Math 223: Homework2

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Problem 1

Classify the points at 0 and ∞ of the following differential equations:

(a) $x^7\, y^{(4)} = y'$

In order to check for singularity points (and classify them), we will rewrite the equation in standard form:
$x^7\, y^{(4)} - y' = 0$
$= y^{(4)} - \frac{y'}{x^7} = 0$
To check for singularity at 0, we can study the behavior as $x \to 0$, which is clearly not finite (therefore a singular point). Now to classify the singular point, we must look at if each term $(x - x_0)^n\, p_0(x)$, $(x - x_0)^{n-1}\, p_1(x)$, $\cdots$, $(x - x_0)\, p_{n-1}(x)$ are analytic in a neighborhood of x_0.

In[∘]:=
```
n = 7;
x0 = 0;
p1[x_] := 1 / (x^7);
p1[x]
(*Series[(x-x0)^(n-1) * p1, {x,x0,10}]*)
```

Out[∘]= $\frac{1}{x^7}$

In[∘]:=
```
Series[(x - x0) ^ (n - 1) * p1[x], {x, x0, 10}]
```

Out[∘]= $\frac{1}{x} + O[x]^{11}$

which is not analytic at x = 0, therefore resulting in an essential singularity. That makes sense since we have a pole of order 7 for $p(x) = \frac{1}{x^7}$, but the differential equation is of order 4.

To check the same thing for $x \to \infty$, we have to use an inversion mapping (i.e. $x = \frac{1}{t}$), and then checking the analyticity of f(t). Since we will be having the same derivatives, we can calculate these values, and keep using them. Taking $f = y\left(\frac{1}{x}\right) = y(t)$

In[∘]:=
```
dt = D[y[1/x], {x, 1}]
```

Out[∘]= $-\frac{y'\left[\frac{1}{x}\right]}{x^2}$

In[∘]:=
```
dt2 = D[y[1/x], {x, 2}]
```

Out[∘]= $\frac{2\, y'\left[\frac{1}{x}\right]}{x^3} + \frac{y''\left[\frac{1}{x}\right]}{x^4}$

In[]:= dt3 = D[y[1/x], {x, 3}]

Out[]= $-\frac{6\, y'\left[\frac{1}{x}\right]}{x^4} - \frac{6\, y''\left[\frac{1}{x}\right]}{x^5} - \frac{y^{(3)}\left[\frac{1}{x}\right]}{x^6}$

In[]:= dt4 = D[y[1/x], {x, 4}]

Out[]= $\frac{24\, y'\left[\frac{1}{x}\right]}{x^5} + \frac{36\, y''\left[\frac{1}{x}\right]}{x^6} + \frac{12\, y^{(3)}\left[\frac{1}{x}\right]}{x^7} + \frac{y^{(4)}\left[\frac{1}{x}\right]}{x^8}$

So rewriting these values in terms of t, we have:

$$\frac{d}{dx} = -t^2\,\frac{d}{dt}$$

$$\frac{d^2}{dx^2} = t^4\,\frac{d^2}{dt^2} + 2\,t^3\,\frac{d}{dt}$$

$$\frac{d^3}{dx} = t^6\,\frac{d^3}{dt^3} - 6\,t^5\,\frac{d^2}{dt^2} - 6\,t^4\,\frac{d}{dt}$$

$$\frac{d^4}{dx} = t^8\,\frac{d^4}{dt^4} + 12\,t^7\,\frac{d^3}{dt^3} + 36\,t^6\,\frac{d^2}{dt^2} + 24\,t^5\,\frac{d}{dt}$$

So now to check for the behavior at $x_0 = \infty$, we use the inversion mapping:

$$y^{(4)} - \frac{y'}{x^7} = 0 \rightarrow t^8\,y^{(4)} + 12\,t^7\,y^{(3)} + 36\,t^6\,y'' + 24\,t^5\,y' + t^7\,y' = 0$$
$$= y^{(4)} + \frac{12}{t}\,y^{(3)} + \frac{36}{t^2}\,y'' + \frac{24}{t^8}\,y' + y'\,\frac{1}{t} = y^{(4)} + \frac{12}{t}\,y^{(3)} + \frac{36}{t^2}\,y'' + \left(\frac{24}{t^3} + \frac{1}{t}\right)y' = 0$$

Given the expression above, we can quickly asses that the singularity at infinity will be a regular point,
since the order of each derivative is higher than the pole. But to concretely check for this, we can check
to see if the coefficients are analytic or not. Let us start by $p_1(t)$ first :

```
n = 7;
t0 = 0;
p1[t_] := (24/t^3 + 1/t);
Series[(t - t0)^(n - 1) * p1[t], {t, t0, 15}]
```

Out[]= $24\,t^3 + t^5 + O[t]^{16}$

so the first set of coefficient is analytical. Now checking for the rest:

```
In[]:= p2[t_] := (36/t^3);
Series[(t - t0)^(n - 2) * p2[t], {t, t0, 15}]
```

Out[]= $36\,t^2 + O[t]^{16}$

```
In[]:= p3[t_] := (12/t);
Series[(t - t0)^(n - 3) * p3[t], {t, t0, 15}]
```

Out[]= $12\,t^3 + O[t]^{16}$

which are all analytic, therefore we have a regular singularity at the point $x \to x_0$, where $x_0 = \infty$.

(b) $x^3\, y^{(3)} = y$

Following our approach in the part (a), we have:

$$y^{(3)} - \frac{y}{x^3} = 0$$

so we have $p_0(x) = \frac{1}{x^3}$, which has a singularity at $x_0 = 0$. Now, to classify the type of singularity at $x_0 = 0$, we look at the analyticity of $(x - x_0)^n\, p_0(x)$:

```
In[]:= n = 3;
x0 = 0;
p0[x_] = 1 / (x^3);
```

```
In[]:= Series[(x - 0)^n * p0[x], {x, x0, 15}]
```

Out[]= 1

which is pretty analytic if you ask me. So the singularity at $x_0 = 0$ for $p_0(x)$ is a regular singular point (poll of order 3). Now to check for $x_0 = \infty$,
we use the derivates of the transformation that we found in (a), which gives us:

$$t^6\, y^{(3)} - 6\, t^5\, y'' - 6\, t^4\, y' + t^3\, y = 0$$
$$= y^{(3)} - \frac{6}{t}\, y'' - \frac{6}{t^2}\, y' + \frac{1}{t^3}\, y = 0$$

here we can see that the pole with the highest degree ($p_0(t)$) has the exact same degree as the differential equation (order 3), which means
that we will have a regular singular point. But to confirm this:

```
In[]:= p0[t_] = 1/t^3;
p1[t_] = -6/t^2;
p2[t_] = -6/t;
t0 = 0;
```

```
In[]:= Series[(t - t0)^n * p0[t], {t, t0, 10}]
```

Out[]= 1

```
In[]:= Series[(t - t0)^(n - 1) * p1[t], {t, t0, 10}]
```

Out[]= -6

```
In[]:= Series[(t - t0) * p2[t], {t, t0, 10}]
```

Out[]= -6

which are all analytic at $t_0 = 0$, implying that we have a regular singular point at $x_0 = \infty$ as well.

(c) $y^{(3)} = x^3\, y$

writing the equation in the normal form gives us $y^{(3)} - x^3 y = 0$, which does not have any singularities at the point $x_0 = 0$, so we have an ordinary point.
To check for the point $x_0 = \infty$, we can rewrite the equation in the terms of the inversion mapping as follows:

$$\frac{y^{(3)}}{t^6} - 6\,\frac{y''}{t^5} - 6\,\frac{y'}{t^4} - \frac{1}{t^3}\,y = 0$$

$$= y^{(3)} - 6\,t^2\,y'' - 6\,t^2\,y' + t^3\,y = 0$$

which has all analytic coefficients, therefore we have an ordinary point at $x_0 = \infty$.

which is clearly not analytic at the point $t_0 = 0$. Therefore we have an irregular singular point at $x_0 = \infty$.

(d) $x^2\,y^{(2)} = e^{1/x}\,y$

As usual, let us write this in the normal form:

$$y^{(2)} - \frac{e^{\frac{1}{x}}}{x^2}\,y = 0$$

and to check for possible singularities at the $x_0 = 0$ and $x_0 = \infty$:

In[∘]:=
```
n = 2;
x0 = 0;
t0 = 0;
px0 = (E^(1/x))/(x^2);
```

In[∘]:=
```
Series[(x - x0)^n * px0, {x, x0, 10}]
```

Out[∘]= $e^{\frac{1}{x}} + O[x]^{12}$

which has is not analytic at $x_0 = 0$, therefore we have an irregular singularity.
Now to check for $x_0 = \infty$, we do the usual inverted mapping of $t = \frac{1}{x}$:

$$\frac{t^4}{t^2}\,y'' + 2\,\frac{t^3}{t^2}\,y' - e^t\,y = y'' + \frac{2}{t}y' - \frac{e^t}{t^2}\,y = 0$$

which we can tell will result in regular singular points, since the order of the ODE matches the order of the highest degree pole.
To check for analyticity:

In[∘]:=
```
n = 2;
t0 = 0;

p0[t_] = -e^t/t^2;

p1[t_] = 2/t;
```

In[∘]:=
```
Series[(t - t0)^n * p0[t], {t, t0, 10}]
```

Out[∘]=

$$-1 - \mathrm{Log}[e]\,t - \frac{1}{2}\,\mathrm{Log}[e]^2\,t^2 - \frac{1}{6}\,\mathrm{Log}[e]^3\,t^3 - \frac{1}{24}\,\mathrm{Log}[e]^4\,t^4 - \frac{1}{120}\,\mathrm{Log}[e]^5\,t^5 - \frac{1}{720}\,\mathrm{Log}[e]^6\,t^6 - \frac{\mathrm{Log}[e]^7\,t^7}{5040} - \frac{\mathrm{Log}[e]^8\,t^8}{40\,320} - \frac{\mathrm{Log}[e]^9\,t^9}{362\,880} - \frac{\mathrm{Log}[e]^{10}\,t^{10}}{3\,628\,800} + O[t]^{11}$$

In[∘]:=
```
Series[(t - t0)^(n - 1) * p1[t], {t, t0, 10}]
```

Out[∘]= 2

which are all analytic as we suspected, therefore we have a regular singularity at $x_0 = \infty$.

(e) $\tan(x)\,y' = y$

writing this in the standard form, we get $y' - \frac{y}{\tan(x)} = 0$. We know we have a singularity for $p_0(x)$, but it should be a regular singularity (do to the behavior of tan(x).
To check the analyticity:

In[•]:=
```
n = 1;
x0 = 0;
p0[x_] = 1/Tan[x];
Series[(x - x0)^n * p0[x], {x, x0, 10}]
```

$$1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} - \frac{x^8}{4725} - \frac{2x^{10}}{93555} + O[x]^{11}$$

which is very analytic! so we have a regular singularity at $x_0 = 0$. For $x_0 = \infty$, as usual we use $x = \frac{1}{t}$, which yields:

$$-t^2 y' - \frac{y}{\tan(x)} = y' + \frac{y}{t^2 \tan(x)} = 0.$$

Now the addition of t^2 to the denominator for a first order ODE is problematic, so we expect an irregular singular point. To check this:

In[•]:=
```
n = 1;
t0 = 0;
p0[t_] = -1/(t^2 Tan[1/t]);
Series[(t - t0)^n * p0[t], {t, t0, 10}]
```

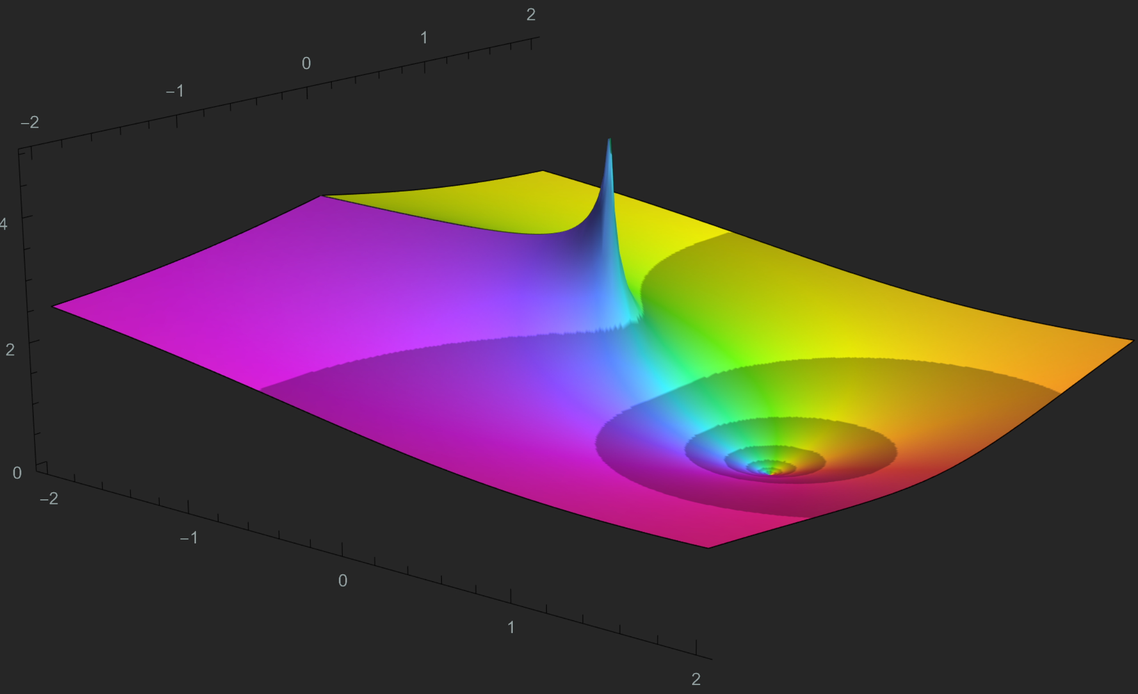
Out[•]= $\mathrm{Cot}\left[\frac{1}{t} + O[t]^{11}\right] \left(-\frac{1}{t} + O[t]^{11}\right)$

which is not analytic at t = 0, therefore we have an irregular singularity at $x_0 = \infty$, as expected.

(f) $y^{(2)} = (\log(x))\, y$

so we have $y'' - (\log(x))\, y = 0$, with log(x) having a singularity at 0. We can immediately suspect to have an irregular singular point do to the logarithm,
since 0 is a branch point. So we have an essential singularity at $x_0 = 0$.

In[•]:=
```
ComplexPlot3D[Log[x], {x, -2 - 2 I, 2 + 2 I},
 ColorFunction → "CyclicLogAbs", Boxed → False]
```

Out[•]=


Again due to the logarithm, we can expect the same thing at $x_0 = \infty$, but solving for it we have:

$$t^4\, y'' + 2\, t^3\, y' + \log(t)\, y = y'' + \frac{2}{t} y' + \frac{(\log(t))}{t^4}\, y = 0$$

In[•]:=
```
n = 2;
t0 = 0;
p0[t_] = Log[t]/t^4;
Series[(t - t0)^n * p0[t], {t, t0, 10}]
```

Out[•]= $\frac{\text{Log}[t]}{t^2} + O[t]^{11}$

which is not analytic, therefore resulting in an irregular singular point at $x_0 = \infty$.

Problem 2

For the initial value problem:

$$(x-1)(x-2)\, y'' + (4x-6)\, y' + 2y = 0,$$
$$y(0) = 0,$$
$$y'(0) = 1$$

where might one expect the series solution about x = 0 to converge? Compute the Taylor expansion about x = 0 of the solution and determine the radius of convergence.

We want to check to see if we have a removable (regular) singularity at the origin, because then the radius of convergence of the solution will be at least as large as the distance to the nearest singularity of the coefficient functions (in the complex plane). So let us first rewrite the ODE in the normal form:

$$y'' + \frac{2(2x-3)}{(x-1)(x-2)}\, y' + \frac{2}{(x-1)(x-2)}\, y = 0$$

Now as we can see, the nearest singularity in the coefficient functions happen to be at $x = 1$, so for now we expect the power series solution about x=0 to converge for (-1,1). But to find the exact radius of convergence, we will find the solution and find the radius of convergence.

```
In[∘]:= sol = DSolve[{(x - 1) * (x - 2) * y''[x] + (4 * x - 6) y'[x] + 2 y[x] == 0,
            y[0] == 0, y'[0] == 1}, y[x], x]
```

Out[∘]= $\left\{\left\{y[x] \to \frac{2x}{2 - 3x + x^2}\right\}\right\}$

so now to expand the solution:

In[∘]:= $y1[x\_] = \frac{2x}{2 - 3x + x^2}$;

```
x0 = 0;
Series[y1[x], {x, x0, 10}]
```

Out[∘]= $x + \frac{3x^2}{2} + \frac{7x^3}{4} + \frac{15x^4}{8} + \frac{31x^5}{16} + \frac{63x^6}{32} + \frac{127x^7}{64} + \frac{255x^8}{128} + \frac{511x^9}{256} + \frac{1023x^{10}}{512} + O[x]^{11}$

which has the closed form $\sum_{n=0}^{\infty}\left(2^{n+1} - 1\right)\frac{x^{n+1}}{2^n}$, which has a radius of convergence of $|x| < 1$. This matches with what we found with Fuch's theorem as well. To double check the convergence:

In[∘]:= $solSeries = \left(2^{q+1} - 1\right)\frac{x^{q+1}}{2^q}$;

```
SumConvergence[solSeries, q]
```

Out[∘]= Abs[x] < 1

confirming that the solutions has an interval of convergence of (-1,1), which was predicted via Fuch's theorem by only looking at the coefficient functions (this is really cool!!!!).

Problem 3

For the following differential equations, where would you expect the series solution about x = 0 to converge? Find series expansions of all the solutions to the following differential equations about x=0. Try to sum in closed form any infinite series that appear.

By Fuch's theorem, if we have a regular singular point, then the series solution will have a radius of convergence at least to the nearest singularity. So in the following problems, we will do:
1) Ensure that we have regular singularity at $x_0 = 0$
2) Use Mathematica to solve each problem, and then expand each solution about 0
3) find the closed form, or estimate the radius of convergence.

(a) $x y'' + y = 0$

First, we want to make sure that we have a regular singularity. Since we have a second order ODE with a first order pole. i.e. $p_0(x) = \frac{1}{x}$. So we have a removable singularity, and we can move forward with finding a series solution. Since the singularity at 0, we Fuch's theorem tells us that the radius of convergence of the solution will be at least as large as the distance to the nearest singularity of the coefficient functions, which is not much in this case. So we will find the radius of convergence explicitly, and see if we have a larger interval of convergence.

To find solve this problem, we can use DSolve to find the closed form of the solution.

In[•]:=
```
solution = DSolve[x * y''[x] + y[x] == 0, y[x], x]
```
Out[•]=
```
{{y[x] → √x BesselJ[1, 2 √x] 𝕔1 + 2 ⅈ √x BesselY[1, 2 √x] 𝕔2}}
```

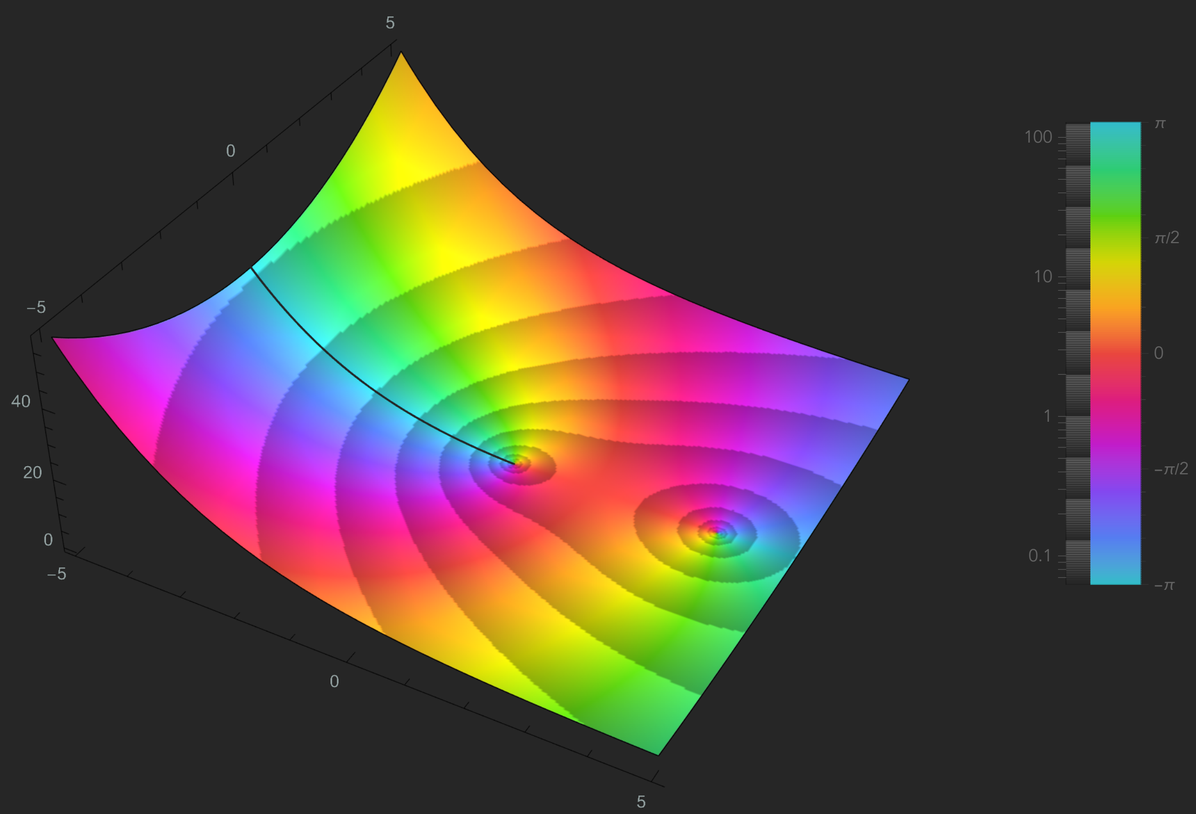
now that we have found the solutions to be Bessel functions (of the first and second kind), we know that they are of the form:

$$y_1 = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+2)} \left(\frac{x}{2}\right)^{2m+1}$$

Now to plot the solutions (in Complex plane):

In[•]:=
```
ComplexPlot3D[√x BesselJ[1, 2 √x], {x, -5 - 5 I, 5 + 5 I},
 ColorFunction → "CyclicLogAbs", PlotLegends → Automatic, Boxed → False]
```

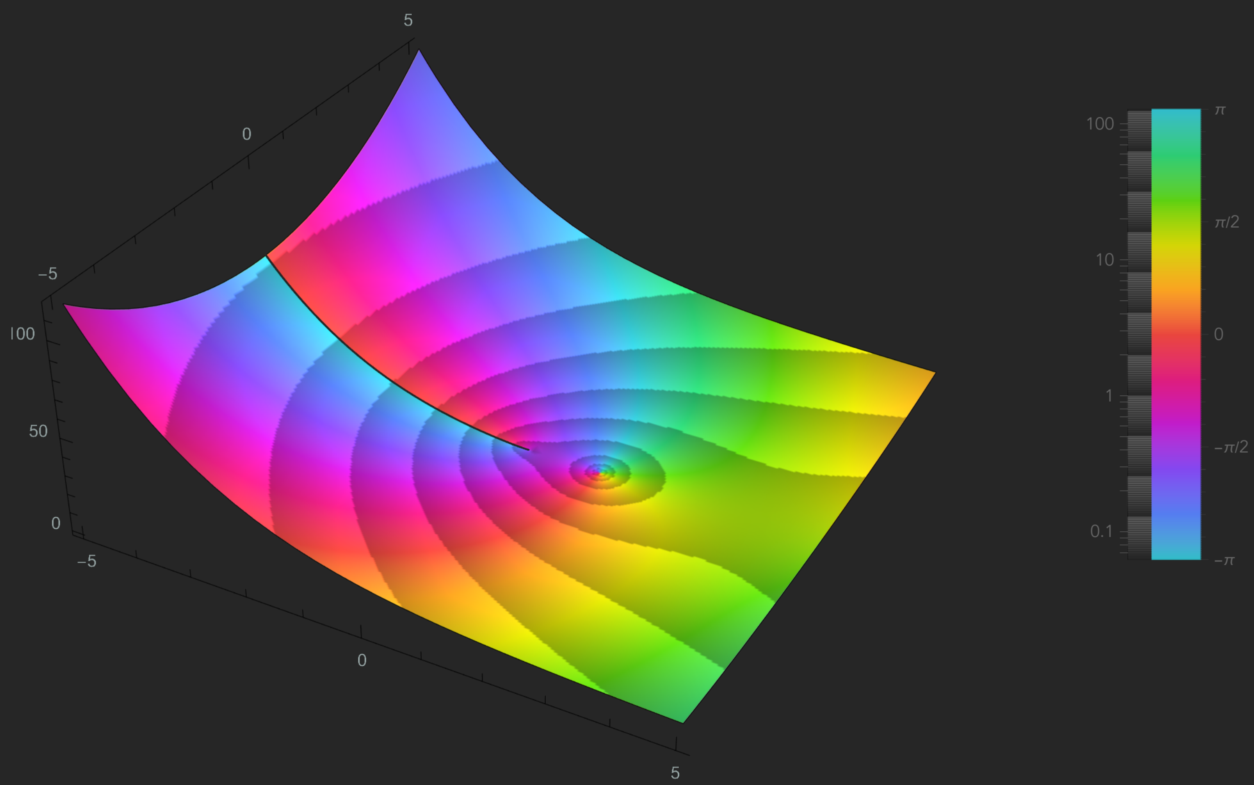
Out[•]=



And to plot the Bessel function of the second kind

In[∘]:=
```
ComplexPlot3D[2 ⅈ √x BesselY[1, 2 √x], {x, -5 - 5 I, 5 + 5 I},
  ColorFunction → "CyclicLogAbs", PlotLegends → Automatic, Boxed → False]
```

Out[∘]=



Now to find the series solution, we could go the messy way of actually expanding the closed form solution, as shown below:

```
In[•]:= firstSol = √x BesselJ[1, 2 √x] ℂ1;
        secondSol = 2 ⅈ √x BesselY[1, 2 √x] ℂ2;
        firstSeries = Series[firstSol, {x, 0, 10}]
        secondSeries = Series[secondSol, {x, 0, 10}]
```

Out[•]=
$$\mathbb{c}_1 x - \frac{\mathbb{c}_1 x^2}{2} + \frac{\mathbb{c}_1 x^3}{12} - \frac{\mathbb{c}_1 x^4}{144} + \frac{\mathbb{c}_1 x^5}{2880} - \frac{\mathbb{c}_1 x^6}{86\,400} + \frac{\mathbb{c}_1 x^7}{3\,628\,800} - \frac{\mathbb{c}_1 x^8}{203\,212\,800} + \frac{\mathbb{c}_1 x^9}{14\,631\,321\,600} - \frac{\mathbb{c}_1 x^{10}}{1\,316\,818\,944\,000} + O[x]^{11}$$

Out[•]=
$$-\frac{2\,i\,\mathbb{c}_2}{\pi} + \frac{2\,i\,\mathbb{c}_2\,(-1 + 2\,\text{EulerGamma} + \text{Log}[x])\,x}{\pi} - \frac{i\,\mathbb{c}_2\,(-5 + 4\,\text{EulerGamma} + 2\,\text{Log}[x])\,x^2}{2\,\pi} + \frac{i\,\mathbb{c}_2\,(-10 + 6\,\text{EulerGamma} + 3\,\text{Log}[x])\,x^3}{18\,\pi} - \frac{i\,\mathbb{c}_2\,(-47 + 24\,\text{EulerGamma} + 12\,\text{Log}[x])\,x^4}{864\,\pi} + \frac{i\,\mathbb{c}_2\,(-131 + 60\,\text{EulerGamma} + 30\,\text{Log}[x])\,x^5}{43\,200\,\pi} - \frac{i\,\mathbb{c}_2\,(-71 + 30\,\text{EulerGamma} + 15\,\text{Log}[x])\,x^6}{648\,000\,\pi} + \frac{i\,\mathbb{c}_2\,(-353 + 140\,\text{EulerGamma} + 70\,\text{Log}[x])\,x^7}{127\,008\,000\,\pi} - \frac{i\,\mathbb{c}_2\,(-1487 + 560\,\text{EulerGamma} + 280\,\text{Log}[x])\,x^8}{28\,449\,792\,000\,\pi} + \frac{i\,\mathbb{c}_2\,(-6989 + 2520\,\text{EulerGamma} + 1260\,\text{Log}[x])\,x^9}{9\,217\,732\,608\,000\,\pi} - \frac{i\,\mathbb{c}_2\,(-1451 + 504\,\text{EulerGamma} + 252\,\text{Log}[x])\,x^{10}}{165\,919\,186\,944\,000\,\pi} + O[x]^{21/2}$$

but perhaps it would be better to use **AsymptoticDSolveValue** to compute the series solution, which will be very close to the expanded version of the solution (above)

```
In[•]:= AsymptoticDSolveValue[x * y''[x] + y[x] == 0, y[x], {x, 0, 10}]
```

Out[•]=
$$\left(x - \frac{x^2}{2} + \frac{x^3}{12} - \frac{x^4}{144} + \frac{x^5}{2880} - \frac{x^6}{86\,400} + \frac{x^7}{3\,628\,800} - \frac{x^8}{203\,212\,800} + \frac{x^9}{14\,631\,321\,600} - \frac{x^{10}}{1\,316\,818\,944\,000}\right)\mathbb{c}_2 + \mathbb{c}_1\left((18\,435\,465\,216\,000 + 18\,435\,465\,216\,000\,x - 23\,044\,331\,520\,000\,x^2 + 5\,120\,962\,560\,000\,x^3 - 501\,427\,584\,000\,x^4 + 27\,951\,920\,640\,x^5 - 1\,009\,967\,616\,x^6 + 25\,619\,328\,x^7 - 481\,788\,x^8 + 6989\,x^9)\,/\,18\,435\,465\,216\,000 - \frac{1}{14\,631\,321\,600}x\,(14\,631\,321\,600 - 7\,315\,660\,800\,x + 1\,219\,276\,800\,x^2 - 101\,606\,400\,x^3 + 5\,080\,320\,x^4 - 169\,344\,x^5 + 4032\,x^6 - 72\,x^7 + x^8)\,\text{Log}[x]\right)$$

as we can see the **AsymptoticDSolveValue** solution is very close to the expanded Bessel functions. We can find an easy pattern on the first solutions and write it as a closed form :

$$y_1 = \sum_{n=0}^{\infty} \mathbb{c}_2 (-1)^n \left(\frac{1}{n!\,(n+1)!}\right) x^{n+1}$$

which will converge for |x|<∞, therefore having a radius of convergence everywhere on ℝ.

With the second solution, it may be a bit harder to find an explicit closed form (at least to me), but we know that it will be of the kind:

$y_2 = \alpha y_1 \log(x) + x\left(1 + \sum_{n=0}^{\infty} \mathbb{c}_1 x^{n+1}\right)$, which the sum will be convergent if |x|<1, so giving us a 1 as the radius of convergence.

(b) $y'' + (e^x - 1)\,y = 0$

Although this equation may seem innocent at first since it is an ordinary point, but we can note that $p_0(x) = e^x - 1$ vanishes for x=0 (making y(x) disappear). So once again, we will be left with the Bessel functions as the analytical solutions, as shown below.

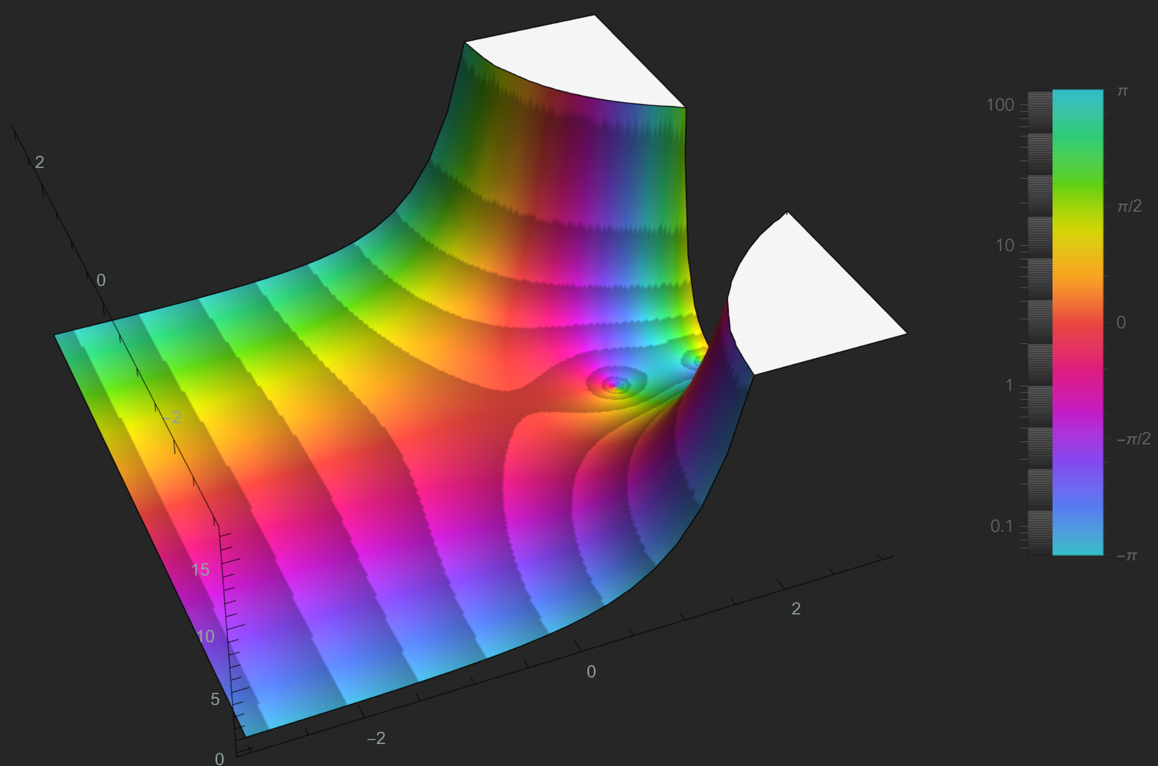
In[ ]:=
```
solution = DSolve[y''[x] + (E^x - 1) * y[x] == 0, y[x], x]
```

Out[ ]= $\left\{\left\{y[x] \to 2\,\text{BesselJ}\left[2, 2\sqrt{e^x}\right] c_1 + 2\,\text{BesselY}\left[2, 2\sqrt{e^x}\right] c_2\right\}\right\}$

and we can plot the solutions for a nice visualization:

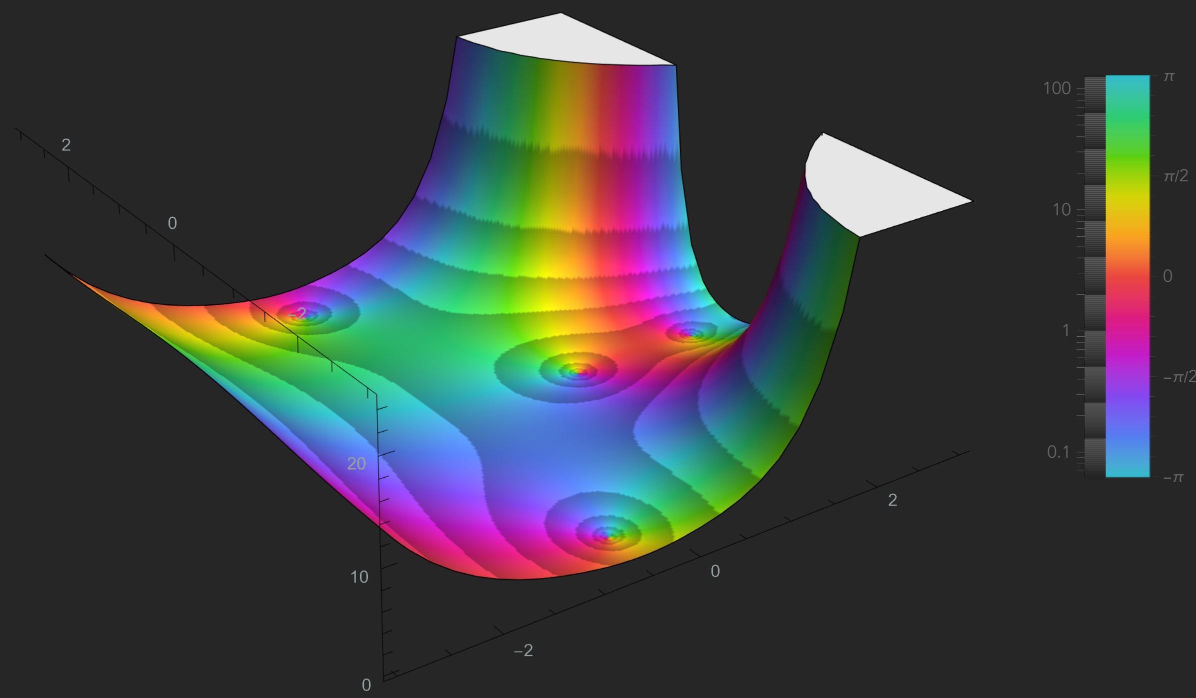
In[ ]:=
```
ComplexPlot3D[2 BesselJ[2, 2 Sqrt[E^x]], {x, -3 - 3 I, 3 + 3 I},
 ColorFunction → "CyclicLogAbs", PlotLegends → Automatic, Boxed → False]
```

Out[ ]=



In[•]:=
```
ComplexPlot3D[2 BesselY[2, 2 √(e^x)], {x, -3 - 3 I, 3 + 3 I},
 ColorFunction → "CyclicLogAbs", PlotLegends → Automatic, Boxed → False]
```

Out[•]=



So according to Fuch's theorem, since we do not have any singularities, then the But we can get the series solution, in order to find the radius of convergence:

In[•]:=
```
AsymptoticDSolveValue[y''[x] + (E^x - 1) * y[x] == 0, y[x], {x, 0, 10}]
```

Out[•]=

$$\left(1-\frac{x^3}{6}-\frac{x^4}{24}-\frac{x^5}{120}+\frac{x^6}{240}+\frac{x^7}{360}+\frac{x^8}{1008}+\frac{23\,x^9}{120\,960}-\frac{x^{10}}{241\,920}\right)\mathbb{c}_1+\left(x-\frac{x^4}{12}-\frac{x^5}{40}-\frac{x^6}{180}+\frac{x^7}{1008}+\frac{x^8}{960}+\frac{11\,x^9}{25\,920}+\frac{31\,x^{10}}{302\,400}\right)\mathbb{c}_2$$

which will have the Bessel functions as the closed forms. So now, we want to estimate the ratio of convergence:

In[•]:=

(c) $(\sin(x))\,y'' - 2\,(\cos(x))\,y' - (\sin(x))\,y = 0$

As before, we first want to confirm that we have a regular singularity. Putting the ODE in normal form we get :

$$y'' - 2\,\frac{\cos(x)}{\sin(x)}\,y' - y = 0$$

so we have a singularity at $x_0 = 0$ (or in fact at every $x_0 = \pm n\pi$ for $n \in \mathbb{N}$). To check if this is a regular singularity:

In[∘]:=
```
n = 2;
x0 = 0;
p1[x_] = Cos[x]/Sin[x] * -2;
Series[(x - x0) ^ (n - 1) * p1[x], {x, x0, 10}]
```

Out[∘]=
$$-2+\frac{2x^2}{3}+\frac{2x^4}{45}+\frac{4x^6}{945}+\frac{2x^8}{4725}+\frac{4x^{10}}{93\,555}+O[x]^{11}$$

which is analytic, therefore giving us a removable singularity.

Computational Solution

First, we will find the solution analytically:

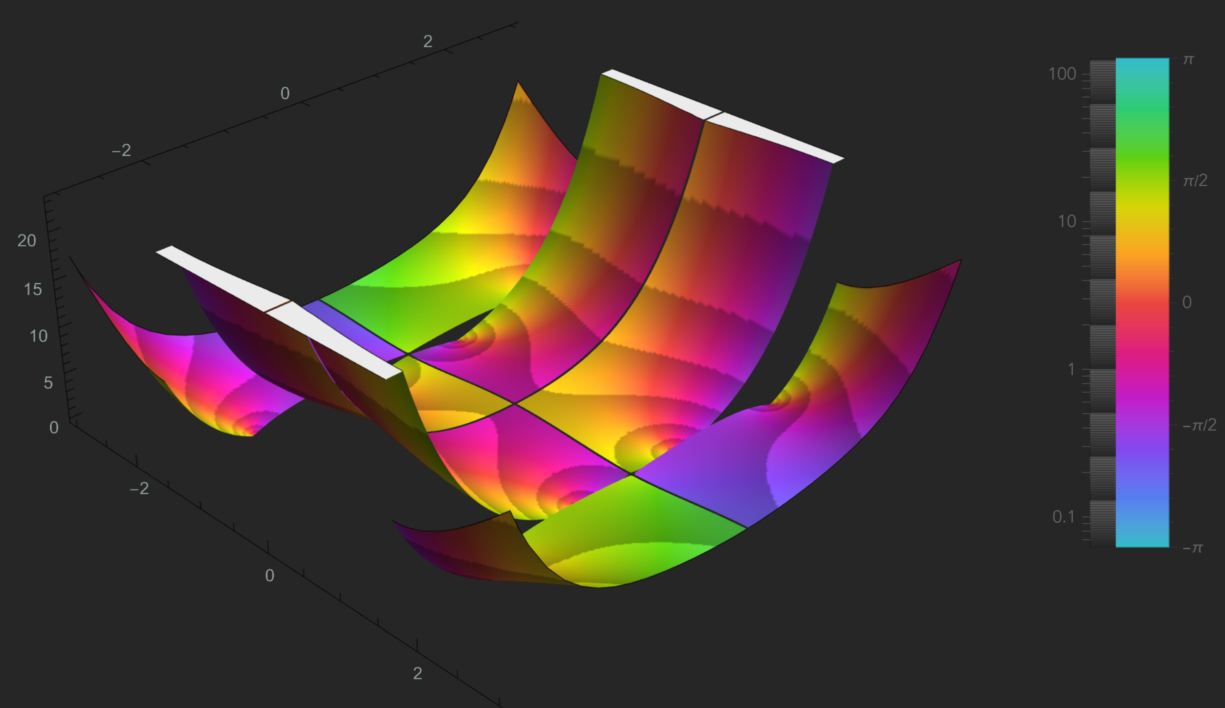
In[∘]:=
```
solution = DSolve[Sin[x] * y''[x] - 2 * Cos[x] * y'[x] - Sin[x] * y[x] == 0, y[x], x]
```

Out[∘]=
$$\left\{\left\{y[x]\to c_1\,\mathrm{Cos}[x]-\frac{c_2\sqrt{-1+\mathrm{Cos}[x]^2}\left(\mathrm{ArcSin}[\mathrm{Cos}[x]]\,\mathrm{Cos}[x]+\sqrt{1-\mathrm{Cos}[x]^2}\right)}{\sqrt{1-\mathrm{Cos}[x]^2}}\right\}\right\}$$

which is very pretty! Let us plot this solution as well:

In[∘]:=
$$\mathrm{ComplexPlot3D}\left[\mathrm{Cos}[x]-\frac{\sqrt{-1+\mathrm{Cos}[x]^2}\left(\mathrm{ArcSin}[\mathrm{Cos}[x]]\,\mathrm{Cos}[x]+\sqrt{1-\mathrm{Cos}[x]^2}\right)}{\sqrt{1-\mathrm{Cos}[x]^2}},\right.$$
```
{x, -3 - 3 I, 3 + 3 I}, ColorFunction → "CyclicLogAbs",
PlotLegends → Automatic, Boxed → False]
```

Out[∘]=



Now let us find the power series representation of the

In[•]:=
```
AsymptoticDSolveValue[
 Sin[x] * y''[x] - 2 * Cos[x] * y'[x] - Sin[x] * y[x] == 0, y[x], {x, 0, 15}]
```

Out[•]=
$$\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}-\frac{x^{10}}{3628800}+\frac{x^{12}}{479001600}-\frac{x^{14}}{87178291200}\right) c_1 + \left(x^3-\frac{x^5}{10}+\frac{x^7}{280}-\frac{x^9}{15120}+\frac{x^{11}}{1330560}-\frac{x^{13}}{172972800}+\frac{x^{15}}{31135104000}-\frac{x^{17}}{7410154752000}\right) c_2$$

Now the first solution is clearly the series for cos(x), as verified below:

In[•]:=
```
Series[Cos[x], {x, 0, 10}]
```

Out[•]=
$$1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}-\frac{x^{10}}{3628800}+O[x]^{11}$$

Now for the second solution, we can see the powers closely resemble sin(x) - xcos(x), which we can verify with the following series using 3 as the coefficient:

In[•]:=
```
Series[3 * (Sin[x] - x * Cos[x]), {x, 0, 15}]
```

Out[•]=
$$x^3-\frac{x^5}{10}+\frac{x^7}{280}-\frac{x^9}{15120}+\frac{x^{11}}{1330560}-\frac{x^{13}}{172972800}+\frac{x^{15}}{31135104000}+O[x]^{16}$$

Now that we have verified the closed form of the series, we can find the radius of convergence:

For the first solution $y_1 = c_1 \cos(x)$, the radius of convergence is all of $\mathbb{C}$.
For the second solution, $y_2 = c2\,(\sin(x) - x\cos(x))$, the radius of convergent is also all of $\mathbb{C}$ (since both sin(x) and xcos(x) converge for all of $\mathbb{C}$).

In[•]:=