Math 223: Homework6

Ali Heydari

March. 4th, 2021

Problem 1

The Fresnel sine integral is given by $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$.

(a) Use integration-by-parts to find the leading behavior of $S(x)$ as $x \to 0^+$ and compare your result with what Mathematica computes, which might be slightly different from what you compute. Comment on what you find and which approximation you believe yields a better approximation overall.

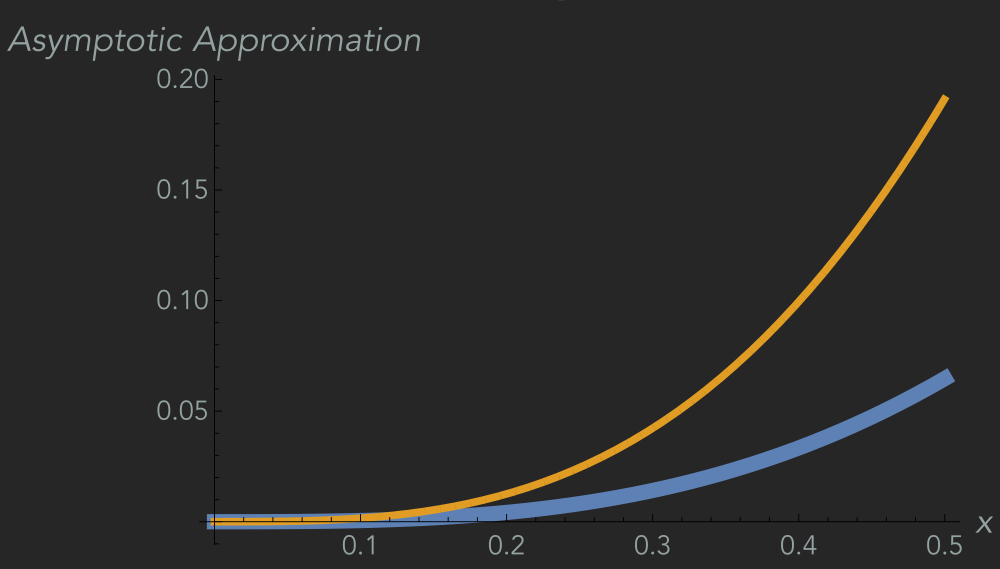
In order to find the asymptotic expansion as $x \to 0^+$, we will perform integration by parts on the integral. We take $u = \sin\left(\frac{\pi t^2}{2}\right)$ and $v = t$, so now we can write the integration by parts as:

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt = = t \sin\left(\frac{\pi t^2}{2}\right)\Big|_0^x - \pi \int_0^x t \cos\left(\frac{\pi t^2}{2}\right) dt = x \sin\left(\frac{\pi x^2}{2}\right) - \pi \int_0^x t \cos\left(\frac{\pi t^2}{2}\right) dt$$

which gives us the first leading term. Now let us plot this to see what we get:

```
In[5]:= Plot[{FresnelS[x], x * Sin[(Pi * x^2)/2]}, {x, 0, 0.5},
 PlotRange → All,
 PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["Asymptotic Approximation", Italic, 18]},
 TicksStyle → Directive[FontSize → 14]]
```

Out[5]=



Which is pretty terrible. So now to get a more accurate approximation, we can find the next term in the expansion using IBP. That means that we have:

$$dv = t^2 \to \frac{t^3}{3},\ u = \cos\left(\frac{\pi t^2}{2}\right)$$

$$du = \pi t \cos\left(\frac{\pi t^2}{2}\right) dt$$

which results in :

$$= t \sin\left(\frac{\pi t^2}{2}\right) - \frac{\pi x^3}{3} \cos\left(\frac{\pi x^2}{2}\right) + \frac{\pi}{3} \int_0^x t^4 \cos\left(\frac{\pi x^2}{2}\right) dt$$

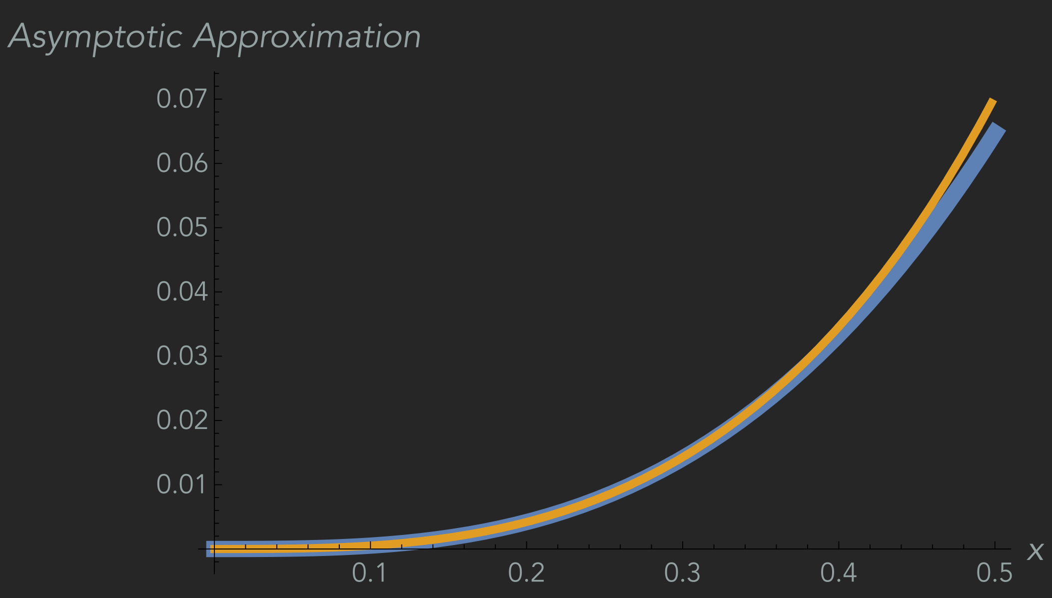
$$S(x) \sim -\frac{\pi x^3}{3} \cos\left(\frac{\pi x^2}{2}\right) + x \sin\left(\frac{\pi x^2}{2}\right) + O\left(x^5\right)$$

Now let us compute what Mathematica gives for the asymptotic expansion:

```
In[•]:= exact = Integrate[Sin[(Pi * t^2) / 2], t]
Out[•]= FresnelS[t]
```

```
In[7]:= Plot[{FresnelS[x], - (Pi / 3) x^3 Cos[(Pi * x^2) / 2] + x * Sin[(Pi * x^2)/2]}, {x, 0, 0.5},
  PlotRange → All,
  PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
  AxesLabel → {Style["x", Italic, 18], Style["Asymptotic Approximation", Italic, 18]},
  TicksStyle → Directive[FontSize → 14]]
```

Out[7]=



(b) Use $S(x) = \int_0^\infty \sin\left(\frac{\pi t^2}{2}\right) dt = \frac{1}{2}$, and integration-by-parts to find the leading behavior of $S(x)$ as $x \to \infty$. Compare your results with what Mathematica computes, and discuss what you find.

In order to find the asymptotic expansion as $x \to \infty$, we will perform integration by parts on the integral. We could use a u-sub for convenience (as the hint suggests), but If we just think about the chain rule that resulted in the integrand, we can see that there must have been a Cosine term with an coefficient of order $\frac{2}{\pi t}$ to cancel out the derivative of the sine function. This means that we must take $dv = \sin\left(\frac{\pi t^2}{2}\right)$ and $u = \sqrt{\frac{2}{\pi t^2}}$. Now letting $\alpha = \sqrt{\frac{2}{\pi}}$ This will give us:

$u = \alpha t^{-1} \Longrightarrow du = -2 \alpha t^{-2} dt$,

And now to make things nicer, we can integrate with respect to $\frac{\pi t^2}{2}$ instead, which means that $\frac{1}{(2\pi)} \int_0^x \sin\left(\frac{\pi t^2}{2}\right) d\left(\frac{\pi t^2}{2}\right) = -\cos\left(\frac{\pi t^2}{2}\right)$ (keep in mind, this trick is just the unmasked version of u-sub).

Now doing the integration by parts yields:

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt = = \frac{1}{2} - \left(\frac{\cos\left(\frac{\pi t^2}{2}\right)}{\sqrt{\frac{\pi t^2}{2}}}\right)_0^x - \frac{1}{2}\int_0^x \frac{\cos\left(\frac{\pi t^2}{2}\right)}{\left(\frac{\pi t^2}{2}\right)^{3/2}} d\left(\frac{\pi t^2}{2}\right)$$

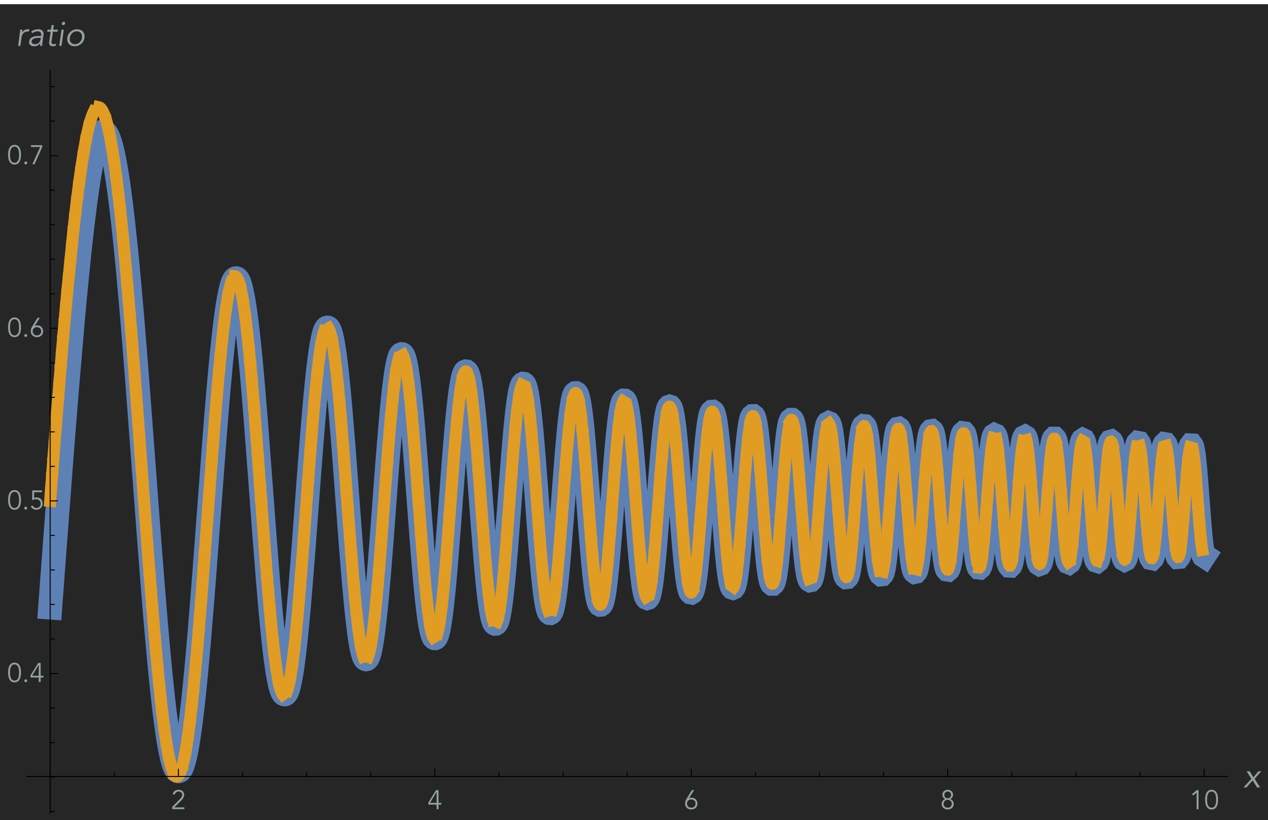
which gives us the first leading term to be:

$$S(x) = = \frac{1}{2} - \left(\frac{\cos\left(\frac{\pi x^2}{2}\right)}{\pi x}\right) + O\left(x^{-3}\right)$$

If we want to find more terms of the expansion, we can keep going with the integration by parts. Now to plot this solution against the exact solution:

```
In[]:= Plot[{FresnelS[x], 1/2 - (Cos[Pi*x^2/2]/(Pi*x))}, {x, 1, 10},
  PlotRange → All,
  PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
  AxesLabel → {Style["x", Italic, 18], Style["ratio", Italic, 18]},
  TicksStyle → Directive[FontSize → 14]]
```

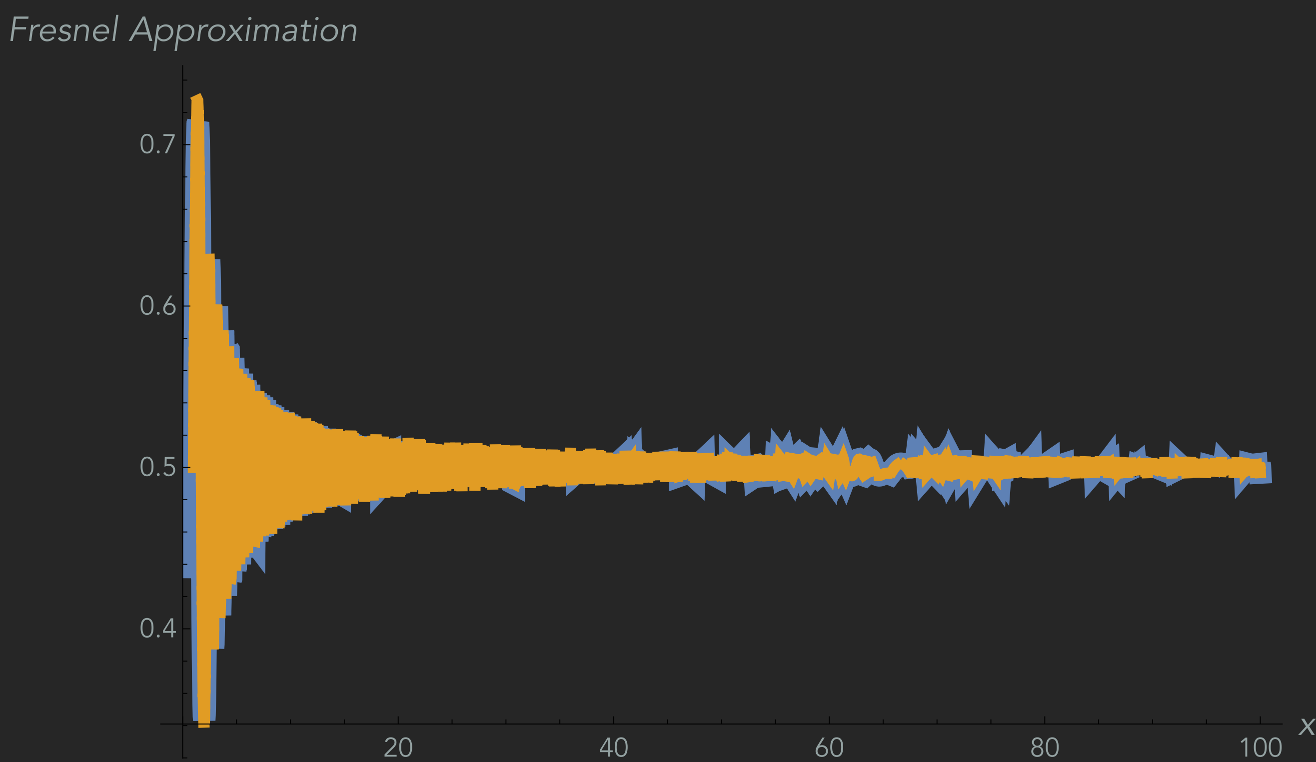
Out[ ]=



this shows that our leading term very well approximates the overall behavior of the Fresnel function, specially as we move further away from 0; this makes sense since our expansion was for $x \to \infty$. One other observation is that Mathematica's exact solution behaves rather chaotic as the values of x get larger, but our approximation seems to capture the behavior just as well as smaller values of x. To demonstrate this, let us consider the plot below:

In[•]:=
```
Plot[{FresnelS[x], 1/2 - (Cos[Pi*x^2/2] / (Pi*x))}, {x, 1, 100},
 PlotRange → All,
 PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["Fresnel Approximation", Italic, 18]},
 TicksStyle → Directive[FontSize → 14]]
```

Out[•]=



which shows the chaotic nature of Mathematica's solution (in blue), while our approximation (in yellow) seems to carry steady on.

Problem 2

Show that $\int_0^1 \frac{e^x - e^{x t}}{1-t}\, d t \sim e^x \log(x) + e^x \gamma + \cdots$, as $x \to \infty$ with γ being the Euler-Mascheroni constant, defined according to $\int_0^\infty \left(\frac{1}{u+1} - e^{-u}\right) \frac{d u}{u}$.

We start by making the substitution $u = x - x t$, which means that $d u = -x$, and $t = -\frac{u}{x} + 1$. So now we start writing the integral:

$$\int_0^1 \frac{e^x - e^{x t}}{1-t}\, d t = \int_x^0 \frac{e^x - e^{-x\left(\frac{u}{x}+1\right)}}{\frac{-u}{x}} \frac{d t}{-x} = \int_x^0 \frac{e^x - e^{-x\left(\frac{u}{x}+1\right)}}{\frac{-u}{x}}\, d t = -\int_0^x \frac{e^x - e^{x-u}}{-u}\, d u$$

Here, we can do the trick where we rewrite 1 as $\frac{u+1}{u+1}$ to the integrand:

$$= e^{x} \int_{0}^{x} \frac{u+1}{u+1} - e^{-u}\, \frac{d\,u}{u} = e^{x} \int_{0}^{x} \left(\frac{u}{u+1} + \frac{1}{u+1} - e^{-u} \right) \frac{d\,u}{u} = e^{x} \left(\int_{0}^{x} \frac{1}{u+1}\, d\,u + \int_{0}^{x} \frac{1}{u+1} - e^{-u}\, \frac{d\,u}{u} \right)$$

Now we can see that we have gotten Euler's gamma from the second integral, so we can rewrite the solution so far to be:

$$= e^{x} \int_{0}^{x} \frac{1}{u+1}\, d\,u + e^{x}\,\gamma = e^{x} \log\,(1+x) + e^{x}\,\gamma$$

Now if we expand the log term using Mathematica

In[2]:=
```
Series[Log[x + 1], x → ∞]
```

Out[2]=
$$\text{Log}[x] + O\left[\frac{1}{x}\right]^{1}$$

So now we can rewrite the asymptotic solution
as:

$$\int_{0}^{1} \frac{e^{x} - e^{x\,t}}{1-t}\, d\,t = e^{x} \int_{0}^{x} \frac{1}{u+1}\, d\,u + e^{x}\,\gamma \sim e^{x} \log\,(x) + e^{x}\,\gamma + O\left(x^{-1}\right)$$

Therefore we have arrived at the desired asymptotic approximation of the integral.