Math 223: Homework 10

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Problem 1

Use boundary-layer theory to find a uniform approximation with error of order ϵ^2 for the problem:

$$\epsilon y'' + y' + y = 0, \qquad 0 \leq x \leq 1, \qquad 0 < \epsilon \ll 1$$

$$y(0) = e,\ y(1) = 1$$

Notice that there is no boundary layer in leading order, but one does appear in next order. Compare your solution with the exact solution to this problem.

To find the boundary layer, we can take a look at the coefficient of the first order derivative, i.e. $a(x) = 1$, which is greater than 0. This means that our boundary layer will be at the left boundary, so now solving for the outer layer:

$$y_{out} = \sum_{i=1}^{\infty} \epsilon^n y_n$$

Now substituting this back into the original perturbed ODE, we get that:

$y_0' \sim y_0$ and $d y_0 \sim -y_0 \, d x$

from which we find that $y_0 = C e^{-x} + O(\epsilon)$ and using the fact that $y(1) = 1 \to 1 = C e^{-1} \Longrightarrow C = e$. We can show this with Mathematica:

```
In[•]:= DSolve[{y0'[x] + y0[x] == 0, y0[0] == E}, y0[x], x]
Out[•]= {{y0[x] → e^(1-x)}}
```

So our solution for up to order ϵ is :

$$y_{out} = e^{1-x} + O(\epsilon)$$

We can now continue to find the order ϵ^2 approximation, meaning y_1:

$$y_1' + y_1 = y_0 = e^{1-x}$$

In[•]:=
```
DSolve[{y1'[x] + y1[x] == Exp[1 - x], y1[1] == 0}, y1[x], x]
```
Out[•]= $\{\{y_1[x] \to e^{1-x}(-1+x)\}\}$

So putting the first two order solutions together, we get:

$$y_{out} = e^{1-x}(1+\epsilon(x-1)) + O\left(\epsilon^2\right)$$

Now to find the distinguished limit we can solve for the stretch coordinate in a general form to be:

$$\frac{dy}{dx} = \frac{1}{\epsilon^p}\frac{dY}{dX}$$

$$\frac{dy^2}{d^2x} = \frac{1}{\epsilon^p}\frac{1}{\epsilon^p}\frac{dY^2}{d^2X}$$

which yields to:

$$\epsilon\left(\epsilon^{-2p}\right)Y'' + \epsilon^{-p}a(x)Y' + b(x)Y = 0$$

and since we want to make sure that the second order derivative does not vanish, we must have that $\epsilon^{-2p+1} \sim \epsilon^{-p}$, which means that $-2p+1 = -p \Longrightarrow p = 1$. That means that $X = \frac{x}{\epsilon}$.

So we find that the stretch coordinate must be $X = \frac{x}{\epsilon}$, which yields :

$$\epsilon^{-1}y'' + \epsilon y' + y = 0$$

assuming the same solution type as outer solution (i.e. $y_{in} = \sum_{i=1}^{\infty}\epsilon^n y_n$), we find the first order solution to be:
$y_0'' \sim -y_0'$

Now we can do the substitution that $u = y_0' \Longrightarrow u' = y_0''$, so now we can solve:

In[•]:=
```
DSolve[{u0'[X] + u0[X] == 0}, u0[X], X]
```
Out[•]= $\{\{u_0[X] \to e^{-X} c_1\}\}$

finding that $u(X) = C e^{-X} \Longrightarrow y(X) = \int C e^{-X} = C\int e^{-X} + C_1$

Now using the boundary condition $y(0) = e$, we can obtain the first order inner solution to be:
$e = -C + C_1 \Longrightarrow C_1 = e$

$$\Longrightarrow y_0 = C\left(1 - e^{-X}\right) + e$$

to obtain the next order solution, we can follow the same procedure and get:

$$y_1'' + y_1' \sim y_0 = C\left(1 - e^{-X}\right) + e$$

In[•]:=
```
DSolve[{u0'[X] + u0[X] == c1 (1 - Exp[-X]) + E}, u0[X], X]
```

Out[•]=

$$\left\{\left\{u_0[X] \to e^{-X}\left(e^{1+X} + e^X c_1 - X c_1\right) + e^{-X} c_2\right\}\right\}$$

so the inner solution now becomes:

$$y_{in} = C\left(1 - e^{-X}\right) + e + \epsilon\left(C X e^{-X} + C_1\left(1 - e^{-X}\right)\right)$$

To determine the constants for the inner solution, we will do the asymptotic matching:

$$\lim_{X\to\infty} y_{in}(0) \sim \lim_{x\to 0} y_{out}(0) \;\to\; C + e + \epsilon C_1 \sim e + \epsilon e$$

meaning that $C = 0$ and $C_1 = e$, so now the inner solution becomes:

$$y_{in} = e\left(1 + \epsilon\left(1 - e^{-\frac{x}{\epsilon}}\right)\right)$$

Finally, to find the uniform solution:

$$y_{uniform} = y_{in} + y_{out} - y_{overlap}$$

$$y_{uniform} = e\left(1 + \epsilon\left(1 - e^{-\frac{x}{\epsilon}}\right)\right) + e^{1-x}\left(1 + \epsilon(x-1)\right) - (e + \epsilon e)$$

and after doing the calculations, we arrive at:

$$= e^{1-x}\left(1 + \epsilon\left(1 - x - e^{\frac{1}{x}}\right)\right) + O\left(\epsilon^2\right)$$

Now to compare with the exact solution:

In[15]:=
```
NumericalSolution =
 NDSolve[{0.01 y''[x] + y'[x] + y[x] == 0, y[0] == E, y[1] == 1}, y[x], {x, 0, 1}]
```

Out[15]= {{y[x] → InterpolatingFunction[Domain: {{0., 1.}} Output: scalar][x]}}

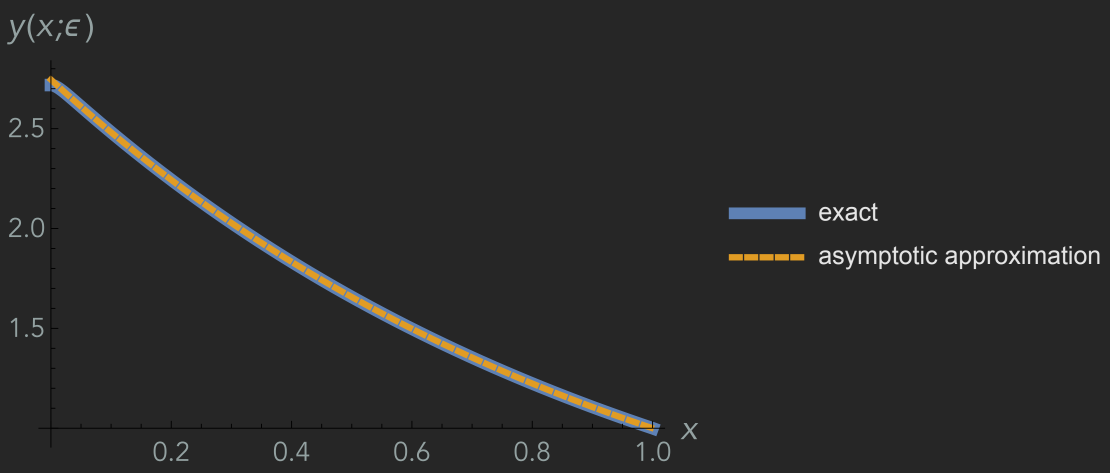
In[14]:=
```
AsymptoticApproxP1[x_, ϵ_] = Exp[1 - x] * (1 + ϵ * (1 - x - Exp[-1 / ϵ]))
```

Out[14]=

$$e^{1-x}\left(1 + \left(1 - e^{-1/\epsilon} - x\right)\epsilon\right)$$

In[17]:=
```
Plot[{y[x] /. NumericalSolution, AsymptoticApproxP1[x, 0.01]}, {x, 0, 1},
 PlotLegends → {"exact", "asymptotic approximation"}, PlotRange -> All,
 PlotStyle → { Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["y(x;ϵ)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14]]
```

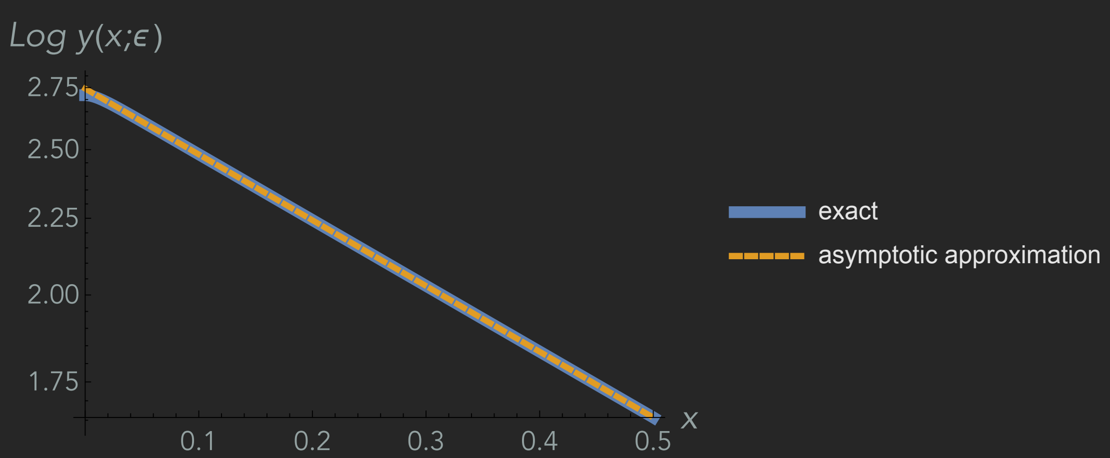
Out[17]=



so we can see that the solutions match nicely! And the solutions match even at the boundary layer, since we continued the approximation for another order. This is perhaps more visible in the Log plot, shown below:

In[25]:=
```
LogPlot[{y[x] /. NumericalSolution, AsymptoticApproxP1[x, 0.01]}, {x, 0, 0.5},
 PlotLegends → {"exact", "asymptotic approximation"}, PlotRange -> All,
 PlotStyle → { Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["Log y(x;ϵ)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14]]
```

Out[25]=



So we have verified that our solution makes sense for this problem!

Problem 2

Use boundary-layer methods to find an approximate solution to the initial-value problem

$$\epsilon y'' + a(x) y' + b(x) y = 0, \qquad x \geq 0, \qquad a(x) > 0, \qquad 0 < \epsilon \ll 1$$

$y(0) = 1, \qquad y'(0) = 1$

Show that the leading-order uniform approximation satisfies $y(0) = 1$, but not $y'(0) = 1$ for arbitrary b. Compare the leading-order uniform approximation with the exact solution to the problem when $a(x)$ and $b(x)$ are constants.

As before, we can take a look at the coefficient of the first order derivative, and since we have that $a(x) > 0$, which means that our boundary layer will be at the left boundary. Solving for the outer layer, we take the following form for y_{out}:

$$y_{out} = \sum_{i=1}^{\infty} \epsilon^n y_n$$

Now substituting this back into the original perturbed ODE, we can look at the first order as

$$a(x) y_0' \sim -b(x) y_0$$

```
In[•]:= DSolve[{a[x] * y₀'[x] + b[x] * y₀[x] == 0}, y₀[x], x, Assumptions → {a[x] > 0, b[x] > 0, x ≥ 0}]
```

Out[•]= $\left\{\left\{y_0[x] \to e^{\int_1^x -\frac{b[K[1]]}{a[K[1]]} dK[1]} c_1\right\}\right\}$

Now we see that this will give us a very ugly solution for y_{out}. So we have that:

$$y_{out} = C_1 e^{-\int_x^1 \frac{b(t)}{a(t)} dt}$$

which is the leading behavior for y_{out}.

Finding y_{in}:

Now for calculating y_{in}, based on the previous example, we take the stretch coordinate $\delta = \frac{x}{\epsilon}$; in other words we can solve for the stretch coordinate in a general form to be:

$$\frac{dy}{dx} = \frac{1}{\epsilon^p} \frac{dY}{dX}$$

$$\frac{dy^2}{d^2 x} = \frac{1}{\epsilon^p} \frac{1}{\epsilon^p} \frac{dY^2}{d^2 X}$$

which yields to:

$$\epsilon\left(\epsilon^{-2p}\right) Y'' + \epsilon^{-p} a(x) Y' + b(x) Y = 0$$

and since we want to make sure that the second order derivative does not vanish, we must have that $\epsilon^{-2p+1} \sim \epsilon^{-p}$, which means that $-2p + 1 = -p \Longrightarrow p = 1$.

Now that we have found the correct distinguished limit, we can proceed:

$\epsilon^{-1} Y'' + \epsilon^{-1} a(x) Y' + b(x) Y = 0$

So solving for the second order, we obtain:

$Y_0'' + a(x) Y_0' = 0$

In[ ]:=
```
DSolve[{u0''[X] + a[x] * u0'[X] == 0, u0[0] == 1, u0'[0] == 1}, u0[X], X] // Expand
```

Out[ ]=
$$\left\{\left\{u_0[X] \to 1 + \frac{1}{a[x]} - \frac{e^{-X a[x]}}{a[x]}\right\}\right\}$$

So we get that the first solution to be $Y_0 = 1 + \frac{1}{a(0)} - \frac{e^{-a(0) X}}{a(0)}$. We will stop at this leading order approximation for y_{in}.

Now to find the overlap and do the asymptotic matching:

$$\lim_{X\to\infty} y_{in}(0) \sim \lim_{x\to 0} y_{out}(0) \to \lim_{X\to\infty} 1 + \frac{1}{a(0)} - \frac{e^{-a(0) X}}{a(0)} \sim \lim_{x\to 0} C_1 e^{-\int_x^1 \frac{b(t)}{a(t)} dt}$$

$$\lim_{X\to\infty} y_{in}(0) \sim \lim_{x\to 0} y_{out}(0) \to \lim_{X\to\infty} 1 \sim \lim_{x\to 0} C_1 e^{-\int_0^1 \frac{b(t)}{a(t)} dt}$$

So then we find that $y_{overlap} = 1$ with $C_1 = \frac{1}{e^{-\int_0^1 \frac{b(t)}{a(t)}}}$

so now that we have found the asymptotic matching, we can write the uniform solution as:

$$y = 1 + \frac{1}{a(0)} - \frac{e^{-a(0) X}}{a(0)} + \frac{1}{e^{-\int_0^1 \frac{b(t)}{a(t)}}} e^{-\int_x^1 \frac{b(t)}{a(t)} dt} - 1$$

replacing X with $\frac{x}{\epsilon}$ we get:

$$y = \frac{1}{a(0)} - \frac{e^{-a(0) \frac{x}{\epsilon}}}{a(0)} + \frac{1}{e^{\int_0^1 \frac{b(t)}{a(t)}}} e^{-\int_x^1 \frac{b(t)}{a(t)} dt}$$

Now to see how well our solutions match, we can set an arbitrary $a(0)$ and $b(x)$, so let $a(x) = b(x) = 2$. So our solution with these choices becomes:

$$y = \frac{1}{2} - \frac{e^{-2 \frac{x}{\epsilon}}}{2} + \frac{1}{e^{\int_0^1 2\, dt}} e^{-\int_x^1 2\, dt} = 1 - \frac{e^{-2 \frac{x}{\epsilon}}}{2} + e^2 \left(e^{x-2}\right)$$

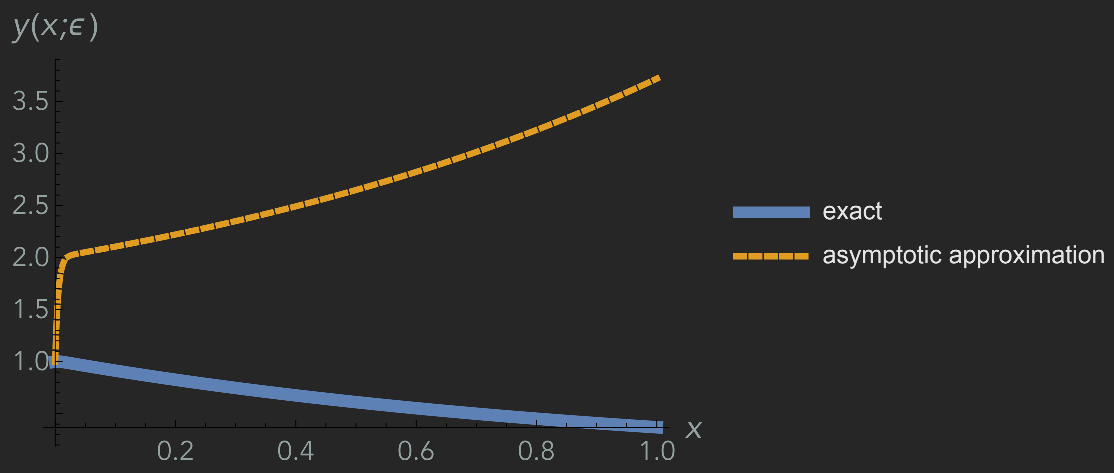
In[7]:=
```
NumericalSolution =
  NDSolve[{0.01 y''[x] + 2 * y'[x] + 2 * y[x] == 0, y[0] == 1, y'[0] == 1}, y[x], {x, 0, 1}]
```

Out[7]= {{y[x] → InterpolatingFunction[Domain: {{0., 1.}} Output: scalar][x]}}

In[10]:=
```
AsymptoticApproxP2[x_, ϵ_] = 1 + E^(2) * E^(x - 2) - Exp[-2 * x / ϵ]
```

Out[10]= $1 + e^{x} - e^{-\frac{2x}{\epsilon}}$

In[11]:=
```
Plot[{y[x] /. NumericalSolution, AsymptoticApproxP2[x, 0.01]}, {x, 0, 1},
  PlotLegends → {"exact", "asymptotic approximation"}, PlotRange -> All,
  PlotStyle → { Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
  AxesLabel → {Style["x", Italic, 18], Style["y(x;ϵ)", Italic, 18]},
  TicksStyle → Directive[FontSize → 14]]
```

Out[11]=


I am surprised by how bad this worked, even though we were given only the initial conditions. To check that our solutions satisfies what the question had asked, i.e. checking that $y(0) = 1$, but $y'(0) \neq 1$:

In[4]:=
```
AsymptoticApproxP2[0, 0.01]
```

Out[4]= 1.

So this matches! now for the derivative:

In[12]:=
```
YPrime[x_, ϵ_] = D[AsymptoticApproxP2[x, ϵ], x]
```

Out[12]= $e^{x} + \frac{2 e^{-\frac{2x}{\epsilon}}}{\epsilon}$

In[6]:=
```
YPrime[0, 0.01]
```

Out[6]= 201.

which is not 1, so clearly this problem does not match the initial condition, as the prompt of the problem suggested it!

Problem 3

The function $y\ (x; \epsilon)$ satisfies:

$$\epsilon y'' + (1+\epsilon) y' + y = 0, \qquad 0 \le x \le 1,$$

and is subject to boundary conditions $y(0) = 0$ and $y(1) = e^{-1}$. Find two terms in the outer approximation, applying only the boundary condition at $x = 1$. Next, find two terms in the inner approximation for the boundary layer near $x = 0$, which can be assumed to have width $O(\epsilon)$, and only applying the boundary condition at $x = 0$. Finally determine the constants of integration in the inner approximation by matching.

As before, we will start by the outer solution: The leading order is given via:

$$y_0' + y_0 = 0$$

which just gives us $y_0 = e^{-x}$. Now moving on to the second order solution, we can find that using:

$$y_1' + y_1 = -y_0'' - y_0'$$

In[∘]:=
```
DSolve[{y₁'[x] + y₁[x] == 0, y₁[0] == 0}, y₁[x], x]
```

Out[∘]=
```
{{y₁[x] → 0}}
```

which is interesting. So now we want to move on to the inner solutions: for the first order inner solution, we find that

$$Y_0'' = -Y_0'$$

In[∘]:=
```
DSolve[{Y₀''[X] + Y₀'[X] == 0, Y₀[0] == 0}, Y₀[X], X]
```

Out[∘]=
```
{{Y₀[X] → e^-X (-1 + e^X) c₁}}
```

$$Y_1'' + Y_1' + Y_0' = -Y_0$$

In[∘]:=
```
DSolve[{Y₁''[X] + Y₁'[X] == -Exp[-X] - (1 - Exp[-X]), Y₁[0] == 0}, Y₁[X], X]
```

Out[∘]=
```
{{Y₁[X] → -e^-X (e^X X + c₁ - e^X c₁)}}
```

Now to find the constant, we can do the asymptotic matching in the limits:

$$\lim_{X\to\infty} y_{in}^0 \sim \lim_{x\to 0} y_{out}^0 \ \to\ C_1 e^{-x}(-1+e^x) = e^{-x} \Longrightarrow C_1 = 1$$

$$\lim_{X\to\infty} y_{in}^1 \sim \lim_{x\to 0} y_{out}^1 \ \to\ \lim_{X\to\infty} \epsilon\left(e^{-X}\left(X e^X\left(e^X + C_1 - C_1 e^X\right)\right)\right) = 0$$

$$\lim_{X\to\infty} y_{in^1 \sim \lim_{x\to 0} y_{out}^1} \ \to\ \lim_{X\to\infty} \epsilon\left(-X - C_1\left(1 - e^{-X}\right)\right) = 0$$

Now to be able to do the matching for the inner solution, we can look at the series expansion of $e^{-X} = e^{-\frac{x}{\epsilon}}$:

$$e^{-\frac{x}{\epsilon}} = 1 - \frac{x}{\epsilon} + O\left(\frac{x^2}{\epsilon}\right)$$

now rewriting the limit we get:

$$\lim_{X\to\infty} \epsilon\left(-X - C_1\left(1-e^{-X}\right)\right) = \epsilon\left(-\frac{x}{\epsilon} - C_1\left(-1-\frac{x}{\epsilon}\right)\right) = -x + C_1\, x = 0 \Longrightarrow C_1 = 1$$

So we find that $y_{\ overlap} = 1$. So for the leading order solution, we get:

$$y = e^{-x} - e^{-\frac{x}{\epsilon}}$$

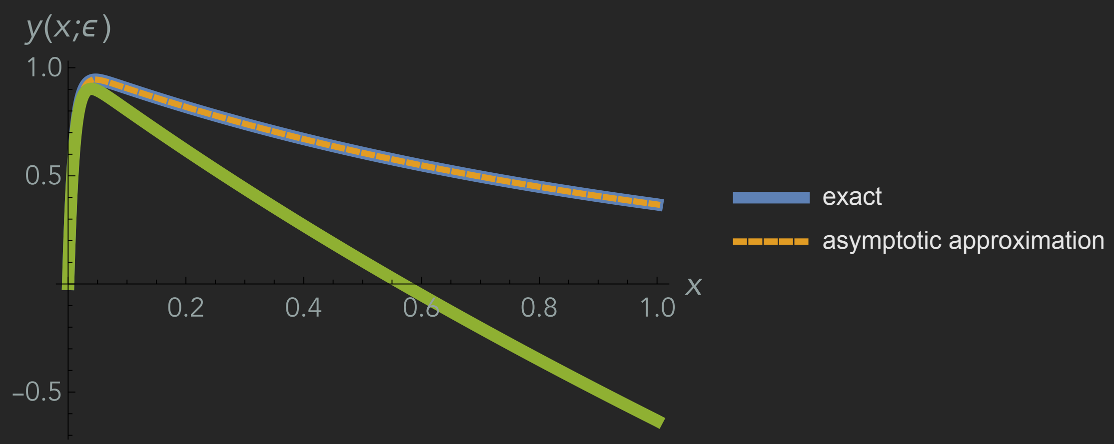
and for the solution with two terms we obtain:

$$y = e^{-x} - e^{-\frac{x}{\epsilon}} - x$$

Now we will plot these solutions against the numerical solution:

```
In[ ]:= NumericalSol = NDSolve[
          {0.01 * y''[x] + (1 + 0.01) * y'[x] + y[x] == 0, y[0] == 0, y[1] == Exp[-1]}, y[x], {x, 0, 1}]

Out[ ]= {{y[x] → InterpolatingFunction[ Domain: {{0., 1. }} Output: scalar ][x]}}

In[ ]:= AsymptoticApprox01[x_, ϵ_] = - Exp[-x / ϵ] + Exp[-x]

Out[ ]= e^-x - e^(-x/ϵ)

In[ ]:= AsymptoticApprox02[x_, ϵ_] = - Exp[-x / ϵ] + Exp[-x] - x

Out[ ]= e^-x - e^(-x/ϵ) - x

In[ ]:= Plot[{y[x] /. NumericalSol, AsymptoticApprox01[x, 0.01], AsymptoticApprox02[x, 0.01]},
         {x, 0, 1}, PlotLegends → {"exact", "asymptotic approximation"},
         PlotRange -> All, PlotStyle → { Directive[Solid, Thickness[0.02]],
           Directive[Dashed, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]]},
         AxesLabel → {Style["x", Italic, 18], Style["y(x;ϵ)", Italic, 18]},
         TicksStyle → Directive[FontSize → 14]]
```

Out[ ]=



so it is very interesting that the leading order solution matches really really well with the exact solution, but that the second order approximation does not match well with the exact solution. So i think the moral of the story in this case is that going higher in order does not necessary provide a better approximation to the problem.