Math 223: Homework4

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Problem 1

Find the behavior as $x \to +\infty$ for the following equations:

(a) $x\, y^{(3)} = y'$

We let the ansatz to be y(t) = $e^{S(t)}$, which means that

$y'(t) = S' e^{S(t)}$

$y''(t) = S'' e^{S(t)} + (S')^2 e^{S(t)}$

$y'''(t) = S''' e^{S(t)} + 3\, S''\, S' e^{S(t)} + (S')^3 e^{S(t)}$

Now to use the method of dominant balance, we first replace our ansatz in the ODE:

In[∘]:=
```
x * y'''[x] - y'[x] == 0 /. y → Function[x, e^S[x]] // FullSimplify
```

Out[∘]= $e^{S[x]} \left(x\, S'[x]^3 + S'[x]\, (-1 + 3\, x\, S''[x]) + x\, S^{(3)}[x]\right) == 0$

Now we make the assumption that:

$x\, S'[x]^3 \sim S'$

and

$S^{(3)} \ll S''(x)\, S'(x) \sim S'(x)^3$

So now solving for S:

In[∘]:=
```
DSolve[x S'[x]^3 - S'[x] == 0, S[x], x]
```

Out[∘]= $\{\{S[x] \to c_1\}, \{S[x] \to -2\sqrt{x} + c_1\}, \{S[x] \to 2\sqrt{x} + c_1\}\}$

As expected, we have found 3 solutions; our hypotheses with these solutions also stand, which means we can proceed. The constant solution can be ignored for now, since it is an exact solution. Therefore we will only consider solutions $S(x) = \pm 2\sqrt{x} + C(x)$. So now our assumptions are:

$C(x) \ll S(x) \ll \sqrt{x}$

In[∘]:=
```
cPrime = D[-2 √x, {x, 1}]
```

Out[∘]= $-\frac{1}{\sqrt{x}}$

$$C'(x) \ll S'(x) \ll \frac{1}{\sqrt{x}}$$

In[∘]:=
```
cDoublePrime = D[-2 √x, {x, 2}]
```

Out[∘]= $\frac{1}{2\, x^{3/2}}$

Now to check if our assumption checks out or not:

In[•]:=
```
Limit[(cDoublePrime) / cPrime^2 , x → ∞]
```
Out[•]= 0

$$C''(x) \ll S'(x) \ll \frac{1}{x^{3/2}}$$

In[•]:=
```
D[-2 Sqrt[x] , {x, 3}]
```
Out[•]= $-\frac{3}{4\,x^{5/2}}$

$$C'''(x) \ll S'(x) \ll \frac{1}{x^{5/2}}$$

Now to get the ODEs in terms of C(x), and solving for it with our assumptions and as $x \to \infty$:

In[1]:=
```
x * y'''[x] - y'[x] == 0 /. y → Function[x, e^(2 x^(1/2) + c[x])] // FullSimplify
```
Out[1]= $\frac{1}{\sqrt{x}} e^{\sqrt{x}+c[x]} \left(3 - 6\sqrt{x} + 2x\left(6x\,c'[x]^2 + 2x^{3/2} c'[x]^3 + 6x\,c''[x] + c'[x]\left(-3 + 4\sqrt{x} + 6x^{3/2} c''[x]\right) + 2x^{3/2} c^{(3)}[x]\right)\right) == 0$

which using our assumptions, we can simplify to obtain:

$$\frac{2}{x} C'(x) - \frac{3}{2x^2} = 0$$

Now solving to find C(x) yields:

In[•]:=
```
DSolve[2/x * c'[x] + 3/2 x^-2 == 0, c[x], x]
```
Out[•]= $\left\{\left\{c[x] \to c_1 - \frac{3\,\mathrm{Log}[x]}{4}\right\}\right\}$

Therefore one of the final solution is:

$$y \sim \alpha\, x^{3/4}\, e^{2\sqrt{x}}$$

Similarly, the other solutions would be a constant solution, and

$$y \sim \beta\, x^{3/4}\, e^{-2\sqrt{x}}.$$

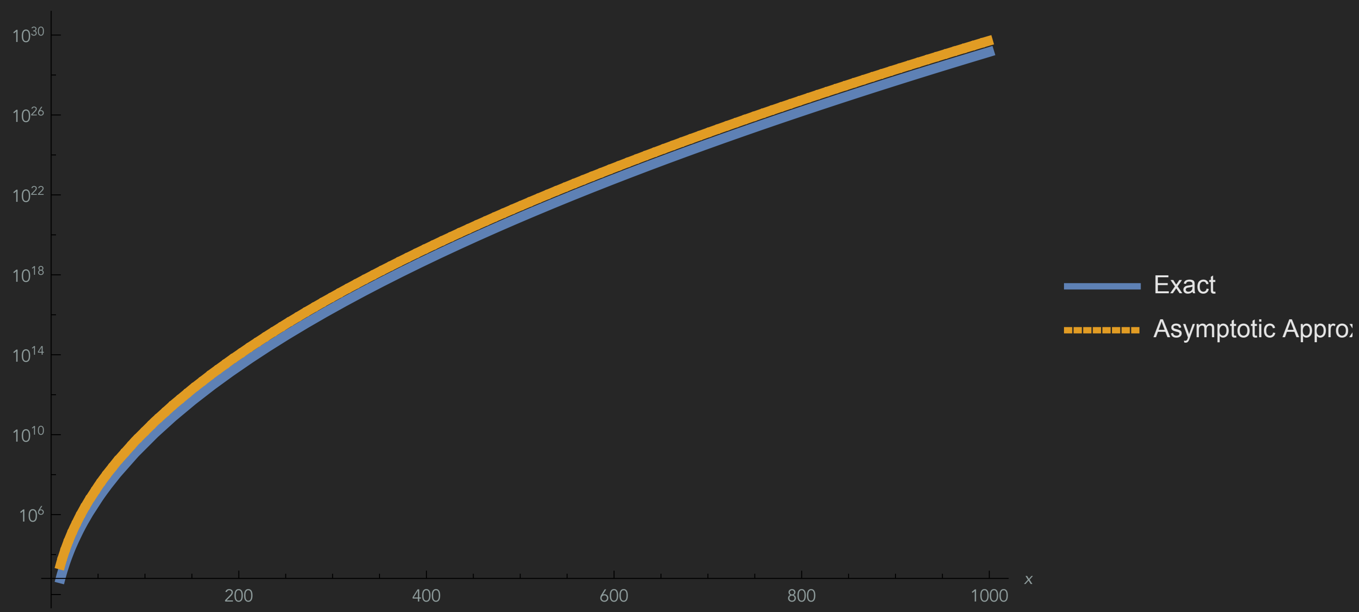
Now to verify our solution, we can plot the exact solution:

In[•]:=
```
DSolve[x * y'''[x] - y'[x] == 0, y[x], x]
```
Out[•]= $\{\{y[x] \to c_3 - x^2\, c_1\, \mathrm{Hypergeometric0F1Regularized}[3, x] + c_2\, \mathrm{MeijerG}[\{\{1\}, \{\}\}, \{\{1, 2\}, \{0\}\}, x]\}\}$

against the computed one that we found:

In[∘]:=
```
LogPlot[{x^2 * Hypergeometric0F1Regularized[3, x], x^(3/4) * E^(2 Sqrt[x])},
 {x, 10, 1000}, PlotStyle → {Directive[Solid, Thickness[0.01]],
   Directive[Dotted, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 PlotLegends → {"Exact", "Asymptotic Approximation"}, AxesLabel → Automatic]
```

Out[∘]=



It is important to note that these solutions are close up to a constant, which means our methods have found really nice approximations of the solutions.

```
(*Plot[{MeijerG[{{1},{}},{{1,2},{0}},x],500*x^(3/4)*E^(-2 Sqrt[x])},
 {x,10,100},PlotStyle→{Directive[Solid,Thickness[0.01]],
   Directive[Dotted,Thickness[0.01]],Directive[Dashed,Thickness[0.01]]},
 PlotLegends→{"Exact", "Asymptotic Approximation"},AxesLabel→Automatic]*)
```

(b) $y'' = \sqrt{x}\, y$

As before, we set the ansatz to be y(t) = $e^{S(t)}$ and using the derivatives found above, we have:

In[∘]:=
```
y''[x] == Sqrt[x] * y[x] /. y → Function[x, E^S[x]] // FullSimplify
```

Out[∘]= $e^{S[x]} \left(-\sqrt{x} + S'[x]^2 + S''[x]\right) == 0$

Now if we assume that $S''(x) \ll (S')^2$, we get

In[∘]:=
```
Solve[-Sqrt[x] + S'[x]^2 == 0, S'[x]]
```

Out[∘]= $\{\{S'[x] \to -x^{1/4}\}, \{S'[x] \to x^{1/4}\}\}$

to verify our assumption:

In[∘]:=
```
sPrimeSquared = (-x^(1/4))^2
sDoublePrime = D[(-x^(1/4)), {x, 1}]
```

Out[∘]= $\sqrt{x}$

Out[∘]= $-\frac{1}{4 x^{3/4}}$

In[∘]:=
```
Limit[sDoublePrime / sPrimeSquared, x → ∞]
```

Out[∘]= 0

* note that this would not be true if $x \to 0$

So now that our assumption holds, we find S(x) for the controlling factor to be:

In[∘]:=
```
DSolve[-√x + S'[x]^2 == 0, S[x], x]
```

Out[∘]= $\left\{\left\{S[x] \to -\frac{4 x^{5/4}}{5} + c_1\right\}, \left\{S[x] \to \frac{4 x^{5/4}}{5} + c_1\right\}\right\}$

So now we have to find the leading behavior:

$S = \pm \frac{4 x^{5/4}}{5} + C(x) \Rightarrow C(x) \ll x^{5/4}$
$C'(x) \ll x^{1/4}$
$C''(x) \ll x^{-3/4}$
now we revisit the original ODE with the leading behavior substitution:

In[∘]:=
```
y''[x] == √x * y[x] /. y → Function[x, e^(4 x^(5/4)/5 + c[x])] // FullSimplify
```

Out[∘]= $\frac{e^{\frac{x^{5/4}}{5} + c[x]} \left(1 + 4 x^{3/4} \left(2 x^{1/4} c'[x] + c'[x]^2 + c''[x]\right)\right)}{x^{1/4}} == 0$

In[∘]:=
```
1 + 4 x^(3/4) (2 x^(1/4) c'[x] + c'[x]^2 + c''[x]) // Expand
```

Out[∘]= $1 + 8 x c'[x] + 4 x^{3/4} c'[x]^2 + 4 x^{3/4} c''[x]$

Using our assumptions about C(x) orders, we can now solve the ODE:

In[∘]:=
```
DSolve[1 + 8 x c'[x] == 0, c[x], x]
```

Out[∘]= $\left\{\left\{c[x] \to c_1 - \frac{\text{Log}[x]}{8}\right\}\right\}$

Therefore one of the asymptotic solutions being:

$$y \sim \alpha x^{-1/8} e^{\frac{4 x^{5/4}}{5}}$$

and another one being

$$y \sim \alpha x^{-1/8} e^{-\frac{4 x^{5/4}}{5}}$$

To verify these solutions, we can plot them against the exact ones:

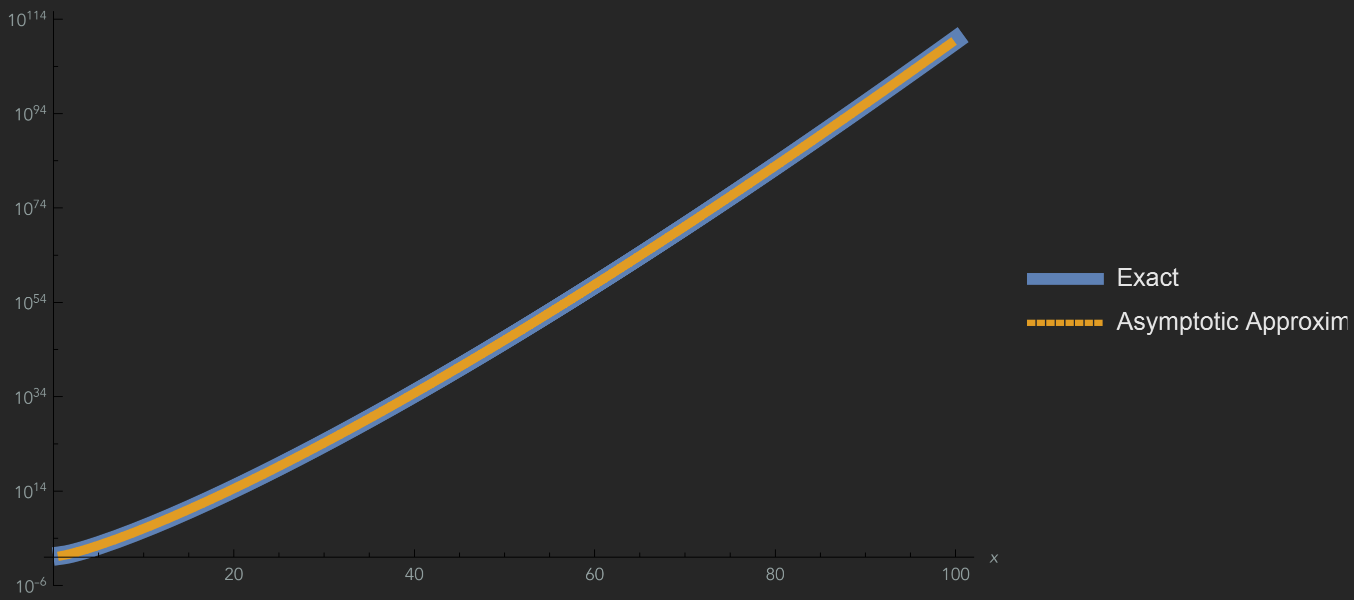
In[∘]:=
```
exact = DSolve[y''[x] == √x y[x], y[x], x]
```

Out[∘]= $\left\{\left\{y[x] \to \left(\frac{2}{5}\right)^{2/5} \sqrt{x}\, \text{BesselI}\left[-\frac{2}{5}, \frac{4 x^{5/4}}{5}\right] c_1\, \text{Gamma}\left[\frac{3}{5}\right] + \left(-\frac{2}{5}\right)^{2/5} \sqrt{x}\, \text{BesselI}\left[\frac{2}{5}, \frac{4 x^{5/4}}{5}\right] c_2\, \text{Gamma}\left[\frac{7}{5}\right]\right\}\right\}$

Plotting the first solution:

In[∘]:=
```
LogPlot[{(2/5)^(2/5) Sqrt[x] BesselI[-2/5, (4 x^(5/4))/5] * 2 * Gamma[3/5], x^(-1/8) * E^(4/5 x^(5/4))},
 {x, 1, 100}, PlotStyle → {Directive[Solid, Thickness[0.02]],
   Directive[Dotted, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 PlotLegends → {"Exact", "Asymptotic Approximation"}, AxesLabel → Automatic]
```

Out[∘]=



Which shows the solutions to match VERY nicely! This is good because we approximated the exponential growth method, which seems to work very nicely.

(c) $y'' = e^{\frac{-3}{x}}\, y$

Following the same procedure as before, we can obtain:

In[∘]:=
```
y''[x] == E^(-3/x) * y[x] /. y → Function[x, e^S[x]] // FullSimplify
```

Out[∘]= $e^{S[x]} \left(S'[x]^2 + S''[x]\right) == e^{-\frac{3}{x}+S[x]}$

Now we make an assumption that $S'' \ll (S'(x))^2$, which simplifies the above expression to:

$(S'(x))^2 = e^{\frac{-3}{2x}} \Rightarrow S'(x) = e^{\frac{-3}{2x}}$

Now to check our assumption:

In[∘]:=
```
sPrimeSquared = (e^(-3/(2 x)))^2
sDoublePrime = D[(e^(-3/(2 x))), {x, 1}]
```

Out[∘]= $e^{-3/x}$

Out[∘]= $\dfrac{3\, e^{-\frac{3}{2}/x}}{2\, x^2}$

In[∘]:=
```
Limit[sDoublePrime / sPrimeSquared, x → ∞]
```

Out[∘]= 0

Which means our assumption holds. So now we can solve for S(x)

In[∘]:= DSolve$\left[S'[x]^2 == e^{\frac{-3}{x}}, S[x], x\right]$

Out[∘]= $\left\{\left\{S[x] \to -e^{-\frac{3}{2}/x}\, x + c_1 - \frac{3}{2}\,\text{ExpIntegralEi}\left[-\frac{3}{2x}\right]\right\}, \left\{S[x] \to e^{-\frac{3}{2}/x}\, x + c_1 + \frac{3}{2}\,\text{ExpIntegralEi}\left[-\frac{3}{2x}\right]\right\}\right\}$

since we have the exponential integral as part of both solutions, we want to expand it (and only use the first few terms):

In[∘]:= Series$\left[\frac{3}{2}\,\text{ExpIntegralEi}\left[-\frac{3}{2x}\right], \{x, \infty, 10\}\right]$

Out[∘]= $\frac{3}{2}\left(\text{EulerGamma} + \text{Log}\left[\frac{3}{2}\right] - \text{Log}[x]\right) - \frac{9}{4x} + \frac{27}{32x^2} - \frac{9}{32x^3} + \frac{81}{1024x^4} - \frac{243}{12800x^5} + \frac{81}{20480x^6} - \frac{729}{1003520x^7} + \frac{2187}{18350080x^8} - \frac{81}{4587520x^9} + \frac{2187}{917504000x^{10}} + O\left[\frac{1}{x}\right]^{11}$

So taking the first term, we now can write S(x) as :

$$S(x) = \left(e^{-\frac{3}{2}/x}\, x + \frac{9}{4x}\right) + C(x)$$

which for ease of notation, we keep as S(x). Moving on to solving for C(x), we first make the following assumptions:

$C(x) \sim e^{-\frac{3}{2}/x}$

and we solve for C(x)

In[∘]:= y''[x] == $E^{\frac{-3}{x}}$ * y[x] /. y → Function$\left[x, e^{S[x]+c[x]}\right]$ // FullSimplify

Out[∘]= $e^{c[x]+S[x]}\left(-e^{-3/x} + (c'[x] + S'[x])^2 + c''[x] + S''[x]\right) == 0$

Which means that :

$-e^{-3/x} + (C'(x) + S'(x))^2 + C''(x) + S''(x) = 0$

$= -e^{-3/x} + C'(x)^2 + S'(x)^2 + 2C'(x)\,S'(x) + C''(x) = 0$

We have that $S'(x) = e^{-\frac{3}{2x}}$, and $(S')^2 = e^{-3/x}$:

$= -e^{-3/x} + e^{-3/x} + C'(x)^2 + 2C'(x)\,S'(x) + C''(x) + S''(x) = 0$

$= C'(x)^2 + 2C'(x)\,S'(x) + C''(x) + S''(x) = 0$

Now if we assume that $C''(x)^2 \ll 2C'(x)\,S'(x)$, we can simplify the above expression to get:

$C'(x)^2 + 2C'(x)\,S'(x) + S''(x) = 0$

finally, we make another assumption that $(C'(x))^2 \ll C(x)\,S(x)$, leaving us with:

$2C'(x)\,S'(x) + S''(x) = 0$

$= 2C'(x) = -\frac{S''(x)}{S'(x)} \Rightarrow 2C(x) = -\log(S'(x))$

and we have that $S'(x) = e^{-\frac{3}{2x}}$, so replacing this gives us:

$C(x) = -\frac{1}{2}\log\left(e^{-\frac{3}{2x}}\right) = \frac{3}{4x}$

Now to verify our assumptions:

(i) $(C'(x))^2 \ll C'(x)\,S'(x)$

(ii) $(C''(x))^2 \ll C'(x)\,S'(x)$

In[ ]:=
```
(* first assumption *)
cPrime = D[3/(4 x), {x, 1}];
sPrime = -E^(-3/(2 x));
Limit[cPrime^2 / (cPrime * sPrime) , x → ∞]
```

Out[ ]= 0

So the first assumption is true.

In[ ]:=
```
(* second assumption *)
cDoublePrime = D[cPrime, {x, 1}];
Limit[cDoublePrime^2 / (cPrime * sPrime) , x → ∞]
```

Out[ ]= 0

and therefore the second assumption is also true.

Putting everything together, we obtain that one solution is:

$$y \sim \alpha x^{-3/4} e^{\left(x e^{\frac{-3}{2x}} + \frac{9}{4x}\right)}$$

and the other solution will be:

$$y \sim \beta x^{-3/4} e^{-\left(x e^{\frac{-3}{2x}} + \frac{9}{4x}\right)}$$

Since we can not find an exact solution with Mathematica, we can not plot our solution against the exact solution. However, we can do an asymptotic solve, and plot that against our approximation instead. Since we focused on $x \to \infty$, we will compare the solution for the exponential growth case:

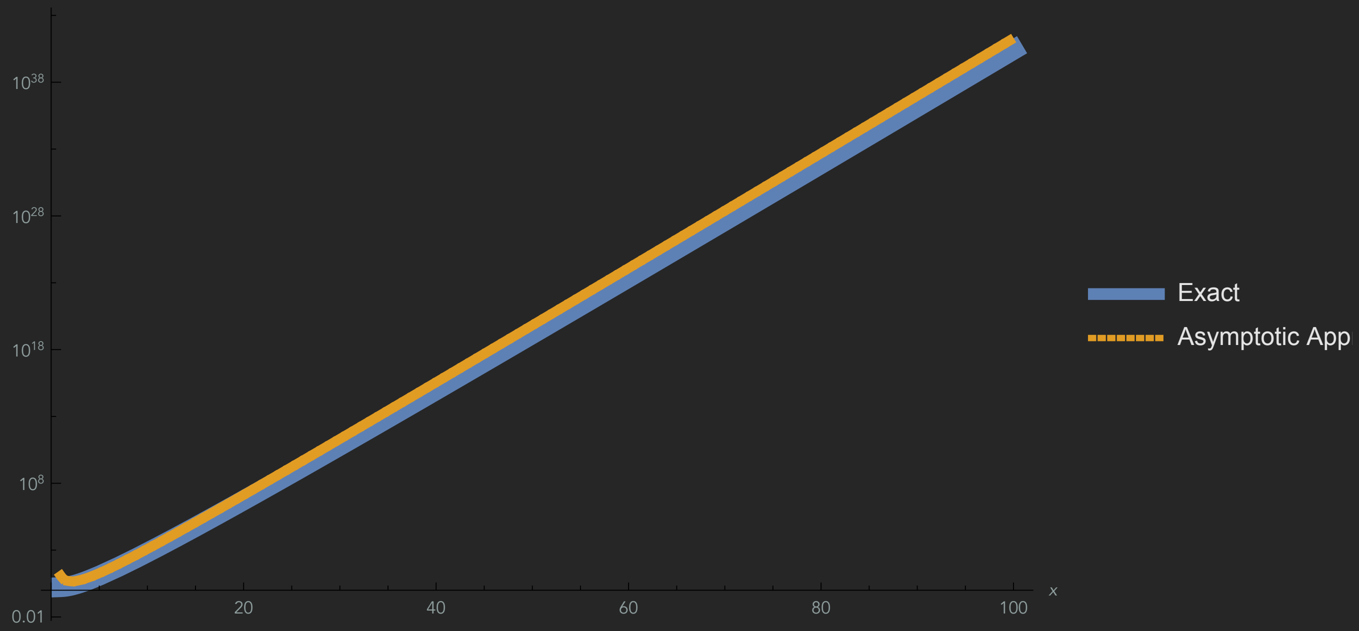
In[19]:=
```
asymSol = AsymptoticDSolveValue[y''[x] == E^(-3/x) * y[x], y[x], x → ∞]
```

Out[19]= $e^{-x}\left(\frac{15\sqrt{x}}{8} + x^{3/2}\right) c_1 + e^{x}\left(-\frac{3}{8\, x^{5/2}} + \frac{1}{x^{3/2}}\right) c_2$

In[28]:=

```
LogPlot[{ⅇ^x (-3/(8 x^(5/2)) + 1/x^(3/2)), x^(-3/4) * E^(x*E^(-3/(2*x)) + 9/(4 x))},
  {x, 1, 100}, PlotStyle → {Directive[Solid, Thickness[0.02]],
    Directive[Dotted, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
  PlotLegends → {"Exact", "Asymptotic Approximation"}, AxesLabel → Automatic]
```

Out[28]=



Which shows the solutions to match nicely. So this semi-verifies our solutions as asymptotic solutions to when $x \to \infty$.