

# Math 223: Homework3

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## Problem 1

Find the behavior as  $x \rightarrow 0^+$  for the following equations:

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(a)  $x^4 y^{(3)} = y$

We let the ansatz to be  $y(t) = e^{S(t)}$ , which means that

$$y'(t) = S' e^{S(t)}$$

$$y''(t) = S'' e^{S(t)} + (S')^2 e^{S(t)}$$

$$y'''(t) = S''' e^{S(t)} + 3S'' S' e^{S(t)} + (S')^3 e^{S(t)}$$

To verify the derivatives we found by hand:

$$\text{In[=]} := x^4 y'''[x] == y[x] \quad / . \quad y \rightarrow \text{Function}[x, e^{S[x]}] // \text{FullSimplify}$$

$$\text{Out[=]} = e^{S[x]} x^4 (S'[x]^3 + 3 S'[x] S''[x] + S^{(3)}[x]) == e^{S[x]}$$

Due to the existence of the irregular singular point at  $x = 0$ , we make the assumption that  $S'' \ll (S')^2$ , which means that now we can solve for  $S'(x)$  as:

$$\text{In[=]} := \text{DSolve}[x^4 (S'[x]^3) == 1, S[x], x] // \text{Simplify}$$

$$\text{Out[=]} = \left\{ \left\{ S[x] \rightarrow -\frac{3}{x^{1/3}} + c_1 \right\}, \left\{ S[x] \rightarrow \frac{3 (-1)^{1/3}}{x^{1/3}} + c_1 \right\}, \left\{ S[x] \rightarrow -\frac{3 (-1)^{2/3}}{x^{1/3}} + c_1 \right\} \right\}$$

To check if our assumption was correct:

$$\text{In[=]} := \text{sDoublePrime} = D[-\frac{3}{x^{1/3}}, \{x, 2\}]$$

$$\text{Out[=]} = -\frac{4}{3 x^{7/3}}$$

$$\text{In[=]} := \text{sPrimeSquared} = \left( D[-\frac{3}{x^{1/3}}, \{x, 1\}] \right)^2$$

$$\text{Out[=]} = \frac{1}{x^{8/3}}$$

$$-\frac{4}{3 x^{7/3}} \ll \frac{1}{x^{8/3}} \text{ as } x \rightarrow 0^+, \text{ our assumption about } S'' \text{ is correct!}$$

So now we have that  $S(x) \sim x^{-1/3}$ , so we can write the controlling factor as  
 $y = e^{3x^{-1/3}}$

for  $x \rightarrow 0^+$ . Now to find the leading behavior, we take  $S(x) = 3x^{-1/3} + c(x)$ :

```
In[6]:= x^4 y'''[x] == y[x] /. y → Function[x, e^{S[x]}] // FullSimplify
Out[6]= e^{S[x]} x^4 (S'[x]^3 + 3 S'[x] S''[x] + S^{(3)}[x]) == e^{S[x]}

In[7]:= Solve[x^4 y'''[x] == y[x] /. y → Function[x, e^{3 x^{-1/3} + c[x]}], c''[x]] // FullSimplify // Expand
Out[7]= {c''[x] → 2/(3 (-x^{8/3} + x^4 c'[x])) + 4 x^{1/3}/(3 (-x^{8/3} + x^4 c'[x])) + 28 x^{2/3}/(27 (-x^{8/3} + x^4 c'[x])) - x^{4/3} c'[x]/(-x^{8/3} + x^4 c'[x]) - 4 x^{5/3} c'[x]/(3 (-x^{8/3} + x^4 c'[x])) + x^{8/3} c'[x]^2/(-x^{8/3} + x^4 c'[x]) - x^4 c'[x]^3/(3 (-x^{8/3} + x^4 c'[x])) - x^4 c^{(3)}[x]/(3 (-x^{8/3} + x^4 c'[x]))}
```

We assume that  $c''$  and higher terms are negligible, so simplifying the above expression just for the manageable terms with  $c(x)$ , we obtain:

$4 - 3x c'[x]^3$ , which we can solve this as

```
In[8]:= DSolve[4 - 3 x * c'[x] == 0, c[x], x] // FullSimplify // Expand
Out[8]= {c[x] → c_1 + 4 Log[x]/3}
```

So now this gives us the solution:

$$y \sim \alpha x^{4/3} e^{x^{-1/3}}$$

for  $\alpha$  some constant, as  $x \rightarrow 0^+$ .

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(b)  $x^6 y'' = e^x y$

Following the same procedure as before, we can obtain:

```
In[9]:= x^6 y''[x] == e^x y[x] /. y → Function[x, e^{S[x]}] // FullSimplify
Out[9]= e^{S[x]} x^6 (S'[x]^2 + S''[x]) == e^{x+S[x]}
```

Solving for  $S'(x)$ :

$Solve[x^6 (S'[x]^2) == e^x, S'[x]] // FullSimplify$

```
Out[10]= {S'[x] → -e^{x/2}/x^3, S'[x] → e^{x/2}/x^3}
```

To check to see if our assumption about  $S''(x)$  is correct:

```
In[11]:= sDoublePrime = D[-e^{x/2}/x^3, x]
```

```
Out[11]= 3 e^{x/2}/x^4 - e^{x/2}/(2 x^3)
```

```
Out[12]= 3 e^{x/2}/x^4 - e^{x/2}/(2 x^3)
```

$$\text{In}[=]: \text{sPrimeSquared} = \left( -\frac{e^{x/2}}{x^3} \right)^2$$

$$\text{Out}[=]: \frac{e^x}{x^6}$$

$$\text{In}[=]: \text{Limit}[\text{sPrimeSquared} / \text{sDoublePrime}, x \rightarrow 0]$$

$$\text{Out}[=]: \infty$$

So it does check out that  $S'' \ll (S')^2$ , but only for the above solution (the other one will go to  $-\infty$ ).

So now to see what polynomial we could use for which means that which we can expand to get the dominant terms:

$$\text{In}[1]:= \text{Series}[-e^{x/2} / x^3, \{x, 0, 2\}]$$

$$\text{Out}[1]:= -\frac{1}{x^3} - \frac{1}{2x^2} - \frac{1}{8x} - \frac{1}{48} - \frac{x}{384} - \frac{x^2}{3840} + O[x]^3$$

taking the first terms, we can write the solution for  $S(x)$  as

$$S'(x) \sim \frac{-1}{x^3}, S(x)' \sim \frac{1}{x^3}$$

which means that  $S(x) = \pm \frac{1}{2x^2}$ . This yields controlling factors being:

$$y \sim e^{2x^{-2}}$$

To determine the leading behavior, we can now let  $S(x) = x^{-2} + c(x)$ , with  $c(x) \ll x^{-2}$ . So now we solve the equation for this:

$$\text{In}[=]: x^6 y''[x] == e^x y[x] /. y \rightarrow \text{Function}[x, e^{x^{-2}+c[x]}] // \text{FullSimplify}$$

$$\text{Out}[=]: e^{\frac{1}{x^2}+c[x]} (-4 + e^x - 6x^2 + 4x^3 c'[x] - x^6 c'[x]^2 - x^6 c''[x]) == 0$$

$$\text{In}[=]: x^6 y''[x] == e^x y[x] /. y \rightarrow \text{Function}[x, e^{x^{-2}+c[x]}] // \text{FullSimplify}$$

$$\text{Out}[=]: e^{\frac{1}{x^2}+c[x]} (-4 + e^x - 6x^2 + 4x^3 c'[x] - x^6 c'[x]^2 - x^6 c''[x]) == 0$$

Now assuming that  $c''(x) \ll (c'(x))^2$ , we end up with  $4x^3 c'[x] - x^6 c'[x]^2$  as the leading manageable terms, so this means that  $c'(x) \sim \frac{1}{x^3}$ . To see how this is:

$$\text{In}[=]: \text{DSolve}\left[ \frac{4 c'[x]}{x^3} - c'[x]^2 == 0, c[x], x \right]$$

$$\text{Out}[=]: \left\{ \{c[x] \rightarrow c_1\}, \left\{ c[x] \rightarrow -\frac{2}{x^2} + c_1 \right\} \right\}$$

Clearly, we can not have the first solution (since the assumption will not hold), so we have:

$$y \sim \alpha e^{(-2x^{-2}+x^{-2})} = \alpha e^{(-x^{-2})}$$

$$(c) x^4 y'' - x^2 y' + \frac{1}{4} y = 0$$

As before, we set the ansatz to be  $y(t) = e^{S(t)}$  and using the derivatives found above, we have:

```
In[5]:= x^4 y'''[x] - x^2 y'[x] + 1/4 y[x] == 0 /. y → Function[x, e^{S[x]}] // FullSimplify
Out[5]= e^{S[x]} \left( (1 - 2 x^2 S'[x])^2 + 4 x^4 S''[x] \right) == 0
```

once again, assuming that  $S'' \ll (S')^2$ , so we now we have:

```
In[6]:= Solve[(1 - 2 x^2 S'[x])^2 == 0, S'[x]]
```

```
Out[6]= \left\{ \left\{ S'[x] \rightarrow \frac{1}{2 x^2} \right\}, \left\{ S'[x] \rightarrow -\frac{1}{2 x^2} \right\} \right\}
```

Now we get the controlling factor to be:

$$y \sim e^{(1/2)x^{-1}}$$

So now to find the leading term, as before we will have:

```
In[7]:= x^4 y'''[x] - x^2 y'[x] + 1/4 y[x] == 0 /.
y → Function[x, e^{-(1/2)*(x^{-1})+c[x]}] // FullSimplify
Out[7]= e^{-\frac{1}{2}x+c[x]} x (-1 + x^3 (c'[x]^2 + c''[x])) == 0
-1 + x^3 (c'[x]^2) == 0
Out[8]= e^{-\frac{1}{2}x+c[x]} x (-1 + x^3 (c'[x]^2 + c''[x])) == 0
```

So we essentially want to find the order for  $-1 + x^3 (c'[x]^2 + c''[x])$ , which results in  $c(x) \sim \sqrt{x}$ . To see how we solved for this:

```
In[9]:= Solve[x^4 y'''[x] - x^2 y'[x] + 1/4 y[x] == 0 /.
y → Function[x, e^{-(1/2)*(x^{-1})+c[x]}], c''[x]] // FullSimplify // Expand
Out[9]= \left\{ \left\{ c''[x] \rightarrow \frac{1}{x^3} - c'[x]^2 \right\} \right\}
In[10]:= DSolve[\frac{1}{x^3} - (c'[x])^2 == 0, c[x], x]
Out[10]= \left\{ \left\{ c[x] \rightarrow -\frac{2}{\sqrt{x}} + C_1 \right\}, \left\{ c[x] \rightarrow \frac{2}{\sqrt{x}} + C_1 \right\} \right\}
```

So now we put everything together and get:

$$y \sim \alpha e^{\frac{\pm 2}{\sqrt{x}} + \frac{1}{-2x}}$$