CHAPTER 16 POLAR COORDINATES

16.1 THE POLAR COORDINATE SYSTEM

16 POLAR COORDINATES

16.1 THE POLAR COORDINATE SYSTEM

As we know, a coordinate system in the plane allows us to associate an ordered pair of numbers with each point in the plane. This simple but powerful idea enables us to study many problems of geometry—especially the properties of curves—by the methods of algebra and calculus. Up to this stage of our work we have considered only the rectangular (or Cartesian) coordinate system, in which the emphasis is placed on the distances of a point from two perpendicular axes. However, it often happens that a curve appears to have a special affinity for the origin, like the path of a planet whose journey around its orbit is determined by the central attracting force of the sun. Such a curve is often best described as the path of a moving point whose position is specified by its direction from the origin and its distance out from the origin. This is exactly what polar coordinates do, as we now explain.

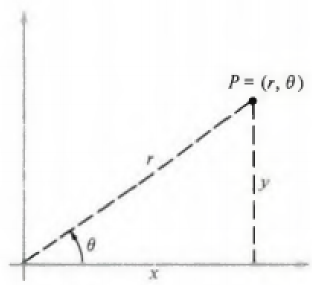

Figure 16.1 Polar coordinates.

A point is located by means of its distance and direction from the origin, as shown in Fig. 16.1. Direction is specified by an angle θ (in radians), measured from the positive x-axis. This angle is understood to be described in the counterclockwise sense if θ is positive and in the clockwise sense if θ is negative, just as in trigonometry. Distance is given by the directed distance r, measured out from the origin along the terminal side of the angle θ. The two numbers r and θ, written in this order as an ordered pair (r, θ), are called *polar coordinates* of the point. The direction $\theta = 0$ (the positive x-axis) is called the *polar axis*.

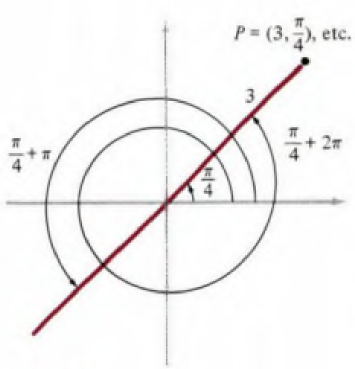

Figure 16.2

Every point has many pairs of polar coordinates. For instance, the point P in Fig. 16.2 has polar coordinates $(3, \pi/4)$, but it also has polar coordinates $(3, \pi/4 + 2\pi)$, $(3, \pi/4 - 4\pi)$, etc. Any multiple of 2π added to or subtracted from the θ-coordinate of a point yields another angle with the same terminal side, and therefore another θ-coordinate of the same point.

The term "directed distance" is intended to suggest that we often meet situations in which r is negative. In this case it is understood that instead of moving out from the origin in the direction indicated by the terminal side of θ, we move *back through the origin* a distance $-r$ in the opposite direction. Thus, another pair of polar coordinates for the point P in Fig. 16.2 is $(-3, \pi/4 + \pi)$. In Fig. 16.3 we plot the two points $Q = (2, \pi/6)$ and $R = (-2, \pi/6)$.

The value $r = 0$ specifies the origin, regardless of the value of θ. For instance, the pairs $(0, 0)$, $(0, \pi/2)$, $(0, -\pi/4)$ are all polar coordinates of the origin.

The fact that a point does not determine a unique pair of polar coordinates is a nuisance, but only a minor nuisance. Nevertheless, it *is* true that when any particular pair of polar coordinates is given, this pair determines the corresponding point without any ambiguity.

Even though it is incorrect to speak of *the* polar coordinates of a point because they are not unique, this error of usage is very common and is tolerated for the sake of euphony.

It is important to know the connection between rectangular and polar coordinates. Figure 16.1 shows at once that

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta. \tag{1}$$

When r and θ are known, these equations tell us how to find x and y. We also have the equations

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x}, \tag{2}$$

which enable us to find r and θ when x and y are known. In using these equations, it is necessary to take a little care to make sure that the sign of r and the choice of θ are consistent with the quadrant in which the given point (x, y) lies.

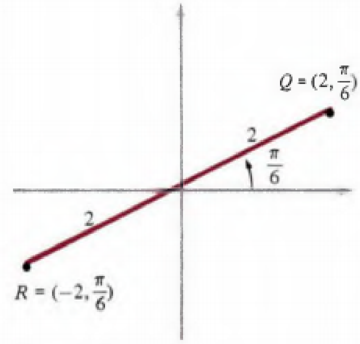

Figure 16.3

一、标题：极坐标系统的基本概念与直角坐标的转换

以下笔记整理自图片中关于"极坐标系统（Polar Coordinates）"的介绍与示例，力求使从未接触过极坐标概念的读者，也能通过本笔记掌握极坐标与直角坐标之间的转换，以及极坐标所涉及的关键思想与细节。

二、详细内容解析

1. 极坐标系统的动机与定义

1. **动机**
   - 在平面内，我们过去通常采用"直角坐标系（Cartesian Coordinates）"来标定点的位置（即 (x, y) 形式）。这样做的好处是便于很多代数与微积分问题的处理。
   - 然而，很多几何或物理情境中，尤其是与"与原点（或中心）距离"以及"绕某个中心旋转"有关的问题时，使用极坐标会更加自然。
   - 例如行星绕太阳运动：行星轨道可视为以太阳为中心，行星与太阳之间的距离及行星所对应的角度更直观地描述了它的运行位置。
2. **极坐标的基本要素**
   - **方向角** θ：从平面中"正 x 轴"出发，以**逆时针**为正方向所测得的角度，单位常用弧度（radians）。θ 可以是任何实数（不必限制在 0 到 2π 之间）。
   - **"有向距离"**r：从原点到目标点的"有向距离"，如果 r 为正值，则表示点位于与θ所代表方向相同的"射线上"；若 r 取负值（在更高阶的使用中会见到），则表示点位于与θ 的方向相反的延长线上，且距离为 $|r|$。
   - 当我们书写极坐标对 (r, θ) 时，即表示：**"该点距离原点 r，方向为与正 x 轴夹角 θ 的射线上。"**

3. **极坐标与点的唯一性**
   - 一个点在平面上，不会只对应唯一的一组极坐标 (r, θ)。造成这一点的主要原因：
     1. **角度可加** 2π：如果 θ 是一个满足要求的角度，那么 $\theta + 2k\pi$（k 为任意整数）依然代表相同方向。
     2. **距离可取负值**：(r, θ) 也可等效为 $(-r, \theta + \pi)$，因为把距离取反再让角度额外加 π 弧度，最终还是同一个点。
   - 尽管如此，一旦"固定"了某个具体的 (r, θ)，它所代表的点在平面上是**唯一**的。只是同一个点可以用多种不相同的极坐标描述，这一点在处理极坐标问题时必须牢记。

2. 图示与示例

1. **Figure 16.1（示意）：极坐标的定义**
   - 图中标出了点 $P = (r, \theta)$ 相对于原点 O，以及正 x 轴与向量 OP 之间的夹角 θ。
   - 距离 r 从原点 O 测量到 P。
   - $\theta = 0$ 表示正 x 轴方向。若 θ 正值，则逆时针旋转；若 θ 负值，则顺时针旋转。
2. **Figure 16.2：同一平面点拥有多组极坐标**
   - 例如，图中给出了点 P 可以有 $(3, 7\pi/4)$ 或 $(3, 7\pi/4 - 2\pi)$ 等等；也可以用 $(-3, 7\pi/4 + \pi)$ 来表示，这些都指向同一个位置。
   - 说明角度每增加或减少 2π，都会回到同一个方向上。
3. **Figure 16.3：示例点 Q 与 R**
   - 在图中，Q 以极坐标 $\left(2, \frac{\pi}{6}\right)$ 表示，含义是"距离原点 2，角度 $\frac{\pi}{6}$。"
   - 而 R 则示意了 $\left(-2, \frac{\pi}{6}\right)$ 的效果：它与 Q 在同一条直线上，但距离被取了负值，角度需要相应调整（通常也可写成 $\left(2, \frac{\pi}{6} + \pi\right)$）。

3. 极坐标与直角坐标的相互转换

1. **从极坐标到直角坐标**
   - 核心公式：
   $$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$
   - 只要知道 r 与 θ，就可以直接计算出 x 和 y。
2. **从直角坐标到极坐标**
   - 若已知点的直角坐标 (x, y)，则：
     1. $r = \sqrt{x^2 + y^2}$
     2. $\tan\theta = \frac{y}{x}$（需要注意 $x = 0$ 的特殊情况，以及根据 x, y 确定 θ 所在象限）
   - 在具体求 θ 时，要注意以下事项：
     - 如果 $x > 0$，则 $\theta = \arctan(\frac{y}{x})$ 即可（结果默认为 $-\frac{\pi}{2}$ 到 $\frac{\pi}{2}$ 或者 $(-\pi, \pi]$ 范围，看函数定义）。
     - 如果 $x < 0$，必须根据 y 的正负情况来**加** π 或在计算时手动调整，以使最终角度符合点所在的象限。
     - r 通常默认取非负值；若问题中允许“有向距离”，可以考虑 $-r$ 与角度相应偏移 π 的处理方式。
3. **关于“the polar coordinates of a point”措辞**
   - 书中提到：实际上一个点有无数组极坐标，因此严格说来不应当把“极坐标”称为该点的**唯一**标记，但在常规使用中，人们经常会说“该点的极坐标是 (r, θ)”而省略“可能多值”的情况，这是一种约定俗成的简化说法。
   - 只要记住遇到极坐标时，同一个点可有多种表示方式，但特定 (r, θ) 对应的点是唯一的。

4. 典型问题与注意事项

1. **角度与距离的符号**
   - 角度 θ 的取值范围在很多场景下会限制在 $[0, 2\pi)$ 或 $(-\pi, \pi]$ 之内，以便有一个"首选"或"主值"表述；但真正应用中，可以无限加减 2π。
   - 距离 r 默认通常为非负数，如果允许使用负距离来表示同一点，则需要配合角度 $\theta + \pi$ 的方式处理。
2. **象限判断**
   - 使用 $\tan\theta = \frac{y}{x}$ 时，必须先确认点 (x, y) 所在的象限来**修正** θ。
   - 例如，如果 (x, y) 在第三象限，单纯使用反正切函数（如 $\arctan\frac{y}{x}$）给出的值会落在第四或第二象限区间，需要再手动加 π 或做相应的计算。

三、本次图片所涉及内容总结

1. **极坐标的核心**：(r, θ) 描述了平面上某点到原点的距离以及与正 x 轴的夹角，且一个点有多组极坐标表示方法。
2. **与直角坐标的转换公式**：

$$x = r\cos\theta, \quad y = r\sin\theta; \quad r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)\ (\text{需根据象限修正}).$$

3. **负距离的扩展含义**：利用"有向距离"概念，可以允许 (r, θ) 中 $r < 0$，对应在几何上朝相反方向移动同样的长度。

Example 1 The rectangular coordinates of a point are $(-1, \sqrt{3})$. Find a pair of polar coordinates for this point.

Solution We have

$$r = \pm\sqrt{1+3} = \pm 2 \qquad \text{and} \qquad \tan\theta = -\sqrt{3}.$$

Since the point is in the second quadrant, we can use our knowledge of trigonometry to choose $r = 2$ and $\theta = 2\pi/3$, so one pair of polar coordinates for the point is $(2, 2\pi/3)$. Another acceptable pair with a negative value of r is $(-2, -\pi/3)$. Students should plot the point and have a clear visual understanding of each of these statements, as suggested by Fig. 16.4.

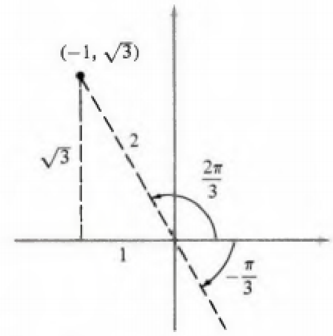

Figure 16.4

Just as in the case of rectangular coordinates, the *graph* of a polar equation

$$F(r, \theta) = 0 \tag{3}$$

is the set of all points $P = (r, \theta)$ whose polar coordinates satisfy the equation. Since the point P has many different pairs of coordinates, it is necessary to state explicitly that P lies on the graph if *any one* of its many different pairs of coordinates satisfies the equation.

Example 2 Show that the points $(1, \pi/2)$ and $(0, \pi/2)$ both lie on the graph of $r = \sin^2\theta$.

Solution The point $(1, \pi/2)$ lies on the graph because the given coordinates satisfy the equation: $1 = \sin^2 \pi/2$. On the other hand, the point $(0, \pi/2)$ lies on the graph even though $0 \neq \sin^2 \pi/2$. The reason for this seemingly strange behavior is that $(0, 0)$ is also a pair of coordinates for the same point, and $0 = \sin^2 0$.

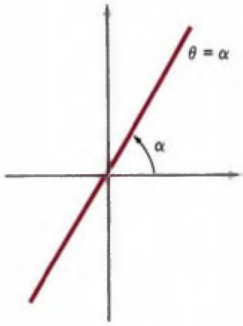

Figure 16.5

In most of the situations we will encounter, equation (3) can be solved for r and takes the form

$$r = f(\theta). \tag{4}$$

If the function $f(\theta)$ is reasonably simple, the graph is fairly easy to sketch. We merely choose a convenient sequence of values for θ, each determining its own direction from the origin, and compute the corresponding values of r that tell us how far out to go in each of these directions. We begin by discussing the simplest possible equations.

Example 3 The equation $\theta = \alpha$, where α is a constant, has as its graph the line through the origin that makes an angle α with the positive x-axis (Fig. 16.5).

Example 4 The equation $r = a$, where a is a positive constant, has as its graph the circle with center at the origin and radius a (Fig. 16.6).

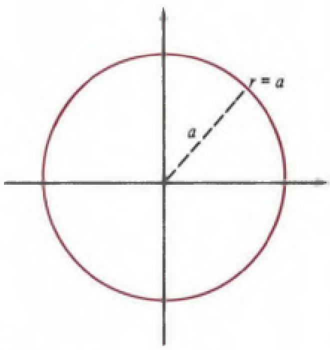

Figure 16.6

Our next example is more complicated, and serves to introduce several important methods.

Example 5 The graph of $r = 2 \cos \theta$ is another circle, but this is not obvious. One way to try to get an idea of the shape of an unknown polar graph is to compute a short table of selected values and plot the corresponding points, as shown in Fig. 16.7.

A better procedure than computing values and plotting points is to sketch the graph as the path of a moving point, by direct analysis of the polar equation, as follows. When $\theta = 0$, $r = 2 \cos 0 = 2$. As θ increases through the first quadrant, from 0 to $\pi/2$, $2 \cos \theta$ decreases from 2 to 0, and we obtain the upper part of the curve shown in Fig. 16.7. As θ increases from $\pi/2$ to π, $2 \cos \theta$ decreases from 0 to -2, and the lower part of the curve is traced out. As θ increases from π to $3\pi/2$, the upper part of the curve is retraced, and as θ increases from $3\pi/2$ to 2π, the lower part is retraced.

θ	r
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

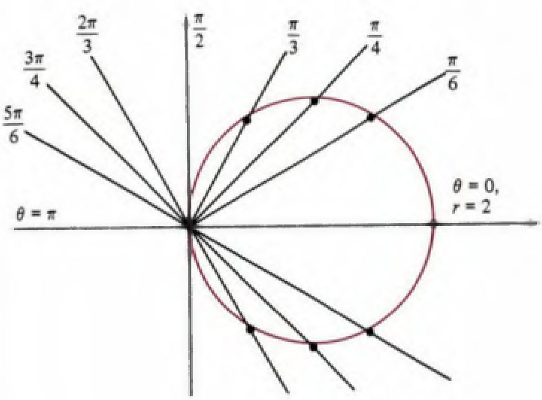

Figure 16.7

It is clear that the resulting graph is some kind of oval, perhaps even a circle. To verify that it really is a circle, we find and recognize the rectangular equation of the curve. To accomplish this, we multiply the given equation $r = 2\cos\theta$ by r and use the change-of-variable equations (1) and (2) to write

$$\begin{aligned} r^2 &= 2r\cos\theta, \\ x^2 + y^2 &= 2x, \\ x^2 - 2x + y^2 &= 0, \\ (x-1)^2 + y^2 &= 1. \end{aligned}$$

This last equation tells us that the graph is a circle with center $(1, 0)$ and radius 1. It should be pointed out that multiplying the given equation by r introduces the origin as a point on the graph; however, since this point is already on the graph, nothing is changed.

The method illustrated here, sketching a polar graph by direct examination of the polar equation $r = f(\theta)$, will often be important in our future work. Briefly, the process is this: We imagine a radius swinging around the origin in the counterclockwise direction, with our curve being traced out by a point attached to this turning radius which is free to move toward the origin or away from it in accordance with the behavior of the function $f(\theta)$. In many of our examples and problems, $f(\theta)$ will be a simple expression involving the trigonometric functions $\sin\theta$ or $\cos\theta$. In these circumstances it will clearly be very useful to have a solid grasp of the way these functions vary as the radius makes one complete revolution, that is, as θ increases from 0 to $\pi/2$, then from $\pi/2$ to π, π to $3\pi/2$, and $3\pi/2$ to 2π.

一、标题：极坐标方程与图形示例的系统讲解

以下笔记基于所附图片，结合了若干极坐标下的示例与其图形特征，旨在帮助读者深入理解如何在极坐标中表示点、如何根据极坐标方程绘制相应曲线，以及如何将极坐标方程转换为直角坐标方程来辨认图形形状。

二、详细内容解析

1. 例题讲解：由直角坐标求极坐标

Example 1：点 $(-1, \sqrt{3})$ 的一组极坐标

- 已知直角坐标：$(x, y) = (-1, \sqrt{3})$。
- 首先计算极径 r：

$$r = \pm\sqrt{x^2 + y^2} = \pm\sqrt{(-1)^2 + (\sqrt{3})^2} = \pm\sqrt{1+3} = \pm 2.$$

- 然后确定极角 θ（即与正 x 轴的夹角，逆时针为正）。由

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}.$$

  $\tan\theta = -\sqrt{3}$ 对应的参考角为 $\frac{\pi}{3}$，但是由于 (x, y) 在**第二象限**（$x < 0, y > 0$），所以实际角度应为

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

  - 这样选取时 r 为正：$r = 2,\ \theta = \frac{2\pi}{3}$。
  - 同一个点也可以用 r 取负来表示：$\left(-2, -\frac{\pi}{3}\right)$（因为 $-\frac{\pi}{3}$ 与 $\frac{2\pi}{3}$ 差了 π，且在几何位置上会落到同一点）。

因此图中给出的两组可接受的极坐标示例：

1. $\left(2, \frac{2\pi}{3}\right)$
2. $\left(-2, -\frac{\pi}{3}\right)$

> **要点：** 当我们说"某点的极坐标"时，往往选择 $r \geq 0$ 且 θ 在某个限定范围（如 $[0, 2\pi)$）作为"主值形式"。但实际上同一点可用多种 (r, θ) 表示。

---

2. 极坐标方程与"曲线"的概念

1. **极坐标方程 $F(r, \theta) = 0$ 的图形**
   - 与直角坐标系中"$F(x, y) = 0$ 的曲线"相似，在极坐标下，方程 $F(r, \theta) = 0$ 也代表**满足此方程的所有点** (r, θ) 的集合。
   - 因为一个点可以有许多不同的极坐标表示（例如角度加减 2π 或者 r 变号再调整 θ），所以更严格地说：**若某一组 (r, θ) 满足方程，则这个几何点就在该方程所描述的图形上。**
2. **Example 2：在曲线 $r = \sin^2 \theta$ 上的两个点**
   - 要说明 $(1, \frac{\pi}{2})$ 与 $(0, \frac{\pi}{2})$ 都在 $r = \sin^2 \theta$ 上。
   - **(1)** 对于 $\theta = \frac{\pi}{2}$ 而 $r = 1$ 时，有

     $$\sin^2\left(\frac{\pi}{2}\right) = \sin^2\left(\frac{\pi}{2}\right) = 1,$$

     所以 $r = 1$ 满足 $r = \sin^2(\frac{\pi}{2}) = 1$。
   - **(2)** 对于 $\theta = \frac{\pi}{2}$ 而 $r = 0$ 的点，看似与 $\sin^2(\frac{\pi}{2}) = 1$ 不符，但关键在于：
     - $(0, \frac{\pi}{2})$ 和 $(0, 0)$ 在平面上都指向**同一个几何点：原点**。
     - 而原点（$r = 0$）也可以通过 $(0, 0)$ 这组极坐标来表示；这时就能满足方程： $0 = \sin^2(0) = 0$。

- 因此，由于原点可有无数种角度描述 $(0, \theta)$，它也能被"视作"满足 $r = \sin^2 \theta$ 的点（如果用 $(0, 0)$ 这组来验证就可以）。

> **要点：** 在极坐标中，$(0, \theta)$ 对任意 θ 都表示**原点**。故某些"矛盾"往往是因为同一几何点有不同的极坐标表达所致。

---

3. 极坐标方程的常见简单形式

1. **Example 3：** $\theta = a$
   - 图形是一条**过原点，且与正 x 轴成角度** a 的射线（严格来说是无穷长的直线，通过原点）。在图中(Figure 16.5)即画出与 x 轴夹角 a 的直线。
2. **Example 4：** $r = a$
   - 若 $a > 0$ 为常数，则代表与原点为圆心、半径 a 的圆（见 Figure 16.6）。
   - 在极坐标下，"$r = a$" 说明所有到原点距离为 a 的点构成此曲线。

---

4. 更多示例：$r = 2\cos\theta$ 及其图形分析

Example 5：方程 $r = 2\cos\theta$

1. **数表法/点列法**
   - 可以选取若干关键角度 θ，计算对应的 r。例如：

$$\begin{aligned} &\theta = 0, \quad r = 2\cos 0 = 2, \\ &\theta = \frac{\pi}{6}, \quad r = 2\cos\left(\frac{\pi}{6}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}, \\ &\theta = \frac{\pi}{4}, \quad r = 2\cos\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}, \\ &\ldots \end{aligned}$$

   依此类推，把 θ 从 0 变到 2π，可以得到一系列 (r, θ) 点，标在极坐标图上，看出其大致形状。

2. **直接几何分析**
   - 我们可以利用**极坐标到直角坐标**的转换：$x = r\cos\theta$，$y = r\sin\theta$。方程 $r = 2\cos\theta$ 两边同乘 r 得：

$$r^2 = 2r\cos\theta \quad \Longrightarrow \quad x^2 + y^2 = 2x.$$

   - 整理得
$$x^2 - 2x + y^2 = 0 \quad \Longrightarrow \quad (x-1)^2 + y^2 = 1.$$
   - 这显然是一个**圆心在** $(1, 0)$、半径为 1 的圆（参见 Figure 16.7 右侧示意图）。
   - 注意：把方程两边同乘 r 会把 $r = 0$ 的点（原点）也自动包含进来；但原点其实已经在圆上：当 $(x, y) = (0, 0)$ 代入 $(x-1)^2 + y^2 = 1$ 也满足 $(0-1)^2 + 0^2 = 1$。

> **几何理解：** 当 θ 从 0 到 2π 变化时，$\cos\theta$ 在 $[1, -1]$ 来回变化，使得 $r = 2\cos\theta$ 从 2 递减到 -2 再回到 2，正负值也会使曲线反复"追踪"圆周的上半与下半部分。

5. 极坐标图形绘制的一般思想

- 书中提到的"方法示例"：**想象一条从原点伸出的可伸缩"半径"随 θ 逆时针旋转。**
- 在旋转过程中，根据方程 $r = f(\theta)$ 的值变化，半径会在不同角度下"伸长或缩短"，从而描绘出整条曲线。
- 对于很多常见的极坐标方程（尤其是涉及 $\sin\theta$、$\cos\theta$ 的简洁函数），一旦理解了 θ 在 $[0, 2\pi]$ 的变化过程，便能大致猜测出图形的形状，再配合将其转化为直角坐标方程作精确辨认。

三、本次图片所涉重点内容总结

1. **由直角坐标求极坐标**：
   - $r = \sqrt{x^2 + y^2}$ 或其负值，θ 由 $\tan\theta = \frac{y}{x}$ 并根据象限加以修正。
   - 一个点有多组极坐标，常选取 $r \geq 0$、$\theta \in [0, 2\pi)$ 或其他规定区间作为"主值"。
2. **极坐标方程** $F(r, \theta) = 0$：
   - 表示所有满足方程的 (r, θ)，可视为与直角坐标曲线类似的几何集合。
   - 必须考虑到同一几何点有多种 (r, θ) 表法。
3. **典型极坐标方程**：
   - $\theta = a$ —— 过原点的直线；
   - $r = a$ —— 圆心在原点的圆，半径 $|a|$；
   - $r = 2\cos\theta$ —— 圆心 $(1, 0)$、半径 1 的圆；
   - $r = \sin^2\theta$ 等函数形式可在 θ 范围内绘制，出现点重复或原点多角度表示的情况。
4. **图形识别技巧**：
   - 直接表格法（离散取点再连线）；
   - 转换为直角坐标方程（用 $x = r\cos\theta$、$y = r\sin\theta$ 及 $r^2 = x^2 + y^2$，必要时同乘 r 等方式），从而发现其本质为圆、直线或更复杂曲线。

16.2 MORE GRAPHS OF POLAR EQUATIONS

16.2 MORE GRAPHS OF POLAR EQUATIONS

We continue our program of getting better acquainted with polar graphs. In this section we concentrate particularly on sketching polar equations $r = f(\theta)$ of the type mentioned earlier, where $f(\theta)$ involves $\sin\theta$ or $\cos\theta$ in some simple way.

We again emphasize the change in point of view that is necessary for sketching polar equations. With rectangular coordinates and $y = f(x)$, we are accustomed to the idea of a point x moving along the horizontal axis and y as the directed distance measured up or down to the corresponding point (x, y) in the plane. We think in terms of "left-right" and "up-down."

With polar coordinates and $r = f(\theta)$, however, we must think of the angle θ swinging around like the hand of a clock turning in the wrong direction. For each θ we measure out from the origin a directed distance $f(\theta)$, and our moving point is farther out or closer in according as $f(\theta)$ is larger or smaller. We must think in terms of "around and around" and "in and out."

Example 1 The curve $r = a(1 + \cos\theta)$ with $a > 0$ is called a *cardioid.* When $\theta = 0$, $\cos\theta = 1$ and $r = 2a$. As θ increases from 0 to $\pi/2$ and on to π, $\cos\theta$ decreases from 1 to 0 to -1, so r decreases steadily from $2a$ to a to 0. This is shown in the upper half of Fig. 16.8. As θ continues to increase through the third and fourth quadrants, we see that $\cos\theta$, and with it r, retraces its values in reverse order, reaching $\cos\theta = 1$ and $r = 2a$ at $\theta = 2\pi$. Since $\cos\theta$ is periodic with period 2π, values of θ less than 0 or greater than 2π give points already sketched. The complete cardioid shown in the figure is evidently symmetric about the x-axis. The strange name this curve bears is accounted for by its fancied resemblance to a heart.

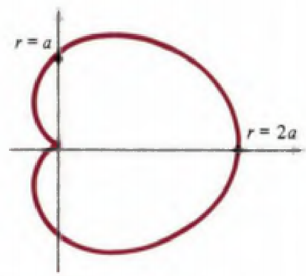

Figure 16.8 A cardioid.

When facing a polar equation, it is a natural temptation to try to return to familiar ground by converting immediately to rectangular coordinates. This is accomplished by using the transformation equations mentioned in Section 16.1,

$$r^2 = x^2 + y^2, \qquad \sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x}.$$

In the case of the cardioid discussed in Example 1, its equation $r = a(1 + \cos\theta)$ becomes

$$r = a\left(1 + \frac{x}{r}\right), \qquad r^2 = a(r + x), \qquad x^2 + y^2 - ax = ar,$$

and finally,

$$(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2).$$

This rectangular equation of the cardioid doesn't really tell us much. Clearly, it is better in this case to think exclusively in the language of polar coordinates. Nevertheless, there is a certain interest in seeing that the cardioid is a fourth-degree curve, in contrast to the second-degree curves discussed in Chapter 15.

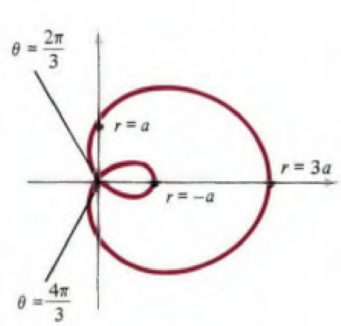

Figure 16.9 A limaçon.

Example 2 The curve $r = a(1 + 2\cos\theta)$ with $a > 0$ is called a *limaçon* (French for "snail"). When $\theta = 0$, $r = 3a$. As θ increases, r decreases, becoming 0 when $2\cos\theta = -1$, that is, when $\theta = 2\pi/3$. As θ continues increasing to π, r continues to decrease through negative values from 0 to $-a$, and the point whose movement we are following traces the lower half of the inner loop shown in Fig. 16.9.

Just as in Example 1, as θ continues to increase through the third and fourth quadrants, r retraces its values in reverse order; the inner loop is completed at $\theta = 4\pi/3$, and the outer loop is completed at $\theta = 2\pi$.

The curves in Examples 1 and 2 are both symmetric about the x-axis. We always have this kind of symmetry when r is a function only of $\cos\theta$, because of the identity $\cos(-\theta) = \cos\theta$. Similarly, if r is a function only of $\sin\theta$, then the curve is symmetric about the y-axis, because of the identity $\sin(\pi - \theta) = \sin\theta$.

We sometimes encounter curves whose equations have the form $r^2 = f(\theta)$. In this case, if θ is an angle for which $f(\theta) < 0$, then there is no corresponding point on the graph, because we must have $r^2 \geq 0$. But if θ is an angle for which $f(\theta) > 0$, then there are *two* corresponding points on the graph, with $r = \pm\sqrt{f(\theta)}$. These points are equally far from the origin in opposite directions, so the graph of $r^2 = f(\theta)$ is always symmetric with respect to the origin.

Example 3 The curve $r^2 = 2a^2 \cos 2\theta$ is called a *lemniscate*. For each θ there are two r's,

$$r = \pm\sqrt{2}a\sqrt{\cos 2\theta}. \tag{1}$$

As θ increases from 0 to $\pi/4$, 2θ increases from 0 to $\pi/2$ and $\cos 2\theta$ decreases from 1 to 0. Accordingly, the two r's in (1) simultaneously trace out the two parts of the curve shown on the left in Fig. 16.10. As θ continues to increase through the second half of the first quadrant and the first half of the second quadrant, 2θ varies through the second and third quadrants and $\cos 2\theta$ is negative, so there is no graph for these θ's. Through the second half of the second quadrant, $\cos 2\theta$ is positive again, and the two r's given by (1) simultaneously complete the two loops begun on the left in the figure. Further investigation reveals that no additional points are obtained, and the complete lemniscate is shown on the right. The name of this curve comes from the Latin word *lemniscus,* meaning a ribbon tied into a bow in the form of a figure eight.*

*The lemniscate was introduced by James Bernoulli in 1694. It played a considerable role in some of the early work of Gauss (in 1797) and Abel (in 1826) on elliptic functions and ruler-and-compass constructions in geometry. See M. Rosen, "Abel's Theorem on the Lemniscate," *Amer. Math. Monthly,* **88** (1981), pp. 387–395.

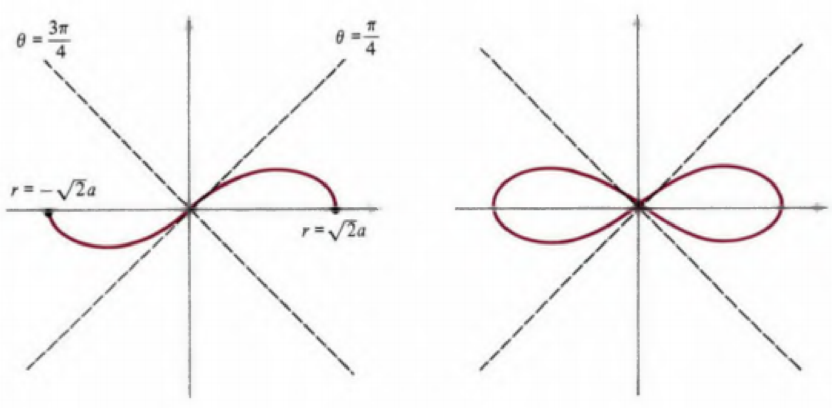

Figure 16.10 A lemniscate.

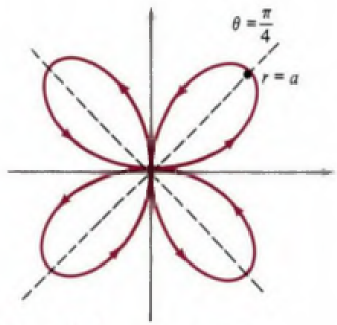

Figure 16.11 A four-leaved rose.

Example 4 The curve $r = a \sin 2\theta$ with $a > 0$ is called a *four-leaved rose,* for reasons that will become clear. To sketch it, we observe that as θ increases from 0 to $\pi/4$, 2θ increases from 0 to $\pi/2$ and r increases from 0 to a; and as θ increases from $\pi/4$ to $\pi/2$, 2θ increases from $\pi/2$ to π and r decreases from a to 0. This gives the leaf in the first quadrant (Fig. 16.11). Values of θ between $\pi/2$ and π (2θ between π and 2π) yield negative r's which trace out the leaf in the fourth quadrant; those between π and $3\pi/2$ (2θ between 2π and 3π) yield positive r's which trace out the leaf in the third quadrant; and those between $3\pi/2$ and 2π (2θ between 3π and 4π) produce negative r's and the leaf in the second quadrant.

We sometimes need to find the points of intersection of two curves that are defined by polar equations. It is natural to try to do this by solving the equations simultaneously. Unfortunately, this may not give *all* the points of intersection. The reason for this is that a point can lie on each of two curves and yet not have a pair of polar coordinates that satisfies both equations simultaneously. An extreme example of this behavior is provided by the two equations

$$r = 1 + \cos^2 \theta \qquad \text{and} \qquad r = -1 - \cos^2 \theta,$$

whose graphs are identical. In this case there are no simultaneous solutions because all the first r's are positive and all the second r's are negative, and yet there are infinitely many points of intersection.

What can be done about finding intersections? The most sensible approach is to depend on drawing good enough graphs of both equations on a single figure to see whether there are any points of intersection. When there are, it is usually possible to find the polar coordinates of these points either by solving simultaneous equations or else by observing where the points are by direct inspection of the figure.

一、标题：常见极坐标曲线及其图形特征（心形线、蜗牛线、二维带结曲线、玫瑰线）

以下笔记根据所附图片，对极坐标方程 $r = f(\theta)$ 的常见典型曲线进行分析，包括它们的生成方式、主要特点以及对称性等。通过这些示例，你可以了解到如何“绕着原点”去理解曲线在极坐标中的变化，并初步掌握简单的绘制思路与几何特征。

二、详细内容解析

1. 心形线（Cardioid）：$r = a(1 + \cos\theta)$

1. **基本形式与由来**

   - 对于常数 $a > 0$，方程

     $$r = a\big(1 + \cos\theta\big)$$

     称为一条**心形线（Cardioid）**。如图（Figure 16.8）所示，整条曲线在极坐标下看起来类似“心形”或“苹果形”。

2. **绘制与对称性**

   - 当 $\theta = 0$ 时，$\cos 0 = 1$，于是 $r = 2a$，这是曲线最远离原点的点（在正 x 轴方向上）。
   - 当 $\theta = \pi$ 时，$\cos\pi = -1$，于是 $r = a(1 - 1) = 0$，曲线经过原点。
   - $\cos\theta$ 在 $[0, 2\pi]$ 内变化会使 r 在 $[0, 2a]$ 之间来回摆动；并且曲线是关于 x 轴对称的，因为 $\cos(-\theta) = \cos\theta$。

3. **直角坐标下的表达**

   - 书中提到可用 $r^2 = x^2 + y^2$，以及 $\sin\theta = \frac{y}{r}$、$\cos\theta = \frac{x}{r}$ 做代换：

     $$r = a\big(1 + \tfrac{x}{r}\big) \implies r^2 = a(r + x) \implies x^2 + y^2 - ax - r^2 = 0,$$

     经过一系列整理得到一个“四次方程”形式。但这个直角坐标方程并不能像圆或椭圆那样一下看出形状，说明对“心形线”而言，**极坐标更直观**。

> **要点：** 心形线的形状在 x 轴上最外侧达 $r = 2a$、另一侧内陷到原点处，并且左右对称（以图中的 x 轴为对称轴）。

---

2. 蜗牛线（Limaçon）：$r = a(1 + k\cos\theta)$

1. **示例：** $r = a\big(1 + 2\cos\theta\big)$

   - 称为 Limaçon（法语里意为“小蜗牛”），如 Figure 16.9 所示。它的形状与心形线类似，但在 $k > 1$ 或不同取值时，会出现“内环(loop)”或凸出的不同形状。

2. **变化过程**

   - 当 $\theta = 0$ 时，$r = a(1 + 2) = 3a$；随 θ 增加，$\cos\theta$ 从 1 降到 -1，因此 r 从 3a 逐步减小到 -a。
   - 当 r 出现负值时，会在几何上形成曲线的“内环”部分。
   - 该曲线同样关于 x 轴对称。上半环与下半环分别在 θ 的区间里被描绘（正负 r 的概念会让曲线有回卷的效果）。

> **要点：** 与心形线类似，Limaçon 一般也对称于 x 轴，因为其方程只含 $\cos\theta$。不同系数 k 值会导致有无“内环”、凹凸不同的形状。

3. 二维“带结”曲线（Lemniscate）：$r^2 = f(\theta)$

1. **示例**：$r^2 = 2\,a^2\cos(2\theta)$
   - 称为 **Lemniscate**（源自拉丁语，意为“带子打成的八字形”），如 Figure 16.10。
   - 若令 $r^2 = 2a^2\cos(2\theta)$，则只要 $\cos(2\theta) > 0$，才有实数 r。当 $\cos(2\theta) < 0$ 时，r^2 为负值，曲线不在该方向上出现点。
2. **形状特点**
   - 将 θ 从 0 增加到 $\pi/4$ 时，$\cos(2\theta)$ 从 1 减到 0，r^2 从 $2a^2$ 到 0，故在第一象限得到曲线的一部分。
   - 再随 θ 进入第二象限，当 $\cos(2\theta)$ 为负时，此区间无实数 r。
   - 继续到 θ 的后续范围里，$\cos(2\theta)$ 再次变为正，就出现曲线另一个分支。
   - 整体看起来类似“∞”或“8”字的形状，对称于原点。

> **要点**：Lemniscate 的典型特征是“两个瓣”对称分布。当 $\cos(2\theta)$ 为正的角度区间内存在实数 r，为负时曲线没有对应点。这样拼接在一起形成双瓣图案。

---

4. 玫瑰线（Rose Curve）：$r = a\sin(n\theta)$ 或 $r = a\cos(n\theta)$

1. **Example 4**：$r = a\sin(2\theta)$，**四瓣玫瑰线**
   - 当 θ 从 0 增至 $\pi/4$ 时，2θ 从 0 到 $\pi/2$，$\sin(2\theta)$ 从 0 增加到 1，于是 r 从 0 到 a，描绘了第一瓣（位于第一象限）。
   - 随后 θ 再增大，2θ 会经历从 $\pi/2$ 到 π 的过程，这时 $\sin(2\theta)$ 由 1 下降到 0，对应曲线“回缩”到原点。
   - 同理，θ 继续向后，2θ 会遍历 π 到 2π 的区间，在第二、第三、第四象限分别画出剩余三瓣。
   - 最终得到 4 个对称的“花瓣”（Figure 16.11），故称**四瓣玫瑰线**。若是 $\sin(3\theta)$ 或 $\cos(3\theta)$ 等，则花瓣数会不同。
2. **负 r 的影响**
   - 当 $\sin(2\theta)$ 为负值时，即可让 r 为正但在相反方向，或直接视作 r 负值与角度偏移π，形成其他瓣。
   - 这些都在极坐标下体现了“花瓣”反复展开、收拢、对称的效果。

> **要点**：玫瑰线通常呈多瓣花状，$\sin(n\theta)$ 或 $\cos(n\theta)$ 里的 n 整数决定瓣数；若 n 为偶数，花瓣数通常是 $2n$；若 n 为奇数，花瓣数通常是 n。

---

5. 关于极坐标曲线交点的问题

- 书中提醒：如果我们想要找两条极坐标曲线的**交点**，仅仅通过联立方程 $F_1(r,\theta)=0$ 和 $F_2(r,\theta)=0$ 并求解，可能**无法**得到所有的几何交点。
- 原因在于：**同一个几何点**可能对应不同的 (r,θ) 表示方式。某些解在联立方程中因正负 r 关系等被忽视。
- 极端示例：$r=1+\cos^2\theta$ 与 $r=-1-\cos^2\theta$ 代表的是**同一条曲线**（只不过一个方程把 r 视为正，另一个把 r 视为负），但把二者联立不可能找到任何 (r,θ) 同时满足，因为一个要求 r 正、另一个要求 r 负。实际上它们却指向同一个几何集合。
- **实用方法**：在纸上先画出两条极坐标曲线的大致形状，再观察交点；或在联立方程时注意到 (r,θ) 多值性的影响，可能需要手动调整角度或 r 符号等。

三、本次图片所涉重点内容总结

1. **本节主旨**：通过更多曲线示例（心形线、Limaçon、lemniscate、玫瑰线等），让我们体会到极坐标方程的多样性，以及“$r=f(\theta)$”在 0 到 2π 范围内如何使曲线向外或向内摆动，从而形成各种美妙对称图案。
2. **对称性判定**：
   - 若方程只含 $\cos\theta$，往往关于 x 轴对称；
   - 若只含 $\sin\theta$，往往关于 y 轴对称；
   - 若含 $\cos(n\theta)$ 或 $\sin(n\theta)$，可能呈现倍数对称或多瓣式“玫瑰”形状。
3. **负 r 引发的“回折”**：极坐标可以允许 r 变为负，从而出现“内环”“双瓣”“花瓣对折”等现象；理解负 r 等效于 $r>0$ 同时角度偏移 π 弧度。
4. **交点谨慎**：同一点可对应不同 (r,θ)。要找曲线交点时，如果只做方程联立，常会漏解或产生不一致，需要结合图形或深入分析。

Remark People who enjoy geometry in school usually take special pleasure in construction problems. As students will perhaps recall, the Greek mathematicians of antiquity learned how to perform a great variety of intricate constructions with only ruler and compass allowed as tools for drawing straight lines and circles: For instance, an angle can be bisected; a segment can be trisected; the perpendicular bisector of a segment can be drawn; regular polygons with n sides, where $n = 3, 4, 5, 6$, can be constructed; etc. All of these constructions and many more have been known since the time of Euclid and Archimedes, and the details form an important part of the study of plane geometry.*

The creation of geometric constructions with ruler and compass alone, when considered as an intellectual game played according to clearly understood rules, was certainly one of the most fascinating and enduring games ever invented. The complicated constructions that turn out to be possible must be seen to be believed. Nevertheless, after ingenious and persistent efforts extending over more than 2000 years, there were three classical Greek construction problems that still remained unsolved at the beginning of the nineteenth century. These problems were:

*See Chapters III, V, and IX of H. Tietze, *Famous Problems of Mathematics* (Graylock Press, 1965).

1 *To trisect an angle,* that is, to divide a given angle into three equal parts
2 *To double a cube,* that is, to construct the edge of a cube with twice the volume of a given cube
3 *To square a circle,* that is, to construct a square whose area equals that of a given circle

In the course of the nineteenth century all three constructions were conclusively proved to be impossible under the stated conditions.

The traditional restriction to the use of ruler and compass alone seems to have originated with the ancient Greek philosophers, but the Greek mathematicians themselves did not hesitate to use other tools. In particular, they invented various bizarre curves for the specific purpose of solving one or another of the classical construction problems. Some of these curves are described in the problems that follow.

一、标题：尺规作图与古希腊三大几何难题

以下笔记基于所附图片中的文字叙述，总结了古希腊数学家在几何尺规作图方面的历史背景，以及“三大几何难题”的主要内容与最终结论。通过阅读，可帮助你理解这三大难题为何在 19 世纪才被彻底证明无法仅用尺规完成。

二、详细内容解析

1. 尺规作图的基本背景

1. **古希腊数学家的传统**
    - 在几何学习中，人们常使用**直尺和圆规**进行各种作图：
        - 能画直线、圆；
        - 可以二等分角、三等分线段、作出垂直平分线；
        - 还能构造正三边形、正方形、正五边形、正六边形等规律多边形；
        - 这些经典作图方法自**欧几里得（Euclid）和阿基米德（Archimedes）**时代就已完善。

2. **将尺规作图视作“思想游戏”**
    - 古希腊数学家和后来的数学爱好者，将仅用尺与圆规来进行作图视为一种高雅的“智力游戏”。
    - 他们希望在**严格规则**的限制下（只能用画线、画圆、找交点的方式）探索“某个几何对象能否构造出来”，或“该如何构造”。

---

2. 三大几何难题

在长达两千多年的历史中，古希腊人引以为豪的各种几何技术都能实现，但仍有三个经典问题在 19 世纪以前始终没有被成功完成：

1. **三等分角（Trisect an angle）**
    - 给定一个角，仅用尺规将其**三等分**（把该角分为相等的三份），不借助其它特殊曲线或工具。
    - 二等分角用尺规可以轻易做到，但三等分角却屡屡失败。
2. **倍立方（Double a cube）**
    - 给定一个立方体，想要构造另一个立方体，其体积是原立方体的**两倍**。
    - 等效说法：给定立方体的边长 a，只用尺规作图来得到边长为 $\sqrt[3]{2}\,a$ 的线段。
    - 这在尺规限制下也始终没有可行的构造。
3. **化圆为方（Square a circle）**
    - 给定一个圆，构造一个正方形，使它的面积与该圆的面积完全相等。
    - 即在尺规作图中得到 $\sqrt{\pi}$ 的长度（当圆半径为 1 时），同样被证明无法实现。

> 这三个问题在 19 世纪被数学家彻底证明**不可能**在"仅用尺规"的前提下完成，主要依据代数和数论的进展（例如对不可解多项式、超越数 π 等相关结果的研究）。

---

3. 古希腊数学家的"工具"与拓展

1. **并非只用尺规**
   - 虽然我们总说古希腊人"只用尺和圆规"，但实际上，当他们面对"无法用尺规解决"的构造难题时，也会想出各种**替代曲线**或工具来"作弊"式地完成作图（例如借助曲线如双曲线、折纸、阿基米德螺线等方法）。
   - 这种做法表明：只要允许使用比尺规更多的几何手段，三大难题均可以"在更宽松的工具体系下"达成。
2. **在 19 世纪前依然未解**
   - 历经两千多年，不计其数的尝试都无法用纯尺规完成这三道作图。最终，19 世纪的数学证明了这些问题确实"用尺规单独不可能"解决，从而将问题彻底定论。

三、本次图片所涉重点内容总结

1. **尺规作图**是古希腊几何研究的核心手段，能完成大量常见的几何构造（如二等分角、正多边形构造等）。
2. **三大几何难题**（三等分角、倍立方、化圆为方）在古希腊时期就被提出，但无数人的尝试都失败。
3. **不可能性结论**：19 世纪的数学研究（代数、超越数理论等）表明，这三大问题在**仅用尺规**的限制下**确实无解**。
4. **历史启示**：为解决这些难题，古人也发明了各种奇妙曲线和方法，证明了研究几何不一定要拘泥于"尺规"两件工具。本质上，这些难题涉及更深层的数论和代数学理论。

16.3 POLAR EQUATIONS OF CIRCLES, CONICS, AND SPIRALS

16.3

POLAR EQUATIONS OF CIRCLES, CONICS, AND SPIRALS

We have already had considerable experience in transforming the rectangular equation of a given curve into an equivalent polar equation for the same curve. Our basic tools for this procedure are the transformation equations listed in Section 16.2.

Consider, for example, the circle (Fig. 16.14, left) with center $(a, 0)$ and radius a:

$$(x-a)^2 + y^2 = a^2 \qquad \text{or} \qquad x^2 + y^2 = 2ax. \tag{1}$$

Since $x^2 + y^2 = r^2$ and $x = r\cos\theta$, this equation becomes

$$r^2 = 2ar\cos\theta,$$

which is equivalent to

$$r = 2a\cos\theta \tag{2}$$

because the origin $r = 0$ lies on the graph of (2).

This example illustrates one way to find the polar equation of a curve, namely, transform its rectangular equation into polar coordinates. Another method that is better whenever it is feasible is to obtain the polar equation directly from some characteristic geometric property of the curve. In the case of the circle just discussed, we use the fact that the angle OPA in the figure on the right is a right angle. Since OPA is a right triangle with r the adjacent side to the acute angle θ, we clearly have

$$r = 2a\cos\theta,$$

which of course is the same equation previously obtained, but derived in a very different way.

We shall use this second and more natural method to find the polar equations of various curves in the following examples.

Example 1 Find the polar equation of the circle with radius a and center at the point C with polar coordinates (b, α), where b is assumed to be positive.

Solution Let $P = (r, \theta)$ be any point on the circle, as shown in Fig. 16.15, and apply the law of cosines to the triangle OPC to obtain

$$a^2 = r^2 + b^2 - 2br\cos(\theta - \alpha).$$

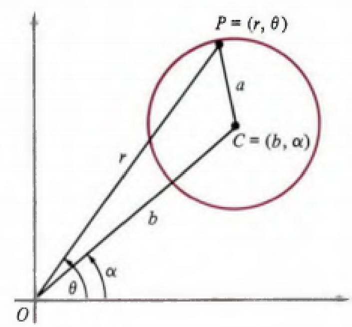

Figure 16.15

This is the polar equation of the circle. For circles that pass through the origin we have $b = a$, and the equation can be written as

$$r = 2a \cos (\theta - \alpha). \tag{3}$$

In particular, when $\alpha = 0$, then (3) reduces to (2), and when $\alpha = \pi/2$, so that the center lies on the y-axis, then $\cos (\theta - \pi/2) = \sin \theta$, and (3) reduces to

$$r = 2a \sin \theta. \tag{4}$$

In this case the right triangle OPA in Fig. 16.16 provides a more direct geometric way of obtaining (4), since here r is the opposite side to the acute angle θ.

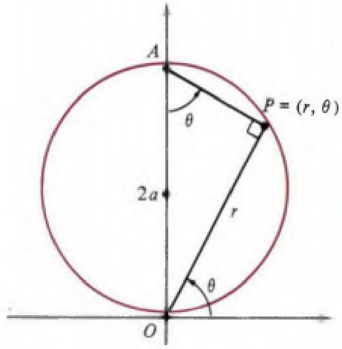

Figure 16.16

Example 2 Let F_1 and F_2 be the two points whose rectangular coordinates are $(a, 0)$ and $(-a, 0)$, as shown in Fig. 16.17. If b is a positive constant, find the polar equation of the locus of a point P that moves in such a way that the product of its distances from F_1 and F_2 is b^2.

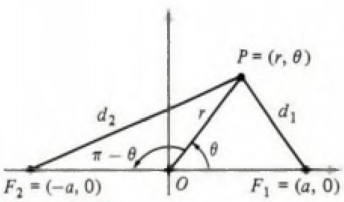

Figure 16.17

Solution If $P = (r, \theta)$ is an arbitrary point on the curve, then the defining condition is

$$d_1 d_2 = b^2 \qquad \text{or} \qquad d_1{}^2 d_2{}^2 = b^4,$$

where $d_1 = PF_1$ and $d_2 = PF_2$. We apply the law of cosines twice, first to the triangle OPF_1,

$$d_1{}^2 = r^2 + a^2 - 2ar \cos \theta, \tag{5}$$

and then to the triangle OPF_2,

$$d_2{}^2 = r^2 + a^2 - 2ar \cos (\pi - \theta). \tag{6}$$

Since $\cos (\pi - \theta) = -\cos \theta$, we can write (6) as

$$d_2{}^2 = r^2 + a^2 + 2ar \cos \theta, \tag{7}$$

and by multiplying (5) and (7) we obtain

$$d_1{}^2 d_2{}^2 = (r^2 + a^2)^2 - (2ar \cos \theta)^2$$

or

$$b^4 = r^4 + a^4 + 2a^2r^2(1 - 2 \cos^2 \theta).$$

The trigonometric identity $2 \cos^2 \theta = 1 + \cos 2\theta$ permits us to write this equation as

$$b^4 = r^4 + a^4 - 2a^2r^2 \cos 2\theta. \tag{8}$$

In the special case $b = a$, the curve passes through the origin, and the equation takes the much simpler form

$$r^2 = 2a^2 \cos 2\theta. \tag{9}$$

We recognize this as the equation of the lemniscate discussed in Example 3 of Section 16.2. When $b > a$, the curve consists of a single loop, but when $b < a$ it breaks into two separate loops. The cases $b < a$ and $b = a$ are illustrated in

Fig. 16.18, along with two cases of $b > a$. Collectively, these curves are called the *ovals of Cassini.*[*]

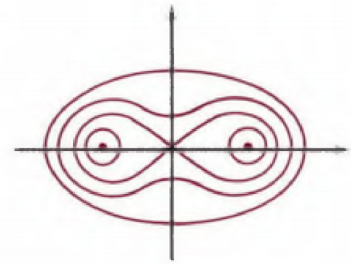

Figure 16.18 The ovals of Cassini.

Polar coordinates are particularly well suited to working with conic sections, as we see in the next example.

Example 3 Find the polar equation of the conic section with eccentricity e if the focus is at the origin and the corresponding directrix is the line $x = -p$ to the left of the origin.

Solution With the notation of Fig. 16.19, the focus-directrix-eccentricity characterization of the conic section is

$$\frac{PF}{PD} = e \qquad \text{or} \qquad PF = e \cdot PD. \tag{10}$$

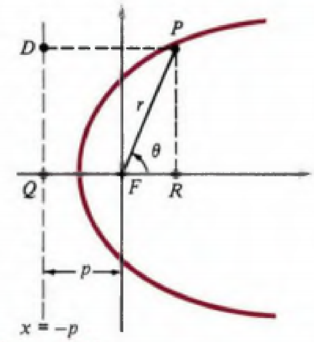

Figure 16.19

We recall that the curve is an ellipse, a parabola, or a hyperbola according as $e < 1$, $e = 1$, or $e > 1$. By examining the figure, we see that $PF = r$ and

$$\begin{aligned} PD = QR &= QF + FR \\ &= p + r\cos\theta, \end{aligned}$$

so (10) is

$$r = e(p + r\cos\theta).$$

This is easily solved for r, which gives

$$r = \frac{ep}{1 - e\cos\theta} \tag{11}$$

as the polar equation of our conic section.

We give two concrete illustrations of the ideas in Example 3.

Example 4 Find the polar equation of the conic with eccentricity $\frac{1}{3}$, focus at the origin, and directrix $x = -4$.

Solution We merely substitute $e = \frac{1}{3}$ and $p = 4$ in equation (11), which yields

$$r = \frac{\frac{1}{3}(4)}{1 - \frac{1}{3}\cos\theta} = \frac{4}{3 - \cos\theta}.$$

The curve is an ellipse. Observe that the denominator here is never zero, so r is bounded for all θ's.

[*]The Italian astronomer Giovanni Domenico Cassini thought of these ovals in 1680 in connection with his efforts to understand the relative motions of the earth and the sun. He proposed them as alternatives to Kepler's ellipses before Newton settled the matter with his theory of the solar system in 1687. Cassini discovered several of the satellites of Saturn, and also the so-called *Cassini division* in Saturn's ring, thereby showing that this ring consists of more than one piece.

一、标题：极坐标下的圆、卵形曲线、及圆锥曲线（焦点-准线定义）

以下笔记基于图片所示的第16.3节内容，系统梳理了如何将常见曲线（包括圆、Cassini 卵形、圆锥曲线等）在直角坐标方程与极坐标方程之间进行转换，以及如何利用几何性质直接得到它们的极坐标形式。通过这些示例，可以让我们更好地理解极坐标对这些曲线的刻画方式。

二、详细内容解析

1. 圆的极坐标方程

1. **一般圆方程及其转换**
   - 在直角坐标系中，一个圆的标准方程若为
     $$(x-a)^2+y^2=a^2 \quad \Longleftrightarrow \quad x^2+y^2=2ax,$$
     其中圆心是 $(a,0)$，半径为 a。
   - 用极坐标的变换关系
     $$x=r\cos\theta, \quad y=r\sin\theta, \quad r^2=x^2+y^2,$$
     可以把 $x^2+y^2=2ax$ 转为
     $$r^2=2a\,r\cos\theta \quad \Longrightarrow \quad r=2a\cos\theta.$$
     该圆恰好经过原点（因为 $r=0$ 时满足方程）。
2. **几何法得到 $r=2a\cos\theta$**
   - 也可无需先写出 $(x-a)^2+y^2=a^2$，而是直接观察图中三角形 OPA：
     - O 为原点，P 为圆上一点 (r,θ)，A 为圆心 $(a,0)$。
     - 右图显示 $\angle OPA$ 是直角，且 $OP=r$ 为斜边，$OA=a$ 为一条底边，三角形中 $\cos\theta=\frac{OA}{OP}=\frac{a}{r}$。
     - 整理得 $r=2a\cos\theta$。

3. **更一般情况：圆心在** (b, α)

- 若圆心的极坐标为 (b, α) 而半径为 a，则可得到

$$r^2 - 2br\cos(\theta - \alpha) + b^2 = a^2,$$

进一步化简可得

$$r = 2b\cos(\theta - \alpha) \quad (\text{当 } a = b \text{ 时，更简单}).$$

- 例如，当 $b = a$ 且 $\alpha = 0$，就回到 $r = 2a\cos\theta$ 的情形；如果 $\alpha = \frac{\pi}{2}$，则得到 $r = 2a\sin\theta$。

> **要点：** 在极坐标中，圆心不在原点的圆若经过原点，会呈现出类似 $r = 2a\cos(\theta - \alpha)$ 或 $r = 2a\sin(\theta - \alpha)$ 这类形式。

---

2. Cassini 卵形（Ovals of Cassini）：两定点乘积距离为常数

1. **问题设定**

- 给定平面上两点 $F_1 = (a, 0)$、$F_2 = (-a, 0)$，并规定某常数 $b > 0$。我们考虑所有满足

$$d_1\, d_2 = b^2$$

的点 P，其中 $d_1 = PF_1$ 和 $d_2 = PF_2$。这就定义了一条曲线。

- 若在直角坐标中，这是著名的 Cassini 卵形（或 Cassini 卵圆）。

2. **极坐标推导**

- 令 $P=(r,\theta)$ 为极坐标形式，且 $F_1=(a,0)$，$F_2=(-a,0)$。
- 通过"余弦定理"来表达 d_1^2 与 θ 的关系：

$$d_1^2=r^2+a^2-2ar\cos\theta,\quad d_2^2=r^2+a^2-2ar\cos(\pi-\theta).$$

  但 $\cos(\pi-\theta)=-\cos\theta$，所以

$$d_2^2=r^2+a^2+2ar\cos\theta.$$

- 再利用 $d_1d_2=b^2\implies d_1^2\,d_2^2=b^4$，即

$$\left(r^2+a^2-2ar\cos\theta\right)\left(r^2+a^2+2ar\cos\theta\right)=b^4.$$

  展开简化可得到

$$(r^2+a^2)^2-(2ar\cos\theta)^2=b^4\quad\implies\quad r^4+a^4+2a^2r^2-4a^2r^2\cos^2\theta=b^4.$$

- 当 $b=a$ 时，方程可归结为

$$r^2=2a^2\cos(2\theta),$$

  这与之前的 Lemniscate 形式（双瓣"∞"形曲线）相吻合。若 $b>a$，则图形是单环卵形；若 $b<a$，则会产生两环。该图在 Figure 16.18 中示意。

> **要点：** Cassini 卵形是一类在极坐标下可用乘积距离关系定义的曲线；当 $b=a$ 时，就退化成 Lemniscate 的形式。

3. 焦点—准线定义下的圆锥曲线

1. **焦点-准线定义**

   - 我们知道椭圆、抛物线、双曲线都可用"焦点-准线-离心率 e"来统一描述：

     $$PF = e \cdot PD,$$

     其中 F 称为焦点，D 为点到准线的垂足，e 为离心率：

     - $e < 1$ 表示椭圆；
     - $e = 1$ 表示抛物线；
     - $e > 1$ 表示双曲线。

2. **示例：焦点在原点，准线为 $x = -p$**

   - 令焦点 $F = (0,0)$，准线为 $x = -p$（即竖直线与 y 轴平行），并设 $P = (r, \theta)$ 是曲线上一点。
   - 几何中，$PF = r$，而到准线的距离 PD 可以通过水平距离计算：准线是 $x = -p$，点 $(r\cos\theta,\, r\sin\theta)$ 与此线的距离即 $\left|r\cos\theta + p\right|$。
   - 给定 $PF = e\,PD$，或者 $r = e\left|\,r\cos\theta + p\right|$。在相应区域（$\theta$ 取值使得分母不为 0 等）可写为：

     $$r = \frac{ep}{1 - e\cos\theta}.$$

     这就是**极坐标下的圆锥曲线方程**（Figure 16.19 所示）。

3. **举例：** $e = \frac{1}{3}, p = 4$

   - 将 $e = \frac{1}{3}$，$p = 4$ 代入

$$r = \frac{\frac{1}{3} \cdot 4}{1 - \frac{1}{3}\cos\theta} = \frac{4}{3 - \cos\theta}.$$

   - 这是一个椭圆（因 $e < 1$），且分母 $3 - \cos\theta$ 永远大于零，不会出现奇异点。从而 r 有界，也符合椭圆形。

> **要点：** 在极坐标里，若焦点放在原点、准线垂直于 x 轴（或与极轴平行），则圆锥曲线可写成

$$r = \frac{ep}{1 - e\cos\theta} \quad \text{或} \quad r = \frac{ep}{1 + e\cos\theta},$$

依离心率 e 的不同，对应椭圆、抛物线、双曲线，极坐标处理它们往往比直角坐标更简洁。

三、本次图片所涉重点内容总结

1. **圆在极坐标中的表达**
   - 若圆心在原点：$r = 常数$；若圆心在 $(a, 0)$ 并过原点：$r = 2a\cos\theta$；更一般地：$r = 2b\cos(\theta - \alpha)$ 或类似形式。
2. **Cassini 卵形**
   - 由两定点 F_1, F_2 定义，满足乘积距离 $d_1 d_2 = b^2$ 不变；在极坐标下经过余弦定理推导得到较复杂的方程。
   - 当 $b = a$（焦点间距为 $2a$），退化成 Lemniscate（双瓣形）。
3. **圆锥曲线（锥度曲线）**
   - 利用焦点-准线定义：$PF = e \cdot PD$。当焦点在原点、准线为 $x = -p$ 时，推导得

$$r = \frac{ep}{1 - e\cos\theta}.$$

   - $e < 1$ 椭圆；$e = 1$ 抛物线；$e > 1$ 双曲线。
4. **极坐标的优点**
   - 对不少曲线（尤其是焦点不在中心、或绕原点具旋转性质的曲线），极坐标可直截了当地表达它们的几何性质，而无需繁琐的直角坐标推导。

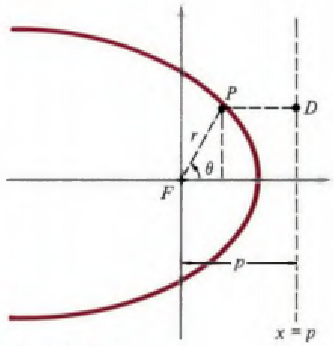

Figure 16.20

Example 5 Given the conic with equation

$$r = \frac{25}{4 - 5\cos\theta},$$

find the eccentricity, locate the directrix, and identify the curve.

Solution We begin by dividing numerator and denominator by 4 to put the equation in the exact form of (11),

$$r = \frac{\frac{25}{4}}{1 - \frac{5}{4}\cos\theta}.$$

This tells us that $e = \frac{5}{4}$ and $ep = \frac{25}{4}$, so $p = 5$. The directrix is the line $x = -5$, and the curve is a hyperbola. Observe that the denominator here is zero when $\cos\theta = \frac{4}{5}$, so r becomes infinite near these directions.

In connection with Example 3, it is worth pointing out that if the directrix is the line $x = p$ to the right of the origin, as in Fig. 16.20, then $PD = p - r\cos\theta$. The equation $PF = e \cdot PD$ now has the form

$$r = e(p - r\cos\theta),$$

and instead of (11) we have

$$r = \frac{ep}{1 + e\cos\theta}.$$

Polar coordinates are very convenient for describing certain spirals.

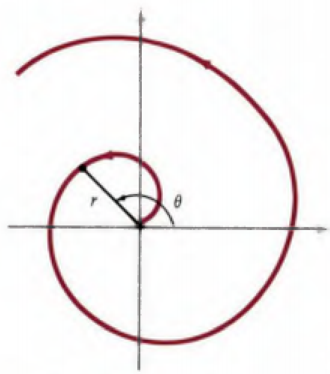

Figure 16.21 The spiral of Archimedes.

Example 6 The *spiral of Archimedes* (Fig. 16.21) can be defined as the locus of a point P that starts at the origin and moves outward at a constant speed along a radius which, in turn, is rotating counterclockwise at a constant speed from its initial position along the polar axis, where both motions start at the same time.* Since both r and θ are proportional to the time t measured from the beginning of the motions, r is proportional to θ and the polar equation of the spiral is $r = a\theta$, where a is a positive constant. In the figure, it is assumed that θ starts at zero and increases into positive values, as implied by the definition. However, if we wish to allow θ to be negative, then there is another part to the spiral which we have deliberately not sketched for the sake of keeping the figure uncluttered.

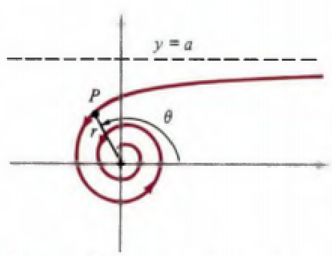

Figure 16.22 A hyperbolic spiral.

Example 7 In the spiral discussed in Example 6, r is directly proportional to θ, $r = a\theta$. We now consider the case in which r is inversely proportional to θ,

$$r = \frac{a}{\theta} \qquad \text{or} \qquad r\theta = a, \tag{12}$$

where a is a positive constant. For positive values of θ, the graph is the curve shown in Fig. 16.22; it is called a *hyperbolic spiral* because of the resemblance of $r\theta = a$ to the equation $xy = a$, which represents a hyperbola in rectangular coordinates.

*These are almost the same words which Archimedes himself uses to define his spiral. See his treatise "On Spirals" in *The Works of Archimedes,* T. L. Heath, ed. (Dover, n.d.), especially p. 154.

The essential features of the graph are easy to see by considering $r = a/\theta$. When $\theta = 0$, there is no r; when θ is small and positive, r is large and positive; and as θ increases to ∞, r decreases to 0. This tells us that a variable point P on the graph comes in from infinity and winds around the origin in the counterclockwise direction in an infinite number of shrinking coils as θ increases indefinitely. To understand the behavior of this curve for small positive θ's, we need to think about what happens to

$$y = r \sin \theta = a \frac{\sin \theta}{\theta}.$$

We know that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1,$$

and therefore

$$\lim_{\theta \to 0} y = \lim_{\theta \to 0} a \frac{\sin \theta}{\theta} = a.$$

It follows that the line $y = a$ is an asymptote of the curve, as shown in the figure.

If θ is allowed to be negative, we get another part of the curve, which again we do not sketch in order to avoid cluttering up the figure. The nature of this other part is easily understood by observing that if r and θ are replaced by $-r$ and $-\theta$, then equation (12) is unaltered. This means that for every point (r, θ) on the curve, the point $(-r, -\theta)$, which is symmetrically located with respect to the y-axis, is also on the curve. Thus, the other part is a second mirror-image spiral that winds in to the origin in the clockwise direction as $\theta \to -\infty$.

一、标题：从焦点-准线超越到螺线：极坐标下的双曲线与阿基米德螺线、双曲螺线

以下笔记基于图片中 Example 5、6、7 的内容，分别解析了一条焦点-准线定义的圆锥曲线（确认为双曲线），以及阿基米德螺线与双曲螺线的极坐标方程与特征。通过这些示例，可更全面地理解极坐标在描述“绕原点”或“以原点为焦点”的曲线时所展现的直观优势。

二、详细内容解析

1. 例 5：给定方程 $r = \frac{25}{4-5\cos\theta}$

1. **化为标准形并判断曲线类型**
   - 原方程：$r = \dfrac{25}{4-5\cos\theta}$。
   - 将分子、分母同除以 4 可写成：
     $$r \;=\; \frac{\frac{25}{4}}{1-\frac{5}{4}\cos\theta}.$$
     与之前所得一般形式
     $$r \;=\; \frac{ep}{1-e\cos\theta}$$
     相对比，立即可识别
     $$e \;=\; \tfrac{5}{4}, \quad ep \;=\; \tfrac{25}{4}, \quad \Longrightarrow \quad p \;=\; 5.$$
   - 因 $e = \frac{5}{4} > 1$，可判定该曲线是**双曲线（hyperbola）**。

2. **焦点与准线的位置**
   - 由上一节的结论可知，若方程是
     $$r = \frac{ep}{1 - e\cos\theta},$$
     焦点位于原点，准线则是 $x = -p$。
   - 因此这里 $p = 5$，故准线为 $x = -5$。
3. **分母零点与曲线不连续性**
   - 当分母 $4 - 5\cos\theta = 0$ 即 $\cos\theta = \frac{4}{5}$ 时，$r \to \infty$，对应双曲线的"渐近走向"。在极坐标图中，这些方向使曲线远离原点而趋于无穷。

> **要点：** 该例清晰展示了如何用极坐标方程识别一条焦点在原点、准线平行于 y 轴的双曲线。分母为零时即意味着沿某些方向 r 发散，展现出双曲线的开口特征。

---

2. 例 6：阿基米德螺线（Spiral of Archimedes）

1. **方程及几何解释**
   - 给定**阿基米德螺线**的极坐标方程：$r = a\theta$，其中 a 为正常数。
   - 这可想象为：点 P 从原点开始，在极轴方向同时做两种匀速运动：
     - θ 以匀速从 0 增加到无穷（逆时针旋转）；
     - r 以与 θ 成正比的速度增长（径向向外）。
   - 当 θ 从 0 开始增大时，r 也从 0 线性增大，故曲线一圈一圈向外扩张，形成著名的阿基米德螺线（Figure 16.21）。

2. **正负 θ 的影响**

   - 书中提到图中只画了 $\theta \geq 0$ 的部分，为了"避免使图线过于复杂"。
   - 若允许 θ 取负值，则在极坐标中可得到螺线的另一分支（向另一侧盘绕）。

3. **曲线特征**

   - 当 $\theta \to \infty$，$r \to \infty$，螺线持续向外伸展。
   - 当 θ 很小（接近 0）时，r 也很小，曲线近似在原点附近卷起。

> **要点：** 阿基米德螺线是最简单的极坐标螺线之一：半径正比于角度。它在很多机械和几何场景中出现，如均匀转动与匀速伸长的组合所描绘的路径。

---

3. 例 7：双曲螺线（Hyperbolic Spiral）

1. **方程：** $r = \frac{a}{\theta}$

   - 例 6 中是 $r = a\theta$（正比关系）；现在考虑**反比**的情形：

$$
r = \frac{a}{\theta} \quad \text{或} \quad r\,\theta = a.
$$

   - 这被称为**"双曲螺线"**，因为在直角坐标中有类似 $xy = a$（双曲线）的对应关系。

2. **曲线行为分析**

   - 当 θ 从 0 向大于 0 增长时：
     - $\theta \to 0^+$，$r \to +\infty$。这意味着曲线从极远处"跑进来"；
     - $\theta \to +\infty$，$r \to 0^+$。曲线无限向原点缠绕。
   - 整体看，曲线从无限远处进场，随着 θ 增加，不断盘绕接近原点。

3. **渐近线 y = a**

- 书中为理解曲线近 $\theta = 0$ 时的走向，考察

$$y = r\sin\theta = \frac{a}{\theta}\,\sin\theta.$$

当 θ 非常小（$\theta \to 0$），$\sin\theta \approx \theta$，于是 $\sin\theta/\theta \to 1$。
故

$$\lim_{\theta\to 0^+} y = \lim_{\theta\to 0^+} a\,\frac{\sin\theta}{\theta} = a.$$

说明在直角坐标中， $y = a$ 是该螺线的水平渐近线。

- 若允许 θ 取负，则曲线还会在另一侧绕行，形成镜面对称的分支。

> **要点：** 双曲螺线因 $r\theta = a$ 呈“反比例”形式，与直角坐标下的双曲线有共通之处；当 $\theta \to 0^+$，曲线向无限远发散并靠近 $y = a$ 这条水平线。

三、本次图片所涉重点内容总结

1. **Example 5：焦点-准线型双曲线**
   - 方程 $r = \dfrac{25}{4 - 5\cos\theta}$ 化为标准形后得 $e = \frac{5}{4}, p = 5$，故是双曲线，准线 $x = -5$。分母变零代表方向上 $r \to \infty$，即双曲线向那方向敞开。
2. **Example 6：阿基米德螺线**
   - 方程 $r = a\theta$，当 θ 从 0 到 ∞ 时，螺线不断向外旋转扩散。若 θ 取负，则出现另一分支。
3. **Example 7：双曲螺线**
   - 方程 $r = \frac{a}{\theta}$。与阿基米德螺线相反，$\theta \to 0^+$ 时 $r \to \infty$，曲线从无穷远处缠绕进入；$\theta \to \infty$ 时 $r \to 0$。
   - 在直角坐标中有水平渐近线 $y = a$。允许负 θ 则得到另一对称分支。

16.4 ARC LENGTH AND TANGENT LINES

16.4 ARC LENGTH AND TANGENT LINES

Consider a curve whose polar equation is $r = f(\theta)$, and let s denote arc length measured along the curve from a specified point in a specified direction (Fig. 16.26). By Section 7.5 we know that the differential element of arc length ds is given by the formula

$$ds^2 = dx^2 + dy^2.$$

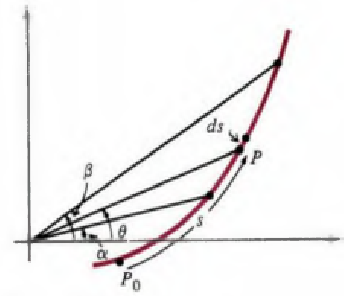

Figure 16.26

But $x = r\cos\theta$ and $y = r\sin\theta$, and by differentiating with respect to θ by the product rule, we obtain

$$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta\,\frac{dr}{d\theta} \qquad \text{and} \qquad \frac{dy}{d\theta} = r\cos\theta + \sin\theta\,\frac{dr}{d\theta},$$

or equivalently, in the notation of differentials,

$$dx = -r\sin\theta\,d\theta + \cos\theta\,dr \qquad \text{and} \qquad dy = r\cos\theta\,d\theta + \sin\theta\,dr. \tag{1}$$

It follows from these formulas that

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= r^2\sin^2\theta\,d\theta^2 - 2r\sin\theta\cos\theta\,dr\,d\theta + \cos^2\theta\,dr^2 \\ &\qquad + r^2\cos^2\theta\,d\theta^2 + 2r\sin\theta\cos\theta\,dr\,d\theta + \sin^2\theta\,dr^2 \\ &= r^2\,d\theta^2 + dr^2. \end{aligned}$$

Thus, we have

$$ds^2 = r^2\,d\theta^2 + dr^2 \tag{2}$$

or

$$\begin{aligned} ds = \sqrt{r^2\,d\theta^2 + dr^2} &= \sqrt{\left(r^2 + \frac{dr^2}{d\theta^2}\right)d\theta^2} \\ &= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta. \end{aligned}$$

This formula enables us to compute arc lengths of polar curves by integration, as suggested by the figure:

$$\text{arc length from } \theta = \alpha \text{ to } \theta = \beta \text{ equals } \int ds = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

Example 1 Find the total length of the cardioid $r = a(1 - \cos\theta)$.

Solution This curve is quite familiar to us and is shown in Fig. 16.29. From the equation of the curve, we have $dr = a\sin\theta\,d\theta$, so formula (2) gives

$$\begin{aligned} ds^2 &= a^2(1 - \cos\theta)^2\,d\theta^2 + a^2\sin^2\theta\,d\theta^2 \\ &= a^2[(1-\cos\theta)^2 + \sin^2\theta]\,d\theta^2 \\ &= 2a^2(1-\cos\theta)\,d\theta^2. \end{aligned}$$

Therefore

$$\begin{aligned} ds &= \sqrt{2}a\sqrt{1-\cos\theta}\,d\theta \\ &= 2a\left|\sin\tfrac{1}{2}\theta\right|\,d\theta, \end{aligned}$$

since $1 - \cos\theta = 2\sin^2\frac{1}{2}\theta$. We know that $\sin\frac{1}{2}\theta \geq 0$ for $0 \leq \theta \leq 2\pi$, so we can drop the absolute value signs and write

$$s = \int ds = \int_0^{2\pi} 2a \sin \frac{1}{2}\theta \, d\theta$$

$$= -4a \cos \frac{1}{2}\theta \Big]_0^{2\pi} = 4a - (-4a) = 8a.$$

The symmetry of this curve about the horizontal axis tells us that we can also obtain the total length by integrating from 0 to π and multiplying by 2,

$$s = 2 \int_0^{\pi} 2a \sin \frac{1}{2}\theta \, d\theta = -8a \cos \frac{1}{2}\theta \Big]_0^{\pi}$$

$$= 0 - (-8a) = 8a.$$

As a matter of routine, we should accustom ourselves to simplifying the calculation of integrals as much as possible by exploiting whatever symmetry is available.

The above formula for ds in polar coordinates can also be used to find areas of surfaces of revolution, as explained in Section 7.6.

Example 2 Find the area of the surface generated when the lemniscate $r^2 = 2a^2 \cos 2\theta$ is revolved about the x-axis.

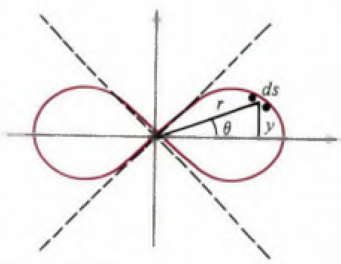

Figure 16.27

Solution An element of arc length ds (Fig. 16.27) generates an element of surface area

$$dA = 2\pi y \, ds,$$

where

$$y = r \sin \theta \qquad \text{and} \qquad ds = \sqrt{r^2 \, d\theta^2 + dr^2},$$

so

$$dA = 2\pi r \sin \theta \sqrt{r^2 \, d\theta^2 + dr^2} = 2\pi \sin \theta \sqrt{r^4 \, d\theta^2 + r^2 \, dr^2}. \tag{3}$$

From the equation of the curve we have

$$r \, dr = -2a^2 \sin 2\theta \, d\theta,$$

so

$$r^4 \, d\theta^2 + r^2 \, dr^2 = (4a^4 \cos^2 2\theta + 4a^4 \sin^2 2\theta) \, d\theta^2$$

$$= 4a^4 \, d\theta^2$$

and (3) becomes

$$dA = 4\pi a^2 \sin \theta \, d\theta.$$

The total surface area is twice the area of the right half, which is generated as ds moves along the part of the lemniscate in the first quadrant, that is, as θ increases from 0 to $\pi/4$. We therefore have

$$A = 2 \int_0^{\pi/4} 4\pi a^2 \sin \theta \, d\theta = -8\pi a^2 \cos \theta \Big]_0^{\pi/4}$$

$$= -8\pi a^2 \left(\frac{\sqrt{2}}{2} - 1\right) = 4\pi a^2 (2 - \sqrt{2}).$$

When working with rectangular coordinates, we specify the direction of a curve $y = f(x)$ at a point by the angle α from the positive x-axis to the tangent line. However, in the case of a polar curve $r = f(\theta)$, it is easier to work with the angle ψ (psi) from the radius vector to the tangent line, as shown in Fig. 16.28. We see from the figure that $\alpha = \theta + \psi$, so $\psi = \alpha - \theta$; and since $\tan \alpha = dy/dx$ and $\tan \theta = y/x$, the subtraction formula for the tangent gives

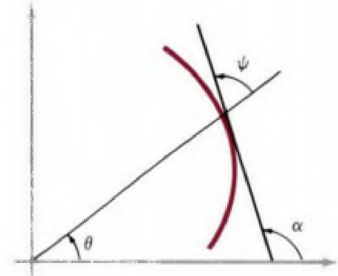

Figure 16.28

$$\begin{aligned}\tan \psi &= \tan (\alpha - \theta) \\ &= \frac{\tan \alpha - \tan \theta}{1 + \tan \alpha \tan \theta} \\ &= \frac{dy/dx - y/x}{1 + (dy/dx) \cdot (y/x)} \\ &= \frac{x\,dy - y\,dx}{x\,dx + y\,dy}.\end{aligned} \tag{4}$$

The reason why ψ is a convenient angle to use with polar coordinates is that (4) can be put into a very simple form. First, the fact that $x^2 + y^2 = r^2$ tells us that $x\,dx + y\,dy = r\,dr$. Next, from (1) we obtain

$$\begin{aligned}x\,dy - y\,dx &= r^2 \cos^2 \theta\, d\theta + r \sin \theta \cos \theta\, dr + r^2 \sin^2 \theta\, d\theta - r \sin \theta \cos \theta\, dr \\ &= r^2\, d\theta.\end{aligned}$$

By substituting these expressions into (4), we find that

$$\tan \psi = \frac{r\,d\theta}{dr} = \frac{r}{dr/d\theta}. \tag{5}$$

This formula is the basic tool for working with tangent lines to polar curves.

一、标题：极坐标曲线的弧长公式与切线分析

以下笔记基于所附图片讲解了如何在极坐标下求曲线的弧长，以及如何确定曲线切线的方向。主要内容包括：

1. **极坐标曲线的弧长微分公式**
2. **典型示例：心形线 $r = a(1 - \cos\theta)$ 的总弧长**
3. **旋转体表面积示例：用 Lemniscate 曲线绕 x 轴旋转得到的表面积**
4. **极坐标曲线的切线角公式**

二、详细内容解析

1. 极坐标下的弧长公式

考虑一条曲线由极坐标方程 $r = f(\theta)$ 给出。若令 θ 为自变量，(x, y) 为相应的笛卡儿坐标，则

$$x = r\cos\theta, \quad y = r\sin\theta.$$

对 θ 做微分，应用乘积法则可得：

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta, \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta.$$

于是微分弧长元 ds 满足

$$ds^2 \ = \ dx^2 + dy^2 \ = \ \left(\tfrac{dx}{d\theta}\right)^2 + \left(\tfrac{dy}{d\theta}\right)^2 \, d\theta^2.$$

代入上式并适当化简后，可以得到一个简洁结论：

$$ds^2 \;=\; r^2\,d\theta^2 \;+\; \left(\tfrac{dr}{d\theta}\right)^2\,d\theta^2,$$

即

$$ds \;=\; \sqrt{\,r^2 + \left(\tfrac{dr}{d\theta}\right)^2\,}\;d\theta.$$

弧长公式： 当 θ 从 α 到 β 时，曲线的弧长 s 为

$$s \;=\; \int_{\theta=\alpha}^{\beta} \sqrt{\,r^2 + \left(\frac{dr}{d\theta}\right)^2\,}\;d\theta.$$

---

2. 示例 1：心形线 $r = a\bigl(1 - \cos\theta\bigr)$ 的总弧长

题意： 求心形线（Cardioid）

$$r = a\bigl(1 - \cos\theta\bigr)$$

从 $\theta = 0$ 到 $\theta = 2\pi$ 的完整曲线长度。

1. **计算 $\frac{dr}{d\theta}$**

$$\frac{dr}{d\theta} = a\,\sin\theta.$$

2. **套用弧长微分公式**

$$ds = \sqrt{r^2 + \left(\tfrac{dr}{d\theta}\right)^2}\ d\theta = \sqrt{a^2(1-\cos\theta)^2 \ + \ a^2\sin^2\theta}\ d\theta.$$

展开后，

$$(1-\cos\theta)^2 + \sin^2\theta = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta = 2\left(1-\cos\theta\right).$$

故

$$ds = a\sqrt{\,2\bigl(1-\cos\theta\bigr)}\ d\theta.$$

又利用三角恒等式 $1-\cos\theta = 2\ \sin^2(\frac{\theta}{2})$，得到

$$ds = a\sqrt{\,4\ \sin^2(\tfrac{\theta}{2})}\ d\theta = 2a\bigl|\sin(\tfrac{\theta}{2})\bigr|\ d\theta.$$

3. **求全程长度**

令 θ 从 0 到 2π，对 $\sin(\frac{\theta}{2})$ 的符号仔细分析，最后结果可简化为

$$s \ = \ \int_0^{2\pi} 2a\left|\sin(\tfrac{\theta}{2})\right|\ d\theta \ = \ 8a.$$

（可将区间分段或直接利用对称性与已知结论。）

> **最终心形线的总弧长**：$\boxed{8a}$。

---

3. 示例 2：Lemniscate 旋转体的表面积

书中进一步说明弧长微分也可用于**旋转面**的面积计算。例如：若将某曲线绕 x 轴旋转，微分表面积为

$$dA \;=\; 2\pi\, y\, ds,$$

其中 $y = r\sin\theta$，$ds = \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$。

- 在截图中，他们以 Lemniscate $r^2 = 2a^2\cos(2\theta)$ 的一部分（θ 从 0 到 $\pi/4$）绕 x 轴旋转，计算所得表面积并做相应对称扩展。
- 通过将 $r\, dr = -2a^2 \sin(2\theta)\, d\theta$ 代入，最后可简洁地算得相应积分结果。
- 结论（省略详尽积分步骤）：得到一个具体数值形式（书中给出的结论形如 $4a^2(\sqrt{2} - 1)$ 之类），展示了**先化简再积分**的常见技巧。

---

4. 极坐标曲线的切线方向

1. **切线与极轴、半径矢量的夹角**
   - 在直角坐标里，我们常用 $\tan(\alpha) = \frac{dy}{dx}$ 来找切线斜率。
   - 在极坐标中，更方便的是研究**半径矢量与切线之间的夹角** ψ。如图所示：α 表示切线与 x 轴夹角，而 θ 表示半径矢量与 x 轴夹角，则

   $$\psi \;=\; \alpha \;-\; \theta.$$

   - 通过三角恒等式以及 $\frac{dy}{dx} = \tan\alpha$、$\tan\theta = \frac{y}{x}$ 等关系，书中最终推得一个简练的结果：

2. **关键公式**

$$\tan\psi \;=\; \frac{r\,d\theta}{dr} \quad\Longrightarrow\quad \psi \;=\; \arctan\Bigl(\frac{r\,d\theta}{dr}\Bigr).$$

换言之,

$$\boxed{\tan\psi \;=\; \frac{r}{dr/d\theta}.}$$

这里 ψ 正是**半径矢量与切线的夹角**。

> **要点:** 在极坐标下讨论切线往往比在直角坐标更简洁，尤其是当曲线自然地用 $r = f(\theta)$ 描述时，直接用 $\tan\psi = \frac{r}{dr/d\theta}$ 即可判断切线指向。

三、本次图片所涉重点内容总结

1. **弧长公式**:

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta, \quad s = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

2. **典型算例**:
   - 心形线 $r = a(1-\cos\theta)$ 的总弧长为 $8a$。
   - Lemniscate 等曲线的**旋转体表面积**可用 $dA = 2\pi\,y\,ds$ 并配合极坐标化简的方式计算。

3. **切线分析**:
   - 用"半径矢量—切线夹角" ψ 进行分析，得到**切线角公式**

$$\tan\psi \;=\; \frac{r}{dr/d\theta}$$

   - 在很多极坐标问题中，此公式能快速求出曲线在给定 θ 处的切线方向或斜率。

Example 3 Find the angle ψ for the cardioid $r = a(1 - \cos\theta)$.

Solution This curve is shown in Fig. 16.29. The equation of the curve gives

$$\frac{dr}{d\theta} = a \sin\theta,$$

so

$$\begin{aligned} \tan\psi = \frac{r}{dr/d\theta} &= \frac{a(1 - \cos\theta)}{a \sin\theta} \\ &= \frac{2\sin^2 \frac{1}{2}\theta}{2 \sin\frac{1}{2}\theta \cos\frac{1}{2}\theta} \\ &= \tan\frac{1}{2}\theta. \end{aligned}$$

We therefore have $\psi = \frac{1}{2}\theta$, and as θ increases from 0 to 2π, ψ increases from 0 to π, as indicated in the figure.

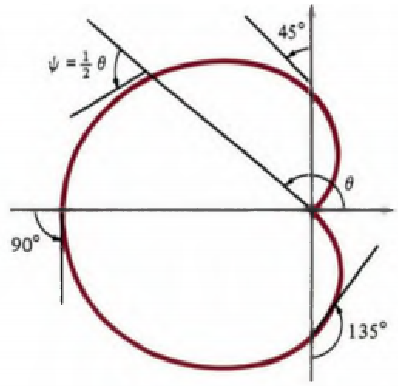

Figure 16.29

As another example of the use of the formula for tan ψ, we consider an interesting curve called the *exponential spiral.*

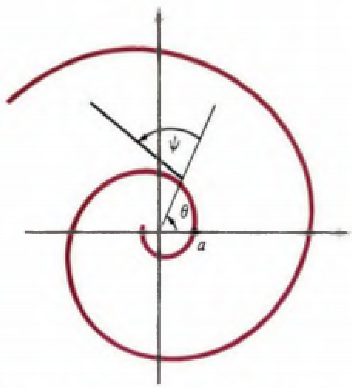

Figure 16.30 The equiangular spiral.

Example 4 Find the angle ψ for the curve $r = ae^{b\theta}$, where $a > 0$ and $b \neq 0$.

Solution If $b > 0$, we see that r increases as θ increases, as shown in Fig. 16.30. Further, it is clear that $r \to \infty$ as $\theta \to \infty$ and $r \to 0$ as $\theta \to -\infty$. The distinctive feature of this curve is that ψ is constant, because

$$\tan \psi = \frac{r}{dr/d\theta} = \frac{ae^{b\theta}}{abe^{b\theta}} = \frac{1}{b}.$$

This enables us to find ψ in the form $\psi = \tan^{-1}(1/b)$. If $b < 0$, the curve spirals in to the origin instead of outward as θ increases. The curve $r = ae^{b\theta}$ is sometimes called the *equiangular spiral* because of the constancy of ψ.

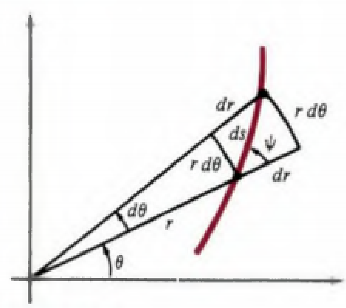

Figure 16.31

The two main facts of this section, formulas (2) and (5), are easy to remember by using Fig. 16.31 as a mnemonic device. In this figure we have an arc of length ds joining two points with polar coordinates r, θ and $r + dr$, $\theta + d\theta$. The outer part of the figure is approximately a rectangle, and the "differential triangle" on the right is approximately a right triangle with hypotenuse ds and with $r\,d\theta$ and dr as the legs opposite and adjacent to the angle ψ. The formulas

$$ds^2 = r^2\,d\theta^2 + dr^2$$

and

$$\tan \psi = \frac{r\,d\theta}{dr}$$

are now self-evident from this triangle, by the theorem of Pythagoras and the right triangle definition of the tangent. Needless to say, this way of reasoning is not a proof, but it is very useful nevertheless. It is also a good example of the true Leibnizian spirit in calculus, in the sense repeatedly explained in Chapter 7.

一、标题：心形线与指数螺线的切线角解析

以下笔记主要聚焦于图片中 Example 3 与 Example 4，展示了在极坐标下如何运用

$$\tan\psi \;=\; \frac{r}{\frac{dr}{d\theta}}$$

来求曲线的"半径矢量-切线"夹角 ψ。通过这些实例，我们可以更直观地掌握极坐标曲线的几何特征。

二、详细内容解析

1. Example 3：心形线 $r = a(1-\cos\theta)$ 的切线角 ψ

1. **已知方程与导数**
   - 心形线方程：$r = a(1-\cos\theta)$。
   - 对 θ 求导：

$$\frac{dr}{d\theta} \;=\; a\sin\theta.$$

2. **代入切线角公式**

$$\tan\psi \;=\; \frac{r}{\frac{dr}{d\theta}} \;=\; \frac{a(1-\cos\theta)}{a\sin\theta} \;=\; \frac{2\sin^2\!\left(\frac{\theta}{2}\right)}{2\sin\!\left(\frac{\theta}{2}\right)\cos\!\left(\frac{\theta}{2}\right)} \;=\; \tan\!\left(\frac{\theta}{2}\right).$$

   （上面用到了三角恒等式：$1-\cos\theta = 2\,\sin^2(\frac{\theta}{2})$，以及 $\sin\theta = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$。）

3. **得到** $\psi = \frac{\theta}{2}$
   - 这说明切线与半径矢量的夹角正好是 $\frac{\theta}{2}$。
   - 当 θ 从 0 增加到 2π，ψ 就从 0 增加到 π，与图中所示情形吻合。

---

2. Example 4：指数螺线 $r = a\,e^{b\theta}$ 的切线角 ψ

1. **曲线方程与导数**

$$r \;=\; a\,e^{b\theta}, \quad \frac{dr}{d\theta} \;=\; ab\,e^{b\theta}.$$

2. **套用公式**

$$\tan\psi \;=\; \frac{r}{\frac{dr}{d\theta}} \;=\; \frac{a\,e^{b\theta}}{ab\,e^{b\theta}} \;=\; \frac{1}{b}.$$

3. **ψ 为常数**
   - 由 $\tan\psi = \frac{1}{b}$ 得 $\psi = \arctan\!\left(\frac{1}{b}\right)$，与 θ 无关。
   - 这意味着不论 θ 如何变化，半径矢量与切线的夹角 ψ 永远相同。
   - 也因此，$r = a\,e^{b\theta}$ 又称**"等角螺线（equiangular spiral）"**：在任何点上，曲线都与引自原点的射线保持相同的夹角。
   - 当 $b > 0$ 时，$\theta \to \infty$ 导致 $r \to \infty$，曲线越绕越远离原点；$\theta \to -\infty$ 时，$r \to 0$ 逼近原点。若 $b < 0$，则相反地螺线"向内"收缩。

三、本次示例重点总结

1. **心形线的切线角**： $\psi = \frac{\theta}{2}$。随着 θ 从 0 到 2π，ψ 从 0 递增到 π。
2. **指数螺线的切线角**： ψ 为常数 $\arctan\!\left(\frac{1}{b}\right)$，表明曲线对原点射线始终保持同一夹角，故又称**等角螺线**。
3. **普适公式**： $\tan\psi = \dfrac{r}{\frac{dr}{d\theta}}$ 在极坐标中非常简洁，适合快速分析曲线几何特征（曲线的开口方向、是否等角、是否随 θ 改变等）。

16.5 AREAS IN POLAR COORDINATES

16.5 AREAS IN POLAR COORDINATES

Our problem here is to find the area A of the region bounded by a polar curve $r = f(\theta)$ and two half-lines $\theta = \alpha$ and $\theta = \beta$, as shown in Fig. 16.33. Our approach is modeled on the "differential element of area" idea of Section 7.1.

In working with areas in rectangular coordinates, we use thin rectangular strips and rely on the fact that the area of a rectangle equals length times width. Here we need the fact (Fig. 16.34) that the area of a sector of a circle of radius r and central angle θ (measured in radians) is $\frac{1}{2}r^2\theta$. In Fig. 16.33 our element of area dA is the area of the very thin sector with radius r and central angle $d\theta$, so

$$dA = \tfrac{1}{2}r^2\, d\theta. \tag{1}$$

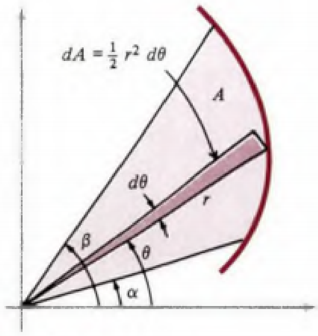

Figure 16.33

In the manner of Section 7.1, we think of the total area A as the result of adding up these elements of area dA as our thin sector sweeps across the region, that is, as θ increases from α to β:

$$A = \int dA = \int_\alpha^\beta \frac{1}{2}\, r^2\, d\theta. \tag{2}$$

Again, the essence of the process of integration is that we calculate the whole of a quantity by cutting it up into a great many convenient small pieces and then adding up these pieces.

We shall give a more mathematically sophisticated approach to formula (2) in Remark 2. First, however, we illustrate its use in several examples. As students will observe in these examples, it is always essential in solving area problems to have a good idea of what the curve looks like, because the correct limits of integration will be determined from the figure.

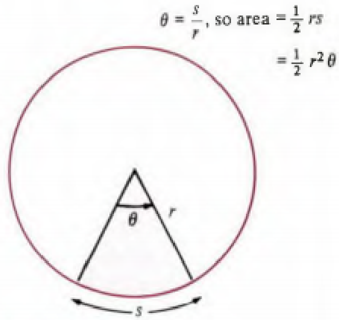

Figure 16.34

Example 1 Use integration to find the area of the circle $r = 2a\cos\theta$.

Solution The complete circle (Fig. 16.35) is swept out as θ increases from $-\pi/2$ to $\pi/2$. By symmetry we can integrate from 0 to $\pi/2$ and multiply by 2,

$$\begin{aligned} A &= 2\int_0^{\pi/2} \frac{1}{2}\, r^2\, d\theta = 2\int_0^{\pi/2} \frac{1}{2}\cdot 4a^2\cos^2\theta\, d\theta \\ &= 4a^2\int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\, d\theta \\ &= 2a^2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big]_0^{\pi/2} = \pi a^2. \end{aligned}$$

Naturally, we expected this answer because our circle has radius a, but it is reassuring to obtain a familiar result by a new method.

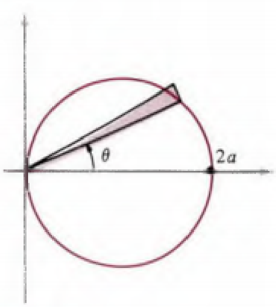

Figure 16.35

Example 2 Find the total area enclosed by the lemniscate $r^2 = 2a^2\cos 2\theta$ (Fig. 16.36).

Solution By symmetry, we calculate the area of the first quadrant part and multiply by 4:

$$\begin{aligned} A &= 4\int_0^{\pi/4} \frac{1}{2}\, r^2\, d\theta = 4\int_0^{\pi/4} \frac{1}{2}\cdot 2a^2\cos 2\theta\, d\theta \\ &= 4a^2\int_0^{\pi/4} \cos 2\theta\, d\theta = 2a^2\sin 2\theta\Big]_0^{\pi/4} = 2a^2. \end{aligned}$$

This problem provides a good illustration of the value of exploiting symmetry; for if we carelessly integrate all the way around from 0 to 2π, forgetting that r^2 is sometimes positive and sometimes negative, then our final answer turns out to be 0, which is obviously wrong.

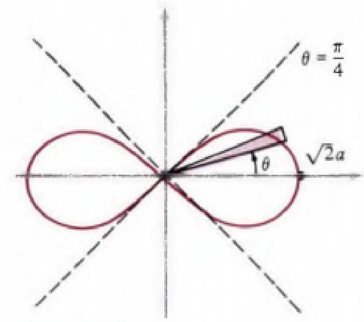

Figure 16.36

Example 3 Find the area inside the circle $r = 6a \cos \theta$ and outside the cardioid $r = 2a(1 + \cos \theta)$.

Solution By equating the r's and solving for θ, we see that the curves intersect in the first quadrant at $\theta = \pi/3$, as shown in Fig. 16.37. The indicated element of area is

$$\begin{aligned} dA &= \tfrac{1}{2}(r_{\text{circle}})^2\, d\theta - \tfrac{1}{2}(r_{\text{cardioid}})^2\, d\theta \\ &= \tfrac{1}{2}[(r_{\text{circle}})^2 - (r_{\text{cardioid}})^2]\, d\theta \\ &= \tfrac{1}{2}[36a^2 \cos^2 \theta - 4a^2(1 + \cos \theta)^2]\, d\theta \\ &= 2a^2(8 \cos^2 \theta - 1 - 2 \cos \theta)\, d\theta. \end{aligned}$$

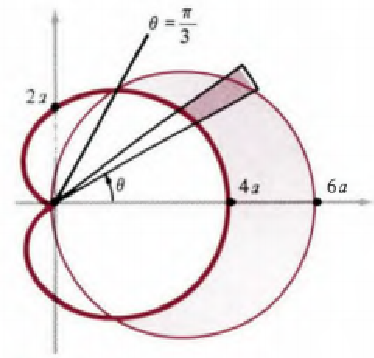

Figure 16.37

By symmetry, the area we seek is double the first quadrant area, so

$$\begin{aligned} A &= 2 \int_0^{\pi/3} 2a^2(8 \cos^2 \theta - 1 - 2 \cos \theta)\, d\theta \\ &= 4a^2 \int_0^{\pi/3} [4(1 + \cos 2\theta) - 1 - 2 \cos \theta]\, d\theta \\ &= 4a^2 \left[3\theta + 2 \sin 2\theta - 2 \sin \theta \right]_0^{\pi/3} = 4\pi a^2. \end{aligned}$$

Remark 1 The ideas of this section have an important application to the astronomy of the solar system. Consider a point P moving along a polar curve $r = f(\theta)$. We can think of P as a planet moving along its orbit, with the sun at the origin. If A is the area swept out by the radius OP from a fixed direction α to a variable direction θ, as shown in Fig. 16.38, then we have

$$dA = \tfrac{1}{2} r^2\, d\theta.$$

If both A and θ are thought of as functions of time t, then we see that

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}.$$

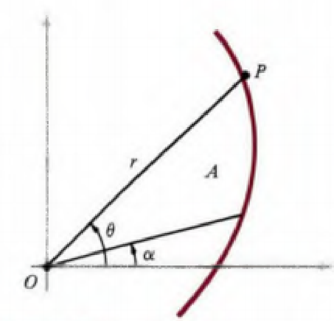

Figure 16.38

The derivative dA/dt is, of course, the rate of change of the area A. Kepler's second law of planetary motion states that a planet moves in such a way that the radius joining the planet to the sun sweeps out area at a constant rate. This means that dA/dt is constant, which in turn means that

$$r^2 \frac{d\theta}{dt} = \text{a constant} \tag{3}$$

for any given planet. Thus, for example, if a planet's orbit takes it in twice as close to the origin, then its angular velocity $d\theta/dt$ must increase by a factor of 4.

This fact has far-reaching implications which we shall examine more thoroughly in the last section of the next chapter.

Remark 2 We briefly reconsider formula (2) for the area A shown in Fig. 16.33. Our purpose is to remind students of the point of view developed in Section 6.4, namely, that a definite integral is defined to be a limit of approximating sums. As usual, we begin with a subdivision of the interval of integration $[\alpha, \beta]$:

$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta.$$

For each $k = 1, 2, \ldots, n$, let m_k and M_k be the minimum and maximum values of $f(\theta)$ on the kth subinterval $[\theta_{k-1}, \theta_k]$ of length $\Delta\theta_k = \theta_k - \theta_{k-1}$. Also, let ΔA_k be the area within the curve $r = f(\theta)$ corresponding to this subinterval. In Fig. 16.39 we show the area ΔA_k squeezed between the areas of the inscribed sector with radius $r = m_k$ and the circumscribed sector with radius $r = M_k$. We therefore have

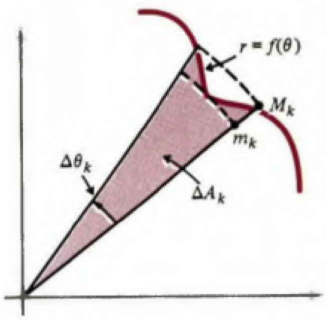

Figure 16.39

$$\tfrac{1}{2}m_k^2\, \Delta\theta_k \le \Delta A_k \le \tfrac{1}{2}M_k^2\, \Delta\theta_k.$$

By adding these inequalities from $k = 1$ to $k = n$, we obtain

$$\sum_{k=1}^{n} \frac{1}{2}\, m_k^2\, \Delta\theta_k \le A \le \sum_{k=1}^{n} \frac{1}{2}\, M_k^2\, \Delta\theta_k,$$

because A is the sum of the ΔA_k's. We now vary the subdivision in the manner described in Section 6.4, so that $\max \Delta\theta_k \to 0$. Then each of these sums approaches the definite integral

$$\int_\alpha^\beta \frac{1}{2} f(\theta)^2\, d\theta$$

for any continuous function $r = f(\theta)$, and since the area A is squeezed between the sums, we legitimately conclude that

$$A = \int_\alpha^\beta \frac{1}{2} f(\theta)^2\, d\theta = \int_\alpha^\beta \frac{1}{2}\, r^2\, d\theta,$$

which is (2).

一、标题：极坐标下的区域面积公式与典型应用

以下笔记基于图片所示的 16.5 节内容，系统讲解了如何使用极坐标来计算由极径曲线 $r = f(\theta)$ 与射线 $\theta = \alpha$、$\theta = \beta$ 所围成的平面区域面积。核心要点包括：

1. 极坐标面积公式的由来与严格性说明；
2. 多个示例（圆、Lemniscate、心形线与圆构成的复合区域等）；
3. Kepler 面积定律在行星运动中与该公式的对应关系。

二、详细内容解析

1. 极坐标面积微元公式

在直角坐标系中，我们常用“微小矩形面积元素”进行积分；而在极坐标中，则采用“微小扇形”进行面积累加。具体而言：

1. **单个扇形的面积**
   - 如图 16.33 或 16.34 所示，若半径为 r，中心角为 $d\theta$（用弧度计），则该扇形面积
     $$dA \;=\; \tfrac{1}{2}\,r^2\,d\theta.$$
2. **总体面积公式**
   - 当 θ 从 α 变化到 β 时，曲线 $r = f(\theta)$ 扫过的区域面积 A 为
     $$\boxed{A \;=\; \int_{\theta=\alpha}^{\beta} \tfrac{1}{2}\,\bigl[r(\theta)\bigr]^2\,d\theta.}$$

- 这是本节的核心结论，也常记为

$$A \ = \ \tfrac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta.$$

2. 示例 1：求圆 $r = 2a\cos\theta$ 的面积

1. **图形识别**

   - 这个圆心在 $(a, 0)$、半径为 a 的圆（见前面章节所示）。
   - 它在极坐标方程中写成：$r = 2a\cos\theta$。

2. **对称性选取积分范围**

   - 因为 $\cos\theta$ 在 $[-\pi/2, \ \pi/2]$ 为非负，且曲线也可对称地覆盖一整圆。通常做法是从 0 到 π 再乘以系数，或从 $-\frac{\pi}{2}$ 到 $\frac{\pi}{2}$ 等。书中用了对称性取 θ 从 0 到 $\frac{\pi}{2}$，再做相应倍数。

3. **积分并得到结果**

   - 最终计算可得该圆面积 πa^2，符合我们对半径为 a 圆的常识验证。

3. 示例 2：Lemniscate $r^2 = 2a^2\cos(2\theta)$ 的面积

1. **曲线概念**

   - Lemniscate 是类似“∞”双瓣的曲线，方程可写成 $r^2 = 2a^2\cos(2\theta)$。在适当的 θ 区间内，$\cos(2\theta)$ 保持非负才有实数 r。

2. **选择第一象限区间**

   - 典型地，当 θ 从 0 到 $\frac{\pi}{4}$ 时，$\cos(2\theta) \geq 0$，曲线占据第一瓣的一半；然后利用对称性乘以 4 或 2。

3. **积分结果**

   - 通过 $\frac{1}{2}\int r^2\,d\theta$ 并代入 $r^2 = 2a^2\cos(2\theta)$，再注意θ区间和倍数，即可得到整个曲线（两瓣）包围的总面积为 $2\,a^2$。

4. 示例 3：复合区域（在 $r = 6a\cos\theta$ 内、且在 $r = 2a(1+\cos\theta)$ 外）

1. **求交点**

   - 先将 $6a\cos\theta = 2a(1+\cos\theta)$ 化简来找曲线的交点 θ，书中得到 $\theta = \frac{\pi}{3}$（见图 16.37）。

2. **区域微元面积**

   - 在交点之前（一象限内），“外圆”与“内心形线”会给出各自 $r(\theta)$。故单个扇形的面积差

$$dA = \tfrac{1}{2}\left[(r_{\text{circle}})^2 - (r_{\text{cardioid}})^2\right]d\theta.$$

3. **倍数对称性**

   - 该区域关于 x 轴对称，故先算第一象限（θ 从 0 到 $\frac{\pi}{3}$），再乘 2。

4. **最终结果**

   - 积分后得 $A = 4\pi\,a^2$。这展示了利用“差面积”公式来处理两个极坐标曲线叠加的典型手法。

5. Kepler 第二定律与极坐标面积

1. **面积随时间变化**

   - 书中 Remark 1 指出：若把 $r = f(\theta)$ 看作行星绕太阳（位于原点）运动的轨道，则半径矢量扫过的面积速率为

$$\frac{dA}{dt} = \tfrac{1}{2}\,r^2\,\frac{d\theta}{dt}.$$

   - Kepler 第二定律（面积不变率）即指出该量是常数： $r^2\,\dot{\theta} = \text{const}$。
   - 这意味着：行星离太阳近时，角速度 $\dot{\theta}$ 必须更大；离太阳远时，角速度减小，以保证“等时段扫过面积相等”。

6. 面积公式的严格性（Remark 2）

- 书中附带一个小型“逼近、上下和定理”的思想（图 16.39）：把 θ 区间分割成很多小段，每段内用最小值 m_k 和最大值 M_k 分别构造内、外“扇形面积”估计。极限过程表明，这些和向同一个极限收敛，即

$$A = \int_{\alpha}^{\beta} \tfrac{1}{2}[f(\theta)]^2\,d\theta.$$

- 这是从积分定义的角度严格论证了该公式的正确性。

三、本次图片所涉重点内容总结

1. **极坐标面积公式**

$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta.}$$

2. **典型示例**

- **圆：** $r = 2a\cos\theta$ 整体面积 πa^2；
- **Lemniscate：** $r^2 = 2a^2\cos(2\theta)$，整个面积 $2\,a^2$；
- **混合区域：** 求“外曲线面积 - 内曲线面积”再乘以对称因子；

3. **天文学应用**

- **Kepler 第二定律**揭示 $\frac{1}{2}r^2\dot{\theta}$ 不变，意味着行星扫过的极坐标面积速率不变；行星轨道在几何上遵循等面积规律。

4. **严谨性**

- 通过上下和逼近说明 $\frac{1}{2}\int r^2 d\theta$ 的正确性，为极坐标面积公式提供更深层的积分基础。

CHAPTER 16 REVIEW: CONCEPTS, FORMULAS

CHAPTER 16 REVIEW: CONCEPTS, FORMULAS

Think through the following.

1 Equations connecting rectangular and polar coordinates.

2 Graphs of polar equations: circles, cardioids, lemniscates, spirals.

3 Arc length formula: $ds^2 = r^2\, d\theta^2 + dr^2$.

4 Tangent line formula: $\tan\psi = r\, d\theta/dr$.

5 Area formula: $dA = \frac{1}{2}r^2\, d\theta$.

一、标题：第16章核心概念与重要公式总览

以下笔记对第16章中的极坐标核心内容和主要公式进行总结，结合此前各节所学知识点，帮助你一站式回顾本章要点。

二、主要概念与知识脉络

1. **直角坐标与极坐标的转换**
   - **极坐标到直角坐标**：

     $$x = r\cos\theta, \quad y = r\sin\theta.$$

   - **直角坐标到极坐标**：

     $$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)\ (\text{需根据象限修正}).$$

   - 同一个几何点通常不只对应唯一的(r, θ)——可通过$r < 0$或$\theta \pm 2\pi$等方式得到等价表述。
2. **常见极坐标曲线**
   - **圆（Circle）**
     - 若圆心在原点：$r = \text{常数}$。
     - 若圆心不在原点、且过原点：$r = 2a\cos(\theta - \alpha)$ 等形式。
   - **心形线（Cardioid）**

     $$r = a\left(1 + \cos\theta\right) \text{ 或 } r = a\left(1 - \cos\theta\right).$$

     呈现出“苹果”或“心形”轮廓。

- **Lemniscate（带结曲线）**

$$r^2 = 2a^2\cos(2\theta),$$

  类似“∞”的双瓣形。

- **螺线（Spirals）**
  - 阿基米德螺线：$r = a\,\theta$，半径随角度呈线性增长。
  - 指数（等角）螺线：$r = a\,e^{b\theta}$，半径随θ呈指数变化，切线与半径矢量夹角不变。

3. **弧长公式**

   当极坐标方程为$r = f(\theta)$时，微元弧长

$$ds \;=\; \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\;d\theta.$$

   完整弧长为

$$s = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\;d\theta.$$

4. **切线方向公式**

   在极坐标里，常研究“半径矢量—切线”的夹角ψ，有

$$\tan\psi \;=\; \frac{r}{\frac{dr}{d\theta}}.$$

   若需切线与x轴的夹角，可再加上θ。

5. **面积公式**

   若曲线$r = f(\theta)$由$\theta = \alpha$旋转到$\theta = \beta$，所围面积为

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\,r^2\,d\theta.$$

- 在求“差面积”或“组合区域”时，可用 $\frac{1}{2}\int\left[r_{外}^2 - r_{内}^2\right]d\theta$ 并加上对称性处理。

三、应用与小结

1. **曲线绘制与识别**
   - 通过函数形式（含$\cos\theta$、$\sin\theta$或其倍角等）可初步判断对称性、曲线形状与分支范围。
   - 常见方法：离散取点、转换到直角坐标(若简洁)、或利用几何性质直接判断。
2. **典型案例**
   - **圆：** $r = 2a\cos\theta$ 面积πa^2。
   - **心形线：** 长度 $8a$，半径矢量到切线的夹角 $\psi = \frac{\theta}{2}$ 等。
   - **Lemniscate：** 双瓣面积 $2a^2$。
   - **指数螺线：** ψ 为常数，呈等角螺旋状。
3. **物理与天文背景：**
   - Kepler第二定律：行星绕焦点运动时，“扇形面积速率”$\frac{1}{2}r^2\dot{\theta}$恒定。与极坐标面积公式相呼应。

四、Chapter 16 的复习要点

- **概念理解**：
  - 极坐标与直角坐标的转换关系；
  - 同一几何点可有多重极坐标表示；
  - 不同常见曲线在极坐标下的典型方程。
- **公式熟记**：
  1. $x = r\cos\theta,\ y = r\sin\theta.$
  2. $r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}.$
  3. 弧长： $ds^2 = r^2\,d\theta^2 + \left(dr\right)^2.$
  4. 切线角： $\tan\psi = \frac{r}{\frac{dr}{d\theta}}.$
  5. 面积： $dA = \frac{1}{2}\,r^2\,d\theta.$
- **灵活运用**：
  - 求曲线长度或围成面积时，多注意对称、正负r区间、交点。θ的积分上下限往往由曲线形状确定。
  - 切线、法线方向可通过$\tan\psi$快速获取。
  - 若是多曲线叠加的面积或"外圆—内心形"之类问题，需先求交点θ，再拆分或做差。