Differential Flatness

The system

$$\dot{x} = f(x,u)$$

$$y = h(x)$$

is said to be differentially flat, with flat output y, if there exists a $\rho_x < \infty$, $\rho_u < \infty$ s.t.

$$x(t) = g_x(y, \dot{y}, \ddot{y}, \dddot{y}, y^{(4)}, \cdot\, y^{(\rho_x)})$$

$$u(t) = g_u(y, \dot{y}, \cdots, y^{(\rho_u)})$$

In other words the state and the input can be expressed in terms of the flat output and a finite number of its derivatives.

Example

Mobile Robot

$$\dot{r}_x = V\cos\psi$$
$$\dot{r}_y = V\sin\psi$$
$$\dot{\psi} = u_1$$
$$\dot{V} = u_2$$

claim: the system is differentially flat with flat outputs $\begin{pmatrix} r_x \\ r_y \end{pmatrix}$

To prove the claim, we must express the states $r_x(t), r_y(t), \psi(t), V(t)$ and the inputs $u_1(t)\ u_2(t)$ in terms of $r_x(t), r_y(t), \dot{r}_x(t), \dot{r}_y(t), \ddot{r}_x(t), \ddot{r}_y(t), \ldots$

Clearly $r_x(t)$ and $r_y(t)$ are expressed in terms of the flat output.

Also, since

$$\dot{r}_x^2 + \dot{r}_y^2 = V^2\cos^2\psi + V^2\sin^2\psi = V^2$$

$$\Rightarrow \boxed{V(t) = \sqrt{\dot{r}_x^2(t) + \dot{r}_y^2(t)}}$$

Also

$$\frac{\dot{r}_y(t)}{\dot{r}_x(t)} = \frac{V(t)\sin\psi(t)}{V(t)\cos\psi(t)} = \tan\psi(t)$$

$$\Rightarrow \boxed{\psi(t) = \tan^{-1}\left(\frac{\dot{r}_y(t)}{\dot{r}_x(t)}\right)}$$

The inputs are given by

$$u_1 = \dot{\psi} = \frac{d}{dt}\left(\tan^{-1}\left(\frac{\dot{r}_y}{\dot{r}_x}\right)\right)$$

$$= \frac{1}{1+\left(\frac{\dot{r}_y}{\dot{r}_x}\right)^2}\left(\frac{\dot{r}_x\ddot{r}_y - \dot{r}_y\ddot{r}_x}{\dot{r}_x^{\,2}}\right)$$

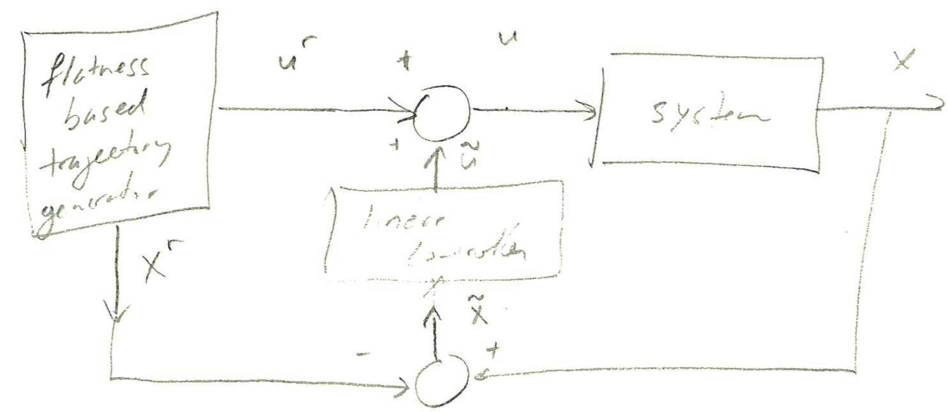
$$\Rightarrow \boxed{u_1 = \frac{\dot{r}_x\ddot{r}_y - \dot{r}_y\ddot{r}_x}{\dot{r}_x^2 + \dot{r}_y^2}}$$

$$u_2 = \dot{v} = \frac{d}{dt}\sqrt{\dot{r}_x^2 + \dot{r}_y^2}$$

$$\Rightarrow \boxed{u_2 = \frac{\dot{r}_x\ddot{r}_x + \dot{r}_y\ddot{r}_y}{\sqrt{\dot{r}_x^2 + \dot{r}_y^2}}}$$

$\therefore$ All of the states and inputs can be expressed in terms of the flat output and its derivatives.

A common feedback scheme for a system based on differential flatness is shown below



The differential flatness property is used to construct a reference trajectory $(x^r(t), u^r(t))$

Setting
$$x = x^r + \tilde{x}$$
$$u = u^r + \tilde{u}$$

the system $\dot{x} = f(x, u)$ can be linearized around the reference trajectory as

$$\dot{x} = \dot{x}^r + \dot{\tilde{x}} = f(x^r + \tilde{x}, u^r + \tilde{u})$$
$$= f(x^r, u^r) + \left.\frac{\partial f}{\partial x}\right|_r \tilde{x} + \left.\frac{\partial f}{\partial u}\right|_r \tilde{u} + H.O.T$$

Since the reference trajectory has been constructed so that

$\dot{x}_r = f(x_r, u_r)$ we have that

$$\dot{\tilde{x}} = A(t)\,\tilde{x} + B(t)\,\tilde{u}$$

where $A(t) = \left.\frac{\partial f}{\partial x}\right|_{x^r, u^r}$, $B(t) = \left.\frac{\partial f}{\partial u}\right|_{x^r, u^r}$

The linear controller is designed as

$$u = -K(t)\,\tilde{x} \quad \text{so that}$$

$$\dot{\tilde{x}} = \big(A(t) - B(t)K(t)\big)\,\tilde{x} \quad \text{is asymptotically stable}$$

Example

Suppose that we would like the mobile robot to follow a figure-8 reference trajectory given by

$$r_x^r(t) = \alpha \cos\left(\frac{\omega}{2} t\right)$$

$$r_y^r(t) = \beta \sin(\omega t)$$

Differentiating we get

$$\dot{r}_x^r = -\frac{\alpha\omega}{2} \sin\left(\frac{\omega}{2} t\right)$$

$$\dot{r}_y^r = \beta\omega \cos(\omega t)$$

$$\ddot{r}_x^r = -\frac{\alpha\omega^2}{4} \cos\left(\frac{\omega t}{2}\right)$$

$$\ddot{r}_y^r = -\beta\omega^2 \sin(\omega t)$$

The Jacobians of the dynamics are given by

$$\left.\frac{\partial f}{\partial x}\right|_r = \begin{pmatrix} 0 & 0 & -V^r \sin\psi^r & \cos\psi^r \\ 0 & 0 & V^r \cos\psi^r & \sin\psi^r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left.\frac{\partial f}{\partial u}\right|_r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We can select $K(t)$ s.t.

$$K(t) = \text{place}\left(A(t),\ B,\ [-a \pm ja,\ -b \pm jb]\right)$$

< See Simulink simulation where

$$\alpha = 2$$

$$\beta = 1$$

$$\omega = 0.5$$

$$a = 10$$

$$b = 20$$

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