Math 223: Homework9

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Problem 1

Use second-order perturbation theory to find approximations to the roots of $x^2 + x + 6\epsilon = 0$.

Since our problem is already in the perturbed form, we can use the perturbation theory and let

$$x \sim \sum_{n=0}^{\infty} x_n \epsilon^n, \quad \epsilon \to 0^+$$

Now we replace this in the original problem, and seek the roots using this expansion:

$$\left(\sum_{n=0}^{\infty} x_n \epsilon^n\right)^2 + \left(\sum_{n=0}^{\infty} x_n \epsilon^n\right) + 6\epsilon = 0.$$

so now we use Mathematica to solve this:

```
In[•]:= SeriesCoefficient[
         Series[(Sum[x[n] ϵ^n, {n, 0, 10}])^2 + Sum[x[n] ϵ^n, {n, 0, 10}] + 6 ϵ, {ϵ, 0, 2}], 0]
Out[•]= x[0] + x[0]^2
```

```
In[•]:= Solve[x[0] + x[0]^2 == 0, x[0]]
Out[•]= {{x[0] → -1}, {x[0] → 0}}
```

Now to find the higher order solutions for each one of the roots:

```
In[•]:= SeriesCoefficient[
          Series[(Sum[x[n] ϵ^n, {n, 0, 10}])^2 + Sum[x[n] ϵ^n, {n, 0, 10}] + 6 ϵ, {ϵ, 0, 2}], 1] /. x[0] → 0
Out[•]= 6 + x[1]
```

```
In[•]:= Solve[6 + x[1] == 0, x[1]]
Out[•]= {{x[1] → -6}}
```

so that means that we have:

$$x_1 = 0 - 6\epsilon + O(\epsilon^2)$$

Now to find the higher order root:

```
In[•]:= SeriesCoefficient[
          Series[(Sum[x[n] ϵ^n, {n, 0, 10}])^2 + Sum[x[n] ϵ^n, {n, 0, 10}] + 6 ϵ, {ϵ, 0, 2}],
          2] /. {x[0] → 0, x[1] → -6}
Out[•]= 36 + x[2]
```

In[•]:=
```
Solve[36 + x[2] == 0, x[2]]
```
Out[•]=
```
{{x[2] → -36}}
```

so we have the first root to be:

$$x_1 = 0 - 6\,\epsilon - 36\,\epsilon + O\left(\epsilon^2\right)$$

similarly for the other root:

In[•]:=
```
SeriesCoefficient[
  Series[(Sum[x[n] ϵ^n, {n, 0, 10}])^2 + Sum[x[n] ϵ^n, {n, 0, 10}] + 6 ϵ, {ϵ, 0, 2}], 1] /. x[0] → -1
```
Out[•]=
```
6 - x[1]
```

In[•]:=
```
Solve[6 - x[1] == 0, x[1]]
```
Out[•]=
```
{{x[1] → 6}}
```

so we have the second root to be:

$$x_2 = -1 + 6\,\epsilon + O\left(\epsilon^2\right)$$

Now to go one order higher:

In[•]:=
```
SeriesCoefficient[
  Series[(Sum[x[n] ϵ^n, {n, 0, 10}])^2 + Sum[x[n] ϵ^n, {n, 0, 10}] + 6 ϵ, {ϵ, 0, 2}],
  2] /. {x[0] → -1, x[1] → 6}
```
Out[•]=
```
36 - x[2]
```

In[•]:=
```
Solve[36 - x[2] == 0, x[2]]
```
Out[•]=
```
{{x[2] → 36}}
```

so we get the second solution to the second order to be:

$$x_2 = -1 + 6\,\epsilon + 36\,\epsilon + O\left(\epsilon^3\right)$$

Problem 2

Compute all of the coefficients in the perturbation series solution to the initial-value problem

$$y' = y + \epsilon\, x\, y \qquad\qquad y(0) = 1,$$

Show that the series converges for all values of ϵ. Also, compute the perturbation series indirectly by expanding the explicit exact solution in powers of ϵ.

We will try to find the solution in the limit as $\epsilon \to 0^+$, and use the notion of perturbation expansion, that is we take:

$$y(x) \sim \sum_{n=0}^{\infty} y_n(x)\,\epsilon^n, \quad \epsilon \to 0^+.$$

We think that this must be a regular perturbation problem, if the perturbation series is a power series in ϵ having a nonvanishing radius of convergence. In other words, the exact solution for small but nonzero $|\epsilon|$ smoothly approaches the unperturbed (or zeroth-order) solution as $\epsilon \to 0$. So we expand the problem in terms of the perturbation, and substituting the perturbation expansion in the ODE gives us:

$$\left(y_0' + \epsilon y_1' + \epsilon^2 y_2' + \epsilon^3 y_3' + \cdots\right) - \left(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right) - \epsilon x \left(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right) = 0$$

to find the regular expression for the initial condition, we have:

$$\left(y_0(0) + y_1(0) + \epsilon^2 y_2(0) + \cdots\right) = 1.$$

which allows us to construct a system of equations for specific orders. So to find the first order approx - imation to this perturbed system, we can solve for:

$$O(1): y_0' - y_0 = 0,\ y_0(0) = 1.$$

In[•]:=
```
DSolve[{y₀'[x] - y₀[x] == 0, y₀[0] == 1}, y₀[x], x]
```

Out[•]= $\{\{y_0[x] \to e^x\}\}$

Next we find the order epsilon approximation:
$O(\epsilon): y_1' - y_1 = x y_0,\ y_1(1) = 0.$

In[•]:=
```
DSolve[{y₁'[x] - y₁[x] == x ⅇ^x, y₁[0] == 0}, y₁[x], x]
```

Out[•]= $\left\{\left\{y_1[x] \to \frac{e^x x^2}{2}\right\}\right\}$

therefore we can write the asymptotic approximation as:

$$y \sim e^x + \frac{\epsilon}{2} x^2 e^x + O\left(\epsilon^2\right)$$

Now to compute a second order accurate solution, we can follow the same procedure to obtain:

$$O\left(\epsilon^2\right): y_2' - y_2 = x y_1,\ y_2(0) = 0.$$

In[•]:=
```
DSolve[{y₂'[x] - y₂[x] == x (ⅇ^x x^2)/2, y₂[0] == 0}, y₂[x], x]
```

Out[•]= $\left\{\left\{y_2[x] \to \frac{e^x x^4}{8}\right\}\right\}$

so our second order accurate solution becomes:

$$y \sim e^x + \frac{\epsilon}{2} x^2 e^x + \frac{\epsilon^2}{8} x^4 e^x + O\left(\epsilon^3\right)$$

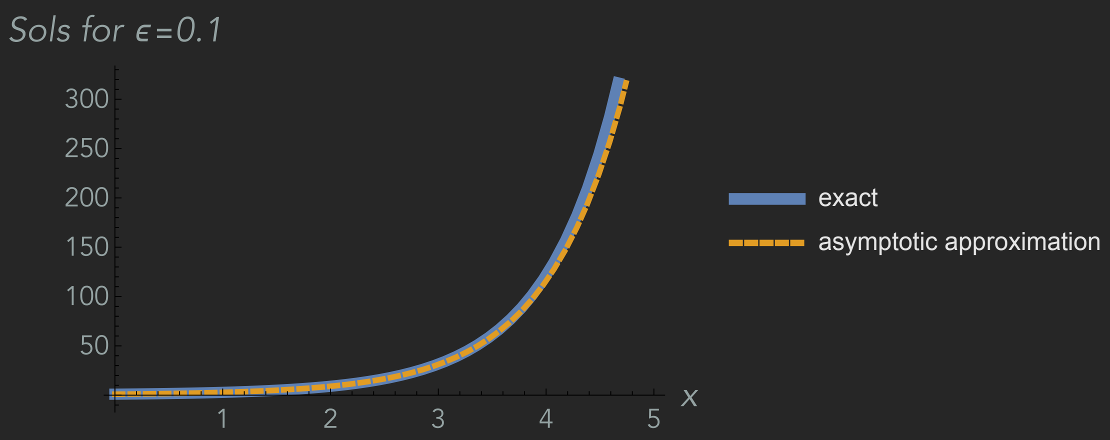
Now to check the validity of our solution:

In[•]:=
```
DSolve[{y'[x] - y[x] - ϵ * x * y[x] == 0, y[0] == 1}, y[x], x]
```

Out[•]= $\left\{\left\{y[x] \to e^{x + \frac{x^2 \epsilon}{2}}\right\}\right\}$

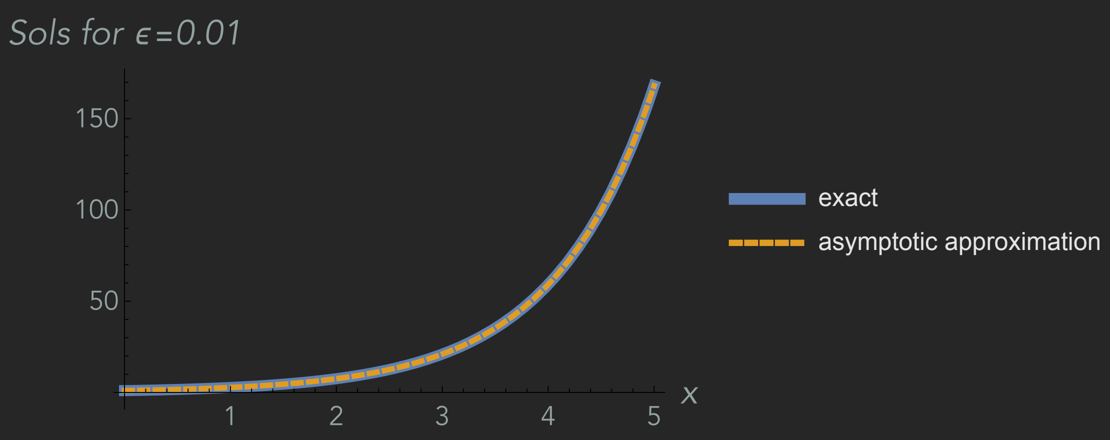
In[ ]:=
```
Plot[{e^(x + x^2 ϵ/2) /. ϵ → 0.1, e^x + ϵ * 1/2 * e^x * x^2 + ϵ^2 (e^x x^4)/8 /. ϵ → 0.1}, {x, 0, 5},
 PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["Sols for ϵ=0.1", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"exact", "asymptotic approximation"}]
```

Out[ ]=



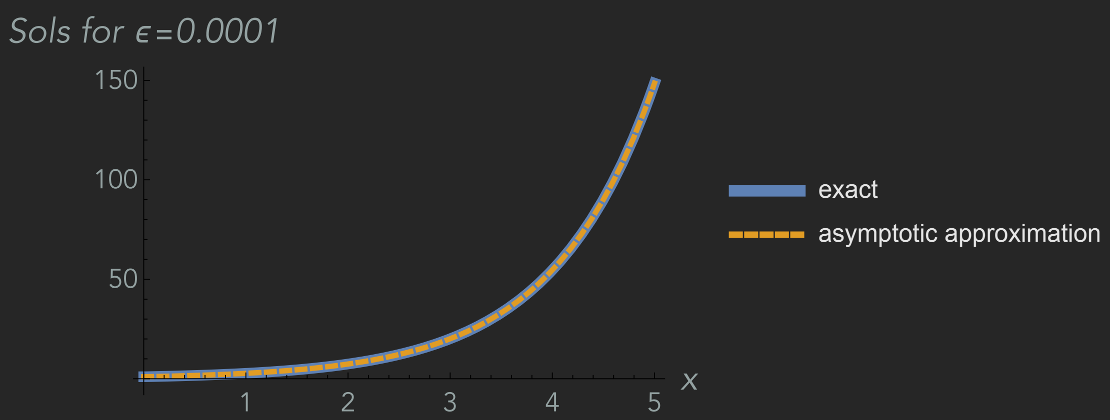
In[ ]:=
```
Plot[{e^(x + x^2 ϵ/2) /. ϵ → 0.01, e^x + ϵ * 1/2 * e^x * x^2 + ϵ^2 (e^x x^4)/8 /. ϵ → 0.01}, {x, 0, 5},
 PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["Sols for ϵ=0.01", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"exact", "asymptotic approximation"}]
```

Out[ ]=



```
Plot[{e^(x + x^2 ϵ/2) /. ϵ → 0.0001, e^x + ϵ * 1/2 * e^x * x^2 + ϵ^2 (e^x x^4)/8 /. ϵ → 0.0001}, {x, 0, 5},
 PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["Sols for ϵ=0.0001", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"exact", "asymptotic approximation"}]
```

Out[•]=



Our approximation seems very descent, given that we have only calculated the approximation up to first order! As shown above, as $\epsilon \to 0$, we can see that our approximation and the exact solution match extremely well! So this concludes finding the solution in perturbed powers of ϵ.

The series that we computed clearly converges for all values of ϵ (using the ratio test), since our series will have the form:

$$y = e^x + \sum_{n=1}^{\infty} a_n \epsilon^n e^x x^{2n}$$

where $a_n = 2n(a_{n-1})$ with $a_0 = 1$.

Now for the second part of the problem, to expand the exact solution in ϵ, we have:

In[•]:= `Series[e^(x + x^2 ϵ/2), {ϵ, 0, 5}]`

Out[•]= $e^x + \frac{1}{2} e^x x^2 \epsilon + \frac{1}{8} e^x x^4 \epsilon^2 + \frac{1}{48} e^x x^6 \epsilon^3 + \frac{1}{384} e^x x^8 \epsilon^4 + \frac{e^x x^{10} \epsilon^5}{3840} + O[\epsilon]^6$

which matches what we found using the perturbation theory, so this again validates our approximate solution we found in the first part.

In[•]:= `Series[e^(x^2 ϵ/2), {ϵ, 0, 5}]`

Out[•]= $1 + \frac{x^2 \epsilon}{2} + \frac{x^4 \epsilon^2}{8} + \frac{x^6 \epsilon^3}{48} + \frac{x^8 \epsilon^4}{384} + \frac{x^{10} \epsilon^5}{3840} + O[\epsilon]^6$

Problem 3

Show that the solution to the initial-value problem

$$y'' + y + \epsilon y = 0 \qquad y(0) = 1, \qquad y'(0) = 0,$$

remains bounded for all real x. Obtain a first-order perturbative approximation to y (x) and and show that it is unbounded as $x \to \infty$. Conclude that the first order approximation of is valid only for $|x| < O(\epsilon^{-1})$.

Following what we did before, we will write a perturbation series such that $y(x) \sim \sum_{n=0}^{\infty} y_n(x)\,\epsilon^n$, $\epsilon \to 0^+$, so we can rewrite the ODE with this substitution to be:

$$\left(y_0'' + \epsilon y_1'' + \epsilon^2 y_2' + \epsilon^3 y_3' + \cdots\right) - \left(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right) - \epsilon\left(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right) = 0$$

So to find the solution for the unperturbed problem we have:

```
In[2]:= DSolve[{y''[x] + y[x] == 0, y[0] == 1, y'[0] == 0}, y[x], x]
Out[2]= {{y[x] → Cos[x]}}
```

So now that we have found the solution to the unperturbed problem, we can find the solution to the order ϵ problem:

$$O(\epsilon): y_1'' + y_1 = -y_0 = -\cos(x)$$

```
In[18]:= DSolve[{y1''[x] + y1[x] == -Cos[x], y1[0] == 0, y1'[0] == 0}, y1[x], x]
```

Out[18]= $\left\{\left\{y1[x] \to \frac{1}{4}\left(2\,\mathrm{Cos}[x] - 2\,\mathrm{Cos}[x]^3 - 2\,x\,\mathrm{Sin}[x] - \mathrm{Sin}[x]\,\mathrm{Sin}[2\,x]\right)\right\}\right\}$

so we get the first order solution to be:

$$y = \cos(x) + \frac{\epsilon}{4}\left(2\cos(x) - 2\cos(x)^3 - 2\,x\sin(x) - \sin(x)\sin(2\,x)\right) + O\left(\epsilon^2\right)$$

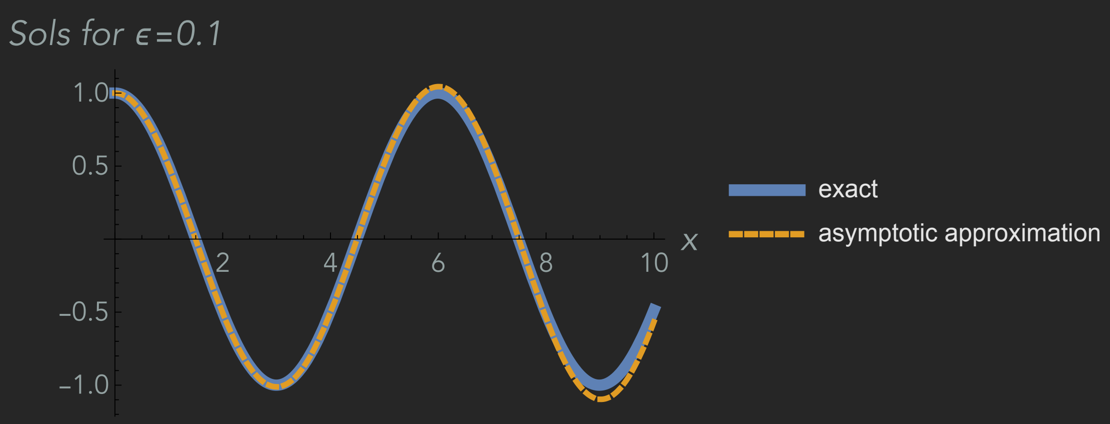
Now to find the exact solution for comparison purposes:

```
In[6]:= DSolve[{y''[x] + y[x] + ϵ * y[x] == 0, y[0] == 1, y'[0] == 0}, y[x], x]
```

Out[6]= $\left\{\left\{y[x] \to \frac{1}{2}\, e^{-x\sqrt{-1-\epsilon}}\left(1 + e^{2\,x\sqrt{-1-\epsilon}}\right)\right\}\right\}$

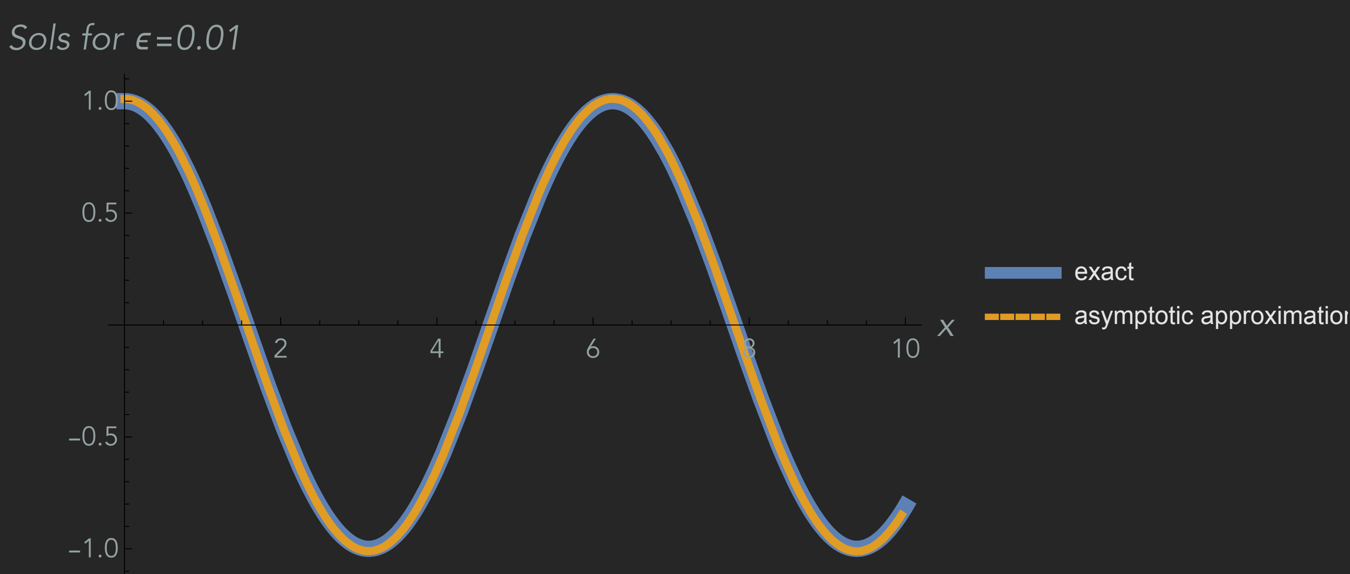
In[19]:=
```
Plot[{1/2 e^(-x Sqrt[-1-ε]) (1 + e^(2 x Sqrt[-1-ε])) /. ε → 0.1,
   Cos[x] + ε/4 (2 Cos[x] - 2 Cos[x]^3 - 2 x Sin[x] - Sin[x] Sin[2 x]) /. ε → 0.1}, {x, 0, 10},
  PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
  AxesLabel → {Style["x", Italic, 18], Style["Sols for ε=0.1", Italic, 18]},
  TicksStyle → Directive[FontSize → 14],
  PlotLegends → {"exact", "asymptotic approximation"}]
```

Out[19]=



In[20]:=
```
Plot[{1/2 e^(-x Sqrt[-1-ε]) (1 + e^(2 x Sqrt[-1-ε])) /. ε → 0.01,
   Cos[x] + ε/4 (6 Cos[x] - 2 Cos[x]^3 - 2 x Sin[x] - Sin[x] Sin[2 x]) /. ε → 0.01}, {x, 0, 10},
  PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
  AxesLabel → {Style["x", Italic, 18], Style["Sols for ε=0.01", Italic, 18]},
  TicksStyle → Directive[FontSize → 14],
  PlotLegends → {"exact", "asymptotic approximation"}]
```

Out[20]=

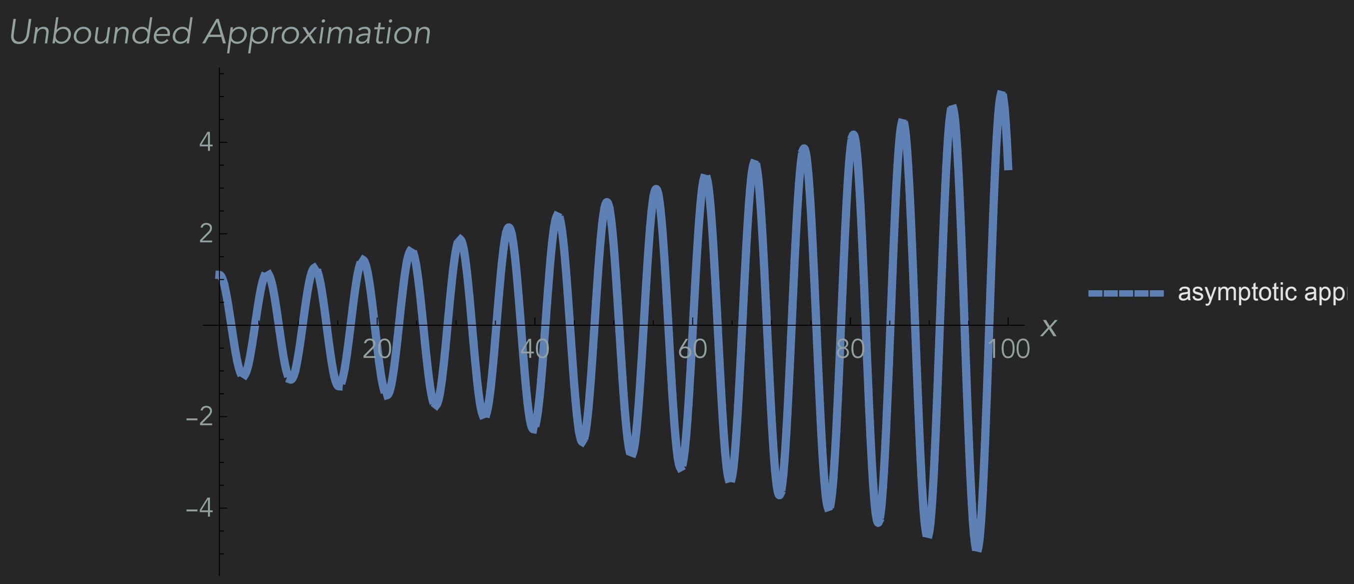


this shows close agreement to between the two solutions, specially given that we only obtained a first order approximate.

To show that our approximate solution is unbounded, we can just plot the solution for a large value of x:

In[15]:=
```
Plot[{Cos[x] + ϵ/4 (6 Cos[x] - 2 Cos[x]^3 - 2 x Sin[x] - Sin[x] Sin[2 x]) /. ϵ → 0.1}, {x, 0, 100},
 PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["Unbounded Approximation", Italic, 18]},
 TicksStyle → Directive[FontSize → 14], PlotLegends → {"asymptotic approximation"}]
```

Out[15]=



which clearly shows that our solution is unbounded as $x \to \infty$. Another way to see this is that we have the term $\epsilon(2x\sin(x))$ in our approximation, which means that after a certain point as x goes to infinity, we will have this unbounded term in the approximation.

Now to show that our first-order solution is valid for when $|x| \propto \frac{1}{\epsilon}$, we can make the following deduction:

$$|y| = \left|\cos(x) + \frac{\epsilon}{4}\left(6\cos(x) - 2\cos(x)^3 - 2x\sin(x) - \sin(x)\sin(2x)\right) + O\left(\epsilon^2\right)\right| \leq \left|\cos(x)\right| + \left| frac\epsilon 4\left(6\,|\cos(x)| - 2\,|\cos(x)^3| - 2\,|x|\,|\sin(x)| - |\sin(x)|\,|\sin(2x)|\right) + O\left(\epsilon^2\right)\right| \leq |1| + \frac{\epsilon}{4}(6 - 2 - 2\,|x| - 1) + O\left(\epsilon^2\right)$$

since we can bound $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$. So now we can see that we have a term that is of the order $\epsilon|x|$. So in order for our approximation to be valid, the solution must remain bounded which means that $|x| \propto \frac{1}{\epsilon}$, or $|x| = O\left(\frac{1}{\epsilon}\right)$ to avoid blowing up.