Math 223: Homework 1

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In[•]:=
```
Clear["Global`*"]
SetOptions[EvaluationNotebook[], Background → Black]
```

Problem 1

Use the **Series** command to compute the Taylor series of $(1 + x)^{1/x}$ about x = 0. Explore different parameter values to make sure you understand how to use this command. Use the **Plot** command to plot a comparison of this function with the polynomial of degree 4 given by the partial sum of the Taylor series over the interval [–1, 4]

Computing the first 10 Taylor series terms about the point x=0

In[•]:=
```
n = 10;
func = (1 + x) ^ (1 / x);
x0 = 0;
expansion1 = Series[func, {x, x0, n}]
```

Out[•]=

$$e - \frac{e\, x}{2} + \frac{11\, e\, x^2}{24} - \frac{7\, e\, x^3}{16} + \frac{2447\, e\, x^4}{5760} - \frac{959\, e\, x^5}{2304} + \frac{238\,043\, e\, x^6}{580\,608} - \frac{67\,223\, e\, x^7}{165\,888} + \frac{559\,440\,199\, e\, x^8}{1\,393\,459\,200} - \frac{123\,377\,159\, e\, x^9}{309\,657\,600} + \frac{29\,128\,857\,391\, e\, x^{10}}{73\,574\,645\,760} + O[x]^{11}$$

Now to play get the 4th order Taylor polynomial:

In[•]:=
```
n = 4;
expansion2 = Series[func, {x, x0, n}];
expansion2 = Normal[%]
```

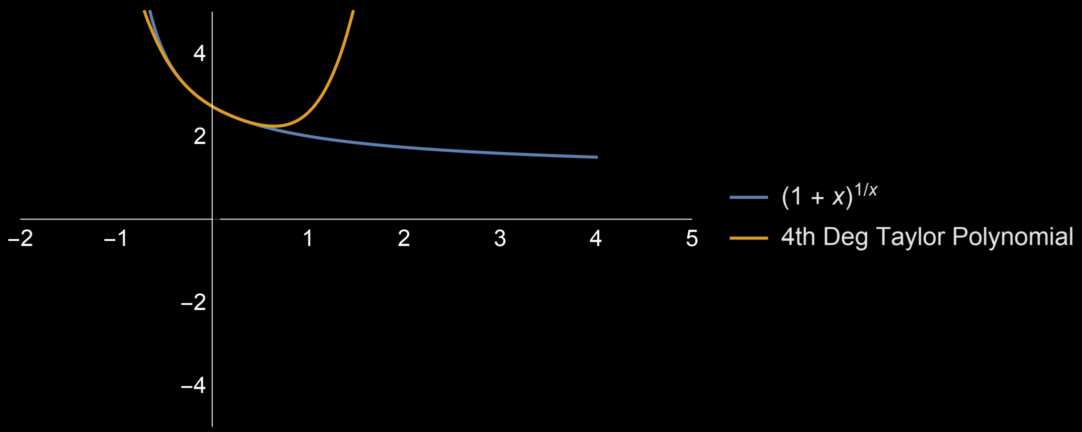
Out[•]=

$$e - \frac{e\, x}{2} + \frac{11\, e\, x^2}{24} - \frac{7\, e\, x^3}{16} + \frac{2447\, e\, x^4}{5760}$$

Now to compare the actual function with the the Taylor polynomial of order 4:

```
In[•]:= Plot[{func, expansion2}, {x, -1, 4}, PlotRange → {{-2, 5}, {-5, 5}},
          PlotLegends → {"(1 + x)^(1/x)", "4th Deg Taylor Polynomial"},
          AxesStyle → Directive[White, 12]]
```

Out[•]=



And the plot makes sense, since we see the expansion roughly match the function around the expanded point.

Problem 2

Use the **Integrate** command to compute $\int_0^{3/2} \sin(x^2)\, dx$. Then, use ***Series*** within the **Integrate** to integrate the polynomial of degree 6 given by the partial sums of the Taylor series of $\text{Sin}(x^2)$ about x=0. Use the **N** command to compute the absolute error made by this approximation.

First, we want to integrate the integral:

```
In[•]:= func2 = Sin[x^2];
        xmin = 0;
        xmax = 3/2;
        Integrate[func2, {x, xmin, xmax}]
```

Out[•]= $\sqrt{\frac{\pi}{2}}$ FresnelS$\left[\frac{3}{\sqrt{2\pi}}\right]$

Now to see the numerical value of this expression:

In[•]:= numApproximation = N$\left[\sqrt{\frac{\pi}{2}}\right.$ FresnelS$\left.\left[\frac{3}{\sqrt{2\pi}}\right]\right]$

Out[•]= 0.778238

This makes sense, so now we move on to integrate the Taylor polynomial (up to degree 6)

```
In[•]:= n = 6;
        x0 = 0;
        taylorApprox = N[Integrate[Series[func2, {x, x0, n}], {x, xmin, xmax}]]
```

Out[•]= 0.718192

Which seems to be close enough to the numerical approximation. In fact, the error is:

```
In[•]:= numApproximation - taylorApprox
```

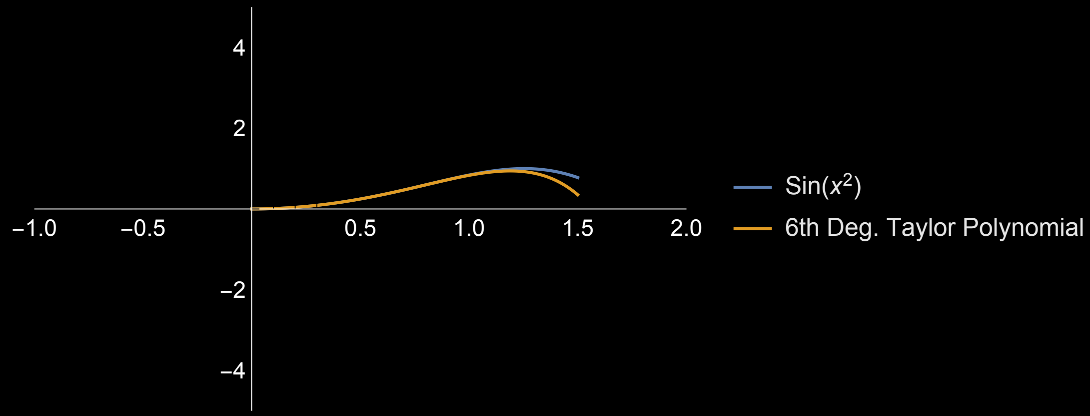
Out[•]= 0.0600458

Now to do another check, we can plot the actual function and the Taylor polynomial (6th degree) to see if the series somewhat approximates the true solution:

In[]:=
```
expansion3 = Series[func2, {x, x0, n}];
expansion3 = Normal[%]
```

Out[]= $x^2 - \frac{x^6}{6}$

In[]:=
```
Plot[{func2, expansion3}, {x, 0, 3/2}, PlotRange → {{-1, 2}, {-5, 5}},
 PlotLegends → {"Sin(x²)", "6th Deg. Taylor Polynomial"}, AxesStyle → Directive[White, 12]]
```

Out[]=


This is good, since we can see the the Taylor polynomial closely approximating the true function!

Problem 3

Use the **DSolve** command to solve the following two-point boundary value problem:

$\epsilon y'' + (1+\epsilon)y' + y = 0$
$y(0) = 0,$
$y(1) = e^{-1}$

Read the documentation to learn how to plot the solution for different values $0 < \epsilon <= 1$. Comment on how the solution changes with ϵ.

First, we solve the ODE:

In[1]:=
```
ode = ϵ * y''[x] + (1 + ϵ) * y'[x] + y[x] == 0
y0 = 0;
y1 = e^-1;
solution = DSolve[{ode, y[0] == y0, y[1] == y1 }, y[x], x]
```

Out[1]= $y[x] + (1+\epsilon)\, y'[x] + \epsilon\, y''[x] == 0$

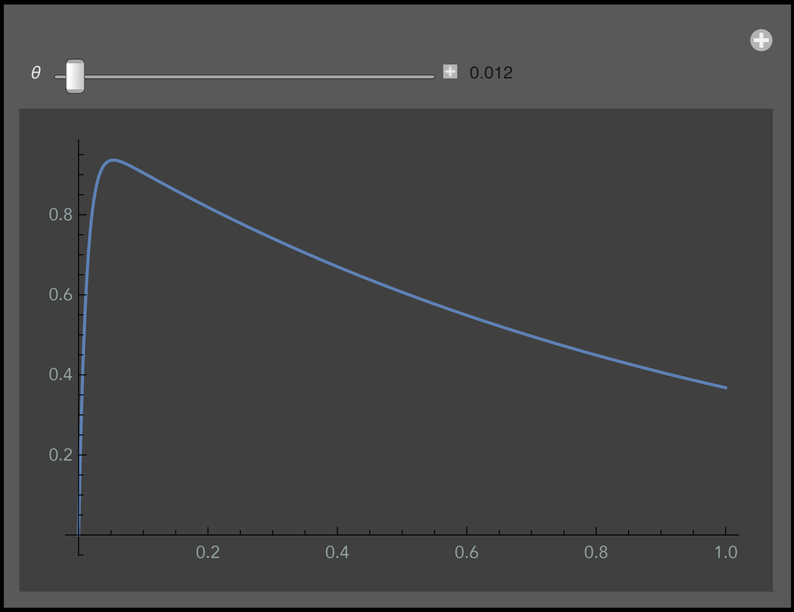
Out[4]= $$\left\{\left\{y[x] \to -\frac{e^{-x+\frac{1}{\epsilon}-\frac{x}{\epsilon}}\left(e^{x} - e^{x/\epsilon}\right)}{-e + e^{\frac{1}{\epsilon}}}\right\}\right\}$$

* the solution is enlarged in order to differentiate between ϵ and e.

To compute the solutions for a different values ϵ, we can use an interactive plot function that allows for adjusting the constants:

In[*]:=
```
Manipulate[Plot[{y[x] /. solution /. ϵ → θ}, {x, 0, 1}, WorkingPrecision → 20],
 {θ, 0 - 10^-8, 1 + 10^-8 , Appearance → "Labeled"}]
```

Out[*]=



In the above plot, we can see that as we modify the values of ϵ towards 0, part of the solution (values of y(x) for $x \in (0,\epsilon)$) tends toward infinity; conversely in the same regime, for values of $x \in (\epsilon,1)$, the solution seems well behaved.

This is supported differential equation (since the term y'' would vanish for small ϵ) and the analytical solution that we obtained (since we have $\frac{1}{\epsilon}$ in the exponent). However, for larger values of ϵ , there is a smoothing that occurs

since the second order term in the ODE does not vanish, and the coefficient of y', $1+\epsilon$, becomes larger and larger. It is important to note that despite the smoothing, solutions at ϵ=1 goes to infinity, since the denominator

$$-e + e^{\frac{1}{\epsilon}}$$

would be 0.