Input-to-state stable analysis on Particle Swarm Optimization

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Outline

Structure

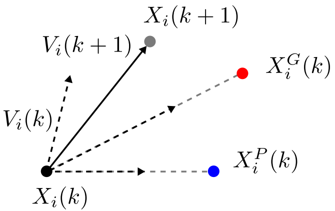
Outline

Particle Swarm Optimization

Introduction

The dynamics of a particle

- social influence $X_i^G(k)$
- cognitive influence $X_i^P(k)$
- memory $V_i(k)$



Understand the dynamics

Introduction

Reasons for analyzing particle dynamics

- inform the setting of parameters
- lead to the proposal of new variants
- allow for the analysis of the behavior

Related work

Introduction

A swarm is in stagnation when the cognitive and social affects are constant, $X_i^P(k) = X_i^P(k+1)$ and $X_i^G(k) = X_i^G(k+1)$

In **stagnation**, dynamics can be analyzed by

- treating stochastic factors as constants [1] [2]
- stochastic analysis [3] [4]
- other systematic analysis [5] [6]

[1] *Clerc et al.*, "The particle swarm - explosion, stability and convergence in a multidimensional complex space"

[2] *Cleghorn et al.*, "A generalized theoretical deterministic particle swarm model"

[3] *Poli et al.*, "Exact analysis of the sampling distribution of particle swarm optimizers during stagnation"

[4] *Jiang et al.*, "Stochastic convergence analysis and parameter selection of the standard particle swarm optimization algorithm"

[5] *Trelea et al.*, "The particle swarm optimization algorithm: convergence analysis and parameter selection"

[6] *Engelbrecht et al.*, "A study of particle swarm optimization particle trajectories"

Related work

Introduction

In **non-stagnation**, dynamics can be analyzed by

- approximate using a continuous-time model [7]
- model the problem into a random search [8]
- running toward a local optima [9]

[7] *Fernandez-Martinez et al.*, "Stochastic stability analysis of the linear continuous and discrete pso models"

[8] *van den Bergh et al.*, "A convergence proof for the particle swarm optimiser"

[9] *Schmitt et al.*, "Particle swarm optimization almost surely finds local optima"

Outline

Update rule

PSO model

Position update of PSO

$$v_{ij}(k+1) = \chi[v_{ij}(k) + \phi^P u_{ij}^P(k)(x_{ij}^P(k) - x_{ij}(k)) \\ + \phi^G u_{ij}^G(k)(x_{ij}^G(k) - x_{ij}(k))],$$

$$x_{ij}(k+1) = x_{ij}(k) + v_{ij}(k+1).$$

Linear form
PSO model

Linear model form of the position update of PSO

$$\begin{bmatrix} v(k+1) \\ x(k+1) - x^* \end{bmatrix} = A(k) \begin{bmatrix} v(k) \\ x(k) - x^* \end{bmatrix} + B(k) \begin{bmatrix} x^G(k) - x^* \\ x^P(k) - x^* \end{bmatrix}$$

with

$$A(k) = \begin{bmatrix} \chi & -\chi\phi^G u^G(k) - \chi\phi^P u^P(k) \\ \chi & 1 - \chi\phi^G u^G(k) - \chi\phi^P u^P(k) \end{bmatrix}$$

and

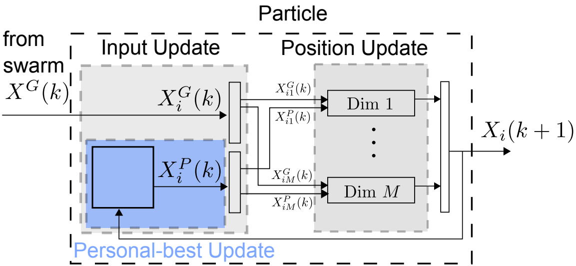
$$B(k) = \begin{bmatrix} \chi\phi^G u^G(k) & \chi\phi^P u^P(k) \\ \chi\phi^G u^G(k) & \chi\phi^P u^P(k) \end{bmatrix}.$$

x^* is a reference point, which can be

- the global optimal position,
- a local optimal position or
- any position that we are interested in using as a reference.

Particle

PSO model



- parallel structure of each dimension
- feedback cascade by the personal-best-update component
- global best as the input

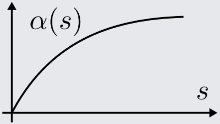
Outline

Define input-to-state stability

Input-to-state stability

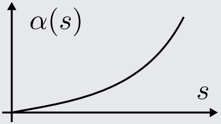
K-function $\alpha(\cdot)$

- continuous
- strictly increasing
- $\alpha(0) = 0$



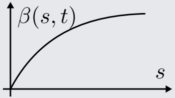
K_∞-function $\alpha(\cdot)$

- K-function
- $\alpha(s) \to \infty$ as $s \to \infty$
- used in proof

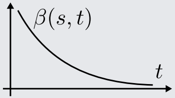


KL-function $\beta(\cdot, \cdot)$

- $\forall t \geq 0$, $\beta(\cdot, t)$ is a K-function;



- $\forall s \geq 0$, $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.



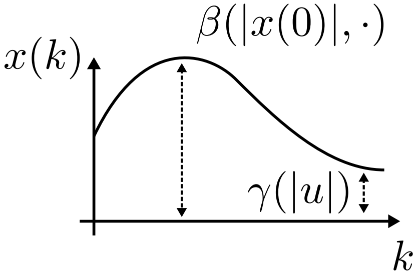
Define input-to-state stability

Input-to-state stability

Input-to-state stability

$|x(k)| \leq \beta(|X(0)|, k) + \gamma(\|u\|)$

- $\beta(\cdot, \cdot)$ is KL-function
- $\gamma(\cdot)$ is K-function



- Since $\beta(|X(0)|, k)$ is a decreasing function, the influence of the initial state $X(0)$ will eventually go to zero.
- If the input u is bounded, $\gamma(\|u\|)$ will also be bounded.
- Thus $|x(k)|$ is bounded.

Condition of Input-to-state stability

Input-to-state stability

Theorem

When $\forall k, |\lambda_{\max}(A(k))| < 1$, the position-update component of PSO is input-to-state stable.

- $A(k)$ is random matrix

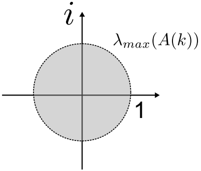
$$\begin{bmatrix} v(k+1) \\ x(k+1) - x^* \end{bmatrix} = \boxed{A(k)} \begin{bmatrix} v(k) \\ x(k) - x^* \end{bmatrix} + B(k) \begin{bmatrix} x^G(k) - x^* \\ x^P(k) - x^* \end{bmatrix}$$

with

$$\boxed{A(k) = \begin{bmatrix} \chi & -\chi\phi^G u^G(k) - \chi\phi^P u^P(k) \\ \chi & 1 - \chi\phi^G u^G(k) - \chi\phi^P u^P(k) \end{bmatrix}}$$

and

$$B(k) = \begin{bmatrix} \chi\phi^G u^G(k) & \chi\phi^P u^P(k) \\ \chi\phi^G u^G(k) & \chi\phi^P u^P(k) \end{bmatrix}.$$



- where $|\lambda_{\max}(A(k))|$ is absolute value of maximum eigenvalue

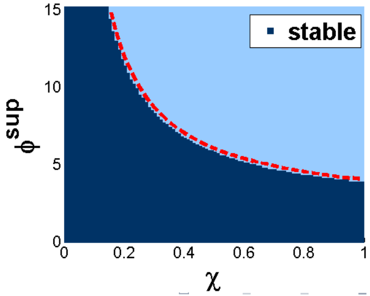
Parameter space of input-to-state stability

Input-to-state stability

Corollary

Let $A(k) = \begin{bmatrix} \chi & -\chi\phi \\ \chi & 1-\chi\phi \end{bmatrix}$, *in which* $\phi \in [0, \phi^{sup}]$ *and* $\phi^{sup} = \phi^P + \phi^G$ *and* $\chi \in (0,1)$. *When* $\phi^{sup} \in \left(0, \frac{2(1+\chi)}{\chi}\right)$, *the position-update component of PSO is input-to-state stable.*

- represent $A(k)$ using ϕ
- parameter space χ and ϕ^{sup}



Boundary of the movement

Input-to-state stability

Corollary

Given a bound on the input $\left| \left[x^G(k) - x^*,\ x^P(k) - x^* \right]^T \right|$ *in the position-update component, we have the bound on the particle position from Equation* (1)*.*

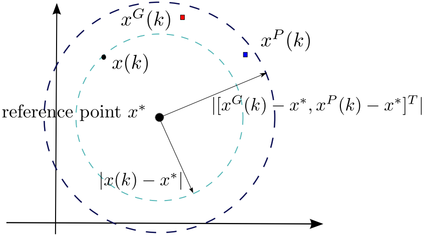
$$\forall k, |x(k) - x^*| \leq \max \left(|x(0) - x^*|, \gamma \left(\left| \left[x^G(k) - x^*,\ x^P(k) - x^* \right]^T \right| \right) \right).$$

The boundary is determined by

- $|x(0) - x^*|$ - the distance from the initial position $x(0)$ to the reference point x^*
- $\gamma \left(\left| \left[x^G(k) - x^*,\ x^P(k) - x^* \right]^T \right| \right)$ - scaled norm of a vector that consists of the distance from the global best position $x^G(k)$ to the reference point x^* and the distance from the personal best position $x^P(k)$ to the reference point x^*

Boundary of the movement

Input-to-state stability



A bound on a particle's position by a reference point x^*.
The ratio of two radii indicates γ.

Moment analysis

Input-to-state stability

- **Mean - first order moment**
  Linear model for the mean of the position update component

$$\begin{bmatrix} E(x(k+1)) - \hat{x} \\ E(x(k)) - \hat{x} \end{bmatrix} = A_m \begin{bmatrix} E(x(k)) - \hat{x} \\ E(x(k-1)) - \hat{x} \end{bmatrix} + B_m \begin{bmatrix} E(x^P(k)) - \hat{x} \\ E(x^G(k)) - \hat{x} \end{bmatrix}$$

Theorem

The mean of the position update component is input-to-state stable, if $|\lambda_{\max}(A_m)| < 1$.

- **Higher order moments**

Outline

When the position update is input-to-state stable

ISS analysis

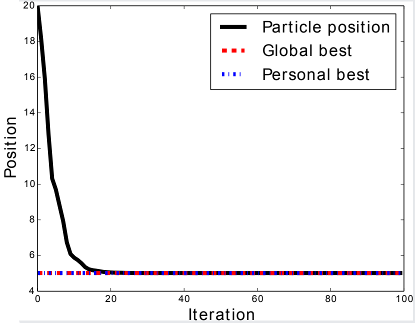
Input-to-State Stable Position-Update Component

- Constant $X^P = X^G$
- Constant $X^P \neq$ constant X^G
- $X^P(k)$ and $X^G(k)$ are bounded
- Converging $X^P(k)$ and $X^G(k)$

When the position update is input-to-state stable

ISS analysis

constant $X^P = X^G$ (Stagnation)



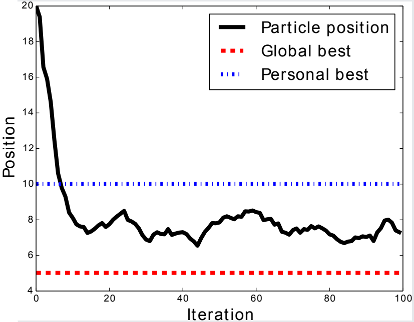
- Converge to X^G
- Simulated using PSO position-update

When the position update is input-to-state stable

ISS analysis

With ISS of position-update component

constant $X^P \neq$ constant X^G (Stagnation)



- Convergence of the mean[a]
- Never stop at one position
- Move in a bound by ISS
- Simulated using PSO position-update

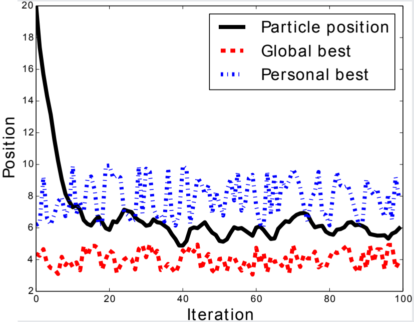
[a] *Poli et al.*, "Exact analysis of the sampling distribution of particle swarm optimizers during stagnation"

When the position update is input-to-state stable

ISS analysis

With ISS of position-update component

$X^P(k)$ and $X^G(k)$ are bounded (Before Stagnation)



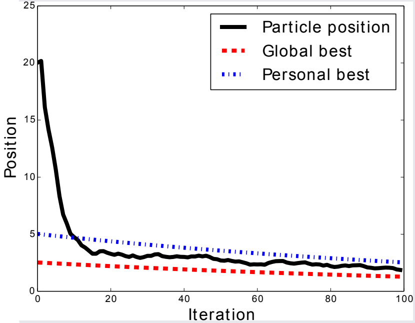
- Converge to a bound
- Simulated using PSO position-update, bounded global-best and personal-best

When the position update is input-to-state stable

ISS analysis

With ISS of position-update component

converging $X^P(k)$ and $X^G(k)$ (Before Stagnation)



- Converge to a bound
- Bound converges

When the position update is input-to-state stable

ISS analysis

- Input-to-State Stable Position-Update Component
  - Stagnation
    - Both $x^G(k)$ and $x^P(k)$ are constant
  - Before stagnation
    - Only $x^G(k)$ is constant
    - Only $x^P(k)$ is constant
    - Both $x^G(k)$ and $x^P(k)$ are not constant

Stagnation
ISS analysis

Prior Research

There exists a distribution.[a]

- Mean of the particle position

$$\hat{x} = \frac{\phi^P x^P + \phi^G x^G}{\phi^P + \phi^G}$$

With ISS of position-update component

There also exists a boundary.

- Boundary of the particle position

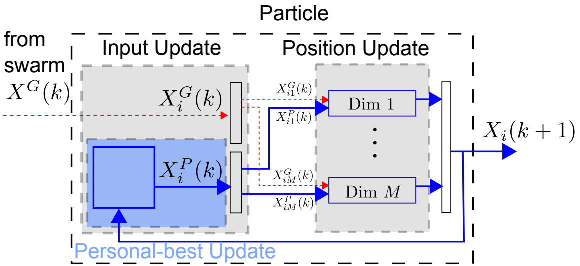
$$\exists T, \forall k > T, |x(k) - \hat{x}| \leq \gamma_d |[x^P - \hat{x},\ x^G - \hat{x}]^T|$$

[a] *Poli et al.*, "Exact analysis of the sampling distribution of particle swarm optimizers during stagnation"

Before stagnation

ISS analysis

when only the $x^G(k)$ is constant

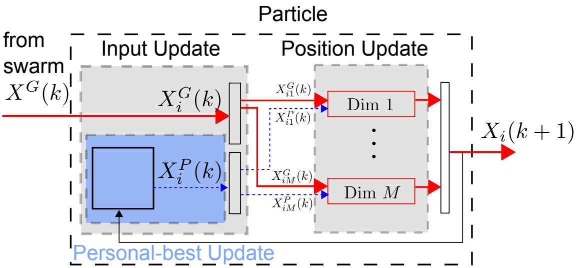


- When personal-best-update component is ISS
- Because of the feedback cascade model
- Small gain theorem applies
- Then the distance to the reference point converges to zero

Before stagnation

ISS analysis

when only the $x^P(k)$ is constant

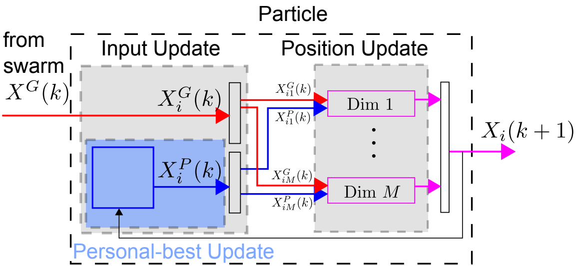


- Personal-best-update component disabled
- The boundary of the position depends on the bound of $x^G(k)$

Before stagnation

ISS analysis

when both the $x^G(k)$ and $x^P(k)$ are not constant



The boundary of the position

- depends on the bound of $x^G(k)$
- depends on the bound of $x^P(k)$

Outline

Summary and futurework

Summary

- Feedback cascade model
  - Input-update component
  - Position-update component
    - Personal-best-update component
    - Global-best-update component
- Input-to-state stability analysis

Futurework

- Including fitness landscape
- Extend from particle behavior to swarm behavior

Thank you