Math 223: Homework5

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Problem 1

(a) Find the first three terms in the local behavior as $x \to 0^+$ of a particular solution y(x) to
$y' + xy = x^{-3}$

So now we make certain assumptions about the non-homogenous part and use that to approximate the particular solution. We then check to see which solution is consistent with that assumption:

* First assumption: $x^{-3} \ll xy$
$y' \sim xy$, so now we can solve this as a homogenous equation:

In[∘]:=
```
DSolve[y'[x] - x * y[x] == 0, y[x], x]
```

Out[∘]= $\left\{\left\{y[x] \to e^{\frac{x^2}{2}} c_1\right\}\right\}$

Now we can check the assumption that we made:

In[∘]:= $\text{Limit}\left[\frac{x^{-3}}{x * e^{\frac{x^2}{2}}}, x \to 0\right]$

Out[∘]= ∞

which clearly does match with our assumption. So now we move on to the next assumption, where we have:

$x\,y \ll x^{-3}$

which means we have $y' \sim x^{-3}$:

In[∘]:= $\text{Limit}\left[\frac{x * -\frac{1}{2} x^{-2}}{x^{-3}}, x \to 0\right]$

Out[∘]= 0

which tells us that the last assumption is consistent. So now we can move on to finding a correction to this solution. To do so, we can find a correction term (like the integration constant, but a function of x) with:

$y \sim -\frac{1}{2x^2} + C(x)$,

In[∘]:= $y'[x] + x\,y[x] == x^{-3}$ /. $y \to \text{Function}\left[x, -\frac{1}{2} x^{-2} + c[x]\right]$ // FullSimplify

Out[∘]= $2\,(x\,c[x] + c'[x]) == \frac{1}{x}$

using the same assumption type as before, we assume that $xC(x) \ll x^{-1}$, implying that :

$C'(x) \sim \frac{1}{2x} \Rightarrow C(x) \sim \frac{\text{Log}[x]}{2}$

so now we can write the particular solution as:

$$y_p(x) = -\frac{1}{2x^2} + \frac{\log(x)}{2} + C$$

which gives us the first three terms of the particular solution. However, since we do not have an initial condition, we can not find the specific constant so we will take it at 0.

Now we can check the accuracy of the approximation by checking against the analytical solution.

In[ ]:=
```
exactSol = DSolve[y'[x] + x * y[x] == x^-3, y[x], x]
```

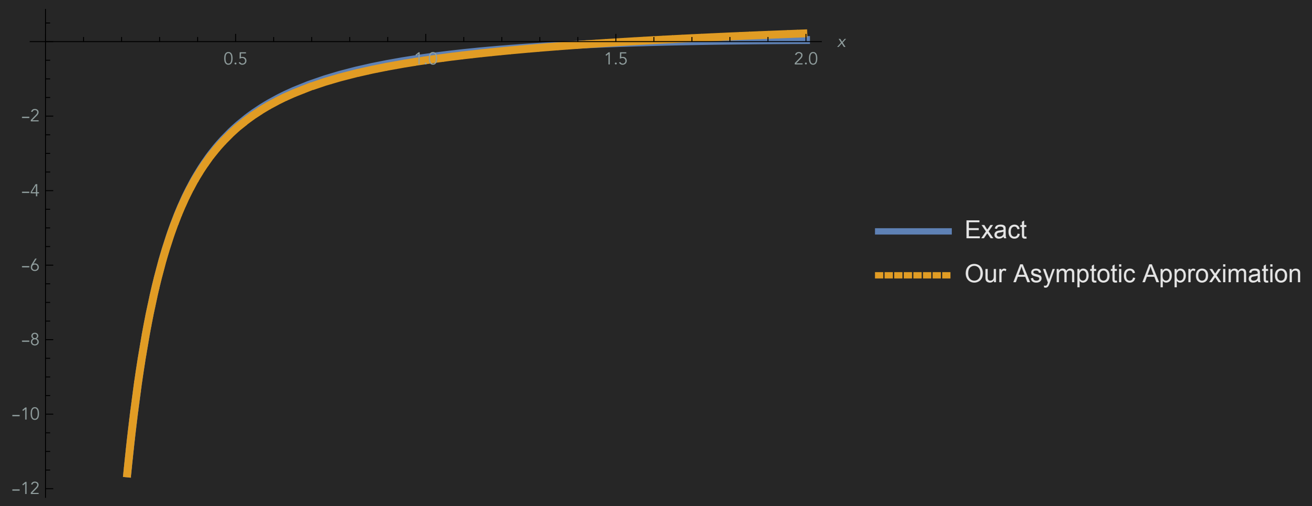
Out[ ]=

$$\left\{\left\{y[x] \to e^{-\frac{x^2}{2}} \mathbb{c}_1 + e^{-\frac{x^2}{2}} \left(-\frac{e^{\frac{x^2}{2}}}{2x^2} + \frac{1}{4}\,\text{ExpIntegralEi}\left[\frac{x^2}{2}\right]\right)\right\}\right\}$$

Plotting the particular solution against the approximation :

In[ ]:=
```
Plot[{e^(-x^2/2) (-e^(x^2/2)/(2 x^2) + 1/4 ExpIntegralEi[x^2/2]), -1/(2 x^2) + Log[x]/2},
  {x, 0, 2}, PlotStyle -> {Directive[Solid, Thickness[0.01]],
    Directive[Dotted, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
  PlotLegends -> {"Exact", "Our Asymptotic Approximation"}, AxesLabel -> Automatic]
```

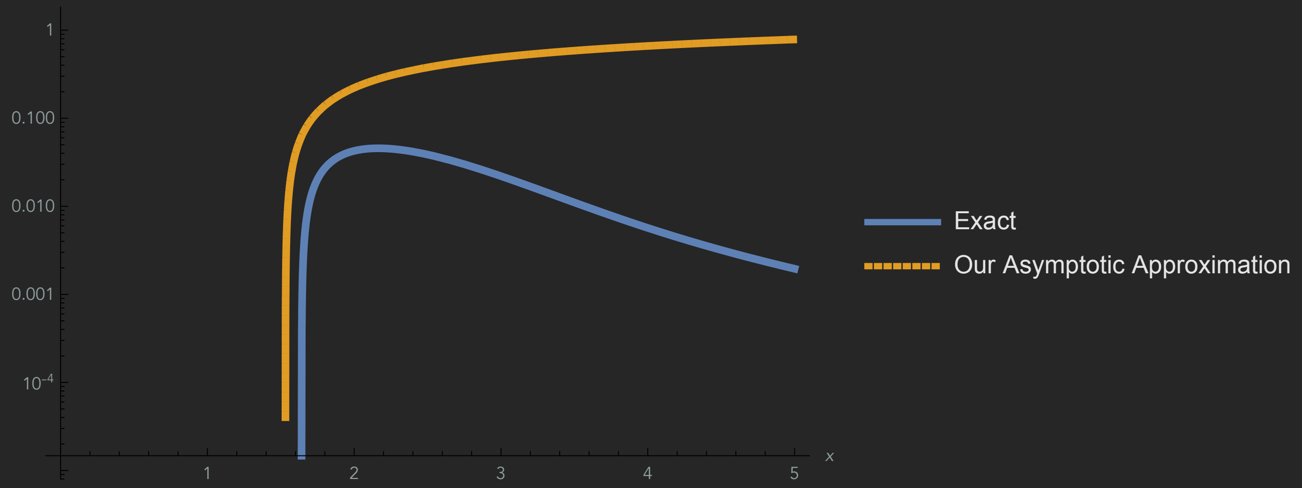
Out[ ]=



which looks good, but if we look at the LogPlot, we can see that our solution is actually not that good:

In[]:=
$$\text{LogPlot}\left[\left\{e^{-\frac{x^2}{2}}\left(-\frac{e^{\frac{x^2}{2}}}{2x^2}+\frac{1}{4}\text{ExpIntegralEi}\left[\frac{x^2}{2}\right]\right), -\frac{1}{2x^2}+\frac{\text{Log}[x]}{2}\right\},\right.$$

```
{x, 0, 5}, PlotStyle → {Directive[Solid, Thickness[0.01]],
  Directive[Dotted, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
PlotLegends → {"Exact", "Our Asymptotic Approximation"}, AxesLabel → Automatic]
```

Out[]=


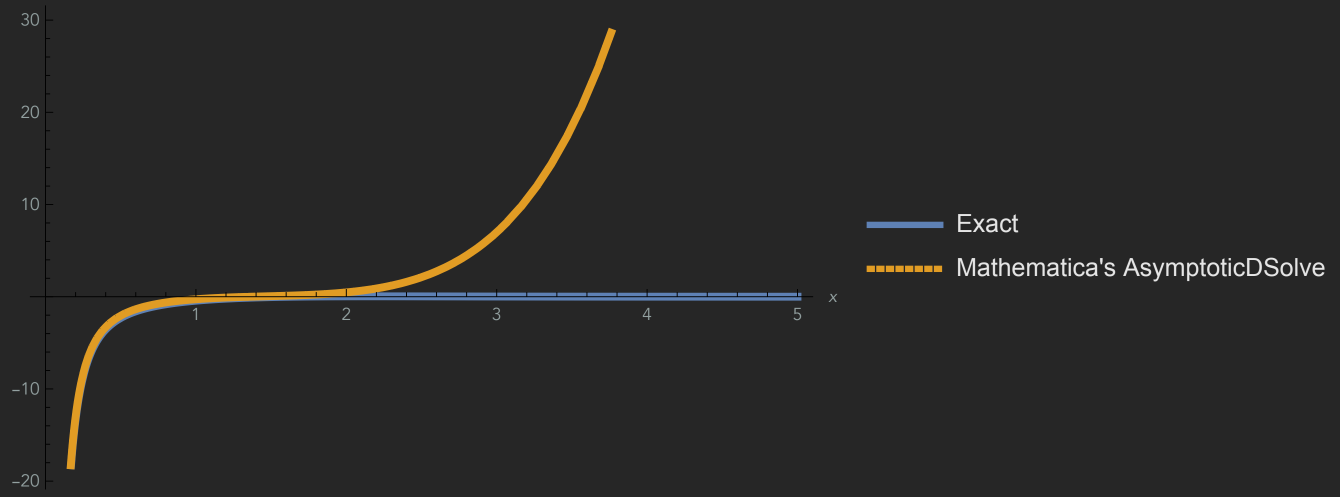
and if we compare Mathematica's Asymptotic DSolve against the exact solution:

In[]:= AsymptoticDSolveValue[y'[x] + x * y[x] == x^{-3}, y[x], {x, 0, 3}]

Out[]=
$$\left(1-\frac{x^2}{2}+\frac{x^4}{8}\right)c_1+\left(1-\frac{x^2}{2}+\frac{x^4}{8}\right)\left(-\frac{1}{2x^2}+\frac{x^2}{16}+\frac{\text{Log}[x]}{2}\right)$$

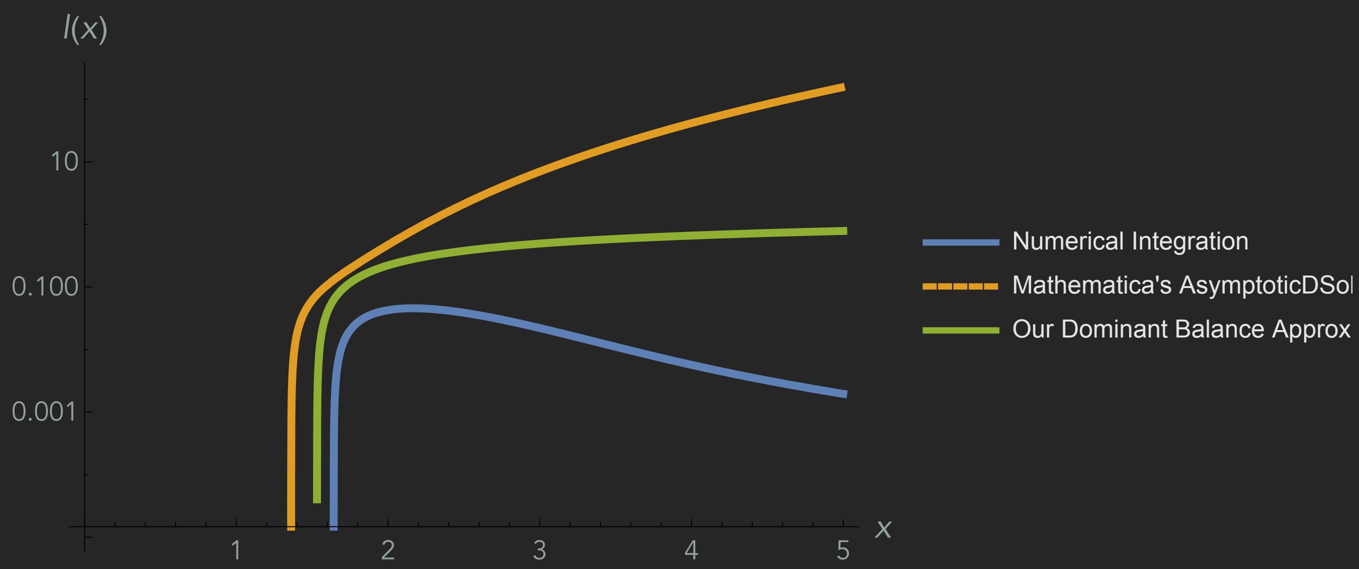
In[]:=
$$\text{Plot}\left[\left\{e^{-\frac{x^2}{2}}\left(-\frac{e^{\frac{x^2}{2}}}{2x^2}+\frac{1}{4}\text{ExpIntegralEi}\left[\frac{x^2}{2}\right]\right), \left(1-\frac{x^2}{2}+\frac{x^4}{8}\right)\left(-\frac{1}{2x^2}+\frac{x^2}{16}+\frac{\text{Log}[x]}{2}\right)\right\},\right.$$

```
{x, 0, 5}, PlotStyle → {Directive[Solid, Thickness[0.01]],
  Directive[Dotted, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
PlotLegends → {"Exact", "Mathematica's AsymptoticDSolve"}, AxesLabel → Automatic]
```

Out[]=


and to compare our dominant balance solution against Mathematica's AsymptoticDSolveValue:

In[]:=
```
LogPlot[{e^(-x^2/2) (-e^(x^2/2)/(2 x^2) + 1/4 ExpIntegralEi[x^2/2]),
  (1 - x^2/2 + x^4/8) (-1/(2 x^2) + x^2/16 + Log[x]/2), -1/(2 x^2) + Log[x]/2}, {x, 0, 5},
 PlotStyle → {Directive[Solid, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["I(x)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14], PlotLegends → {"Numerical Integration",
   "Mathematica's AsymptoticDSolve", "Our Dominant Balance Approximation"}]
```

Out[]=


so I guess compared to this, our method of dominant balance performs better qualitatively.

(b) Find the leading behavior of $\int_0^1 \frac{e^{-xt}}{(1+t)^2}\, dt$ as $x \to 0^+$

The idea here is to find the asymptotic behavior of the integral by considering the asymptotic behavior of the integrand (as individual functions). So we notice that the integral has the form:

$$\int_a^b \zeta(x, t)\, dt, \text{ as } x \to 0^+,$$

in other words, x happens to be a parameter since the integration is in terms of change in t. Therefore, if we can find an asymptotic relation for ζ just in terms of t, i.e. if we find some holomorphic function ψ(t) that is asymptotically equivalent, then we can express the integrand as

$\zeta(x, t) = \psi_0(t) + \psi_1 (x-0)^n + \psi_2 (x-0)^n + \cdots$ for some n uniformly in $a \leq t \leq b$.

Once we have found such asymptotic expansion, then we can integrate term by term until the desired accuracy is achieved.

Looking back at the given problem, we can check to see if the expansion will have a constant term as the leading term:

In[∘]:=
```
Series[Exp[-x * t]/(1 + t)^2, {x, 0, 5}]
```

Out[∘]=
$$\frac{1}{(1+t)^2}-\frac{t\,x}{(1+t)^2}+\frac{t^2\,x^2}{2\,(1+t)^2}-\frac{t^3\,x^3}{6\,(1+t)^2}+\frac{t^4\,x^4}{24\,(1+t)^2}-\frac{t^5\,x^5}{120\,(1+t)^2}+O[x]^6$$

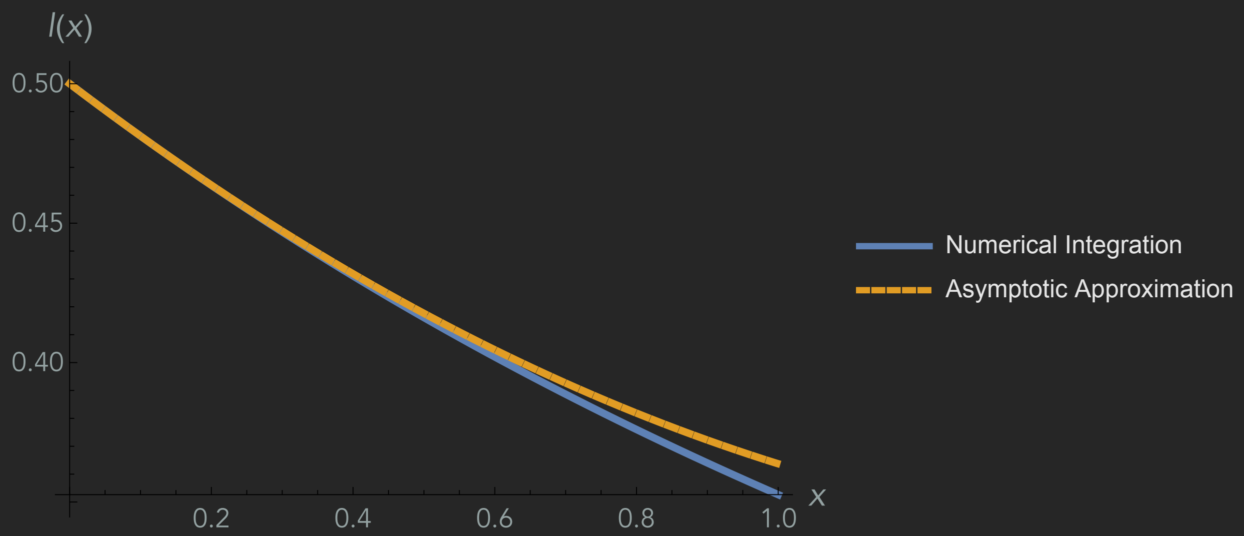
Which is does not. So we can just take the integral of this expansion, say the first two terms, and for an interval of $a=0\leq t\leq b=1$:

In[∘]:=
```
behavior = Integrate[Collect[Series[Exp[-x * t]/(1 + t)^2, {x, 0, 2}], x], {t, 0, 1}]
```

Out[∘]=
$$\frac{1}{4}\,(2+x\,(2+3\,x-4\,(1+x)\,\mathrm{Log}[2]))$$

now to check the solution:

In[∘]:=
```
Plot[{NIntegrate[Exp[-x * t]/(1 + t)^2, {t, 0, 1}], behavior}, {x, 0, 1},
 PlotStyle → {Directive[Solid, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["I(x)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"Numerical Integration", "Asymptotic Approximation"}]
```

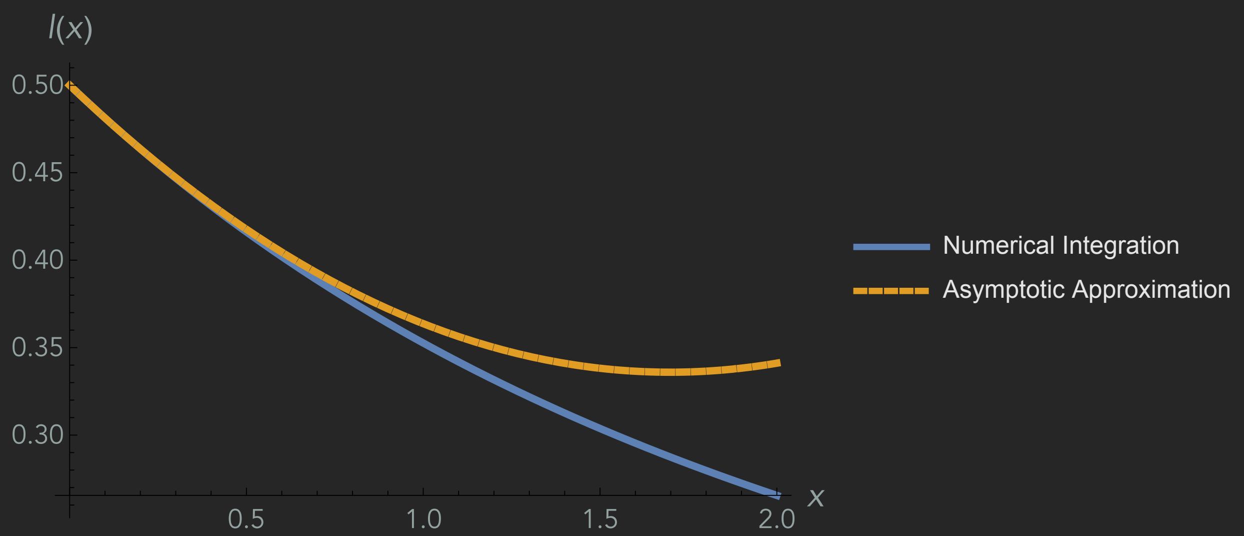
Out[∘]=


And if we want to make this more accurate, we can just take more terms:

In[∘]:=
```
betterApprox = Integrate[Collect[Series[Exp[-x * t]/(1 + t)^2, {x, 0, 5}], x], {t, 0, 1}]
```

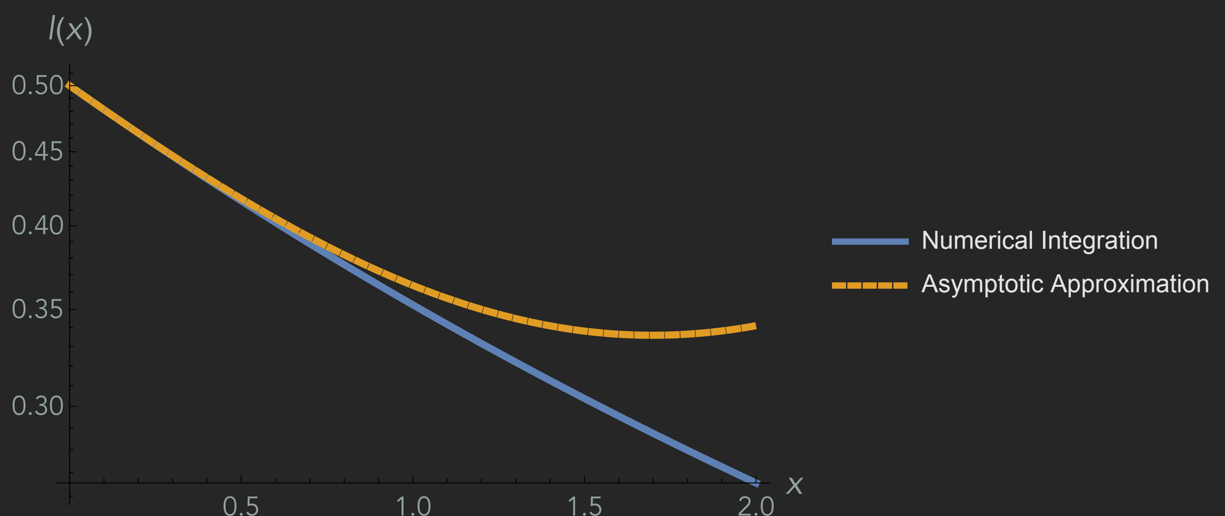
Out[∘]=
$$\frac{1}{2}+\frac{1}{1440}x\,(-720\,(-1+\mathrm{Log}[4])+x\,(x\,(x\,(170+41\,x-60\,(4+x)\,\mathrm{Log}[2])-240\,(-2+\mathrm{Log}[8]))-360\,(-3+\mathrm{Log}[16])))$$

```
Plot[{NIntegrate[Exp[-x * t]/(1 + t)^2, {t, 0, 1}], behavior}, {x, 0, 2},
 PlotStyle → {Directive[Solid, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["I(x)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"Numerical Integration", "Asymptotic Approximation"}]
```



To see the difference in log-scale:

```
LogPlot[{Integrate[Exp[-x * t]/(1 + t)^2, {t, 0, 1}],
  NIntegrate[Exp[-x * t]/(1 + t)^2, {t, 0, 1}], behavior}, {x, 0, 2},
 PlotStyle → {Directive[Solid, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["I(x)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14], PlotLegends →
  {"Exact Integration", "Numerical Integration", "Asymptotic Approximation"}]
```



which seems good for $x \in [0, 1]$, but it does much worse as we get further away from 0. Also, it is important to note that these results are for a very small value of t, and increasing these window could also yield worse approximations.

(c) Find the asymptotic series for $\int_0^x t^{-1/4} e^{-t}\, dt,\ x \to 0^+$

To find the asymptotic series, we can first rewrite the given integrand with the substitution $s = t^{1/4}$:

$$\int_0^x t^{-1/4} e^{-t}\, dt = 4 \int_0^{x^{\frac{1}{4}}} s^2 e^{-s^4}\, ds$$

Now we can integrate this expansion instead, term by term:

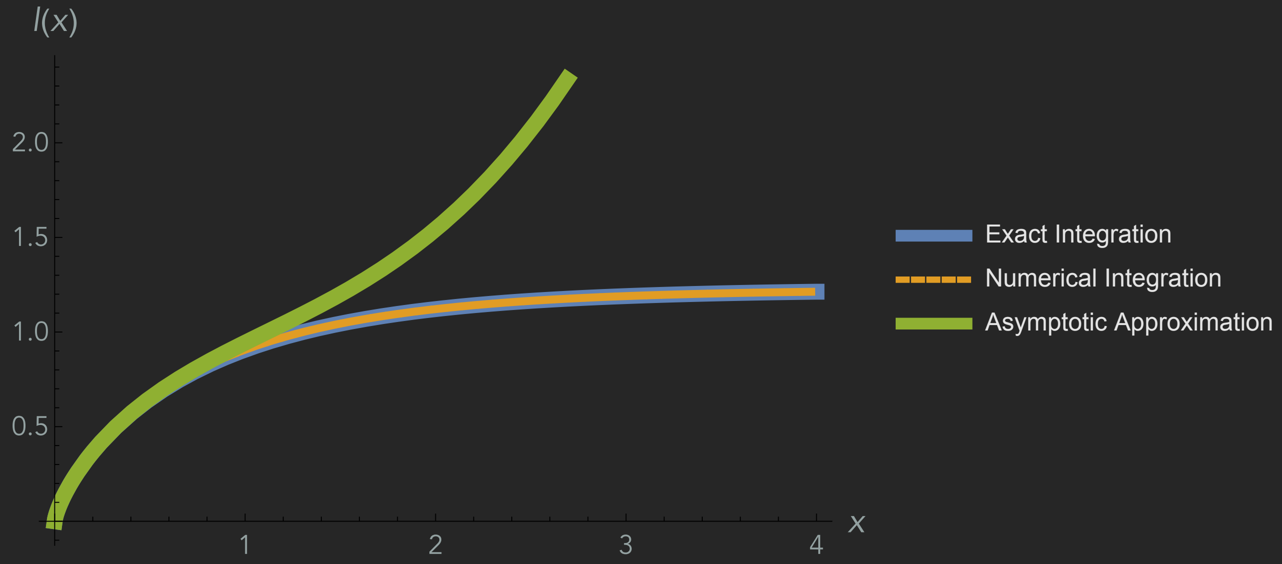
In[1]:=
```
approxInt = Integrate[Collect[Series[4 s^2 * Exp[-s^4], {s, 0, 10}], s], {s, 0, x^(1/4)}]
```

Out[1]=
$$\frac{4\, x^{3/4}}{3} - \frac{4\, x^{7/4}}{7} + \frac{2\, x^{11/4}}{11}$$

now to check our solution qualitatively:

In[2]:=
```
Plot[{Integrate[t^(-1/4) e^-t, {t, 0, x}], NIntegrate[t^(-1/4) e^-t, {t, 0, x}], approxInt}, {x, 0, 4},
 PlotStyle → {Directive[Solid, Thickness[0.02]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["I(x)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14], PlotLegends →
  {"Exact Integration", "Numerical Integration", "Asymptotic Approximation"}]
```

Out[2]=



and to make it even more accurate, we can take more terms:

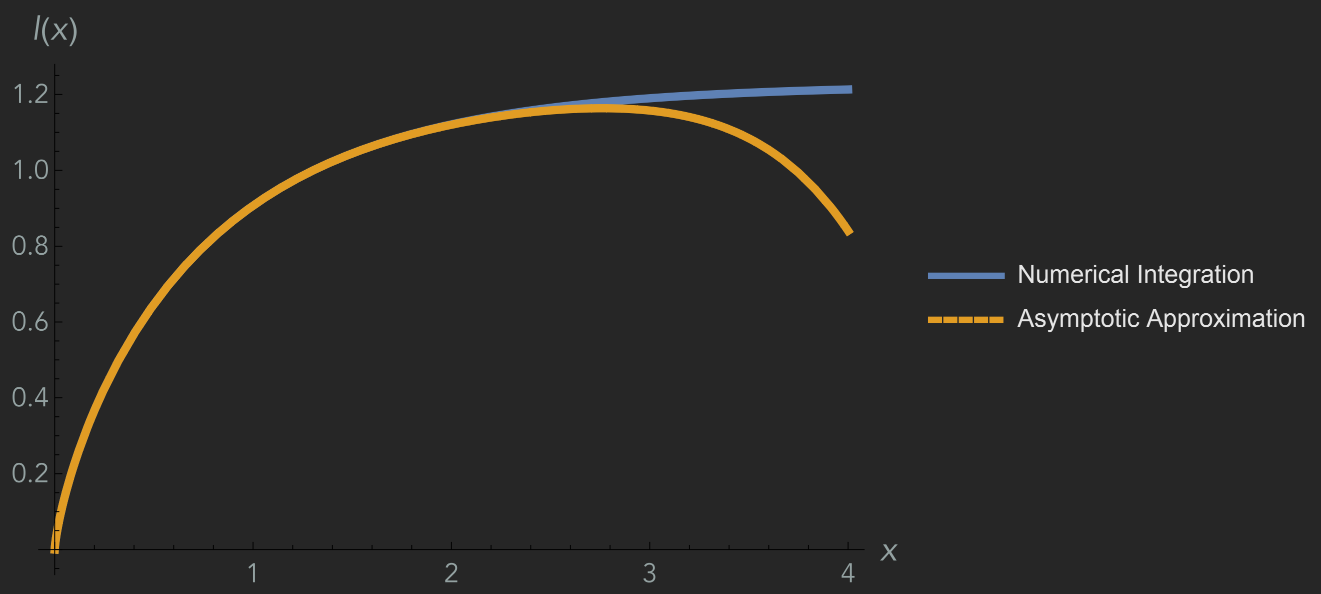
In[4]:=
```
betterApproxInt = Integrate[Collect[Series[4 s^2 * Exp[-s^4], {s, 0, 30}], s], {s, 0, x^(1/4)}]
```

Out[4]=
$$\frac{4\, x^{3/4}}{3} - \frac{4\, x^{7/4}}{7} + \frac{2\, x^{11/4}}{11} - \frac{2\, x^{15/4}}{45} + \frac{x^{19/4}}{114} - \frac{x^{23/4}}{690} + \frac{x^{27/4}}{4860} - \frac{x^{31/4}}{39\,060}$$

which we can once again plot:

In[ ]:=
```
Plot[{NIntegrate[t^(-1/4) e^(-t), {t, 0, x}], betterApproxInt}, {x, 0, 4},
 PlotStyle → {Directive[Solid, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["I(x)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14],
 PlotLegends → {"Numerical Integration", "Asymptotic Approximation"}]
```

Out[ ]=



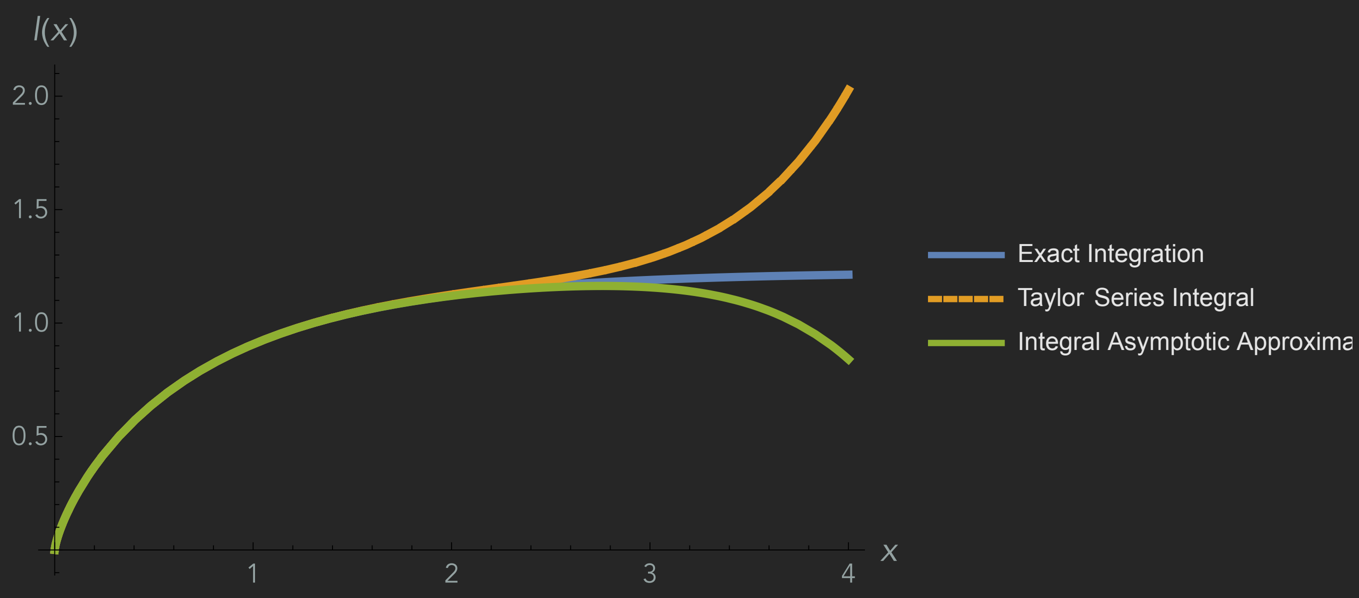
In[ ]:=
```
taylorSol = Collect[Integrate[ Collect[Series[t^(-1/4) e^(-t), {t, 0, 6}], t], {t, 0, x}] , x]
```

Out[ ]=
$$\frac{4\,x^{3/4}}{3}-\frac{4\,x^{7/4}}{7}+\frac{2\,x^{11/4}}{11}-\frac{2\,x^{15/4}}{45}+\frac{x^{19/4}}{114}-\frac{x^{23/4}}{690}+\frac{x^{27/4}}{4860}$$

finally as a last check, we can see if the solution that we get matches the one we would get if we just did a Taylor expansion, and integrated (which should be!)

In[5]:=
```
Plot[{Integrate[t^-1/4 e^-t, {t, 0, x}],
  Collect[Integrate[ Collect[Series[t^-1/4 e^-t, {t, 0, 6}], t], {t, 0, x}] , x],
  betterApproxInt}, {x, 0, 4},
 PlotStyle → {Directive[Solid, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 AxesLabel → {Style["x", Italic, 18], Style["I(x)", Italic, 18]},
 TicksStyle → Directive[FontSize → 14], PlotLegends →
  {"Exact Integration", "Taylor Series Integral", "Integral Asymptotic Approximation"}]
```

Out[5]=



which shows agreement between the series we derived using u-sub and the integral, and the approximation we would obtain if we had taken the Taylor series first and then had integrated the solution. Note that the Taylor series integration (orange line) has fewer terms than the integral asymptotics approximation (green line), hence the difference in the end behaviors. For both method, we can qualitatively say that the results match closely as $x \to 0^+$, as desired.