Math 223: Homework3

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Problem 1

Find the behavior as $x \to 0^+$ for the following equations:

(a) $x^4 y^{(3)} = y$

We let the ansatz to be y(t) = $e^{S(t)}$, which means that

$y'(t) = S' e^{S(t)}$

$y''(t) = S'' e^{S(t)} + (S')^2 e^{S(t)}$

$y'''(t) = S''' e^{S(t)} + 3 S'' S' e^{S(t)} + (S')^3 e^{S(t)}$

To verify the derivatives we found by hand:

```
In[•]:= x^4 y'''[x] == y[x] /. y → Function[x, e^S[x]] // FullSimplify
```

Out[•]= $e^{S[x]}\, x^4 \left(S'[x]^3 + 3\, S'[x]\, S''[x] + S^{(3)}[x]\right) == e^{S[x]}$

Due to the existence of the irregular singular point at x = 0, we make the assumption that $S'' \ll (S')^2$, which means that now we can solve for $S'(x)$ as:

```
In[•]:= DSolve[ x^4 (S'[x]^3) == 1, S[x], x] // Simplify
```

Out[•]= $\left\{\left\{S[x] \to -\frac{3}{x^{1/3}} + c_1\right\}, \left\{S[x] \to \frac{3\,(-1)^{1/3}}{x^{1/3}} + c_1\right\}, \left\{S[x] \to -\frac{3\,(-1)^{2/3}}{x^{1/3}} + c_1\right\}\right\}$

To check if our assumption was correct:

```
In[•]:= sDoublePrime = D[-3/x^(1/3), {x, 2}]
```

Out[•]= $-\frac{4}{3\, x^{7/3}}$

```
In[•]:= sPrimeSquared = (D[-3/x^(1/3), {x, 1}])^2
```

Out[•]= $\frac{1}{x^{8/3}}$

$-\frac{4}{3\, x^{7/3}} \ll \frac{1}{x^{8/3}}$ as $x \to 0^+$, our assumption about S'' is correct!

So now we have that $S(x) \sim x^{-1/3}$, so we can write the controlling factor as

$y = e^{3 x^{-1/3}}$

for $x \to 0^+$. Now to find the leading behavior, we take $S(x) = 3 x^{-1/3} + c(x)$:

In[∘]:= $x^4 y'''[x] == y[x]$ /. y → Function$[x, e^{S[x]}]$ // FullSimplify

Out[∘]= $e^{S[x]} x^4 \left(S'[x]^3 + 3 S'[x] S''[x] + S^{(3)}[x]\right) == e^{S[x]}$

In[∘]:= Solve$\left[x^4 y'''[x] == y[x] \text{ /. } y \to \text{Function}\left[x, e^{3 x^{-1/3}+c[x]}\right], c''[x]\right]$ // FullSimplify // Expand

Out[∘]= $$\left\{\left\{c''[x] \to \frac{2}{3\left(-x^{8/3}+x^4 c'[x]\right)} + \frac{4 x^{1/3}}{3\left(-x^{8/3}+x^4 c'[x]\right)} + \frac{28 x^{2/3}}{27\left(-x^{8/3}+x^4 c'[x]\right)} - \frac{x^{4/3} c'[x]}{-x^{8/3}+x^4 c'[x]} - \frac{4 x^{5/3} c'[x]}{3\left(-x^{8/3}+x^4 c'[x]\right)} + \frac{x^{8/3} c'[x]^2}{-x^{8/3}+x^4 c'[x]} - \frac{x^4 c'[x]^3}{3\left(-x^{8/3}+x^4 c'[x]\right)} - \frac{x^4 c^{(3)}[x]}{3\left(-x^{8/3}+x^4 c'[x]\right)}\right\}\right\}$$

We assume that c'' and higher terms are negligible, so simplifying the above expression just for the manageable terms with c(x), we obtain:

$4 - 3 x c'[x]^3$, which we can solve this as

In[∘]:= DSolve[4 - 3 x * c'[x] == 0, c[x], x] // FullSimplify // Expand

Out[∘]= $\left\{\left\{c[x] \to c_1 + \frac{4 \operatorname{Log}[x]}{3}\right\}\right\}$

So now this gives us the solution:

$$y \sim \alpha x^{4/3} e^{x^{-1/3}}$$

for α some constant, as $x \to 0^+$.

(b) $x^6 y'' = e^x y$

Following the same procedure as before, we can obtain:

In[2]:= $x^6 y''[x] == e^x y[x]$ /. y → Function$[x, e^{S[x]}]$ // FullSimplify

Out[2]= $e^{S[x]} x^6 \left(S'[x]^2 + S''[x]\right) == e^{x+S[x]}$

Solving for $S'(x)$:

In[∘]:= Solve$\left[x^6 \left(S'[x]^2\right) == e^x, S'[x]\right]$ // FullSimplify

Out[∘]= $\left\{\left\{S'[x] \to -\frac{e^{x/2}}{x^3}\right\}, \left\{S'[x] \to \frac{e^{x/2}}{x^3}\right\}\right\}$

To check to see if our assumption about $S''(x)$ is correct:

In[∘]:= sDoublePrime = D$\left[-\frac{e^{x/2}}{x^3}, x\right]$

Out[∘]= $\frac{3 e^{x/2}}{x^4} - \frac{e^{x/2}}{2 x^3}$

Out[∘]= $\frac{3 e^{x/2}}{x^4} - \frac{e^{x/2}}{2 x^3}$

```
In[•]:= sPrimeSquared = (-e^(x/2)/x^3)^2
```

Out[•]= $\frac{e^{x}}{x^{6}}$

```
In[•]:= Limit[sPrimeSquared / sDoublePrime, x → 0]
```

Out[•]= ∞

So it does check out that $S'' << (S')^2$, but only for the above solution (the other one will go to -∞).

So now to see what polynomial we could use for which means that which we can expand to get the dominant terms:

```
In[1]:= Series[-e^(x/2) / x^3, {x, 0, 2}]
```

Out[1]= $-\frac{1}{x^3}-\frac{1}{2x^2}-\frac{1}{8x}-\frac{1}{48}-\frac{x}{384}-\frac{x^2}{3840}+O[x]^3$

taking the first terms, we can write the solution for $S(x)$ as

$$S'(x) \sim \frac{-1}{x^3},\ S(x)' \sim \frac{1}{x^3}$$

which means that $S(x) = \pm \frac{1}{2x^2}$. This yields controlling factors being:

$$y \sim e^{2x^{-2}}$$

To determine the leading behavior, we can now let $S(x) = x^{-2} + c(x)$, with $c(x) \ll x^{-2}$. So now we solve the equation for this:

```
In[•]:= x^6 y''[x] == e^x y[x] /. y → Function[x, e^(x^-2+c[x])] // FullSimplify
```

Out[•]= $e^{\frac{1}{x^2}+c[x]}\left(-4+e^{x}-6x^{2}+4x^{3}c'[x]-x^{6}c'[x]^{2}-x^{6}c''[x]\right)==0$


```
In[•]:= x^6 y''[x] == e^x y[x] /. y → Function[x, e^(x^-2+c[x])] // FullSimplify
```

Out[•]= $e^{\frac{1}{x^2}+c[x]}\left(-4+e^{x}-6x^{2}+4x^{3}c'[x]-x^{6}c'[x]^{2}-x^{6}c''[x]\right)==0$

Now assuming that $c''(x) \ll (c'(x))^2$, we end up with $x^3 c'[x] - x^6 c'[x]^2$ as the leading manageable terms, To see how this is:

```
In[6]:= DSolve[3 x^2 - 2 (x^3 c'[x]) == 0, c[x], x]
```

Out[6]= $\left\{\left\{c[x]\to c_1+\frac{3\,\mathrm{Log}[x]}{2}\right\}\right\}$

Clearly, we can not have the first solution (since the assumption will not hold), so we have:

$$y \sim \alpha x^{\frac{3}{2}}\, e^{(-2x^{-2})} = \alpha x^{\frac{3}{2}}\, e^{(-2x^{-2})}$$

(c) $x^4 y'' - x^2 y' + \frac{1}{4} y = 0$

As before, we set the ansatz to be $y(t) = e^{S(t)}$ and using the derivatives found above, we have:

```
In[•]:= x^4 y''[x] - x^2 y'[x] + 1/4 y[x] == 0 /. y → Function[x, e^S[x]] // FullSimplify
```

Out[•]= $e^{S[x]}\left(\left(1-2x^{2}S'[x]\right)^{2}+4x^{4}S''[x]\right)==0$

once again, assuming that $S'' \ll (S')^2$, so we now we have:

In[∘]:= `Solve[(1 - 2 x^2 S'[x])^2 == 0, S'[x]]`

Out[∘]= $\left\{\left\{S'[x] \to \frac{1}{2 x^2}\right\}, \left\{S'[x] \to \frac{1}{2 x^2}\right\}\right\}$

Now we get the controlling factor to be:

$$y \sim e^{(1/2)\, x^{-1}}$$

So now to find the leading term, as before we will have:

In[∘]:= $x^4\, y''[x] - x^2\, y'[x] + \frac{1}{4}\, y[x] == 0$ /.
 $y \to \text{Function}\left[x, e^{-(1/2)*(x^{-1})+c[x]}\right]$ // FullSimplify

Out[∘]= $e^{-\frac{1}{2x}+c[x]}\, x\left(-1 + x^3\left(c'[x]^2 + c''[x]\right)\right) == 0$

Out[∘]= $e^{-\frac{1}{2x}+c[x]}\, x\left(-1 + x^3\left(c'[x]^2 + c''[x]\right)\right) == 0$

So we essentially want to find the order for $-1 + x^3\left(c'[x]^2 + c''[x]\right)$, which results in $c(x) \sim \sqrt{x}$ .To see how we solved for this:

In[∘]:= $\text{Solve}\left[x^4\, y''[x] - x^2\, y'[x] + \frac{1}{4}\, y[x] == 0 \text{ /. } y \to \text{Function}\left[x, e^{-(1/2)*(x^{-1})+c[x]}\right], c''[x]\right]$ //
 FullSimplify // Expand

Out[∘]= $\left\{\left\{c''[x] \to \frac{1}{x^3} - c'[x]^2\right\}\right\}$

In[∘]:= $\text{DSolve}\left[\frac{1}{x^3} - (c'[x])^2 == 0, c[x], x\right]$

Out[∘]= $\left\{\left\{c[x] \to -\frac{2}{\sqrt{x}} + c_1\right\}, \left\{c[x] \to \frac{2}{\sqrt{x}} + c_1\right\}\right\}$

So now we put everything together and get:

$$y \sim \alpha\, e^{\frac{\pm 2}{\sqrt{x}} + \frac{1}{-2\, x}}$$

However, I was expecting to get a dominating behavior term (like a polynomial in terms of x). But, this is not too bad since the term c(x) grows slower than S(x), so S(x) can still be the controlling factor.