

Math 223: Homework8

Ali Heydari

April 1st, 2021

Problem 1

Find the asymptotic behavior of

$$S(x) = \int_0^2 (\sin t + t) e^{ixt} dt$$

as $x \rightarrow \infty$ up to terms involving $\frac{1}{x^2}$.

Since $\phi(t)$ is monotonic on the interval $[0,2]$, we can find the solution through iterative; specifically,

$I(x) = \int_a^b f(t) e^{ixt} dt \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{(ix)^{n+1}} [f^{(n)}(b) e^{ixb} - f^{(n)}(a) e^{ixa}]$, $x \rightarrow +\infty$, where $f(t) = \sin(t) + t$. Going up to two terms, we have:

$$I(x) \sim \frac{1}{ix} [\sin(2) e^{2ix} + 2e^{2ix}] + \frac{1}{(ix)^2} [\cos(2) e^{2ix} + 2e^{2ix} - \cos(0) e^{i \times 0}] + \sum_{n=3}^{\infty} \frac{(-1)^n}{(ix)^{n+1}} [f^{(n)}(2) e^{ix2} - f^{(n)}(0) e^{ix0}]$$

$$I(x) \sim \frac{1}{ix} \sin(2) e^{2ix} - 2e^{2ix} - \frac{1}{x^2} [(\cos(2) + 1) e^{2ix} + 2]$$

So now we save this solution for future use:

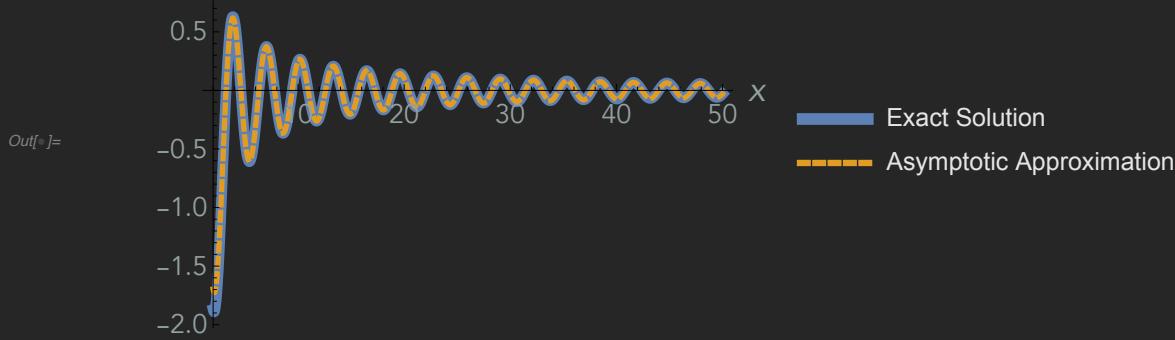
$$\text{In}[1]:= \text{approxSol}[x_] = -\frac{\frac{i}{x} (2 + \text{Sin}[2]) e^{2ix}}{x} + \frac{e^{2ix}}{x^2} (1 + \text{Cos}[2]) - \frac{2}{x^2}$$
$$\text{Out}[1]= -\frac{2}{x^2} + \frac{e^{2ix} (1 + \text{Cos}[2])}{x^2} - \frac{i e^{2ix} (2 + \text{Sin}[2])}{x}$$

Now to compare our solution with the exact solution:

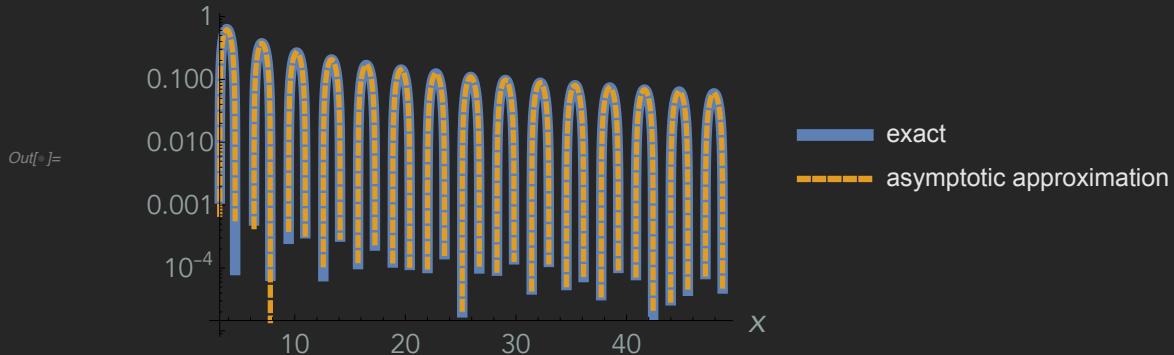
$$\text{In}[2]:= \text{exactSol}[x_] = \text{Integrate}[\text{Sin}[t] * E^{(I*x*t) + t * E^{(I*x*t)}}, \{t, 0, 2\}]$$
$$\text{Out}[2]= \frac{-1 + e^{2ix} (1 - 2ix)}{x^2} + \frac{-1 + e^{2ix} (\text{Cos}[2] - ix \text{Sin}[2])}{-1 + x^2}$$

Resulting in:

```
In[=] Plot[{Re[exactSol[x]], Re[approxSol[x]]}, {x, 2, 50}, PlotRange -> All,
PlotStyle -> {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
AxesLabel -> {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]},
TicksStyle -> Directive[FontSize -> 14],
PlotLegends -> {"Exact Solution", "Asymptotic Approximation"}]
```

Solutions ($\text{Re}[I(x)]$)

```
In[=] LogPlot[{Re[exactSol[x]], Re[approxSol[x]]}, {x, 2, 50}, PlotRange -> All,
PlotStyle -> {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
AxesLabel -> {Style["x", Italic, 18], Style["Log Plot of Re[I(x)]", Italic, 18]},
TicksStyle -> Directive[FontSize -> 14],
PlotLegends -> {"exact", "asymptotic approximation"}]
```

Log Plot of $\text{Re}[I(x)]$ 

these plots show great agreement as $x \rightarrow \infty$, as desired. This is a good verification that our approach worked well and that we have characterized the asymptotic behavior of the solution well.

Problem 2

Use the method of stationary phase to find the leading behaviors of the following integrals as $x \rightarrow \infty$.

(a) $\int_0^1 \tan(t) e^{ixt^4} dt$

We know that the method of stationary phase will give us the leading asymptotic behavior of generalized Fourier integrals that have stationary points, i.e. given $\int_a^b f(t) e^{ix\phi(t)} dt$, $\phi' = 0$ at some point in the interval. We can also see that $\phi'(0) = \phi''(0) = \phi'''(0)$, but $\phi^{(4)}(0) = 24 \neq 0$. Knowing this, we can see that the stationary point of $\phi(t) = t^4$ will be at $t = 0$, since $\phi'(0) = 0$. However, we want to do a substitu-

tion for the change of variables, so that we can actually perform the method of stationary phase. That is:

let $u = t^2 \Rightarrow dt = \frac{1}{2t} du$ and $t = \sqrt{u}$. So now we rewrite the integral as:

$$\int_0^1 \tan(t) e^{ixt^4} dt = \frac{1}{2} \int_0^1 \frac{\tan(\sqrt{u})}{\sqrt{u}} e^{ixu^2} du$$

Now we take $f(t) = \frac{\tan(\sqrt{u})}{\sqrt{u}}$, which means that the Taylor expansion of this function will have a leading constant plus polynomials in terms of u . That is:

```
In[6]:= Series[Tan[Sqrt[u]] / Sqrt[u], {u, 0, 5}]
```

```
Out[6]= 1 +  $\frac{u}{3} + \frac{2u^2}{15} + \frac{17u^3}{315} + \frac{62u^4}{2835} + \frac{1382u^5}{155925} + O[u]^{11/2}$ 
```

$$\int_0^1 \frac{\tan(\sqrt{u})}{\sqrt{u}} e^{ixu^2} du \sim \int_0^{0+\epsilon} e^{ix\left(\frac{1}{2}(u-0)^2\right)} du = \int_0^{0+\epsilon} e^{ixu^2} du \sim \int_0^{\infty} e^{ixu^2} du$$

which is almost what we had before, except now we have taken care of the $\tan(x)$ term. So now we take the limits of integration to infinity, and obtain:

$$\frac{1}{2} \int_0^{\infty} e^{ixu^2} du$$

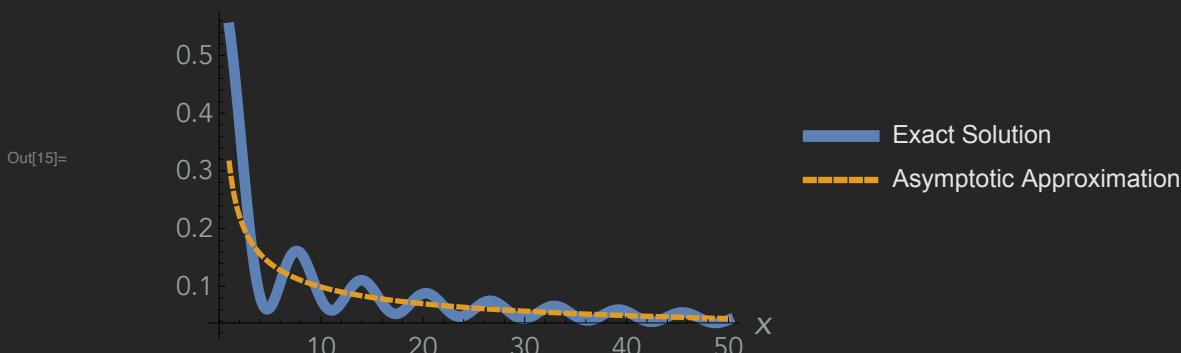
```
In[14]:= approxSol = 1/2 * Integrate[E^(I*x*u^2), {u, 0, \infty}, Assumptions \rightarrow Im[x] > 0]
```

```
Out[14]=  $\frac{\sqrt{\pi}}{4 \sqrt{-i x}}$ 
```

Now we want to check out solution with what he numerical solution computes:

```
In[15]:= Plot[{Re[NIntegrate[(Tan[t]) * E^(I*x*t^4), {t, 0, 1}]], Re[approxSol]}, {x, 1, 50}, PlotRange \rightarrow All, PlotStyle \rightarrow {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]}, AxesLabel \rightarrow {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]}, TicksStyle \rightarrow Directive[FontSize \rightarrow 14], PlotLegends \rightarrow {"Exact Solution", "Asymptotic Approximation"}]
```

Solutions (Re[I(x)])



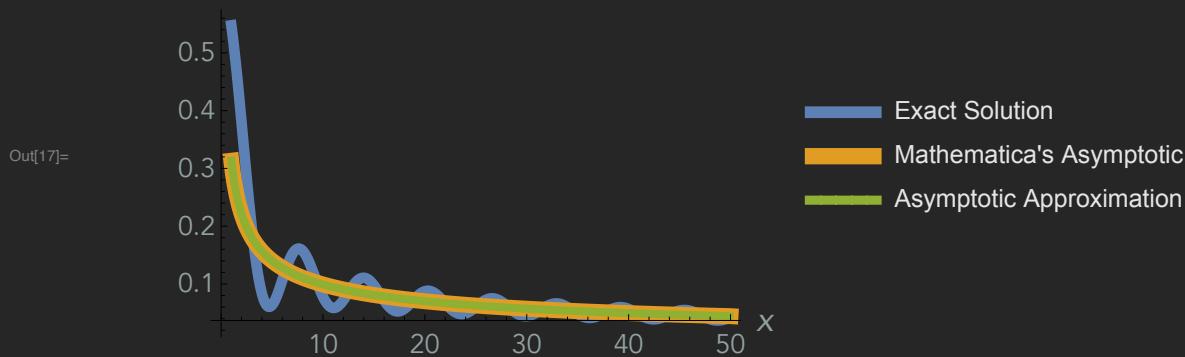
Which is pretty terrible actually... so let's see what Mathematica's solution computes:

```
In[16]:= mathematicaAsymp = AsymptoticIntegrate[(Tan[t]) * E^(I*x*t^4), {t, 0, 1}, x → ∞]
Out[16]=  $\frac{1}{4} e^{\frac{i\pi}{4}} \sqrt{\pi} \sqrt{\frac{1}{x}}$ 
```

comparing all three now:

```
In[17]:= Plot[{Re[NIntegrate[(Tan[t]) * E^(I*x*t^4), {t, 0, 1}]], Re[mathematicaAsymp], Re[approxSol]}, {x, 1, 50}, PlotRange → All, PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Solid, Thickness[0.03]], Directive[Dashed, Thickness[0.015]]}, AxesLabel → {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]}, TicksStyle → Directive[FontSize → 14], PlotLegends → {"Exact Solution", "Mathematica's Asymptotic", "Asymptotic Approximation"}]
```

Solutions ($Re[I(x)]$)



So we do just as bad (or just as well) as Mathematica's asymptotic solve. This is somewhat comforting at least ...

$$(b) \int_{\frac{1}{2}}^2 (1+t) e^{ix(\frac{t^3}{3}-t)} dt$$

Once again we want to use the method of stationary phase for finding the leading behavior of this integral. We have that $\phi(t) = \frac{t^3}{3} - t$, resulting in the stationary points to be at -1 and 1 (since $\phi'(t) = t^2 - 1$), but since we are on the positive axis, we only care for 1.

Following the process of Stationary Phase method, we can rewrite the original integral as:

$$\int_{\frac{1}{2}}^2 (1+t) e^{ix(\frac{t^3}{3}-t)} dt \sim \int_{\frac{1}{2}}^{1-\epsilon} (1+t) e^{ix(\frac{t^3}{3}-t)} dt + \int_{1+\epsilon}^{1+\epsilon} (1+t) e^{ix(\frac{t^3}{3}-t)} dt + \int_{1+\epsilon}^2 (1+t) e^{ix(\frac{t^3}{3}-t)} dt$$

using the Riemann-Lebesgue lemma, we can see that $\int_{\frac{1}{2}}^{1-\epsilon} (1+t) e^{ix(\frac{t^3}{3}-t)} dt$ and $\int_{1+\epsilon}^2 (1+t) e^{ix(\frac{t^3}{3}-t)} dt$ will

tend to 0, so all we are left with now is $\int_{1-\epsilon}^{1+\epsilon} (1+t) e^{ix(\frac{t^3}{3}-t)} dt$. So now we must find $\phi^{(n)} \neq 0$, which

would be the second derivative since $\phi''(t) = 2t \rightarrow \phi''(1) = 2$. So now we expand $\phi(t)$ to get:

$$\phi(t) = \left(\frac{1}{3} - 1\right) + 0 + \frac{2}{2}(t-1)^2 + O(t^3) \sim -\frac{2}{3} + (t-1)^2$$

Similarly, taking $f'(t) = 1 + t$, we take the exact value at the stationary point, i.e. $f(1) = 2$. So now we have:

$$\int_{\frac{1}{2}}^2 (1+t) e^{ix\left(\frac{i^3}{3}-t\right)} dt \sim 2 \int_{1-\epsilon}^{1+\epsilon} e^{ix\left(-\frac{2}{3}+(t-1)^2\right)} dt \sim 2 \int_{-\infty}^{\infty} e^{ix\left(-\frac{2}{3}+(t-1)^2\right)} dt$$

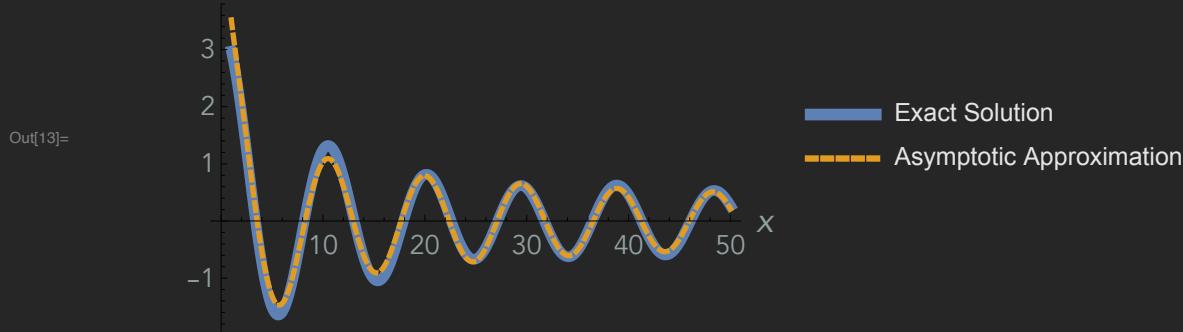
So now we can integrate this value to see what our approximation will yield:

```
approx = 2 * Integrate[E^(I*x*(-2/3 + (t-1)^2)), {t, -∞, ∞}, Assumptions → Im[x] > 0]
Out[11]= 
$$\frac{2 e^{-\frac{2 i x}{3}} \sqrt{\pi}}{\sqrt{-\frac{1}{3} x}}$$

```

```
In[13]:= Plot[{Re[NIntegrate[(1+t)*E^(I*x*((t^3/3)-t)), {t, 1/2, 2}]], Re[approx]}, {x, 1, 50}, PlotRange → All, PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]}, AxesLabel → {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]}, TicksStyle → Directive[FontSize → 14], PlotLegends → {"Exact Solution", "Asymptotic Approximation"}]
```

Solutions (Re[I(x)])



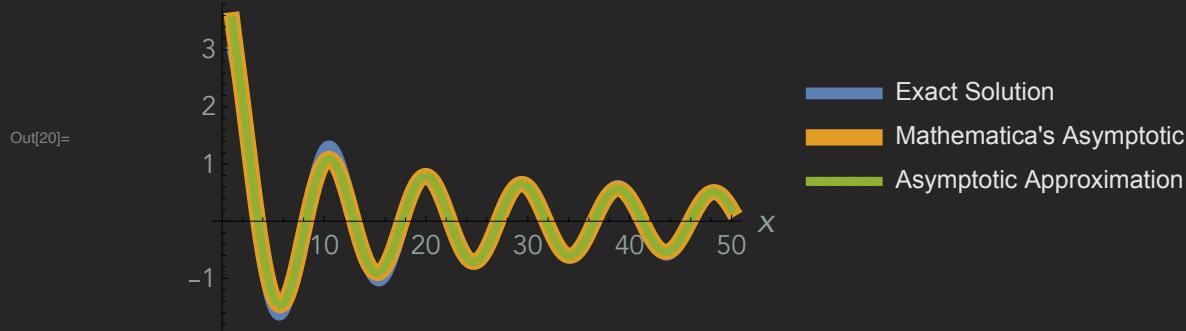
Which shows much better agreement between our asymptotic solution and the exact (numerical) solution. To compare our solutions with the Mathematica's Asymptotic solution:

```
In[18]:= mathematicaAsymp = AsymptoticIntegrate[(1+t)*E^(I*x*((t^3/3)-t)), {t, 1/2, 2}, x → ∞]
Out[18]= 
$$\frac{2 e^{\frac{i \pi}{4}-\frac{2 i x}{3}} \sqrt{\pi}}{\sqrt{x}}$$

```

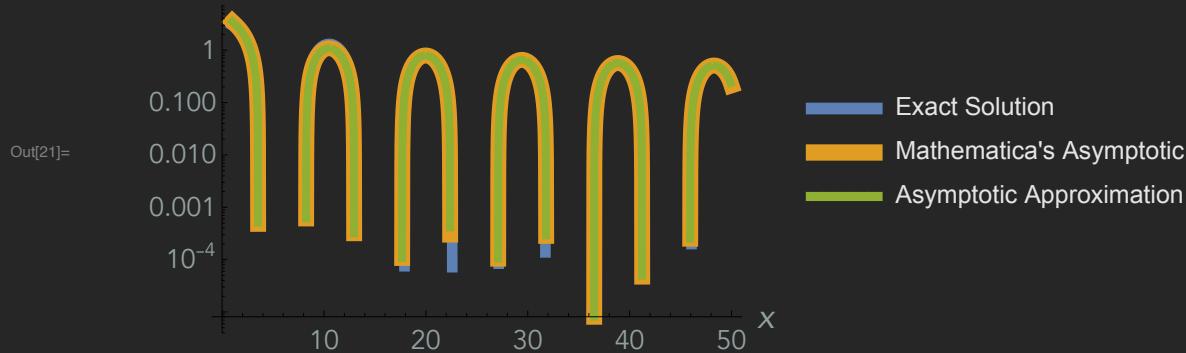
```
In[20]:= Plot[{Re[NIntegrate[(1 + t) * E^(I * x * ((t^3 / 3) - t)), {t, 1/2, 2}]},  
Re[mathematicaAsymp], Re[approx], {x, 1, 50}, PlotRange -> All,  
PlotStyle -> {Directive[Solid, Thickness[0.02]],  
Directive[Solid, Thickness[0.03]], Directive[Dashed, Thickness[0.015]]},  
AxesLabel -> {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]},  
TicksStyle -> Directive[FontSize -> 14], PlotLegends ->  
{"Exact Solution", "Mathematica's Asymptotic", "Asymptotic Approximation"}]
```

Solutions ($\text{Re}[I(x)]$)



```
In[21]:= LogPlot[{Re[NIntegrate[(1 + t) * E^(I * x * ((t^3 / 3) - t)), {t, 1/2, 2}]},  
Re[mathematicaAsymp], Re[approx], {x, 1, 50}, PlotRange -> All,  
PlotStyle -> {Directive[Solid, Thickness[0.02]],  
Directive[Solid, Thickness[0.03]], Directive[Dashed, Thickness[0.015]]},  
AxesLabel -> {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]},  
TicksStyle -> Directive[FontSize -> 14], PlotLegends ->  
{"Exact Solution", "Mathematica's Asymptotic", "Asymptotic Approximation"}]
```

Solutions ($\text{Re}[I(x)]$)



Which show both our solution and Mathematica's solution to match very closely with the numerical solution, which is a good sign!