Math 223: Homework 8

Ali Heydari

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Problem 1

Find the asymptotic behavior of

$$S(x) = \int_0^2 (\sin t + t)\, e^{ixt}\, dt$$

as $x \to \infty$ up to terms involving $\frac{1}{x^2}$.

Since $\phi(t)$ is monotonic on the interval [0,2], we can find the solution through iterative; specifically, $I(x) = \int_a^b f(t)\, e^{ixt}\, dt \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{(ix)^{n+1}} \left[f^{(n)}(b)\, e^{ixb} - f^{(n)}(a)\, e^{ixa} \right], \quad x \to +\infty$, where $f(t) = \sin(t) + t$. Going up to two terms, we have:

$$I(x) \sim \frac{1}{ix}\left[\sin(2)\, e^{2ix} + 2\, e^{2ix}\right] + \frac{1}{(ix)^2}\left[\cos(2)\, e^{2ix} + 2\, e^{2ix} - \cos(0)\, e^{ix0}\right] + \sum_{n=3}^{\infty} \frac{(-1)^n}{(ix)^{n+1}}\left[f^{(n)}(2)\, e^{ix2} - f^{(n)}(0)\, e^{ix0}\right]$$

$$I(x) \sim \frac{1}{ix} \sin(2)\, e^{2ix} - 2\, e^{2ix} - \frac{1}{x^2}\left[(\cos(2) + 1)\, e^{2ix} + 2\right]$$

So now we save this solution for future use:

```
In[]:= approxSol[x_] = - (I (2 + Sin[2]) E^(2 I x))/x + E^(2 I x)/x^2 (1 + Cos[2]) - 2/x^2
```

$$\text{Out[]=} \quad -\frac{2}{x^2} + \frac{e^{2ix}(1 + \text{Cos}[2])}{x^2} - \frac{i\, e^{2ix}(2 + \text{Sin}[2])}{x}$$

Now to compare our solution with the exact solution:

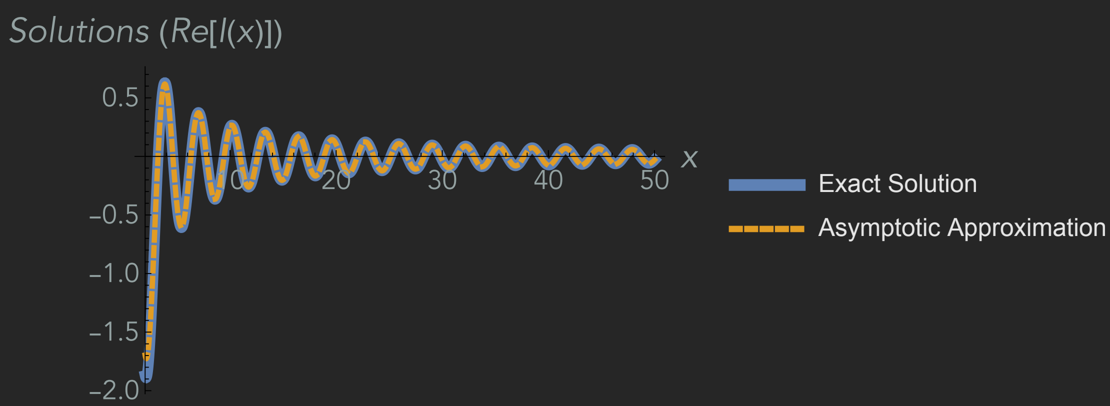
```
In[]:= exactSol[x_] = Integrate[Sin[t] * E^(I * x * t) + t * E^(I * x * t), {t, 0, 2}]
```

$$\text{Out[]=} \quad \frac{-1 + e^{2ix}(1 - 2ix)}{x^2} + \frac{-1 + e^{2ix}(\text{Cos}[2] - i\, x\, \text{Sin}[2])}{-1 + x^2}$$

Resulting in:

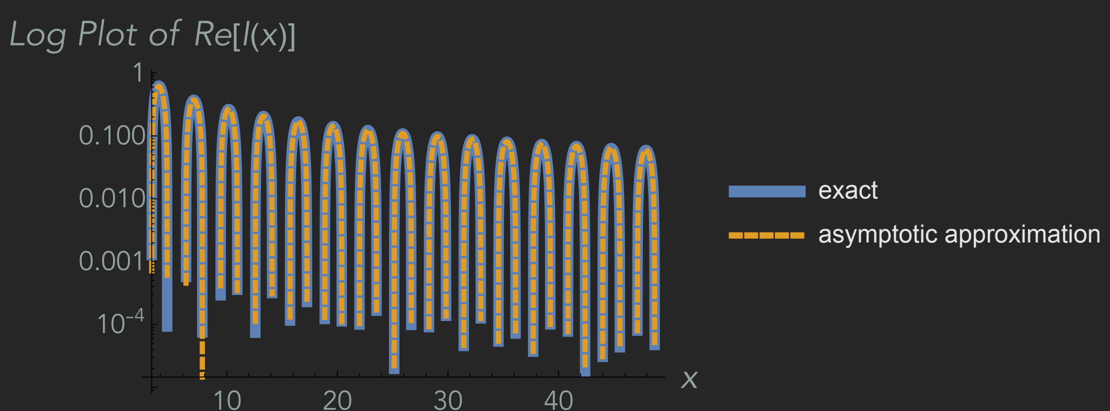
In[]:=
```
Plot[{Re[exactSol[x]], Re[approxSol[x]]}, {x, 2, 50}, PlotRange → All,
 PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"Exact Solution", "Asymptotic Approximation"}]
```

Out[]=



In[]:=
```
LogPlot[{Re[exactSol[x]], Re[approxSol[x]]}, {x, 2, 50}, PlotRange → All,
 PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["Log Plot of Re[I(x)]", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"exact", "asymptotic approximation"}]
```

Out[]=



these plots show great agreement as $x \to \infty$, as desired. This is a good verification that our approach worked well and that we have characterized the asymptotic behavior of the solution well.

Problem 2

Use the method of stationary phase to find the leading behaviors of the following integrals as $x \to \infty$.

(a) $$\int_0^1 \tan(t)\, e^{i x t^4}\, dt$$

We know that the method of stationary phase will give us the leading asymptotic behavior of generalized Fourier integrals that have stationary points, i.e. given $\int_a^b f(t)\, e^{i x \phi(t)}\, dt$, $\phi' = 0$ at some point in the interval. We can also see that $\phi'(0) = \phi''(0) = \phi'''(0)$, but $\phi^{(4)}(0) = 24 \neq 0$. Knowing this, we can see that the stationary point of $\phi(t) = t^4$ will be at $t = 0$, since $\phi'(0) = 0$. However, we want to do a substitu -

tion for the change of variables, so that we can actually perform the method of stationary phase. That is:

let $u = t^2 \Longrightarrow d t = \frac{1}{2t} d u$ and $t = \sqrt{u}$. So now we rewrite the integral as:

$$\int_0^1 \tan(t)\, e^{i x t^4}\, d t = \frac{1}{2} \int_0^1 \frac{\tan\left(\sqrt{u}\right)}{\sqrt{u}} e^{i x u^2}\, d s$$

Now we take $f(t) = \frac{\tan\left(\sqrt{u}\right)}{\sqrt{u}}$, which means that the Taylor expansion of this function will have a leading constant plus polynomials in terms of s. That is:

In[6]:=
```
Series[Tan[Sqrt[u]] / Sqrt[u], {u, 0, 5}]
```

Out[6]= $1 + \frac{u}{3} + \frac{2 u^2}{15} + \frac{17 u^3}{315} + \frac{62 u^4}{2835} + \frac{1382 u^5}{155925} + O[u]^{11/2}$

$$\int_0^1 \frac{\tan\left(\sqrt{u}\right)}{\sqrt{u}} e^{i x u^2}\, d u \sim \int_0^{0+\epsilon} e^{i x \left(\frac{2}{2}(u-0)^2\right)}\, d u = \int_0^{0+\epsilon} e^{i x u^2}\, d u \sim \int_0^{\infty} e^{i x u^2}\, d u$$

which is almost what we had before, except now we have taken care of the tan (x) term. So now we take the limits of integration to infinity, and obtain:

$$\frac{1}{2} \int_0^{\infty} e^{i x u^2}\, d t$$

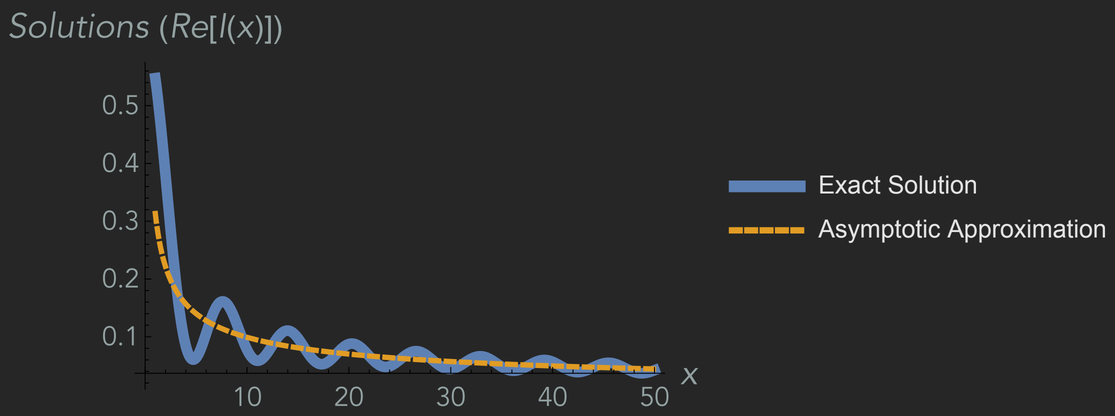
In[14]:=
```
approxSol = 1 / 2 * Integrate[E^(I * x * u^2), {u, 0, ∞}, Assumptions → Im[x] > 0]
```

Out[14]= $\frac{\sqrt{\pi}}{4 \sqrt{-i\, x}}$

Now we want to check out solution with what he numerical solution computes:

In[15]:=
```
Plot[{Re[NIntegrate[(Tan[t]) * E^(I * x * t^4), {t, 0, 1}]], Re[approxSol]},
 {x, 1, 50}, PlotRange → All,
 PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"Exact Solution", "Asymptotic Approximation"}]
```

Out[15]=



Which is pretty terrible actually... so let's see what Mathematica's solution computes:

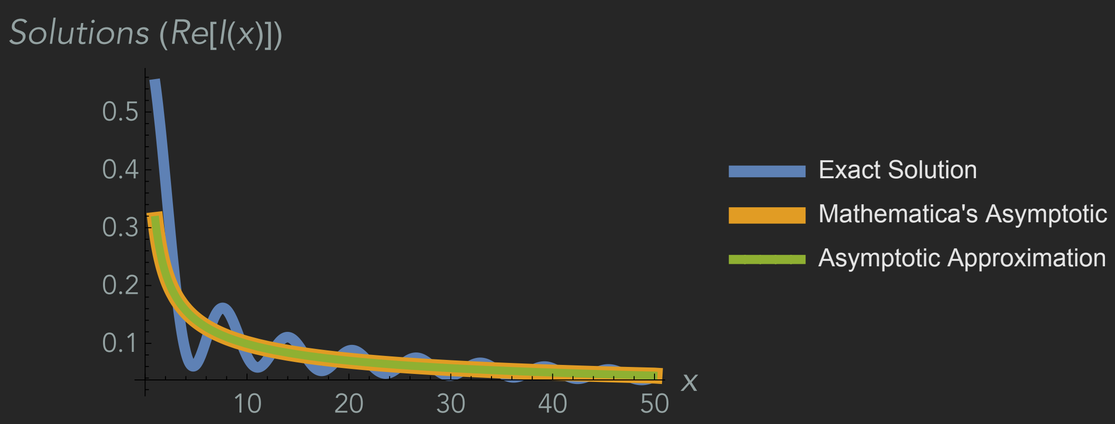
In[16]:=
```
mathematicaAsymp = AsymptoticIntegrate[(Tan[t]) * E^(I * x * t^4), {t, 0, 1}, x → ∞]
```

Out[16]= $\frac{1}{4}\,e^{\frac{i\pi}{4}}\sqrt{\pi}\sqrt{\frac{1}{x}}$

comparing all three now:

In[17]:=
```
Plot[{Re[NIntegrate[(Tan[t]) * E^(I * x * t^4), {t, 0, 1}]],
   Re[mathematicaAsymp], Re[approxSol]}, {x, 1, 50}, PlotRange → All,
  PlotStyle → {Directive[Solid, Thickness[0.02]],
    Directive[Solid, Thickness[0.03]], Directive[Dashed, Thickness[0.015]]},
  AxesLabel → {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]},
  TicksStyle → Directive[FontSize → 14], PlotLegends →
   {"Exact Solution", "Mathematica's Asymptotic", "Asymptotic Approximation"}]
```

Out[17]=



So we do just as bad (or just as well) as Mathematica's asymptotic solve. This is somewhat comforting at least ...

(b) $\int_{\frac{1}{2}}^{2}(1+t)\,e^{i x\left(\frac{t^3}{3}-t\right)}\,dt$

Once again we want to use the method of stationary phase for finding the leading behavior of this integral. We have that $\phi(t)=\frac{t^3}{3}-t$, resulting in the stationary points to be at -1 and 1 (since $\phi'(t)=t^2-1$), but since we are on the positive axis, we only care for 1.

Following the process of Stationary Phase method, we can rewrite the original integral as:

$$\int_{\frac{1}{2}}^{2}(1+t)\,e^{i x\left(\frac{t^3}{3}-t\right)}\,dt \sim \int_{\frac{1}{2}}^{1-\epsilon}(1+t)\,e^{i x\left(\frac{t^3}{3}-t\right)}\,dt+\int_{1-\epsilon}^{1+\epsilon}(1+t)\,e^{i x\left(\frac{t^3}{3}-t\right)}\,dt+\int_{1+\epsilon}^{2}(1+t)\,e^{i x\left(\frac{t^3}{3}-t\right)}\,dt$$

using the Riemann-Lebesgue lemma, we can see that $\int_{\frac{1}{2}}^{1-\epsilon}(1+t)\,e^{i x\left(\frac{t^3}{3}-t\right)}\,dt$ and $\int_{1+\epsilon}^{2}(1+t)\,e^{i x\left(\frac{t^3}{3}-t\right)}\,dt$ will tend to 0, so all we are left with now is $\int_{1-\epsilon}^{1+\epsilon}(1+t)\,e^{i x\left(\frac{t^3}{3}-t\right)}\,dt$. So now we must find $\phi^{(n)}\neq 0$, which would be the second derivative since $\phi''(t)=2t\to\phi''(1)=2$. So now we expand $\phi(t)$ to get:
$\phi(t)=\left(\frac{1}{3}-1\right)+0+\frac{2}{2}(t-1)^2+O\left(t^3\right)\sim-\frac{2}{3}+(t-1)^2$

Similarly, taking $f\ (t) = 1 + t$, we take the exact value at the stationary point, i.e. f(1) = 2. So now we have:

$$\int_{\frac{1}{2}}^{2} (1+t)\, e^{i\,x\left(\frac{t^3}{3}-t\right)}\, dt \sim 2\int_{1-\epsilon}^{1+\epsilon} e^{i\,x\left(-\frac{2}{3}+(t-1)^2\right)}\, dt \sim 2\int_{-\infty}^{\infty} e^{i\,x\left(-\frac{2}{3}+(t-1)^2\right)}\, dt$$

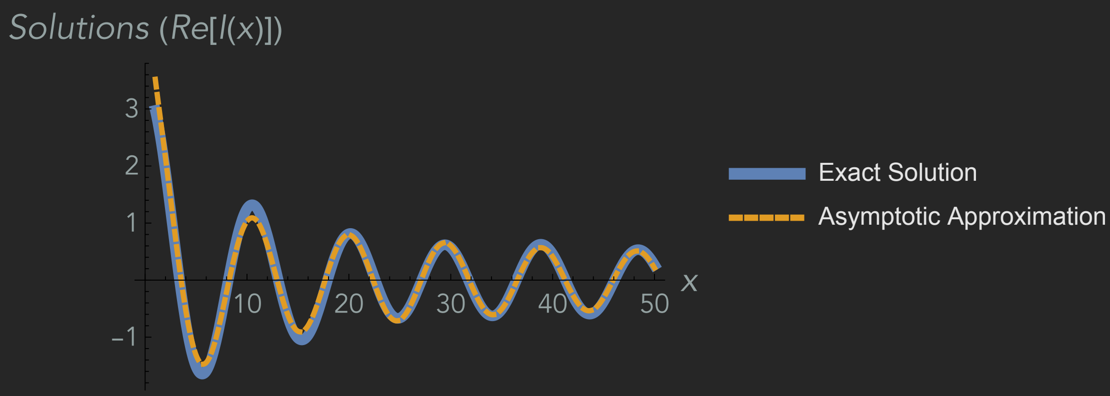
So now we can integrate this value to see what our approximation will yield:

```
approx = 2 * Integrate[E^(I * x * (-2/3 + (t - 1)^2)), {t, -∞, ∞}, Assumptions → Im[x] > 0]
```

Out[11]= $\dfrac{2\, e^{-\frac{2\,i\,x}{3}}\sqrt{\pi}}{\sqrt{-i\,x}}$

```
In[13]:= Plot[{Re[NIntegrate[(1 + t) * E^(I * x * ((t^3 / 3) - t)), {t, 1/2, 2}]], Re[approx]},
  {x, 1, 50}, PlotRange → All,
  PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
  AxesLabel → {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]},
  TicksStyle → Directive[FontSize → 14],
  PlotLegends → {"Exact Solution", "Asymptotic Approximation"}]
```

Out[13]=



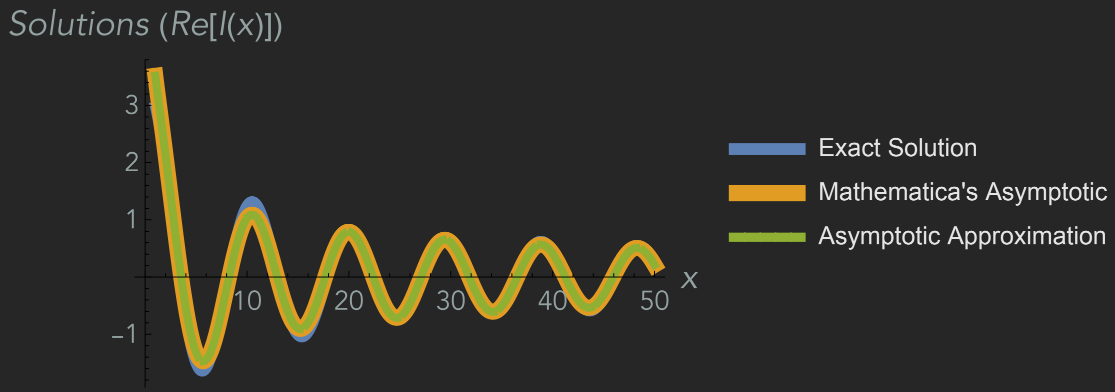
Which shows much better agreement between our asymptotic solution and the exact (numerical) solution. To compare our solutions with the Mathematica's Asymptotic solution:

```
In[18]:= mathematicaAsymp = AsymptoticIntegrate[(1 + t) * E^(I * x * ((t^3 / 3) - t)), {t, 1/2, 2}, x → ∞]
```

Out[18]= $\dfrac{2\, e^{\frac{i\pi}{4}-\frac{2\,i\,x}{3}}\sqrt{\pi}}{\sqrt{x}}$

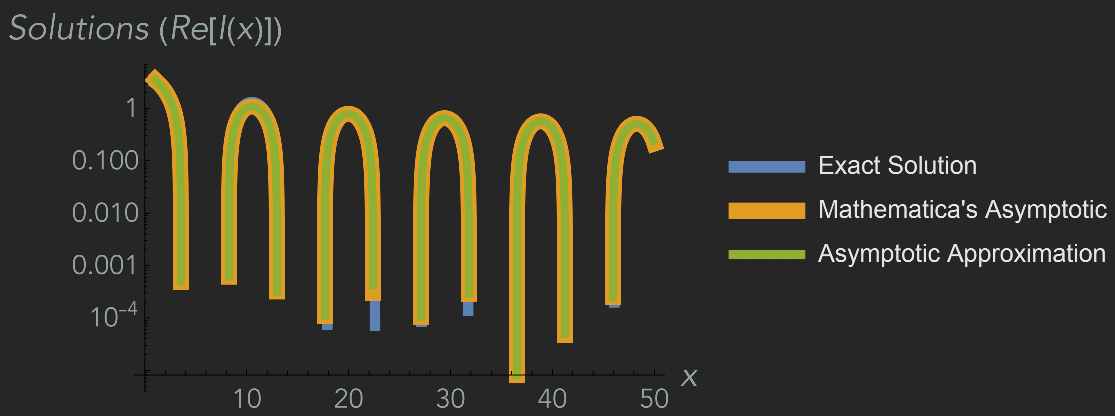
In[20]:=
```
Plot[{Re[NIntegrate[(1 + t) * E^(I * x * ((t^3 / 3) - t)), {t, 1 / 2, 2}]],
   Re[mathematicaAsymp], Re[approx]}, {x, 1, 50}, PlotRange → All,
  PlotStyle → {Directive[Solid, Thickness[0.02]],
    Directive[Solid, Thickness[0.03]], Directive[Dashed, Thickness[0.015]]},
  AxesLabel → {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]},
  TicksStyle → Directive[FontSize → 14], PlotLegends →
   {"Exact Solution", "Mathematica's Asymptotic", "Asymptotic Approximation"}]
```

Out[20]=



In[21]:=
```
LogPlot[{Re[NIntegrate[(1 + t) * E^(I * x * ((t^3 / 3) - t)), {t, 1 / 2, 2}]],
   Re[mathematicaAsymp], Re[approx]}, {x, 1, 50}, PlotRange → All,
  PlotStyle → {Directive[Solid, Thickness[0.02]],
    Directive[Solid, Thickness[0.03]], Directive[Dashed, Thickness[0.015]]},
  AxesLabel → {Style["x", Italic, 18], Style["Solutions (Re[I(x)])", Italic, 18]},
  TicksStyle → Directive[FontSize → 14], PlotLegends →
   {"Exact Solution", "Mathematica's Asymptotic", "Asymptotic Approximation"}]
```

Out[21]=



Which show both our solution and Mathematica's solution to match very closely with the numerical solution, which is a good sign!