Math 223: Homework 11

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Problem 1

Determine the averaged equations for the van der Pol oscillator,

$$y'' + \epsilon\left(y^2 - 1\right)y' + y = 0$$

Use the averaged equations to show that it has a stable limit cycle that is nearly circular with radius $r = 2 + O(\epsilon)$.

Using the general average theory we derived in class, we can Now we can use the definition of y and y' to find what h must be; that is, we know that $y' = -r\sin(\tau+\phi) = -r\sin(\theta)$ and $y = r^2\cos^2(\tau+\phi) = r^2\cos^2(\theta)$. So we can write h as : $h = y'\left(y^2-1\right) = \langle\left(r^2\cos^2(\theta)-1\right)(-r\sin(\theta))\langle$ = $\langle r\sin^2(\theta)\rangle$ - $\langle r^3\cos^2(\theta)\sin^2(\theta)\rangle$

now using Mathematical to find the running average:

```
In[•]:= Integrate[1 / (2 * π) ((r^2 * Cos[θ]^2) - 1) * (-r * Sin[θ]^2), {θ, 0, 2 * π}]
```

Out[•]= $-\frac{1}{8} r \left(-4 + r^2\right)$

which gives us:

$-\frac{1}{8}r^3 + \frac{1}{2}r$

and to find $r\ \phi'$:

$r\phi'$ = $\langle h\cos(\theta)\rangle$ = $\langle\left(r^2\cos^2(\theta)-1\right)(\cos(\theta))(-r\sin(\theta)\rangle$ = $\langle r\sin(\theta)\cos(\theta)\rangle$ - $\langle r^3\cos^3(\theta)\sin(\theta)\rangle$

once again we can use Mathematica to find the average:

```
In[•]:= Integrate[((r^2 * Cos[θ])^2 - 1) * (-r * Sin[θ] * Cos[θ]), {θ, 0, 2 * π}]
```

Out[•]= 0

As expected (since we are integrating an odd function from 0 to 2 π) we find that $r\ \phi'$ = 0. So this means that ϕ' = 0 $\Longrightarrow \phi = C$.

In[•]:= $\text{DSolve}\left[y'[x] == -\frac{1}{8} y[x]^3 + \frac{1}{2} y[x], y[x], x\right]$

Out[•]= $\left\{\left\{y[x] \to -\frac{2\, e^{x/2}}{\sqrt{e^x + e^{8 c_1}}}\right\}, \left\{y[x] \to \frac{2\, e^{x/2}}{\sqrt{e^x + e^{8 c_1}}}\right\}\right\}$

So we know that our solution is $r(\tau) = \pm\frac{2\, e^{\tau/2}}{\sqrt{e^{\tau} + e^{8 c_1}}} + O(\epsilon)$. Given that we do not have an initial condition, we have to determine which one of the solutions gives us a stable limit cycle, which in this case will be the positive solution (since the negative radius is meaningless). Also, the other fixed point at 0 is not an attractor, so indeed this limit cycle will be a stable limit cycle.

Now to find the limit cycle with the given radius, we can let $\tau \to \infty$, as the following:

In[ ]:=
```
Limit[2 e^(τ/2) / Sqrt[e^τ + e^(8 c1)], τ → ∞]
```
$$\text{Limit}\left[\frac{2\, e^{\tau/2}}{\sqrt{e^{\tau} + e^{8 c_1}}}, \tau \to \infty\right]$$

Out[ ]= 2

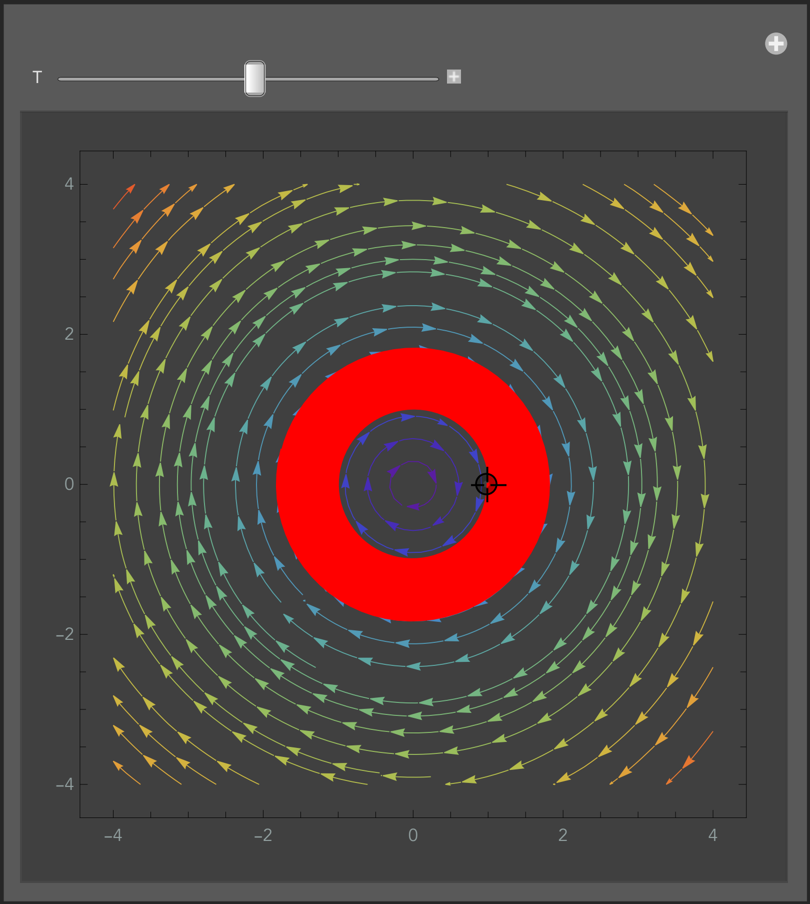
So we can see that as $\tau \to \infty$, we have that $r(\tau) = 2 + O(\epsilon)$. Our first order asymptotic solution now becomes:

$$y_0 \sim r(\tau) \cos(\tau + \phi) = 2 \cos(\tau + C) + O(\epsilon).$$

We can show this using the StreamPlot of Mathematica :

In[44]:=
```
splot = StreamPlot[{v, -u - 0.01 * (u^2 - 1) * v},
   {u, -4, 4}, {v, -4, 4}, StreamColorFunction → "Rainbow"];
Manipulate[Show[splot, ParametricPlot[Evaluate[
     First[{u[t], v[t]} /. NDSolve[{u'[t] == v[t], v'[t] == -u[t] - 0.01 * (u[t]^2 - 1) * v[t],
         Thread[{u[0], v[0]} == point]}, {u, v}, {t, 0, T}]]], {t, 0, T}, PlotStyle → Red]],
 {{T, 20}, 1, 500}, {{point, {1, 0}}, Locator}, SaveDefinitions → True]
```

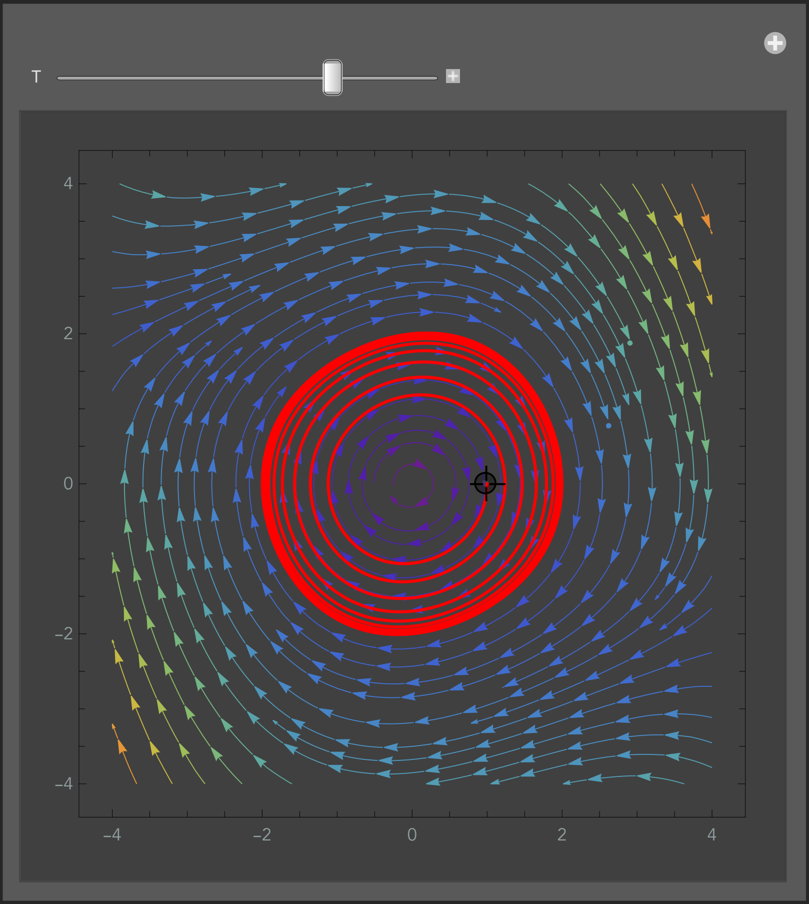
Out[45]=



In this case, we can see that we do get the behavior of a stable limit cycle, as expected. I am curious to see how well this holds if we increase the value of ϵ:

In[40]:=
```
splot2 = StreamPlot[{v, -u - 0.1 * (u^2 - 1) * v},
   {u, -4, 4}, {v, -4, 4}, StreamColorFunction → "Rainbow"];
Manipulate[Show[splot2, ParametricPlot[Evaluate[
     First[{u[t], v[t]} /. NDSolve[{u'[t] == v[t], v'[t] == -u[t] - 0.1 * (u[t]^2 - 1) * v[t],
         Thread[{u[0], v[0]} == point]}, {u, v}, {t, 0, T}]]], {t, 0, T}, PlotStyle → Red]],
  {{T, 10}, 1, 100}, {{point, {1, 0}}, Locator}, SaveDefinitions → True]
```

Out[41]=



We can see that for a larger value of ϵ the limit cycle is still there, as it should be.

Problem 2

Find and solve the averaged equations for

$$\frac{d\,y^2}{d\,t^2} + y + \epsilon\left(y^2\right)\frac{d\,y}{d\,t} = 0\,,\ \ y\,(0) = 1,\ y'\,(0) = 0$$

Describe the long-term behavior of the solution based on your results.

Using the same theory we derived before, we know that $y = r\cos\,(\tau + \phi) = r\cos\,(\theta)$, so this means that h = $\langle\left(r^2\cos^2(\theta)\right)\left(-r\sin\,(\theta)\right)\rangle$ So r' becomes:

$r' = \langle h\sin\,(\theta)\rangle$ = $\langle\left(-r^3\sin\,(\theta)\cos^2(\theta)\right)\sin\,(\theta)\rangle$=-$r^3\langle\sin^2(\theta)\cos^2(\theta)\rangle$. To find the running average of this, we can use Mathematica:

In[ ]:=
```
Integrate[1 / (2 * π) (-r^3 * Cos[θ]^2 * Sin[θ]^2), {θ, 0, 2 * π}]
```

Out[ ]= $-\frac{r^3}{8}$

which means that we have $r\ ' = -\frac{r^3}{8}$. Similarly, we can find $r\ \ \phi'$, which is given as:

$r' \phi = \langle\left(-r^3 \sin(\theta)\cos^2(\theta)\right)\cos(\theta)\rangle = \langle\left(-r^3 \sin(\theta)\cos^3(\theta)\right)\rangle$ Once again we can use Mathematica:

In[ ]:=
```
Integrate[1 / (2 * π) (r^2 * Cos[θ]^3 * Sin[θ]), {θ, 0, 2 * π}]
```

Out[ ]= 0

so we find that $r\ \ \phi' = 0$, implying that $\phi = C$.

So no we want to find r, given the initial conditions, for which we use Mathematica.

In[ ]:=
```
DSolve[{y'[x] == -1/8 y[x]^3, y[0] == 1}, y[x], x]
```

Out[ ]= $\left\{\left\{y[x] \to \frac{2}{\sqrt{4+x}}\right\}\right\}$

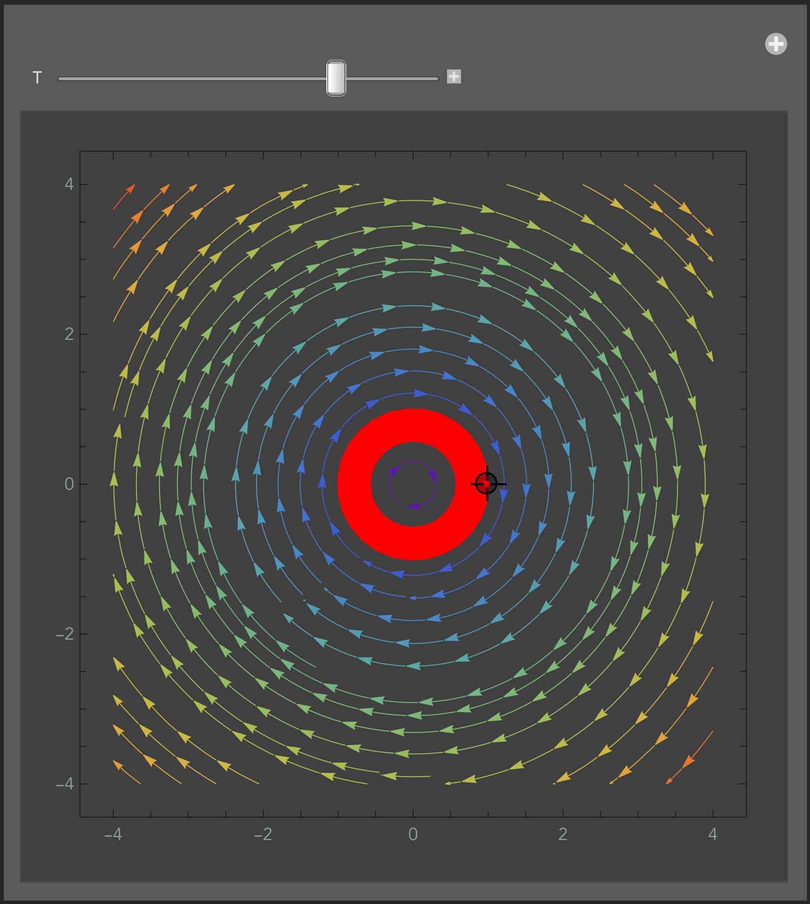
So we have that $r\ \ (\tau) = \frac{2}{\sqrt{4+\tau}} + O(\epsilon)$. So now to find the limit cycle's stability and radius, we can let $\tau \to \infty$, which gives us 0. This means that we should not expect a stable limit cycle time increases.

So our first order solution is:

$y_0 = r(\tau)\cos(\phi + \tau) = \frac{2}{\sqrt{4+\tau}}\cos(\tau + C) + O(\epsilon)$

In[50]:=
```
splot3 = StreamPlot[{v, -u - 0.01 * (u^2) * v},
   {u, -4, 4}, {v, -4, 4}, StreamColorFunction → "Rainbow"];
Manipulate[Show[splot, ParametricPlot[Evaluate[
     First[{u[t], v[t]} /. NDSolve[{u'[t] == v[t], v'[t] == -u[t] - 0.01 * (u[t]^2) * v[t],
         Thread[{u[0], v[0]} == point]}, {u, v}, {t, 0, T}]]], {t, 0, T}, PlotStyle → Red]],
 {{T, 20}, 1, 1000}, {{point, {1, 0}}, Locator}, SaveDefinitions → True]
```

Out[51]=



As we predicted earlier, we do not have a stable limit cycle, and the trajectory seems to be closing in and being attracted to 0, without a limit cycle; this was expected because we found $r \qquad = 0$. Let us now plot the true solution against what we find to see how well we approximated the oscillatory solution:

In[48]:=
```
NumericalSolution =
 NDSolve[{y''[t] + y[t] + 0.01 * (y[t]^2) * y'[t] == 0, y[0] == 1, y'[0] == 0}, y[t], {t, 0, 100}]
```

Out[48]= {{y[t] → InterpolatingFunction[Domain: {{0., 100. }} Output: scalar][t]}}

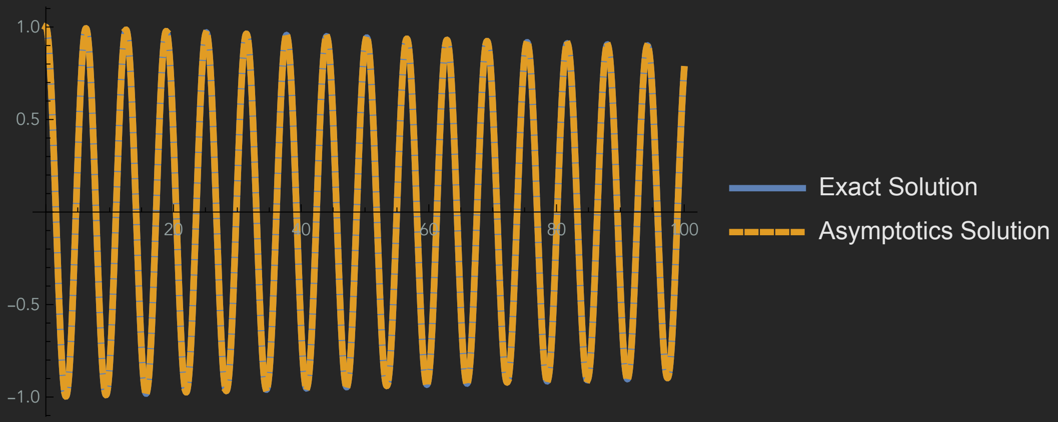
In[14]:=
```
AsymptoticApproximation[t_, ε_] = (2 * Cos[t] / (Sqrt[4 + (ε * t)]))
```

Out[14]=

$$\frac{2\,\mathrm{Cos}[t]}{\sqrt{4 + t\,\varepsilon}}$$

In[49]:=
```
Plot[{y[t] /. NumericalSolution, AsymptoticApproximation[t, 0.01]}, {t, 0, 100},
 PlotStyle → {Directive[Solid, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 PlotLegends → {"Exact Solution", "Asymptotics Solution"}, PlotRange → All]
```

Out[49]=



In this case, we do not have a limit cycle for the true solution. Our averaging methodAs we can see, our asymptotic approximation closely matches the true solution, and it seems like the oscillations are dampening (extremely slowly!!) as time goes to infinity.