Zürcher Hochschule
für Angewandte Wissenschaften
School of Engineering

Chapter 6:

System Modelling in Frequency Domain

Contents

References

[1] L.F.Chaparro , „Signals and Systems using Matlab", Academic Press, 2015

[2] I.Rennert, Signale und Systeme, Fachbuchverlag Leipzig, 2013

1. Introduction

In this chapter we will learn how to describe systems in the frequency domain (frequency response), and relate to the description forms in the time domain, which we presented in chapter 5 (differential equation, block diagram and impulse response).

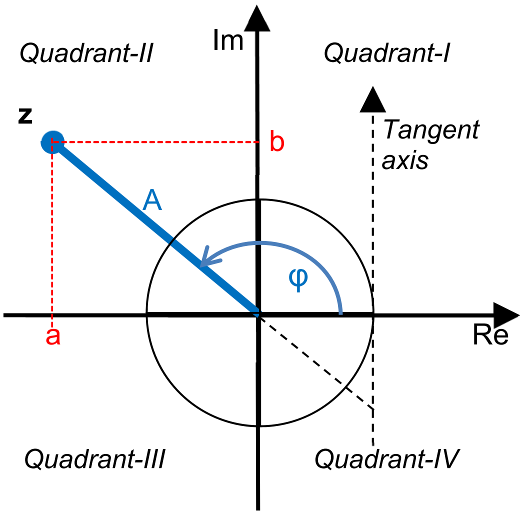
Furthermore we will analyse in detail a few reference systems: 1st and 2nd order filters.

2. Frequency Response

In chapter 5 we mentioned that the frequency response is the output when testing a linear system using a sine or cosine wave as test signal. In particular the frequency response gives the changes in amplitude and phase for cosine waves with all possible frequency values.

Before we discuss how to get this frequency response out of the differential equation, let us first consider some mathematics of complex numbers. In fact instead of taking a cosine wave as test signal it is easier to take a complex exponential, because integrating and differentiating an exponential function is much easier. Plus we know from the Euler formula, that once we know how the system affects exp(jωt) and exp(-jωt), we know as well what is the effect for the cosine wave.

A complex number z can be split into either its real and complex parts (cartesian coordinates), or in amplitude and phase (polar coordinates). The relationship between these representations is recalled below:



Complex number notations:

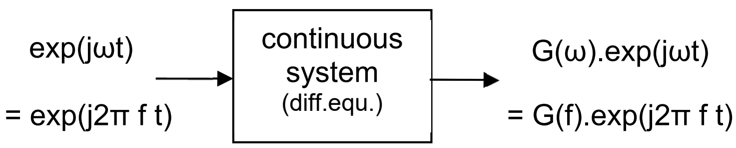
$$z = a + jb = A \cdot e^{j\varphi} \qquad (1a)$$

$$A = \sqrt{a^2 + b^2}$$

$$\varphi = \operatorname{arctan2}(b, a) \qquad (1b)$$

In order to calculate the phase out of the a and b parts, use the function arcus-tangent-2 which is able to differentiate among the 4 quadrants.

The idea is to express the change in amplitude and phase as a multiplication with a complex number. Since the value of the complex number changes for each frequency we have a complex function that we call G(f) or G(ω), depending on whether we are using frequency f in Hz or angular frequency ω in rad/s.



$$G(\omega) = G(f) = |G(f)| \cdot \exp[j\angle(G(f))] = A(f) \cdot \exp[j \cdot \varphi(f)]$$

amplitude and phase changes *(varying with freq)*

Figure 6-1 Calculation method of frequency response from system's differential equation

Example 6-1

Let us calculate the frequency response G(ω) for the passive RC low pass filter discussed in previous chapters. The departure point is the differential equation describing the system:

$$\tau \cdot \dot{y}(t) + y(t) = x(t) \qquad \text{with} \qquad \tau = R \cdot C$$

Using the solution approach shown in figure 6-1:

$$x(t) = e^{j\omega t} \qquad \leftrightarrow \qquad y(t) = G(\omega) \cdot e^{j\omega t}$$

so

$$\dot{y}(t) = (j\omega) \cdot G(\omega) \cdot e^{j\omega t}$$

Then, replacing these definitions in the differential equation and isolating for G(ω), gives us:

$$\tau \cdot (j\omega) \cdot G(\omega) \cdot e^{j\omega t} + G(\omega) \cdot e^{j\omega t} = e^{j\omega t} \qquad \Rightarrow \qquad G(\omega) \cdot \{j\omega\tau + 1\} = 1$$

Frequency response for passive RC low pass filter:

$$G(\omega) = \frac{1}{j\omega\tau + 1} \quad \text{or with f as frequency variable} \quad G(f) = \frac{1}{j2\pi f\tau + 1} \qquad (2)$$

Since we are interested in the amplitude and phase changes represented by the complex function G(ω), we need to split G(ω) into its amplitude (also called magnitude or absolute value) and phase (or angle) components:

$$G(\omega) = |G(\omega)| \cdot \exp\left[j\langle G(\omega)\right]$$

$$|G(\omega)| = \frac{|1|}{|j\omega\tau + 1|} = \frac{1}{|j\omega\tau + 1|}$$

$$\langle G(\omega) = \{\arctan(0) - \arctan(\omega\tau)\} = -\arctan(\omega\tau)$$

In order to get this result you need to remember that when you have a ratio (quotient) of two complex numbers, you can calculate the polar representation by:

$$G(\omega) = \frac{N(\omega)}{D(\omega)} \qquad \text{then}$$

$$G(\omega) = |G(\omega)| \cdot \exp\left[j\langle G(\omega)\right] = \frac{|N(\omega)|}{|D(\omega)|} \cdot \frac{\exp\left[j\langle N(\omega)\right]}{\exp\left[j\langle D(\omega)\right]} = \frac{|N(\omega)|}{|D(\omega)|} \cdot \exp\left[j\langle N(\omega) - j\langle D(\omega)\right]$$

Often it is easier to interpret a function if we make a graphical representation of it. The frequency response is often represented in a graphic with logarithmic scales called Bode diagram. The frequency axis is a logarithmic scale of either ω in (rad/s) or f in (Hz). The upper diagram is the amplitude in (dB) and the lower the phase in degrees (°).

The unit [dB] is a relative unit expressing the gain with respect to a reference. In this case the magnitude IG(ω)I represents the gain of the output signal with respect to the input signal.

The logarithmic function in equation 3 below is a tenth-logarithm (basis equals 2).

Magnitude of frequency response in dB:

$$|G(\omega)|_{dB} = 20 \cdot \log(|G(\omega)|) \tag{3}$$

The figure 6-2 shows a bode diagram for equation 2. Fill out the table below to understand the meaning of the asymptotes and to find out the value of the tau (τ) constant used in this plot.

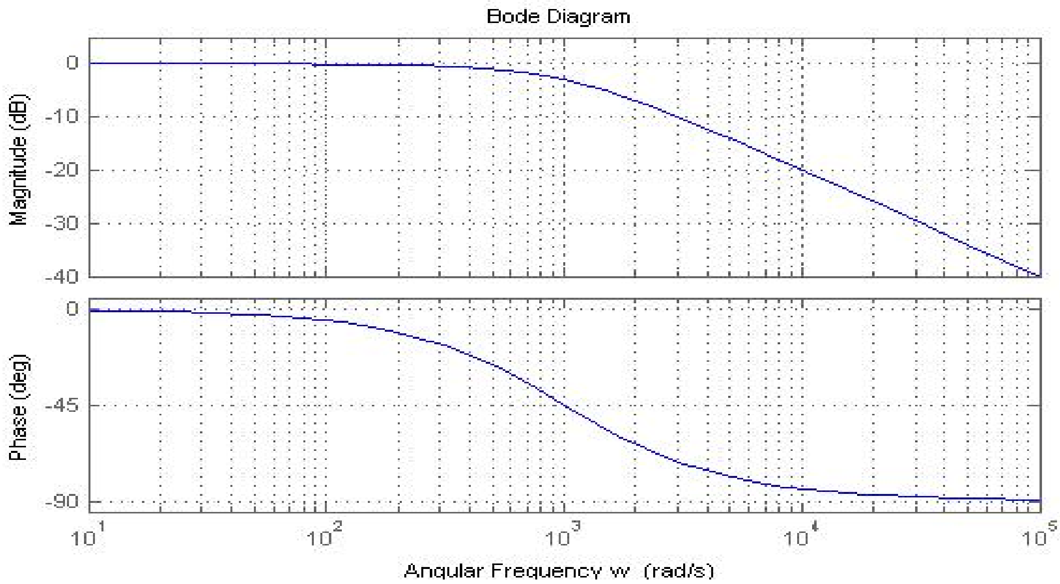


Figure 6-2 Bode Diagram for RC low pass filter: G(jω) = 1/(jωτ+1)

Term	Regions	Magnitude abs{G(jω)}	Phase Phase{G(jω)}
1/(jωτ+1)	ω << 1/τ	Asymptote:	
	ω = 1/τ	Point:	
	ω >> 1/τ	ω=10/ τ : ω=100/ τ Asymptote:	

Table 6-1 Calculation of Bode Asymptotes and Reference points

The asymptotes are the slopes which approximate the linear behaviour of the plot in a certain region. For instance our plot above has an asymptote of 0dB/decade for low frequencies and another of -20dB/decade for high frequencies, and a kink or break-point of -3dB at $\omega = 1/\tau$. The slopes are expressed in terms of decade (frequency increase by factor 10) or octave (frequency increase by factor 2).

Question 6-1

Fill out the table with the logarithmic expressions, and then the table calculating the slopes in the corresponding units. *Hint:* you do not need a calculator!

Powers of 2	Powers of 10
$20\log_{10}(2) = 6\ dB$	$20\log_{10}(10) = 20\ dB$
$20\log_{10}(4) =$	$20\log_{10}(100) =$
$20\log_{10}(8) =$	$20\log_{10}(1000) =$
$20\log_{10}\left(\frac{1}{2}\right) =$	$20\log_{10}\left(\frac{1}{10}\right) =$
$20\log_{10}\left(\frac{1}{4}\right) =$	$20\log_{10}\left(\frac{1}{100}\right) =$
$20\log_{10}\left(\sqrt{2}\right) =$	**Mixed**
$20\log_{10}\left(\frac{1}{\sqrt{2}}\right) =$	$20\log_{10}(5) =$

Slopes
$+20\ dB/decade = \ldots\ldots\ldots\ldots\ dB/octave$
$+40\ dB/decade = \ldots\ldots\ldots\ldots\ dB/octave$
$-20\ dB/decade = \ldots\ldots\ldots\ldots\ dB/octave$

Question 6-2

Calculate and draw the Bode diagram for the following basic terms of 1st order.
Hint: for each term, make a table and corresponding plot in logarithmic scale.

$G_A(\omega) = j\omega\tau_A \quad ; \quad \tau_A = 10ms$	$G_B(\omega) = \dfrac{1}{j\omega\tau_B} \quad ; \quad \tau_B = 1ms$
$G_C(\omega) = j\omega\tau_C + 1 \quad ; \quad \tau_C = 10\mu s$	$G_D(\omega) = \dfrac{1}{j\omega\tau_D + 1} \quad ; \quad \tau_B = 1\mu s$

3. Reference Systems

In order to get an overview of the several system representations or views we discussed in this chapter, let us check these views for three types of continuous systems, which are often used as reference systems. The tables 6-2 and 6-3 summarize these views and in the following questions you will see how to calculate them.

In these tables we also introduce a notation for the differential equations of 1st and 2nd order, using new coefficients. The coefficients are chosen to reflect physical characteristics of the system responses.

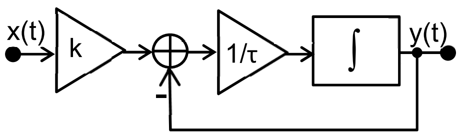
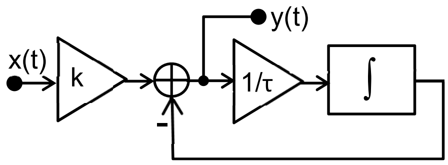
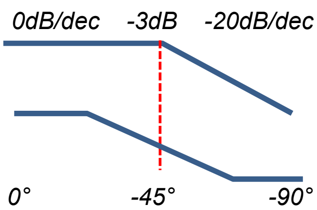
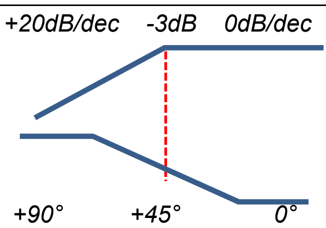
View of LTI System	1st Order Low Pass Filter	1st Order High Pass Filter
Differential Equation	$\tau \cdot \dot{y}(t) + y(t) = k \cdot x(t)$	$\tau \cdot \dot{y}(t) + y(t) = k \cdot \tau \cdot \dot{x}(t)$
Block diagram		
Step response *with zero initial conditions and for t ≥ 0*	$h(t) = k \cdot \left[1 - e^{-t/\tau}\right]$	$h(t) = k \cdot e^{-t/\tau}$
Impulse response *with zero initial conditions and for t ≥ 0*	$g(t) = +\frac{k}{\tau} \cdot e^{-t/\tau}$ **Parameters:** **k** : gain ; **τ (tau)**: time constant	$g(t) = \delta(t) - \frac{k}{\tau} \cdot e^{-t/\tau}$
Frequency response	$G(\omega) = \frac{k}{j\omega\tau + 1} = \frac{k}{j2\pi f\tau + 1}$	$G(\omega) = k \cdot \frac{j\omega\tau}{j\omega\tau + 1} = k \cdot \frac{j2\pi f\tau}{j2\pi f\tau + 1}$
Bode Diagram	Amp: Phase: 	Amp: Phase: 

Table 6-2 Summary of views for 1st order LTI Systems (LPF and HPF)

Question 6-3

Re-visit the passive RC example and verify the correspondence to the views of the LPF 1st order. Check why τ (tau) is called time constant, and k static gain.

Hint: for the step response, you could use the mathematical method to solve the differential equation as a sum of the homogene and particular solutions. Here in SiSy, we learned two other approximation methods: calculate the homogene response (using hypothesis of exponential output) and use it as approximation of the impulse response, or calculate it numerically describing the system in Matlab.

Question 6-4

In the passive RC, swap the R and C, recalculate the views and check the correspondence to the views of the HPF 1st order.

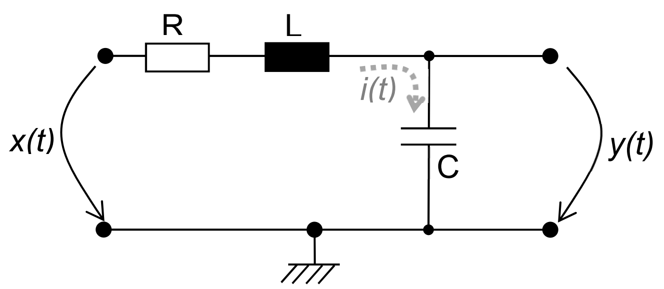
In chapter 5 section 3 (*System Modelling with Test Signals*) we saw three examples of 2nd order systems, which can all be represented with the notation introduced in table 6-3, using the constants: k, ω_0, d .

The electrical analogy to these systems is a passive RLC circuit discussed in question 6-5 below. The advantage of this notation is that the parameters k, ω_0 and d, directly reflect physical characteristics of the system response, namely: gain, natural frequency and damping. In the step response, two other parameters appear: ω_e (eigen-angular-frequency) and σ_e (decay constant), also present in the free or homogene response of the system.

Question 6-5

Calculate the differential equation for the passive RLC circuit below and verify the correspondence to the views of the LPF 2nd order.

Hint: you do not need to calculate the step response here manually, we will simulate it with Matlab for the moment, and learn later (GRT), methods that make easier the "hand" calculation (with Laplace Transformation).



x(t) : input voltage [V]

y(t) : output voltage [V]

i(t) : internal current [A]

R [Ohm] = [Ω]
C [Faraday] = [F]

In this chapter we learned different representations of continuous and discrete linear systems both in the time and in the frequency domain. These representations and their relationship are shown in the figures of annex 6-A. Some of the views in these figures will be the subject of later courses, like the frequency response of discrete systems, or the state space representations.

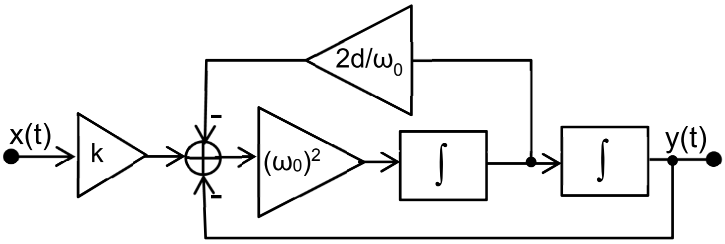
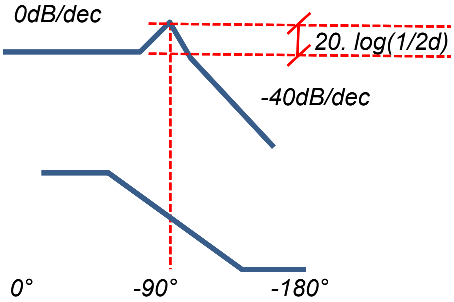
View of LTI System	2nd Order Low Pass Filter
Differential Equation	$\ddot{y}(t)+(2d\omega_0)\cdot\dot{y}(t)+(\omega_0^2)\cdot y(t)=(k\omega_0^2)\cdot x(t)$
Block diagram	
Frequency response	$G(\omega) = \dfrac{k\omega_0^2}{(j\omega)^2 + (2d\omega_0)(j\omega) + \omega_0^2} = \dfrac{k}{\left(j\frac{\omega}{\omega_0}\right)^2 + (2d)\left(j\frac{\omega}{\omega_0}\right) + 1}$ **Parameters:** **k** : gain ; **d** : damping constant; **ω_0** : natural angular frequency [rad/s] Alternative with **Q** : quality factor ; and **Ω**: normalised frequency: $G(\omega) = \dfrac{k}{(j\Omega)^2 + \left(\frac{1}{Q}\right)(j\Omega) + 1}$ where $Q = \frac{1}{2d}$ and $\Omega = \frac{\omega}{\omega_0}$
Bode Diagram	Amp: Phase: 
Characteristic Equation	$s^2+(2d\omega_0)\cdot s+(\omega_0^2)=0$
Roots of the characteristic equation	$s_{1,2}=-d\omega_0 \pm j\omega_0\sqrt{1-d^2}=-\sigma_e \pm j\omega_e$ d > 1 over-damped d = 1 critically damped 0 < d < 1 under-damped
Step response *with zero initial conditions and for t ≥ 0*	$h(t)=k\cdot\left[1-Ae^{-\sigma_e t}\cdot\sin(\omega_e t+B)\right]$; $A=\dfrac{1}{\sqrt{1-d^2}}$; $B=\arccos(d)$

Table 6-3 **Summary of views for 2nd order LTI Systems (LPF)**

4. Vocabulary

transfer function: Übertragungsfunktion

…to be continued…

ABBREVIATIONS:

LTD = LT(I)D : linear time-invariant and discrete system
LTI : linear time-invariant continuous system

5. Matlab Commands for LTI Systems

For example, if you want to define a LTI sys of 2nd order, more specifically a LPF with the reference parameters k, ω0 and d, then you can do:

```
% Example: LTI System Definition in Matlab
% =====================================
clear all, close all, clc;


% Referenz System 2te Ordnung

% Parameters
w0 = 2*pi*1e3;
d = 0.05;
k = 1;

% System Definition
num = [k*w0^2];
den = [1 2*d*w0 w0^2];

sys = tf(num,den)

figure(1)
bode(sys),grid on


figure(2)
subplot(211),step(sys),grid on
subplot(212),impulse(sys),grid on
```

In order to visualise the frequency response and the answers to test signals step & impulse (in the time domain).

6. Annexes

(6-A) Overview of LTI system representations

(6-A) Overview of LTI system representations (*Überblick LTI Beschreibungen)*

Die Abbildung unten gibt einen Überblick von Darstellungsarten für lineare, zeitinvariante, und **kontinuierliche Systeme** (auf Englisch *linear time invariant LTI*).