Icarus," in his collection *The Nine Billion Names of God* (New American Library, 1974). See also Chapter 2, "The Little Planets," in Fletcher G. Watson's *Between the Planets* (Doubleday Anchor Books, 1962), especially p. 29.

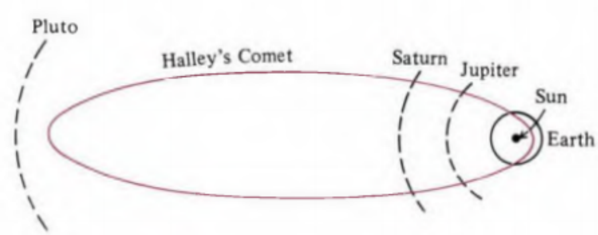

Figure 15.22 Orbit of Halley's Comet, drawn approximately to scale.

Halley's Comet is only 0.59 AU away from the sun. Its previous visits to the near neighborhood of the sun have been traced back step by step by means of historical records to the year 11 B.C., and perhaps even earlier.*

*For a more detailed account of these remarkable events, see P. L. Brown, *Comets, Meteorites and Men* (Taplinger, 1974); or N. Calder, *The Comet Is Coming!* (Viking, 1980).

1. 标题：行星与彗星的椭圆轨道与离心率——从几乎圆形到高度扁长的天体运动

本次课件最后部分（图 15.20 至图 15.22）介绍了太阳系中行星及一些特殊天体（如小行星 Icarus、哈雷彗星）的轨道离心率与椭圆轨道特征。通过表格、示意图和历史实例，展示了开普勒、牛顿所开创的“行星绕太阳呈椭圆轨道”理论在天文学中的重要地位，以及离心率的大小对轨道形状及天体运行距离变化的影响。

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2. 详细知识点与深入解析

（A）行星轨道离心率与近似圆形

1. **行星轨道的基本事实**
   - 根据表格（图 15.20）：水星 (Mercury) 的离心率约 $e = 0.21$，金星 (Venus) $e = 0.01$，地球 (Earth) $e = 0.02$，火星 (Mars) $e = 0.09$，木星 (Jupiter) $e = 0.05$，土星 (Saturn) $e = 0.06$，天王星 (Uranus) $e = 0.05$，海王星 (Neptune) $e = 0.01$，冥王星 (Pluto) $e = 0.25$。
   - 除去冥王星和水星，其他主要行星的轨道离心率都很小（通常在 $0.01 \sim 0.09$ 之间），因此它们绕太阳的轨道几乎接近圆形。
   - 17 世纪时，开普勒先以经验规律发现行星轨道为椭圆，随后牛顿在其万有引力理论中予以数学论证。
2. **冥王星和水星的相对偏长轨道**
   - 水星的离心率达到 0.21，冥王星约 0.25，因而在近日点和远日点距离上有更大差别；这也导致它们在轨道运行过程中所受太阳辐射强度变化更剧烈。

(B) 更高离心率的天体：Icarus 小行星

1. **Icarus 的轨道特征**

 - 1949 年在帕洛马山天文台 (Mount Palomar) 发现，直径约 1 英里，小行星 Icarus 的离心率高达 $e = 0.83$。
 - 它在近日点时半径可比地球轨道更靠近太阳（甚至介于水星与太阳之间），而在远日点时又能到达地球轨道之外，因而是高度椭圆的一种特殊小行星。
 - 文中提到其近日点温度可能达 900°F（约 480°C），凸显了高离心率轨道带来的极端温差。

(C) 哈雷彗星 (Halley's Comet)——极端椭圆轨道的典范

1. **基本参数与周期**

 - 哈雷彗星的离心率约 $e = 0.98$，半长轴约 17 天文单位 (AU)，远日点在 35 AU 左右；轨道形状极其细长，近日点则非常接近太阳。
 - 它绕太阳一周大约需要 76 年：1910 年、1985–1986 年先后两次回归，下一次将于 2061 年左右出现。

2. **历史观测与预测**

 - 1682 年人们观测到该彗星，牛顿的好友哈雷 (Edmund Halley) 成功预测它将于 1758 年回归（在哈雷本人去世多年后验证）。
 - 这是牛顿引力理论在天体力学上最令人信服的经典胜利之一：同一颗彗星在不同时期的再回归可精确追溯到公元前 11 年或更早。

3. **近日点与极端接近太阳**

 - 在近日点时，哈雷彗星只离太阳约 0.59 AU，比金星轨道还要更靠近太阳。
 - 高度扁长的椭圆轨道导致彗星在远日点时远离太阳数十倍天文单位，而近日点时则穿越行星轨道区域。

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3. 本次内容总结

1. **离心率与轨道形状**
   - 本节通过实例（行星、小行星 Icarus、哈雷彗星）说明了从近似圆形轨道（如金星离心率 0.01）到极度扁长轨道（如哈雷彗星 0.98）都有其天体存在。
   - 这直接体现了开普勒定律与牛顿万有引力原理中“椭圆轨道”的普适性。
2. **轨道力学的应用与历史意义**
   - 古人对彗星的周期性认识最初只停留在观测记录，直到开普勒、牛顿等人奠定了行星运动的数学模型，哈雷对彗星回归时间的预测才成为可能。
   - 如今，通过航天器探测、观测数据修正等，我们对高离心率天体的轨道和“近日点冲击”有了更多了解。
3. **太阳系多样性**
   - 太阳系并不只有近圆形轨道，还包含了各种大小天体在不同离心率椭圆轨道上运行，形成丰富多彩而又有规律的天体运动体系。

15.4 HYPERBOLAS

15.4 HYPERBOLAS

The ideas of Section 15.1 allow us to define a *hyperbola* as the locus of a point P that moves in such a way that the difference of its distances from two fixed points F and F' (called the *foci*) is constant. If this constant is denoted by $2a$, with $a > 0$, then a little thought will show that the locus consists of two *branches,* as shown in Fig. 15.25, where the right branch is the locus of the equation $PF' - PF = 2a$ and the left branch is the locus of the equation $PF - PF' = 2a$. The defining condition for the complete hyperbola can therefore be written as

$$PF' - PF = \pm 2a. \tag{1}$$

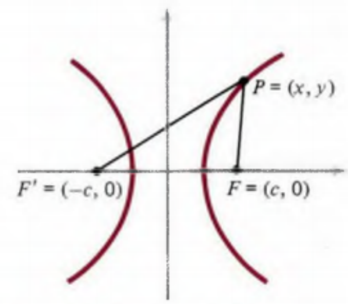

Figure 15.25 A hyperbola.

To find a simple equation for the hyperbola, we take the x-axis along the segment FF' and the y-axis as the perpendicular bisector of this segment. If $2c$ denotes the distance between F and F', then $F = (c, 0)$ and $F' = (-c, 0)$, as shown in Fig. 15.25, and (1) becomes

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a.$$

By moving the second radical to the right side, squaring, and simplifying, we obtain the focal radius formulas

$$PF = \sqrt{(x - c)^2 + y^2} = \pm\left(\frac{c}{a}x - a\right) \tag{2}$$

and

$$PF' = \sqrt{(x + c)^2 + y^2} = \pm\left(\frac{c}{a}x + a\right), \tag{3}$$

where (3) follows from (2) because $PF' = \pm 2a + PF$. As in (1), the plus signs here correspond to the right branch of the curve, and the minus signs to the left branch. By squaring and simplifying, either of these equations gives

$$\left(\frac{c^2 - a^2}{a^2}\right)x^2 - y^2 = c^2 - a^2$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1. \tag{4}$$

To simplify this equation still further, we begin by observing that in the triangle $PF'F$ with P on the right branch we have $PF' < PF + FF'$, because one side of a triangle is less than the sum of the other two sides. Therefore $PF' - PF < FF'$, or $2a < 2c$, so $a < c$ and $c^2 - a^2$ is a positive number which we denote by b^2,

$$b^2 = c^2 - a^2. \tag{5}$$

This enables us to write (4) as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{6}$$

which is the standard form of the equation of the hyperbola shown in Fig. 15.25.

We now turn to a careful consideration of equation (6) and the light it sheds on the nature of the hyperbola it represents. Our discussion will reveal several additional features of this curve that are not obvious from the definition and that are indicated in greater detail in Fig. 15.26.

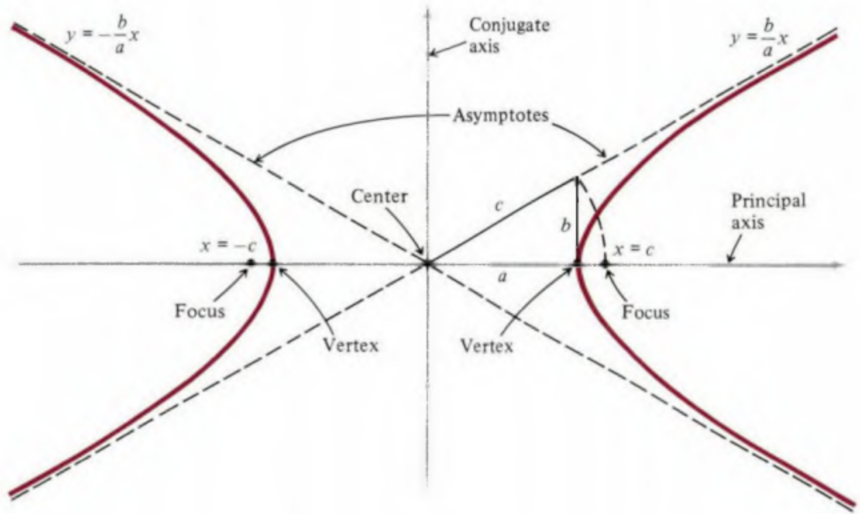

Figure 15.26 Features of a hyperbola.

Since the equation contains only even powers of x and y, the hyperbola is symmetric with respect to both coordinate axes. They are therefore called the *axes* of the curve, and their intersection is called the *center*. The left-right, up-down symmetry is perhaps the only feature of the hyperbola that is easy to see directly from the definition.

When $y = 0$, the equation gives $x = \pm a$, but when $x = 0$, y is imaginary. Therefore the axis through the foci, called the *principal axis,* intersects the curve at two points called the *vertices,* which are located at a distance a on each side of the center; but the other axis, called the *conjugate axis,* does not intersect the curve at all. The hyperbola thus consists of two separate parts, its symmetrical *branches,* on opposite sides of the conjugate axis.

These facts are easier to see if equation (6) is solved for y,

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}. \tag{7}$$

This formula shows that there are no points of the graph in the vertical strip $-a < x < a$, because for these x's the quantity inside the radical is negative. When $x = \pm a$, (7) yields $y = 0$; these two points are the vertices. And now, as x increases from a or decreases from $-a$, we get two distinct values of y that increase numerically as x moves farther to the right or left; this behavior produces the upper and lower arms of each branch of the curve.

A very significant feature of the graph can be observed by writing (7) in the form

$$y = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}}. \tag{8}$$

When x is numerically large, the quantity inside the radical in (8) is nearly 1, and for this reason it appears that the hyperbola is very close to the pair of straight lines

$$y = \pm\frac{b}{a}x. \tag{9}$$

We can verify this guess as follows. In the first quadrant, if x is large, then the vertical distance from the hyperbola up to the corresponding line is

$$\begin{aligned}\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2} &= \frac{b}{a}(x - \sqrt{x^2 - a^2}) \\ &= \frac{b}{a}\,\frac{(x - \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}} \\ &= \frac{ab}{x + \sqrt{x^2 - a^2}}.\end{aligned}$$

This clearly approaches zero as $x \to \infty$. The lines (9) are therefore called the *asymptotes* of the hyperbola. The asymptotes provide a convenient guide for sketching a hyperbola whose equation is given: Simply plot the vertices, draw the asymptotes, and fill in the two branches of the curve in a reasonable way, as suggested by the figure.

The triangle shown in the first quadrant of Fig. 15.26 is a convenient mnemonic device for remembering the main geometric features of a hyperbola. Its base a is the distance from the center to the vertex on the right; its height b is the distance from this vertex up to the asymptote in the first quadrant, whose slope is b/a; and since (5) tells us that

$$c^2 = a^2 + b^2,$$

the hypotenuse c of this triangle is also the distance from the center to a focus.

The ratio c/a is called the *eccentricity* of the hyperbola, and is denoted by e:

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} = \sqrt{1 + \left(\frac{b}{a}\right)^2}.$$

It is clear that $e > 1$. When e is near 1, then b is small compared with a, and the hyperbola lies in a small angle between the asymptotes. When e is large, then b is large compared with a, the angle between the asymptotes is large, and the hyperbola is rather flat at the vertices.

To understand the significance of the eccentricity, we consider again formulas (2) and (3) for the right and left focal radii PF and PF'. These formulas can be written as

$$PF = \pm(ex - a) = \pm e\left[x - \frac{a}{e}\right] \tag{10}$$

and

$$PF' = \pm(ex + a) = \pm e\left(x + \frac{a}{e}\right) = \pm e\left[x - \left(-\frac{a}{e}\right)\right], \tag{11}$$

where the plus signs apply to the right branch of the curve (see Fig. 15.27) and the minus signs to the left branch. If P lies on the right branch, as shown in the figure, then the quantities in brackets can be interpreted as the distances PD and PD' from P to the lines $x = a/e$ and $x = -a/e$, respectively. The same statement is true if P lies on the left branch, if the effect of the minus signs is properly taken into account. Therefore, in all cases formulas (10) and (11) can be written in the form

$$\frac{PF}{PD} = e \qquad \text{and} \qquad \frac{PF'}{PD'} = e. \tag{12}$$

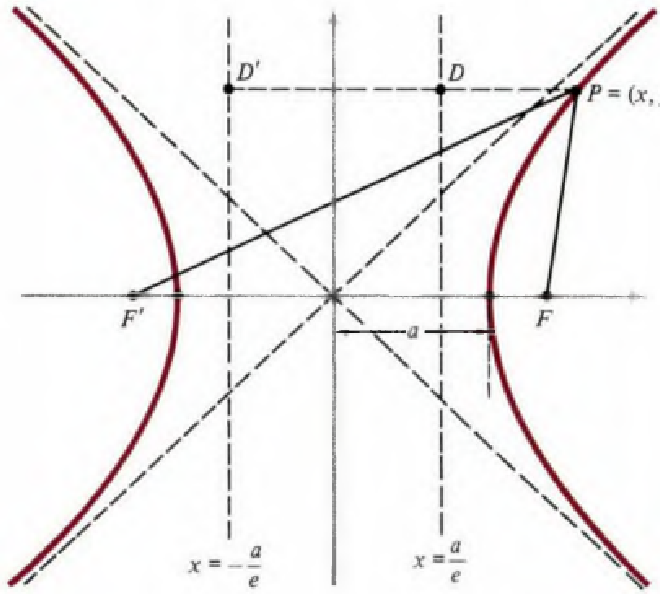

Figure 15.27

Each of the lines $x = a/e$ and $x = -a/e$ is called a *directrix* of the hyperbola. Equations (12) show that *a hyperbola can be characterized as the locus of a point that moves in such a way that the ratio of its distance from a fixed point (a focus) to its distance from a fixed line (the corresponding directrix) equals a constant* $e > 1$. Just as in the case of ellipses, this way of characterizing hyperbolas will be needed in our future work.

By interchanging the roles of x and y in the preceding discussion, we find that the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \tag{13}$$

represents a hyperbola with vertical principal axis, vertices at $(0, \pm a)$, and foci at $(0, \pm c)$, where $c^2 = a^2 + b^2$. This time the asymptotes are the lines

$$y = \pm \frac{a}{b} x,$$

as we easily see by writing (13) in a form solved for y,

$$y = \pm \frac{a}{b} \sqrt{x^2 + b^2} = \pm \frac{a}{b} x \sqrt{1 + \frac{b^2}{x^2}}.$$

Notice that the axis containing the foci of a hyperbola is not determined by the relative size of a and b, as it was in the case of an ellipse, but rather by which term is subtracted from which in the standard form of the equation. The numbers a and b can therefore be of any relative size. In particular they can be equal, in which case the asymptotes are perpendicular to each other and the hyperbola is called *rectangular*. The equations

$$x^2 - y^2 = a^2 \qquad \text{and} \qquad y^2 - x^2 = a^2$$

represent rectangular hyperbolas.

1. 标题：双曲线的焦点定义、标准方程与渐近线——从“距离差”到“焦点-准线”刻画

在第 15.4 节中，课件主要围绕**双曲线（Hyperbola）**展开，介绍了其“焦点-定差”定义、与椭圆相似但却又相反的几何构造，以及在坐标系下如何得到其标准方程和渐近线。此外，还说明了双曲线的另一种“焦点—准线”定义，并给出了不同方向主轴的双曲线方程。以下笔记将帮助我们系统掌握双曲线最核心的几何性质。

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2. 详细知识点与深入解析

（A）双曲线的“焦点-定差”定义

1. **定义**
   - 类比椭圆 “$\text{距离和} = \text{常数}$” 的定义，双曲线可定义为：
     > 平面上一点 P 的轨迹，使得其到两个固定点（焦点 F 和 F'）的距离差为常数。
   - 约定常数为 $2a$，则两个分支对应的方程为：
     $$\left| PF - PF' \right| = 2a.$$
   - 也可以拆分成两条分支：
     - 右分支满足 $PF' - PF = 2a$，
     - 左分支满足 $PF - PF' = 2a$。
2. **几何图像**
   - 在图 15.25 中，双曲线有左右两支，分别对应距离差为 $+2a$ 和 $-2a$。
   - 当点 P 非常靠近一个焦点时，另一焦点到 P 的距离很大，故能满足距离差为固定常数。
   - 与椭圆相反，双曲线呈开放状，两支相互对称地分布在焦点两侧。

(B) 坐标下的标准方程推导

1. **焦点选取与方程化简**

    - 将中心置于原点 $(0,0)$、焦点分别放在 $(c,0)$ 和 $(-c,0)$，主轴在 x-轴上。
    - 定义：$\left|PF' - PF\right| = 2a$。
    - 令点 P 为 (x,y)，则

    $$PF = \sqrt{(x-c)^2 + y^2}, \quad PF' = \sqrt{(x+c)^2 + y^2}.$$

    - 设"右分支"方程：

    $$PF' - PF = 2a. \tag{1}$$

    - 通过移项、平方、再移项、再平方的过程（与椭圆类似，但中间推导符号相反），可得：

    $$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1.$$

    - 由于在三角形关系中有 $PF' < PF + FF'$，以及 $c^2 > a^2$（因为双曲线焦点比顶点更远），可令

    $$b^2 = c^2 - a^2 \quad \Longrightarrow \quad c^2 = a^2 + b^2. \tag{5}$$

    - 最终化为**标准形式**：

    $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{6}$$

    其中 $c^2 = a^2 + b^2$。

2. **主轴、顶点、中心**

    - 如图 15.26 所示，"主轴（Principal Axis）"指穿过两个焦点的那条轴，中心在 $(0,0)$。
    - 顶点（Vertices）位于 $(\pm a, 0)$。
    - 另一条与主轴垂直的轴称为**共轭轴（Conjugate Axis）**，它与曲线本身不相交。

3. **对称性**

   - 因为方程只含有 x^2 和 y^2，对 x 轴和 y 轴都对称。双曲线有左右对称和上下对称四个象限分支，但真正的曲线只有左右两支；上下方向上则是“虚部不相交”的区域，配合渐近线表现出来。

(C) 渐近线 (Asymptotes)

1. **渐近线的来历**

   - 将方程 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 改写得到

   $$y^2 = \frac{b^2}{a^2}(x^2 - a^2).$$

   或

   $$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}.$$

   - 当 x 的绝对值非常大时，$\sqrt{x^2 - a^2} \approx |x|$，所以

   $$y \approx \pm\frac{b}{a}x.$$

   - 因而，当 $|x| \to \infty$，双曲线靠近两条直线

   $$y = \pm\frac{b}{a}\,x. \tag{9}$$

   - 这些直线称为**渐近线**。双曲线的两支在远离中心时就越来越接近渐近线，却永远不与之相交。

2. **几何意义**
   - 渐近线为我们快速绘制双曲线提供了参考：只要找准顶点、焦点、和渐近线的斜率，即可比较准确地画出双曲线的形状。
3. **当 $a = b$ 时的“正交”双曲线**
   - 如果 $a = b$，则渐近线互相垂直（$\pm 45^\circ$）。此时方程可写成

     $$x^2 - y^2 = a^2 \quad \text{或} \quad y^2 - x^2 = a^2,$$

     被称为**矩形双曲线（rectangular hyperbola）**。

(D) 离心率与焦点-准线定义

1. **离心率（Eccentricity）**
   - 对双曲线，也定义 $c^2 = a^2 + b^2$。其离心率为

     $$e = \frac{c}{a}.$$

   - 因为 $c > a$，所以 $e > 1$。若 e 越大，表示双曲线越“扁平”，两分支与主轴越开阔；若 e 刚大于 1 不多，则双曲线在顶点处张角较小，整体较“细长”。
2. **焦点—准线距离比定义**
   - 与椭圆 $(e < 1)$ 和抛物线 $(e = 1)$ 一脉相承，双曲线同样可用“焦点—准线”刻画：

     $$\frac{\text{点 } P \text{到焦点 } F \text{的距离}}{\text{点 } P \text{到准线的距离}} \;=\; e, \quad e > 1.$$

   - 在图 15.27 中，准线为平行于共轭轴的竖直线：$x = \pm \frac{a}{e}$。
   - 这说明了：**双曲线可以定义为这样一类点，其到焦点与到相应准线的距离之比为常数 > 1。**

(E) 垂直主轴情形

1. **更换 x 与 y 的角色**
   - 如果主轴垂直，则可令方程呈
   $$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \tag{13}$$
   - 此时焦点在 $(0, \pm c)$ 处，且仍满足 $c^2 = a^2 + b^2$。
   - 渐近线变为
   $$y = \pm \frac{a}{b}\, x.$$
   - 其他定理如焦点、顶点位置、离心率计算都类似，只是坐标轴方向对调。
2. **图形特征**
   - 这种情形下的两支分布在上下方向；常称这时的主轴是 y-轴，共轭轴是 x-轴。

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3. 本次内容总结

1. **双曲线的基本定义与标准方程**
   - 双曲线通过"到两焦点距离之差为常数（$2a$）"来定义；在坐标系中推导出 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 这一标准形式，并有 $c^2 = a^2 + b^2$。
2. **渐近线与图形构造**
   - 双曲线远离中心时逐渐逼近直线 $y = \pm \frac{b}{a}\, x$。顶点位于 $(\pm a, 0)$ 或 $(0, \pm a)$，形成左右（或上下）两支。
3. **离心率与焦点-准线定义**
   - $e = \dfrac{c}{a} > 1$，双曲线可用"$\frac{PF}{PD} = e$"刻画，其准线为 $x = \pm \dfrac{a}{e}$（或 $y = \pm \frac{a}{e}$）等形式。

4. **垂直主轴与矩形双曲线**
   - 若把 y 作为主轴，则方程 $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$，其余性质以同理可得；当 $a = b$ 时，渐近线互相垂直即成为矩形双曲线。

通过本节的学习，我们了解到双曲线与椭圆、抛物线一样，都有各自的焦点、准线与标准方程，也都能从圆锥切割视角进行统一理解——这是圆锥曲线的重要收尾部分。

Example 1 Find the equation of the hyperbola with foci $(\pm 6, 0)$ and the lines $5y = \pm 2\sqrt{5}\, x$ as asymptotes.

Solution First, the location of the foci tells us that the principal axis is the x-axis. We see that $c = 6$ and $b/a = \frac{2}{5}\sqrt{5}$, so $a = (\sqrt{5}/2)b$. Since $a^2 + b^2 = c^2$, we have $\frac{5}{4}b^2 + b^2 = 36$, so $b^2 = \frac{4}{9} \cdot 36 = 16$ and $a^2 = \frac{5}{4}b^2 = \frac{5}{4} \cdot 16 = 20$. This shows that

$$\frac{x^2}{20} - \frac{y^2}{16} = 1$$

is the equation of the hyperbola.

Example 2 Determine the principal axis of the hyperbola $6y^2 - 9x^2 = 36$ and find its vertices, foci, and asymptotes.

Solution The equation can be put in the standard form

$$\frac{y^2}{6} - \frac{x^2}{4} = 1,$$

so the principal axis is the y-axis, $a^2 = 6$, $b^2 = 4$, and $c^2 = a^2 + b^2 = 10$. Hence the vertices are $(0, \pm\sqrt{6})$, the foci are $(0, \pm\sqrt{10})$, and the asymptotes are $y = \pm(\sqrt{6}/2)x$.

Just as in the case of the ellipse, we can easily write the equation of a hyperbola with center (h, k) and principal axis parallel to one of the coordinate axes. The equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \text{or} \qquad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1,$$

according as the principal axis is horizontal or vertical. This suggests that we consider equations of the form

$$Ax^2 + By^2 + Cx + Dy + E = 0,$$

where A and B have opposite signs. Such an equation will usually represent a hyperbola, but in certain special cases it may represent a pair of intersecting straight lines. The next example illustrates these possibilities.

Example 3 Identify the graph of

$$16x^2 - 9y^2 - 64x - 18y + E = 0$$

for various values of E.

Solution The procedure is to complete the square on the x and y terms, which yields

$$16(x^2 - 4x) - 9(y^2 + 2y) = -E$$

and

$$16(x - 2)^2 - 9(y + 1)^2 = 55 - E.$$

There are now three cases.

CASE 1 $55 - E > 0$; for example, $E = -89$, so that $55 - E = 144$. In this case we have

$$\frac{(x-2)^2}{9} - \frac{(y+1)^2}{16} = 1,$$

which is a hyperbola with center $(2, -1)$ and horizontal principal axis.

CASE 2 $55 - E < 0$; for example, $E = 199$, so that $55 - E = -144$. Here we have

$$\frac{(y+1)^2}{16} - \frac{(x-2)^2}{9} = 1,$$

which is a hyperbola with center $(2, -1)$ and vertical principal axis.

CASE 3 $55 - E = 0$; $E = 55$. This time our equation becomes

$$16(x-2)^2 - 9(y+1)^2 = 0$$

or

$$4(x-2) = \pm 3(y+1).$$

This represents the two lines

$$y + 1 = \pm \tfrac{4}{3}(x-2),$$

which are the asymptotes in the first two cases.

1. 标题：焦点已知、渐近线已知的双曲线求方程——从关键参数到标准形式

在这道例题中（Example 1），我们要找的双曲线已给定了焦点 $(\pm 6, 0)$ 以及渐近线方程 $5y = \pm 2\sqrt{5}\,x$。本笔记将详述解题思路，展示如何利用焦点位置和渐近线斜率来确定标准方程。

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2. 详细知识点与过程解析

（A）已知信息与双曲线的主轴方向

1. **焦点 ± 6 在 x-轴上**
   - 两焦点分别是 $F = (+6, 0)$ 和 $F' = (-6, 0)$，说明**主轴在 x-轴** 上，且双曲线中心位于原点 $(0, 0)$。
   - 记双曲线的一般标准形为
     $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$
     其中焦点坐标为 $(\pm c, 0)$。由此可知 $c = 6$。
2. **渐近线：**$5y = \pm 2\sqrt{5}\,x$
   - 将其化为斜率形式：
     $$y = \pm \frac{2\sqrt{5}}{5}\,x \;=\; \pm \frac{2}{\sqrt{5}}\,x.$$
   - 对双曲线 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$，渐近线通常是
     $$y = \pm \frac{b}{a}\,x.$$
   - 因而我们可知
     $$\frac{b}{a} = \frac{2}{\sqrt{5}}.$$

- 整理得

$$b = \frac{2}{\sqrt{5}}\,a \;=\; \tfrac{2a}{\sqrt{5}}.$$

(B) 利用 $c^2 = a^2 + b^2$ 关系确定 a、b

1. **焦距关系**

- 对双曲线，有
$$c^2 = a^2 + b^2.$$
- 此处 $c = 6$，所以 $c^2 = 36$。
- 又由上面得到 $b = \frac{2a}{\sqrt{5}}$，故

$$b^2 = \frac{4a^2}{5}.$$

- 将 b^2 带入 $c^2 = a^2 + b^2$:

$$36 = a^2 + \frac{4a^2}{5}.$$

$$36 = a^2\left(1 + \frac{4}{5}\right) = a^2 \cdot \frac{9}{5}.$$

$$a^2 = \frac{36 \times 5}{9} = 20.$$

- 因而
$$a = \sqrt{20} = 2\sqrt{5}.$$
- 进而
$$b^2 = \frac{4a^2}{5} = \frac{4 \times 20}{5} = 16, \quad b = 4.$$

(C) 写出双曲线方程

1. **标准方程**
   - 由于主轴为 x-轴，标准形为
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$
   - 现已知 $a^2 = 20,\, b^2 = 16$，故
$$\frac{x^2}{20} - \frac{y^2}{16} = 1.$$
   - 这正是要求的双曲线方程。
2. **验证渐近线**
   - 此方程的渐近线应为
$$y = \pm\frac{b}{a}\,x \;=\; \pm\frac{4}{2\sqrt{5}}\,x \;=\; \pm\frac{2}{\sqrt{5}}\,x,$$
     等价于题目给出的 $5y = \pm 2\sqrt{5}\,x$。与题意吻合。
3. **小结**
   - 焦点：$\pm(6,0)$
   - 渐近线：$y = \pm\frac{2}{\sqrt{5}}\,x$
   - 关键参数：$c = 6,\; a^2 = 20,\; b^2 = 16$
   - 最终方程：$\dfrac{x^2}{20} - \dfrac{y^2}{16} = 1.$

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3. 本次内容总结

1. **核心步骤**
   - **根据焦点在 x-轴** → 选用 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 形式;
   - **渐近线斜率** → 得出 $\frac{b}{a}$;
   - **焦距公式** $c^2 = a^2 + b^2$ → 联立求解 a^2, b^2。
2. **结果**
   - 双曲线方程: $\dfrac{x^2}{20} - \dfrac{y^2}{16} = 1$。
   - 该方程的焦点正好是 $(\pm 6, 0)$，渐近线是 $5y = \pm 2\sqrt{5}\,x$。

这样的求解过程是典型的"已知焦点+渐近线→求双曲线方程"思路。在实际题目中，只要按此逻辑依次锁定中心位置、主轴方向、渐近线斜率与焦点距离关系，即可顺利完成对双曲线方程的求解。

Remark 1 Hyperbolas have the following reflection property: The tangent line at any point P on a hyperbola bisects the angle between the focal radii PF and PF'. This means that $\alpha = \beta$ in the notation of Fig. 15.28 (see Problem 21). As a consequence of this, if the hyperbola is revolved about its principal axis to form a surface of revolution, and if the convex sides of each part are silvered to make them reflecting surfaces, then any ray of light that approaches a convex side along a line pointing toward a focus (Fig. 15.28, right) is reflected toward the other focus.

This property of hyperbolas is the essential principle in the design of reflecting telescopes of the Cassegrain type (Fig. 15.29). As the figure shows, one focus of the hyperbolic mirror is at the focus of the parabolic mirror and the other is at the vertex of the parabolic mirror, where an eyepiece or camera is located.

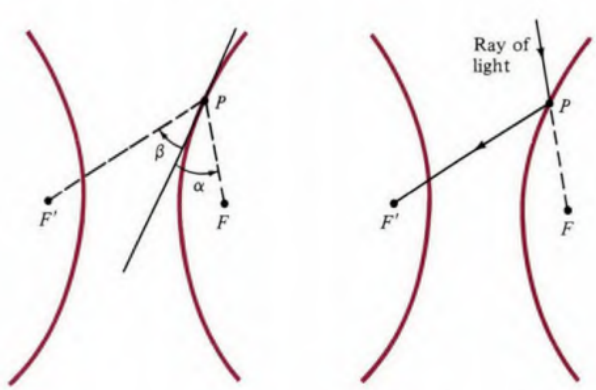

Figure 15.28 The reflection property.

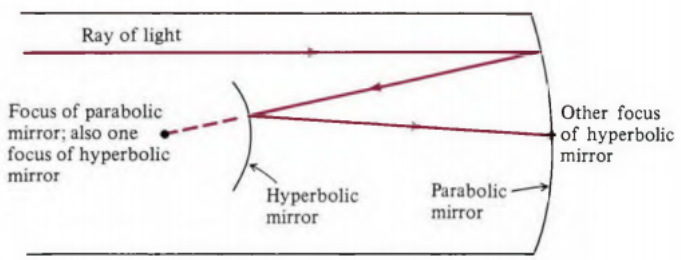

Figure 15.29 Design of Cassegrain telescope.

Faint parallel rays of starlight are therefore reflected off the parabolic mirror toward its focus, then are intercepted by the hyperbolic mirror and reflected back toward the eyepiece or camera.

Remark 2 There are two kinds of comets. Some are permanent members of the solar system, like Halley's Comet described in Section 15.3, and travel forever around the sun in elliptical orbits with the sun at one focus. Others enter the solar system at high speeds from outer space, swing around the sun in hyperbolic orbits with the sun at one focus, and then escape into outer space again. The crucial factor is the total energy E of the comet itself, which is the sum of the kinetic energy due to its motion and the potential energy due to the gravitational attraction of the sun. It turns out that if $E < 0$, the orbit is an ellipse, and if $E > 0$, the orbit is a hyperbola. (The case $E = 0$ corresponds to a parabolic orbit, but this is exceedingly unlikely.)

1. 标题：双曲线的反射性质与Cassegrain望远镜设计——从几何到天文应用

本节（图 15.28 与图 15.29）主要讨论了**双曲线的反射性质**及其在**Cassegrain型反射望远镜**中的重要应用；并进一步在备注中介绍了太阳系彗星绕太阳运动时"椭圆轨道"和"双曲线轨道"的能量判据。以下是对图片及文字内容的系统整理与深度讲解。

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2. 详细知识点与深入解析

（A）双曲线的反射性质

1. **基本描述：切线的角平分性质**
   - 在图 15.28 中，取双曲线上一任意点 P，记双曲线的两个焦点为 F 和 F'。
   - 设过点 P 的切线与线段 PF 的夹角为 α，与线段 PF' 的夹角为 β。
   - **反射定理**：该切线会"等分"从焦点 F 到点 P 再到焦点 F' 的两条半径所形成的夹角，即 $\alpha = \beta$。
   - 直观理解：从焦点 F 射向点 P 的光线，若在点 P 处被双曲线"镜面"反射，则会经过另一焦点 F'。
2. **几何后果：绕主轴旋转的三维镜面**
   - 假设我们把双曲线绕其主轴旋转，得到一个三维旋转曲面（如图 15.28 右侧示意）。
   - 如果把该曲面的内壁表面做成镜面，那么**任意射向其中一个焦点的光线**，最终都会在镜面上沿反射定理转向**另一个焦点**。
   - 这类特性在"旋转椭圆面""旋转抛物面"等情形也有类似原理，但双曲面的焦点—焦点反射路径常被用于**Cassegrain望远镜**等高端光学系统的设计之中。

（B）Cassegrain型望远镜：抛物面镜 + 双曲面镜的巧妙组合

1. **基本结构（图 15.29）**
    - Cassegrain望远镜在正前方有一个**大口径抛物面主镜**，它聚焦来自遥远天体的几乎平行光线到其焦点。
    - 在该焦点处，还有一个**二次镜**——它的反射面呈**双曲面**。
    - 双曲镜一个焦点与抛物镜的焦点重合，另一个焦点则位于望远镜的目镜或相机所在位置（通常在望远镜的后部、主镜后面留下的一个“洞”）。
2. **光路原理**
    - 平行星光被抛物面主镜反射，收拢到主镜焦点。
    - 在主镜焦点处（兼作双曲面焦点），光线继续向前，遇到处于此焦点位置的双曲镜“凸侧”并被反射。
    - 利用**双曲面的焦点—焦点反射特性**，光最终会被引向双曲面另一个焦点，也就是望远镜后部的目镜或探测器位置。
    - 如图所示，最后的光路能在较短的镜筒内实现长焦距、高放大倍率的观测效果。
3. **优点与应用**
    - Cassegrain望远镜在天文观测、军用望远镜以及高端实验室光学系统中被广泛采用。
    - 双曲面的焦点位置调节与主镜组合，可有效减少像差并节省空间，便于大型或专业级光学系统设计。

（C）备注 2：彗星的椭圆轨道与双曲线轨道

1. **两类彗星轨道**
    - 课件指出：**有的彗星**（如哈雷彗星）属于太阳系“永久成员”，绕太阳做**椭圆**轨道运行；
    - **另一些彗星**可能来自外太空，以很高速度掠过太阳，运动轨迹是“高能量”的**双曲线轨道**，然后再次远离进入宇宙深空。
2. **能量判据**
    - 关键在于**彗星的总能量** E（动能 + 位能）：
        - 若 $E<0$，则天体会被太阳引力束缚成**椭圆轨道**，永远绕日运行；
        - 若 $E>0$，则对应**双曲线轨道**，天体在近太阳时急速掠过后最终逃逸；
        - 若 $E=0$，则是**抛物线**情况，但极不常见。
    - 这与牛顿—开普勒理论对天体运动轨道的分类相一致，也呼应了前面在圆锥曲线中通过离心率 $e<1,=1,>1$ 区分椭圆、抛物线、双曲线轨道的观点。

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3. 本次内容总结

1. **双曲线的反射定理**
   - 在任意点 P 上的切线，会等分从 P 到焦点 F 和 F' 的两条线段夹角($\alpha = \beta$)；因此射向一个焦点的光线将被反射到另一个焦点。
2. **Cassegrain望远镜设计原理**
   - 通过在抛物面主镜焦点处放置一个双曲面次镜，使得经过主镜初次聚焦的光线再次被双曲面"焦点—焦点"反射引向目镜或相机，从而在有限镜筒内实现长焦距、清晰成像。
3. **彗星轨道与能量**
   - 若彗星的总能量 $E < 0$，它在太阳系内呈椭圆轨道被引力束缚（如哈雷彗星）；若 $E > 0$，轨道为双曲线，天体仅短暂经过内太阳系后离开。
   - 这展现了圆锥曲线在天文学中的极端重要性：**同一引力定律下，行星、彗星可呈现椭圆或双曲线（及极少数抛物线）运动。**

本节由双曲线的几何反射性质出发，继而衔接到光学系统和天体物理，让我们更深刻地体会到几何曲线与自然现象之间的紧密联系。

15.5 THE FOCUS-DIRECTRIX-ECCENTRCITY DEFINITIONS

15.5
THE FOCUS-DIRECTRIX-ECCENTRICITY DEFINITIONS

Students have already seen that there are several distinct but equivalent ways of defining the conic sections, each with its own merits. We began with the definition by means of a given cone and a slicing plane that cuts through the cone more or less steeply, yielding our three types of curves by varying the degree of steepness. This three-dimensional approach is vivid and geometric, and provides a clear visual impression of what the curves look like. However, for the purpose of obtaining Cartesian equations for use in precise quantitative studies, we needed two-dimensional characterizations, and for this the focal properties discussed at the end of Section 15.1 turned out to be convenient. The concepts of eccentricity and directrix emerged in the course of our detailed work on ellipses and hyperbolas, and we saw that each of these curves can be given yet another two-dimensional characterization by means of a focus, a directrix, and an eccentricity. Our purpose in this brief section is to show that all three of the conic sections—parabolas, ellipses, and hyperbolas—can in this way be given unified definitions that depend directly on our original concept of these curves as sections of a cone.*

Our discussion is based on Fig. 15.32, which shows a cone with vertex angle α and a slicing plane with tilting angle β. This tilting angle can be defined as the angle between the axis of the cone and a normal line to the plane, but it plays its main role in our argument as the indicated acute angle of the right triangle PQD. The figure is drawn to illustrate the case of an ellipse, but the argument is valid for the other cases as well.

We begin at the beginning. Let there be inscribed in the cone a sphere which is tangent to the slicing plane at a point F, and tangent to the cone along a circle C. If d is the line in which the slicing plane intersects the plane of the circle C, we shall prove that the conic section has F as its focus and d as its directrix, and the facts about the eccentricity will emerge in the course of our discussion.

To this end, let P be a point on the conic section, let Q be the point where the line through P and parallel to the axis of the cone intersects the plane of C, let R be the point where the generator through P intersects C, and let D be the foot of the perpendicular from P to the line d. Then PR and PF are two segments which are tangent to the sphere from the same point P, and therefore have the same length,

$$PR = PF. \tag{1}$$

*For reasons that will soon be clear, circles must be excluded from this discussion, because the necessary geometric constructions are not possible when the slicing plane is perpendicular to the axis of the cone.

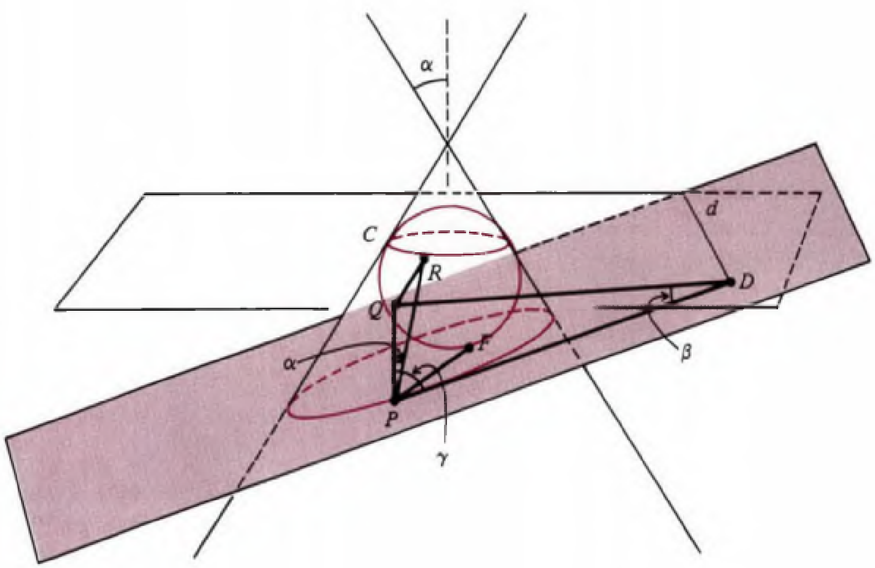

Figure 15.32

Also, from the right triangle PQR we have

$$PQ = PR \cos \alpha;$$

and from the right triangle PQD we have

$$PQ = PD \sin \beta.$$

It follows that

$$PR \cos \alpha = PD \sin \beta,$$

so

$$\frac{PR}{PD} = \frac{\sin \beta}{\cos \alpha}.$$

In view of (1) this means that

$$\frac{PF}{PD} = \frac{\sin \beta}{\cos \alpha}.$$

This can be written in the slightly more convenient form

$$\frac{PF}{PD} = \frac{\cos \gamma}{\cos \alpha}, \tag{2}$$

where γ is the other acute angle in the right triangle PQD. If we now define the eccentricity e by

$$e = \frac{\cos \gamma}{\cos \alpha},$$

then this number is constant for a given cone and a given slicing plane, and (2) becomes

$$\frac{PF}{PD} = e \begin{cases} < 1 & \text{for an ellipse,} \\ = 1 & \text{for a parabola,} \\ > 1 & \text{for a hyperbola,} \end{cases}$$

where the statements on the right are easily verified by inspecting the figure. Thus, for a parabola, we see that PD is parallel to a generator of the cone, so $\gamma = \alpha$ and $e = 1$; for an ellipse, we have $\gamma > \alpha$, so $\cos \gamma < \cos \alpha$ and $e < 1$; and for a hyperbola, we have $\gamma < \alpha$, so $\cos \gamma > \cos \alpha$ and $e > 1$.

The words "parabola," "ellipse," and "hyperbola" come from three Greek words meaning "a comparison," "a deficiency," and "an excess," referring to the fact that for the corresponding curves we have $e = 1$, $e < 1$, and $e > 1$. One should also compare these words with the words "parable," "ellipsis," and "hyperbole" in modern English.

1. 标题：焦点–准线–离心率统一定义：从圆锥切割到平面刻画的几何融合

在第 15.5 节中，课件提出了如何在三维圆锥切割的几何背景下，通过“焦点（focus）、准线（directrix）与离心率（eccentricity）”对所有圆锥曲线（椭圆、抛物线、双曲线）进行统一的平面定义。以下笔记帮助我们理解这一思路：先从切割圆锥的三维构造出发，再用简单的三角关系推导出“$\frac{PF}{PD} = e$”这一焦点-准线形式，并揭示何时 $e < 1$、$e = 1$、$e > 1$ 分别对应椭圆、抛物线、双曲线。

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2. 详细知识点与深入解析

（A）回顾：多种圆锥曲线定义

1. **三维圆锥切割定义**
   - 通过改变切割平面的倾斜角度，可得到“不过顶点”的三类截线：椭圆、抛物线、双曲线（以及垂直情况下的圆等特殊情形）。
   - 这种方法形象、生动，却不便于直接得到在平面上处理所需的代数方程。
2. **焦点-定长和/差定义**
   - 椭圆：$PF + PF' = 2a$。
   - 双曲线：$|PF - PF'| = 2a$。
   - 抛物线：$PF = PD$（到焦点与到准线距离相等）。
   - 这些定义使得在平面上分析圆锥曲线的几何性质非常直观，但仍希望寻找更一般化、统一的公式（尤其是抛物线、椭圆、双曲线之间的“距离比”视角）。

（B）课件构造：球面、切割平面和几何量度

1. **图 15.32 的圆锥、球面与切割平面**
   - 如图所示，在圆锥内部放置一个与圆锥、切割平面都相切的球。焦点 F 可视作球面与切割平面的一处切点（类似之前椭圆、双曲线证明焦点性质时的思路）。
   - 在切割平面上，得到的圆锥曲线就有一个焦点 F 和一个"准线" d。此时离心率 e 将通过圆锥的**顶角 α**和**平面倾角 β**或 γ 来确定。
2. **几何量：** α, β, γ
   - 在图中，α 是圆锥的顶角与对称轴之间的夹角，也可视为"生成线与轴的夹角"之一。
   - β 是切割平面相对于某些基准（如与圆锥轴线法线）的倾斜角。
   - 通过对比三角关系，可得 γ 为另一锐角，用来表达切割平面在另一几何位置上的倾角。

（C）推导关键：$\frac{PF}{PD} = \frac{\sin\beta}{\cos\alpha}$ 或 $\frac{\cos\gamma}{\cos\alpha}$

1. **辅助线段及三角形**
   - 令 P 为截线上一点、Q 为生成线与平面交点、R, D 分别是投影或垂足等几何构造点。
   - 通过一系列直角三角形（PQR、PQD 等）的关系，课件里先得到

     $$PR = PF, \quad PR\cos\alpha = PD\sin\beta, \quad \dots$$
   - 最终推导出

     $$\frac{PF}{PD} = \frac{\sin\beta}{\cos\alpha} \quad 或 \quad \frac{\cos\gamma}{\cos\alpha},$$

     视具体选取而定。
2. **离心率 e 的定义**
   - 令离心率 $e = \frac{\sin\beta}{\cos\alpha}$（或 $= \frac{\cos\gamma}{\cos\alpha}$），这是在给定圆锥（$\alpha$固定）和给定倾斜切割（$\beta$、$\gamma$固定）后得到的不变数。
   - 由此就可以解释：当 $e < 1$ 时曲线为椭圆，$e = 1$ 为抛物线，$e > 1$ 为双曲线。

（D）何时 $e<1,\ =1,\ >1$：椭圆、抛物线与双曲线

1. **几何含义**
   - 若β（或γ）取值使得$\sin\beta<\cos\alpha$，则 $\frac{\sin\beta}{\cos\alpha}<1$，对应**椭圆**。
   - 若$\sin\beta=\cos\alpha$，则 $\frac{\sin\beta}{\cos\alpha}=1$，对应**抛物线**（切割平面与圆锥母线平行）。
   - 若$\sin\beta>\cos\alpha$，则 $\frac{\sin\beta}{\cos\alpha}>1$，对应**双曲线**。
2. **与“焦点-准线”距离之比统一**
   - 在平面上，这就等价于 $\dfrac{PF}{PD}=e<1$（椭圆）、$=1$（抛物线）、>1（双曲线）。
   - 整个圆锥曲线家族于是可由一个统一的“$e=\frac{PF}{PD}$”来分门别类：
     - $0\le e<1$：椭圆（含极端 $e=0$ 为圆）
     - $e=1$：抛物线
     - $e>1$：双曲线

（E）命名由来：parabola, ellipse, hyperbola

课件最后提到这三个术语源自希腊语含义：

- **parabola** 来自“比较”（意指与某个状态“相等”），
- **ellipse** 来自“缺失”（小于那个状态），
- **hyperbola** 来自“过量”（大于那个状态）。
  恰与 $e=1, e<1, e>1$ 相吻合，体现了几何—语言的对应关系。

3. 本次内容总结

1. **统一思路**
   - 通过在圆锥+切割平面+内部球面的精妙几何构造，证明了“焦点-准线比 = 常数 e” 这一特征对抛物线、椭圆、双曲线都适用；也就把三类圆锥曲线统一到同一个公式 $\frac{PF}{PD} = e$。
2. **几何参数与圆锥倾斜角度**
   - 对于固定的圆锥（顶角 α不变），只有当切割平面不同倾斜度 β 变化时，才会改变 $\sin\beta$ 与 $\cos\alpha$ 的关系，从而得到不同曲线类型。
3. **椭圆、抛物线、双曲线的分界**
   - 若 $e < 1$ 对应椭圆，$e = 1$ 对应抛物线，$e > 1$ 对应双曲线。
   - 这与之前在平面上由“距离和=常数”/“距离差=常数”等定义相吻合，并在概念上更具统一性。

借助这一“焦点–准线–离心率”框架，我们完成了对圆锥曲线在三维（圆锥）与二维（平面）之间的最终统一说明，也为后续更深入的解析几何和应用奠定了坚实的理论基础。

15.6(OPTIONAL) SECOND-DEGREE EQUATIONS. ROTATION OF AXES

15.6

(OPTIONAL) SECOND-DEGREE EQUATIONS. ROTATION OF AXES

The general equation of the second degree in x and y is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

where at least one of the coefficients A, B, C is different from zero. The latter requirement, of course, guarantees that the degree of the equation really is 2, rather than 1 or 0. In the preceding sections we have found that circles, parabolas, ellipses, and hyperbolas are all curves whose equations are special cases of (1). Thus, for example, the circle

$$(x-h)^2+(y-k)^2=r^2$$

can be obtained from (1) by taking

$$A=C=1, \qquad B=0, \qquad D=-2h, \qquad E=-2k,$$
$$F=h^2+k^2-r^2,$$

and the parabola

$$x^2=4py$$

by taking

$$A=1, \qquad E=-4p, \qquad B=C=D=F=0.$$

In addition to the conic sections mentioned here, we have also noted various "exceptional cases" that can arise as graphs of (1) from special choices of the coefficients. Thus, the graph of

$$x^2+y^2=0$$

is a point, and the graph of

$$x^2+y^2+1=0$$

is the empty set. Further, the graph of

$$x^2=0$$

is a single line, namely, the y-axis, and the graph of

$$x^2-y^2=0, \qquad \text{or equivalently} \qquad (x+y)(x-y)=0,$$

is a pair of lines, namely, $x+y=0$ and $x-y=0$. Our purpose in this section is to investigate the full range of possibilities of the curves represented by (1). Briefly, we shall find that the eight graphs we have just listed exhaust all possibilities:

The graph of every second-degree equation of the form (1) *is a circle, a parabola, an ellipse, a hyperbola, a point, the empty set, a single line, or a pair of lines.*

The main problem before us is posed by the so-called *mixed term Bxy* in (1), because when this term is present we have no idea how to identify the graph. No such terms have arisen in our previous work on the conic sections. The reason for this is that in every case we have been careful to choose the coordinate axes in a simple and natural position, so that at least one axis is parallel to an axis of symmetry of the curve under discussion. In order to see what can happen when a curve is placed in a skew position relative to the axes, let us find the equation of the hyperbola (see Fig. 15.33) with foci $F=(2,2)$ and $F'=(-2,-2)$, where $PF'-PF=\pm 4$. We have

$$\sqrt{(x+2)^2+(y+2)^2}-\sqrt{(x-2)^2+(y-2)^2}=\pm 4,$$

and when we move the second radical to the right side, square, solve for the radical that still remains, and square again, this reduces to

$$xy=2. \tag{2}$$

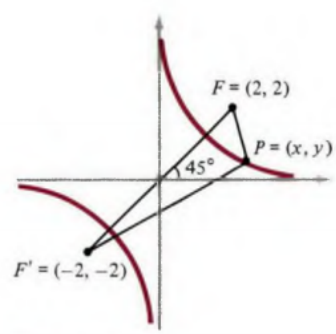

Figure 15.33

This is really a very simple equation, but nevertheless it does provide a special case of (1) in which the mixed term is present. The asymptotes of the hyperbola (2) are evidently the x- and y-axes, and its principal axis is the line $y = x$, which makes a 45° angle with the x-axis. It will become clear that a mixed term is present only when a curve is "tilted" in this way with respect to the coordinate axes, and also that this term can be removed by rotating the axes to "untilt" the curve. In the case of (2), it is easy to see by looking at the figure that this curve can be untilted by rotating the axes through a 45° angle in the counterclockwise direction.

To construct the machinery that is necessary for carrying out an arbitrary rotation of axes, we start with the xy-system and rotate these axes counterclockwise through an angle θ to obtain the $x'y'$-system, as shown in Fig. 15.34. A point P in the plane will then have two pairs of rectangular coordinates, (x, y) and (x', y'). To see how these coordinates are related, we observe from the figure that

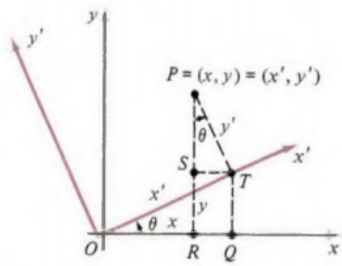

Figure 15.34 Rotation of axes.

$$\begin{aligned} x &= OR = OQ - RQ = OQ - ST \\ &= x' \cos \theta - y' \sin \theta \end{aligned}$$

and

$$\begin{aligned} y &= RP = RS + SP = QT + SP \\ &= x' \sin \theta + y' \cos \theta. \end{aligned}$$

We write these equations together for convenient reference,

$$\begin{aligned} x &= x' \cos \theta - y' \sin \theta, \\ y &= x' \sin \theta + y' \cos \theta; \end{aligned} \tag{3}$$

they are called the *equations for rotation of axes*. For example, if $\theta = 45°$, then, since $\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2} = 1/\sqrt{2}$, we have

$$x = \frac{x' - y'}{\sqrt{2}}, \qquad y = \frac{x' + y'}{\sqrt{2}}. \tag{4}$$

And for another example, if $\theta = 30°$, then since $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{1}{2}\sqrt{3}$, we have

$$x = \frac{\sqrt{3}x' - y'}{2}, \qquad y = \frac{x' + \sqrt{3}y'}{2}. \tag{5}$$

As a simple illustration of the use of these equations, we substitute (4) into (2) and obtain

$$\frac{x'^2 - y'^2}{2} = 2 \qquad \text{or} \qquad \frac{x'^2}{4} - \frac{y'^2}{4} = 1.$$

This is immediately recognizable as a rectangular hyperbola whose principal axis is the x'-axis. Of course, we already knew this from the way (2) was obtained. However, if we had started with (2) without knowing anything about the nature of its graph, then this procedure for removing the mixed term would have enabled us to identify the curve without difficulty.

In the case of equation (2), the 45° rotation represented by equations (4) worked. But how could we have known this in advance? Can we be sure that a suitable rotation will always remove the xy term if one is present? And if so, how do we find a suitable angle of rotation?

To answer these questions we return to the general second-degree equation (1),

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

and we apply the general rotation (3) through an unspecified angle θ, which yields

$$A(x' \cos \theta - y' \sin \theta)^2 + B(x' \cos \theta - y' \sin \theta)(x' \sin \theta + y' \cos \theta) + C(x' \sin \theta + y' \cos \theta)^2 + D(x' \cos \theta - y' \sin \theta) + E(x' \sin \theta + y' \cos \theta) + F = 0.$$

When we collect coefficients for the various terms, we get a new equation of the same form,

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0, \tag{6}$$

with new coefficients related to the old ones by the following formulas:

$$\begin{aligned}
A' &= A \cos^2 \theta + B \sin \theta \cos \theta + C \sin^2 \theta, \\
B' &= -2A \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) + 2C \sin \theta \cos \theta, \\
C' &= A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta, \\
D' &= D \cos \theta + E \sin \theta, \\
E' &= -D \sin \theta + E \cos \theta, \\
F' &= F.
\end{aligned} \tag{7}$$

We have written down all these formulas for future reference, but for the moment we are only interested in B'. If we start out with a second-degree equation (1) in which the mixed term is present, $B \neq 0$, then we can always find an angle θ of rotation such that the new mixed term is eliminated. To find a suitable angle θ, we simply put $B' = 0$ in (7) and solve for θ. To do this most easily, we use the double-angle formulas

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

and

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

to write

$$B' = B \cos 2\theta + (C - A) \sin 2\theta.$$

Then $B' = 0$ if we choose θ so that

$$\cot 2\theta = \frac{A - C}{B}. \tag{8}$$

Since we are assuming that $B \neq 0$, it is clear that this is always possible, and furthermore that θ can always be chosen in the first quadrant, $0 < \theta < \pi/2$.

1. 标题：二次方程的一般形式与坐标轴旋转——如何去除 xy 项并识别圆锥曲线

在第 15.6 节（可选内容）中，课件探讨了**二次方程一般形式**

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

所表示的曲线可能是圆、椭圆、双曲线、抛物线，也可能退化为点、直线或空集。并深入讲解了当出现"混合项" Bxy 时，**通过坐标轴旋转**可以将其消去，从而更轻松地识别所对应的曲线类型。以下是对图片与文字的系统整理与详细讲解。

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2. 详细知识点与深入解析

（A）二次方程的一般形式与各种圆锥曲线

1. **一般二次方程：**

   $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

   其中至少有一个 A、B、C 不为 0，才能保证它是二次方程而非一次或常数方程。

2. **特殊情形举例**
   - **圆**：$(x-h)^2 + (y-k)^2 = r^2$
     通过令 $A = C = 1$、$B = 0$、$D = -2h$、$E = -2k$、$F = h^2 + k^2 - r^2$ 等可得。
   - **抛物线**：$x^2 = 4py$ 或类似形式，通过相应系数设置得到。
   - **椭圆、双曲线**：前面章节已展示标准形如何对应到上述系数。
   - **退化情形**：某些选取会让曲线变成一个点、空集或一条直线乃至两条相交直线。

3. **问题的关键：混合项** Bxy
   - 当方程中出现 $B \neq 0$ 时，说明该曲线的"主轴"很可能不平行于 x 轴或 y 轴，而是呈某种倾斜状态。
   - 在之前学习圆锥曲线时，为了简化，我们通常选取恰当的坐标系，使曲线的主轴与坐标轴对齐，因而 $B = 0$。现在如果"原坐标系"不是对齐好的，就可能会出现 $B \neq 0$。

（B）坐标轴的旋转与混合项的消除

1. **旋转坐标系的几何背景**
   - 如图 15.33 所示，如果曲线的对称轴（或渐近线等）与原来的 x 轴、y 轴成某个夹角 θ，我们可以**逆时针旋转**坐标轴，使新的 x' 轴与原曲线的主轴平行。
   - 这样，曲线在 (x', y') 坐标系下通常就没有混合项 $x'y'$ 了。
2. **坐标旋转变换公式**
   - 在平面上，若将 (x, y) 逆时针旋转 θ 变为 (x', y')，则

$$\begin{cases} x = x'\cos\theta \ - \ y'\sin\theta, \\ y = x'\sin\theta \ + \ y'\cos\theta. \end{cases} \tag{3}$$

   - 这组公式可以将任何点从新坐标系 (x', y') 转换回原坐标系 (x, y)；反之亦然。
3. **将二次方程代入旋转公式**
   - 若我们把 (x, y) 用 $(x'\cos\theta - y'\sin\theta,\ x'\sin\theta + y'\cos\theta)$ 替换回去，就得到在 (x', y')-系下的新方程：

$$A'x'^2 \ + \ B'x'y' \ + \ C'y'^2 \ + \ D'x' \ + \ E'y' \ + \ F' = 0,$$

     其中 A', B', C', D', E', F' 是旧系数与 θ 的某些组合。

4. **如何选 θ 使混合项消失?**
   - 新方程中的"混合项"系数为 B'。课件给出
   $$B' = B\cos 2\theta + (C - A)\sin 2\theta. \quad (\text{或用另一种等价表达})$$
   - **若要消去 $x'y'$ 项**，只需令 $B' = 0$。得到的条件是：
   $$\tan 2\theta \;=\; \frac{B}{A - C}, \quad \text{或} \quad \cot 2\theta \;=\; \frac{A - C}{B}.$$
   - 由于我们假设 $B \neq 0$，所以总能找到这样一个 θ。从而，坐标轴旋转后就没有混合项了。
5. **旋转后曲线更容易辨认**
   - 消去 B' 后，我们可以看新的二次方程是 $\frac{x'^2}{\alpha^2} \pm \frac{y'^2}{\beta^2} = 1$（椭圆或双曲线），或 $x'^2 = 4py'$（抛物线），等等，从而迅速判定曲线类别。

(C) 示例：方程 $xy = 2$ 的旋转

1. **原方程**
   $$xy = 2 \quad \Longleftrightarrow \quad xy - 2 = 0.$$
   其中 $A = 0$，$B = 1$，$C = 0$，$D = E = 0$，$F = -2$。这是一个**带有混合项** xy 但无 x^2 或 y^2 的特殊二次方程。
2. **几何认识**
   - 从图 15.33 可以看出，这是**一条矩形双曲线**，它的渐近线恰好是 $x = 0$ 与 $y = 0$；而"主轴"是被旋转 45° 的坐标系。
   - 若用公式 $\theta = 45^\circ$ 作旋转，就能把 $xy = 2$ 化成无混合项的双曲线标准形。

3. **操作示例**

   - 若令 $\theta = 45^\circ$，则

$$\begin{cases} x = \frac{x'-y'}{\sqrt{2}}, \\ y = \frac{x'+y'}{\sqrt{2}}. \end{cases}$$

   - 代入 $xy = 2$ 后，做代数简化，可得

$$\frac{x'^2}{4} - \frac{y'^2}{4} = 1, \quad \text{或} \quad \frac{x'^2}{2} - \frac{y'^2}{2} = 2.$$

   - 这就显现出它是一条**"方程形如 $\frac{x'^2}{a^2} - \frac{y'^2}{a^2} = 1$" 的双曲线**，主轴对应于 x'-轴。

（D）二次方程曲线类型的完整分类

课件总结道：

1. **在不退化的情况下，所有形如 $Ax^2 + Bxy + Cy^2 + \cdots = 0$ 的二次方程图形必定是**

   - **椭圆**（含圆为特殊椭圆）、
   - **双曲线**、
   - **抛物线**、
     这三种基本圆锥曲线之一；

2. **退化情况**包括：

   - 一条直线或两条相交直线、
   - 一点（当椭圆退化）、
   - 空集（无实际点）。

当 $B \neq 0$ 时，可通过**旋转坐标系**将混合项消去并看出曲线的真实面目；若 $B = 0$ 且满足其他条件，我们常直接用“配平方”等方法识别曲线类型。

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3. 本次内容总结

1. **一般二次方程**
   - 形式：$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$，可代表圆锥曲线（椭圆、双曲线、抛物线）或退化为点、直线、空集。
   - 出现混合项 Bxy 往往暗示曲线的主轴不是与坐标轴对齐，而是“倾斜”的。
2. **轴旋转去除 xy**
   - 通过把坐标系逆时针旋转一个角度 θ，我们可把方程改写到 (x', y')-系中，使新的混合项系数 $B' = 0$。
   - 这样就能转化到我们熟悉的标准形，快速判断曲线是椭圆、抛物线还是双曲线。
3. **判别与应用**
   - 选取合适的 θ 满足 $\tan 2\theta = \frac{B}{A-C}$，便可完成去混合项操作。
   - 在几何和物理应用中，若某圆锥曲线“斜放”在原坐标系里，通过旋转坐标系能让分析更简洁、图形更显结构化。

因此，本节为我们提供了一个**“坐标旋转”**的强大工具，用来识别和简化任何一般二次方程所代表的曲线——这是高等解析几何的重要方法之一。

CHAPTER 15 REVIEW: DEFINITIONS, PROPERTIES

CHAPTER 15 REVIEW: DEFINITIONS, PROPERTIES

Think through the following.

1 Conic sections from a cone.
2 Parabola, focus, directrix, axis, vertex.
3 Equation of parabola — standard form.
4 Reflection property of parabolas.
5 Ellipse, foci, major and minor axes, eccentricity.
6 Equation of ellipse — standard form.
7 Reflection property of ellipses.
8 Hyperbola, foci, principal axis, asymptotes, eccentricity.
9 Equation of hyperbola — standard form.
10 Reflection property of hyperbolas.
11 Focus-directrix-eccentricity definitions.

1. 标题：全面掌握圆锥曲线——从三维切割到焦点-准线定义及坐标方程

本章（第 15 章）系统介绍了圆锥曲线（抛物线、椭圆和双曲线）的几何起源、历史背景、标准方程、焦点性质与反射定理，以及在更广义的二次方程视角下如何通过坐标旋转消去混合项，最终得到它们的统一分类。以下按各要点依次回顾本章的主要内容、核心性质与典型应用示例。

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2. 详细知识点与深入解析

（A）圆锥曲线的三维来源与基本分类

1. **圆锥曲面与“nappes”**
   - 将一个点 V 与底面圆 C 上的所有点相连，即可生成上下两个对顶的圆锥面（称为两张“nappes”）。
   - 当一个平面不经过顶点而切割该圆锥，就可得到三类截线：**椭圆、抛物线、双曲线**。平面平行于某生成线时切得抛物线；若较“轻微”倾斜只割到一张 nappe 则得椭圆；如果割穿两张 nappe 则得双曲线。
2. **退化情况**
   - 当切割平面恰好通过顶点，可产生退化圆锥曲线：一条直线、两条相交直线或一个点。
3. **历史背景**
   - 伽利略利用抛物线刻画抛体运动；开普勒发现行星绕太阳运动为椭圆；牛顿从万有引力定律出发解释行星椭圆轨道、彗星可呈抛物线或双曲线掠过太阳等。

（B）抛物线：焦点—准线、标准方程与反射定理

1. **焦点—准线定义**
   - 在平面几何中，抛物线可定义为到**定点**（焦点 F）与到**定直线**（准线 d）距离相等的点集：

$$PF = PD.$$

2. **标准方程**
   - 选焦点 $(0, p)$、准线 $y = -p$ 并设 (x, y) 在抛物线上，推导得

$$x^2 = 4py \quad 或 \quad y^2 = 4px \quad （视开口方向而定）.$$

   - 配平方技巧可把一般形 $Ax^2 + Bxy + Cy^2 + \cdots = 0$ 化为抛物线标准形式。

3. **抛物线的反射性质**
   - 任意点 P 上的切线等分从焦点来的射线与切线反射后平行光线之间的夹角，故光线从焦点出发经抛物线反射后变为平行于对称轴。
   - 应用于探照灯、雷达天线等的抛物面反射镜设计。

（C）椭圆：焦点—定长和、主轴、离心率与反射性质

1. **焦点—定长和定义**
   - 椭圆为满足 $PF + PF' = 2a$ 的点集；几何作图常用"一根线段拉紧两端在焦点"勾画椭圆。

2. **主轴、次轴与椭圆标准方程**
   - 以中心为原点，焦点在 $(\pm c, 0)$；由 $\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$ 化简得

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad 其中\ c^2 = a^2 - b^2.$$

   - 顶点在 $(\pm a, 0)$，短轴半径为 b，离心率 $e = c/a < 1$。

3. **反射性质**
   - 从一焦点 F 发出的光线，在椭圆上反射后会通过另一个焦点 F'；该特性在某些"耳语廊"及医疗体外碎石术中有巧妙应用。

4. **圆为椭圆特例**
   - 当焦点合一（$c = 0$）则 $e = 0$，椭圆退化成圆。

（D）双曲线：焦点—定差、渐近线、离心率与反射性质

1. **焦点—定差定义**
   - 双曲线为满足 $|PF - PF'| = 2a$ 的点集，共有左右（或上下）两支。
2. **标准方程**
   - 若中心在原点、焦点在 $(\pm c, 0)$，则推导得

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{并有 } c^2 = a^2 + b^2.$$

   - 两支分别位于 $\pm x$-方向，渐近线方程为 $y = \pm \frac{b}{a} x$。
3. **离心率、反射性质**
   - 双曲线离心率 $e = \frac{c}{a} > 1$。
   - 反射定理：切线会等分从焦点来的射线夹角；若将双曲线绕其轴生成旋转曲面，可使从一个焦点入射的光线反射到另一个焦点——应用于 Cassegrain 望远镜的次镜设计。
4. **矩形双曲线**
   - 若 $a = b$ 则渐近线互相垂直（“正交”），称矩形双曲线，如 $xy = \text{常数}$。

（E）焦点-准线-离心率统一定义

1. **概念**
   - 抛物线：$\frac{PF}{PD} = 1$，
   - 椭圆：$\frac{PF}{PD} = e < 1$，
   - 双曲线：$\frac{PF}{PD} = e > 1$。
   - 其中 F 是焦点，D 是到准线的垂足，e 称为离心率。
2. **三维圆锥切割推导**
   - 选定圆锥顶角 α、切割平面倾斜角 β，借助球面辅助线可证 $\frac{PF}{PD} = \dfrac{\sin\beta}{\cos\alpha}$。若其比值小于 1（椭圆）、等于 1（抛物线）、大于 1（双曲线），与切割倾斜度相关。

（F）二次方程一般形式与坐标轴旋转

1. **一般二次方程**

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

   - 包含所有圆锥曲线及退化情形（点、直线、空集）。

2. **混合项 Bxy 与坐标旋转**
   - 若 $B \neq 0$，曲线“轴”或对称线并不与 x、y 轴对齐。通过**旋转坐标系**角度 θ 可消去 $B'x'y'$ 项，进而发现其真实形态（椭圆、双曲线或抛物线）。
   - 旋转变换公式：

$$\begin{cases} x = x'\cos\theta - y'\sin\theta, \\ y = x'\sin\theta + y'\cos\theta. \end{cases}$$

     选 θ 使得新方程无混合项。

3. **完整分类**
   - 不退化情形：必是椭圆（含圆）、抛物线或双曲线。
   - 退化情形：可能是一点、两条直线、空集等。

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3. 本章内容总结

1. **三维到二维**
   - 起初从“圆锥 + 切割平面”获取视觉化定义：一条平面曲线（椭圆、抛物线、或双曲线）。
   - 进一步在平面内用“焦点—准线”或“焦点—定长和/差”方式刻画。
2. **标准方程与几何性质**
   - 椭圆、抛物线、双曲线都能在平面坐标下化为特定的二次方程形式，并可利用配平方或坐标旋转消去混合项来识别曲线类型与关键参数（焦点、顶点、准线、离心率）。
3. **反射定理与应用**
   - 三类圆锥曲线各有焦点反射特性：
     - **抛物线**：焦点出发的光线反射后平行；
     - **椭圆**：焦点 F 出发，反射后通过另一个焦点 F'；
     - **双曲线**：一焦点入射的光线反射后通过另一焦点。
   - 广泛应用于灯塔、雷达、望远镜、声学房间等。
4. **离心率与统一视角**
   - 用 $\frac{PF}{PD} = e$ 统一区分：$e < 1$ 椭圆、$e = 1$ 抛物线、$e > 1$ 双曲线。
   - 这一视角凸显了圆锥曲线在几何与物理中的深层一致性。
