Chapter 18 VECTORS IN THREE-DIMENSIONAL SPACE. SURFACES

18.1

18 VECTORS IN THREE-DIMENSIONAL SPACE. SURFACES

18.1 COORDINATES AND VECTORS IN THREE-DIMENSIONAL SPACE

In the preceding seventeen chapters we have discussed many aspects of the calculus of functions of a *single* variable. The geometry of these functions is two-dimensional because the graph of a function of a single variable is a curve in the plane. Most of the remainder of this book is concerned with the calculus of functions of *several* (two or more) independent variables. The geometry of functions of two variables is three-dimensional, because in general the graph of such a function is a curved surface in space.

In this chapter we discuss the analytic geometry of three-dimensional space. Our treatment will emphasize vector algebra, partly because this approach provides a more direct and intuitive understanding of the equations of lines and planes, and partly because the concepts of dot and cross products as developed in the next two sections are indispensable in many other parts of mathematics and physics.

Rectangular coordinates in the plane can be generalized in a natural way to rectangular coordinates in space. The position of a point in space is described by giving its location relative to three mutually perpendicular *coordinate axes* passing through the *origin O*. We always draw the x-, y-, and z-axes as shown in Fig. 18.1, with equal units of length on all three axes and with arrows indicating the

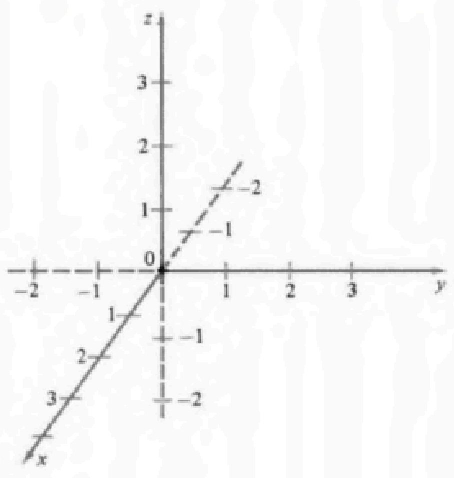

Figure 18.1 Coordinate axes.

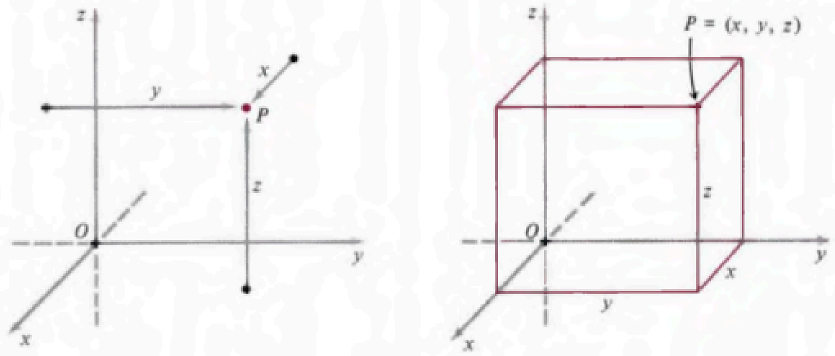

Figure 18.2 Locating a point by its rectangular coordinates.

positive directions. Each pair of axes determines a *coordinate plane*: the x-axis and y-axis determine the xy-plane, etc. The configuration of axes in this figure is called *right-handed,* because if the thumb of the right hand points in the direction of the positive z-axis, then the curl of the fingers gives the positive direction of rotation in the xy-plane, from the positive x-axis to the positive y-axis.

Since many people have trouble visualizing space figures from plane drawings, we point out that Fig. 18.1 can be thought of as part of a rectangular room drawn in perspective, with the origin O at the far left corner of the floor. The xy-plane is the floor, and has the normal appearance of the xy-plane if we look down on it from a point on the positive z-axis; the yz-plane is the back wall of the room, in the plane of the paper; and the xz-plane is the wall on the left side of the room.

A point P in space (see Fig. 18.2) is said to have *rectangular* (or *Cartesian*) *coordinates* x, y, z if:

x is its signed distance from the yz-plane;

y is its signed distance from the xz-plane;

z is its signed distance from the xy-plane.

Just as in plane analytic geometry, we write $P = (x, y, z)$ and identify the point P with the ordered triple of its coordinates. On the right in the figure we attempt to strengthen the illusion of three dimensions by completing the box that has O and P as opposite vertices.*

The three coordinate planes divide all of space into eight cells called *octants.* The cell emphasized in Fig. 18.2, where x, y, and z are all positive numbers, is called the *first octant.* (No one bothers to number the other seven octants.)

Even before plunging into a general study of the equations of lines and planes in Section 18.4, we can notice a few obvious facts. The xy-plane is the set of all points $(x, y, 0)$; it consists precisely of those points in space whose z-coordinate is 0, so its equation is

$$z = 0.$$

Similarly, the equation of the yz-plane is $x = 0$, and the equation of the xz-plane is $y = 0$.

*The technical term for the object shown on the right is "rectangular parallelepiped." We prefer the simpler word "box."

1. 标题与概述

标题： 三维空间中的坐标系与向量初步

概述：

本部分介绍了如何从二维平面坐标系推广到三维坐标系，讨论了三条相互垂直且交于同一点（原点 O ）的坐标轴 x 、 y 、 z ，并说明了如何利用它们描述空间中任意一点的位置。文中还提到，三维空间的几何可通过坐标平面划分为若干区域（称为卦限

或“八分体”），以及在三维解析几何中常见的几个基础概念，如平面方程、点在空间中的坐标表示、坐标轴的正向定义（右手定则）等。

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2. 逐点详解图片与核心知识点

下面我们依据图片内容，对每个知识点做详细解释。

(a）三维空间与函数图像的维度

- **单变量函数与二维曲线：** 在前面章节讨论的是单变量函数（例如 $y = f(x)$ ），其图像是平面上的一条曲线，属于二维几何。
- **多变量函数与三维曲面：** 当考虑两个或更多自变量时，对应的函数图像就会是三维空间中（或更高维）的曲面。例如，一个有两个自变量 x, y 的函数 $z = f(x, y)$ ，图像是三维空间中的一个曲面。
- **本章的主要内容：** 将聚焦在三维空间的解析几何，包括向量代数的基本运算、三维中直线与平面的方程等。这些知识在后续学习中非常重要，比如点积与叉积（内积、外积）在数学和物理里的应用。

(b）三维坐标系的引入（图 18.1）

1. **坐标轴与正方向：**
   - 在图 18.1 中，我们看到三条互相垂直且通过同一个原点 O 的轴，分别是 x -轴、 y -轴与 z -轴。它们的方向设定采用“**右手定则**（right-handed）”：
     - 大拇指指向正 z -轴，其他手指从正 x -轴转向正 y -轴时，所指的方向就是正向旋转方向。
   - 这三条轴上的单位长度相等，且端点都用箭头标明正方向。
2. **三个坐标平面：**
   - ** xy -平面：** 由 x -轴与 y -轴共同张成（即当 $z = 0$ 时所在的平面）。
   - ** yz -平面：** 由 y -轴与 z -轴共同张成（即当 $x = 0$ 时所在的平面）。
   - ** xz -平面：** 由 x -轴与 z -轴共同张成（即当 $y = 0$ 时所在的平面）。
3. **三维坐标轴在图中的可视化：**
   - 文中提到可以把图 18.1 理解为一个房间：
     - xy -平面相当于房间的地面。
     - yz -平面相当于房间正面的墙。
     - xz -平面相当于房间左侧的墙。
     - 原点 O 就好比位于地面房间左下角的交点处。

(c）空间中点的坐标（图 18.2）

1. **点 P 的直角坐标定义：**
   在三维空间中，若一点 P 的直角（或称笛卡儿）坐标为 (x, y, z)，则这三个数有以下含义：
   - x 是点到 yz -平面的有符号距离。
   - y 是点到 xz -平面的有符号距离。
   - z 是点到 xy -平面的有符号距离。

   这里的“有符号距离”指的是根据点在平面哪一侧来判断正负：
   - 如果 P 在正 x -轴一侧（即远离 $x = 0$ 的正方向），则 x 为正；若在负轴一侧则为负；其他坐标类似。

2. **如何写出点的坐标：**
   - 与平面解析几何中将点写成 (x, y) 一样，我们在三维中将点记为 (x, y, z) 。
   - **图 18.2** 中还画了一个长方体（严格名称：矩形平行六面体，或矩直平行六面体）来帮助读者想象三维的“盒子”结构：顶点 O 与 P 分别是长方体对角线的两个端点。
3. **八分体（八个卦限）：**
   - 三个坐标平面：xy -平面、 yz -平面、 xz -平面，共同将整个三维空间分为 8 个区域（每个区域称为一个“octant”）。
   - 当 x, y, z 都大于 0 时，就在**第一卦限**（或称第一象限的三维对应区），有时称之为“第一八分体”。
   - 其余的各个区域则根据 x, y, z 分别为正或负来划分。
4. **坐标平面的方程：**
   - xy -平面：所有满足 $z = 0$ 的点组成；
   - yz -平面：所有满足 $x = 0$ 的点组成；
   - xz -平面：所有满足 $y = 0$ 的点组成。

（d）补充说明

- **右手坐标系：** 文字中强调了此图是右手坐标系。当把右手的大拇指指向正 z -轴时，其他手指从正 x -轴弯向正 y -轴，就得到该坐标系的旋转方向。
- **命名简化：** 对于图中所示的长方体，技术上它可称为“rectangular parallelepiped（矩直平行六面体）”，但为了简化，常直接称其为“盒子”或者长方体。

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3. 本次内容总结

本部分内容主要为三维解析几何的开篇介绍，核心包含以下要点：

1. **三维空间与多变量函数：** 从单变量到多变量，图像从二维曲线扩展成三维曲面。
2. **三维坐标系基本要素：** 三条互相垂直的坐标轴（ x, y, z ）、三个坐标平面及其标准方程；通过有符号距离把空间中任意点的坐标写成 (x, y, z) 。
3. **八分体的概念：** 三个坐标平面将空间分成 8 个部分，每个部分内点的 (x, y, z) 都有特定的正负组合。
4. **右手定则：** 用来判断坐标轴的正方向和正向旋转方向，确保是一个右手系。

这些知识将为后续讨论三维空间中直线、平面、向量运算（点积、叉积）打下基础。

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4. 报错或不清晰说明

- 本次提供的图片内容整体清晰，没有出现无法辨认的地方或缺失的重点。若有其他不明之处，可再补充提问。

至此，本部分的内容笔记完成。希望通过这些要点的详细分析，能帮助你充分理解三维空间中坐标系与向量引入的基本思想，为后续学习三维几何与多变量微积分做准备。

18.1.2

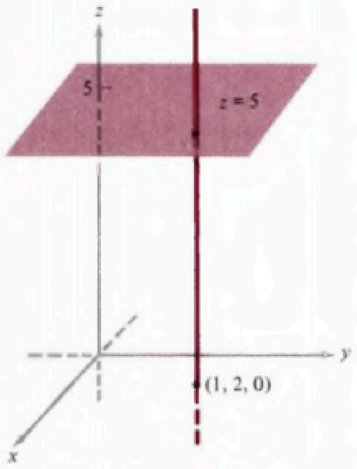

Figure 18.3 Horizontal plane.

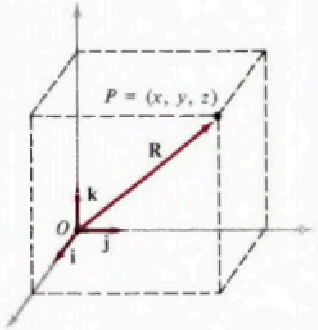

Figure 18.4 The position vector of P.

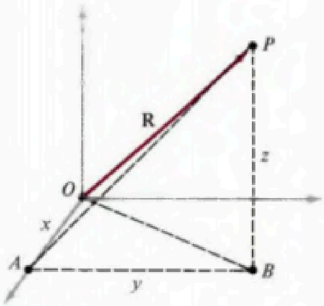

Figure 18.5

The z-axis is the set of all points $(0, 0, z)$. It is therefore represented by the pair of equations

$$x = 0, \qquad y = 0. \tag{1}$$

These are the equations of the yz-plane and the xz-plane, respectively, so equations (1) taken together characterize the z-axis as the intersection of these two coordinate planes. Similarly, the equations of the x-axis are $y = 0$, $z = 0$; and the equations of the y-axis are $x = 0$, $z = 0$.

There is nothing special about the number 0 in these remarks. For instance, the equation of the horizontal plane 5 units above the xy-plane is $z = 5$; and the equations of the vertical line that passes through the point $(1, 2, 0)$ in the xy-plane are $x = 1$, $y = 2$. See Fig. 18.3.

Almost all of the ideas about vectors that were presented in Section 17.3 are valid in three-dimensional space and require no further discussion. This remark applies to the concept of a vector, to the definition of equality for vectors, and to the definitions of addition and scalar multiplication. In all this material there is no need at all to suppose that the vectors lie in a plane.

The only real difference is that a vector in space has three components rather than two. In computing with vectors in the plane, we used the unit vectors **i** and **j** in the positive x- and y-directions. In order to compute with vectors in three-dimensional space, we introduce a third unit vector **k** in the positive z-direction, as shown in Fig. 18.4. If $P = (x, y, z)$ is any point in space, the position vector $\mathbf{R} = \overrightarrow{OP}$ can be written in the form

$$\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

and the numbers x, y, and z are called its **i**-, **j**-, and **k**-*components.*

The *length* of the vector **R** is given by the formula

$$|\mathbf{R}| = \sqrt{x^2 + y^2 + z^2}. \tag{2}$$

This can be proved by a double application of the theorem of Pythagoras, as illustrated in Fig. 18.5:

$$\begin{aligned} |\mathbf{R}|^2 = OP^2 &= OB^2 + BP^2 \\ &= OA^2 + AB^2 + BP^2 \\ &= x^2 + y^2 + z^2. \end{aligned}$$

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are any two points in space (Fig. 18.6), the distance between them is the length of the vector $\overrightarrow{P_1P_2}$ from P_1 to P_2. Since

$$\begin{aligned} \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1 &= (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) - (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \\ &= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}, \end{aligned}$$

we can use (2) to obtain

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{3}$$

This is the important *distance formula*; it has many uses.

Since a sphere is the set of all points P at a given distance r from a given fixed point P_0 (Fig. 18.7), the equation of a sphere can be written as

$$|\overrightarrow{P_0P}| = r. \tag{4}$$

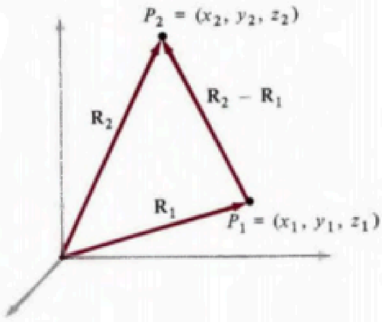

Figure 18.6

If $P = (x, y, z)$ and $P_0 = (x_0, y_0, z_0)$, then (3) enables us to write (4) in the equivalent form

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2. \tag{5}$$

This is the standard equation of the sphere with center $P_0 = (x_0, y_0, z_0)$ and radius r.

Example If we complete the squares in the equation

$$x^2 + y^2 + z^2 + 4x - 2y - 6z + 8 = 0, \tag{6}$$

it becomes

$$(x + 2)^2 + (y - 1)^2 + (z - 3)^2 = 6.$$

By comparing this with (5), we see at once that (6) is the equation of the sphere with center $(-2, 1, 3)$ and radius $\sqrt{6}$.

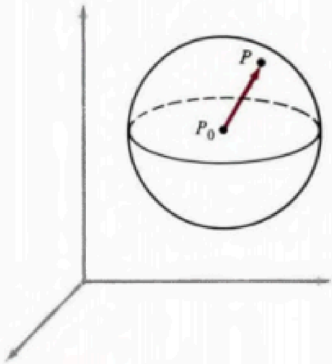

Figure 18.7 Sphere.

1. 标题与概述
标题： 三维空间中的坐标应用与向量长度、球面方程
概述：
本次内容延续了对三维空间坐标的讨论，首先说明了坐标轴和坐标平面的方程如何推广到任意垂直平面或直线，例如水平面 $z = 5$，或穿过某点 $(1, 2, 0)$ 的垂直线 $x = 1,\ y = 2$。随后引入了三维向量（拥有 x, y, z 三个分量）及其长度公式，并推导得到了空间中任意两点的距离公式。最后讨论了球面的标准方程，通过“配方法”将给定的二次方程化为球面方程，指出球心与半径。

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2. 逐点详解图片与核心知识点

(a) 坐标轴与平面的进一步说明

1. ** z -轴的方程：**
   - 文中指出， z -轴由所有满足 $(x = 0,\ y = 0)$ 的点组成，故可写成两条方程：
   $$x = 0, \quad y = 0.$$
   - 类似地， y -轴则由 $\{x = 0,\ z = 0\}$ 表示， x -轴由 $\{y = 0,\ z = 0\}$ 表示。
2. **平面的推广举例：**
   - 当我们说“水平面” $z = 5$ 时，它与 xy -平面（ $z = 0$ ）平行，只是平移到 $z = 5$ 的位置，如图 18.3 所示。
   - 当我们说一条竖直线穿过点 $(1, 2, 0)$ 时，则该线与 xy -平面垂直，并且满足 $\{x = 1,\ y = 2\}$， z 可任意取值。

这些实例说明：任意形如 $z = \text{常数}$ 的平面都是水平面；形如 $x = \text{常数}$ 或 $y = \text{常数}$ 的平面通常是竖直面。如果再加上一个常数等式，就能描述更多位置的平面或线。

(b) 三维向量与其表示（图 18.4 与图 18.5）

1. **三维向量的引入**：
   - 之前在二维平面中，我们用单位向量 $\mathbf{i}$ 和 $\mathbf{j}$ 分别代表 x -方向与 y -方向。在三维空间，需要再加一个单位向量 $\mathbf{k}$ ，用来表示 z -方向的单位向量。
   - 若空间中一点 $P=(x,y,z)$ ，则从原点 O 到 P 的**位置向量** $\overrightarrow{OP}$ （记为 $\mathbf{R}$ ）可写作

$$\mathbf{R}=x\,\mathbf{i}\;+\;y\mathbf{j}\;+\;z\,\mathbf{k}.$$

   - 其中 x,y,z 分别称为向量 $\mathbf{R}$ 的 i -分量、 j -分量、 k -分量。
2. **向量的长度公式（图 18.5）**：
   - 三维向量 $\mathbf{R}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$ 的长度（也记作 $\|\mathbf{R}\|$ ）是

$$\|\mathbf{R}\|=\sqrt{x^2+y^2+z^2}.\quad (2)$$

   - 该结论可看作两次应用勾股定理：
     1. 先在 xy -平面内求得 $\sqrt{x^2+y^2}$ ，
     2. 再把这个结果与 z 作为两边，在三维中再次应用勾股定理，得到最终的三维向量长度。

(c) 空间中两点距离公式（图 18.6）

1. **向量差表示**：
   - 若 $P_1=(x_1,y_1,z_1)$ 、 $P_2=(x_2,y_2,z_2)$ ，则从 P_1 指向 P_2 的向量可写作

$$\overrightarrow{P_1P_2}=\mathbf{R}_2-\mathbf{R}_1=(x_2-x_1)\,\mathbf{i}+(y_2-y_1)\,\mathbf{j}+(z_2-z_1)\,\mathbf{k}.$$

2. **距离的长度**：
   - 两点 P_1,P_2 间的距离就是 $\overrightarrow{P_1P_2}$ 的长度，即

$$|\overrightarrow{P_1P_2}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.\quad (3)$$

   - 这就是**三维空间点与点之间的距离公式**，是二维距离公式的自然推广。

(d) 球面的方程（图 18.7）

1. **球面方程的推导**：
   - 若一个点 $P=(x,y,z)$ 与某中心 $P_0=(x_0,y_0,z_0)$ 的距离为固定值 r ，则表示点 P 位于以 P_0 为球心、半径为 r 的球面上。
   - 由距离公式

$$|\overrightarrow{P_0P}\,|=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}=r,$$

     将其平方就得到球面的标准方程

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2.\quad (5)$$

2. **配方法示例**：
   - 给定一个二次方程

$$x^2 + y^2 + z^2 + 4x - 2y - 6z + 8 = 0,$$

通过"**配平方**"将其改写：

$$x^2 + 4x \;\to\; (x+2)^2 - 4, \quad y^2 - 2y \;\to\; (y-1)^2 - 1, \quad z^2 - 6z \;\to\; (z-3)^2 - 9.$$

将所有项组合后得到：

$$(x+2)^2 + (y-1)^2 + (z-3)^2 = 6.$$

- 其中左边配好方后把多余的常数移动到右边，计算出右侧为 6。
- 由此可知该球面的中心是 $(-2,\,1,\,3)$，半径是 $\sqrt{6}$。

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3. 本次内容总结

1. 坐标轴和坐标平面的方程可应用到任意位置：只需令 x, y, z 固定为某常数即可描述平行平面或直线；
2. 三维向量与二维向量类似，只是多了一个 $\mathbf{k}$ 分量。位置向量的长度由三维勾股定理给出 $\sqrt{x^2 + y^2 + z^2}$；
3. 利用向量差，得到了两点间的距离公式 $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$；
4. 球面方程是距离公式在三维空间的又一重要应用，即 $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$。完成配方后更容易识别球心位置与半径大小。

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4. 报错或不清晰说明

- 从本次图片内容看，均可清楚辨认，无明显缺失或无法阅读之处。
- 若对配平方过程或三维向量推导有疑问，可再进一步提问。

以上即为本部分完整笔记。通过这些要点，可以更好地理解三维空间中坐标方程、向量长度以及球面方程的基础知识，为后续学习空间中的直线、平面方程或更深入的向量运算奠定了必要的基础。

18.2

18.2 THE DOT PRODUCT OF TWO VECTORS

Up to this point in our work we have not defined the product of two vectors **A** and **B**. There are two different ways of doing this, both of which have important uses in geometry and physics. Since there is no reason to choose one of these definitions in preference to the other, we keep both, using a dot for one definition and a cross for the other. The *dot product* (or *scalar product*) of **A** and **B** is denoted by $\mathbf{A} \cdot \mathbf{B}$ and is a number. The *cross product* (or *vector product*) is denoted by $\mathbf{A} \times \mathbf{B}$ and is a vector. These two kinds of multiplication are totally different. We discuss the first in this section and the second in Section 18.3.

The *dot product* $\mathbf{A} \cdot \mathbf{B}$ of two vectors **A** and **B** is defined to be the product of their lengths and the cosine of the angle between them. This definition can be written as

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta, \tag{1}$$

where θ $(0 \leq \theta \leq \pi)$ is the angle between **A** and **B** when they are placed so that their tails coincide (Fig. 18.8). It is clear from the definition that $\mathbf{A} \cdot \mathbf{B}$ is a *scalar* (or number), not a vector.

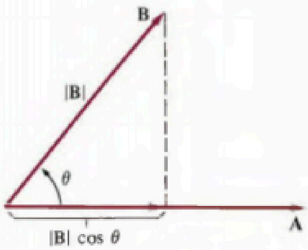

Figure 18.8 Scalar projection.

As Fig. 18.8 shows, the number $|\mathbf{B}| \cos \theta$ is the *scalar projection* of **B** on **A**, denoted by $\text{proj}_{\mathbf{A}} \mathbf{B}$. Definition (1) can therefore be interpreted geometrically as follows:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= |\mathbf{A}|(|\mathbf{B}| \cos \theta) = |\mathbf{A}| \text{ proj}_{\mathbf{A}} \mathbf{B} \\ &= (\text{length of } \mathbf{A}) \times (\text{scalar projection of } \mathbf{B} \text{ on } \mathbf{A}). \end{aligned}$$

By interchanging the roles of **A** and **B**, we also have

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= |\mathbf{B}|(|\mathbf{A}| \cos \theta) = |\mathbf{B}| \text{ proj}_{\mathbf{B}} \mathbf{A} \\ &= (\text{length of } \mathbf{B}) \times (\text{scalar projection of } \mathbf{A} \text{ on } \mathbf{B}). \end{aligned}$$

The *vector projection* of **B** on **A** is also indicated in the figure. Both types of projections are useful in applications.

It is easy to see from the definition (1) that the dot product has the properties

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}, \qquad \text{the commutative law,} \tag{2}$$

and

$$(c\mathbf{A}) \cdot \mathbf{B} = c(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot (c\mathbf{B}). \tag{3}$$

It also has the property

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}, \qquad \text{the distributive law,} \tag{4}$$

but this is not quite as evident as (2) and (3). To establish (4), we observe from Fig 18.9 that

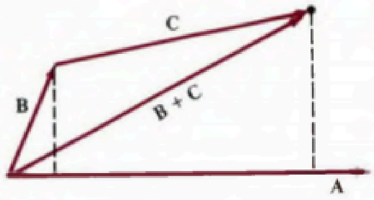

Figure 18.9

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= |\mathbf{A}|[\text{proj}_{\mathbf{A}}(\mathbf{B} + \mathbf{C})]\\ &= |\mathbf{A}|(\text{proj}_{\mathbf{A}}\, \mathbf{B} + \text{proj}_{\mathbf{A}}\, \mathbf{C})\\ &= |\mathbf{A}|\text{proj}_{\mathbf{A}}\, \mathbf{B} + |\mathbf{A}|\text{proj}_{\mathbf{A}}\, \mathbf{C}\\ &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}.\end{aligned}$$

If we combine (4) with the commutative law (2), we also have

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}. \tag{5}$$

Properties (4) and (5) permit us to multiply out sums of vectors by the ordinary procedures of elementary algebra, as in

$$(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{C} + \mathbf{D}) = \mathbf{A} \cdot \mathbf{C} + \mathbf{A} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{D}.$$

Another simple consequence of the definition (1) is the fact that

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 \tag{6}$$

for any vector $\mathbf{A}$.

Example 1 In the notation of Fig. 18.10, the cosine law of trigonometry states that

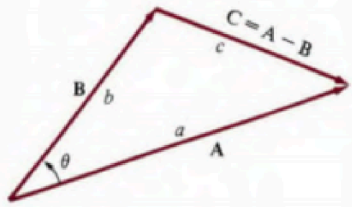

Figure 18.10

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

This can be proved very easily by using property (6) to write

$$\begin{aligned}c^2 = |\mathbf{C}|^2 = |\mathbf{A} - \mathbf{B}|^2 &= (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})\\ &= \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} - 2\mathbf{A} \cdot \mathbf{B}\\ &= |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2\mathbf{A} \cdot \mathbf{B}\\ &= a^2 + b^2 - 2ab\cos\theta.\end{aligned}$$

If we apply the definition (1) to the mutually perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ introduced in Section 18.1, we obtain

$$\begin{aligned}\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1,\\ \mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0.\end{aligned} \tag{7}$$

These facts enable us to find a convenient formula for computing the dot product of any two vectors given in $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form,

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{and} \qquad \mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}.$$

If we expand $\mathbf{A} \cdot \mathbf{B}$ by using (7) together with the general properties previously discussed, we get

$$\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3, \tag{8}$$

since six of the nine terms in the expansion vanish. Thus, to compute $\mathbf{A} \cdot \mathbf{B}$, we simply multiply their respective $\mathbf{i}$-, $\mathbf{j}$-, and $\mathbf{k}$-components, and add.

If $\mathbf{A}$ and $\mathbf{B}$ are nonzero vectors, the definition (1) can be written in the form

$$\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}. \tag{9}$$

This formula displays the main significance of the dot product in geometry: It provides a simple way to find the angle between two vectors and, in particular, to decide when two vectors are perpendicular. Indeed, if we agree that the zero vector is perpendicular to every vector, then by (9) we see at once that

$$\mathbf{A} \perp \mathbf{B} \quad \text{if and only if} \quad \mathbf{A} \cdot \mathbf{B} = 0.$$

Formula (8) makes it possible for us to use the dot product in these ways as a convenient computational tool.

Example 2 Find the cosine of the angle θ between the vectors $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = -3\mathbf{i} + 4\mathbf{j}$.

Solution It is clear that

$$|\mathbf{A}| = \sqrt{1 + 4 + 4} = 3, \qquad |\mathbf{B}| = \sqrt{9 + 16} = 5, \qquad \mathbf{A} \cdot \mathbf{B} = -3 + 8 + 0 = 5.$$

Therefore by (9) we have

$$\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{5}{3 \cdot 5} = \frac{1}{3}.$$

If we want the angle θ itself, we can use a calculator to find that $\theta \cong 70.5^\circ$.

Example 3 Compute the cosine of the angle θ between $\mathbf{A}$ and $\mathbf{B}$ if $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = -\mathbf{i} + c\mathbf{k}$, and find a value of c for which $\mathbf{A} \perp \mathbf{B}$.

Solution We have

$$|\mathbf{A}| = \sqrt{1 + 4 + 4} = 3, \qquad |\mathbf{B}| = \sqrt{1 + c^2}, \qquad \mathbf{A} \cdot \mathbf{B} = -1 + 2c,$$

so

$$\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{2c - 1}{3\sqrt{1 + c^2}}.$$

When $c = \frac{1}{2}$, this quantity has the value 0, and hence the vectors are perpendicular.

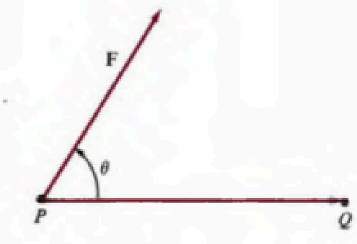

Figure 18.11

The simplest physical illustration of the use of the dot product is furnished by the concept of work. We recall that the work W done by a constant force F exerted along the line of motion in moving a particle through a distance d is given by $W = Fd$. But what if the force is a constant vector $\mathbf{F}$ pointing in some direction other than the line of motion from P to Q, as shown in Fig. 18.11? Only the vector component of $\mathbf{F}$ in the direction of the line of motion does work, so in this case we have

$$W = (|\mathbf{F}| \cos \theta)|\overrightarrow{PQ}| = |\mathbf{F}||\overrightarrow{PQ}| \cos \theta = \mathbf{F} \cdot \overrightarrow{PQ},$$

1. 标题与概况

标题： 向量点积的定义、性质与应用

概况：

本节引入了在三维（乃至任意维）向量中常用的两种“乘法”形式之一——**点积（dot product）**，也称**数量积（scalar product）**。点积把两个向量映射为一个实数，具有重要的几何与物理意义：

- 几何上，点积和向量夹角以及向量长度联系紧密，可用来判断向量正交、求向量间夹角等。

- 物理上，点积可解释力和位移之间的“功”等概念。

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2. 逐点详解图片与核心知识点

(a) 点积的定义（图 18.8）

1. **定义公式：**
给定非零向量 $\mathbf{A}$ 和 $\mathbf{B}$，令 θ 表示它们在起点重合时的夹角（$0 \le \theta \le \pi$）。则定义

$$\mathbf{A} \cdot \mathbf{B} \;=\; |\mathbf{A}|\,|\mathbf{B}|\;\cos\theta.$$

   - 从定义可知， $\mathbf{A} \cdot \mathbf{B}$ 是一个**实数**，而非向量。

2. **几何解释：**
   - $|\mathbf{B}|\cos\theta$ 可以视为 $\mathbf{B}$ 在 $\mathbf{A}$ 方向上的有向“投影长度”（图 18.8 所示），因此

$$\mathbf{A} \cdot \mathbf{B} \;=\; |\mathbf{A}| \times (\text{B 在 A 上的投影长度}).$$

   - 同理，也可以看作 $|\mathbf{B}| \times (\text{A 在 B 上的投影长度})$。

3. **与投影的关系：**
   - 记 $\mathrm{proj}_{\mathbf{A}}\mathbf{B} = |\mathbf{B}|\cos\theta$ 表示“$\mathbf{B}$ 在 $\mathbf{A}$ 方向上的标量投影”，则

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\;\mathrm{proj}_{\mathbf{A}}\mathbf{B}, \quad \mathbf{B} \cdot \mathbf{A} = |\mathbf{B}|\;\mathrm{proj}_{\mathbf{B}}\mathbf{A}.$$

(b) 点积的运算性质（图 18.9）

根据定义和几何解释，点积满足以下重要性质：

1. **交换律（commutative law）**

$$\mathbf{A} \cdot \mathbf{B} \;=\; \mathbf{B} \cdot \mathbf{A}.$$

2. **数乘结合律（与标量相乘）**

$$(c\,\mathbf{A}) \cdot \mathbf{B} \;=\; c\,(\mathbf{A} \cdot \mathbf{B}) \;=\; \mathbf{A} \cdot (c\,\mathbf{B}).$$

3. **分配律（distributive law）**

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) \;=\; \mathbf{A} \cdot \mathbf{B} \;+\; \mathbf{A} \cdot \mathbf{C}.$$

   - 直观解释：可以利用“投影”的概念说明， $\mathbf{B} + \mathbf{C}$ 相当于在 A 方向上先投影 B，再投影 C，二者相加。

4. ** $\mathbf{A} \cdot \mathbf{A}$ 与向量长度的关系**

$$\mathbf{A} \cdot \mathbf{A} \;=\; |\mathbf{A}|^2.$$

   - 这表明向量与自身的点积正好是它的长度平方，在几何上也符合 $|\mathbf{A}|\,|\mathbf{A}|\cos 0 = |\mathbf{A}|^2$（因为夹角 $\theta = 0$ 时 $\cos 0 = 1$）。

由以上性质可推出一些重要推论。例如，结合分配律和交换律，可以像普通代数一样对向量和标量做“乘法展开”。

(c) 与余弦定律的联系（图 18.10）

- **余弦定律**：在三角形中，若三边长度分别为 a, b, c，且夹角 θ 为 a 与 b 之间的角，则

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

- **向量解释**：若设 $\mathbf{C} = \mathbf{A} - \mathbf{B}$，则

$$c^2 = |\mathbf{C}|^2 = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} - 2\,\mathbf{A} \cdot \mathbf{B} = a^2 + b^2 - 2ab\cos\theta.$$

  - 由此可知，点积与余弦定律是相容的。

(d) 在标准基向量下的坐标计算（图 18.10-18.11）

1. **单位向量 $\mathbf{i}, \mathbf{j}, \mathbf{k}$ 的点积**：

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \quad \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{i} \cdot \mathbf{k} = 0.$$

   - 这是因为 $\mathbf{i}, \mathbf{j}, \mathbf{k}$ 两两正交且都为单位长度。

2. **向量分量形式**：
   若

$$\mathbf{A} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}, \quad \mathbf{B} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k},$$

   则它们的点积简化为

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3. \quad (8)$$

   - 这是由于大部分交叉项因基向量正交性而变为零。

3. **求夹角公式（图 18.11）**：
   如果 $\mathbf{A}$ 和 $\mathbf{B}$ 非零，则由定义可得

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|}. \quad (9)$$

   - 特别地，$\mathbf{A}$ 与 $\mathbf{B}$ 垂直（正交）当且仅当 $\mathbf{A} \cdot \mathbf{B} = 0$。
   - 此公式可快速判定向量垂直与否，也可用来求向量间的夹角。

(e) 点积的应用：力与功（图 18.11）

1. **功（Work）的概念**：

   在物理中，若一个恒定力 $\mathbf{F}$ 作用在物体上，并让物体在某位移 $\overrightarrow{PQ}$ 下运动，则**所做的功**可写为

$$W = \mathbf{F} \cdot \overrightarrow{PQ}. \quad (10)$$

   - 其中 $\overrightarrow{PQ}$ 表示从点 P 到 Q 的位移向量。
   - 若 $\mathbf{F}$ 与位移方向不一致，需要取力在位移方向上的分量，点积恰好给出这个效果。

2. **更一般的情形**：
   - 当力随位置变化、或者作用点在曲线上移动时，则需要将功分割成微小位移的积分形式。但(10) 仍是后续所有高级应用的起点。

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3. 本次内容总结

1. **定义角度：** 点积通过 $|\mathbf{A}|\ |\mathbf{B}|\ \cos\theta$ 将向量长度和夹角联系在一起。
2. **代数性质：** 点积满足交换律、结合律（数乘）与分配律，可视为对向量的“内积”或“数量乘法”，其结果是标量。
3. **分量计算：** 在正交单位基 $\mathbf{i}, \mathbf{j}, \mathbf{k}$ 下， $\mathbf{A} \cdot \mathbf{B}$ 仅需对应分量相乘并相加，非常方便。
4. **几何与物理应用：**
    - 可用来判断向量是否正交、求向量间夹角；
    - 可用来表示力与位移产生的功等物理量。

综上，点积在三维几何、物理（力学与功）、以及后续向量微积分中具有基础而广泛的应用。

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4. 报错或不清晰说明

- 本次图片内容完整、清晰，无难以辨认之处。
- 若需更深入探讨配合实际例题或者更高阶的物理应用，可再提问。

以上即为本部分完整笔记。通过这些要点与实例，可对向量点积的定义、运算性质及应用场景有系统的了解，为后续学习更复杂的向量运算与空间解析几何奠定重要基础。

18.3

18.3 THE CROSS PRODUCT OF TWO VECTORS

Many problems in geometry require us to find a vector that is perpendicular to each of two given vectors $\mathbf{A}$ and $\mathbf{B}$. A routine way of doing this is provided by the *cross product* (or *vector product*) of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \times \mathbf{B}$. This cross product is very different from the dot product $\mathbf{A} \cdot \mathbf{B}$ — for one thing, $\mathbf{A} \times \mathbf{B}$ is a vector, while $\mathbf{A} \cdot \mathbf{B}$ is a scalar. First we define this new product, then we describe its algebraic properties so that we can compute it with reasonable ease, and finally we illustrate some of its uses.

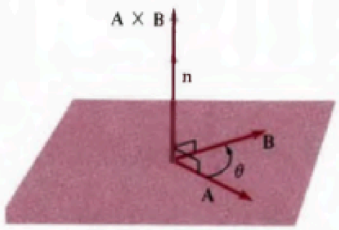

Figure 18.16

Consider two nonzero vectors $\mathbf{A}$ and $\mathbf{B}$. Suppose that one of these vectors is translated, if necessary, so that their tails coincide, and let θ be the angle from $\mathbf{A}$ to $\mathbf{B}$ (*not* from $\mathbf{B}$ to $\mathbf{A}$), with $0 \leq \theta \leq \pi$. If $\mathbf{A}$ and $\mathbf{B}$ are not parallel, so that $0 < \theta < \pi$, then these two vectors determine a plane, as shown in Fig. 18.16. We now choose the unit vector $\mathbf{n}$ which is normal (perpendicular) to this plane and whose direction is determined by the *right-hand thumb rule*. This means that if the right hand is placed so that the thumb is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ and the fingers curl from $\mathbf{A}$ to $\mathbf{B}$ in the direction of the angle θ, then $\mathbf{n}$ points in the same direction as the thumb of this hand. This gives the direction of the vector $\mathbf{A} \times \mathbf{B}$ that we are defining. Not only do the vectors $\mathbf{A}$ and $\mathbf{B}$ determine the plane under consideration, but they also determine a parallelogram in this plane, of area $|\mathbf{A}||\mathbf{B}| \sin \theta$ (see Fig. 18.17). We take the area of this parallelogram as the magnitude of the vector $\mathbf{A} \times \mathbf{B}$. With these preliminaries, we can now state the definition of the cross product of $\mathbf{A}$ and $\mathbf{B}$, in this order, as follows:

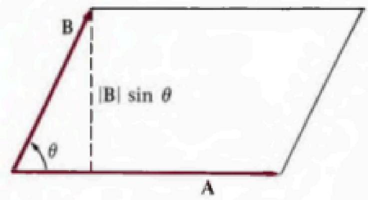

Figure 18.17

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin \theta\, \mathbf{n}. \tag{1}$$

Observe that if $\mathbf{A}$ or $\mathbf{B}$ is $\mathbf{0}$, or if $\mathbf{A}$ and $\mathbf{B}$ are parallel, then they do not determine a plane, and hence the unit normal vector $\mathbf{n}$ is not defined. But in these cases $|\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$, or $\sin \theta = 0$, so by (1) we have $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ and the determination of $\mathbf{n}$ is not necessary. If we agree that the zero vector is to be considered as parallel to every vector, then it is easy to see that

$$\mathbf{A} \text{ is parallel to } \mathbf{B} \text{ if and only if } \mathbf{A} \times \mathbf{B} = \mathbf{0}.$$

In particular, we have

$$\mathbf{A} \times \mathbf{A} = \mathbf{0}$$

for every $\mathbf{A}$. If instead of $\mathbf{A} \times \mathbf{B}$ we consider $\mathbf{B} \times \mathbf{A}$, then the direction of the angle θ is reversed, and we must flip the right hand over so that the thumb points in the opposite direction. This means that $\mathbf{n}$ is replaced by $-\mathbf{n}$, and therefore

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}. \tag{2}$$

This shows that the cross product is not commutative, and we must pay close attention to the order of the factors.

If we keep (2) in mind and apply the definition (1) to the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (Fig. 18.18), then we easily see that

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \\ \mathbf{j} \times \mathbf{k} &= -\mathbf{k} \times \mathbf{j} = \mathbf{i}, \\ \mathbf{k} \times \mathbf{i} &= -\mathbf{i} \times \mathbf{k} = \mathbf{j}, \end{aligned} \tag{3}$$

and also that

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

For example, the right-hand thumb rule says that the direction of $\mathbf{i} \times \mathbf{j}$ is the same as the direction of $\mathbf{k}$. But the area of the parallelogram determined by $\mathbf{i}$ and $\mathbf{j}$ is 1, and since $\mathbf{k}$ itself has length 1, we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}.$$

The products (3) are easy to remember by visualizing the figure. Another way to remember them is to arrange $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in cyclic order,

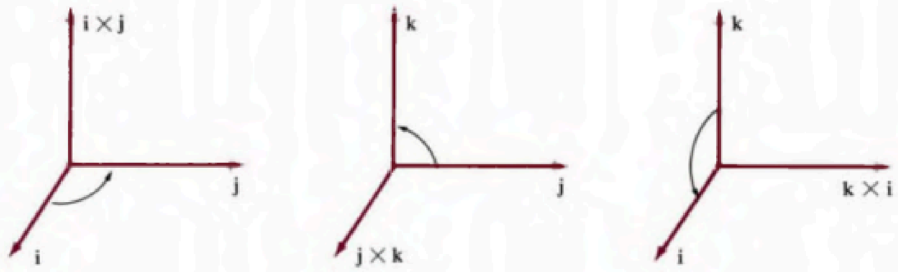

Figure 18.18

$$\begin{tikzcd} \mathbf{i} \arrow[r] & \mathbf{j} \arrow[r] & \mathbf{k}, \arrow[ll, bend left] \end{tikzcd}$$

and to observe that

$$\text{(each unit vector)} \times \text{(the next one)} = \text{(the third one)}.$$

Our next objective is to develop a convenient formula for calculating $\mathbf{A} \times \mathbf{B}$ in terms of the components of $\mathbf{A}$ and $\mathbf{B}$, where

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{and} \qquad \mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}. \tag{4}$$

In order to multiply out the product

$$\mathbf{A} \times \mathbf{B} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \,?$$

we need to know that the cross product possesses the following algebraic properties:

$$(c\mathbf{A}) \times \mathbf{B} = c(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times (c\mathbf{B}), \tag{5}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \tag{6}$$

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}. \tag{7}$$

Property (5) is easily established directly from the definition (1). Property (7) follows from (6) by using (2),

$$\begin{aligned} (\mathbf{A} + \mathbf{B}) \times \mathbf{C} &= -[\mathbf{C} \times (\mathbf{A} + \mathbf{B})] \\ &= -(\mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B}) \\ &= -\mathbf{C} \times \mathbf{A} - \mathbf{C} \times \mathbf{B} \\ &= \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}. \end{aligned}$$

The real difficulty here is with the distributive law (6). There is no simple proof of this fact; and rather than hold up our progress by pausing to insert a complicated proof here, we simply take (6) for granted and continue on to our immediate objective. A proof of (6) is given in Remark 2 for the use of any students who may wish to examine it.

We continue with our task of multiplying out the cross product of the vectors (4). Remembering to pay close attention to the order of the factors, we have

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1\mathbf{i} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_2\mathbf{j} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &\qquad + a_3\mathbf{k} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \end{aligned}$$

so

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= a_1b_1\mathbf{i} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k} \\ &\qquad + a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_2\mathbf{j} \times \mathbf{j} + a_2b_3\mathbf{j} \times \mathbf{k} \\ &\qquad + a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} + a_3b_3\mathbf{k} \times \mathbf{k}. \end{aligned}$$

By using (3), we now obtain the rather awkward formula

$$\mathbf{A} \times \mathbf{B} = \mathbf{i}(a_2 b_3 - a_3 b_2) - \mathbf{j}(a_1 b_3 - a_3 b_1) + \mathbf{k}(a_1 b_2 - a_2 b_1). \tag{8}$$

(The slightly strange way of writing the signs here has a purpose that will become clear below.)

1. 标题与概况

标题： 向量叉积（外积）的定义、方向与分量公式

概况：

在三维几何和物理中，我们常常需要一个同时垂直于两个已知向量 $\mathbf{A}$ 和 $\mathbf{B}$ 的向量。**叉积**（cross product）或**向量积**正是为此而定义。它与点积截然不同：

- **点积** 结果是一个标量（数字），用于测量向量间的“平行程度”；
- **叉积** 结果是一个新的向量，用于测量向量间的“垂直程度”。其长度代表由 $\mathbf{A}$ 和 $\mathbf{B}$ 所围平行四边形的面积，方向由右手法则给定。

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2. 逐点详解图片与核心知识点

(a) 几何定义与右手定则（图 18.16 与 18.17）

1. **几何定义（公式 (1)）：**
   给定两个非零向量 $\mathbf{A}$ 和 $\mathbf{B}$，若它们形成的夹角为 θ（$0 < \theta < \pi$），则**叉积** $\mathbf{A} \times \mathbf{B}$ 定义为

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\,\sin\theta\,\mathbf{n},$$

   其中：
   - $|\mathbf{A}||\mathbf{B}|\sin\theta$ 是**平行四边形**在平面 $\mathbf{A},\mathbf{B}$ 上的面积（图 18.17 中阴影部分）。
   - $\mathbf{n}$ 是**与平面** $\mathbf{A},\mathbf{B}$ 垂直的单位向量，其方向由**右手定则**确定：将右手手指从向量 $\mathbf{A}$ 弯向向量 $\mathbf{B}$，大拇指指向即为 $\mathbf{A} \times \mathbf{B}$ 的方向。
2. **特别情况：**
   - 若 $\mathbf{A}$ 或 $\mathbf{B}$ 为零向量，或二者平行（$\theta = 0$ 或 $\theta = \pi$，$\sin\theta = 0$），则 $\mathbf{A} \times \mathbf{B} = \mathbf{0}$。
   - 物理上可理解：当两个向量“重叠”或“伸直”时，其“垂直作用”消失。
3. **右手定则与顺序：**
   - 如果换成 $\mathbf{B} \times \mathbf{A}$，方向会相反，因此

$$\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B}).$$

   - 这说明**叉积不满足交换律**，而是满足“反交换律”。

(b) 单位基向量的叉积（图 18.18，公式 (3)）

在三维空间的标准基向量 $\mathbf{i},\mathbf{j},\mathbf{k}$ 中，利用右手定则可得：

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

以及它们的逆序乘积则出现负号：

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

另外，每个基向量与自身的叉积为零，例如 $\mathbf{i} \times \mathbf{i} = \mathbf{0}$。

(c）叉积的代数性质（公式 (2)、(5)-(7)）

1. **反交换律：**

$$\mathbf{A}\times\mathbf{B} \;=\; -(\mathbf{B}\times\mathbf{A}). \quad (2)$$

2. **数乘与结合律：**

$$(c\,\mathbf{A})\times\mathbf{B} \;=\; c\,(\mathbf{A}\times\mathbf{B}) \;=\; \mathbf{A}\times(c\,\mathbf{B}). \quad (5)$$

3. **分配律：**

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) \;=\; \mathbf{A}\times\mathbf{B} \;+\; \mathbf{A}\times\mathbf{C}, \quad (\mathbf{A}+\mathbf{B})\times\mathbf{C} \;=\; \mathbf{A}\times\mathbf{C} \;+\; \mathbf{B}\times\mathbf{C}. \quad (6)\,(7)$$

   - 与点积一样，叉积也有相应的分配律，可将向量和标量“展开”处理。

(d）组件公式（分量形式的计算，公式 (8)）

1. **一般向量表示：**

$$\mathbf{A} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}, \quad \mathbf{B} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}.$$

2. **叉积分量展开：**
将它们做叉积，可根据基向量交叉规律逐项计算，最终可总结成

$$\mathbf{A}\times\mathbf{B} = \bigl(a_2b_3 - a_3b_2\bigr)\,\mathbf{i} \;-\; \bigl(a_1b_3 - a_3b_1\bigr)\,\mathbf{j} \;+\; \bigl(a_1b_2 - a_2b_1\bigr)\,\mathbf{k}. \quad (8)$$

   - 一种常见记忆方式是利用行列式符号：

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

   - 注意中间会出现符号正负交替。

3. **几何含义：**
   - 该结果的方向与 $\mathbf{A},\mathbf{B}$ 所在平面垂直，符号“正负”可由右手法则判断；
   - 模长为 $|\mathbf{A}\times\mathbf{B}| \;=\; |\mathbf{A}||\mathbf{B}|\sin\theta$，对应平行四边形面积。

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3. 本次内容总结

1. **叉积的几何意义**
   - 方向：同时垂直于 $\mathbf{A}$ 和 $\mathbf{B}$，使用右手定则判断；
   - 长度：表示由 $\mathbf{A}$ 与 $\mathbf{B}$ 围成的平行四边形的面积，是 $|\mathbf{A}||\mathbf{B}|\sin\theta$。
2. **运算性质**
   - 不满足交换律，而是具有反交换性： $\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A})$；
   - 具有分配律和标量结合律，可在向量表达式中“正常”展开；
   - 与单位基向量 $\mathbf{i},\mathbf{j},\mathbf{k}$ 交叉时可简化为一套固定规则，可推广到三维分量的行列式形式。
3. **特殊情况**
   - $\mathbf{A}\times\mathbf{0} = \mathbf{0}$； $\mathbf{A}\times\mathbf{A} = \mathbf{0}$；当 $\mathbf{A}$ 与 $\mathbf{B}$ 平行时， $\mathbf{A}\times\mathbf{B} = \mathbf{0}$。
   - 在空间几何、物理（转动、力矩、磁场等）中，叉积具有关键作用。

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4. 报错或不清晰说明

- 本次截图内容相对完整清晰，无明显模糊不清之处。
- 若对行列式记忆方式、右手法则或分配律的证明细节有疑问，可再行补充讨论。

通过以上笔记，可掌握**叉积**的定义、方向判断及运算公式，为后续学习空间向量几何、物理力学（如力矩、角动量）奠定基础。

18.3.2

It is not necessary to memorize formula (8), because there is an equivalent version involving determinants that is easy to remember. We recall that a determinant of order 2 is defined by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

For example,

$$\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3 \cdot 5 - (-2) \cdot 4 = 23.$$

A determinant of order 3 can be defined in terms of determinants of order 2:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{9}$$

Here we see that each number in the first row on the left is multiplied by the determinant of order 2 that remains when that number's row and column are deleted. We particularly notice the minus sign attached to the middle term on the right side of formula (9).

Even though a determinant of order 3 can be expanded along any row or column, we use only expansions along the first row, as in (9). For example,

$$\begin{aligned} \begin{vmatrix} 3 & 2 & -1 \\ 4 & 3 & 3 \\ -2 & 7 & 1 \end{vmatrix} &= 3 \begin{vmatrix} 3 & 3 \\ 7 & 1 \end{vmatrix} - 2 \begin{vmatrix} 4 & 3 \\ -2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 3 \\ -2 & 7 \end{vmatrix} \\ &= 3(3 \cdot 1 - 3 \cdot 7) - 2[4 \cdot 1 - 3 \cdot (-2)] + (-1)[4 \cdot 7 - 3 \cdot (-2)] \\ &= -54 - 20 - 34 = -108. \end{aligned}$$

Formula (8) for the vector product of $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is clearly equivalent to

$$\mathbf{A} \times \mathbf{B} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}. \tag{10}$$

Motivated by (9), we now write (10) in the form

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{11}$$

This is the concise and easily remembered formula for $\mathbf{A} \times \mathbf{B}$ that we have been seeking. The "symbolic determinant" here is to be evaluated by expanding along its first row, just as in equation (9). We emphasize that the components of the first vector $\mathbf{A}$ in $\mathbf{A} \times \mathbf{B}$ form the second row of the determinant in (11), and that the components of the *second* vector $\mathbf{B}$ form the *third* row of this determinant.*

*Some authors *define* $\mathbf{A} \times \mathbf{B}$ by formula (11). This approach has several disadvantages, one of which is that considerable effort is needed before the geometric nature of $\mathbf{A} \times \mathbf{B}$ (that is, its length and direction) can be understood. We prefer to define $\mathbf{A} \times \mathbf{B}$ directly, in terms of its length and direction, and to consider formula (11) as simply a convenient tool for making calculations. Definitions of vector operations that avoid dependence on explicit representations of vectors in terms of any particular coordinate system are called *invariant* or *coordinate-free*.

Example 1 Calculate the cross product of $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$.

Solution By formula (11) we have

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ 1 & 5 & -3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 4 \\ 5 & -3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix}$$
$$= -17\mathbf{i} + 10\mathbf{j} + 11\mathbf{k}.$$

As a routine check to help guard ourselves against computational errors, we observe that our answer is perpendicular to $\mathbf{A}$ because $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A} = -34 - 10 + 44 = 0$, and is perpendicular to $\mathbf{B}$ because $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{B} = -17 + 50 - 33 = 0$.

Example 2 Find all unit vectors perpendicular to both of the vectors $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{B} = -4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$.

Solution Since $\mathbf{A} \times \mathbf{B}$ is automatically perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, we compute

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -4 & 3 & -5 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 3 \\ 3 & -5 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 3 \\ -4 & -5 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix}$$
$$= -4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

We next convert this into a unit vector in the same direction by dividing by its own length, which is $\sqrt{16 + 4 + 4} = \sqrt{24} = 2\sqrt{6}$:

$$\frac{-4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}{2\sqrt{6}} = \frac{-2\mathbf{i} - \mathbf{j} + \mathbf{k}}{\sqrt{6}}.$$

And finally, we introduce a plus-or-minus sign,

$$\pm \frac{-2\mathbf{i} - \mathbf{j} + \mathbf{k}}{\sqrt{6}},$$

because there are two possible directions.

1. 标题与概览

标题： 行列式形式的叉积公式与具体计算示例

概览：

在前面我们学习了向量叉积的几何定义（长度和方向）及其分配律、反交换律等性质。本部分则进一步介绍了**行列式**在叉积计算中的便捷应用，并通过两个示例说明如何借助行列式快速求叉积以及如何得到单位法向量。

2. 详细内容与知识点解析

（a）行列式与叉积公式的衔接

1. **二阶行列式**
   - 对于 2×2 的矩阵

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 \; - \; a_2 b_1.$$

   - 这是我们常用的"顺乘 - 逆乘"规则：首项是左上乘右下，减去左下乘右上。
2. **三阶行列式（公式 (9)）**
   - 例如若有

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

   则可沿着第一行展开（也可以沿其他行或列展开），具体过程为

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \; - \; a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \; + \; a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

   注意中间项会带有**负号。**
3. **用三阶行列式表示叉积**
   - 在前面我们已有"分量展开公式"

$$\mathbf{A} \times \mathbf{B} = \left(a_2 b_3 - a_3 b_2\right) \mathbf{i} - \left(a_1 b_3 - a_3 b_1\right) \mathbf{j} + \left(a_1 b_2 - a_2 b_1\right) \mathbf{k}.$$

   - 这与三阶行列式

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

   完全等价（公式 (11)）。这是记忆与计算叉积的一种**简便方法**：只要会熟练展开三阶行列式，就能迅速得到叉积的分量。

（b）示例计算

示例 1

题目： 已知向量

$$\mathbf{A} = 2\mathbf{i} \; - \; \mathbf{j} \; + \; 4\mathbf{k}, \quad \mathbf{B} = \mathbf{i} \; + \; 5\mathbf{j} \; - \; 3\mathbf{k}.$$

求 $\mathbf{A} \times \mathbf{B}$。

解：
根据行列式形式（公式 (11)），

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ 1 & 5 & -3 \end{vmatrix}.$$

1. **沿第一行展开**

- $\mathbf{i}$ 对应的 2×2 子行列式是
  $\begin{vmatrix} -1 & 4 \\ 5 & -3 \end{vmatrix} = (-1)(-3) - (4)(5) = 3 - 20 = -17,$
  所以 $\mathbf{i}$ 方向系数是 ** -17 **。
- $\mathbf{j}$ 对应的子行列式是
  $\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} = (2)(-3) - (4)(1) = -6 - 4 = -10,$
  但注意要带有**负号**（展开时第二项前面为"−"），因此 $\mathbf{j}$ 方向系数为 $-(-10) = +10$ 。
- $\mathbf{k}$ 对应的子行列式是
  $\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} = (2)(5) - (-1)(1) = 10 + 1 = 11.$

2. **综合得出**

$$\mathbf{A} \times \mathbf{B} = -17\,\mathbf{i} \;+\; 10\,\mathbf{j} \;+\; 11\,\mathbf{k}.$$

3. **检验结果**
   - 验证 $\mathbf{A} \times \mathbf{B}$ 与 $\mathbf{A}$ 垂直：做点积 $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A}$，若为 0 则说明正交；实际计算可以得到 0。
   - 验证与 $\mathbf{B}$ 也垂直：$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{B} = 0$ 。

因此结果 $\mathbf{A} \times \mathbf{B} = \langle -17, 10, 11 \rangle$ 确定无误。

示例 2

题目： 找到所有既垂直于
$\mathbf{A} = 2\mathbf{i} \;-\; \mathbf{j} \;+\; 3\mathbf{k}$
又垂直于
$\mathbf{B} = -4\mathbf{i} \;+\; 3\mathbf{j} \;-\; 5\mathbf{k}$
的单位向量。

解：

1. **叉积求公共法向量**
   - 由于 $\mathbf{A} \times \mathbf{B}$ 同时与 $\mathbf{A}$ 和 $\mathbf{B}$ 垂直，故先计算

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -4 & 3 & -5 \end{vmatrix}.$$

   - 沿第一行展开：
     - $\mathbf{i}$ 方向：
       $\begin{vmatrix} -1 & 3 \\ 3 & -5 \end{vmatrix} = (-1)(-5) - (3)(3) = 5 - 9 = -4.$
     - $\mathbf{j}$ 方向：
       $\begin{vmatrix} 2 & 3 \\ -4 & -5 \end{vmatrix} = (2)(-5) - (3)(-4) = -10 + 12 = 2,$
       但要记得前面有"−"，故 $\mathbf{j}$ 系数为 -2 。
     - $\mathbf{k}$ 方向：
       $\begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} = (2)(3) - (-1)(-4) = 6 - 4 = 2.$

   因此

$$\mathbf{A} \times \mathbf{B} = -4\,\mathbf{i} \;-\; 2\,\mathbf{j} \;+\; 2\,\mathbf{k}.$$

2. **转为单位向量**
    - 先计算模长：
      $\sqrt{(-4)^2+(-2)^2+2^2}=\sqrt{16+4+4}=\sqrt{24}=2\sqrt{6}.$
    - 将 $\mathbf{A}\times\mathbf{B}$ 除以其长度，即得**方向相同**的单位向量：

$$\frac{\mathbf{A}\times\mathbf{B}}{\|\mathbf{A}\times\mathbf{B}\|}=\frac{-4\,\mathbf{i}-2\,\mathbf{j}+2\,\mathbf{k}}{2\sqrt{6}}=\frac{-2\,\mathbf{i}-\mathbf{j}+\mathbf{k}}{\sqrt{6}}.$$

3. **符号** $\pm$
    - 在三维中，"同时垂直于 $\mathbf{A}$ 和 $\mathbf{B}$"的方向有两个：一个是上面算出的方向，另一个正好相反。
    - 因此所有符合条件的单位向量为

$$\pm\frac{-2\,\mathbf{i}-\mathbf{j}+\mathbf{k}}{\sqrt{6}}.$$

---

3. 本次内容总结

1. **行列式与叉积**：
    - 将叉积分量公式写成"行列式"的形式（即 $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$），是一个非常实用且易记的**速算方法**。
    - 只需掌握二阶、三阶行列式的展开规则，即可轻松应对三维向量叉积运算。
2. **计算与校验**：
    - 计算好每个基向量对应的 2×2 子行列式，并注意中间带负号的项。
    - 完成后可用"与原向量做点积=0"来检查结果是否满足正交性，从而减少运算失误。
3. **与实际应用**：
    - 一旦得到 $\mathbf{A}\times\mathbf{B}$，可以将它"规格化"（除以模长）成为单位向量；
    - 在空间几何中，这个**单位法向量**常被用于表示平面法线、曲面法线、或在物理中表示转动轴、力矩方向等。

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4. 报错或不清晰处

- 本次图片中行列式与算例均清晰可辨，未见缺失。
- 如果对行列式展开或符号顺序仍有疑问，可再行提问探讨。

以上即为此部分的完整笔记总结。掌握了行列式形式的叉积公式，能大幅提升三维向量运算的效率和准确性，也为今后更复杂的几何与物理问题（如法向量、角动量、力矩等）奠定了良好基础。

18.3.3

Example 3 Find the area of the triangle whose vertices are $P = (2, -1, 3)$, $Q = (1, 2, 4)$, and $R = (3, 1, 1)$.

Solution Two sides of the triangle are represented by the vectors

$$\mathbf{A} = \overrightarrow{PQ} = (1-2)\mathbf{i} + (2+1)\mathbf{j} + (4-3)\mathbf{k} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k},$$

$$\mathbf{B} = \overrightarrow{PR} = (3-2)\mathbf{i} + (1+1)\mathbf{j} + (1-3)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}.$$

The vector

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 1 & 2 & -2 \end{vmatrix} = -8\mathbf{i} - \mathbf{j} - 5\mathbf{k}$$

has magnitude $\sqrt{64 + 1 + 25} = \sqrt{90} = 3\sqrt{10}$, and this is equal to the area of the parallelogram with $\mathbf{A} = \overrightarrow{PQ}$ and $\mathbf{B} = \overrightarrow{PR}$ as adjacent sides. The area of the given triangle is clearly half the area of this parallelogram, and is therefore $\frac{3}{2}\sqrt{10}$.

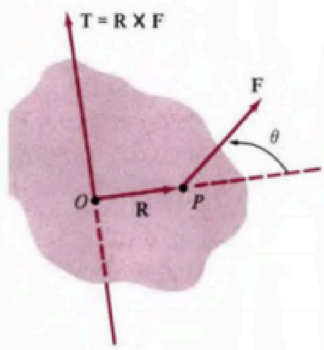

Figure 18.19 Torque vector.

Remark 1 The cross product arises quite naturally in many situations in physics. For example, if a force $\mathbf{F}$ is applied to a body at a point P (Fig. 18.19), and if $\mathbf{R}$ is the vector from a fixed origin O to P, then this force tends to rotate the body about an axis through O and perpendicular to the plane of $\mathbf{R}$ and $\mathbf{F}$. The *torque vector* $\mathbf{T}$ defined by

$$\mathbf{T} = \mathbf{R} \times \mathbf{F}$$

specifies the direction and magnitude of this rotational effect, since $|\mathbf{R}||\mathbf{F}| \sin \theta$ is the moment of the force about the axis, namely, the product of the length of the lever arm and the scalar component of $\mathbf{F}$ perpendicular to $\mathbf{R}$.

As another example, we mention the force $\mathbf{F}$ exerted on a moving charged particle by a magnetic field $\mathbf{B}$. It turns out that

$$\mathbf{F} = q\mathbf{V} \times \mathbf{B},$$

where $\mathbf{V}$ is the velocity of the charged particle and q is the magnitude of its charge. This is the primary fact that causes the aurora borealis, or northern lights, which are produced by blasts of charged particles from the sun streaming through the magnetic field of the earth. This basic principle of electromagnetism also underlies the design and operation of cyclotrons and TV sets.

Remark 2 We now return to the problem of establishing the distributive law (6). We prove (6) only for unit vectors $\mathbf{A}$, because once this has been done, an application of (5) allows us to obtain (6) immediately for vectors $\mathbf{A}$ of arbitrary length.

With a unit vector $\mathbf{A}$ and an arbitrary vector $\mathbf{V}$, $\mathbf{A} \times \mathbf{V}$ can be constructed by performing the following two operations, shown on the left side of Fig. 18.20: First, project $\mathbf{V}$ on the plane perpendicular to $\mathbf{A}$ to obtain a vector $\mathbf{V}'$ of length $|\mathbf{V}| \sin \theta$; then rotate $\mathbf{V}'$ in this plane through an angle of 90° in the positive direction to obtain $\mathbf{V}''$, which is $\mathbf{A} \times \mathbf{V}$ since $\mathbf{A}$ is a unit vector. Each of these operations transforms a triangle into a triangle; so if we start with the three vectors $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{B} + \mathbf{C}$ shown on the right, the final three vectors $\mathbf{B}''$, $\mathbf{C}''$, and $(\mathbf{B} + \mathbf{C})''$ still form a triangle, and therefore $(\mathbf{B} + \mathbf{C})'' = \mathbf{B}'' + \mathbf{C}''$. But this means that

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C},$$

and the argument is complete.

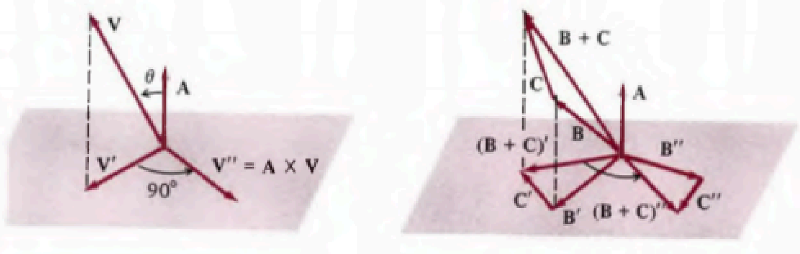

Figure 18.20 The distributive law.

1. 标题与概览

标题： 利用叉积求三角形面积与叉积在物理中的典型应用

概览：

在本节示例中，我们看到如何用顶点坐标构造向量，再通过它们的叉积求出三角形面积。此外，还提及了叉积在物理学中的两个典型应用场景：**力矩（扭矩）与带电粒子在磁场中的受力**，说明了叉积所蕴含的垂直性和几何意义。

2. 详细内容与知识点解析

（a）示例：利用叉积求三角形面积

题目： 已知三角形的三个顶点

$$P=(2,\,-1,\,3),\quad Q=(1,\,2,\,4),\quad R=(3,\,1,\,1).$$

求该三角形的面积。

1. **步骤 1：构造向量边**
   通常可选取三角形的两条边作为向量，便于利用叉积公式来求面积。令

   $$\mathbf{A}=\overrightarrow{PQ}=Q-P,\quad \mathbf{B}=\overrightarrow{PR}=R-P.$$

   - 计算 $\overrightarrow{PQ}$：

   $$Q-P=(1-2,\,2-(-1),\,4-3)=(-1,\,3,\,1)\ \rightarrow\ -\mathbf{i}+3\mathbf{j}+\mathbf{k}.$$

   - 计算 $\overrightarrow{PR}$：

   $$R-P=(3-2,\,1-(-1),\,1-3)=(1,\,2,\,-2)\ \rightarrow\ \mathbf{i}+2\mathbf{j}-2\mathbf{k}.$$

2. **步骤 2：求叉积 $\mathbf{A}\times\mathbf{B}$**
   - 采用行列式形式：

   $$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&3&1\\1&2&-2\end{vmatrix}.$$

   - 沿第一行展开：
     1. $\mathbf{i}$ 方向：$\begin{vmatrix}3&1\\2&-2\end{vmatrix}=3\times(-2)-1\times2=-6-2=-8.$
     2. $\mathbf{j}$ 方向：$\begin{vmatrix}-1&1\\1&-2\end{vmatrix}=(-1)(-2)-(1)(1)=2-1=1,$ 但前面有负号，故系数是 -1 。
     3. $\mathbf{k}$ 方向：$\begin{vmatrix}-1&3\\1&2\end{vmatrix}=(-1)(2)-(3)(1)=-2-3=-5.$
   - 综合：

   $$\mathbf{A}\times\mathbf{B}=-8\,\mathbf{i}\ -\ \mathbf{j}\ -\ 5\,\mathbf{k}.$$

3. **步骤 3：求平行四边形面积与三角形面积**
   - 由几何定义可知，$\|\mathbf{A}\times\mathbf{B}\|$ 即为由 $\mathbf{A}$ 与 $\mathbf{B}$ 两条边所围平行四边形的面积。
   - 计算模长：

   $$\sqrt{(-8)^2+(-1)^2+(-5)^2}=\sqrt{64+1+25}=\sqrt{90}=3\sqrt{10}.$$

   - 该平行四边形的面积为 $3\sqrt{10}$ 。
   - 由于三角形面积是相应平行四边形面积的一半，故三角形面积

   $$=\tfrac{1}{2}\times\|\mathbf{A}\times\mathbf{B}\|=\tfrac{1}{2}\times3\sqrt{10}=\tfrac{3}{2}\sqrt{10}.$$

（b）叉积的典型物理应用

1. **力矩（Torque）**
   - 如图 18.19 所示，若对某刚体上一点 P 施加一个力 $\mathbf{F}$，并且 $\mathbf{R} = \overrightarrow{OP}$ 表示从旋转轴或参考原点 O 到作用点 P 的位置向量，则扭矩（或力矩）定义为

     $$\mathbf{T} = \mathbf{R} \times \mathbf{F}.$$

   - 扭矩的大小 $\|\mathbf{R} \times \mathbf{F}\|$ 代表力使刚体绕某轴转动的“转动力”强弱；方向与平面 $\mathbf{R}, \mathbf{F}$ 垂直。
2. **带电粒子在磁场中受力**
   - 若带电量为 q 的粒子以速度 $\mathbf{v}$ 穿过磁场 $\mathbf{B}$，则其所受的洛伦兹力

     $$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}.$$

   - 该力**垂直于**速度方向，故只改变粒子运动的方向而不改变其速率。在自然界中，这种现象与极光的形成等都有重要联系。

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3. 本次内容总结

1. **三角形面积的求解**
   - 通过向量 $\mathbf{PQ}, \mathbf{PR}$ 的叉积，可迅速得到对应平行四边形面积，再将其减半即为三角形面积。
   - 计算中行列式展开的正负号和顺序需仔细小心，可用点积检验结果正确性（若有需要）。
2. **叉积在物理中的两大常见应用**
   - **力矩 $\mathbf{T} = \mathbf{R} \times \mathbf{F}$：** 体现叉积在转动力学中的意义；
   - **洛伦兹力 $\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$：** 体现叉积在电磁学中的意义。
   - 这些场景均利用了叉积“长度 = 两向量长度乘以 $\sin\theta$”及“方向垂直于原向量平面”的特征。

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4. 报错或不清晰说明

- 题目与示例中的坐标、向量均清晰无误，计算过程中无明显难以辨认之处。
- 若需更进一步探讨扭矩、洛伦兹力或叉积的其他应用及几何解释，可再行提问。

通过以上笔记，可掌握**如何用叉积快速求三角形（乃至多边形分割法）面积**，并了解其在物理中的典型应用场景，为后续更深入的空间向量学习和物理分析奠定扎实基础。

18.4

18.4
LINES AND PLANES

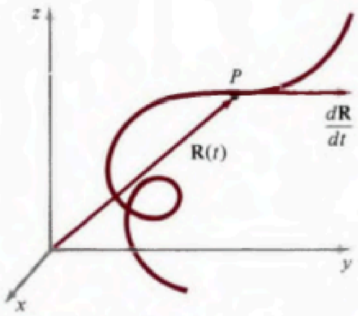

Figure 18.21 A space curve.

Since all the machinery of vector *algebra* is now in place, it might be expected that we would next turn to the *calculus* of vector functions in three-dimensional space. However, a full study of this subject belongs to a later course in advanced calculus or vector analysis, and is not part of our purpose in this book.

Nevertheless, on a few occasions we will need to consider the position vector $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ of a point P that moves along a space curve, as shown in Fig. 18.21. The derivative of this function is defined in the obvious way,

$$\frac{d\mathbf{R}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{R}(t + \Delta t) - \mathbf{R}(t)}{\Delta t},$$

and has all the properties we expect on the basis of our experience in Chapter 17. In particular, $d\mathbf{R}/dt$ is tangent to the path at the point P, and is the velocity of P if the parameter t is time, and the unit tangent vector if t is arc length.

With these brief remarks we put aside the calculus of vector functions, and turn to the main subject of the rest of this chapter, namely, the analytic geometry of lines, planes, and curved surfaces in three-dimensional space. We shall find that the vector algebra discussed in the preceding sections is a very valuable tool for this work.

As we know, in plane analytic geometry a single first-degree equation,

$$ax + by + c = 0,$$

is the equation of a straight line (assuming that a and b are not both zero). However, we shall see that in the geometry of three dimensions such an equation represents a plane, and therefore it is not possible to represent a line in space by any single first-degree equation.

We begin with the study of lines. A line in space can be given geometrically in three ways: as the line through two points, as the intersection of two planes, or as the line through a point in a specified direction. The third way is the most important for us.

Suppose L is the line in space that passes through a given point $P_0 = (x_0, y_0, z_0)$ and is parallel to a given nonzero vector

$$\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k},$$

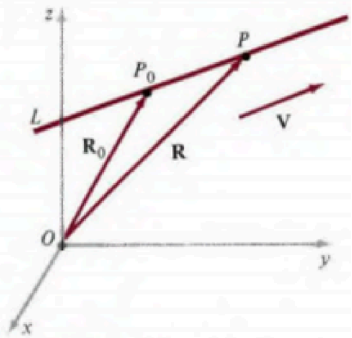

Figure 18.22 A line in space.

as shown in Fig. 18.22. Then another point $P = (x, y, z)$ lies on the line L if and only if the vector $\overrightarrow{P_0P}$ is parallel to the vector $\mathbf{V}$. That is, P lies on L if and only if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{V}$, so that

$$\overrightarrow{P_0P} = t\mathbf{V} \tag{1}$$

for some real number t. If $\mathbf{R}_0 = \overrightarrow{OP_0}$ and $\mathbf{R} = \overrightarrow{OP}$ are the position vectors of P_0 and P, then $\overrightarrow{P_0P} = \mathbf{R} - \mathbf{R}_0$ and (1) gives

$$\mathbf{R} = \mathbf{R}_0 + t\mathbf{V}, \tag{2}$$

which is the *vector equation* of L. As t varies from $-\infty$ to ∞, the point P traverses the entire infinite line L, moving in the direction of $\mathbf{V}$.

If we write (2) in the form

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0k + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

and equate the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, we get the three scalar equations

$$\begin{aligned} x &= x_0 + at, \\ y &= y_0 + bt, \\ z &= z_0 + ct. \end{aligned} \tag{3}$$

These are the *parametric equations* of the line L through the point $P_0 = (x_0, y_0, z_0)$ and parallel to the vector $\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Observe that the parametric equations of a straight line are not unique. The numbers x_0, y_0, and z_0 can be replaced by the coordinates of any other point on L, and a, b, and c can be replaced by the components of any other nonzero vector parallel to L, and the resulting parametric equations will be completely equivalent to equations (3) in the sense that they describe the same line.

In order to obtain the Cartesian equations of the line, we eliminate the parameter from equations (3) by equating the three expressions obtained by solving for t. This gives

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{4}$$

These are called the *symmetric equations* of the line L. If any one of the constants a, b, c is zero in a denominator of (4), then the corresponding numerator must also be zero. This is easy to see from the parametric form (3), which shows, for example, that if

$$x = x_0 + at \qquad \text{and} \qquad a = 0,$$

then $x = x_0$. Thus, when one of the denominators in (4) vanishes, we interpret this as meaning that the corresponding numerator must also vanish. With this interpretation, equations (4) can always be used, even though division by zero is normally forbidden.

1. 标题与概括
标题： 三维空间中直线的向量与参数方程
概括：
在三维解析几何中，简单的一次方程（如 $ax+by+c=0$ ）只能表示一个平面，而不能像在二维平面那样表示一条直线。要在三维中刻画一条直线，常用到向量与参数这两种工具。本节先介绍了直线在三维空间的三种几何描述方式，重点放在"给定一点和一个方向向量"来写出直线的**向量方程**和**参数方程**，然后再将参数消去得到**对称式方程**。

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2. 详细知识点解析

(a) 三维中一条直线的三种常见描述方式

1. **通过两点：** 给定直线上的两点 P_1 与 P_2，即可确定唯一直线。
2. **两平面的交线：** 若有两个不重合的平面，其公共交线也是确定的一条直线。
3. **一点与方向向量：** 若已知直线上一点 P_0 和与直线平行的非零向量 $\mathbf{V}$，则直线可被唯一确定。

> 本节重点在第三种：由"**一个已知点 + 平行向量**"来写出直线的方程。

(b) 向量方程与参数方程（图 18.22）

设直线 L 经过已知点

$$P_0=(x_0,\ y_0,\ z_0)$$

并且与向量

$$\mathbf{V}=a\,\mathbf{i}+b\,\mathbf{j}+c\,\mathbf{k}$$

平行。三维空间中任意一点 $P=(x,y,z)$ 在什么条件下属于这条直线?

1. **向量平行条件:**
   如果 P 在直线上，那么 $\overrightarrow{P_0P}$ 与 $\mathbf{V}$ 必须平行。用符号表示即

   $$\overrightarrow{P_0P}=t\,\mathbf{V},\quad (1)$$

   其中 t 是任意实数。
2. **从位置向量角度:**
   - 用 $\mathbf{R}_0=\overrightarrow{OP_0}$ 表示点 P_0 的位置向量;
   - 用 $\mathbf{R}=\overrightarrow{OP}$ 表示点 P 的位置向量。
   - 则 $\overrightarrow{P_0P}=\mathbf{R}-\mathbf{R}_0$。代入 (1) 得:

     $$\mathbf{R}-\mathbf{R}_0=t\,\mathbf{V}\implies\mathbf{R}=\mathbf{R}_0+t\,\mathbf{V}.\quad (2)$$

     这就是**直线 L 的向量方程。**
3. **参数方程（标量形式）：**
   将 (2) 写成分量形式:

   $$x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}=x_0\,\mathbf{i}+y_0\,\mathbf{j}+z_0\,\mathbf{k}+t\,(a\,\mathbf{i}+b\,\mathbf{j}+c\,\mathbf{k}),$$

   对应到各坐标分量，得到:

$$\begin{cases} x = x_0 + a\,t, \\ y = y_0 + b\,t, \\ z = z_0 + c\,t. \end{cases} \quad (3)$$

这三式合称为**直线的参数方程**。其中：

- t 是实数参数，在 $(-\infty, +\infty)$ 上变化。
- 向量 $\mathbf{V} = (a, b, c)$ 即为**方向向量**，它决定了直线沿哪个方向延伸。
- 点 (x_0, y_0, z_0) 则为该直线上的已知定点。

> **注意**：若更换同一直线上其他点作为 P_0，或改用与 $\mathbf{V}$ 平行的向量，得到的参数方程会不同，但描述的几何"同一条线"不变。

(c) 对称式方程 (4)

1. **从参数方程到对称式**：
   若要消去参数 t，可从方程

$$\begin{cases} x = x_0 + a\,t, \\ y = y_0 + b\,t, \\ z = z_0 + c\,t \end{cases}$$

   分别解出

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

   这就是**对称式方程**（或称**对称式**）：

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \quad (4)$$

   - 假设 $a, b, c \neq 0$。若其中某个分量为0，则表示该坐标不随 t 变化。此时可用"分子也必须为 0"的理解方式继续使对称式成立。
2. **特殊情况**：
   - 若方向向量 $\mathbf{V}$ 中某分量为0（例如 $a = 0$），说明直线在 x-坐标上不变，即 $x = x_0$。对称式中 $\frac{x-x_0}{0}$ 表面上是"不可除"，但我们约定：若分母=0，则相应分子也=0，等价于 $x = x_0$。这样对称式写法仍保持统一。

(d) 小结与注意要点

1. **三种方程的关系**：
   - **向量方程** $\mathbf{R} = \mathbf{R}_0 + t\,\mathbf{V}$ 最直观地表示"原点到直线上任意点的向量 = 常向量 + 参数 $\times$ 方向向量"。
   - **参数方程** $(x, y, z) = (x_0 + at,\ y_0 + bt,\ z_0 + ct)$ 是向量方程的坐标展开版，常用于代入其他计算或联立方程时。
   - **对称式方程** $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ 不显式包含参数，写起来简洁。但要小心分母为零的情况。
2. **与二维的区别**：
   - 在二维里，一条直线用一阶方程 $ax + by + c = 0$ 即可表示；
   - 在三维里，形如 $ax + by + c = 0$ 的方程却表示**平面**而非直线。因为三维中要限制两个自由度才能变成一条线，而一个线性方程只限制了一个自由度。
3. **实用提示**：
   - 当需要**联立**多条线或面时，使用参数形式往往更灵活；
   - 当直线并没有方向向量分量为 0 时，对称式很简洁；若出现分量为 0 的特殊情况，参数形式就更方便且不易出错。

3. 本次内容总结

- **直线在三维中的方程**：不可能只用一个线性方程来表示，需要“点 + 方向”或“平面交线”等方式。
- **主要形式**：
  1. **向量方程**： $\mathbf{R} = \mathbf{R}_0 + t\,\mathbf{V}$ ，最能体现几何含义；
  2. **参数方程**： $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$ ；
  3. **对称式方程**： $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ 。
- **分母为零**：对应坐标不随参数变化，即该坐标恒等于某常数。可通过“分子也为 0”继续使对称式有效。

这些知识为三维解析几何的进一步内容（如求平面方程、求直线与平面的交点等）奠定了基础。结合之前的向量运算，可以更灵活地处理空间几何问题。

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4. 报错或不清晰说明

- 本次图片内容与文字都较为完整清晰，没有出现明显缺失或无法识别的部分。
- 若对分母为 0 的处理规则、或者如何与其他几何对象联立方程有疑问，可再做追问。

以上即为本部分的完整笔记，希望能帮助你清晰掌握三维直线方程的写法与对应几何含义。

18.4.2

Example 1 A line L goes through the points $P_0 = (3, -2, 1)$ and $P_1 = (5, 1, 0)$. Find the parametric equations and the symmetric equations of L. Also find the points at which this line pierces the three coordinate planes.

Solution The line L is parallel to the vector $\overrightarrow{P_0P_1} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, so by using P_0 as the known point on the line, equations (3) give the parametric equations

$$x = 3 + 2t,$$
$$y = -2 + 3t,$$
$$z = 1 - t.$$

By eliminating t, we obtain the symmetric equations

$$\frac{x-3}{2} = \frac{y+2}{3} = \frac{z-1}{-1}.$$

To find the point at which L pierces the xy-plane, we set $z = 0$ in the third parametric equation and see that $t = 1$. With this value of t, $x = 5$ and $y = 1$, so the point is $(5, 1, 0)$. Similarly, $x = 0$ implies that $t = -\frac{3}{2}$, so the point in the yz-plane is $(0, -\frac{13}{2}, \frac{5}{2})$; and $y = 0$ implies $t = \frac{2}{3}$, so the point in the xz-plane is $(\frac{13}{3}, 0, \frac{1}{3})$.

Now we turn to the study of planes. A plane can also be characterized in several ways: as the plane through three noncollinear points, as the plane through a line and a point not on the line, or as the plane through a point and perpendicular to a specified direction. Again, the third approach is the most convenient for us.

Consider the plane that passes through a given point $P_0 = (x_0, y_0, z_0)$ and is perpendicular to a given nonzero vector

$$\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, \tag{5}$$

as shown in Fig. 18.23. Another point $P = (x, y, z)$ lies on this plane if and only if the vector P_0P is perpendicular to the vector $\mathbf{N}$, which means that

$$\mathbf{N} \cdot \overrightarrow{P_0P} = 0. \tag{6}$$

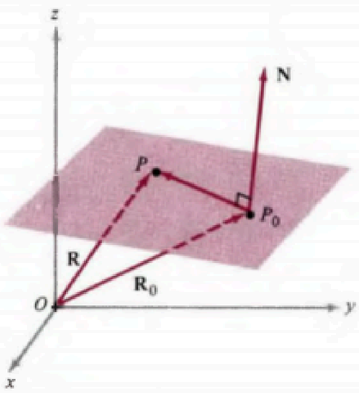

Figure 18.23 A plane in space.

If $\mathbf{R}_0 = \overrightarrow{OP_0}$ and $\mathbf{R} = \overrightarrow{OP}$ are the position vectors of P_0 and P, so that $\overrightarrow{P_0P} = \mathbf{R} - \mathbf{R}_0$, then (6) becomes

$$\mathbf{N} \cdot (\mathbf{R} - \mathbf{R}_0) = 0. \tag{7}$$

This is the *vector equation* of the plane under discussion.

Since $\mathbf{R} - \mathbf{R}_0 = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$, (7) can be written out in the scalar form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{8}$$

This is the *Cartesian equation* of the plane through the point $P_0 = (x_0, y_0, z_0)$ with normal vector $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. For example, the equation of the plane through $P_0 = (5, -3, 1)$ with normal vector $\mathbf{N} = 4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ is

$$4(x-5)+3(y+3)-2(z-1)=0$$

or

$$4x+3y-2z=9.$$

Observe that the coefficients of x, y, and z in the last equation are the components of the normal vector. This is always the case, for equation (8) can be written in the form

$$ax+by+cz=d, \tag{9}$$

where $d=ax_0+by_0+cz_0$; and the coefficients of x, y, and z in this equation are clearly the components of the normal vector (5). Conversely, every *linear equation* in x, y, and z of the form (9) represents a plane with normal vector $\mathbf{N}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ if the coefficients a, b, and c are not all zero. To see this, we notice that if (for instance) $a \neq 0$, then this permits us to choose y_0 and z_0 arbitrarily and solve the equation $ax_0+by_0+cz_0=d$ for x_0. With these values, (9) can be written as

$$ax+by+cz=ax_0+by_0+cz_0$$

or

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0,$$

and this is immediately recognizable as the equation of the plane through (x_0, y_0, z_0) with normal vector $\mathbf{N}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$.

Example 2 Find an equation for the plane through the three points $P_0=(3, 2, -1)$, $P_1=(1, -1, 3)$, and $P_2=(3, -2, 4)$.

Solution To use equation (8), we must find a vector $\mathbf{N}$ that is normal to the plane. This is easy to do by using the cross product. We compute

$$\mathbf{N}=\overrightarrow{P_0P_1}\times\overrightarrow{P_0P_2}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -3 & 4 \\ 0 & -4 & 5 \end{vmatrix}=\mathbf{i}+10\mathbf{j}+8\mathbf{k}.$$

Since $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$ lie in the plane, their cross product $\mathbf{N}$ is normal to the plane. Using equation (8) with P_0 as the given point, our plane has the equation

$$(x-3)+10(y-2)+8(z+1)=0$$

or

$$x+10y+8z=15,$$

after simplification.

Example 3 Find the point at which the line

$$\frac{x-2}{1}=\frac{y+3}{2}=\frac{z-4}{2}$$

pierces the plane $x+2y+2z=22$.

Solution To find parametric equations for the line, we introduce t as the common ratio in the given symmetric equations,

$$\frac{x-2}{1} = \frac{y+3}{2} = \frac{z-4}{2} = t,$$

which gives

$$x = 2 + t, \qquad y = -3 + 2t, \qquad z = 4 + 2t.$$

We want the value of t for which the variable point (x, y, z) on the line lies on the given plane. By substituting these equations into the equation of the plane, we obtain

$$(2 + t) + 2(-3 + 2t) + 2(4 + 2t) = 22,$$

so $t = 2$ at the point where the line pierces the plane. By substituting $t = 2$ back in the parametric equations of the line, we find that the desired point is $(4, 1, 8)$.

1. 标题与概述
标题： 三维空间中平面的方程与直线-平面交点求法
概述：
在前文已经掌握了三维直线的参数方程及对称式方程后，本节进一步探讨**平面**在三维空间的表示方式。我们先从“已知平面上一点与法向量”出发，写出平面的一般方程，再说明如何利用**叉积**在给定三个非共线点的情况下找到法向量，从而确定平面方程。最后，通过一个示例演示如何求给定直线与平面的交点。

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2. 逐点详解图片与知识点

（a）示例 1：已知两点确定直线，求其方程与三坐标平面的交点

书中示例先回顾了**如何由两点确定直线**。已知：

$$P_0 = (3,\ -2,\ 1), \quad P_1 = (5,\ 1,\ 0).$$

- **方向向量**：$\overrightarrow{P_0P_1} = (5-3,\ 1-(-2),\ 0-1) = (2,\ 3,\ -1)$ 。
- **参数方程**（令 t 为参数）：

$$\begin{cases} x = 3 + 2t, \\ y = -2 + 3t, \\ z = 1 - t. \end{cases}$$

- **对称式方程**（消去 t ）：

$$\frac{x-3}{2} = \frac{y+2}{3} = \frac{z-1}{-1}.$$

三坐标平面的交点指该直线与 xy -平面（ $z=0$ ）、 yz -平面（ $x=0$ ）、 xz -平面（ $y=0$ ）三者的交点。

- 例：求与 xy -平面的交点 —— 在参数方程中令 $z=0$ 得 $1-t=0$ ，可解 $t=1$ ，进而求 $(x,y) = (5, -2+3) = (5,1)$ ，故交点是 $(5,1,0)$ 。
- 同理处理 $x=0$ 、 $y=0$ 分别可得到相应交点。

（b）平面的三种几何定义（引入法向量）

与直线类似，平面在三维也可有多种描述方法：

1. 通过**三点**（且三点不共线）来确定平面；
2. 通过**一条线和线外一点**来确定；
3. 通过**一点与一个法向量**（在几何或物理中通常称“垂直于平面”方向的向量）来确定。

书中重点使用第三种：**给定平面上一点 $P_0 = (x_0, y_0, z_0)$ 与一个法向量 $\mathbf{N} = (a, b, c)$** 。

（c）平面的向量方程与标量方程 (图 18.23)

1. **向量方程**

   若平面 Π 上一点 $P = (x, y, z)$ ，则 $\overrightarrow{P_0P}$ 必须与法向量 $\mathbf{N}$ 垂直，写成点积为零：

$$\mathbf{N} \cdot \overrightarrow{P_0P} = 0 \quad \Longrightarrow \quad \mathbf{N} \cdot (\mathbf{R} - \mathbf{R}_0) = 0,$$

   其中 $\mathbf{R}_0$ 是 P_0 的位置向量，$\mathbf{R}$ 是 P 的位置向量。

2. **标量方程(8)**
   将 $\mathbf{N} = (a, b, c)$ 与 $\mathbf{R} - \mathbf{R}_0 = (x - x_0,\, y - y_0,\, z - z_0)$ 做点积，为 0 得：

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

   若将常数项移到右边，可写成

$$a\,x + b\,y + c\,z = d,$$

   其中 $d = a\,x_0 + b\,y_0 + c\,z_0$ 。这便是**平面的标准形式**：$ax + by + cz = d$ 。

> 由此可见，平面方程中 x, y, z 的**系数就是法向量的分量**。

（d）示例 2：通过三点确定平面方程

题目： 给定三点

$$P_0 = (3,\,2,\,-1), \quad P_1 = (1,\,-1,\,3), \quad P_2 = (3,\,-2,\,4),$$

求该平面的方程。

1. **思路：** 先求在平面内的两个向量 $\overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}$ ，再用它们做**叉积**得到与平面垂直的向量 $\mathbf{N}$ 。
2. **计算向量与叉积：**
   - $\mathbf{P_0P_1} = (1 - 3,\, -1 - 2,\, 3 - (-1)) = (-2,\, -3,\, 4)$ 。
   - $\mathbf{P_0P_2} = (3 - 3,\, -2 - 2,\, 4 - (-1)) = (0,\, -4,\, 5)$ 。
   - 叉积 $\mathbf{N} = \mathbf{P_0P_1} \times \mathbf{P_0P_2}$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -3 & 4 \\ 0 & -4 & 5 \end{vmatrix} = \mathbf{i} \cdot [(-3) \cdot 5 - 4 \cdot (-4)] \;-\; \mathbf{j} \cdot [(-2) \cdot 5 - 4 \cdot 0] \;+\; \mathbf{k} \cdot [(-2) \cdot (-4) - (-3) \cdot 0].$$

     - $\mathbf{i}$ -系数：$(-3) \times 5 - 4 \times (-4) = -15 + 16 = 1.$
     - $\mathbf{j}$ -系数：$(-2) \times 5 - 4 \times 0 = -10,$ 但要带上负号，故 $-(-10) = +10.$
     - $\mathbf{k}$ -系数：$(-2) \times (-4) - (-3) \times 0 = 8.$

  - 所以 $\mathbf{N}=(1,\,10,\,8)$。
3. **写出平面方程**
   用公式(8)： $\mathbf{N}\cdot(\mathbf{R}-\mathbf{R}_0)=0$，把 $\mathbf{R}_0=(3,2,-1)$ 代入：

$$1\,(x-3)+10\,(y-2)+8\,(z+1)=0.$$

   展开可得

$$x-3+10y-20+8z+8=0\quad\Longrightarrow\quad x+10y+8z=15.$$

   这就是平面方程的简洁形式。

（e）示例 3：求直线与平面的交点

题目： 已知直线

$$\frac{x-2}{1}=\frac{y+3}{2}=\frac{z-4}{2},$$

与平面

$$x+2y+2z=22,$$

求它们的交点。

1. **将对称式转成参数方程：**
   令公共比值为 t，则：

$$x=2+1\cdot t=2+t,\quad y=-3+2t,\quad z=4+2t.$$

2. **把 (x,y,z) 代入平面方程**
   平面方程为 $x+2y+2z=22$。代入即

$$(2+t)+2[-3+2t]+2[4+2t]=22.$$

   - 化简：

$$2+t+2(-3+2t)+2(4+2t)=2+t+[-6+4t]+[8+4t]=(2-6+8)+(t+4t+4t)=4+9t=22.$$

     故 $9t=18$，得 $t=2$。
3. **求交点坐标：**
   - $x=2+(2)=4$
   - $y=-3+2(2)=-3+4=1$
   - $z=4+2(2)=4+4=8$
     所以交点为 $(4,\,1,\,8)$。

---

3. 本次内容总结

1. **平面方程的核心：法向量**
   - 若 $\mathbf{N}=(a,b,c)$ 是平面 Π 的法向量，则平面方程必可写为 $ax+by+cz=d$。
   - 要想求得 $\mathbf{N}$，常用手段就是在平面内找两向量做叉积。
2. **通过点与法向量写方程**
   - 标量形式： $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$。

    - 有时将常数项移到右边，得 $ax + by + cz = d$ 。
3. **直线与平面交点**
    - 常见解法：用直线参数方程 (x, y, z) 代入平面方程，解参数值 t ，再回代求出交点坐标。
    - 若方程无解则说明直线与平面平行且不相交；若满足整个方程组恒成立则说明直线整条都在平面上。

---

4. 报错或不清晰说明

- 以上截图内容整体完整，示例计算各步也较清晰。
- 若在具体叉积符号或参数消去过程中有疑问，可再行提问。

通过这些示例，可以系统掌握**如何在三维中确定平面**以及**如何求直线与平面的交点**；结合先前的直线方程知识，就能解决较多三维解析几何的基本问题了。

18.4.3

Example 4 Find the cosine of the angle between the two planes $x + 4y - 4z = 9$ and $x + 2y + 2z = -3$. Also, find parametric equations for the line of intersection of these planes.

Solution Clearly the angle θ between two planes is the angle between their normals (Fig. 18.24). By inspecting the equations of the given planes, we see at once that their normals are

$$\mathbf{N}_1 = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}, \qquad \mathbf{N}_2 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$

We therefore use the dot product to obtain

$$\cos\theta = \frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{|\mathbf{N}_1||\mathbf{N}_2|} = \frac{1}{3\sqrt{33}}.$$

From this we can find the angle θ if we wish, by tables or otherwise.

To find parametric equations for the line of intersection, we need a vector $\mathbf{V}$ parallel to this line and a point on the line. We find $\mathbf{V}$ by computing the cross product of $\mathbf{N}_1$ and $\mathbf{N}_2$,

$$\mathbf{V} = \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -4 \\ 1 & 2 & 2 \end{vmatrix} = 16\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}.$$

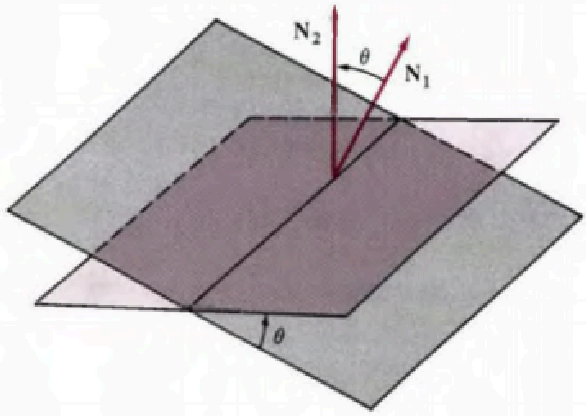

Figure 18.24

Since any vector parallel to the line will do, we divide by 2 and use the slightly simpler vector $8\mathbf{i} - 3\mathbf{j} - \mathbf{k}$. To find a point on the line, we can set $z = 0$ and solve the resulting system in the unknowns x and y,

$$x + 4y = 9,$$
$$x + 2y = -3.$$

This yields $x = -15$, $y = 6$. The desired point is therefore $(-15, 6, 0)$, and the parametric equations of the line are

$$x = -15 + 8t,$$
$$y = 6 - 3t,$$
$$z = -t.$$

We repeat that there is nothing unique about these equations, for we could have found a point on the line in many other ways and there are many different vectors parallel to the line.

As we remarked at the beginning of this section, any two intersecting planes determine a straight line in space. The equations of the two planes are satisfied simultaneously only by points on the line of intersection. From this point of view, a pair of linear equations considered as a simultaneous system can be interpreted as representing a line, namely, the line of intersection of the two planes represented by the individual equations. (Of course, the planes must actually intersect, and not be parallel or identical.) Thus, in Example 4 the pair of simultaneous equations

$$x + 4y - 4z = 9,$$
$$x + 2y + 2z = -3,$$

represents the line discussed in that example. We also point out that the symmetric equations (4) are equivalent to the three simultaneous equations

$$b(x - x_0) - a(y - y_0) = 0,$$
$$c(x - x_0) - a(z - z_0) = 0,$$
$$c(y - y_0) - b(z - z_0) = 0.$$

These are the equations of three planes that intersect in the line L represented by (4). The first has normal vector $b\mathbf{i} - a\mathbf{j}$, which is parallel to the xy-plane, so the first plane is perpendicular to the xy-plane. Similarly, the second plane is perpendicular to the xz-plane and the third is perpendicular to the yz-plane. Any pair of these equations represents the line L, which is the intersection of the corresponding pair of planes.

1. 标题与概览

标题： 求两平面间的夹角及其交线方程

概览：

本节通过一个示例说明了如何计算两平面的夹角，以及如何求它们在三维空间中的交线。核心要点包括：

1. **夹角**：两平面的夹角等于它们**法向量**之间的夹角，可用点积公式求出余弦值。
2. **交线**：两平面的交线方向向量可由其法向量做**叉积**得到，随后只需在两个方程中找一个特解点，即可写出交线的参数方程。

---

2. 详解示例与核心知识点

（a）示例：给定平面方程，求它们的夹角与交线

题目：
两平面方程分别为

$$\Pi_1: \quad x + 4y \; - \; 4z \; = \; 9, \qquad \Pi_2: \quad x + 2y \; + \; 2z \; = \; -3.$$

1. 求两平面的夹角 θ。
2. 求这两平面的交线方程（即它们在空间中的交线）。

（b）求平面夹角

1. **提取法向量**
   - 从平面方程 Π_1 可见，系数 $(1,\,4,\,-4)$ 即是 Π_1 的法向量：
   $$\mathbf{N}_1 = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}.$$
   - 从平面方程 Π_2 可见，系数 $(1,\,2,\,2)$ 即是 Π_2 的法向量：
   $$\mathbf{N}_2 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$
2. **用点积求夹角余弦**
   - 点积定义：$\mathbf{N}_1 \cdot \mathbf{N}_2 = |\mathbf{N}_1| \; |\mathbf{N}_2| \; \cos\theta$。
   - 首先计算 $\mathbf{N}_1 \cdot \mathbf{N}_2$：
   $$(1)\cdot(1) + (4)\cdot(2) + (-4)\cdot(2) = 1 + 8 - 8 = 1.$$
   - 然后各自模长：
   $$|\mathbf{N}_1| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}, \quad |\mathbf{N}_2| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3.$$
   - 因此
   $$\cos\theta = \frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{|\mathbf{N}_1| \; |\mathbf{N}_2|} = \frac{1}{\sqrt{33}\,\cdot\,3} = \frac{1}{3\sqrt{33}}.$$
   - 若需要 θ，可再取反余弦：$\theta = \arccos\!\left(\frac{1}{3\sqrt{33}}\right)$。

几何含义： 两平面的夹角与它们法向量的夹角相同（图 18.24 所示 θ 即为 $\mathbf{N}_1, \mathbf{N}_2$ 的夹角）。

（c）求交线方程

两平面相交于一条直线。为写出这条交线的参数方程，需要：

1. 一个**方向向量**：它应当同时垂直于 $\mathbf{N}_1$ 与 $\mathbf{N}_2$ 的“法向量们”，故可取
$$\mathbf{V} = \mathbf{N}_1 \times \mathbf{N}_2.$$

2. 线上的**任意一点**。可通过在这两平面方程中赋值或消去变量来寻找一个解。

---

1. 求方向向量 $\mathbf{V}$

- 先做叉积

$$\mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -4 \\ 1 & 2 & 2 \end{vmatrix}.$$

  沿第一行展开：

  - $\mathbf{i}$ -系数：$4 \cdot 2 - (-4) \cdot 2 = 8 + 8 = 16$.
  - $\mathbf{j}$ -系数：$(1 \cdot 2 - (-4) \cdot 1) = 2 + 4 = 6$ 但带负号 => -6 .
  - $\mathbf{k}$ -系数：$(1 \cdot 2 - 4 \cdot 1) = 2 - 4 = -2.$
    所以

$$\mathbf{V} = (16,\ -6,\ -2).$$

- 若想简化，可除以 2 得 $\mathbf{V} = (8,\ -3,\ -1)$ 。方向相同即可。

---

2. 找交线上一点

要找一组 (x, y, z) 同时满足

$$\begin{cases} x + 4y - 4z = 9, \\ x + 2y + 2z = -3. \end{cases}$$

技巧：常常可以令某个坐标为方便的数值（如 0）来求解。这里，作者把 $z = 0$ 代入，快速找到 (x, y) :

- 令 $z = 0$ ，则系统变为

$$\begin{cases} x + 4y = 9, \\ x + 2y = -3. \end{cases}$$

- 相减得 $2y = 12 \Rightarrow y = 6$ 。带回 $x + 2(6) = -3 \Rightarrow x + 12 = -3 \Rightarrow x = -15$ 。
- 因此，当 $z = 0$ 时 $(x, y) = (-15, 6)$ ，可取到一点 $(-15, 6, 0)$ 在交线上。

---

3. 写参数方程

令参数为 t 。用所求点作"起点"、 $\mathbf{V} = (8,\ -3,\ -1)$ 作方向：

$$\begin{cases} x = -15 + 8t, \\ y = 6 - 3t, \\ z = 0 - 1t = -t. \end{cases}$$

这就是交线 ℓ 的参数方程。

(d) 结果总结与说明

1. **平面夹角：**

$$\cos\theta = \frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{|\mathbf{N}_1|\ |\mathbf{N}_2|} = \frac{1}{3\sqrt{33}} \quad \Longrightarrow \quad \theta = \arccos\left(\tfrac{1}{3\sqrt{33}}\right).$$

2. **交线参数方程：** 任选化简的方向向量后，可写为

$$\begin{cases} x = -15 + 8t, \\ y = 6 - 3t, \\ z = -t. \end{cases}$$

多解性的说明： 找交线上“某个点”并不唯一（可令其他坐标为 0 或者其他常数也行），方向向量也不唯一（乘任何非零常数都可）。但几何上描述的还是同一条线。

---

3. 本次内容总结

1. **两平面夹角 = 两法向量夹角**
   - 点积公式：$\cos\theta = \dfrac{\mathbf{N}_1 \cdot \mathbf{N}_2}{|\mathbf{N}_1|\ |\mathbf{N}_2|}$ 。
2. **两平面交线**
   - **方向向量**取 $\mathbf{N}_1 \times \mathbf{N}_2$；
   - **线上一点**可通过联立平面方程或设定一个坐标值后求解；
   - 再写出标准的参数方程即可。

这样，就能系统解决“两平面相交”的基本问题：既可求夹角，也可求交线的方程。

---

4. 报错或不清晰说明

- 本节示例中提供的数值与演算结果清晰可见，没有缺失。
- 若需进一步探讨比如如何判断平面是否平行、不相交或重合，可再行提问。

以上笔记即为两平面夹角及其交线方程的完整解析，结合前面向量知识即可熟练运用到三维解析几何的更多场景之中。

18.5

18.5
CYLINDERS AND SURFACES OF REVOLUTION

We know that the graph of an equation $f(x, y) = 0$ is usually a curve in the xy-plane. In just the same way, the graph of an equation

$$F(x, y, z) = 0 \tag{1}$$

is usually a surface in xyz-space. The simplest surfaces are planes, and we saw in Section 18.4 that the equation of a plane is a linear equation that can be written in the form

$$ax + by + cz + d = 0;$$

that is, it contains only first-degree terms in the variables x, y, and z. In this section and the next we examine a few other simple surfaces containing terms of higher degree that often appear in multivariable calculus.

Cylinders are the next surfaces after planes in order of complexity. To understand what these surfaces are, we consider a plane curve C and a line L not parallel to the plane of C. By a *cylinder* we mean the geometric figure in space that is generated (or swept out) by a straight line moving parallel to L and passing through C (Fig. 18.25).* The moving line is called the *generator* of the cylinder. The cylinder can be thought of as consisting of infinitely many parallel lines,

*This concept includes the familiar *right circular* cylinders of elementary geometry, for which the curve C is a circle and the line L is perpendicular to the plane of the circle. In geometry the adjectives are often omitted, because no other kinds of cylinders are considered. However, it should be noticed that when C is itself a straight line, the cylinder is a plane, so cylinders also include planes as special cases.

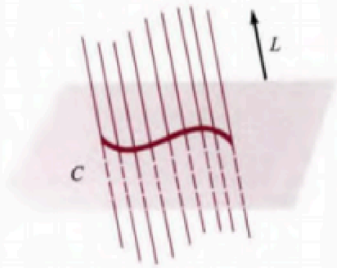

Figure 18.25 A general cylinder.

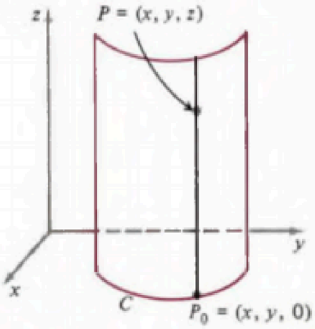

Figure 18.26

called *rulings,* corresponding to various positions of the generator. This is suggested in the figure.

For example, suppose that the given curve C is the curve

$$f(x, y) = 0 \tag{2}$$

in the xy-plane, and let the generator be parallel to the z-axis, as shown in Fig. 18.26. *Then exactly the same equation* (2) *is the equation of the cylinder in three-dimensional space.* The reason for this is that the point $P = (x, y, z)$ lies on the cylinder if and only if the point $P_0 = (x, y, 0)$ lies on the curve C, and this happens if and only if $f(x, y) = 0$. The essential feature of (2) as the equation of the cylinder is that it is an equation of the form (1) from which the variable z is missing. To express this in another way, the fact that we are dealing with a cylinder whose rulings are parallel to the z-axis means that for a point $P = (x, y, z)$, the value of z has no bearing on whether P lies on the cylinder or not; and since only the variables x and y are relevant to this issue, only the variables x and y can be present in the equation of the cylinder—that is, z must be missing from this equation.

Example 1 Sketch the cylinder

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

Solution This appears to be the equation of an ellipse in the xy-plane. However, it is stated that this is a cylinder, and since the variable z is missing from the equation, the rulings of this cylinder are parallel to the z-axis. In Fig. 18.27, the ellipse in the xy-plane is drawn first, then two vertical rulings, then a horizontal elliptical cross section above the xy-plane. In spite of the limitations of our figure (which we hope students will try to overcome by an active use of imagination), it should be remembered that all rulings on a cylinder extend to infinity in both directions. This surface is called an *elliptic cylinder.*

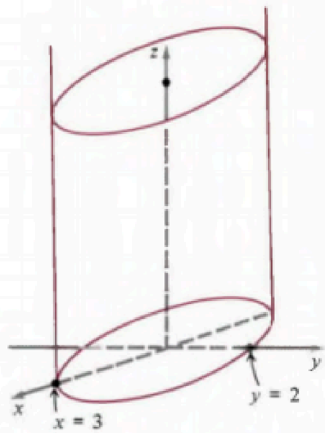

Figure 18.27 Elliptic cylinder.

It is clear that this discussion can be carried through for a cylinder with rulings parallel to any coordinate axis. We therefore have the conclusion that *any equation in rectangular coordinates x, y, z with one variable missing represents a cylinder whose rulings are parallel to the axis corresponding to the missing variable.*

Example 2 Sketch the cylinder $z = x^2$.

Solution In the xz-plane, this is the equation of a parabola with vertex at the origin that opens in the positive z-direction. However, we know that we are dealing with a cylinder, and since the variable y is missing from the equation, the rulings of this cylinder are parallel to the y-axis. In Fig. 18.28 the parabola in the xz-plane is drawn first, then several rulings, and then a second parabolic cross section located to the right of the xz-plane. This surface can be described as a *parabolic cylinder.*

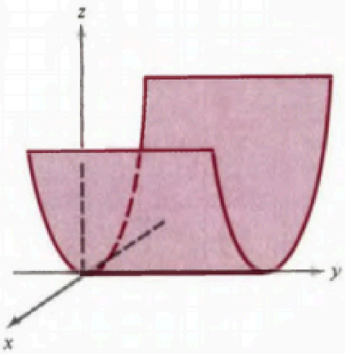

Figure 18.28 Parabolic cylinder.

Another way to generate a surface by using a plane curve C is to revolve the curve (in space) about a line L in its plane. The resulting surface is called a *sur-*

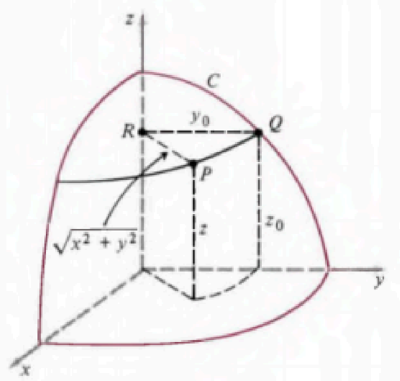

Figure 18.29

face of revolution with *axis L*. In Chapter 7 we became acquainted with surfaces of revolution by calculating their areas as an application of definite integrals. We now consider the equations of these surfaces.

Suppose, for example, that the curve C lies in the yz-plane and has equation

$$f(y, z) = 0. \tag{3}$$

As this curve is revolved about the z-axis, a typical point $P = (x, y, z)$ on the resulting surface comes from a point Q on C, as shown in Fig. 18.29. Since Q lies on C, its coordinates (y_0, z_0) satisfy (3),

$$f(y_0, z_0) = 0. \tag{4}$$

But the relation of P to Q tells us that $z_0 = z$ and $y_0 = \sqrt{x^2 + y^2}$, so (4) yields

$$f(\sqrt{x^2 + y^2}, z) = 0 \tag{5}$$

as the equation of the surface of revolution. Briefly, as Q swings out to the point P on the surface, the distances QR and PR to the z-axis are equal, and we get equation (5) by replacing y in (3) by $\sqrt{x^2 + y^2}$. Equation (5) assumes that $y \geq 0$ on C. If y is positive on some parts of C and negative on others, we must replace y in (3) by $\pm\sqrt{x^2 + y^2}$ to get

$$f(\pm\sqrt{x^2 + y^2}, z) = 0$$

as the equation of the complete surface. The awkward radical with its plus-or-minus sign can usually be eliminated by squaring.

Example 3 If the line $z = 3y$ in the yz-plane is revolved about the z-axis, the resulting surface of the revolution is clearly a right circular cone of two nappes with vertex at the origin and axis the z-axis (Fig. 18.30). To get the equation of this cone, we replace y in the equation $z = 3y$ by $\pm\sqrt{x^2 + y^2}$ and then rationalize by squaring:

$$z = \pm 3\sqrt{x^2 + y^2}, \qquad z^2 = 9(x^2 + y^2).$$

If we had merely replaced y by $\sqrt{x^2 + y^2}$ to obtain

$$z = 3\sqrt{x^2 + y^2},$$

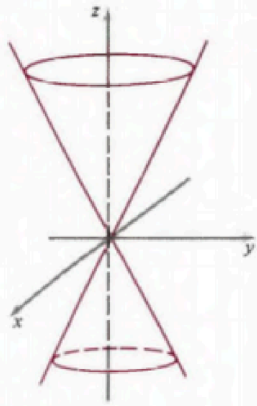

Figure 18.30 Cone.

1. 标题与概览

标题： 圆柱曲面与旋转曲面在三维空间中的表示

概览：

本节讨论了继平面之后的一类常见空间曲面，即**圆柱**与**旋转曲面**。其中，圆柱的本质是“在平面曲线的基础上沿与该平面垂直（或不平行）的方向平移（或保持平行）而形成的曲面”；旋转曲面则是把某条（或某些）平面曲线绕一条指定的直线旋转所形成。通过适当的坐标变量省略或替换，可以写出这些曲面的方程并进行可视化理解。

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2. 详细知识点解析

(a) 从平面曲线到空间曲面

1. **从 F(x,y)=0 到 F(x,y,z)=0：**
   - 在二维中，方程 $F(x,y)=0$ 通常表示平面上的一条曲线（例如椭圆、抛物线等）。
   - 在三维中，方程 $F(x,y,z)=0$ 通常表示空间中的一个曲面。最简单的例子是**线性方程** $ax+by+cz+d=0$ ，它表示一个平面。
   - 本节则关注含有二次或其他形式，但"缺少某个变量"或"在旋转中引入新变量关系"的曲面。

(b) 圆柱曲面（Cylinders）

1. **一般定义：**
   给定一个平面曲线 C 和一条不平行于该平面的直线 L 。在空间里，让一条直线与 L 保持平行，同时它的一个端点滑过曲线 C ，那么由该移动直线所"扫出"的集合就形成一个**圆柱曲面**（见图 18.25）。
   - 这些"移动的直线"称为**母线（generator or ruling）**。
   - 如果该平面曲线 C 本身是圆，那么就得到我们熟悉的**圆柱**；如果是椭圆曲线，则得到**椭圆柱**，依此类推。
2. **在坐标中的特征：**
   如果母线是与 z-轴平行，那么曲面的方程往往是**不包含** z 的。也就是形如

$$f(x,y)=0,$$

   但此方程却在三维空间里代表一个圆柱曲面（母线方向为 z-轴），因为 z 的取值对满足曲面方程毫无影响。
   - 例如：
     - ** $\frac{x^2}{9}+\frac{y^2}{4}=1$ ** 不再只是二维椭圆，而是在三维中（省略 z ）代表一个"椭圆柱" (图 18.27)。
     - ** $z=x^2$ ** 如果视之为"缺少 y "，则它对应的母线平行于 y-轴，得到的是一个"抛物柱" (图 18.28)。
   - 一般而言，若方程缺少某个变量，就可理解该变量方向上的"平移"或"延伸"不会影响曲线本身的限制，从而形成无穷多条平行母线构成的圆柱曲面。

(c) 旋转曲面（Surfaces of Revolution）

1. **由曲线绕某轴旋转：**
   - 若给定一条平面曲线 C （位于某平面上）以及一条用于旋转的直线 L （通常称为旋转轴），则将 C 绕 L 作一周旋转，就在三维中得到一个旋转曲面。
   - 常见例子：圆在绕某直径旋转可生成球面，抛物线或直线绕某轴旋转可生成抛物面或圆锥面等。
2. **以 z 轴为旋转轴（示意图 18.29）：**
   - 假设曲线 C 位于 yz-平面，即它的方程可写为

$$f(y,z)=0. \quad (3)$$

   - 当它绕着 z-轴旋转时，空间中某点 $P=(x,y,z)$ 在该旋转面上，若投影到 yz-平面的点是 $Q=(0,y_0,z_0)$ ，则此时满足 $y_0^2+x^2=r^2$ （因为以 z-轴为中心绕圈， r 即是与 (y,z) 相关的半径），并且 (y_0,z_0) 在曲线 C 上。
   - 具体来说：

$$y_0=\sqrt{x^2+y^2}, \quad z_0=z.$$

   将 $\{y_0,z_0\}$ 代入方程 (3)，即可得到**旋转曲面**方程形如

$$f(\pm\sqrt{x^2+y^2},\, z) = 0.$$

- 实际运算中，往往通过平方等方式去掉正负号，得到更简洁的方程。

3. **示例：旋转直线得圆锥面（图 18.30）**
   - 在 yz-平面里，有一条直线 $z=3y$。将它绕 z-轴旋转，即得到**圆锥**。
   - 解析：令 $y_0=\sqrt{x^2+y^2}$ 且 $z_0=z$，把 $z=3y$ 改写成 $z_0=3y_0$。
   - 则在三维中：$z=3\sqrt{x^2+y^2}$（只对应上半部），若要完整双侧锥面可写成 $\pm 3\sqrt{x^2+y^2}$ 或经平方：

   $$z^2 = 9(x^2+y^2).$$

   这就生成一个**正圆锥**，上下两"瓣"称为两张 (nappe)。

（d）总结：识别圆柱与旋转曲面

1. **识别圆柱：**
   - 当方程**缺少一个变量**时，就可以看作圆柱的一种形式；母线平行于那个"缺失变量"的坐标轴。
   - 不论该方程在平面上对应的是椭圆、抛物线还是其他曲线，只要少一个变量，就对应在三维形成一个柱面。
2. **识别旋转曲面：**
   - 通常看见类似 $\sqrt{x^2+y^2}$ 或者 x^2+y^2 出现在方程里，并与另一个坐标（如 z）组合成有规律的关系，就很可能是以 z-轴为轴的旋转曲面（球面、圆锥面、圆环面、旋转抛物面等）。
   - 相似地，若以 x-轴或 y-轴为轴，也可出现 $\sqrt{y^2+z^2}$ 或 $\sqrt{x^2+z^2}$ 之类结构。

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3. 本次内容总结

1. **圆柱（Cylinders）**
   - 基本形式：$f(x,y)=0$（不含 z），或者缺少其他某一变量时，可视为母线平行该变量所在的坐标轴；
   - 实例：椭圆柱 $\frac{x^2}{9}+\frac{y^2}{4}=1$（母线 $\parallel z$-轴）、抛物柱 $z=x^2$（母线 $\parallel y$-轴）等。
2. **旋转曲面（Surfaces of Revolution）**
   - 将平面曲线绕某直线旋转 360° 后在三维产生的曲面；
   - 常见：
     - 直线旋转成**圆锥**(上、下两部分)；
     - 圆旋转成**球面**；
     - 抛物线旋转成**抛物面**；
   - 典型特征：方程里往往带有 $\sqrt{x^2+y^2}$ 或相应平方 x^2+y^2，来体现与某旋转轴的距离。

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4. 报错或不清晰说明

- 本次截图中的方程示例和图示都较为直观，未发现难以辨认之处。
- 如果对如何区分"母线平行于哪个坐标轴"或在旋转方程中"正负根"做何种物理/几何解释仍有疑问，可进一步提问。

这便是**圆柱与旋转曲面**的主要内容：只要把握"缺少一个变量"对应圆柱，以及"包含 $\sqrt{x^2+y^2}$ 等"往往代表围绕 z 轴（或其他坐标轴）旋转，就能在后续多变量微积分与空间几何中灵活识别并应用这些曲面。

18.6

18.6 QUADRIC SURFACES

In Section 15.6 we learned that the graph of a second-degree equation in the variables x and y is always a conic section—a parabola, an ellipse, a hyperbola, or perhaps some degenerate form of one of these curves, such as a point, the empty set, or a pair of straight lines.

In three-dimensional space the most general equation of the second degree is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0. \qquad (1)$$

We assume that not all of the coefficients $A, B, \ldots, F$ are zero, so that the degree of the equation is really 2 instead of 1 or 0. The graph of such an equation is called a *quadric surface.* We have already encountered several quadric surfaces, such as spheres and parabolic, elliptic, and hyperbolic cylinders, but there are a number of others as well. Indeed, if we set aside the familiar case of cylinders, then by suitable rotations and translations of the coordinate axes—which we do not discuss—it is possible to simplify any equation of the form (1) and thereby show that there are exactly six distinct kinds of nondegenerate quadric surfaces:

1 The ellipsoid.
2 The hyperboloid of one sheet.
3 The hyperboloid of two sheets.
4 The elliptic cone.
5 The elliptic paraboloid.
6 The hyperbolic paraboloid.

In the following we give an example of each type of surface in which the equation appears in as simple a form as possible.

Students should become familiar with these surfaces and their equations, and in particular should try to understand how the shape of each surface is related to the special features of its equation. For the purpose of visualizing and sketching a surface, it is often useful to examine its *sections,* which are the curves of intersection of the surface with planes

$$x = k, \qquad y = k, \qquad z = k$$

parallel to the coordinate planes. We point out explicitly that every second-degree section of every quadric surface is a conic section. Sections that are closed curves are usually the easiest to sketch, and therefore we look for elliptic sections and sketch these first. Symmetry considerations should also be kept in mind.

In the following examples, the numbers a, b, and c are all assumed to be positive. We comment informally rather than exhaustively on the surface considered in each example.

Example 1 The *ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{2}$$

is shown in Fig. 18.32. Since only even powers of x, y, and z occur in the equation, this surface is symmetric about each coordinate plane. The sections in the xz- and yz-planes are the ellipses

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \qquad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

with a common vertical axis. The section in a horizontal plane $z = k$ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2},$$

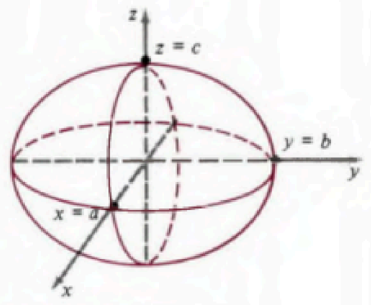

Figure 18.32 Ellipsoid.

and this decreases in size as k varies from 0 to c or $-c$. The numbers a, b, and c are the intercepts on the coordinate axes, and are called the *semiaxes.* If two of the semiaxes are equal, the ellipsoid is called a *spheroid*—an *oblate* spheroid if it is flattened like a "flying saucer," and a *prolate* spheroid if it is elongated like a football. Of course, if $a = b = c$, then the ellipsoid is a sphere.

Example 2 The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{3}$$

is a *hyperboloid of one sheet* (Fig. 18.33). If we write the equation in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} + 1, \tag{4}$$

then we see that all its horizontal sections in planes $z = k$ are ellipses, and that these ellipses grow larger as their planes move up or down from the xy-plane, the smallest ellipse being the one in the xy-plane. The section of the surface in the yz-plane is the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

It is this hyperbola that binds together the horizontal elliptical sections into a smooth surface. The phrase "of one sheet" is used because this surface consists of one piece, in contrast to the hyperboloid discussed in the next example, which consists of two pieces. Observe that equation (3) is obtained from (2) by changing the sign of the third term on the left; we get the same kind of surface no matter which of these terms has its sign changed.

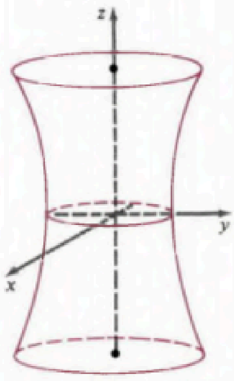

Figure 18.33 Hyperboloid of one sheet.

1. 标题与概览

标题： 二次曲面（Quadric Surfaces）的基本类型与示例

概览：

在二维解析几何中，我们熟知任意二次方程 $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ 表示的曲线可归结为圆、椭圆、抛物线、双曲线等（或其退化形式）。类似地，在三维空间中，“二次方程”

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

所刻画的曲面称为“二次曲面”。若我们经过适当坐标平移与旋转，可把方程化简成几类“标准形式”。本节介绍其中**六种非退化类型**的二次曲面：

1. **椭球面（ellipsoid）**
2. **单叶双曲面（hyperboloid of one sheet）**
3. **双叶双曲面（hyperboloid of two sheets）**
4. **椭圆锥面（elliptic cone）**
5. **椭圆抛物面（elliptic paraboloid）**
6. **双曲抛物面（hyperbolic paraboloid）**

下文首先给出两个示例：椭球面与单叶双曲面的标准方程和图像特征，并概括其横截面及对称性。

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2. 详细知识点解析

(a）一般二次曲面方程与简化思路

1. **一般二次曲面方程：**

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J\ =\ 0.$$

要满足 $\deg=2$，意味着至少一个二次项系数不为零。通过**平移坐标原点**（去除一次项）和**旋转坐标轴**（去除 xy,xz,yz 等交叉项），往往可将此方程化为某个比较标准的形式。如前面几何中所见，类似于“对角化”过程。

2. **六种主要非退化二次曲面：**
   - 它们的**标准方程**通常只剩下 $\pm x^2/a^2\pm y^2/b^2\pm z^2/c^2=1$ 或者 $\pm x^2/a^2\pm y^2/b^2=\pm z$ 等结构。
   - 一旦出现退化情况（系数特殊组合），可能退化为平面、直线、点、空集等。

(b）示例 1：椭球面（ellipsoid）

1. **标准方程 (2)：**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad a,b,c>0.$$

若 $a=b=c$，则得到**球面**。若不都相等，则形状类似“椭圆形球体”，也称椭球。

2. **对称与切片：**
   - 该方程中的 x^2,y^2,z^2 都是二次且带**正号**，因此在三坐标平面上具有对称性：关于 $x=0$、$y=0$、$z=0$ 对称。
   - 以某个固定 $z=k$ 的平面与椭球面相交，得到的是椭圆截面（若 $k^2<c^2$），并随着 $|k|$ 变大而截面逐渐缩小。
   - 同理，$x=\text{const.}$ 或 $y=\text{const.}$ 的切面也是椭圆（或可能退化为点，若截面太“高”或“远”）。

3. **几何外形：**
   - 如果 $a>b\approx c$，就像橄榄球形；
   - 如果 $a<b<c$，则更扁或更拉长，可能像飞碟形（扁椭球）或长椭球等。

(c）示例 2：单叶双曲面（hyperboloid of one sheet）

1. **标准方程 (3)：**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

与椭球面方程类似，只是有一个坐标项带**负号**。这个负号使得曲面呈现“腰鼓”形状，中间收窄、上下左右无界延伸。

2. **分层切片：**
   - **水平切片（$z=k$ 固定）**：

   $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}.$$

   右边大于 1，所以这是**椭圆**，并且随 $|k|$ 增大而“半径”增大，即越远离 xy-平面，截面越大。
   - **竖直切片（如 $y=0$）**：

   $$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1,$$

这是双曲线的一部分。类似地， $x=0$ 截面也是双曲线。

3. **“一片”与“两片”对比：**
   - 之所以叫“one sheet”，是因为整条曲面在几何上是一个**连续整体**，像一个螺栓中间的“腰身”；
   - 后面将讨论的“two sheets”则会分成上下两个完全分离的部分。

（d）后续的其他二次曲面

除了以上示例外，还包括：

- **双叶双曲面（hyperboloid of two sheets）**：方程形如 $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$，表现为上下（或左右）两个分离的碟形部分。
- **椭圆锥面（elliptic cone）**：形如 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$。相当于双曲面在 $=0$ 时的极端情形。
- **椭圆抛物面（elliptic paraboloid）**： $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$ （或类似），外形像碗状。
- **双曲抛物面（hyperbolic paraboloid）**： $\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$ 等；外形常称“马鞍面”。

在学习中，多对比截面曲线来理解各自形状。

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3. 本次内容总结

1. **二次曲面的通式**
   - 在三维中，任意真正二次曲面 $\deg=2$ 的方程可以通过适当平移旋转坐标系化为某种标准形式，最终归纳出六大非退化类型（不含柱面及退化情况）。
2. **椭球与单叶双曲面**
   - 二者在标准方程中有三个二次项，但椭球都为正号（球为特殊椭球），单叶双曲面有一个负号。
   - 可通过截面分析（水平切片或纵切片）来判断曲面的形状及其开口方向。
3. **其他类型**
   - 包括双叶双曲面、椭圆锥面、椭圆抛物面、双曲抛物面等，互相可视作对二次项正负号、或对常数项符号的更改，从而在几何形态上发生显著差异。

通过对这些标准方程及其切面分析的掌握，便能在后续多变量微积分和空间几何中快速判定或描绘出二次曲面的形状与特征。

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4. 报错或不清晰说明

- 本次文档中展示的示例（椭球面、单叶双曲面）方程及图示清晰可见。
- 若需更具体的例题（如如何用已知点和切面信息来定二次曲面方程），或对坐标旋转/平移过程有疑问，可再行提问。

以上即为二次曲面的基本介绍与示例。透过对六大非退化类型的熟悉及对称性、截面形状的分析，能在三维解析几何中对其进行识别与研究。

18.6.2

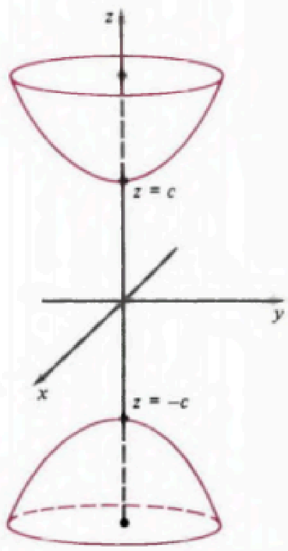

Figure 18.34 Hyperboloid of two sheets.

Example 3 The *hyperboloid of two sheets*

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{5}$$

is shown in Fig. 18.34. This equation is obtained from (2) by changing the signs of the first two terms on the left. (The reason for this choice is explained below.) If we write the equation in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1, \tag{6}$$

then we see that all its horizontal sections in planes $z = k$ with $k \geq c$ or $k \leq -c$ are ellipses or single points, while sections in planes $z = k$ with $|k| < c$ are empty. The section in the yz-plane is the hyperbola

$$\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1,$$

and it is this hyperbola that unifies the horizontal sections into a smooth surface —of "two sheets." Observe that (6) is identical with (4) except for the presence of the minus sign on the right, and it is this sign that makes all the difference between the surfaces in these two examples; for the right side of (4) is positive for all z's, whereas the right side of (6) is negative for $|z| < c$.

Example 4 The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \tag{7}$$

is an *elliptic cone* (Fig. 18.35). This surface intersects the xz-plane and the yz-plane in the pairs of intersecting straight lines

$$z = \pm\frac{c}{a}x \qquad \text{and} \qquad z = \pm\frac{c}{b}y,$$

respectively. It intersects the xy-plane at the origin alone. All horizontal sections in planes $z = k$ with $k \neq 0$ are ellipses. (In Chapter 15 it was convenient to distinguish circles from ellipses; here we include circles among the ellipses.) It is clear from the form of (7) that if (x, y, z) is a point on the surface, then (tx, ty, tz) is also on the surface for any number t. This tells us that the entire surface can be thought of as generated by a moving line through the origin O and a variable point P on any horizontal elliptical section. When $a = b$, the cone is the familiar right circular cone.

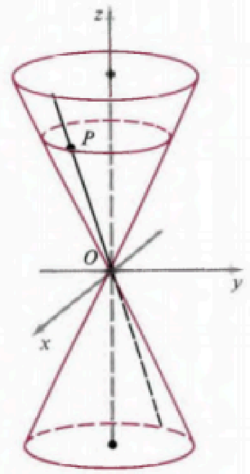

Figure 18.35 Elliptic cone.

Example 5 The *elliptic paraboloid*

$$z = ax^2 + by^2. \tag{8}$$

is shown in Fig. 18.36. The vertical sections of this surface in the xz-plane and yz-plane are the parabolas

$$z = ax^2 \qquad \text{and} \qquad z = by^2,$$

respectively. The horizontal section in the plane $z = k$ is an ellipse if $k > 0$, the origin alone if $k = 0$, and empty if $k < 0$.

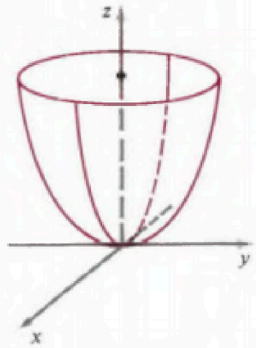

Figure 18.36 Elliptic paraboloid.

Example 6 In Fig. 18.37 we sketch the *hyperbolic paraboloid*

$$z = by^2 - ax^2. \tag{9}$$

The section in the yz-plane is the parabola $z = by^2$ opening upward, and that in the xz-plane is the parabola $z = -ax^2$ opening downward. In all planes $y = k$ parallel to the xz-plane, the sections are downward-opening parabolas that are identical with one another and can be thought of as hanging from their vertices at various points along the parabola $z = by^2$; this is emphasized in the way we

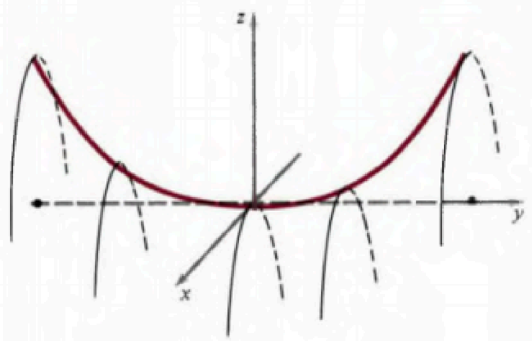

Figure 18.37 Hyperbolic paraboloid.

have drawn the figure. Near the origin the surface rises in the y-direction and falls in the x-direction, and thus has the general shape of a saddle or a mountain pass. For this reason, the surface is often called a *saddle surface*, with the origin as the *saddle point*. It is clear from (9) that in the horizontal plane $z = k$, the section is a hyperbola with principal axis in the y-direction if $k > 0$, and a hyperbola with principal axis in the x-direction if $k < 0$; if $k = 0$, the section is a pair of intersecting straight lines through the origin.

1. 标题与概述
标题： 二次曲面的典型类型（下）——双叶双曲面、椭圆锥面、椭圆抛物面及双曲抛物面
概述：
在上一部分我们讨论了椭球面和单叶双曲面这两种二次曲面。本次内容继续介绍其余四种常见的**非退化二次曲面**：

1. **双叶双曲面（Hyperboloid of two sheets）**
2. **椭圆锥面（Elliptic cone）**
3. **椭圆抛物面（Elliptic paraboloid）**
4. **双曲抛物面（Hyperbolic paraboloid）**

我们会分析它们的标准方程、外形特征以及典型的切面（截面）形状，帮助我们在三维几何中识别和理解它们。

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2. 详细知识点与示例解析

(a) 双叶双曲面（Hyperboloid of Two Sheets）

1. **标准方程 (5)**

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

在书中常见的另一种等价形式为

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1,$$

或者把“正负”分配给不同的变量，只要最终有两个负号、一正号（或类似）且右边为 $= 1$。

2. **几何形状：**
   - 与“单叶双曲面”不同之处在于，这个曲面分成**上下（或左右）两个分离的部分**，类似两个对口放置的碗（图 18.34 所示）。
   - 没有一条连续的“腰部”将上下贯通，而是被分隔成两个独立的“叶子（sheets）”。
3. **水平截面：**
   - 令 $z = k$，则方程变为

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}.$$

   - 当 $|k| < c$ 时，右侧正好是负数，方程没有实解（因为左侧是负数或 0，而右侧是正负号矛盾），说明在中间区域没有截面；
   - 当 $|k| > c$ 时，右侧变为某个负值加 1，为正数，得**椭圆**截面或退化为一点（视具体数值而定）。

- 可见，$z=\pm c$ 是分界面，中间部分($-c<z<c$)并没有真实点，所以曲面就分上下两块。

4. **与“单叶”对比：**
   - 书中对比如下：单叶双曲面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ 整个曲面在 z-方向上是**贯通**的；双叶双曲面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ 则是分裂成两个部分。
   - 核心在于右边是“1 还是 -1”，以及正负号如何排列。

(b) 椭圆锥面（Elliptic Cone）

1. **标准方程 (7)**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,$$

   亦可把 $\pm$ 号分配给不同变量，只要保证两正一负，并且右边 $=0$ 即可。

2. **几何形状：**
   - 这是“锥形”曲面，顶点通常在原点，且向四面无界延伸（图 18.35）。当 $a=b$ 时就是“正圆锥”，否则是“椭圆”横截面的锥面。
   - 在 $z>0$ 与 $z<0$ 区域分别是一张锥面，二者共顶点而不相交（除原点）。

3. **切面：**
   - 例如固定 $z=k\neq 0$，则

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2},$$

     这是一个椭圆；但注意当 $k=0$ 时就退化成原点，可见只有“最底部或顶部”是一个点截面。
   - 与 xz-平面的交线（置 $y=0$）或与 yz-平面的交线（置 $x=0$）往往是成对的直线，如图中所示：$z=\pm\frac{c}{a}x$ 、$z=\pm\frac{c}{b}y$。

(c) 椭圆抛物面（Elliptic Paraboloid）

1. **标准方程 (8)**

$$z=a\,x^2+b\,y^2,$$

   也可出现 (x,y,z) 排列交换，或在右侧换成 $-a\,x^2-b\,y^2$ 等，只要具备“**两个平方项**（都为正或都为负）**=** 一个线性变量”这种形式。

2. **几何形状：**
   - 形如碗状或锅状，若系数为正则开口向上（图 18.36 所示），若为负则开口向下。
   - 当 $a=b$ 时就是**正圆抛物面**，否则椭圆形；

3. **切面：**
   - **竖直切面**：若固定 $x=\text{const.}$，则方程中只剩 $z=b\,y^2+(\text{常数项})$，即抛物线；若固定 $y=\text{const.}$，同理也是抛物线。
   - **水平切面**：令 $z=k$，则

$$k=a\,x^2+b\,y^2 \quad\Longrightarrow\quad \frac{x^2}{\frac{k}{a}}+\frac{y^2}{\frac{k}{b}}=1 \quad (\text{若 } k>0).$$

     这是椭圆（或圆、若 $a=b$）。若 $k<0$，无实解，所以在 $z<0$ 区域无点（对“开口向上”的情形）。这说明曲面只占据半空间某一侧。

（d）双曲抛物面（Hyperbolic Paraboloid）

1. **标准方程 (9)**

$$z = b\,y^2 \ - \ a\,x^2,$$

   或一般形如

$$z = \pm\bigl(A\,x^2 - B\,y^2\bigr),$$

   都表示著名的“马鞍面” (图 18.37)。

2. **“马鞍面”特征**：
   - 在 xz-平面上（$y=0$）看到的是抛物线 $z=-a\,x^2$，开口向下；
   - 在 yz-平面上（$x=0$）看到的是抛物线 $z=b\,y^2$，开口向上；
   - 因此在某个方向上凸起，另一个方向上凹陷，形成一个鞍状或山口形。

3. **水平切面** $z=k$：

$$k = b\,y^2 - a\,x^2.$$

   - 若 $k>0$，方程可改写

$$\frac{y^2}{\frac{k}{b}} - \frac{x^2}{\frac{k}{a}} = 1,$$

     这是以 y-轴为主轴的双曲线；
   - 若 $k<0$，则双曲线主轴在 x-方向；
   - 若 $k=0$，则就是两条直线 $b\,y^2 = a\,x^2$ 或 $y^2 = \frac{a}{b}x^2$，所以截面退化为一对交于原点的直线。

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3. 本次内容总结

1. **双叶双曲面**： 分成两块互不相连，常见方程 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$；中间区域无点。
2. **椭圆锥面**： 二次项之和等于 0，形如 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$；顶点在原点，具有上下两部分尖锥。
3. **椭圆抛物面**： 一个变量 = 两个平方和(或差)，形如 $z=a\,x^2+b\,y^2$，呈碗状开口。
4. **双曲抛物面**： “马鞍面”，形如 $z=b\,y^2-a\,x^2$；在垂直切面中方向各异，一面向上一面向下。

至此，我们把常见的“六大非退化二次曲面”全部列举完毕：椭球面、单叶双曲面、双叶双曲面、椭圆锥面、椭圆抛物面、双曲抛物面。通过分析其**方程形式**与**切面**形状，就能快速识别并理解它们的空间几何特征。

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4. 报错或不清晰说明

- 书中给出的图例（图 18.34 至图 18.37）都较为清晰展示了各曲面的形状和典型截面；
- 若需要更具体的实例（如如何根据给定信息推导出某二次曲面的具体方程，或如何利用配方法简化原方程）可再行提问。

以上即为这四种二次曲面的基本笔记。结合前面两种（椭球面与单叶双曲面），便能掌握所有主要非退化二次曲面的标准形态与几何特征。

18.7

18.7

CYLINDRICAL AND SPHERICAL COORDINATES

In plane analytic geometry we used a rectangular coordinate system for some types of problems and a polar coordinate system for others. We saw that there are many situations in which one system is more convenient than the other. The same is true for the study of geometry and calculus in space. We now describe two other three-dimensional coordinate systems, in addition to the now-familiar rectangular coordinate system, that are often useful for dealing with special kinds of problems.

Consider a point P in space whose rectangular coordinates are (x, y, z). The *cylindrical coordinates* of this point are obtained by replacing x and y with the corresponding polar coordinates r and θ, and allowing z to remain unchanged. That is, we place a z-axis on top of a polar coordinate system and describe the location of a point in space by the three coordinates (r, θ, z). We will always assume that this cylindrical coordinate system is superimposed on a rectangular coordinate system in the manner shown in Fig. 18.39, so that the transformation equations connecting the two sets of coordinates of a given point are

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

and

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad z = z.$$

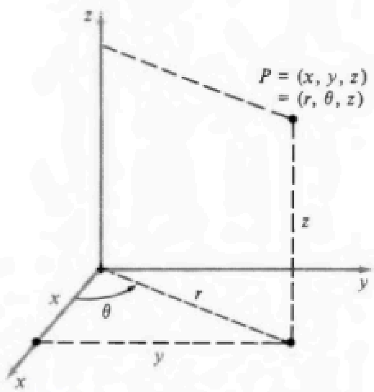

Figure 18.39 Cylindrical coordinates.

It is easy to see that the graph of the equation $r =$ a constant is a right circular cylinder whose axis is the z-axis; this is the reason for the term "cylindrical coordinates." Similarly, the graph of $\theta =$ a constant is a plane containing the z-axis, and the graph of $z =$ a constant is a horizontal plane.

Example 1 Find cylindrical coordinates for the points P_1 and P_2 whose rectangular coordinates are $(3, 3, 7)$ and $(2\sqrt{3}, 2, 5)$, respectively.

Solution For P_1 we have $r = \sqrt{9 + 9} = 3\sqrt{2}$, $\tan\theta = 1$, $z = 7$, so a set of cylindrical coordinates is $(3\sqrt{2}, \pi/4, 7)$. For P_2 we have $r = \sqrt{12 + 4} = 4$, $\tan\theta = 1/\sqrt{3} = \frac{1}{3}\sqrt{3}$, $z = 5$, so a set of cylindrical coordinates is $(4, \pi/6, 5)$.

Example 2 Describe the surfaces

(a) $r + z = 3$, and

(b) $r(2\cos\theta + 5\sin\theta) + 3z = 0$.

Solution (a) The intersection of the surface $r + z = 3$ with the yz-plane is the straight line $y + z = 3$, because $r = y$ in the yz-plane. But θ is missing from the given equation, so the desired surface is symmetric about the z-axis, and is therefore the cone generated by revolving the line $y + z = 3$ about the z-axis. More generally, it follows from our discussion of surfaces of revolution in Section 18.5 that if a curve $f(y, z) = 0$ is revolved about the z-axis, then the cylindrical equation of the resulting surface is $f(r, z) = 0$.

(b) Since $r\cos\theta = x$ and $r\sin\theta = y$, the given equation transforms into $2x + 5y + 3z = 0$, which is the plane through the origin with normal vector $\mathbf{N} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$.

Example 3 Find a cylindrical equation for (a) the spheroid $x^2 + y^2 + 2z^2 = 4$, and (b) the hyperbolic paraboloid $z = x^2 - y^2$.

Solution The equation in (a) transforms at once into $r^2 + 2z^2 = 4$. For (b), we have

$$\begin{aligned} z &= x^2 - y^2 \\ &= r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta) \\ &= r^2\cos 2\theta, \end{aligned}$$

so $z = r^2\cos 2\theta$ is the desired equation.

In physics, cylindrical coordinates are particularly convenient for studying situations in which there is axial symmetry, that is, symmetry about a line in space. As examples we mention two important classes of problems: those dealing with the flow of heat in solid cylindrical rods, and those concerned with the movements of a vibrating circular membrane—for instance, a drumhead.

Again consider a point P in space whose rectangular coordinates are (x, y, z). The *spherical coordinates* of P are the numbers (ρ, ϕ, θ) shown in Fig. 18.40. Here ρ (the Greek letter *rho*) is the distance from the origin O to P, so $\rho \geq 0$. The angle ϕ is the angle down from the positive z-axis to the radial line OP, and it is understood that ϕ is restricted to the interval $0 \leq \phi \leq \pi$. Finally, the angle θ has exactly the same meaning in spherical coordinates as it has in cylindrical coordinates; that is, θ is the angle from the positive x-axis to the line OP', where P' is the projection of P on the xy-plane. It is clear from the figure that $OP' = \rho\sin\phi$, and since $x = OP'\cos\theta$ and $y = OP'\sin\theta$, we have the transformation equations

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi,$$

and

$$\rho^2 = x^2 + y^2 + z^2, \qquad \tan\phi = \frac{\sqrt{x^2 + y^2}}{z}, \qquad \tan\theta = \frac{y}{x}.$$

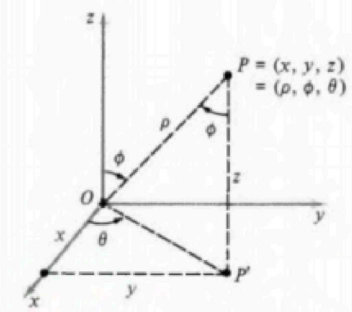

Figure 18.40 Spherical coordinates.

The term "spherical coordinates" is used because the graph of the equation $\rho = a$ (constant) is a sphere with center at the origin and radius a. The graph of $\phi = \alpha$ (constant) is the upper nappe of a cone with vertex at the origin and vertex angle α, if $0 < \alpha < \pi/2$. The graph of $\theta = \theta_0$ (constant) is a plane containing the z-axis, just as in cylindrical coordinates.

Example 4 Find an equation in spherical coordinates for the sphere $x^2 + y^2 + z^2 - 2az = 0$, where $a > 0$.

Solution Since $\rho^2 = x^2 + y^2 + z^2$ and $z = \rho \cos \phi$, the given equation can be written as

$$\rho^2 - 2a\rho \cos \phi = 0 \qquad \text{or} \qquad \rho(\rho - 2a \cos \phi) = 0.$$

The graph of this equation is the graph of $\rho = 0$ together with the graph of $\rho - 2a \cos \phi = 0$. But the graph of $\rho = 0$ (namely, the origin) is part of the graph of $\rho = 2a \cos \phi$, so the desired equation is

$$\rho = 2a \cos \phi.$$

This is the sphere of radius a that is tangent to the xy-plane at the origin, as shown in Fig. 18.41.

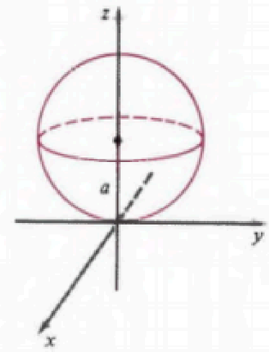

Figure 18.41

Example 5 What is the graph of the spherical equation $\rho = 2a \sin \phi$?

Solution The variable θ is missing from this equation, so we have a surface of revolution about the z-axis. In the yz-plane the equation $\rho = 2a \sin \phi$ represents a circle of radius a, as shown in Fig. 18.42. Since the graph we are seeking is obtained by revolving this circle about the z-axis, this graph is a torus (doughnut) in which the hole has radius zero.

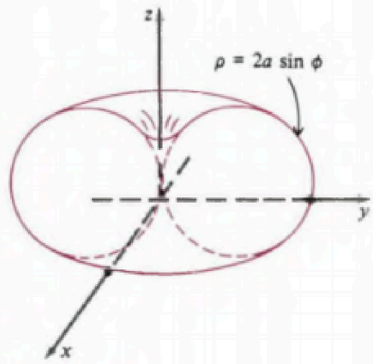

Figure 18.42

There are many physical uses of spherical coordinates, ranging from problems about heat conduction to problems in the theory of gravitation. We shall discuss some of these applications in Chapter 20.

1. 标题与概览
标题： 三维中的柱面坐标与球面坐标
概览：
在三维解析几何中，除了我们熟悉的直角坐标系 (x, y, z) 以外，为了更方便地处理具有对称性或旋转特征的空间问题，还常用到**柱面坐标（cylindrical coordinates）**和**球面坐标（spherical coordinates）**。本节首先介绍柱面坐标 (r, θ, z) 的定义与基本应用，然后讲解球面坐标 (ρ, ϕ, θ) 的变换关系和一些典型示例。

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2. 柱面坐标 (r, θ, z)

(a）定义与坐标变换

1. **定义：**
   在二维极坐标中，我们用 (r, θ) 表示平面上的任意点，其中 $r \geq 0$ 、θ 是与 x -轴正向的夹角。将这一想法推广到三维，就得到**柱面坐标**：

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

   同时还满足

$$r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}, \quad z = z.$$

- 其中 θ 的取值一般在 $[0, 2\pi)$ 或 $(-\pi, \pi]$ 等;
- r 表示点到 z-轴的水平距离，θ 是从 x-轴到点在 xy-平面的投影之角度；z 保持与直角坐标相同。

2. **几何意义：**
   - $r = \text{常数}$ 表示一个以 z-轴为轴心的**圆柱面**；
   - $\theta = \text{常数}$ 表示一个包含 z-轴的**垂直平面**；
   - $z = \text{常数}$ 表示一个**水平平面**。

(b) 示例与应用

- **示例 1（坐标转换）**
  给定直角坐标系下的点 $P_1 = (3, 3, 7)$ 与 $P_2 = (2\sqrt{3}, 2, 5)$，求柱面坐标。
  - 对 P_1：

$$r = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}, \quad \tan\theta = \frac{y}{x} = \frac{3}{3} = 1 \implies \theta = \frac{\pi}{4}, \quad z = 7.$$

    故 P_1 的柱面坐标可写作 $\left(3\sqrt{2}, \frac{\pi}{4}, 7\right)$。
  - 对 P_2：

$$r = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{12 + 4} = 4, \quad \tan\theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \implies \theta = \frac{\pi}{6}, \quad z = 5.$$

    故 P_2 的柱面坐标为 $(4, \frac{\pi}{6}, 5)$。
- **示例 2（描述曲面）**
  1. $r + z = 3$
     - 观察到它缺少 θ，即对 θ 不设限，则是绕 z-轴的对称曲面。若在 yz-平面（令 $x = 0 \implies r = y$）看，就得到直线 $y + z = 3$。可视为这条直线绕 z-轴旋转得到的锥面或相关曲面。
  2. $(r)(2\cos\theta + 5\sin\theta) + 3z = 0$
     - 以 $x = r\cos\theta, y = r\sin\theta$ 替换后，可得 $2x + 5y + 3z = 0$。这是一个经过原点的平面，法向量是 $\langle 2, 5, 3 \rangle$。
- **示例 3（将方程化为柱面坐标）**
  1. 球面 $x^2 + y^2 + 2z^2 = 4$：改成 $r^2 + 2z^2 = 4$。
  2. 双曲抛物面 $z = x^2 - y^2$：改写为

$$z = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos(2\theta).$$

由此可见，柱面坐标常在处理**绕 z-轴对称**的问题时非常方便，比如圆柱、某些旋转曲面等。

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3. 球面坐标 (ρ, ϕ, θ)

(a) 定义与变换关系

1. **定义：**
   在三维空间取原点 O、z-轴为垂直方向，则点 $P = (x, y, z)$ 的**球面坐标** (ρ, ϕ, θ) 定义为：

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad (\rho \geq 0), \quad \phi = \text{极角} = \text{从正 } z\text{-轴向下测得的角度}, \quad 0 \leq \phi \leq \pi, \quad \theta = \text{与 } (x\text{-轴}) \text{的水平夹角}, \quad [0, 2\pi).$$

   - 其中 ϕ 有时也叫“倾角”或“极角”，θ 与柱面坐标相同含义。
2. **坐标变换：**

$$
\begin{cases} x = \rho\,\sin\phi\cos\theta, \\ y = \rho\,\sin\phi\sin\theta, \\ z = \rho\,\cos\phi. \end{cases} \quad\Longleftrightarrow\quad \begin{cases} \rho^2 = x^2 + y^2 + z^2, \\ \tan\theta = \frac{y}{x}, \\ \phi = \arccos\Bigl(\frac{z}{\sqrt{x^2+y^2+z^2}}\Bigr). \end{cases}
$$

3. **几何意义：**
   - $\rho = \text{常数}$ 是以原点为球心的球面；
   - $\phi = \text{常数}$ 是以 z-轴为对称轴的圆锥面（或其一部分），如 $\phi = \frac{\pi}{2}$ 则是 xy-平面；
   - $\theta = \text{常数}$ 与柱面坐标类似，为包含 z-轴的垂直平面。

（b）示例：球面坐标的应用

- **示例 4：方程 $x^2 + y^2 + z^2 - 2az = 0$ 转化为球面坐标**
  - 已知 $x^2 + y^2 + z^2 = \rho^2$，且 $z = \rho\cos\phi$。
  - 将方程写成 $\rho^2 - 2a(\rho\cos\phi) = 0$，或

$$
\rho(\rho - 2a\cos\phi) = 0.
$$

  - $\rho = 0$ 仅表示原点；主要是 $\rho = 2a\cos\phi$。这表示一个半径为 a、且与 xy-平面相切于原点的球（图 18.41 所示），因为可以解出 $\rho = 2a\cos\phi \implies \rho = 2a\cos\phi \le 2a$.
- **示例 5：方程 $\rho = 2a\sin\phi$**
  - θ 未出现，说明它对 θ 具有旋转对称性；
  - 在 yz-平面（$\theta = \frac{\pi}{2}$ 或 $\frac{3\pi}{2}$ 等）时，$\rho = 2a\sin\phi$ 可以视作半圆投影。
  - 旋转得到的是一个**环面（torus）**，其实是“绕 z-轴旋转”的圆，见图 18.42。此时内、外半径的尺寸由参数 a 决定。

在物理中，球面坐标广泛用于研究球对称（如重力或电场）问题；柱面坐标则用于研究圆柱对称（如圆柱导热、振动膜片）的问题。这些坐标系在后续多变量微积分和偏微分方程应用中十分常见。

---

4. 总结

1. **柱面坐标（r, θ, z）**
   - 与极坐标类似，只是多了一个与直角坐标相同的 z；
   - 适合描述**绕 z-轴对称**的曲面、区域或物理问题；
   - 不包含某个变量（如 θ）往往表示绕 z-轴旋转的几何图形。
2. **球面坐标（ρ, ϕ, θ）**
   - ρ 为原点到点的距离，ϕ 为与正 z-轴的夹角，θ 为与 x-轴的水平夹角；
   - $\rho = \text{常数}$ 表示一个球面；$\phi = \text{常数}$ 表示一个圆锥面；$\theta = \text{常数}$ 表示一个垂直平面；
   - 适合处理**球对称**或一般旋转对称的三维问题。
3. **方程变换及图形理解**
   - 从 (x, y, z) 到 (r, θ, z) 或 (ρ, ϕ, θ) 的代换往往能简化方程，使对称性更明显；
   - 例如球的方程在球面坐标中极其简单：$\rho = \text{常数}$；
   - 在柱面坐标中，θ-独立的方程往往表示绕 z-轴对称的曲面。

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5. 报错或不清晰说明

- 本次内容和书中例题皆可清晰辨认，没有明显的缺失或无法识别之处；
- 若对某些几何曲面如何在柱面/球面坐标下简化方程有更多细节疑问，可再行提问。

以上即为三维柱面坐标与球面坐标的核心知识点总结。它们为后续学习多变量微积分、矢量场积分（特别是柱坐标系与球坐标系的体积分或曲面积分）奠定了必要的坐标工具基础。

CHAPTER 18 REVIEW: DEFINTIONS,EQUATIONS

CHAPTER 18 REVIEW: DEFINITIONS, EQUATIONS

Think through the following.

1 Rectangular (or Cartesian) coordinates.
2 Equation of a sphere.
3 Dot product: definition and formula.
4 Cross product: definition and formula.
5 Equations of a line: parametric and symmetric.
6 Equation of a plane.
7 Cylinder.
8 Surface of revolution.
9 The six quadric surfaces: graphs and equations.
10 Cylindrical coordinates.
11 Spherical coordinates.

1. 标题与内容概览
标题： 第18章核心概念回顾——坐标体系、向量运算与典型曲面方程
概览：
本章围绕三维解析几何的基础内容展开，从最基本的直角坐标到不同坐标系，再到向量点积、叉积以及各种典型曲面的方程，构筑了空间几何与微积分的重要基础。以下依次回顾章节所列的 11 个知识点。

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2. 详细回顾与关键方程

1. Rectangular (or Cartesian) coordinates

- **定义：** 在三维空间中，以互相垂直且交于原点 O 的 x-轴、y-轴、z-轴为基准，任一点 P 表示为 (x, y, z)。
- **坐标平面：** xy-平面（$z=0$）、yz-平面（$x=0$）、xz-平面（$y=0$）；
- **八分体：** 三维中由三个坐标平面将空间分成 8 个区域（octants）。

2. Equation of a sphere

- **标准方程：** 若球心为 (x_0, y_0, z_0)，半径 r，则

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2.$$

- **示例：** 当球心在原点 $(0,0,0)$、半径为 R 时，方程即 $x^2+y^2+z^2=R^2$。

3. Dot product: definition and formula

- **定义：** 对于向量 $\mathbf{A}, \mathbf{B}$，夹角为 θ，有

$$\mathbf{A}\cdot\mathbf{B}=|\mathbf{A}|\ |\mathbf{B}|\ \cos\theta.$$

- **分量表达：** 若 $\mathbf{A}=(a_1,a_2,a_3)$、$\mathbf{B}=(b_1,b_2,b_3)$，则

$$\mathbf{A}\cdot\mathbf{B}=a_1b_1+a_2b_2+a_3b_3.$$

- **判定正交**：若 $\mathbf{A}\cdot\mathbf{B}=0$，则 $\mathbf{A}\perp\mathbf{B}$。

4. Cross product: definition and formula

- **定义**：

$$\mathbf{A}\times\mathbf{B}=|\mathbf{A}|\,|\mathbf{B}|\,\sin\theta\,\mathbf{n},$$

 其中 $\mathbf{n}$ 为与平面 $\mathbf{A},\mathbf{B}$ 垂直且用右手定则确定方向的单位向量。

- **分量表达（行列式）**：

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ a_1&a_2&a_3\\ b_1&b_2&b_3\end{vmatrix}=\big(a_2b_3-a_3b_2,\,a_3b_1-a_1b_3,\,a_1b_2-a_2b_1\big).$$

- **反交换律**：$\mathbf{A}\times\mathbf{B}=-(\mathbf{B}\times\mathbf{A})$。

5. Equations of a line: parametric and symmetric

- **给定一点** (x_0,y_0,z_0) **与方向向量** $\mathbf{v}=(a,b,c)$：
 1. **向量方程**：$\mathbf{R}=\mathbf{R}_0+t\,\mathbf{v}$。
 2. **参数方程**：

$$x=x_0+a\,t,\quad y=y_0+b\,t,\quad z=z_0+c\,t.$$

 3. **对称式**：

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c},$$

 （若分量为 0，须相应特殊处理）。

6. Equation of a plane

- **平面方程**：若平面过点 (x_0,y_0,z_0)，法向量为 $\mathbf{N}=(A,B,C)$，则

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0,$$

 或写作 $Ax+By+Cz+D=0$，其中 $D=-(Ax_0+By_0+Cz_0)$。

7. Cylinder

- **定义**：将一条平面曲线 C 沿与平面不平行的直线方向平移或延展，即可形成圆柱面（广义的 cylinder）。
- **方程特征**：若母线与 z-轴平行，则方程中常**缺少** z；例如 $\frac{x^2}{9}+\frac{y^2}{4}=1$ 表示椭圆柱。

8. Surface of revolution

- **定义**：若平面曲线 C 围绕一条直线（旋转轴）转动 360°，在三维空间生成的曲面；
- **常见示例**：圆旋转生成球面，直线旋转生成圆锥，抛物线旋转生成抛物面等；方程中常出现 $\sqrt{x^2+y^2}$ (绕 z-轴) 或类似结构。

9. The six quadric surfaces: graphs and equations

非退化的二次曲面主要六种：

1. **椭球面** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$。
2. **单叶双曲面** $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$。
3. **双叶双曲面** $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$。
4. **椭圆锥面** $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$。
5. **椭圆抛物面** $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$。
6. **双曲抛物面** $z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$。

10. Cylindrical coordinates

- **定义与变换**：

$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z,$$

  其中 $r\geq 0$，$\theta\in[0,2\pi)$。
- **几何含义**：$r=\text{常数}$ 为绕 z-轴的圆柱，$\theta=\text{常数}$ 为包含 z-轴的平面，$z=\text{常数}$ 为水平面。

11. Spherical coordinates

- **定义与变换**：

$$\rho=\sqrt{x^2+y^2+z^2},\quad \phi=\arccos\left(\frac{z}{\rho}\right),\quad \theta=\arctan\left(\frac{y}{x}\right),$$

  或在正向：

$$x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi.$$

- **几何含义**：$\rho=\text{常数}$ 表示球面，$\phi=\text{常数}$ 表示圆锥面，$\theta=\text{常数}$ 表示垂直平面。

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3. 总结

- 本章全面介绍了三维解析几何的基础：从最基本的**坐标系**（直角、柱面、球面）到**向量运算**（点积、叉积），再到**空间几何图形**（线、面、柱面、旋转面和二次曲面）。
- 掌握这些概念和方程形式，对于理解三维空间里的曲线曲面、并在后续多变量微积分或空间向量分析中处理积分、投影、对称性问题极为关键。

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4. 报错或不清晰说明

- 本章各知识点的定义与方程在书中均有配图与示例，整体无明显缺失。
- 若对坐标变换过程、二次曲面分类或向量运算细节有更多疑惑，可作进一步提问或结合具体练习加深理解。

以上即为第18章关键内容的回顾与总结。希望能帮助你系统梳理三维解析几何与向量基础，为接下来更深入的学习夯实基础。