Boundary and multiple dimensions

0.1 How the boundary is like

As we have the update equation as $X(k+1) = A(k)X(k) + B(k)U(k)$, we can derive

$$X(k+1) = (\prod_{k}^{i=0} A(i))X(0) + \sum_{i=0}^{k}[(\prod_{j=0}^{i-1} A(j))B(i)U(i)] \tag{1}$$

by recursively applying it.

By the property of matrix norm, we have

$$|X(k+1)| \leq (\prod_{i=0}^{k} \|A(i)\|)|X(0)| + \sum_{i=0}^{k}[(\prod_{j=0}^{i-1} \|A(j)\|)\|B(i)\||U(i)|]. \tag{2}$$

$\forall i \in [0,k]$, let $\|A(i)\| \leq \|A\|$, $\|B(i)\| \leq \|B\|$ and $|U(i)| \leq |U|$, we have

$$\begin{aligned} |X(k+1)| &\leq (\|A\|)^{k+1}|X(0)| + \sum_{i=0}^{k}[(\|A\|)^{i}\|B\||U|] \\ &= (\|A\|)^{k+1}|X(0)| + \frac{1-(\|A\|)^{k+1}}{1-\|A\|}\|B\||U| \end{aligned} \tag{3}$$

The boundary will be a function of $bound(\|A\|, \|B\|, |X(0)|, |U|, k)$. Thus the minimum boundary is $\min_k bound(\|A\|, \|B\|, |X(0)|, |U|, k)$.

When we have $\|A\| < 1$, $(\|A\|)^{k+1} \to 0$ and $\frac{1-(\|A\|)^{k+1}}{1-\|A\|} \to \frac{1}{1-\|A\|}$ as $k \to \infty$.

0.2 Multiple dimensions

Let the update rule of i-th dimension case be $[v_i(k+1), x_i(k+1)]^T = A_i(k)[v_i(k), x_i(k)]^T + B_i(k)[x_i^G(k), x_i^P(k)]^T$, we can have

$$\begin{aligned} \begin{bmatrix} [v_1(k+1), x_1(k+1)]^T \\ [v_2(k+1), x_2(k+1)]^T \\ \vdots \\ [v_N(k+1), x_N(k+1)]^T \end{bmatrix} &= \begin{bmatrix} A_1(k) & \mathbf{0}^2 & \cdots & \mathbf{0}^2 \\ \mathbf{0}^2 & A_2(k) & \cdots & \mathbf{0}^2 \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0}^2 & \mathbf{0}^2 & \cdots & A_N(k) \end{bmatrix} \begin{bmatrix} [v_1(k), x_1(k)]^T \\ [v_2(k), x_2(k)]^T \\ \vdots \\ [v_N(k), x_N(k)]^T \end{bmatrix} \\ &+ \begin{bmatrix} B_1(k) & \mathbf{0}^2 & \cdots & \mathbf{0}^2 \\ \mathbf{0}^2 & B_2(k) & \cdots & \mathbf{0}^2 \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0}^2 & \mathbf{0}^2 & \cdots & B_N(k) \end{bmatrix} \begin{bmatrix} [x_1^G(k), x_1^P(k)]^T \\ [x_2^G(k), x_2^P(k)]^T \\ \vdots \\ [x_N^G(k), x_N^P(k)]^T \end{bmatrix} \\ &= A(k)X(k) + B(k)U(k) \end{aligned} \tag{4}$$

for N-dimension particle.

This is a parallel interconnection of the update rule of each single dimension. Any states of two dimensions are not coupled. If the update rule of each single dimension is input-to-state stable, the entire state of the system is input-to-state stable.

By $\lambda^{\max}(A) \leq \max\{\lambda^{\max}(A_1), \cdots, \lambda^{\max}(A_N)\}$ and $\lambda^{\max}(B) \leq \max\{\lambda^{\max}(B_1), \cdots, \lambda^{\max}(B_N)\}$, we can have $\|A\| \leq \max\{\|A_1\|, \cdots, \|A_N\|\}$ and $\|B\| \leq \max\{\|B_1\|, \cdots, \|B_N\|\}$. The properties can be reserved.