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PREFACE

This book is designed in the light of the new guidelines and syllabi – 2003 for the Higher Secondary Mathematics, prescribed for the Second Year, by the Government of Tamil Nadu.

The 21st century is an era of Globalisation, and technology occupies the prime position. In this context, writing a text book on Mathematics assumes special significance because of its importance and relevance to Science and Technology.

As such this book is written in tune with the existing international standard and in order to achieve this, the team has exhaustively examined internationally accepted text books which are at present followed in the reputed institutions of academic excellence and hence can be relevant to secondary level students in and around the country.

This text book is presented in two volumes to facilitate the students for easy approach. Volume I consists of Applications of Matrices and Determinants, Vector Algebra, Complex numbers and Analytical Geometry which is dealt with a novel approach. Solving a system of linear equations and the concept of skew lines are new ventures. Volume II includes Differential Calculus – Applications, Integral Calculus and its Applications, Differential Equations, Discrete Mathematics (a new venture) and Probability Distributions.

The chapters dealt with provide a clear understanding, emphasizes an investigative and exploratory approach to teaching and the students to explore and understand for themselves the basic concepts introduced.

Wherever necessary theory is presented precisely in a style tailored to act as a tool for teachers and students.

Applications play a central role and are woven into the development of the subject matter. Practical problems are investigated to act as a catalyst to motivate, to maintain interest and as a basis for developing definitions and procedures.

The solved problems have been very carefully selected to bridge the gap between the exposition in the chapter and the regular exercise set. By doing these exercises and checking the complete solutions provided, students will be able to test or check their comprehension of the material.

Fully in accordance with the current goals in teaching and learning Mathematics, every section in the text book includes worked out and exercise (assignment) problems that encourage geometrical visualisation, investigation, critical thinking, assimilation, writing and verbalization.

We are fully convinced that the exercises give a chance for the students to strengthen various concepts introduced and the theory explained enabling them to think creatively, analyse effectively so that they can face any situation with conviction and courage. In this respect the exercise problems are meant only to students and we hope that this will be an effective tool to develop their talents for greater achievements. Such an effort need to be appreciated by the parents and the well-wishers for the larger interest of the students.

Learned suggestions and constructive criticisms for effective refinement of the book will be appreciated.

K.SRINIVASAN
Chairperson
Writing Team.

SYLLABUS

(1) **APPLICATIONS OF MATRICES AND DETERMINANTS :** ***Adjoint, Inverse*** – Properties, Computation of inverses, solution of system of linear equations by matrix inversion method. ***Rank of a Matrix*** – Elementary transformation on a matrix, consistency of a system of linear equations, Cramer's rule, Non-homogeneous equations, homogeneous linear system, rank method.

(20 periods)

(2) **VECTOR ALGEBRA :** ***Scalar Product*** – Angle between two vectors, properties of scalar product, applications of dot products. ***Vector Product*** – Right handed and left handed systems, properties of vector product, applications of cross product. ***Product of three vectors*** – Scalar triple product, properties of scalar triple product, vector triple product, vector product of four vectors, scalar product of four vectors. ***Lines*** – Equation of a straight line passing through a given point and parallel to a given vector, passing through two given points (derivations are not required). angle between two lines. ***Skew lines*** – Shortest distance between two lines, condition for two lines to intersect, point of intersection, collinearity of three points. ***Planes*** – Equation of a plane (derivations are not required), passing through a given point and perpendicular to a vector, given the distance from the origin and unit normal, passing through a given point and parallel to two given vectors, passing through two given points and parallel to a given vector, passing through three given non-collinear points, passing through the line of intersection of two given planes, the distance between a point and a plane, the plane which contains two given lines, angle between two given planes, angle between a line and a plane. ***Sphere*** – Equation of the sphere (derivations are not required) whose centre and radius are given, equation of a sphere when the extremities of the diameter are given. **(28 periods)**

(3) **COMPLEX NUMBERS :** Complex number system, ***Conjugate*** – properties, ordered pair representation. ***Modulus*** – properties, geometrical representation, meaning, polar form, principal value, conjugate, sum, difference, product, quotient, vector interpretation, solutions of polynomial equations, De Moivre's theorem and its applications. ***Roots of a complex number*** – nth roots, cube roots, fourth roots. **(20 periods)**

(4) **ANALYTICAL GEOMETRY :** ***Definition of a Conic*** – General equation of a conic, classification with respect to the general equation of a conic, classification of conics with respect to eccentricity. ***Parabola*** – Standard equation of a parabola

(derivation and tracing the parabola are not required), other standard parabolas, the process of shifting the origin, general form of the standard equation, some practical problems. ***Ellipse*** – Standard equation of the ellipse (derivation and tracing the ellipse are not required), $x^2/a^2 + y^2/b^2 = 1$, ($a > b$), Other standard form of the ellipse, general forms, some practical problems, ***Hyperbola*** – standard equation (derivation and tracing the hyperbola are not required), $x^2/a^2 - y^2/b^2=1$, Other form of the hyperbola, parametric form of conics, chords. ***Tangents and Normals*** – Cartesian form and ***Parametric form*****,** equation of chord of contact of tangents from a point (x_1, y_1), ***Asymptotes, Rectangular hyperbola*** – standard equation of a rectangular hyperbola.

(30 periods)

(5) **DIFFERENTIAL CALCULUS – APPLICATIONS I :** ***Derivative as a rate measure*** – rate of change – velocity – acceleration – related rates – Derivative as a measure of slope – tangent, normal and angle between curves. Maxima and Minima. ***Mean value theorem*** – Rolle's Theorem – Lagrange Mean Value Thorem – Taylor's and Maclaurin's series, l' Hôpital's Rule, stationary points – increasing, decreasing, maxima, minima, concavity convexity, points of inflexion.

(28 periods)

(6) **DIFFERENTIAL CALCULUS – APPLICATIONS II :** ***Errors and approximations*** – absolute, relative, percentage errors, curve tracing, partial derivatives – Euler's theorem. **(10 periods)**

(7) **INTEGRAL CALCULUS AND ITS APPLICATIONS :** Properties of definite integrals, reduction formulae for $\sin^n x$ and $\cos^n x$ (only results), Area, length, volume and surface area **(22 periods)**

(8) **DIFFERENTIAL EQUATIONS :** Formation of differential equations, order and degree, solving differential equations (1st order) – variable separable homogeneous, linear equations. Second order linear equations with constant co-efficients $f(x) = e^{mx}$, $\sin mx$, $\cos mx$, x, x^2. **(18 periods)**

(9A) **DISCRETE MATHEMATICS :** ***Mathematical Logic*** – Logical statements, connectives, truth tables, Tautologies.

(9B) **GROUPS :** ***Binary Operations*** – Semi groups – monoids, groups (Problems and simple properties only), order of a group, order of an element. **(18 periods)**

(10) **PROBABILITY DISTRIBUTIONS :** Random Variable, Probability density function, distribution function, mathematical expectation, variance, Discrete Distributions – Binomial, Poisson, Continuous Distribution – Normal distribution

(16 periods)

Total : 210 Periods

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1. APPLICATIONS OF MATRICES AND DETERMINANTS

1.1. Introduction :

The students are already familiar with the basic definitions, the elementary operations and some basic properties of matrices. The concept of division is not defined for matrices. In its place and to serve similar purposes, the notion of the inverse of a matrix is introduced. In this section, we are going to study about the inverse of a matrix. To define the inverse of a matrix, we need the concept of adjoint of a matrix.

1.2 Adjoint :

Let $A = [a_{ij}]$ be a square matrix of order n. Let A_{ij} be the cofactor of a_{ij}. Then the nth order matrix $[A_{ij}]^T$ is called the adjoint of A. It is denoted by adj$A$. Thus the adj$A$ is nothing but the transpose of the cofactor matrix $[A_{ij}]$ of A.

Result : If A is a square matrix of order n, then A (adj$A$) = $|A|\, I_n$ = (adj A) A, where I_n is the identity matrix of order n.

Proof : Let us prove this result for a square matrix A of order 3.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Then adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\left.\begin{array}{r}\text{The } (i, j)^{\text{th}} \\ \text{element of } A \text{ (adj } A)\end{array}\right\} = a_{i1} A_{j1} + a_{i2} A_{j2} + a_{i3} A_{j3} = \Delta = |A| \text{ if } i = j$$

$$= 0 \text{ if } i \neq j$$

$$\therefore A \text{ (adj } A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I_3$$

Similarly we can prove that (adj A)A = $|A|\, I_3$

$\therefore$ A (adj A) = | A | I_3 = (adj A) A

In general we can prove that A (adj A) = | A | I_n = (adj A) A.

Example 1.1 : Find the adjoint of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solution: The cofactor of a is d, the cofactor of b is $-c$, the cofactor of c is $-b$ and the cofactor of d is a. The matrix formed by the cofactors taken in order is the cofactor matrix of A.

$$\therefore \text{ The cofactor matrix of } A \text{ is } = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Taking transpose of the cofactor matrix, we get the adjoint of A.

$$\therefore \text{ The adjoint of } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 1.2 : Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: The cofactors are given by

$$\text{Cofactor of } 1 = A_{11} = \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 3$$

$$\text{Cofactor of } 1 = A_{12} = -\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = -9$$

$$\text{Cofactor of } 1 = A_{13} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$$

$$\text{Cofactor of } 1 = A_{21} = -\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -4$$

$$\text{Cofactor of } 2 = A_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

$$\text{Cofactor of } -3 = A_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3$$

$$\text{Cofactor of } 2 = A_{31} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5$$

$$\text{Cofactor of } -1 = A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = 4$$

$$\text{Cofactor of } 3 = A_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$\text{The Cofactor matrix of } A \text{ is } [A_{ij}] = \begin{bmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{bmatrix}$$

$$\therefore\ adj\ \text{A} = (A_{ij})^T = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

Example 1.3 : If $A = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$, verify the result A (adj A) = (adj A) A = $|A|\, I_2$

Solution: $\text{A} = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix},\ |A| = \begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} = 2$

$$\text{adj } A = \begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix}$$

$$A\,(\text{adj } A) = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2I_2 \qquad \dots (1)$$

$$(\text{adj } A)\,A = \begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2I_2 \qquad \dots (2)$$

From (1) and (2) we get

$\therefore A\,(\text{adj } A) = (\text{adj } A)\,A = |A|\, I_2.$

Example 1.4 : If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, verify A (adj A) = (adj A) A = $|A|\, I_3$

Solution: In example 1.2, we have found

$$\text{adj } A = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix} = 1(6-3) - 1\,(3+6) + 1(-1-4) = -11$$

$$A\,(\text{adj } A) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix}$$

$$= -11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -11\, I_3 = |A|\, I_3 \qquad \ldots(1)$$

$$(\text{adj } A)\, A = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix}$$

$$= -11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -11\, I_3 = |A|\, I_3 \qquad \ldots(2)$$

From (1) and (2) we get

$$A(\text{adj } A) = (\text{adj } A)\, A = |A|\, I_3$$

1.3 Inverse :

Let A be a square matrix of order n. Then a matrix B, if it exists, such that $AB = BA = I_n$ is called inverse of the matrix A. In this case, we say that A is an invertible matrix. If a matrix A possesses an inverse, then it must be unique. To see this, assume that B and C are two inverses of A, then

$$AB = BA = I_n \qquad \ldots(1)$$

$$AC = CA = I_n \qquad \ldots(2)$$

Now $AB = I_n$

$$\Rightarrow \quad C(AB) = CI_n \Rightarrow (CA)B = C \quad (\because \text{ associative property})$$

$$\Rightarrow \quad I_nB = C \Rightarrow B = C$$

i.e., The inverse of a matrix is unique. Next, let us find a formula for computing the inverse of a matrix.

We have already seen that, if A is a square matrix of order n, then

$$A(\text{adj } A) = (\text{adj } A)A = |A|\, I_n$$

If we assume that A is non-singular, then $|A| \neq 0$.

Dividing the above equation by $|A|$, we get

$$A\left\{\frac{1}{|A|}(\text{adj } A)\right\} = \left\{\frac{1}{|A|}(\text{adj } A)\right\}A = I_n.$$

From this equation it is clear that the inverse of A is nothing but $\frac{1}{|A|}$ (adj A). We denote this by A^{-1}.

Thus we have the following formula for computing the inverse of a matrix through its adjoint.

If A is a non-singular matrix, there exists an inverse which is given by $A^{-1} = \frac{1}{|A|}$ (adj A).

1.3.1 Properties :

1. Reversal Law for Inverses :

If A, B are any two non-singular matrices of the same order, then AB is also non-singular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

i.e., the inverse of a product is the product of the inverses taken in the reverse order.

Proof : Since A and B are non-singular, $|A| \neq 0$ and $|B| \neq 0$.

We know that $|AB| = |A| \; |B|$

$|A| \neq 0, \; |B| \neq 0 \;\Rightarrow\; |A| \; |B| \neq 0 \;\Rightarrow\; |AB| \neq 0$

Hence AB is also non-singular. So AB is invertible.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$
$$= AIA^{-1} = AA^{-1} = \text{I}$$

Similarly we can show that $(B^{-1}A^{-1})(AB) = I$

$$\therefore (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

$\therefore B^{-1}A^{-1}$ is the inverse of AB.

$$\therefore (AB)^{-1} = B^{-1}A^{-1}$$

2. Reversal Law for Transposes (without proof) :

If A and B are matrices conformable to multiplication, then $(AB)^T = B^TA^T$.

i.e., the transpose of the product is the product of the transposes taken in the reverse order.

3. For any non-singular matrix A, $(A^T)^{-1} = (A^{-1})^T$

Proof : We know that $AA^{-1} = I = A^{-1}A$

Taking transpose on both sides of $AA^{-1} = I$, we have $(AA^{-1})^T = I^T$

By reversal law for transposes we get

$$(A^{-1})^T A^T = I \qquad \ldots (1)$$

Similarly, by taking transposes on both sides of $A^{-1}A = I$, we have

$$A^T(A^{-1})^T = I \qquad \ldots (2)$$

From (1) & (2)

$$(A^{-1})^T A^T = A^T (A^{-1})^T = I$$

$\therefore (A^{-1})^T$ is the inverse of A^T

i.e., $\quad (A^T)^{-1} = (A^{-1})^T$

1.3.2 Computation of Inverses

The following examples illustrate the method of computing the inverses of the given matrices.

Example 1.5 : Find the inverses of the following matrices :

(i) $\begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Solution:

(i) Let $A = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$, Then $|A| = \begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} = 2 \neq 0$

A is a non-singular matrix. Hence it is invertible. The matrix formed by the cofactors is

$$[A_{ij}] = \begin{bmatrix} -4 & -1 \\ -2 & -1 \end{bmatrix}$$

$$\text{adj } A = [A_{ij}]^T = \begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,(\text{adj } A) = \frac{1}{2}\begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

(ii) Let $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$. then $|A| = \begin{vmatrix} 2 & -1 \\ -4 & 2 \end{vmatrix} = 0$

A is singular. Hence A^{-1} does not exist.

(iii) Let $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$. Then $|A| = \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix}$

$$= \cos^2\alpha + \sin^2\alpha = 1 \neq 0$$

$\therefore$ A is non singular and hence it is invertible

$$\text{Adj } A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,(\text{Adj } A) = \frac{1}{1}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

(iv) Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$. Then $|A| = \begin{vmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{vmatrix} = 2 \neq 0$

A is non-singular and hence A^{-1} exists

$$\text{Cofactor of } 3 = A_{11} = \begin{vmatrix} -2 & 0 \\ 2 & -1 \end{vmatrix} = 2$$

$$\text{Cofactor of } 1 = A_{12} = -\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = 2$$

$$\text{Cofactor of } -1 = A_{13} = \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} = 6$$

$$\text{Cofactor of } 2 = A_{21} = -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1$$

$$\text{Cofactor of } -2 = A_{22} = \begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix} = -2$$

$$\text{Cofactor of } 0 = A_{23} = -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -5$$

$$\text{Cofactor of } 1 \;=\; A_{31} \;=\; \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -2$$

$$\text{Cofactor of } 2 \;=\; A_{32} \;=\; -\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -2$$

$$\text{Cofactor of } -1 \;=\; A_{33} \;=\; \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} = -8$$

$$[A_{ij}] = \begin{bmatrix} 2 & 2 & 6 \\ -1 & -2 & -5 \\ -2 & -2 & -8 \end{bmatrix}; \text{ adj } A = \begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,(\text{adj } A) \;=\; \frac{1}{2}\begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & -1 & -1 \\ 3 & -\frac{5}{2} & -4 \end{bmatrix}$$

Example 1.6 : If $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

$|A| = -1 \neq 0$ and $|B| = 1 \neq 0$

So A and B are invertible.

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1 \neq 0. \text{ So } AB \text{ is invertible.}$$

$$\text{adj } A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,(\text{adj } A) \;=\; \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\text{adj } B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \ (\text{adj } B) = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\text{adj } AB = \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|} \ (\text{adj } AB) = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \qquad \ldots (1)$$

$$B^{-1} A^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \qquad \ldots (2)$$

From (1) and (2) we have $(AB)^{-1} = B^{-1} A^{-1}$.

EXERCISE 1.1

(1) Find the adjoint of the following matrices :

(i) $\begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

(2) Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ and verify the result

A (adj A) = (adj A)A = $|A| . I$

(3) Find the adjoint of the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ and verify the result

A (adj A) = (adj A)A = $|A| . I$

(4) Find the inverse of each of the following matrices :

(i) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(5) If $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ verify that

(i) $(AB)^{-1} = B^{-1} A^{-1}$ (ii) $(AB)^T = B^T A^T$

(6) Find the inverse of the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ and verify that $A^3 = A^{-1}$

(7) Show that the adjoint of $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is $3A^T$.

(8) Show that the adjoint of $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ is A itself.

(9) If $A = \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$, prove that $A^{-1} = A^T$.

(10) For $A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$, show that $A = A^{-1}$

1.3.3 Solution of a system of linear equations by Matrix Inversion method :

Consider a system of n linear non-homogeneous equations in n unknowns $x_1, x_2, x_3 \ldots\ldots\ldots x_n$.

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots\ldots + a_{1n} x_n = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots + a_{2n} x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1} x_1 + a_{n2} x_2 + \ldots\ldots\ldots\ldots\ldots + a_{nn} x_n = b_n$$

This is of the form $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ an_1 & an_2 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_n \end{bmatrix}$

Thus we get the matrix equation $AX = B$ … (1) where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix};\ X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix};\ B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_n \end{bmatrix}$$

If the coefficients matrix A is non-singular, then A^{-1} exists. Pre-multiply both sides of (1) by A^{-1} we get

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B \text{ is the solution of (1)}$$

Thus to determine the solution vector X we must compute A^{-1}. Note that this solution is unique.

Example 1.7 : Solve by matrix inversion method $x + y = 3$, $2x + 3y = 8$

Solution:

The given system of equations can be written in the form of

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$AX = B$$

Here $|A| = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \neq 0$

Since A is non-singular, A^{-1} exists.

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

The solution is $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x = 1, \quad y = 2$$

Example 1.8 : Solve by matrix inversion method $2x - y + 3z = 9$, $x + y + z = 6$, $x - y + z = 2$

Solution : The matrix equation is

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$A X = B$, where $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = -2 \neq 0$$

A is a non-singular matrix and hence A^{-1} exists.

The cofactors are $A_{11} = 2$, $A_{12} = 0$, $A_{13} = -2$

$A_{21} = -2$, $A_{22} = -1$, $A_{23} = 1$, $A_{31} = -4$, $A_{32} = +1$, $A_{33} = 3$

The matrix formed by the cofactors is

$$[A_{ij}] = \begin{bmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{bmatrix}$$

$$\text{The adjoint of } A = \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \text{adj } A$$

$$\text{Inverse of } A = \frac{1}{|A|}\ (\text{adj } A)$$

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

The solution is given by $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore\ x = 1,\ y = 2, z = 3$

EXERCISE 1.2

Solve by matrix inversion method each of the following system of linear equations :

(1) $2x - y = 7$, $\quad 3x - 2y = 11$

(2) $7x + 3y = -1$, $\quad 2x + y = 0$

(3) $x + y + z = 9$, $\quad 2x + 5y + 7z = 52$, $\quad 2x + y - z = 0$

(4) $2x - y + z = 7$, $\quad 3x + y - 5z = 13$, $\quad x + y + z = 5$

(5) $x - 3y - 8z + 10 = 0$, $\quad 3x + y = 4$, $\quad 2x + 5y + 6z = 13$

1.4 Rank of a Matrix :

With each matrix, we can associate a non-negative integer, called its rank. The concept of rank plays an important role in solving a system of homogeneous and non-homogeneous equations.

To define rank, we require the notions of submatrix and minor of a matrix. A matrix obtained by leaving some rows and columns from the matrix A is called a submatrix of A. In particular A itself is a submatrix of A, because it is obtained from A by leaving no rows or columns. The determinant of any square submatrix of the given matrix A is called a minor of A. If the square submatrix is of order r, then the minor is also said to be of order r.

Definition :

The matrix A is said to be of rank r, if

(i) A has atleast one minor of order r which does not vanish.

(ii) Every minor of A of order $(r + 1)$ and higher order vanishes.

In other words, the rank of a matrix is the order of any highest order non vanishing minor of the matrix.

The rank of A is denoted by the symbol $\rho(A)$. The rank of a null matrix is defined to be zero.

The rank of the unit matrix of order n is n. The rank of an $m \times n$ matrix A cannot exceed the minimum of m and n. i.e., $\rho(A) \leq \min \{m, n\}$.

Example 1.9 : Find the rank of the matrix $\begin{bmatrix} 7 & -1 \\ 2 & 1 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 7 & -1 \\ 2 & 1 \end{bmatrix}$. This is a second order matrix.

$\therefore$ The highest order of minor of A is also 2.

The minor is given by $\begin{vmatrix} 7 & -1 \\ 2 & 1 \end{vmatrix} = 9 \neq 0$

$\therefore$ The highest order of non-vanishing minor of A is 2. Hence $\rho(A) = 2$.

Example 1.10 : Find the rank of the matrix $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$.

The highest order minor of A is given by $\begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} = 0$. Since the second order minor vanishes $\rho(A) \neq 2$. We have to try for atleast one non-zero first order minor, i.e., atleast one non-zero element of A. This is possible because A has non-zero elements $\therefore$ $\rho(A) = 1$.

Example 1.11 : Find the rank of the matrix $\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

The highest order minor of A is

$$\begin{vmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix} = -2 \begin{vmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \\ 5 & 1 & -1 \end{vmatrix} = 0$$

Since the third order minor vanishes, $\rho(A) \neq 3$

$$\begin{vmatrix} -2 & 4 \\ 5 & 1 \end{vmatrix} = -22 \neq 0$$

$\therefore$ A has atleast one non-zero minor of order 2. $\therefore$ $\rho(A) = 2$

Example 1.12 : Find the rank of the matrix $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

This is a matrix of order 3×4

$\therefore$ A has minors of highest order 3. They are given by

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 5 & -1 & 7 \end{vmatrix} = 0 \; ; \quad \begin{vmatrix} 1 & 1 & 3 \\ 2 & -1 & 4 \\ 5 & -1 & 11 \end{vmatrix} = 0 \; ;$$

$$\begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 5 & 7 & 11 \end{vmatrix} = 0 \; ; \quad \begin{vmatrix} 1 & 1 & 3 \\ -1 & 3 & 4 \\ -1 & 7 & 11 \end{vmatrix} = 0$$

All the third order minors vanish. $\therefore$ $\rho(A) \neq 3$

Next, we have to try for atleast one non-zero minor of order 2. This is possible, because A has a 2^{nd} order minor $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3 \neq 0$ $\therefore$ $\rho(A) = 2$

Note : In the above examples, we have seen that the determination of the rank of a matrix involves the computation of determinants. The computation of determinants may be greatly reduced by means of certain elementary transformations of its rows and columns. These transformations will greatly facilitate our dealings with the problem of the determination of the rank and other allied problems.

1.4.1. Elementary transformations on a Matrix:

(i) Interchange of any two rows (or columns)

(ii) Multiplication of each element of a row (or column) by any non-zero scalar.

(iii) Addition to the elements of any row (or column) the same scalar multiples of corresponding elements of any other row (or column)

the above elementary transformations taken inorder can be represented by means of symbols as follows :

(i) $R_i \leftrightarrow \mathrm{R}_j$ $(C_i \leftrightarrow C_j)$; (ii) $R_i \to kR_i$ $(C_i \to kC_i)$

(ii) $R_i \to R_i + kR_j$ $(C_i \to C_i + kC_j)$

Two matrices A and B of the same order are said to be equivalent if one can be obtained from the other by the applications of a finite sequence of elementary transformation "The matrix A is equivalent to the matrix B" is symbolically denoted by $A \sim B$.

Result (without proof) :

"Equivalent matrices have the same rank"

Echelon form of a matrix :

A matrix A (of order $m \times n$) is said to be in echelon form (triangular form) if

(i) Every row of A which has all its entries 0 occurs below every row which has a non-zero entry.

(ii) The first non-zero entry in each non-zero row is 1.

(iii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

By elementary operations one can easily bring the given matrix to the echelon form.

Result (without proof) :

The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

Note :

(1) The above result will not be affected even if condition (ii) given in the echelon form is omitted. (i.e.) the result holds even if the non-zero entry in each non-zero row is other than 1.

(2) The main advantage of echelon form is that the rank of the given matrix can be found easily. In this method we don't have to compute determinants. It is enough, if we find the number of non-zero rows.

In the following examples we illustrate the method of finding the rank of matrices by reducing them to the echelon form.

Example 1.13 : Find the rank of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & -5 & 6 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

The last equivalent matrix is in echelon form. The number of non-zero rows is 2. $\therefore\ \rho(A) = 2$

Example 1.14 : Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 4 & 6 & -2 \\ 3 & 6 & 9 & -3 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 4 & 6 & -2 \\ 3 & 6 & 9 & -3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

This equivalent matrix is in the echelon form. Since the number of non-zero rows of the matrix in this echelon form is 1, $\rho(A) = 1$.

Example 1.15 : Find the rank of the matrix $\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 6 & 7 \\ 0 & 1 & 2 & 1 \end{bmatrix} \; C_1 \leftrightarrow C_3$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -5 & -10 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \; R_2 \to R_2 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \; R_2 \to -\frac{1}{5} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; R_3 \to R_3 - R_2$$

The last equivalent matrix is in the echelon form.

The number of non-zero rows in this matrix is two. $\therefore$ $\rho(A) = 2$

Example 1.16 : Find the rank of the matrix $\begin{bmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & -5 \\ 3 & 1 & -5 & -1 \\ 1 & 5 & -7 & 2 \end{bmatrix} \; R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -5 \\ 0 & 7 & -8 & 14 \\ 0 & 7 & -8 & 7 \end{bmatrix} \; \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -5 \\ 0 & 7 & -8 & 14 \\ 0 & 0 & 0 & -7 \end{bmatrix} \; R_3 \to R_3 - R_2$$

The last equivalent matrix is in the echelon form.

It has three non-zero rows. $\therefore$ $\rho(A) = 3$

EXERCISE 1.3

Find the rank of the following matrices :

(1) $\begin{bmatrix} 1 & 1 & -1 \\ 3 & -2 & 3 \\ 2 & -3 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 6 & 12 & 6 \\ 1 & 2 & 1 \\ 4 & 8 & 4 \end{bmatrix}$ (3) $\begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix}$

(4) $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & -3 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$ (5) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$ (6) $\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$

1.5 Consistency of a system of linear equations :

The system of linear equations arises naturally in many areas of Science, Engineering, Economics and Commerce. The analysis of electronic circuits, determination of the output of a chemical plant, finding the cost of chemical reaction are some of the problems which depend on the solutions of simultaneous linear equations. So, finding methods of solving such equations acquire considerable importance. In this connection methods using matrices and determinants play an important role.

We have already seen the idea of solving a system of linear equations by the matrix inversion method. This method is applicable provided the number of equations is equal to the number of unknowns, and the coefficient matrix is non-singular. Also the solution obtained under this method is unique. But this is not so in all cases. For many of the problems the number of equations need not be equal to the number of unknowns. In such cases, we see that any one of the following three possibilities can occur. The system has (1) unique solution (2) more than one solution (3) no solution at all.

Cases (1) and (3) have no significant role to play in higher studies. Although there exist many solutions, in some cases all the points in the solution are not attractive. Some provide greater significance than others. We have to select the best point among them. In this section we are going to discuss the following two methods.

(1) Cramer's rule method (or Determinant method)

(2) Rank method

These methods not only decide the existence of a solution but also help us to find the solution (if it exists) of the given system.

1.5.1 The Geometry of Solution sets :

The solution set of a system of linear equations is the intersection of the solution sets of the individual equations. That is, any solution of a system must be a solution of each of the equations in that system.

The equation $ax = b$ ($a \neq 0$) has only one solution, namely $x = b/a$ and it represents a point on the line. Similarly, a single linear equation in two unknowns has a line in the plane as its solution set and a single linear equation in three unknowns has a plane in space as its solution set.

Illustration I : (No. of unknowns $\geq$ No. of equations)

Consider the solution of the following three different problems.

(i) $2x = 10$ (ii) $2x + y = 10$ (iii) $2x + y - z = 10$

Solution (i) $2x = 10 \Rightarrow x = 5$

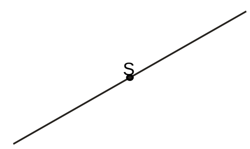


Fig. 1.1

Solution (ii) $2x + y = 10$

We have to determine the values of two unknown from a single equation. To find the solution we can assign arbitrary value to x and solve for y, or, choose an arbitrary value to y and solve for x.

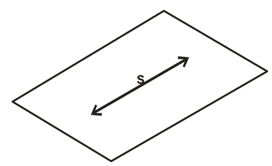


Fig. 1.2

Suppose we assign x an arbitrary value k, we obtain

$x = k$ and $y = (10 - 2k)$

These formulae give the solution set interms of the parameter 'k'. Particular numerical solution can be obtained by substituting values for 'k'. For example when $k = 1, 2, 5, -3, \frac{1}{2}$, we get $(1, 8)$, $(2, 6)$, $(5, 0)$, $(-3, 16)$ and $\left(\frac{1}{2}, 9\right)$ as the respective solutions.

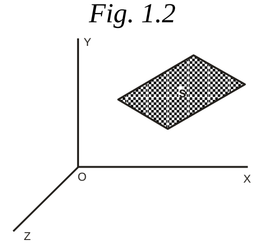


Fig. 1.3

Solution (iii) $2x + y - z = 10$

In this case, we have to determine three unknowns x, y and z from a single equation. We can assign arbitrary values to any two variables and solve for the third variable. We assign arbitrary values 's' and 't' to x and y respectively, and solve for z.

We get $x = s$, $y = t$ and $z = 2s + t - 10$ is the solution set.

For different values of s and t we get different solutions.

1.5.2 Cramer's Rule Method : (Determinant Method)

Gabriel Cramer (1704 – 1752), a Swiss mathematician wrote on philosophy of law and government, and history of mathematics. He served in a public office, participated in artillery and fortifications activity for the government instructed workers on techniques of cathedral repair and undertook excavations of cathedral archives. Cramer, a bachelor, received numerous honours for his achievements.

His theorem provides a useful formula for the solution of certain linear system of n equations in n unknowns. This formula, known as Cramer's Rule, is of marginal interest for computational purposes, but it is useful for studying the mathematical properties of a solution without actually solving the system.

Theorem 1.1 (without proof) **: Cramer's Rule :** If $AX = B$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det A},\quad x_2 = \frac{\det(A_2)}{\det(A)},\ \ldots\ x_n = \frac{\det(A_n)}{\det(A)}$$

Where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix. $B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$

Cramer's Rule for Non homogeneous equations of 2 unknowns :

Let us start with the system of two linear equations in two unknowns 'x' and 'y'.

$$a_{11}x + a_{12}y = b_1 \qquad \ldots \text{(i)}$$

$$a_{21}x + a_{22}y = b_2 \qquad \ldots \text{(ii)}$$

$$\text{Let}\quad \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\therefore\ x\,.\,\Delta = x\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11}x & a_{12} \\ a_{21}x & a_{22} \end{vmatrix}$$

$$= \begin{vmatrix} b_1 - a_{12}y & a_{12} \\ b_2 - a_{22}y & a_{22} \end{vmatrix} \quad \text{(by equation (i) and (ii))}$$

$$= \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} - y \begin{vmatrix} a_{12} & a_{12} \\ a_{22} & a_{22} \end{vmatrix} \text{ (by properties of determinants)}$$

$$= \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} - y \,.\, 0 \text{ (by properties of determinants)}$$

$$x \,.\, \Delta = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = \Delta_x \text{ (say)}$$

Similarly $$y.\, \Delta = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = \Delta_y \text{ (say)}$$

Δ_x, Δ_y are the determinants which can also be obtained by replacing 1st and 2nd column respectively by the column of constants containing b_1 and b_2 i.e. by $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ Thus, we have, $x\Delta = \Delta_x \Rightarrow x = \dfrac{\Delta_x}{\Delta}$

$$y\Delta = \Delta_y \Rightarrow y = \frac{\Delta_y}{\Delta} \text{ provided } \Delta \neq 0$$

Since Δ, Δ_x, Δ_y are unique, there exists a unique solution for the above system of equations. i.e., the system is consistent and has a unique solution.

The method stated above to solve the system of equation is known as **Cramer's Rule.**

Cramer's rule is applicable when $\Delta \neq 0$.

If $\Delta = 0$, then the given system may be consistent or inconsistent.

Case 1 : If $\Delta = 0$ and $\Delta_x = 0$, $\Delta_y = 0$ and atleast one of the coefficients a_{11}, a_{12}, a_{21}, a_{22} is non-zero, then the system is consistent and has infinitely many solutions.

Case 2 : If $\Delta = 0$ and atleast one of the values Δ_x, Δ_y is non-zero, then the system is inconsistent i.e. it has no solution.

To illustrate the possibilities that can occur in solving systems of linear equations with two unknowns, consider the following three examples. Solve :

(1) $x + 2y = 3$	(2) $x + 2y = 3$	(3) $x + 2y = 3$
$x + y = 2$	$2x + 4y = 6$	$2x + 4y = 8$

Solution (1) :

Unique solution

We have $\Delta = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$

$\Delta_x = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -1$

$\Delta_y = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1$

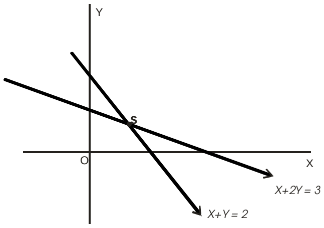


Fig. 1.4

Since $\Delta \neq 0$, the system has unique solution. By Cramer's rule

$x = \dfrac{\Delta_x}{\Delta} = 1$; $y = \dfrac{\Delta_y}{\Delta} = 1$ $\quad \therefore (x, y) = (1, 1)$

Solution (2) :

Infinitely many solution

We have $\Delta = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

$\Delta_x = \begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 0$

$\Delta_y = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0$

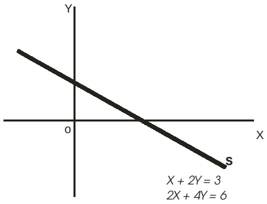


Fig. 1.5

Since $\Delta = 0$ and $\Delta_x = 0$, $\Delta_y = 0$ and atleast one of $a_{11}, a_{12}, a_{21}, a_{22}$ is non zero, it has infinitely (case 1) many solutions. The above system is reduced to a single equation $x + 2y = 3$. To solve this equation, assign $y = k$

$\therefore\ x = 3 - 2y = 3 - 2k$

The solution is $x = 3 - 2k, y = k$; $k \in \boldsymbol{R}$

For different value of k we get different solution. In particular (1, 1), (– 1, 2), (5 – 1) and (8, – 2.5) are some solutions for k = 1, 2, – 1 and – 2.5 respectively

Solution (3) :

No Solution

$\Delta = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$;

$\Delta_x = \begin{vmatrix} 3 & 2 \\ 8 & 4 \end{vmatrix} = -4$; $\Delta_y = \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} = 2$

Since $\Delta = 0$ and $\Delta_x \neq 0$, $\Delta_y \neq 0$ (case 2 : atleast one of the value of Δ_x, Δ_y, non-zero), the system is inconsistent.

i.e. it has no solution.

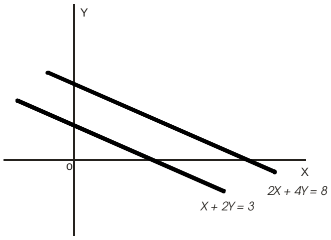


Fig. 1.6

1.5.3 Non homogeneous equations of three unknowns :

Consider the system of linear equations

$a_{11}x + a_{12}y + a_{13}z = b_1$; $a_{21}x + a_{22}y + a_{23}z = b_2$; $a_{31}x + a_{32}y + a_{33}z = b_3$

Let us define Δ, Δ_x, Δ_y and Δ_z as already defined for two unknowns.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

As we discussed earlier for two variables, we give the following rule for testing the consistency of the above system.

Case 1 : If $\Delta \neq 0$, then the system is consistent, and has a unique solution. Using **Cramer's Rule** can solve this system**.**

Case 2 : If $\Delta = 0$, we have three important possibilities.

Subcase 2(a) : If $\Delta = 0$ and atleast one of the values of Δ_x, Δ_y and Δ_z is non-zero, then the system has no solution i.e. Equations are inconsistent.

Subcase 2(b) : If $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$ and atleast one of the 2×2 minor of Δ is non zero, then the system is consistent and has infinitely many solution. In this case, the system of three equations is reduced to two equations. It can be solved by taking two suitable equations and assigning an arbitrary value to one of the three unknowns and then solve for the other two unknowns.

Subcase 2(c) : If $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$ and all their (2×2) minors are zero but atleast one of the elements of Δ is non zero ($a_{ij} \neq 0$) then the system is consistent and it has infinitely many solution. In this case, system is reduced to a single equation. To solve we can assign arbitrary values to any two variables and can determine the value of third variable.

Subcase 2(d) : If $\Delta = 0$, $\Delta_x = \Delta_y = \Delta_z = 0$, all 2×2 minors of $\Delta = 0$ and atleast one 2×2 minor of Δ_x or Δ_y or Δ_z is non zero then the system is inconsistent.

Theorem 1.2 (without proof) :

If a non-homogeneous system of linear equations with more number of unknowns than the number of equations is consistent, then it has infinitely many solutions.

To illustrate the different possibilities when we solve the above type of system of equations, consider the following examples.

(1) $2x + y + z = 5$
$x + y + z = 4$
$x - y + 2z = 1$

(2) $x + 2y + 3z = 6$
$x + y + z = 3$
$2x + 3y + 4z = 9$

(3) $x + 2y + 3z = 6$
$2x + 4y + 6z = 12$
$3x + 6y + 9z = 18$

(4) $x + 2y + 3z = 6$
$x + y + z = 3$
$2x + 3y + 4z = 10$

(5) $x + 2y + 3z = 6$
$2x + 4y + 6z = 12$
$3x + 6y + 9z = 24$

Solution (1) :

$$2x + y + z = 5 \;;\quad x + y + z = 4 \;;\quad x - y + 2z = 1$$

We have

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 3$$

$$\Delta_x = \begin{vmatrix} 5 & 1 & 1 \\ 4 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 3$$

Unique solution

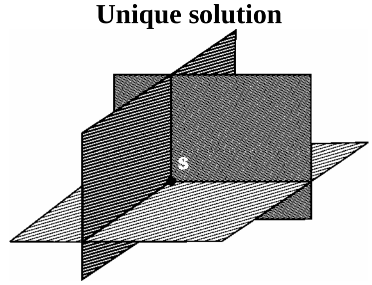


Fig. 1.7

$$\Delta_y = \begin{vmatrix} 2 & 5 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 6 \;;\qquad \Delta_z = \begin{vmatrix} 2 & 1 & 5 \\ 1 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 3$$

$$\Delta = 3, \Delta_x = 3, \Delta_y = 6, \Delta_z = 3$$

$\because \Delta \neq 0$, The system has unique solution. By Cramer's rule.

$$x = \frac{\Delta_x}{\Delta} = \frac{3}{3} = 1, \quad y = \frac{\Delta_y}{\Delta} = \frac{6}{3} = 2, \quad z = \frac{\Delta z}{\Delta} = 1$$

$\therefore$ The solution is $x = 1,\ y = 2,\ z = 1$

$$(x, y, z) = (1, 2, 1)$$

Solution (2) :

$$x + 2y + 3z = 6 \ ; \ \ x + y + z = 3 \ ; \ \ 2x + 3y + 4z = 9$$

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 0 \ ; \qquad \Delta_x = \begin{vmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 9 & 3 & 4 \end{vmatrix} = 0$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 1 & 3 & 1 \\ 2 & 9 & 4 \end{vmatrix} = 0 \ ; \qquad \Delta_z = \begin{vmatrix} 1 & 2 & 6 \\ 1 & 1 & 3 \\ 2 & 3 & 9 \end{vmatrix} = 0$$

Since $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$ but atleast one of the 2×2 minors of Δ is non-zero $\left(\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \neq 0 \right)$, the system is consistent (by case 2(*b*)) and has infinitely many solution.

The system is reduced to 2 equations. $\therefore$ Assigning an arbitrary value to one of unknowns, say $z = k$, and taking first two equations.

We get $x + 2y + 3k = 6$

$$x + y + k = 3$$

i.e.,

$$x + 2y = 6 - 3k$$

$$x + y = 3 - k$$

$$\Delta = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$$

Infinitely many solution

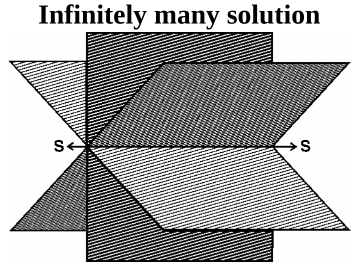


Fig. 1.8

$$\Delta_x = \begin{vmatrix} 6 - 3k & 2 \\ 3 - k & 1 \end{vmatrix} = 6 - 3k - 6 + 2k = -k$$

$$\Delta_y = \begin{vmatrix} 1 & 6 - 3k \\ 1 & 3 - k \end{vmatrix} = 3 - k - 6 + 3k = 2k - 3$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-k}{-1} = k$$

$$y = \frac{\Delta_y}{\Delta} = \frac{2k - 3}{-1} = 3 - 2k$$

The solution is $x = k, y = 3 - 2k$ and $z = k$

i.e. $(x, y, z) = (k, 3 - 2k, k)$. $k \in R$

Particularly, for $k = 1, 2, 3, 4$ we get

(1, 1, 1), (2, – 1, 2), (3, – 3, 3), (4, – 5, 4) respectively as solution.

Solution (3) :

$x + 2y + 3z = 6$; $2x + 4y + 6z = 12$; $3x + 6y + 9z = 18$

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix} = 0 \quad ; \quad \Delta_x = \begin{vmatrix} 6 & 2 & 3 \\ 12 & 4 & 6 \\ 18 & 6 & 9 \end{vmatrix} = 0$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 12 & 6 \\ 3 & 18 & 9 \end{vmatrix} = 0 \quad ; \quad \Delta_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 12 \\ 3 & 6 & 18 \end{vmatrix} = 0$$

Here $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$.

Also all their 2×2 minors are zero, but atleast one of a_{ij} of Δ is non- zero.

$\therefore$ It has infinitely many solution (by case 2(c)). The system given above is reduced to one equation i.e. $x + 2y + 3z = 6$

Infinitely many solution

Fig. 1.9

Assigning arbitrary values to two of the three unknowns say $y = s$, $z = t$

We get $x = 6 - 2y - 3z = 6 - 2s - 3t$

$\therefore$ The solution is $x = 6 - 2s - 3t$, $y = s$, $z = t$

i.e. $(x, y, z) = (6 - 2s - 3t, s, t)$ $s, t \in R$

For different value s, t we get different solution.

Solution (4) :

$x + 2y + 3z = 6$; $x + y + z = 3$; $2x + 3y + 4z = 10$

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

$$\Delta_x = \begin{vmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{vmatrix} = -1$$

No Solution

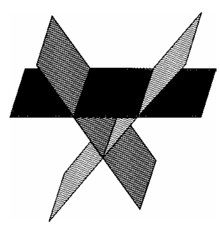


Fig. 1.10

Since $\Delta = 0$, $\Delta_x \neq 0$ (atleast one of the values of Δ_x, Δ_y, Δ_z non-zero) The system is inconsistent (by case 2(a)).

$\therefore$ It has no solution.

Solution (5) :

$x + 2y + 3z = 6$; $2x + 4y + 6z = 12$; $3x + 6y + 9z = 24$

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix} = 0 \;; \qquad \Delta_x = \begin{vmatrix} 6 & 2 & 3 \\ 12 & 4 & 6 \\ 24 & 6 & 9 \end{vmatrix} = 0$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 12 & 6 \\ 3 & 24 & 9 \end{vmatrix} = 0 \;; \qquad \Delta_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 12 \\ 3 & 6 & 24 \end{vmatrix} = 0$$

Here $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$.

All the 2×2 minors of Δ are zero, but we see that atleast one of the 2×2 minors of Δ_x or Δ_y or Δ_z is non zero. i.e.

$$\left(\begin{vmatrix} 12 & 4 \\ 24 & 6 \end{vmatrix} \neq 0 \quad \text{minor of 3 in } \Delta_x \right)$$

$\therefore$ by case 2(d), the system is inconsistent and it has no solution.

No solution

Fig. 1.11

Example 1.17 : Solve the following system of linear equations by determinant method.

(1) $x + y = 3$, $2x + 3y = 7$ (2) $2x + 3y = 8$, $4x + 6y = 16$ (3) $x - y = 2$, $3y = 3x - 7$

Solution (1) : $x + y = 3$; $2x + 3y = 7$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1, \;; \quad \because \Delta \neq 0 \text{ It has unique solution}$$

$$\Delta_x = \begin{vmatrix} 3 & 1 \\ 7 & 3 \end{vmatrix} = 9 - 7 = 2 \;; \quad \Delta_y = \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = 7 - 6 = 1$$

$$\Delta = 1, \quad \Delta_x = 2, \quad \Delta_y = 1$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} = \frac{2}{1} = 2 \;; \quad y = \frac{\Delta_y}{\Delta} = \frac{1}{1} = 1$$

solution is $(x, y) = (2, 1)$

Solution (2) : $2x + 3y = 8$; $4x + 6y = 16$

$$\Delta = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$$

$$\Delta_x = \begin{vmatrix} 8 & 3 \\ 16 & 6 \end{vmatrix} = 48 - 48 = 0$$

$$\Delta_y = \begin{vmatrix} 2 & 8 \\ 4 & 16 \end{vmatrix} = 32 - 32 = 0$$

Since $\Delta = 0$, and $\Delta_x = \Delta_y = 0$ and atleast one of the coefficients a_{ij} of $\Delta \neq 0$, the system is consistent and has infinitely many solutions.

All 2×2 minor are zero and atleast (1×1) minor is non zero. The system is reduced to a single equation. We assign arbitrary value to x (or y) and solve for y (or x).

Suppose we assign $x = t$, from equation (1)

we get $y = \frac{1}{3}(8 - 2t)$.

$\therefore$ The solution set is $\quad (x, y) = \left(t, \frac{8 - 2t}{3}\right), \quad t \in R$

In particular $\quad (x, y) = (1, 2) \quad$ for $t = 1$

$(x, y) = (-2, 4) \quad$ for $t = -2$

$(x, y) = \left(-\frac{1}{2}, 3\right) \quad$ for $t = -\frac{1}{2}$

Solution (3) : $x - y = 2$; $3y = 3x - 7$

$$\Delta = \begin{vmatrix} 1 & -1 \\ 3 & -3 \end{vmatrix} = 0,$$

$$\Delta_x = \begin{vmatrix} 2 & -1 \\ 7 & -3 \end{vmatrix} = 1$$

Since $\Delta = 0$ and $\Delta_x \neq 0$ (atleast one of the values Δ_x or $\Delta_y \neq 0$)

the system is inconsistent. $\therefore$ It has no solution.

Example 1.18 : Solve the following non-homogeneous equations of three unknowns.

(1)		(2)		(3)	
	$x + 2y + z = 7$		$x + y + 2z = 6$		$2x + 2y + z = 5$
	$2x - y + 2z = 4$		$3x + y - z = 2$		$x - y + z = 1$
	$x + y - 2z = -1$		$4x + 2y + z = 8$		$3x + y + 2z = 4$

(4) $x + y + 2z = 4$
$2x + 2y + 4z = 8$
$3x + 3y + 6z = 12$

(5) $x + y + 2z = 4$
$2x + 2y + 4z = 8$
$3x + 3y + 6z = 10$

Solution (1) : $x + 2y + z = 7,\quad 2x - y + 2z = 4,\quad x + y - 2z = -1$

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{vmatrix} = 15 \qquad \therefore\ \Delta \neq 0 \text{ it has unique solution.}$$

$$\Delta_x = \begin{vmatrix} 7 & 2 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -2 \end{vmatrix} = 15 \quad ; \quad \Delta_y = \begin{vmatrix} 1 & 7 & 1 \\ 2 & 4 & 2 \\ 1 & -1 & -2 \end{vmatrix} = 30$$

$$\Delta_z = \begin{vmatrix} 1 & 2 & 7 \\ 2 & -1 & 4 \\ 1 & 1 & -1 \end{vmatrix} = 30$$

$\Delta = 15,\quad \Delta_x = 15,\quad \Delta_y = 30,\quad \Delta_z = 30$

By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} = 1,\quad y = \frac{\Delta_y}{\Delta} = 2,\quad z = \frac{\Delta_z}{\Delta} = 2$$

Solution is $(x, y, z) = (1, 2, 2)$

Solution (2) : $x + y + 2z = 6,\quad 3x + y - z = 2,\quad 4x + 2y + z = 8$

$$\Delta = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 0, \quad \Delta_x = \begin{vmatrix} 6 & 1 & 2 \\ 2 & 1 & -1 \\ 8 & 2 & 1 \end{vmatrix} = 0,$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 2 \\ 3 & 2 & -1 \\ 4 & 8 & 1 \end{vmatrix} = 0, \quad \Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 3 & 1 & 2 \\ 4 & 2 & 8 \end{vmatrix} = 0$$

Since $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$, also atleast one of the (2×2) minors of Δ is not zero, the system is consistent and has infinitely many solution.

Take two suitable equations and assign arbitrary value to one of the three unknowns. We solve for the other two unknowns.

Let $z = k \in R$

$\therefore$ equation (1) and (2) becomes

$$x + y = 6 - 2k$$
$$3x + y = 2 + k$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = 1 - 3 = -2$$

$$\Delta_x = \begin{vmatrix} 6-2k & 1 \\ 2+k & 1 \end{vmatrix} = 6 - 2k - 2 - k = 4 - 3k$$

$$\Delta_y = \begin{vmatrix} 1 & 6-2k \\ 3 & 2+k \end{vmatrix} = 2 + k - 18 + 6k = 7k - 16$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} = \frac{4-3k}{-2} = \frac{1}{2}(3k-4)$$

$$y = \frac{\Delta_y}{\Delta} = \frac{7k-16}{-2} = \frac{1}{2}(16-7k)$$

$\therefore$ The solution set is

$$(x, y, z) = \left(\frac{3k-4}{2}, \frac{16-7k}{2}, k\right) \qquad k \in R$$

Particular Numerical solutions for $k = -2$ and 2 are

$(-5, 15, -2)$ and $(1, 1, 2)$ respectively

Solution (3) : $2x + 2y + z = 5, \quad x - y + z = 1, \quad 3x + y + 2z = 4$

$$\Delta = \begin{vmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 0 \quad ; \quad \Delta_x = \begin{vmatrix} 5 & 2 & 1 \\ 1 & -1 & 1 \\ 4 & 1 & 2 \end{vmatrix} \neq 0$$

Since $\Delta = 0$ and $\Delta_x \neq 0$ (atleast one of the values of Δ_x, Δ_y, Δ_z non zero) the system is inconsistent. i.e. it has no solution.

Solution (4) : $x + y + 2z = 4, \quad 2x + 2y + 4z = 8, \quad 3x + 3y + 6z = 12$

$$\Delta = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{vmatrix} = 0 \qquad \Delta_x = \begin{vmatrix} 4 & 1 & 2 \\ 8 & 2 & 4 \\ 12 & 3 & 6 \end{vmatrix} = 0$$

$$\Delta_y = \begin{vmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ 3 & 12 & 6 \end{vmatrix} = 0, \quad \Delta_z = \begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 8 \\ 3 & 3 & 12 \end{vmatrix} = 0$$

Since $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$ also all 2×2 minors of Δ, Δ_x, Δ_y and Δ_z are zero, by case 2(c), it is consistent and has infinitely many solutions. ($\because$ all 2×2 minors zero and atleast one of a_{ij} of $\Delta \neq 0$, the system is reduced to single equation).

Let us take $x = s$ and $y = t$, we get from equation (1)

$z = \frac{1}{2}\ (4 - s - t)\ \ \therefore$ the solution set is

$$(x, y, z) = \left(s,\ t,\ \frac{4-s-t}{2}\right),\quad s, t \in R$$

Particular numerical solution for

$$(x, y, z) = (1, 1, 1) \qquad \text{when } s = t = 1$$

$$(x, y, z) = \left(-1,\ 2,\ \frac{3}{2}\right) \qquad \text{when } s = -1, t = 2$$

Solution (5) : $x + y + 2z = 4,\qquad 2x + 2y + 4z = 8,\qquad 3x + 3y + 6z = 10$

$$\Delta = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{vmatrix} = 0 \qquad \Delta_y = \begin{vmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ 3 & 10 & 6 \end{vmatrix} = 0$$

$$\Delta_x = \begin{vmatrix} 4 & 1 & 2 \\ 8 & 2 & 4 \\ 10 & 3 & 6 \end{vmatrix} = 0, \qquad \Delta_z = \begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 8 \\ 3 & 3 & 10 \end{vmatrix} = 0$$

$\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$. Also all 2×2 minors of $\Delta = 0$, but not all the minors of Δ_x, Δ_y and Δ_z are zero.

Therefore the system is inconsistent. i.e. it has no solution.

Example 1.19 : A bag contains 3 types of coins namely Re. 1, Rs. 2 and Rs. 5. There are 30 coins amounting to Rs. 100 in total. Find the number of coins in each category.

Solution :

Let x, y and z be the number of coins respectively in each category Re. 1, Rs. 2 and Rs. 5. From the given information

$$x + y + z = 30 \qquad \text{(i)}$$

$$x + 2y + 5z = 100 \qquad \text{(ii)}$$

Here we have 3 unknowns but 2 equations. We assign arbitrary value k to z and solve for x and y.

(i) and (ii) become

$$x + y = 30 - k$$

$$x + 2y = 100 - 5k \qquad k \in R$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1,\ \Delta_x = \begin{vmatrix} 30-k & 1 \\ 100-5k & 2 \end{vmatrix} = 3k - 40,\quad \Delta_y = \begin{vmatrix} 1 & 30-k \\ 1 & 100-5k \end{vmatrix} = 70 - 4k$$

By Cramer's Rule

$$x = \frac{\Delta_x}{\Delta} = 3k - 40, \qquad y = \frac{\Delta_y}{\Delta} = 70 - 4k$$

The solution is $(x, y, z) = (3k - 40, 70 - 4k, k)$ $k \in R$.

Since the number of coins is a non-negative integer, $k = 0, 1, 2 \ldots$

Morever $3k - 40 \geq 0$, and $70 - 4k \geq 0 \Rightarrow 14 \leq k \leq 17$

$\therefore$ The possible solutions are (2, 14, 14), (5, 10, 15), (8, 6, 16) and (11, 2, 17)

1.5.4 Homogeneous linear system :

A system of linear equations is said to be homogeneous if the constant terms are all zero; that is, the system has the form

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots\ldots + a_{1n} x_n = 0$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots + a_{2n} x_n = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots\ldots + a_{mn} x_n = 0$$

Every homogeneous system of linear equations is always consistent, since all such systems have $x_1 = 0, x_2 = 0 \ldots\ldots x_n = 0$ as a solution. This solution is called **trivial solution**. If there are other solution they are called **non trivial solutions**. Because a homogeneous linear system always has the trivial solution, there are only two possibilities.

(i) (The system has only) the trivial solution

(ii) (The system has) infinitely many solutions in addition to the trivial solution.

As an illustration, consider a homogeneous linear system of two equations in two unknowns.

$$x + y = 0$$
$$x - y = 0$$

the graph of these equations are lines through the origin and the trivial solution corresponding to the point of intersection at the origin.

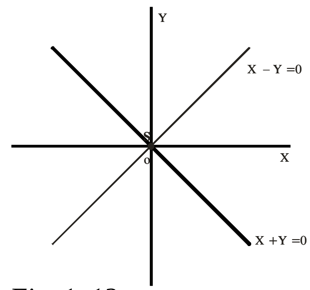


Fig. 1. 12

For the following system

$$x - y = 0$$
$$2x - 2y = 0$$

the graph shows, that the system has infinitely many solutions.

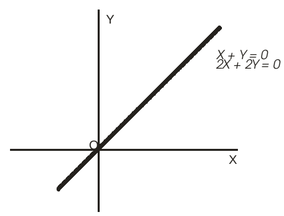


Fig. 1.13

There is one case in which a homogeneous system is assured of having non-trivial solutions, namely, whenever the system involves more number of unknowns than the number of equations.

Theorem 1.3 : (without proof)

A homogeneous system of linear equations with more number of unknowns than the number of equations has infinitely many solutions.

Example 1.20 :

Solve :
$$\begin{aligned} x + y + 2z &= 0 \\ 2x + y - z &= 0 \\ 2x + 2y + z &= 0 \end{aligned}$$

Solution :

$$\Delta = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & -1 \\ 2 & 2 & 1 \end{vmatrix} = 3$$

$\because \Delta \neq 0$, the system has unique solution.

$\therefore$ The above system of homogeneous equation has only trivial solution. i.e., $(x, y, z) = (0, 0, 0)$.

Example 1.21 :

Solve :
$$\begin{aligned} x + y + 2z &= 0 \\ 3x + 2y + z &= 0 \\ 2x + y - z &= 0 \end{aligned}$$

Solution :

$$\Delta = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 0$$

Since $\Delta = 0$, it has infinitely many solutions. Also atleast one 2×2 minors of $\Delta \neq 0$, the system is reduced to 2 equations.

$\therefore$ Assigning arbitrary value to one of the unknowns, say $z = k$ and taking first and last equations. (Here we can take any two equations)

we get
$$\begin{aligned} x + y &= -2k \\ 2x + y &= k \end{aligned}$$

$$\therefore \Delta = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1, \quad \Delta_x = \begin{vmatrix} -2k & 1 \\ k & 1 \end{vmatrix} = -3k, \quad \Delta_y = \begin{vmatrix} 1 & -2k \\ 2 & k \end{vmatrix} = 5k$$

By Cramer's Rule

$x = 3k, \quad y = -5k$

$\therefore$ Solution is $(x, y, z) = (3k, -5k, k)$

EXERCISE 1.4

Solve the following non-homogeneous system of linear equations by determinant method :

(1) $3x + 2y = 5$
$x + 3y = 4$

(2) $2x + 3y = 5$
$4x + 6y = 12$

(3) $4x + 5y = 9$
$8x + 10y = 18$

(4) $x + y + z = 4$
$x - y + z = 2$
$2x + y - z = 1$

(5) $2x + y - z = 4$
$x + y - 2z = 0$
$3x + 2y - 3z = 4$

(6) $3x + y - z = 2$
$2x - y + 2z = 6$
$2x + y - 2z = -2$

(7) $x + 2y + z = 6$
$3x + 3y - z = 3$
$2x + y - 2z = -3$

(8) $2x - y + z = 2$
$6x - 3y + 3z = 6$
$4x - 2y + 2z = 4$

(9) $\frac{1}{x}+\frac{2}{y}-\frac{1}{z}=1$; $\frac{2}{x}+\frac{4}{y}+\frac{1}{z}=5$; $\frac{3}{x}-\frac{2}{y}-\frac{2}{z}=0$

(10) A small seminar hall can hold 100 chairs. Three different colours (red, blue and green) of chairs are available. The cost of a red chair is Rs.240, cost of a blue chair is Rs.260 and the cost of a green chair is Rs.300. The total cost of chair is Rs.25,000. Find atleast 3 different solution of the number of chairs in each colour to be purchased.

1.5.5 Rank method :

Let us consider a system of "m" linear algebraic equation, in "n" unknowns $x_1, x_2, x_3, \ldots x_n$ as in section 1.2.

The equations can be written in the form of matrix equation as $AX = B$

Where the $m \times n$ matrix A is called the coefficient matrix.

A set of values $x_1, x_2, x_3 \ldots x_n$ which satisfy the above system of equations is called a solution of the system.

The system of equations is said to be consistent, if it has atleast one solution. A consistent system may have one or infinite number of solutions, when the system possesses only one solution then it is called a unique solution. The system of equations is said to be inconsistent if it has no solution.

The $m \times (n + 1)$ matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} & b_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} & b_m \end{bmatrix}$$ is called the augmented matrix of the system and it is denoted by $[A, B]$. The condition for the consistency of a system of simultaneous linear equations can be given interms of the coefficient and augmented matrices.

The system of simultaneous linear equations $AX = B$ is consistent if and only if the matrices A and $[A, B]$ are of the same rank.

The solution of a given system of linear equations is not altered by interchanging any two equations or by multiplying any equation by a non-zero scalar or by adding a multiple of one equation to another equation. By applying elementary row operations to the augmented matrix the given system of equations can be reduced to an equivalent system and this reduced form is used to test for consistency and to find the solutions.

Steps to be followed for testing consistency :

(i) Write down the given system of equations in the form of a matrix equation $AX = B$.

(ii) Find the augmented matrix $[A, B]$ of the system of equations.

(iii) Find the rank of A and rank of $[A, B]$ by applying only elementary row operations. Column operations should not be applied.

(iv) (a) If the rank of $A \neq$ rank of $[A, B]$ then the system is inconsistent and has no solution.

(b) If the rank of A = rank of $[A, B] = n$, where n is the number of unknowns in the system then A is a non-singular matrix and the system is consistent and it has a unique solution.

(c) If the rank of A = rank of $[A, B] < n$, then also the system is consistent but has an infinite number of solutions.

Example 1.22 : Verify whether the given system of equations is consistent. If it is consistent, solve them.

$$2x + 5y + 7z = 52, \qquad x + y + z = 9, \qquad 2x + y - z = 0$$

Solution : The given system of equations is equivalent to the single matrix equation.

$$\begin{bmatrix} 2 & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 52 \\ 9 \\ 0 \end{bmatrix}$$

$$A\ X = B$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 2 & 5 & 7 & 52 \\ 1 & 1 & 1 & 9 \\ 2 & 1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 3 & 5 & 34 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -4 & -20 \end{bmatrix} R_3 \to R_3 + 3R_2$$

The last equivalent matrix is in the echelon form. It has three non-zero rows.

$\therefore\ \rho(A, B) = 3$

Also $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -4 \end{bmatrix}$

Since there are three non-zero rows, $\rho(A) = 3$

$\rho(A) = \rho[A, B] = 3 =$ number of unknowns.

$\therefore$ The given system is consistent and has a unique solution.

To find the solution, we see that the given system of equations is equivalent to the matrix equation.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -18 \\ -20 \end{bmatrix}$$

$$x + y + z = 9 \qquad \ldots (1)$$

$$-y - 3z = -18 \qquad (2)$$

$$-4z = -20 \qquad \ldots (3)$$

(3) $\Rightarrow$ $z = 5$; (2) $\Rightarrow y = 18 - 3z = 3$; (1) $\Rightarrow x = 9 - y - z \Rightarrow x = 9-3-5 = 1$

$\therefore$ Solution is $x = 1, \quad y = 3, \quad z = 5$

Example 1.23 :

Examine the consistency of the equations

$2x - 3y + 7z = 5, \qquad 3x + y - 3z = 13, \qquad 2x + 19y - 47z = 32$

Solution :

The given system of equations can be written in the form of a matrix equation as

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

$$A\ X = B$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix} R_1 \to \frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 0 & \frac{11}{2} & -\frac{27}{2} & \frac{11}{2} \\ 0 & 22 & -54 & 27 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 0 & \frac{11}{2} & -\frac{27}{2} & \frac{11}{2} \\ 0 & 0 & 0 & 5 \end{bmatrix} R_3 \to R_3 - 4R_2$$

The last equivalent matrix is in the echelon form. It has three non-zero rows. $\therefore$ $\rho[A, B] = 3$ and $\rho(A) = 2$

$\rho(A) \neq \rho[A, B]$

$\therefore$ The given system is inconsistent and hence has no solution.

Note : This problem can be solved by not dividing R_1 by 2 also. i.e., $R_2 \to 2R_2 - 3R_1$

Example 1.24 :

Show that the equations $x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 7z = 30$ are consistent and solve them.

Solution : The matrix equation corresponding to the given system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$$A\,X = B$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 3R_2$$

In the last equivalent matrix, there are two non-zero rows.

$\therefore$ $\rho(A, B) = 2$ and $\rho(A) = 2$

$$\rho(A) = \rho(A, B)$$

$\therefore$ The given system is consistent. But the value of the common rank is less than the number of unknowns. The given system has an infinite number of solutions.

The given system is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$$x + y + z = 6 \qquad \ldots (1)$$

$$y + 2z = 8 \qquad \ldots (2)$$

(2) $\Rightarrow$ $y = 8 - 2z$; (1) $\Rightarrow$ $x = 6 - y - z = 6 - (8 - 2z) - z = z - 2$

Taking $z = k$, we get $x = k - 2$, $y = 8 - 2k$; $k \in R$

Putting $k = 1$, we have one solution as $x = -1$, $y = 6$, $z = 1$. Thus by giving different values for k we get different solutions. Hence the given system has infinite number of solutions.

Example 1.25 :

Verify whether the given system of equations is consistent. If it is consistent, solve them :

$$x - y + z = 5, \quad -x + y - z = -5, \quad 2x - 2y + 2z = 10$$

Solution : The matrix equation corresponding to the given system is

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$$

$$A\,X = B$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 1 & -1 & 1 & 5 \\ -1 & 1 & -1 & -5 \\ 2 & -2 & 2 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 + R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{matrix}$$

In the last equivalent matrix, there is only one non-zero row

$\therefore\ \rho[A, B] = 1$ and $\rho(A) = 1$

Thus $\rho(A) = \rho[A, B] = 1$. $\therefore$ the given system is consistent. Since the common value of the rank is less than the number of unknowns, there are infinitely many solutions. The given system is equivalent to the matrix equation.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

$x - y + z = 5$; Taking $y = k_1$, $z = k_2$, we have $x = 5 + k_1 - k_2$. for various values of k_1 and k_2 we have infinitely many solutions. $k_1, k_2 \in R$

Example 1.26 : Investigate for what values of λ, μ the simultaneous equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have (i) no solution (ii) a unique solution and (iii) an infinite number of solutions.

Solution :

The matrix equations corresponding to the given system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$A\,X = B$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{matrix}$$

Case (i) : $\lambda - 3 = 0$ and $\mu - 10 \neq 0$ i.e. $\lambda = 3$ and $\mu \neq 10$.

In this case $\rho(A) = 2$ while $\rho[A, B] = 3$ $\therefore$ $\rho(A) \neq \rho[A, B]$

$\therefore$ The given system is inconsistent and has no solution.

Case (ii) : $\lambda - 3 \neq 0$ i.e., $\lambda \neq 3$ and μ can take any value in R.

In this case $\rho(A) = 3$ and $\rho[A, B] = 3$

$\rho(A) = \rho[A, B] = 3$ = number of unknowns.

$\therefore$ The given system is consistent and has a unique solution.

Case (iii) :

$\lambda - 3 = 0$ and $\mu - 10 = 0$ i.e., $\lambda = 3$ and $\mu = 10$

In this case $\rho(A) = \rho[A, B] = 2 <$ number of unknowns.

$\therefore$ The given system is consistent but has an infinite number of solutions.

1.5.6 Homogeneous linear Equations :

A system of homogeneous linear equations is given by

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots\ldots\ldots\ldots + a_{1n} x_n = 0$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots\ldots\ldots\ldots + a_{2n} x_n = 0$

…………………………………………………

…………………………………………………

$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \ldots\ldots\ldots\ldots + a_{mn} x_n = 0$

and the corresponding augmented matrix is

$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & 0 \end{bmatrix} = [A, O]$$

Since rank of A = rank of $[A, O]$ is always true, we see that the system of homogeneous equations is always consistent.

Note that $x_1 = 0, x_2 = 0, x_3 = 0 \ldots x_n = 0$ is always a solution of the system. This solution is called a trivial solution. If the rank of A = rank of $[A, B] < n$ then the system has non trivial solutions including trivial solution. If $\rho(A) = n$ then the system has only trivial solution.

Example 1.27 : Solve the following homogeneous linear equations.

$x + 2y - 5z = 0, \quad 3x + 4y + 6z = 0, \quad x + y + z = 0$

Solution : The given system of equations can be written in the form of matrix equation

$$\begin{bmatrix} 1 & 2 & -5 \\ 3 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A\,X = B$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 1 & 2 & -5 & 0 \\ 3 & 4 & 6 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & -2 & 21 & 0 \\ 0 & -1 & 6 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & -2 & 21 & 0 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix} R_3 \to R_3 - 2R_2$$

This is in the echelon form.

Clearly $\rho[A, B] = 3$. and. $\rho(A) = 3$

$$\therefore \rho(A) = \rho[A, B] = 3 = \text{number of unknowns.}$$

$\therefore$ The given system of equations is consistent and has a unique solution. i.e., trivial solution.

$$\therefore \; x = 0, \; y = 0 \text{ and } z = 0$$

Note : Since $\rho(A) = 3$, $|A| \neq 0$ i.e. A is non-singular ;

$\therefore$ The given system has only trivial solution $x = 0, y = 0, z = 0$

Example 1.28 : For what value of μ the equations

$x + y + 3z = 0, \quad 4x + 3y + \mu z = 0, \quad 2x + y + 2z = 0$ have a (i) trivial solution, (ii) non-trivial solution.

Solution : The system of equations can be written as $AX = B$

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 3 & \mu \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A, B] = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 4 & 3 & \mu & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & \mu-12 & 0 \\ 0 & -1 & -4 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & \mu-12 & 0 \\ 0 & 0 & 8-\mu & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

Case (i) : If $\mu \neq 8$ then $8 - \mu \neq 0$ and hence there are three non-zero rows.

$\therefore \ \rho[A] = \rho[A, B] = 3$ = the number of unknowns.

$\therefore$ The system has the trivial solution $x = 0, \ y = 0, \ z = 0$

Case (ii) :

If $\mu = 8$ then.

$\rho[A, B] = 2$ and $\rho(A) = 2$

$\therefore \rho(A) = \rho[A, B] = 2 <$ number of unknowns.

The given system is equivalent to

$$x + y + 3z = 0 \ ; \quad y + 4z = 0$$

$$\therefore \ y = -4z \ ; \qquad x = z$$

Taking $z = k$, we get $x = k, y = -4k, z = k \quad [k \in R - \{0\}]$

which are non-trivial solutions.

Thus the system is consistent and has infinitely many non-trivial solutions.

Note : In case (ii) the system also has trivial solution. For only non-trivial solutions we removed $k = 0$.

EXERCISE 1.5

(1) Examine the consistency of the following system of equations. If it is consistent then solve the same.

(i) $4x + 3y + 6z = 25$ $x + 5y + 7z = 13$ $2x + 9y + z = 1$

(ii) $x - 3y - 8z = -10$ $3x + y - 4z = 0$ $2x + 5y + 6z - 13 = 0$

(iii) $x + y + z = 7$ $x + 2y + 3z = 18$ $y + 2z = 6$

(iv) $x - 4y + 7z = 14$ $3x + 8y - 2z = 13$ $7x - 8y + 26z = 5$

(v) $x + y - z = 1$ $2x + 2y - 2z = 2$ $-3x - 3y + 3z = -3$

(2) Discuss the solutions of the system of equations for all values of λ.

$x + y + z = 2,$ $2x + y - 2z = 2,$ $\lambda x + y + 4z = 2$

(3) For what values of k, the system of equations

$kx + y + z = 1,$ $x + ky + z = 1,$ $x + y + kz = 1$ have

(i) unique solution (ii) more than one solution (iii) no solution

2. VECTOR ALGEBRA

2.1 Introduction :

We have already studied two operations ‘addition’ and ‘subtraction’ on vectors in class XI. In this chapter we will study the notion of another operation, namely product of two vectors. The product of two vectors results in two different ways, viz., a scalar product and a vector product. Before defining these products we shall define the angle between two vectors.

2.2 Angle between two vectors :

Let two vectors $\vec{a}$ and $\vec{b}$ be represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Then the angle between $\vec{a}$ and $\vec{b}$ is the angle between their directions when these directions both converge or both diverge from their point of intersection.

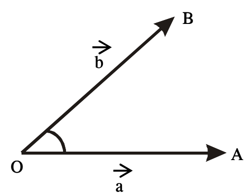


Fig. 2. 1

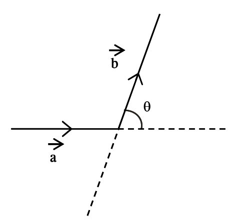


Fig. 2. 2

It is evident that if θ is the numerical measure of the angle between two vectors, then $0 \le \theta \le \pi$.

2.3 The Scalar product or Dot product

Let $\vec{a}$ and $\vec{b}$ be two non zero vectors inclined at an angle θ. Then the scalar product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \cdot \vec{b}$ and is defined as the scalar $|\vec{a}|\,|\vec{b}| \cos\theta$.

$$\text{Thus } \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = ab\cos\theta$$

Note : Clearly the scalar product of two vectors is a scalar quantity. Therefore the product is called scalar product. Since we are putting dot between $\vec{a}$ and $\vec{b}$, it is also called dot product.

Geometrical Interpretation of Scalar Product

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. From B draw BL ⊥r to OA.

OL is called the projection of $\vec{b}$ on $\vec{a}$.

From ΔOLB, $\cos\theta = \dfrac{OL}{OB}$

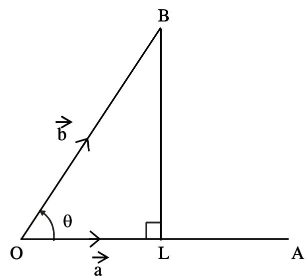


Fig. 2.3

$$\Rightarrow OL = (OB)(\cos\theta)$$

$$\Rightarrow OL = \left|\vec{b}\right| (\cos\theta) \qquad \ldots (1)$$

Now by definition $\vec{a} \cdot \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \cos\theta$

$$= \left|\vec{a}\right| (OL) \qquad [\because \text{ using } (1)]$$

$$\therefore \vec{a} \cdot \vec{b} = \left|\vec{a}\right| \left[\text{projection of } \vec{b} \text{ on } \vec{a}\right]$$

$$\text{Projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\left|\vec{a}\right|} = \frac{\vec{a}}{\left|\vec{a}\right|} \cdot \vec{b} = \hat{a} \cdot \vec{b}$$

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\left|\vec{b}\right|} = \vec{a} \cdot \frac{\vec{b}}{\left|\vec{b}\right|} = \vec{a} \cdot \hat{b}$$

2.3.1 Properties of Scalar Product :

Property 1 :

The scalar product of two vectors is commutative

(i.e.,) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ for any two vectors $\vec{a}$ and $\vec{b}$

Proof :

Let $\vec{a}$ and $\vec{b}$ be two vectors and θ the angle between them.

$$\vec{a} \cdot \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \cos\theta \qquad \ldots (1)$$

$$\therefore \vec{b} \cdot \vec{a} = \left|\vec{b}\right| \left|\vec{a}\right| \cos\theta$$

$$\vec{b} \cdot \vec{a} = |\vec{a}|\,|\vec{b}| \cos\theta \qquad \ldots (2)$$

From (1) and (2)

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

Thus dot product is commutative.

Property 2 : Scalar Product of Collinear Vectors :

(i) When the vectors $\vec{a}$ and $\vec{b}$ are collinear and are in the same direction, then $\theta = 0$

Thus $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = |\vec{a}|\,|\vec{b}|\,(1) = ab \qquad \ldots (1)$

(ii) When the vectors $\vec{a}$ and $\vec{b}$ are collinear and are in the opposite direction, then $\theta = \pi$

Thus

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = |\vec{a}|\,|\vec{b}|(\cos\pi) \qquad \ldots (1)$$

$$= |\vec{a}|\,|\vec{b}|\,(-1) = -ab$$

Property 3 : Sign of Dot Product

The dot product $\vec{a} \cdot \vec{b}$ may be positive or negative or zero.

(i) If the angle between the two vectors is acute (i.e., $0 < \theta < 90°$) then $\cos\theta$ is positive. In this case dot product is positive.

(ii) If the angle between the two vectors is obtuse (i.e., $90 < \theta < 180$) then $\cos\theta$ is negative. In this case dot product is negative.

(iii) If the angle between the two vectors is $90°$ (i.e., $\theta = 90°$) then $\cos\theta = \cos 90° = 0$. In this case dot product is zero.

Note : If $\vec{a} \cdot \vec{b} = 0$, we have the following three possibilities

$\vec{a} \cdot \vec{b} = 0 \;\Rightarrow\; |\vec{a}|\,|\vec{b}| \cos\theta = 0$

(i) $|\vec{a}| = 0$ (i.e.,) $\vec{a}$ is a zero vector and $\vec{b}$ any vector.

(ii) $|\vec{b}| = 0$ (i.e.,) $\vec{b}$ is a zero vector and $\vec{a}$ any vector.

(iii) $\cos\theta = 0$ (i.e.,) $\theta = 90°$ (i.e.,) $\vec{a} \perp \vec{b}$

Important Result :

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors, then $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

Property 4 : Dot product of equal vectors :

$$\vec{a}\,.\,\vec{a} = |\vec{a}|\,|\vec{a}| \cos 0 = |\vec{a}|\,|\vec{a}| = |\vec{a}|^2 = a^2$$

Convention : $\left(\vec{a}\right)^2 = \vec{a}\,.\,\vec{a} = |\vec{a}|^2 = \vec{a}^2 = a^2$

Property 5 :

(i) $\vec{i}\,.\,\vec{i} = \vec{j}\,.\,\vec{j} = \vec{k}\,.\,\vec{k} = 1$

(ii) $\vec{i}\,.\,\vec{j} = \vec{j}\,.\,\vec{i} = \vec{j}\,.\,\vec{k} = \vec{k}\,.\,\vec{j} = \vec{k}\,.\,\vec{i} = \vec{i}\,.\,\vec{k} = 0$

$$\vec{i}\,.\,\vec{i} = |\vec{i}|\,|\vec{i}| \cos 0 = (1)\,(1)\,(1) = 1$$

$$\vec{i}\,.\,\vec{j} = |\vec{i}|\,|\vec{j}| \cos 90 = (1)\,(1)\,(0) = 0$$

Property 6 :

If m is any scalar and $\vec{a}$, $\vec{b}$ are any two vectors, then

$$\left(m\vec{a}\right).\,\vec{b} = m\left(\vec{a}\,.\,\vec{b}\right) = \vec{a}\,.\left(m\vec{b}\right)$$

Property 7 :

If m, n are scalars and $\vec{a}$, $\vec{b}$ are two vectors then

$$m\vec{a}\,.\,n\vec{b} = mn\left(\vec{a}\,.\,\vec{b}\right) = \left(mn\vec{a}\right).\,\vec{b} = \vec{a}\,.\left(mn\vec{b}\right)$$

Property 8 :

The scalar product is distributive over addition.

$\vec{a}\,.\left(\vec{b} + \vec{c}\right) = \vec{a}\,.\,\vec{b} + \vec{a}\,.\,\vec{c}$, for any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$

Proof :

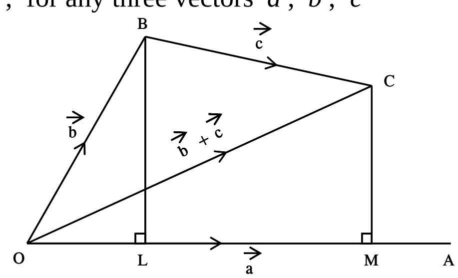


Fig. 2.4

Let $\overrightarrow{OA} = \vec{a}$

$\overrightarrow{OB} = \vec{b}$

$\overrightarrow{BC} = \vec{c}$

Then $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$

$= \vec{b} + \vec{c}$

Draw BL $\perp$ OA and $CM \perp OA$

$$\therefore\ OL = \text{Projection of } \vec{b} \text{ on } \vec{a}$$

$$LM = \text{Projection of } \vec{c} \text{ on } \vec{a}$$

$$OM = \text{Projection of } \left(\vec{b} + \vec{c}\right) \text{ on } \vec{a}$$

We have $\vec{a}\,.\,\vec{b} = \left|\vec{a}\right| \left(\text{Projection of } \vec{b} \text{ on } \vec{a}\right)$

$$\Rightarrow\ \vec{a}\,.\,\vec{b} = \left|\vec{a}\right| (OL) \qquad \ldots (1)$$

Also $\quad \vec{a}\,.\,\vec{c} = \left|\vec{a}\right| \left(\text{Projection of } \vec{c} \text{ on } \vec{a}\right)$

$$\Rightarrow\ \vec{a}\,.\,\vec{c} = \left|\vec{a}\right| (LM) \qquad \ldots (2)$$

Now $\quad \vec{a}\,.\left(\vec{b} + \vec{c}\right) = \left|\vec{a}\right| \left(\text{Projection of } \left(\vec{b} + \vec{c}\right) \text{ on } \vec{a}\right)$

$$= \left|\vec{a}\right| (OM) = \left|\vec{a}\right| (OL + LM)$$

$$= \left|\vec{a}\right| (OL) + \left|\vec{a}\right| (LM)$$

$$= \vec{a}\,.\,\vec{b} + \vec{a}\,.\,\vec{c} \qquad \text{[by using (1) and (2)]}$$

Hence $\quad \vec{a}\,.\left(\vec{b} + \vec{c}\right) = \vec{a}\,.\,\vec{b} + \vec{a}\,.\,\vec{c}$

Corollary : $\vec{a}\,.\left(\vec{b} - \vec{c}\right) = \vec{a}\,.\,\vec{b} - \vec{a}\,.\,\vec{c}$

Property 9 :

(i) For any two vectors $\vec{a}$ and $\vec{b}$,

$$\left(\vec{a} + \vec{b}\right)^2 = \left(\vec{a}\right)^2 + 2\,\vec{a}\,.\,\vec{b} + \left(\vec{b}\right)^2 = a^2 + 2\,\vec{a}\,.\,\vec{b} + b^2$$

Proof : $\quad \left(\vec{a} + \vec{b}\right)^2 = \left(\vec{a} + \vec{b}\right).\left(\vec{a} + \vec{b}\right)$

$$= \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} \qquad \text{(by distribution law)}$$

$$= \left(\vec{a}\right)^2 + \vec{a}.\vec{b} + \vec{a}.\vec{b} + \left(\vec{b}\right)^2 \quad \left(\because\ \vec{a}.\vec{b} = \vec{b}.\vec{a}\right)$$

$$= \left(\vec{a}\right)^2 + 2\,\vec{a}.\vec{b} + \left(\vec{b}\right)^2 = a^2 + 2\,\vec{a}.\vec{b} + b^2$$

(ii) $\quad \left(\vec{a} - \vec{b}\right)^2 = \left(\vec{a}\right)^2 - 2\,\vec{a}\,.\,\vec{b} + \left(\vec{b}\right)^2 = a^2 - 2\,\vec{a}\,.\,\vec{b} + b^2$

(iii) $\left(\vec{a} + \vec{b}\right) . \left(\vec{a} - \vec{b}\right) = \left(\vec{a}\right)^2 - \left(\vec{b}\right)^2 = a^2 - b^2$

Proof : $\left(\vec{a} + \vec{b}\right) . \left(\vec{a} - \vec{b}\right) = \vec{a} . \vec{a} - \vec{a} . \vec{b} + \vec{b} . \vec{a} - \vec{b} . \vec{b}$

$$= \left(\vec{a}\right)^2 - \left(\vec{b}\right)^2 = a^2 - b^2$$

Property 10 : Scalar product in terms of components :

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ $\quad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\vec{a} . \vec{b} = \left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right) . \left(b_1\vec{i} + b_2\vec{j} + b_3\vec{k}\right)$$

$$= a_1b_1\left(\vec{i} . \vec{i}\right) + a_1b_2\left(\vec{i} . \vec{j}\right) + a_1b_3\left(\vec{i} . \vec{k}\right) + a_2b_1\left(\vec{j} . \vec{i}\right) + a_2b_2\left(\vec{j} . \vec{j}\right) + a_2b_3\left(\vec{j} . \vec{k}\right) + a_3b_1\left(\vec{k} . \vec{i}\right) + a_3b_2\left(\vec{k} . \vec{j}\right) + a_3b_3\left(\vec{k} . \vec{k}\right)$$

$$= a_1b_1(1) + a_1b_2(0) + a_1b_3(0) + a_2b_1(0) + a_2b_2(1) + a_2b_3(0) + a_3b_1(0) + a_3b_2(0) + a_3b_3(1)$$

$$= a_1b_1 + a_2b_2 + a_3b_3$$

Thus, the scalar product of two vectors is equal to the sum of the products of their corresponding components.

Property 11 : Angle between two vectors :

Let $\vec{a}, \vec{b}$ be two vectors inclined at an angle θ.

Then $\vec{a} . \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \cos\theta$

$$\Rightarrow \cos\theta = \frac{\vec{a} . \vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|} \quad \Rightarrow \theta = \cos^{-1}\left[\frac{\vec{a} . \vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}\right]$$

If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

Then $\vec{a} . \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\left|\vec{a}\right| = \sqrt{a_1^2 + a_2^2 + a_3^2} \; ; \; \left|\vec{b}\right| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

$$\therefore \theta = \cos^{-1}\left[\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right]$$

Property 12 : For any two vectors $\vec{a}$ and $\vec{b}$

$$\left|\vec{a} + \vec{b}\right| \le \left|\vec{a}\right| + \left|\vec{b}\right| \text{ (Triangle inequality)}$$

We have $\left|\vec{a} + \vec{b}\right|^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\left(\vec{a} \cdot \vec{b}\right)$

$$\Rightarrow \left|\vec{a} + \vec{b}\right|^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\left|\vec{a}\right|\left|\vec{b}\right| \cos\theta$$

$$\le \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\left|\vec{a}\right|\left|\vec{b}\right| \qquad [\because \cos\theta \le 1]$$

$$\Rightarrow \left|\vec{a} + \vec{b}\right|^2 \le \left(\left|\vec{a}\right| + \left|\vec{b}\right|\right)^2$$

$$\Rightarrow \left|\vec{a} + \vec{b}\right| \le \left|\vec{a}\right| + \left|\vec{b}\right|$$

Example 2.1 : Find $\vec{a} \cdot \vec{b}$ when

(i) $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$ and $\vec{b} = 4\vec{i} - 4\vec{j} + 7\vec{k}$

(ii) $\vec{a} = \vec{j} + 2\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{k}$

(iii) $\vec{a} = \vec{j} - 2\vec{k}$ and $\vec{b} = 2\vec{i} + 3\vec{j} - 2\vec{k}$

Solution :

(i) $\vec{a} \cdot \vec{b} = \left(\vec{i} - 2\vec{j} + \vec{k}\right) \cdot \left(4\vec{i} - 4\vec{j} + 7\vec{k}\right)$
$= (1)(4) + (-2)(-4) + (1)(7) = 19$

(ii) $\vec{a} \cdot \vec{b} = \left(\vec{j} + 2\vec{k}\right) \cdot \left(2\vec{i} + \vec{k}\right) = (0)(2) + (1)(0) + (2)(1) = 2$

(iii) $\vec{a} \cdot \vec{b} = \left(\vec{j} - 2\vec{k}\right) \cdot \left(2\vec{i} + 3\vec{j} - 2\vec{k}\right)$
$= (0)(2) + (1)(3) + (-2)(-2) = 7$

Example 2.2 : For what value of m the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other

(i) $\vec{a} = m\vec{i} + 2\vec{j} + \vec{k}$ and $\vec{b} = 4\vec{i} - 9\vec{j} + 2\vec{k}$

(ii) $\vec{a} = 5\vec{i} - 9\vec{j} + 2\vec{k}$ and $\vec{b} = m\vec{i} + 2\vec{j} + \vec{k}$

Solution :

(i) Given : $\vec{a} \perp \vec{b}$

$$\therefore\ \vec{a}\ .\ \vec{b} = 0 \Rightarrow \left(m\vec{i} + 2\vec{j} + \vec{k}\right).\left(4\vec{i} - 9\vec{j} + 2\vec{k}\right) = 0$$

$$\Rightarrow\ 4m - 18 + 2 = 0 \ \Rightarrow\ m = 4$$

(ii) $\left(5\vec{i} - 9\vec{j} + 2\vec{k}\right).\left(m\vec{i} + 2\vec{j} + \vec{k}\right) = 0$

$$\Rightarrow\ 5m - 18 + 2 = 0 \ \Rightarrow\ m = \frac{16}{5}$$

Example 2.3 : If $\vec{a}$ and $\vec{b}$ are two vectors such that $\left|\vec{a}\right| = 4$, $\left|\vec{b}\right| = 3$ and $\vec{a}\ .\ \vec{b} = 6$. Find the angle between $\vec{a}$ and $\vec{b}$

Solution :

$$\cos\theta = \frac{\vec{a}\ .\ \vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|} = \frac{6}{(4)\ (3)} = \frac{1}{2} \qquad \Rightarrow\ \theta = \frac{\pi}{3}$$

Example 2.4 : Find the angle between the vectors

$3\vec{i} - 2\vec{j} - 6\vec{k}$ and $4\vec{i} - \vec{j} + 8\vec{k}$

Solution : Let $\vec{a} = 3\vec{i} - 2\vec{j} - 6\vec{k}$; $\vec{b} = 4\vec{i} - \vec{j} + 8\vec{k}$

Let 'θ' be the angle between the vectors

$$\vec{a}\ .\ \vec{b} = 12 + 2 - 48 = -34$$

$$\left|\vec{a}\right| = 7,\ \left|\vec{b}\right| = 9$$

$$\cos\theta = \frac{\vec{a}\ .\ \vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|} = \frac{-34}{7\times 9}$$

$$\theta = \cos^{-1}\left(-\frac{34}{63}\right)$$

Example 2.5 : Find the angle between the vectors $\vec{a}$ and $\vec{b}$ where $\vec{a} = \vec{i} - \vec{j}$ and $\vec{b} = \vec{j} - \vec{k}$

Solution :

$$\cos\theta = \frac{\vec{a}\ .\ \vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|} = \frac{\left(\vec{i} - \vec{j}\right).\left(\vec{j} - \vec{k}\right)}{\left|\vec{i} - \vec{j}\right|\ \left|\vec{j} - \vec{k}\right|}$$

$$\Rightarrow\ \cos\theta = \frac{(1)\ (0)\ +\ (-1)\ (1)\ +\ (0)\ (-1)}{\sqrt{2}\times\sqrt{2}}$$

$$\Rightarrow\ \cos\theta = -\frac{1}{2} \ \Rightarrow\ \theta = \frac{2\pi}{3}$$

Example 2.6 : For any vector $\vec{r}$

prove that $\vec{r} = \left(\vec{r}\,.\,\vec{i}\right)\vec{i} + \left(\vec{r}\,.\,\vec{j}\right)\vec{j} + \left(\vec{r}\,.\,\vec{k}\right)\vec{k}$

Solution : Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be an arbitrary vector.

$$\vec{r}\,.\,\vec{i} = \left(x\vec{i} + y\vec{j} + z\vec{k}\right).\,\vec{i} = x$$

$$\vec{r}\,.\,\vec{j} = \left(x\vec{i} + y\vec{j} + z\vec{k}\right).\,\vec{j} = y$$

$$\vec{r}\,.\,\vec{k} = \left(x\vec{i} + y\vec{j} + z\vec{k}\right).\,\vec{k} = z$$

$$\left(\vec{r}\,.\,\vec{i}\right)\vec{i} + \left(\vec{r}\,.\,\vec{j}\right)\vec{j} + \left(\vec{r}\,.\,\vec{k}\right)\vec{k} = x\vec{i} + y\vec{j} + z\vec{k} = \vec{r}$$

Example 2.7 : Find the projection of the vector

$7\vec{i} + \vec{j} - 4\vec{k}$ on $2\vec{i} + 6\vec{j} + 3\vec{k}$

Solution : Let $\vec{a} = 7\vec{i} + \vec{j} - 4\vec{k}$; $\vec{b} = 2\vec{i} + 6\vec{j} + 3\vec{k}$

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}\,.\,\vec{b}}{\left|\vec{b}\right|} = \frac{\left(7\vec{i} + \vec{j} - 4\vec{k}\right).\left(2\vec{i} + 6\vec{j} + 3\vec{k}\right)}{\left|2\vec{i} + 6\vec{j} + 3\vec{k}\right|}$$

$$= \frac{14 + 6 - 12}{\sqrt{4 + 36 + 9}} = \frac{8}{7}$$

Example 2.8 : For any two vectors $\vec{a}$ and $\vec{b}$

prove that $\left|\vec{a} + \vec{b}\right|^2 + \left|\vec{a} - \vec{b}\right|^2 = 2\left(\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2\right)$

Solution : $\left|\vec{a} + \vec{b}\right|^2 = \left(\vec{a} + \vec{b}\right)^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\vec{a}\,.\,\vec{b}$ … (1)

$$\left|\vec{a} - \vec{b}\right|^2 = \left(\vec{a} - \vec{b}\right)^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 - 2\vec{a}\,.\,\vec{b} \quad \ldots (2)$$

Adding (1) and (2)

$$\left|\vec{a} + \vec{b}\right|^2 + \left|\vec{a} - \vec{b}\right|^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\vec{a}\,.\,\vec{b} + \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 - 2\vec{a}\,.\,\vec{b}$$

$$= 2\left|\vec{a}\right|^2 + 2\left|\vec{b}\right|^2 = 2\left(\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2\right)$$

Example 2.9 : If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that $\sin\frac{\theta}{2} = \frac{1}{2}\left|\hat{a} - \hat{b}\right|$

Solution : $\left|\hat{a} - \hat{b}\right|^2 = \hat{a}^2 + \hat{b}^2 - 2\hat{a}\,.\,\hat{b} = 1 + 1 - 2\left|\hat{a}\right|\left|\hat{b}\right|\cos\theta$

$$= 2 - 2\cos\theta = 2(1 - \cos\theta) = 2\left(2\sin^2\frac{\theta}{2}\right)$$

$$\therefore \left|\hat{a} - \hat{b}\right| = 2\sin\frac{\theta}{2} \Rightarrow \sin\frac{\theta}{2} = \frac{1}{2}\left|\hat{a} - \hat{b}\right|$$

Example 2.10 : If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $\left|\vec{a}\right| = 3$, $\left|\vec{b}\right| = 5$ and $\left|\vec{c}\right| = 7$, find the angle between $\vec{a}$ and $\vec{b}$

Solution : $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\vec{a} + \vec{b} = -\vec{c}$$

$$\left(\vec{a} + \vec{b}\right)^2 = \left(-\vec{c}\right)^2$$

$$\Rightarrow \left(\vec{a}\right)^2 + \left(\vec{b}\right)^2 + 2\vec{a}\,.\,\vec{b} = \left(\vec{c}\right)^2$$

$$\Rightarrow \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta = \left|\vec{c}\right|^2$$

$$\Rightarrow 3^2 + 5^2 + 2(3)(5)\cos\theta = 7^2$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Example 2.11 : Show that the vectors

$2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} - 3\vec{j} - 5\vec{k}$, $-3\vec{i} + 4\vec{j} + 4\vec{k}$ form the sides of a right angled triangle.

Solution : Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$; $\vec{b} = \vec{i} - 3\vec{j} - 5\vec{k}$; $\vec{c} = -3\vec{i} + 4\vec{j} + 4\vec{k}$

We see that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\therefore$ $\vec{a}, \vec{b}, \vec{c}$ forms a triangle

Further $\vec{a}\,.\,\vec{b} = \left(2\vec{i} - \vec{j} + \vec{k}\right)\,.\,\left(\vec{i} - 3\vec{j} - 5\vec{k}\right)$

$$= 2 + 3 - 5 = 0$$

$\therefore$ $\vec{a} \perp \vec{b}$ $\therefore$ The vectors form the sides of a right angled triangle.

EXERCISE 2.1

(1) Find $\vec{a} \cdot \vec{b}$ when $\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 6\vec{i} - 3\vec{j} + 2\vec{k}$

(2) If $\vec{a} = \vec{i} + \vec{j} + 2\vec{k}$ and $\vec{b} = 3\vec{i} + 2\vec{j} - \vec{k}$ find $\left(\vec{a} + 3\vec{b}\right) \cdot \left(2\vec{a} - \vec{b}\right)$

(3) Find λ so that the vectors $2\vec{i} + \lambda\vec{j} + \vec{k}$ and $\vec{i} - 2\vec{j} + \vec{k}$ are perpendicular to each other.

(4) Find the value of m for which the vectors $\vec{a} = 3\vec{i} + 2\vec{j} + 9\vec{k}$ and $\vec{b} = \vec{i} + m\vec{j} + 3\vec{k}$ are (i) perpendicular (ii) parallel

(5) Find the angles which the vector $\vec{i} - \vec{j} + \sqrt{2}\,\vec{k}$ makes with the coordinate axes.

(6) Show that the vector $\vec{i} + \vec{j} + \vec{k}$ is equally inclined with the coordinate axes.

(7) If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that

(i) $\cos\dfrac{\theta}{2} = \dfrac{1}{2}\left|\hat{a} + \hat{b}\right|$ (ii) $\tan\dfrac{\theta}{2} = \dfrac{\left|\hat{a} - \hat{b}\right|}{\left|\hat{a} + \hat{b}\right|}$

(8) If the sum of two unit vectors is a unit vector prove that the magnitude of their difference is $\sqrt{3}$.

(9) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three mutually perpendicular unit vectors, then prove that $\left|\vec{a} + \vec{b} + \vec{c}\right| = \sqrt{3}$

(10) If $\left|\vec{a} + \vec{b}\right| = 60$, $\left|\vec{a} - \vec{b}\right| = 40$ and $\left|\vec{b}\right| = 46$ find $\left|\vec{a}\right|$.

(11) Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vector such that $\vec{u} + \vec{v} + \vec{w} = \vec{0}$.

If $\left|\vec{u}\right| = 3$, $\left|\vec{v}\right| = 4$ and $\left|\vec{w}\right| = 5$ then find $\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}$

(12) Show that the vectors $3\vec{i} - 2\vec{j} + \vec{k}$, $\vec{i} - 3\vec{j} + 5\vec{k}$ and $2\vec{i} + \vec{j} - 4\vec{k}$ form a right angled triangle.

(13) Show that the points whose position vectors

$4\vec{i} - 3\vec{j} + \vec{k}$, $2\vec{i} - 4\vec{j} + 5\vec{k}$, $\vec{i} - \vec{j}$ form a right angled triangle.

(14) Find the projection of

(i) $\vec{i} - \vec{j}$ on z-axis (ii) $\vec{i} + 2\vec{j} - 2\vec{k}$ on $2\vec{i} - \vec{j} + 5\vec{k}$

(iii) $3\vec{i} + \vec{j} - \vec{k}$ on $4\vec{i} - \vec{j} + 2\vec{k}$

2.3.2 Geometrical Applicaton of dot product

Cosine formulae :

Example 2.12 : With usual notations :

(i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$; (ii) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$ (iii) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Solution (i) :

From the diagram

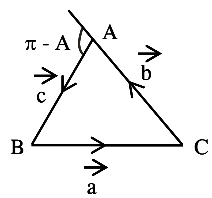


Fig. 2.5

$$\vec{AB} + \vec{BC} + \vec{CA} = \vec{0} \Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\vec{a} = -\left(\vec{b} + \vec{c}\right)$$

$$\left(\vec{a}\right)^2 = \left(\vec{b} + \vec{c}\right)^2$$

$$\Rightarrow a^2 = b^2 + c^2 + 2\vec{b} \cdot \vec{c}$$

$$\Rightarrow \quad a^2 = b^2 + c^2 + 2bc\cos(\pi - A)$$

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$

$$2bc\cos A = b^2 + c^2 - a^2$$

$$\boxed{\cos A = \frac{b^2 + c^2 - a^2}{2bc}}$$

Similarly we can prove the results (ii) & (iii)

Projection Formulae :

Example 2.13 : With usual notations

(i) $a = b\cos C + c\cos B$ (ii) $b = a\cos C + c\cos A$ (iii) $c = a\cos B + b\cos A$

Solution (i) :

From the diagram

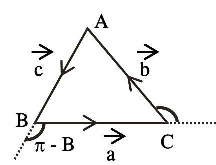


Fig. 2.6

$$\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$$

$$\Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\vec{a} = -\vec{b} - \vec{c}$$

$$\vec{a} \cdot \vec{a} = -\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$$

We have

$$a^2 = -ab\cos(\pi - C) - ac\cos(\pi - B)$$
$$a^2 = -ab(-\cos C) - ac(-\cos B)$$
$$\Rightarrow \quad a^2 = ab\cos C + ac\cos B$$
$$\Rightarrow \quad \boxed{a = b\cos C + c\cos B}$$

Similarly (ii) and (iii) can be proved.

Example 2.14 : Angle in a semi-circle is a right angle. Prove by vector method.

Solution : Let AB be the diameter of the circle with centre O.

Let P be any point on the semi-circle.

To prove $\underline{|APB} = 90°$

We have $OA = OB = OP$ (radii)

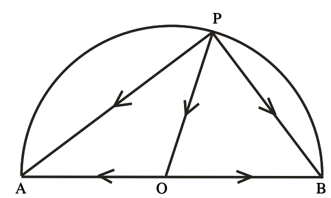


Fig. 2.7

Now $\quad \vec{PA} = \vec{PO} + \vec{OA}$

Also $\quad \vec{PB} = \vec{PO} + \vec{OB}$

$$= \vec{PO} - \vec{OA}$$

$$\therefore \vec{PA} \cdot \vec{PB} = \left(\vec{PO} + \vec{OA}\right) \cdot \left(\vec{PO} - \vec{OA}\right)$$
$$= \left(\vec{PO}\right)^2 - \left(\vec{OA}\right)^2$$
$$= PO^2 - OA^2 = 0$$
$$\therefore \vec{PA} \perp \vec{PB} \Rightarrow \underline{|APB} = \frac{\pi}{2}$$

Hence angle in a semi-circle is a right angle.

Example 2.15 : Diagonals of a rhombus are at right angles. Prove by vector methods.

Solution : Let $ABCD$ be a rhombus. Let $\vec{AB} = \vec{a}$ and $\vec{AD} = \vec{b}$

We have $AB = BC = CD = DA$

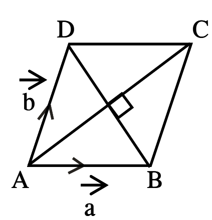


Fig. 2.8

i.e., $\quad \left|\vec{a}\right| = \left|\vec{b}\right| \quad \ldots (1)$

$$\vec{AC} = \vec{AB} + \vec{BC} = \vec{a} + \vec{b}$$

Also $\quad \vec{BD} = \vec{BC} + \vec{CD}$

$$= \vec{AD} - \vec{AB} = \vec{b} - \vec{a}$$

$$\therefore \overrightarrow{AC} \cdot \overrightarrow{BD} = \left(\vec{a} + \vec{b}\right) \cdot \left(\vec{b} - \vec{a}\right)$$
$$= \left(\vec{b} + \vec{a}\right) \cdot \left(\vec{b} - \vec{a}\right)$$
$$= \left(\vec{b}\right)^2 - \left(\vec{a}\right)^2 = 0 \quad \left(\because \left|\vec{a}\right| = \left|\vec{b}\right|\right)$$

Thus $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0 \Rightarrow \overrightarrow{AC} \perp \overrightarrow{BD}$

Hence the diagonals of a rhombus are at right angles.

Example 2.16 : Altitudes of a triangle are concurrent – prove by vector method.

Solution :

Let ABC be a triangle and let AD, BE be its two altitudes intersecting at O.

In order to prove that the altitudes are concurrent it is sufficient to prove that CO is perpendicular to AB.

Taking O as the origin, let the position vectors of A, B, C be $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively.

Then $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$; $\overrightarrow{OC} = \vec{c}$

Now $AD \perp BC$

$$\Rightarrow \qquad \overrightarrow{OA} \perp \overrightarrow{BC}$$

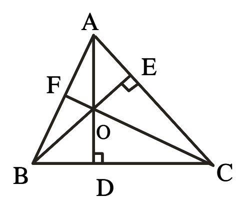


Fig. 2.9

$$\Rightarrow \qquad \overrightarrow{OA} \cdot \overrightarrow{BC} = 0$$
$$\Rightarrow \qquad \vec{a} \cdot \left(\vec{c} - \vec{b}\right) = 0$$
$$\Rightarrow \quad \vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b} = 0 \qquad \ldots(1)$$

$BE \perp CA \Rightarrow \overrightarrow{OB} \perp \overrightarrow{CA}$

$$\Rightarrow \overrightarrow{OB} \cdot \overrightarrow{CA} = 0 \Rightarrow \vec{b} \cdot \left(\vec{a} - \vec{c}\right) = 0$$
$$\Rightarrow \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{c} = 0 \qquad \ldots (2)$$

Adding (1) and (2), we get

$$\vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c} = 0 \Rightarrow \left(\vec{a} - \vec{b}\right) \cdot \vec{c} = 0$$

$$\Rightarrow \; \overrightarrow{BA} \,.\, \overrightarrow{OC} = 0 \;\; \Rightarrow \;\; \overrightarrow{OC} \perp \overrightarrow{AB}$$

Hence the three altitudes are concurrent.

Example 2.17 : Prove that $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Solution :

Take the points P and Q on the unit circle with centre at the origin O. Assume that OP and OQ make angles A and B with x-axis respectively.

$$\therefore \angle POQ = \angle POx - \angle QOx = A - B$$

Clearly the coordinates of P and Q are $(\cos A, \sin A)$ and $(\cos B, \sin B)$.

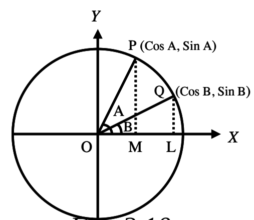


Fig. 2.10

Take the unit vectors $\vec{i}$ and $\vec{j}$ along x and y axes.

$$\therefore \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = \cos A\,\vec{i} + \sin A\,\vec{j}$$

$$\overrightarrow{OQ} = \overrightarrow{OL} + \overrightarrow{LQ} = \cos B\,\vec{i} + \sin B\,\vec{j}$$

By value, $$\overrightarrow{OP} \,.\, \overrightarrow{OQ} = \left(\cos A\,\vec{i} + \sin A\,\vec{j}\right) . \left(\cos B\,\vec{i} + \sin B\,\vec{j}\right) \quad ..(1)$$

$$= \cos A \cos B + \sin A \sin B$$

By definition , $$\overrightarrow{OP} \,.\, \overrightarrow{OQ} = \left|\overrightarrow{OP}\right| \left|\overrightarrow{OQ}\right| \cos(A - B) = \cos(A - B) \qquad .. (2)$$

From (1) and (2) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

2.3.4 Application of Scalar Product in Physics

Work done by force :

The work done by a force is a scalar quantity and its measure is equal to the product of the magnitude of the force and the resolved part of the displacement in the direction of the force.

Let a particle be placed at O and a force $\vec{F}$ represented by $\overrightarrow{OB}$ be acting on the particle at O. Due to the application of force $\vec{F}$, the particle is displaced in the direction of $\overrightarrow{OA}$. Here $\overrightarrow{OA}$ is the displacement and OL is the displacement in the direction of $\vec{F}$

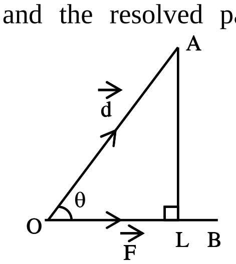


Fig. 2.11

In right angled $\Delta\, OLA$

$$OL = OA\cos\theta = \left|\vec{d}\right|\cos\theta \quad \left\{\begin{array}{l}\text{where } \theta \text{ is the angle}\\ \text{between } \vec{F} \text{ and } \vec{d}\end{array}\right.$$

The work done by a force = (Magnitude of force) (displacement in the direction of force)

$$= \left|\vec{F}\right| OL = \left|\vec{F}\right| \left|\vec{d}\right| \cos\theta$$

$$\text{Work done by the force} = \vec{F} \cdot \vec{d}$$

Note : If a number of forces are acting on a particle, then the sum of the works done by the separate forces is equal to the work done by the resultant force.

Example 2.18 : Find the work done in moving a particle from the point *A*, with position vector $2\vec{i} - 6\vec{j} + 7\vec{k}$, to the point *B*, with position vector $3\vec{i} - \vec{j} - 5\vec{k}$, by a force $\vec{F} = \vec{i} + 3\vec{j} - \vec{k}$

Solution :

$$\vec{F} = \vec{i} + 3\vec{j} - \vec{k}\ ;\quad \vec{OA} = 2\vec{i} - 6\vec{j} + 7\vec{k}\ ;\ \vec{OB} = 3\vec{i} - \vec{j} - 5\vec{k}$$

$$\vec{d} = \vec{AB} = \vec{OB} - \vec{OA} = \vec{i} + 5\vec{j} - 12\vec{k}$$

$$\text{Work done} = \vec{F} \cdot \vec{d}$$

$$= \left(\vec{i} + 3\vec{j} - \vec{k}\right) \cdot \left(\vec{i} + 5\vec{j} - 12\vec{k}\right)$$

$$= (1)(1) + 3(5) + 12 = 28$$

Example 2.19 : The work done by the force $\vec{F} = a\vec{i} + \vec{j} + \vec{k}$ in moving the point of application from (1, 1, 1) to (2, 2, 2) along a straight line is given to be 5 units. Find the value of *a*.

Solution : $\vec{F} = a\vec{i} + \vec{j} + \vec{k}\ ;\ \vec{OA} = \vec{i} + \vec{j} + \vec{k}\ ;\ \vec{OB} = 2\vec{i} + 2\vec{j} + 2\vec{k}$

Work done = 5 units

$$\vec{d} = \vec{AB} = \vec{OB} - \vec{OA} = \vec{i} + \vec{j} + \vec{k}$$

$$\text{Work done} = \vec{F} \cdot \vec{d}$$

$$5 = \left(a\vec{i} + \vec{j} + \vec{k}\right) \cdot \left(\vec{i} + \vec{j} + \vec{k}\right)$$

$$5 = a + 1 + 1 \Rightarrow \boxed{a = 3}$$

EXERCISE 2.2

Prove by vector method

(1) If the diagonals of a parallelogram are equal then it is a rectangle.

(2) The mid point of the hypotenuse of a right angled triangle is equidistant from its vertices

(3) The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.

(4) $\cos (A + B) = \cos A \cos B - \sin A \sin B$

(5) Find the work done by the force $\vec{F} = 2\vec{i} + \vec{j} + \vec{k}$ acting on a particle, if the particle is displaced from the point with position vector $2\vec{i} + 2\vec{j} + 2\vec{k}$ to the point with position vector $3\vec{i} + 4\vec{j} + 5\vec{k}$.

(6) A force of magnitude 5 units acting parallel to $2\vec{i} - 2\vec{j} + \vec{k}$ displaces the point of application from (1, 2, 3) to (5, 3, 7). Find the work done.

(7) The constant forces $2\vec{i} - 5\vec{j} + 6\vec{k}$, $-\vec{i} + 2\vec{j} - \vec{k}$ and $2\vec{i} + 7\vec{j}$ act on a particle which is displaced from position $4\vec{i} - 3\vec{j} - 2\vec{k}$ to position $6\vec{i} + \vec{j} - 3\vec{k}$. Find the work done.

(8) Forces of magnitudes 3 and 4 units acting in the directions $6\vec{i} + 2\vec{j} + 3\vec{k}$ and $3\vec{i} - 2\vec{j} + 6\vec{k}$ respectively act on a particle which is displaced from the point (2, 2, – 1) to (4, 3, 1). Find the work done by the forces.

2.4 Vector product :

2.4.1 Right-handed and left handed systems :

Consider a set of three linearly independent vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ through the origin O. As they are linearly independent no two of them have parallel directions and not all of them lie on the same plane. Let θ be the smaller angle (i.e. $0 < \theta < \pi$) between $\vec{a}$ and $\vec{b}$. Let an observer walk from $\vec{a}$ to $\vec{b}$ through the angle θ keeping O always to his left. If the observer's head is on the same side of the plane of $\vec{a}$ and $\vec{b}$ as the vector $\vec{c}$, we say $\vec{a}$, $\vec{b}$, $\vec{c}$ is a right handed system or right handed triple (or) triad.

If $\vec{c}$ has the opposite direction, $\vec{a}$, $\vec{b}$, $\vec{c}$ is a left handed system.

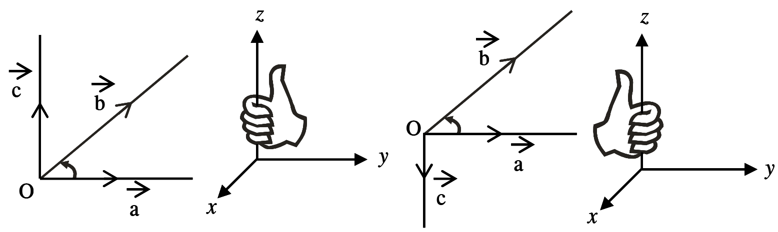


Fig. 2. 12

Definition : The vector product of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a} \times \vec{b}$ and it is defined as a vector whose magnitude is $|\vec{a}|\,|\vec{b}|$ sin θ where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and whose direction is perpendicular to both $\vec{a}$ and $\vec{b}$ in such a way that $\vec{a}$, $\vec{b}$ and this direction constitute a right handed system.

In other words,

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$, $\hat{n}$ form a right handed system.

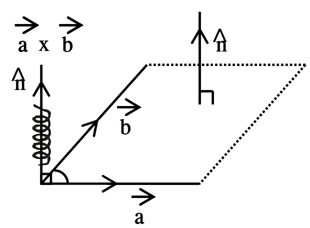


Fig. 2.13

Note :

(1) $\vec{a}$, $\vec{b}$, $\hat{n}$ form a right handed system means that if we rotate $\vec{a}$ into $\vec{b}$, then $\hat{n}$ will point in the direction perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ in which a right handed screw will move if it is turned in the same manner.

(2) $\vec{a} \times \vec{b}$ is read as $\vec{a}$ cross $\vec{b}$ since we are putting cross between $\vec{a}$ and $\vec{b}$.

2.4.2 Geometrical interpretation of Vector product :

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$

Let θ be the angle between $\vec{a}$ and $\vec{b}$

Complete the parallelogram *OACB* with $\vec{OA}$ and $\vec{OB}$ as adjacent sides.

Draw $BN \perp OA$.

In right angled triangle *ONB*

$BN = |\vec{b}| \sin\theta$

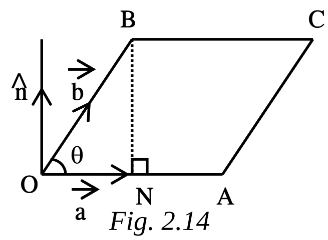


Fig. 2.14

Now
$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\,\hat{n}$$
$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$
$$= (OA)\ (BN)$$
$$= \text{Base} \times \text{height}$$
$$= \text{Area of parallelogram } OACB$$
$$\therefore |\vec{a} \times \vec{b}| = \begin{cases} \text{Area of parallelogram with} \\ \vec{a} \text{ and } \vec{b} \text{ as adjacent sides} \end{cases}$$

Also, area of $\Delta OAB = \frac{1}{2}$ area of a parallelogram *OACB*
$$= \frac{1}{2}\left|\vec{OA} \times \vec{OB}\right| = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$$

Vector area of $\Delta OAB = \frac{1}{2}\left(\vec{a} \times \vec{b}\right)$

Some important results :

Result : (1) The area of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$

(2) The vector area of a parallelogram with adjacent sides is $\vec{a} \times \vec{b}$

(3) The area of a triangle with sides $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$

(4) The area of a triangle *ABC* is $\frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$ (or) $\frac{1}{2}\left|\vec{BC} \times \vec{BA}\right|$ (or) $\frac{1}{2}\left|\vec{CA} \times \vec{CB}\right|$

2.4.3 Properties of Vector Product :

Property (1) : Non-Commutativity of Vector product :

Vector product is not commutative (i.e.) if $\vec{a}$ and $\vec{b}$ are any two vectors, then $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ however $\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$.

Let $\vec{a}$ and $\vec{b}$ be two non-zero, non parallel vectors and let θ be the angle between them. Then

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$ where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$

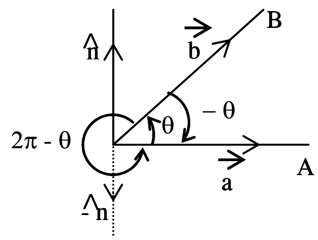


Fig. 2.15

$$\vec{b} \times \vec{a} = |\vec{b}|\,|\vec{a}| \sin(\theta)\,(-\hat{n}) = -|\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n} = -\left(\vec{a} \times \vec{b}\right)$$

Note that $\vec{b}$, $\vec{a}$ and $-\hat{n}$ form a right handed system.

$$\text{Hence}\quad \vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$$

$$\text{But}\ \vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$$

Property (2) :

Vector product of Collinear (Parallel) Vectors :

If the vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel then $\vec{a} \times \vec{b} = \vec{0}$

The vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel, then θ = 0, π

$$\sin\theta = 0 \quad \text{for } \theta = 0, \pi$$

$$\text{Thus}\ \vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n}$$

$$= |\vec{a}|\,|\vec{b}|\,(O)\,\hat{n} = \vec{0}$$

Result : The vector product of two non-zero vectors is zero vector if and only if they are parallel (collinear)

i.e., $\vec{a} \times \vec{b} = \vec{o} \Leftrightarrow \vec{a}$ is parallel to $\vec{b}$, where $\vec{a}$, $\vec{b}$ are non zero vectors.

Proof (i) :

$$\text{Suppose}\ \vec{a} \times \vec{b} = \vec{O}$$

then $\left|\vec{a}\right| \left|\vec{b}\right| \sin\theta\ \hat{n} = \vec{O}$ But $\left|\vec{a}\right| \neq 0$ & $\left|\vec{b}\right| \neq 0, \hat{n} \neq \vec{O}$

$\Rightarrow \sin\theta = 0 \quad \Rightarrow \theta = 0$ or π

$\Rightarrow \vec{a}$ and $\vec{b}$ are collinear (parallel)

conversely if $\vec{a} \parallel \vec{b}$ then

$$\theta = O \text{ or } \pi$$
$$\Rightarrow \sin\theta = O$$
$$\Rightarrow \vec{a} \times \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \sin\theta\ \hat{n} = \vec{o}$$
$$\Rightarrow \vec{a} \times \vec{b} = \vec{o}$$

Note : If $\vec{a} \times \vec{b} = \vec{o}$, we have the following three possibilities.

(i) $\vec{a}$ is a zero vector and $\vec{b}$ is any vector.

(ii) $\vec{b}$ is a zero vector and $\vec{a}$ is any vector

(iii) $\vec{a}$ and $\vec{b}$ are parallel (collinear)

Property (3) :

Cross Product of Equal Vectors :

$$\vec{a} \times \vec{a} = \left|\vec{a}\right| \left|\vec{a}\right| \sin\theta\ \hat{n}$$
$$= \left|\vec{a}\right| \left|\vec{a}\right| (0)\ \hat{n}$$
$$= \vec{o}$$

$\therefore\ \vec{a} \times \vec{a} = \vec{o}$ for every non-zero vector $\vec{a}$

Property (4) :

Cross product of Unit Vectors $\vec{i}, \vec{j}, \vec{k}$

By the above property

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{O}$$

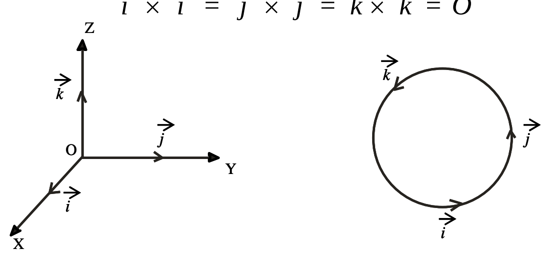


Fig. 2. 16

Also $\quad \vec{i} \times \vec{j} = \left|\vec{i}\right| \left|\vec{j}\right| \sin 90^\circ \vec{k} = (1)\,(1)\,(1)\,\vec{k} = \vec{k}$

Similarly $\quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}$

and $\quad \vec{j} \times \vec{i} = -\vec{k}, \quad \vec{k} \times \vec{j} = -\vec{i}, \quad \vec{i} \times \vec{k} = -\vec{j}$

Property (5) :

If m is any scalar and $\vec{a}, \vec{b}$ are two vectors inclined at angle θ, then

$$m\vec{a} \times \vec{b} = m\left(\vec{a} \times \vec{b}\right) = \vec{a} \times m\vec{b}$$

Property (6) : Distributivity of vector product over vector addition

Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors. then

(i) $\quad \vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (Left distributivity)

(ii) $\quad \left(\vec{b} + \vec{c}\right) \times \vec{a} = \left(\vec{b} \times \vec{a}\right) + \left(\vec{c} \times \vec{a}\right)$ (Right distributivity)

Result :

Vector Product in the determinant form

Let $\quad \vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and

$\quad\quad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ be the two vectors

Then $\quad \vec{a} \times \vec{b} = \left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right) \times \left(b_1\vec{i} + b_2\vec{j} + b_3\vec{k}\right)$

$$= a_1b_1\left(\vec{i} \times \vec{i}\right) + a_1b_2\left(\vec{i} \times \vec{j}\right) + a_1b_3\left(\vec{i} \times \vec{k}\right)$$

$$+ a_2b_1\left(\vec{j} \times \vec{i}\right) + a_2b_2\left(\vec{j} \times \vec{j}\right) + a_2b_3\left(\vec{j} \times \vec{k}\right)$$

$$+ a_3b_1\left(\vec{k} \times \vec{i}\right) + a_3b_2\left(\vec{k} \times \vec{j}\right) + a_3b_3\left(\vec{k} \times \vec{k}\right)$$

$$= a_1b_2\vec{k} + a_1b_3\left(-\vec{j}\right) + a_2b_1\left(-\vec{k}\right) + a_2b_3\vec{i}$$

$$+ a_3b_1\vec{j} + a_3b_2\left(-\vec{i}\right)$$

$$= (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Property (7) : Angle between two vectors :

Let $\vec{a}$, $\vec{b}$ be two vectors inclined at an angle θ.

$$\text{Then } \vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \;\hat{n}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \left|\,|\vec{a}|\,|\vec{b}| \sin\theta \;\hat{n}\right|$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$

$$\Rightarrow \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|} \Rightarrow \theta = \sin^{-1}\left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}\right)$$

Note :

In this case θ is always acute. Thus if we try to find the angle using vector product, we get only the acute angle.

Hence in problems of finding the angle, the use of dot product is preferable since it specifies the position of the angle θ.

Property (8) : Unit vectors perpendicular to two given vectors

(i.e.) Unit vectors normal to the plane of two given vectors.

Let $\vec{a}$, $\vec{b}$ be two non-zero, non-parallel vectors and θ be the angle between them.

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \;\hat{n} \qquad \ldots (1)$$

Where $\hat{n}$ is a unit vector perpendicular to the both of $\vec{a}$ and $\vec{b}$

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta \qquad \ldots (2)$$

From (1) and (2) $\qquad \hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

Note that $-\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is also a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

Unit vectors perpendicular to $\vec{a}$ and $\vec{b}$ are

$$\therefore \pm\hat{n} = \pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

Vectors of magnitude μ normal to the plane containing $\vec{a}$ and $\vec{b}$ is given by $\pm\dfrac{\mu\left(\vec{a}\times\vec{b}\right)}{|\vec{a}\times\vec{b}|}$

Example 2.20 :

If $\vec{a}$, $\vec{b}$ are any two vectors, then $|\vec{a}\times\vec{b}|^2 + \left(\vec{a}\,.\,\vec{b}\right)^2 = |\vec{a}|^2\,|\vec{b}|^2$

Solution :

Let θ be the angle between $\vec{a}$ and $\vec{b}$

$$\therefore\quad \vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\ \hat{n}$$

$$|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$$

$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2\sin^2\theta$$

$$\left(\vec{a}\,.\,\vec{b}\right)^2 = |\vec{a}|^2\,|\vec{b}|^2\cos^2\theta$$

$$|\vec{a}\times\vec{b}|^2 + \left(\vec{a}\,.\,\vec{b}\right)^2 = |\vec{a}|^2\,|\vec{b}|^2(\sin^2\theta + \cos^2\theta) = |\vec{a}|^2\,|\vec{b}|^2$$

Example 2.21 : Find the vectors of magnitude 6 which are perpendicular to both the vectors $4\vec{i} - \vec{j} + 3\vec{k}$ and $-2\vec{i} + \vec{j} - 2\vec{k}$

Solution :

Let $\quad \vec{a} = 4\vec{i} - \vec{j} + 3\vec{k}\ ;\ \vec{b} = -2\vec{i} + \vec{j} - 2\vec{k}$

$$\text{Then}\qquad \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = -\vec{i} + 2\vec{j} + 2\vec{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = 3$$

$$\text{Required vectors} = 6\left[\pm\left(\frac{\vec{a} \times \vec{b}}{\left|\vec{a} \times \vec{b}\right|}\right)\right]$$

$$= \pm\left(-2\vec{i} + 4\vec{j} + 4\vec{k}\right)$$

Example 2.22 : If $\left|\vec{a}\right| = 13$, $\left|\vec{b}\right| = 5$ and $\vec{a}\,.\,\vec{b} = 60$ then find $\left|\vec{a} \times \vec{b}\right|$

Solution :

$$\left|\vec{a} \times \vec{b}\right|^2 + \left(\vec{a}\,.\,\vec{b}\right)^2 = \left|\vec{a}\right|^2 \left|\vec{b}\right|^2$$

$$\left|\vec{a} \times \vec{b}\right|^2 = \left|\vec{a}\right|^2 \left|\vec{b}\right|^2 - \left(\vec{a}\,.\,\vec{b}\right)^2$$

$$= (13)^2 (5)^2 - (60)^2 = 625$$

$$\Rightarrow \left|\vec{a} \times \vec{b}\right| = 25$$

Example 2.23 : Find the angle between the vectors $2\vec{i} + \vec{j} - \vec{k}$ and $\vec{i} + 2\vec{j} + \vec{k}$ by using cross product.

Solution :

$$\text{Let} \quad \vec{a} = 2\vec{i} + \vec{j} - \vec{k} \;;\; \vec{b} = \vec{i} + 2\vec{j} + \vec{k}$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$

$$\therefore \theta = \sin^{-1}\left(\frac{\left|\vec{a} \times \vec{b}\right|}{\left|\vec{a}\right|\left|\vec{b}\right|}\right)$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\vec{i} - 3\vec{j} + 3\vec{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{3^2 + (-3)^2 + 3^2} = 3\sqrt{3}$$

$$\left|\vec{a}\right| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$$

$$\left|\vec{b}\right| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

$$\therefore\ \sin\theta = \left(\frac{\left|\vec{a}\times\vec{b}\right|}{\left|\vec{a}\right|\left|\vec{b}\right|}\right) = \left(\frac{3\sqrt{3}}{\sqrt{6}\ \sqrt{6}}\right) = \left(\frac{\sqrt{3}}{2}\right)$$

$$\theta = \frac{\pi}{3}$$

Example 2.24 : If $\vec{p} = -3\vec{i} + 4\vec{j} - 7\vec{k}$ and $\vec{q} = 6\vec{i} + 2\vec{j} - 3\vec{k}$ then find $\vec{p}\times\vec{q}$. Verify that $\vec{p}$ and $\vec{p}\times\vec{q}$ are perpendicular to each other and also verify that $\vec{q}$ and $\vec{p}\times\vec{q}$ are perpendicular to each other.

Solution :

$$\vec{p}\times\vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 4 & -7 \\ 6 & 2 & -3 \end{vmatrix}$$

$$= 2\vec{i} - 51\vec{j} - 30\vec{k}$$

Now $\quad \vec{p}\,.\left(\vec{p}\times\vec{q}\right) = \left(-3\vec{i} + 4\vec{j} - 7\vec{k}\right).\left(2\vec{i} - 51\vec{j} - 30\vec{k}\right)$

$$= -6 - 204 + 210 = 0$$

Hence $\vec{p}$ and $\vec{p}\times\vec{q}$ are perpendicular to each other.

Now $\quad \vec{q}\,.\left(\vec{p}\times\vec{q}\right) = \left(6\vec{i} + 2\vec{j} - 3\vec{k}\right).\left(2\vec{i} - 51\vec{j} - 30\vec{k}\right)$

$$= 12 - 102 + 90 = 0$$

Hence $\vec{q}$ and $\vec{p}\times\vec{q}$ are perpendicular to each other.

Example 2.25 : If the position vectors of three points A, B and C are respectively $\vec{i} + 2\vec{j} + 3\vec{k}$, $4\vec{i} + \vec{j} + 5\vec{k}$ and $7\left(\vec{i} + \vec{k}\right)$. Find $\vec{AB}\times\vec{AC}$. Interpret the result geometrically.

Solution :

$$\vec{OA} = \vec{i} + 2\vec{j} + 3\vec{k}\ ,\quad \vec{OB} = 4\vec{i} + \vec{j} + 5\vec{k}\ \ ;\ \vec{OC} = 7\vec{i} + 7\vec{k}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = \left(4\vec{i} + \vec{j} + 5\vec{k}\right) - \left(\vec{i} + 2\vec{j} + 3\vec{k}\right)$$

$$\vec{AB} = 3\vec{i} - \vec{j} + 2\vec{k}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = 6\vec{i} - 2\vec{j} + 4\vec{k}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 6 & -2 & 4 \end{vmatrix} = \vec{0}$$

The vectors $\vec{AB}$ and $\vec{AC}$ are parallel. But they have the point A as a common point.

$\therefore$ $\vec{AB}$ and $\vec{AC}$ are along the same line.

$\therefore$ A, B, C are collinear.

EXERCISE 2.3

(1) Find the magnitude of

$\vec{a} \times \vec{b}$ if $\vec{a} = 2\vec{i} + \vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$

(2) If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\vec{a}.\vec{b} = 9$ then find $|\vec{a} \times \vec{b}|$

(3) Find the unit vectors perpendicular to the plane containing the vectors $2\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{j} + \vec{k}$

(4) Find the vectors whose length 5 and which are perpendicular to the vectors $\vec{a} = 3\vec{i} + \vec{j} - 4\vec{k}$ and $\vec{b} = 6\vec{i} + 5\vec{j} - 2\vec{k}$

(5) Find the angle between two vectors $\vec{a}$ and $\vec{b}$ if $|\vec{a} \times \vec{b}| = \vec{a}.\vec{b}$

(6) If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $\vec{a} \times \vec{b} = 3\vec{i} - 2\vec{j} + 6\vec{k}$ find the angle between $\vec{a}$ and $\vec{b}$.

(7) If $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{b} = -\vec{i} + 3\vec{k}$ then find $\vec{a} \times \vec{b}$. Verify that $\vec{a}$ and $\vec{b}$ are perpendicular to $\vec{a} \times \vec{b}$ separately.

(8) For any three vectors $\vec{a}, \vec{b}, \vec{c}$ show that

$\vec{a} \times \left(\vec{b} + \vec{c}\right) + \vec{b} \times \left(\vec{c} + \vec{a}\right) + \vec{c} \times \left(\vec{a} + \vec{b}\right) = \vec{0}$

(9) Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors such that $\vec{a} \, . \, \vec{b} = \vec{a} \, . \, \vec{c} = 0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$. Prove that $\vec{a} = \pm 2\left(\vec{b} \times \vec{c}\right)$

(10) If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$,

show that $\vec{a} - \vec{d}$ and $\vec{b} - \vec{c}$ are parallel.

2.4.4 Geometrical applications of cross product :

Example 2.26 : Prove that the area of a quadrilateral *ABCD* is $\frac{1}{2}\left|\vec{AC} \times \vec{BD}\right|$ where *AC* and *BD* are its diagonals.

Solution :

$$\left.\begin{array}{l}\text{Vector Area of}\\ \text{quadrilateral } ABCD\end{array}\right\} = \text{Vector area of } \Delta ABC + \text{Vector area of } \Delta ACD$$

$$= \frac{1}{2}\left(\vec{AB} \times \vec{AC}\right) + \frac{1}{2}\left(\vec{AC} \times \vec{AD}\right)$$

$$= -\frac{1}{2}\left(\vec{AC} \times \vec{AB}\right) + \frac{1}{2}\left(\vec{AC} \times \vec{AD}\right)$$

$$= \frac{1}{2}\vec{AC} \times \left(-\vec{AB} + \vec{AD}\right)$$

$$= \frac{1}{2}\vec{AC} \times \left(\vec{BA} + \vec{AD}\right)$$

$$= \frac{1}{2}\vec{AC} \times \vec{BD}$$

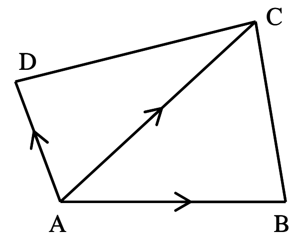


Fig. 2.17

The area of the quadrilateral *ABCD* $= \frac{1}{2}\left|\vec{AC} \times \vec{BD}\right|$

Deduction :

Area of a parallelogram $= \frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$, where $\vec{d}_1$ and $\vec{d}_2$ are the diagonals.

Example 2.27 :

If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices *A, B, C* of a triangle *ABC*, then prove that the area of triangle *ABC* is $\frac{1}{2}\left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right|$Deduce the condition for points $\vec{a}, \vec{b}, \vec{c}$ to be collinear.

Solution : Area of $\Delta\, ABC = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$

Now $\vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a}$

and $\vec{AC} = \vec{OC} - \vec{OA} = \vec{c} - \vec{a}$

$$\text{Hence, area of } \Delta ABC = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| = \frac{1}{2}\left|\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right)\right|$$

$$= \frac{1}{2}\left|\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}\right|$$

$$= \frac{1}{2}\left|\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}\right|$$

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right|$$

If the points A, B, C are collinear, then the area of $\Delta ABC = 0$

$$\Rightarrow \frac{1}{2}\left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right| = 0$$

$$\Rightarrow \left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right| = 0$$

$$\text{(or)} \quad \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$$

Thus $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$ is the required condition of collinearity of the points with positions $\vec{a}$, $\vec{b}$, $\vec{c}$.

Example 2.28 : With usual notation prove that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Solution : Let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$, $\vec{AB} = \vec{c}$

By the area property of triangles $\frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\left|\vec{b} \times \vec{c}\right| = \frac{1}{2}\left|\vec{c} \times \vec{a}\right|$

$\Rightarrow \left|\vec{a} \times \vec{b}\right| = \left|\vec{b} \times \vec{c}\right| = \left|\vec{c} \times \vec{a}\right|$

$ab \sin(\pi - C) = bc \sin(\pi - A) = ca \sin(\pi - B)$

$\Rightarrow ab \sin C = bc \sin A = ca \sin B$

Divide by abc

$$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$

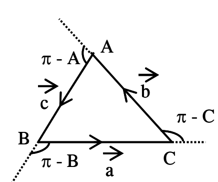


Fig. 2.18

Take the reciprocals, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Example 2.29 : Prove that $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Solution :

Take the points P and Q on the unit circle with centre at the origin O. Assume that OP and OQ make angles A and B with x-axis respectively.

$\therefore \underline{\lfloor POQ} = \underline{\lfloor POx} + \underline{\lfloor QOx} = A + B$

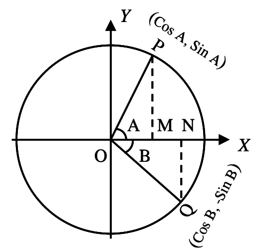


Fig. 2.19

Clearly the coordinates of P and Q are $(\cos A, \sin A)$ and $(\cos B, -\sin B)$.

Take the unit vectors $\vec{i}$ and $\vec{j}$ along x and y axes respectively.

$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = \cos A\,\vec{i} + \sin A\,\vec{j}$$

$$\overrightarrow{OQ} = \overrightarrow{ON} + \overrightarrow{NQ} = \cos B\,\vec{i} + \sin B\left(-\vec{j}\right) \quad \because \left|\overrightarrow{NQ}\right| = \sin B$$

$$= \cos B\,\vec{i} - \sin B\,\vec{j}$$

$$\overrightarrow{OQ} \times \overrightarrow{OP} = \left|\overrightarrow{OQ}\right| \left|\overrightarrow{OP}\right| \sin(A + B)\,\vec{k} = \sin(A + B)\,\vec{k} \qquad \ldots (1)$$

$$\overrightarrow{OQ} \times \overrightarrow{OP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos B & -\sin B & 0 \\ \cos A & \sin A & 0 \end{vmatrix} = \vec{k}\,[\sin A \cos B + \cos A \sin B] \qquad \ldots (2)$$

From (1) and (2)

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Example 2.30 : Show that the area of a parallelogram having diagonals $3\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{i} - 3\vec{j} + 4\vec{k}$ is $5\sqrt{3}$.

Solution : Let $\vec{d}_1 = 3\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{d}_2 = \vec{i} - 3\vec{j} + 4\vec{k}$

$$\text{Area of the parallelogram} = \frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$$

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = -2\vec{i} - 14\vec{j} - 10\vec{k}$$

$$\Rightarrow \left|\vec{d}_1 \times \vec{d}_2\right| = \sqrt{(-2)^2 + (-14)^2 + (-10)^2}$$

$$= \sqrt{300} = 10\sqrt{3}$$

$$\text{Area of the parallelogram} = \frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right| = \frac{1}{2}10\sqrt{3} = 5\sqrt{3} \text{ sq. units}$$

2.4.5 Applications of Vector Product in Physics

The moment of a force about a point :

Let a force $\vec{F}$ be applied at a point P of a rigid body. Then the moment of force $\vec{F}$ about a point O measures the tendency (amount) of $\vec{F}$ to turn the body about point O. If this tendency of rotation about O is in anti-clockwise direction the moment is positive, otherwise it is negative.

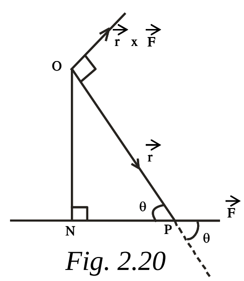


Fig. 2.20

Let $\vec{F}$ be the force and P be a point on the line of action of $\vec{F}$. Let $\vec{r}$ be the position vector of P relative to O.

The magnitude of the moment of the force $\vec{F}$ about O is the product of the magnitude of $\vec{F}$ and the length of the perpendicular from O to the line of action of the force.

$\therefore$ Magnitude of the moment $= \left|\vec{F}\right| (ON)$

In right angled triangle ONP

$$\sin\theta = \frac{ON}{OP} = \frac{ON}{\left|\vec{r}\right|}$$

$$\left|\vec{r}\right| \sin\theta = ON$$

$\therefore$ Magnitude of the moment $= \left|\vec{F}\right| (ON)$

$$= \left|\vec{r}\right| \left|\vec{F}\right| \sin\theta$$

$$= \left|\vec{r} \times \vec{F}\right|$$

$\therefore$ Moment (or) Torque of force $\vec{F}$ about the point O is defined as the vector $\vec{M} = \vec{r} \times \vec{F}$

Example 2.31 :

A force given by $3\vec{i} + 2\vec{j} - 4\vec{k}$ is applied at the point $(1, -1, 2)$. Find the moment of the force about the point $(2, -1, 3)$.

Solution :

We have

$$\vec{F} = 3\vec{i} + 2\vec{j} - 4\vec{k}$$

$$\vec{OP} = \vec{i} - \vec{j} + 2\vec{k}$$

$$\vec{OA} = 2\vec{i} - \vec{j} + 3\vec{k}$$

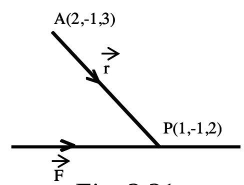


Fig. 2.21

$$\vec{r} = \vec{AP} = \vec{OP} - \vec{OA}$$

$$= \left(\vec{i} - \vec{j} + 2\vec{k}\right) - \left(2\vec{i} - \vec{j} + 3\vec{k}\right)$$

$$\vec{r} = -\vec{i} - \vec{k}$$

The moment $\vec{M}$ of the force $\vec{F}$ about the point A is given by

$$\vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix} = 2\vec{i} - 7\vec{j} - 2\vec{k}$$

EXERCISE 2.4

(1) Find the area of parallelogram $ABCD$ whose vertices are $A(-5, 2, 5)$, $B(-3, 6, 7)$, $C(4, -1, 5)$ and $D(2, -5, 3)$

(2) Find the area of the parallelogram whose diagonals are represented by $2\vec{i} + 3\vec{j} + 6\vec{k}$ and $3\vec{i} - 6\vec{j} + 2\vec{k}$

(3) Find the area of the parallelogram determined by the sides

$\vec{i} + 2\vec{j} + 3\vec{k}$ and $-3\vec{i} - 2\vec{j} + \vec{k}$

(4) Find the area of the triangle whose vertices are (3, – 1, 2), (1, – 1, – 3) and (4, – 3, 1)

(5) Prove by vector method that the parallelograms on the same base and between the same parallels are equal in area.

(6) Prove that twice the area of a parallelogram is equal to the area of another parallelogram formed by taking as its adjacent sides the diagonals of the former parallelogram.

(7) Prove that $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

(8) Forces $2\vec{i} + 7\vec{j}$, $2\vec{i} - 5\vec{j} + 6\vec{k}$, $-\vec{i} + 2\vec{j} - \vec{k}$ act at a point P whose position vector is $4\vec{i} - 3\vec{j} - 2\vec{k}$. Find the moment of the resultant of three forces acting at P about the point Q whose position vector is $6\vec{i} + \vec{j} - 3\vec{k}$.

(9) Show that torque about the point $A(3, -1, 3)$ of a force $4\vec{i} + 2\vec{j} + \vec{k}$ through the point $B(5, 2, 4)$ is $\vec{i} + 2\vec{j} - 8\vec{k}$.

(10) Find the magnitude and direction cosines of the moment about the point (1, – 2, 3) of a force $2\vec{i} + 3\vec{j} + 6\vec{k}$ whose line of action passes through the origin.

2.5 Product of three vectors :

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors. By inserting dot and cross between $\vec{a}$, $\vec{b}$, $\vec{c}$ in the same alphabetical order we introduce the following :

$$\left(\vec{a} \cdot \vec{b}\right) \cdot \vec{c}\ ,\ \left(\vec{a} \cdot \vec{b}\right) \times \vec{c},\ \left(\vec{a} \times \vec{b}\right) \cdot \vec{c} \text{ and } \left(\vec{a} \times \vec{b}\right) \times \vec{c}$$

Consider $\left(\vec{a} \cdot \vec{b}\right) \cdot \vec{c}$

Here $\vec{a} \cdot \vec{b}$ is a scalar quantity and dot product is not defined between a scalar and vector quantity. Therefore $\left(\vec{a} \cdot \vec{b}\right) \cdot \vec{c}$ is not meaningful.

Similary $\left(\vec{a} \cdot \vec{b}\right) \times \vec{c}$ is not meaningful.

But $\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$ is meaningful, because $\vec{a} \times \vec{b}$ is a vector and $\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$ is the dot product between the vectors $\vec{a} \times \vec{b}$ and $\vec{c}$.

Similarly $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ is meaningful.

2.5.1 Scalar Triple Product :

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. Then the product $\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$ is called a scalar triple product.

Geometrical Interpretation of Scalar Triple Product :

Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors. Consider a parallelopiped having co-terminus edges *OA*, *OB* and *OC* such that $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$.

Then $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

Let ϕ be the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$.

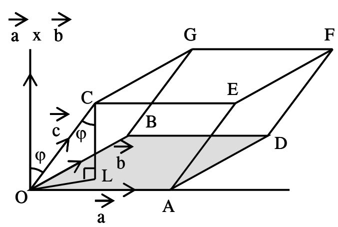


Fig. 2.22

Let *CL* be perpendicular to the base *OADB*. Here *CL* is the height of the parallelopiped.

Here *C*L *and* $\vec{a} \times \vec{b}$ are perpendicular to the same plane

$\Rightarrow$ *CL* is parallel to $\vec{a} \times \vec{b}$. $\Rightarrow \angle OCL = \phi$

In right angled triangle *OLC*, $CL = \left|\vec{c}\right| \cos\phi$

$\therefore$ Height of the paralleopiped $CL = \left|\vec{c}\right| \cos\phi$

Base area of the parallelopiped = $\begin{cases}\text{Area of the parallelogram} \\ \text{with } \vec{a} \text{ and } \vec{b} \text{ as adjacent sides}\end{cases}$

Base area of the parallelopiped = $\left|\vec{a} \times \vec{b}\right|$

Now, $\left(\vec{a} \times \vec{b}\right) . \vec{c} = \left|\vec{a} \times \vec{b}\right| \left|\vec{c}\right| \cos \phi$

$= [\text{base area}] \ [\text{height}]$

$$\left(\vec{a} \times \vec{b}\right) . \vec{c} = \begin{cases} \text{Volume of the parallelopiped with} \\ \text{co-terminous edges } \vec{a}, \vec{b}, \vec{c} \end{cases}$$

Thus, the scalar triple product $\left(\vec{a} \times \vec{b}\right) . \vec{c}$ represents the volume of the paralleopiped whose co-terminous edges $\vec{a}, \vec{b}, \vec{c}$ form a right handed system of vectors.

2.5.2 Properties of Scalar Triple Product :

Property (1) :

$$\left(\vec{a} \times \vec{b}\right) . \vec{c} = \left(\vec{b} \times \vec{c}\right) . \vec{a} = \left(\vec{c} \times \vec{a}\right) . \vec{b} \quad \text{[Cyclic order]}$$

Proof :

Let $\vec{a}, \vec{b}, \vec{c}$ represent the co-terminous edges of a parallelopiped such that they form a right handed system. Then the volume V of the parallelopiped is given by $V = \left(\vec{a} \times \vec{b}\right) . \vec{c}$

Clearly $\vec{b}, \vec{c}, \vec{a}$ as well as $\vec{c}, \vec{a}, \vec{b}$ form a right handed system of vectors and represent the co-terminous edges of the same parallelopiped.

$\therefore\ V = \left(\vec{b} \times \vec{c}\right) . \vec{a}$ and $V = \left(\vec{c} \times \vec{a}\right) . \vec{b}$

$$\therefore V = \left(\vec{a} \times \vec{b}\right) . \vec{c} = \left(\vec{b} \times \vec{c}\right) . \vec{a} = \left(\vec{c} \times \vec{a}\right) . \vec{b} \qquad \ldots (1)$$

Since dot product is commutative (1) gives

$$V = \vec{c} . \left(\vec{a} \times \vec{b}\right) = \vec{a} . \left(\vec{b} \times \vec{c}\right) = \vec{b} . \left(\vec{c} \times \vec{a}\right) \qquad \ldots (2)$$

From (1) and (2)

$$\left(\vec{a} \times \vec{b}\right) . \vec{c} = \vec{a} . \left(\vec{b} \times \vec{c}\right)$$

The dot and cross are interchangeable in a scalar triple product.

In view of this property, the scalar triple product is written in the following notation.

$$\left(\vec{a} \times \vec{b}\right) . \vec{c} = \vec{a} . \left(\vec{b} \times \vec{c}\right) = \left[\vec{a} \ \vec{b} \ \vec{c}\right]$$

$$\therefore \left[\vec{a}, \vec{b}, \vec{c}\right] = \left[\vec{b} \ \vec{c} \ \vec{a}\right] = \left[\vec{c} \ \vec{a} \ \vec{b}\right]$$

Property (2) :

The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude.

(i.e.) $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = -\left[\vec{b}\ \vec{a}\ \vec{c}\right] = -\left[\vec{c}\ \vec{b}\ \vec{a}\right] = -\left[\vec{a}\ \vec{c}\ \vec{b}\right]$

Proof :

We have $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left(\vec{a} \times \vec{b}\right) . \vec{c}$

$$= -\left(\vec{b} \times \vec{a}\right) . \vec{c} \quad \because \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\left[\vec{a},\ \vec{b},\ \vec{c}\right] = -\left[\vec{b}\ \vec{a}\ \vec{c}\right] \qquad \ldots (1)$$

Similarly we can prove other results.

Property (3) : The scalar triple product of three vector is zero if any two of them are equal.

Proof : Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors.

When $\vec{a} = \vec{b}$,

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left(\vec{a} \times \vec{b}\right) . \vec{c} = \left(\vec{b} \times \vec{b}\right) . \vec{c}$$

$$= \vec{o} . \vec{c} = 0 \qquad \because \vec{b} \times \vec{b} = \vec{o}$$

Similarly we can prove for $\vec{b} = \vec{c}$ and for $\vec{c} = \vec{a}$

Property (4) :

For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and scalar $\left[\lambda\vec{a}\ \vec{b}\ \vec{c}\right] = \lambda\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

Proof : $\left[\lambda\vec{a}\ \vec{b}\ \vec{c}\right] = \left(\lambda\vec{a} \times \vec{b}\right) . \vec{c} = \lambda\left(\vec{a} \times \vec{b}\right) . \vec{c}$

$$= \lambda\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

Property (5) :

The scalar triple product of three vectors is zero if any two of them are parallel or collinear.

Proof : Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors such that $\vec{a}$ is parallel or collinear to $\vec{b}$. Then $\vec{a} = \lambda\vec{b}$ for some scalar λ.

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left[\lambda\vec{b}\ \vec{b}\ \vec{c}\right] = \lambda\ (0) = 0$$

Property (6) (without proof) :

The necessary and sufficient condition for three non-zero, non-collinear vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ to be coplanar is $\left[\vec{a}\ \ \vec{b}\ \vec{c}\right] = 0$

i.e., $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow$ $\left[\vec{a}\ \ \vec{b}\ \vec{c}\right] = 0$

Note : Three possibilities for $\left[\vec{a}\ \ \vec{b}\ \vec{c}\right]$ to be zero are

(i) atleast one of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is a zero vector.

(ii) any two of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are parallel.

(iii) The vector $\vec{a}$, $\vec{b}$, $\vec{c}$ are co-planar.

But for cases (i) and (ii), the case (iii) is trivially true.

Result :

Scalar Triple Product in terms of components :

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$,

$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$,

Then $\left[\vec{a}\ \ \vec{b}\ \vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Proof : We have $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

$$\therefore \left[\vec{a},\ \vec{b},\ \vec{c}\right] = \left(\vec{a} \times \vec{b}\right) . \vec{c} = \left(\vec{a} \times \vec{b}\right) . \left(c_1\vec{i} + c_2\vec{j} + c_3\vec{k}\right)$$

$$= (a_2b_3 - a_3b_2)\,c_1 - (a_1b_3 - a_3b_1)c_2 + (a_1b_2 - a_2b_1)c_3$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Distributivity of Cross product over Vector addition :

Result :

For any three vectors $\vec{a}, \vec{b}, \vec{c}$

we have $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

This can be proved by determinant form of cross product.

Example 2.32 : If the edges $\vec{a} = -3\vec{i} + 7\vec{j} + 5\vec{k}$, $\vec{b} = -5\vec{i} + 7\vec{j} - 3\vec{k}$

$\vec{c} = 7\vec{i} - 5\vec{j} - 3\vec{k}$ meet at a vertex, find the volume of the parallelopiped.

Solution :

$$\text{Volume of the parallelopiped} = [\vec{a}, \vec{b}, \vec{c}]$$

$$= \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} = -264$$

The volume cannot be negative

$\therefore$ Volume of parallelopiped = 264 cu. units.

Note : Box product may be negative.

Example 2.33 : For any three vectors $\vec{a}\ \vec{b}\ \vec{c}$ prove that

$$[\vec{a} + \vec{b},\ \vec{b} + \vec{c},\ \vec{c} + \vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$$

Solution : $[\vec{a} + \vec{b},\ \vec{b} + \vec{c},\ \vec{c} + \vec{a}]$

$$= \{(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})\} \cdot (\vec{c} + \vec{a})$$

$$= \{(\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{b}) + (\vec{b} \times \vec{c})\} \cdot (\vec{c} + \vec{a})$$

$$= \{\vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c}\} \cdot (\vec{c} + \vec{a})$$

$$= (\vec{a} \times \vec{b}) \cdot \vec{c} + (\vec{a} \times \vec{c}) \cdot \vec{c} + (\vec{b} \times \vec{c}) \cdot \vec{c}$$

$$+ (\vec{a} \times \vec{b}) \cdot \vec{a} + (\vec{a} \times \vec{c}) \cdot \vec{a} + (\vec{b} \times \vec{c}) \cdot \vec{a}$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{b}\ \vec{c}\ \vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$$

Example 2.34 : If $\vec{x} \cdot \vec{a} = 0$, $\vec{x} \cdot \vec{b} = 0$, $\vec{x} \cdot \vec{c} = 0$ and $\vec{x} \neq \vec{0}$ then show that $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

Solution :

$\vec{x} \cdot \vec{a} = 0$ and $\vec{x} \cdot \vec{b} = 0$ implies $\vec{a}$ and $\vec{b}$ are $\perp$r to $\vec{x}$

$\therefore$ $\vec{a} \times \vec{b}$ is parallel to $\vec{x}$

$$\therefore \vec{x} = \lambda\left(\vec{a} \times \vec{b}\right)$$

Now $\vec{x} \cdot \vec{c} = 0 \Rightarrow \lambda\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = 0 \Rightarrow \left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

$\Rightarrow$ $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar

2.5.3 Vector Triple Product :

Definition :

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be any three vectors, then the product $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$ and $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ are called vector triple products of $\vec{a}$, $\vec{b}$, $\vec{c}$

Result :

For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$

$$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = \left(\vec{a} \cdot \vec{c}\right)\vec{b} - \left(\vec{b} \cdot \vec{c}\right)\vec{a}$$

This result can be proved by taking $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$;

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$; $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

Property (1) :

The vector triple product $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ is a linear combination of those two vectors which are within brackets.

Property (2) :

The vector triple product $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ is perpendicular to $\vec{c}$ and lies in the plane which contains $\vec{a}$ and $\vec{b}$.

Property (3) :

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \cdot \vec{c}\right)\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c}$$

Example 2.35 : If $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ then

Prove that $(\vec{c} \times \vec{a}) \times \vec{b} = \vec{o}$

Proof : Given :

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$$

$$(\vec{a} \,.\, \vec{c})\, \vec{b} - (\vec{a} \,.\, \vec{b})\, \vec{c} = (\vec{a} \,.\, \vec{c})\, \vec{b} - (\vec{b} . \vec{c})\, \vec{a}$$

$$\Rightarrow (\vec{a} \,.\, \vec{b})\vec{c} = (\vec{b} . \vec{c})\, \vec{a}$$

$$\Rightarrow (\vec{c} \,.\, \vec{b})\vec{a} - (\vec{a} \,.\, \vec{b})\vec{c} = \vec{0}$$

$$\Rightarrow \quad (\vec{c} \times \vec{a}) \times \vec{b} = \vec{0}$$

Example 2.36 : If $\vec{a} = 3\vec{i} + 2\vec{j} - 4\vec{k}$, $\vec{b} = 5\vec{i} - 3\vec{j} + 6\vec{k}$, $\vec{c} = 5\vec{i} - \vec{j} + 2\vec{k}$, find (i) $\vec{a} \times (\vec{b} \times \vec{c})$ (ii) $(\vec{a} \times \vec{b}) \times \vec{c}$ and show that they are not equal.

Solution :

(i)
$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -3 & 6 \\ 5 & -1 & 2 \end{vmatrix} = 20\vec{j} + 10\vec{k}$$

$$\therefore \quad \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -4 \\ 0 & 20 & 10 \end{vmatrix} = 100\vec{i} - 30\vec{j} + 60\vec{k}$$

(ii)
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -4 \\ 5 & -3 & 6 \end{vmatrix} = -38\vec{j} - 19\vec{k}$$

$$\therefore \quad (\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -38 & -19 \\ 5 & -1 & 2 \end{vmatrix} = -95\vec{i} - 95\vec{j} + 190\vec{k}$$

From (i) and (ii) $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

2.5.4 Vector product of four vectors ;

For the four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ the vector product of the two vectors $\left(\vec{a} \times \vec{b}\right)$ and $\left(\vec{c} \times \vec{d}\right)$ namely $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right)$ is called vector product of four vectors.

Example 2.37 : Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be any four vectors then

(i) $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \left[\vec{a}\ \vec{b}\ \vec{d}\right] \vec{c} - \left[\vec{a}\ \vec{b}\ \vec{c}\right] \vec{d}$

(ii) $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \left[\vec{a}\ \vec{c}\ \vec{d}\right] \vec{b} - \left[\vec{b}\ \vec{c}\ \vec{d}\right] \vec{a}$

Solution :

(i) $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \vec{x} \times \left(\vec{c} \times \vec{d}\right)$ where $\vec{x} = \vec{a} \times \vec{b}$

$$= \left(\vec{x} \cdot \vec{d}\right) \vec{c} - \left(\vec{x} \cdot \vec{c}\right) \vec{d}$$

$$= \left\{\left(\vec{a} \times \vec{b}\right) \cdot \vec{d}\right\} \vec{c} - \left\{\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}\right\} \vec{d}$$

$$= \left[\vec{a}\ \vec{b}\ \vec{d}\right] \vec{c} - \left[\vec{a}\ \vec{b}\ \vec{c}\right] \vec{d}$$

Similarly we can prove other result by taking $\vec{x} = \vec{c} \times \vec{d}$

Note : (1) If the four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are coplanar then $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \vec{o}$.

(2) Let $\vec{a}$, $\vec{b}$ be lie on one plane and $\vec{c}$, $\vec{d}$ lie on another plane. These planes are perpendicular then $\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right) = \vec{o}$

Example 2.38 :

Prove that $\left[\vec{a} \times \vec{b},\ \vec{b} \times \vec{c},\ \vec{c} \times \vec{a}\right] = \left[\vec{a},\ \vec{b},\ \vec{c}\right]^2$

Solution :

$$\left[\vec{a} \times \vec{b},\ \vec{b} \times \vec{c},\ \vec{c} \times \vec{a}\right] = \left\{\left(\vec{a} \times \vec{b}\right) \times \left(\vec{b} \times \vec{c}\right)\right\} \cdot \left(\vec{c} \times \vec{a}\right)$$

$$= \left\{\left[\vec{a}\ \vec{b}\ \vec{c}\right] \vec{b} - \left[\vec{a}\ \vec{b}\ \vec{b}\right] \vec{c}\right\} \cdot \left(\vec{c} \times \vec{a}\right)$$

$$= \left[\vec{a}\ \vec{b}\ \vec{c}\right] \left\{\vec{b} \cdot \left(\vec{c} \times \vec{a}\right)\right\} \text{ since } \left[\vec{a}\ \vec{b}\ \vec{b}\right] = 0$$

$= \left[\vec{a}\ \vec{b}\ \vec{c}\right]\left[\vec{b}\ \vec{c}\ \vec{a}\right]$

$= \left[\vec{a}\ \vec{b}\ \vec{c}\right]^2$ since $\left[\vec{b}\ \vec{c}\ \vec{a}\right] = \left[\vec{a}\ \vec{b}\ \vec{c}\right]$

2.5.5 Scalar product of four vectors :

For four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ the scalar product of the two vectors namely $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ is called scalar product of four vectors.

i.e. $\left(\vec{a} \times \vec{b}\right) . \left(\vec{c} \times \vec{d}\right)$

Result : Determinant form of $\left(\vec{a} \times \vec{b}\right) . \left(\vec{c} \times \vec{d}\right)$

i.e. $\left(\vec{a} \times \vec{b}\right) . \left(\vec{c} \times \vec{d}\right) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

Proof :

$$\begin{aligned}
\left(\vec{a} \times \vec{b}\right) . \left(\vec{c} \times \vec{d}\right) &= \left(\vec{a} \times \vec{b}\right) . \vec{x} \quad \text{where } \vec{x} = \vec{c} \times \vec{d} \\
&= \vec{a} . \left(\vec{b} \times \vec{x}\right) \quad \text{(interchange dot and cross)} \\
&= \vec{a} . \left[\vec{b} \times \left(\vec{c} \times \vec{d}\right)\right] \\
&= \vec{a} . \left[\left(\vec{b} . \vec{d}\right)\vec{c} - \left(\vec{b} . \vec{c}\right)\vec{d}\right] \\
&= \left(\vec{b} . \vec{d}\right)\left(\vec{a} . \vec{c}\right) - \left(\vec{b} . \vec{c}\right)\left(\vec{a} . \vec{d}\right) \\
&= \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}
\end{aligned}$$

EXERCISE 2.5

(1) Show that vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar if and only if $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$ are coplanar.

(2) The volume of a parallelopiped whose edges are represented by $-12\vec{i} + \lambda\vec{k}$, $3\vec{j} - \vec{k}$, $2\vec{i} + \vec{j} - 15\vec{k}$ is 546. Find the value of λ.

(3) Prove that $\left|\left[\vec{a}\ \vec{b}\ \vec{c}\right]\right| = abc$ if and only if $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular.

(4) Show that the points (1, 3, 1), (1, 1, – 1), (– 1, 1, 1) (2, 2, – 1) are lying on the same plane. (Hint : It is enough to prove any three vectors formed by these four points are coplanar).

(5) If $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = -2\vec{i} + 5\vec{k}$, $\vec{c} = \vec{j} - 3\vec{k}$

Verify that $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \,.\, \vec{c}\right)\vec{b} - \left(\vec{a} \,.\, \vec{b}\right)\vec{c}$

(6) Prove that $\vec{a} \times \left(\vec{b} \times \vec{c}\right) + \vec{b} \times \left(\vec{c} \times \vec{a}\right) + \vec{c} \times \left(\vec{a} \times \vec{b}\right) = \vec{o}$

(7) If $\vec{a} = 2\vec{i} + 3\vec{j} - 5\vec{k}$, $\vec{b} = -\vec{i} + \vec{j} + 2\vec{k}$ and

$\vec{c} = 4\vec{i} - 2\vec{j} + 3\vec{k}$, show that $\left(\vec{a} \times \vec{b}\right) \times \vec{c} \neq \vec{a} \times \left(\vec{b} \times \vec{c}\right)$

(8) Prove that $\left(\vec{a} \times \vec{b}\right) \times \vec{c} = \vec{a} \times \left(\vec{b} \times \vec{c}\right)$ iff $\vec{a}$ and $\vec{c}$ are collinear. (where vector triple product is non-zero).

(9) For any vector $\vec{a}$

prove that $\vec{i} \times \left(\vec{a} \times \vec{i}\right) + \vec{j} \times \left(\vec{a} \times \vec{j}\right) + \vec{k} \times \left(\vec{a} \times \vec{k}\right) = 2\vec{a}$

(10) Prove that $\left(\vec{a} \times \vec{b}\right) . \left(\vec{c} \times \vec{d}\right) + \left(\vec{b} \times \vec{c}\right) . \left(\vec{a} \times \vec{d}\right) + \left(\vec{c} \times \vec{a}\right) . \left(\vec{b} \times \vec{d}\right) = 0$

(11) Find $\left(\vec{a} \times \vec{b}\right) . \left(\vec{c} \times \vec{d}\right)$ if $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\vec{b} = 2\vec{i} + \vec{k}$, $\vec{c} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{d} = \vec{i} + \vec{j} + 2\vec{k}$

(12) Verify $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \left[\vec{a}\ \vec{b}\ \vec{d}\right]\vec{c} - \left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{d}$

for $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ in problem 11.

2.6 Lines :

2.6.1 Equation of a line :

Parametric and non parametric vector equations :

Let P be an any point with position vector $\vec{r}$ on the given line. A relation satisfied by $\vec{r}$ for all points on the line is then found using certain conditions. This relation is called the vector equation of the line.

Parametric vector equations :

If $\vec{r}$ is expressed in terms of some fixed vectors and variable scalars, (parameters) the relation is then called a parametric vector equation.

Non-parametric vector equation : If no parameter is involved, the equation is called a non-parametric vector equation.

Vector and Cartesian Equations of Straight lines :

A straight line is uniquely determined in space if

(i) a point on it and its direction are given

(ii) two points on it are given.

Note : Eventhough the syllabus does not require the derivations (2.6.2, 2.6.3) and it needs only the results, the equations are derived for better understanding the results.

2.6.2 Equation of a straight line passing through a given point and parallel to a given vector :

Vector form :

Let the line pass through a given point *A* whose position vector is $\vec{a}$ w.r.to *O* and parallel to the given vector $\vec{v}$. Let *P* be any point on the line and its position vector w.r.to *O* be $\vec{r}$.

We have $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OP} = \vec{r}$

$\overrightarrow{AP}$ and $\vec{v}$ are parallel.

$\therefore \overrightarrow{AP} = t\vec{v}$ for some scalar t

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\vec{r} = \vec{a} + t\vec{v}$ … (1)

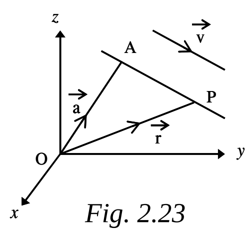


Fig. 2.23

This represents the vector equation of the given straight line.

Note : $\vec{r} = \vec{a} + t\vec{v}$, where t is a variable scalar (i.e., a parameter) is the parametric vector equation of the line.

Corollary : If the straight line is given to be passing through the origin, then the equation (1) becomes $\vec{r} = t\vec{v}$

Cartesian form : Let the co-ordinates of the fixed point *A* be (x_1, y_1, z_1) and the direction ratios of the parallel vector be *l*, *m*, *n*. Then

$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$; $\vec{v} = l\vec{i} + m\vec{j} + n\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{r} = \vec{a} + t\vec{v} \Rightarrow x\vec{i} + y\vec{j} + z\vec{k} = \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right) + t\left(l\vec{i} + m\vec{j} + n\vec{k}\right)$$

Equating the coefficients of $\vec{i}$, $\vec{j}$, $\vec{k}$ we get

$$\left.\begin{array}{l} x = x_1 + tl \\ y = y_1 + tm \\ z = z_1 + tn \end{array}\right\} \text{These are the parametric equations of the line}$$

$$\Rightarrow \frac{x - x_1}{l} = t, \quad \frac{y - y_1}{m} = t, \quad \frac{z - z_1}{n} = t$$

Eliminating t, we get $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$

This is the cartesian equation of the line passing through a point (x_1, y_1, z_1) and parallel to a vector whose drs are l, m, n.

Non-parametric vector equation :

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{a}$$

$$\text{But} \quad \overrightarrow{AP} \,||\, \vec{v} \Rightarrow \overrightarrow{AP} \times \vec{v} = \vec{0}$$

$$\Rightarrow \left(\vec{r} - \vec{a}\right) \times \vec{v} = \vec{0}$$

$$\Rightarrow \vec{r} \times \vec{v} - \vec{a} \times \vec{v} = \vec{0}$$

$$\Rightarrow \vec{r} \times \vec{v} = \vec{a} \times \vec{v}$$

This is the non-parametric vector equation of the line.

2.6.3 Equation of a straight line passing through two given points:

Vector Form :

Let the line pass through two given points A and B whose position vectors are $\vec{a}$ and $\vec{b}$ respectively.

Let P be any point on the line and its position vector be $\vec{r}$

We have

$$\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b} \text{ and } \overrightarrow{OP} = \vec{r}$$

$\overrightarrow{AP}$ and $\overrightarrow{AB}$ are parallel vectors.

$\therefore \overrightarrow{AP} = t\,\overrightarrow{AB}$ for some scalar t

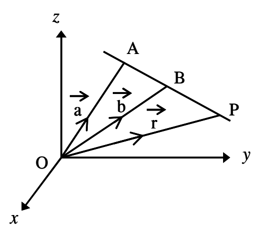


Fig. 2.24

$$= t\left(\overrightarrow{OB} - \overrightarrow{OA}\right) = t\left(\vec{b} - \vec{a}\right)$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$\vec{r} = \vec{a} + t\left(\vec{b} - \vec{a}\right) \quad \text{(or)} \qquad \ldots (1)$$

$$\vec{r} = (1 - t)\,\vec{a} + t\,\vec{b}$$

This represents the vector equation of the given straight line.

Note : $\vec{r} = (1 - t)\,\vec{a} + t\,\vec{b}$ where t is a variable scalar (i.e., a parameter) is the parametric vector equation of the required line.

Cartesian form :

Let the co-ordinates of the fixed points A be (x_1, y_1, z_1) and B be (x_2, y_2, z_2)

$$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \;;\; \vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k} \;;\; \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Substituting these values in equation (1) we get

$$x\vec{i} + y\vec{j} + z\vec{k} = \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right) + t\left[\left(x_2\vec{i} + y_2\vec{j} + z_2\vec{k}\right) - \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right)\right]$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$

$$\left.\begin{aligned} x &= x_1 + t(x_2 - x_1) \\ y &= y_1 + t(y_2 - y_1) \\ z &= z_1 + t(z_2 - z_1) \end{aligned}\right\} \text{These are the parametric equations of the line}$$

$$\Rightarrow \frac{x - x_1}{x_2 - x_1} = t, \quad \frac{y - y_1}{y_2 - y_1} = t, \quad \frac{z - z_1}{z_2 - z_1} = t$$

Eliminating t, we get

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

This is the cartesian equation of the required line.

Note : $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ are the d.r.s of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2)

Non-parametric vector equation :

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{a}$$

Since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear vectors

$$\Rightarrow \overrightarrow{AP} \times \overrightarrow{AB} = \vec{0}$$

$$\Rightarrow \left(\vec{r} - \vec{a}\right) \times \left(\vec{b} - \vec{a}\right) = \vec{0}$$

This is the non parametric vector equation.

2.6.4 Angle between two lines :

Let $\vec{r} = \vec{a}_1 + t\vec{u}$ and $\vec{r} = \vec{a}_2 + s\vec{v}$ be the two lines in space.

These two lines are in the direction of $\vec{u}$ and $\vec{v}$.

"Angle between the two lines is defined as the angle between their directions".

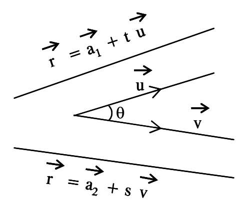


Fig. 2.25

If θ is the angle between the given lines then $\theta = \cos^{-1}\left(\dfrac{\vec{u}\cdot\vec{v}}{\left|\vec{u}\right|\left|\vec{v}\right|}\right)$

Cartesian form : If the equations of the lines are in Cartesian form

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \text{ and } \frac{x - x_1}{a_2} = \frac{y - y_1}{b_2} = \frac{z - z_1}{c_2}$$

Where a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines, then angle between them is

$$\theta = \cos^{-1}\left[\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right]$$

Note : When two lines are perpendicular then $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Example 2.39 : Find the vector and cartesian equations of the straight line passing through the point A with position vector $3\vec{i} - \vec{j} + 4\vec{k}$ and parallel to the vector $-5\vec{i} + 7\vec{j} + 3\vec{k}$

Solution : We know that vector equation of the line through the point with position vector $\vec{a}$ and parallel to $\vec{v}$ is given by $\vec{r} = \vec{a} + t\vec{v}$ where t is a scalar.

Here $\vec{a} = 3\vec{i} - \vec{j} + 4\vec{k}$

$\vec{v} = -\ 5\vec{i} + 7\vec{j} + 3\vec{k}$

$\therefore$ Vector equation of the line is

$$\vec{r} = \left(3\vec{i} - \vec{j} + 4\vec{k}\right) + t\left(-\ 5\vec{i} + 7\vec{j} + 3\vec{k}\right) \qquad \ldots (1)$$

The cartesian equation of the line passing through (x_1, y_1, z_1) and parallel to a vector whose d.r.s are l, m, n is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Here $(x_1, y_1, z_1) = (3, -1, 4)$

$(l, m, n) = (-5, 7, 3)$

$\therefore$ The required equation is $\frac{x - 3}{-5} = \frac{y + 1}{7} = \frac{z - 4}{3}$

Example 2.40 : Find the vector and cartesian equations of the straight line passing through the points $(-5, 2, 3)$ and $(4, -3, 6)$

Solution : Vector equation of the straight line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{r} = \vec{a} + t\left(\vec{b} - \vec{a}\right)$$

Here $\vec{a} = -5\vec{i} + 2\vec{j} + 3\vec{k}$

$\vec{b} = 4\vec{i} - 3\vec{j} + 6\vec{k}$

$\vec{b} - \vec{a} = 9\vec{i} - 5\vec{j} + 3\vec{k}$

$\therefore$ Vector equation of the line is

$\vec{r} = \left(-5\vec{i} + 2\vec{j} + 3\vec{k}\right) + t\left(9\vec{i} - 5\vec{j} + 3\vec{k}\right)$ or

$\vec{r} = (1 - t)\left(-5\vec{i} + 2\vec{j} + 3\vec{k}\right) + t\left(4\vec{i} - 3\vec{j} + 6\vec{k}\right)$

Cartesian Form :

The required equation is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Here $(x_1, y_1, z_1) = (-5, 2, 3)$; $(x_2, y_2, z_2) = (4, -3, 6)$

$\therefore \frac{x + 5}{9} = \frac{y - 2}{-5} = \frac{z - 3}{3}$ is the cartesian equation of the line

Example 2.41 : Find the angle between the lines

$$\vec{r} = 3\vec{i} + 2\vec{j} - \vec{k} + t\left(\vec{i} + 2\vec{j} + 2\vec{k}\right) \text{ and}$$

$$\vec{r} = 5\vec{j} + 2\vec{k} + s\left(3\vec{i} + 2\vec{j} + 6\vec{k}\right)$$

Solution : Let the given lines be in the direction of $\vec{u}$ and $\vec{v}$

Then $\vec{u} = \vec{i} + 2\vec{j} + 2\vec{k}$, $\vec{v} = 3\vec{i} + 2\vec{j} + 6\vec{k}$

Let θ be the angle between the given lines

$$\therefore \cos\theta = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

$$\vec{u}\cdot\vec{v} = 19 \;;\; |\vec{u}| = 3 \;;\; |\vec{v}| = 7$$

$$\cos\theta = \frac{19}{21} \quad \Rightarrow \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

EXERCISE 2.6

(1) Find the d.c.s of a vector whose direction ratios are 2, 3, – 6.

(2) (i) Can a vector have direction angles 30°, 45°, 60°.

(ii) Can a vector have direction angles 45°, 60°, 120°?

(3) What are the d.c.s of the vector equally inclined to the axes?

(4) A vector $\vec{r}$ has length $35\sqrt{2}$ and direction ratios (3, 4, 5) , find the direction cosines and components of $\vec{r}$.

(5) Find direction cosines of the line joining (2, – 3, 1) and (3, 1, – 2).

(6) Find the vector and cartesian equation of the line through the point (3, – 4, – 2) and parallel to the vector $9\vec{i} + 6\vec{j} + 2\vec{k}$.

(7) Find the vector and cartesian equation of the line joining the points (1, – 2, 1) and (0, – 2, 3).

(8) Find the angle between the following lines.

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-4}{6} \text{ and } x + 1 = \frac{y+2}{2} = \frac{z-4}{2}$$

(9) Find the angle between the lines

$$\vec{r} = 5\vec{i} - 7\vec{j} + \mu\left(-\vec{i} + 4\vec{j} + 2\vec{k}\right)$$

$$\vec{r} = -2\vec{i} + \vec{k} + \lambda\left(3\vec{i} + 4\vec{k}\right)$$

2.6.5 Skew lines

Consider two straight lines in the space. There are three possibilities.

(i) either they are intersecting

(ii) (or) parallel

(iii) (or) neither intersecting nor parallel

Two lines in space which are either intersecting or parallel are called coplanar lines.

i.e., a plane can be defined passing through (the two lines completely lie on the plane) two intersecting lines or through two parallel lines.

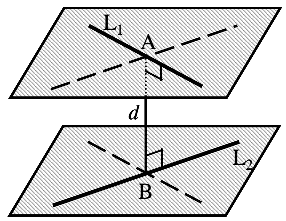


Fig. 2.26

Therefore, two lines lie on the same plane are called **coplanar lines.**

Two lines L_1 and L_2 in space, which are neither intersecting nor parallel are called **skew lines.** (See Fig. 2.26)

i.e., two lines in space which are not coplanar are called skew lines.

Shortest distance between two lines

(i) Trivially the shortest distance between two intersecting lines is zero.

(ii) Parallel lines

Theorem : (without proof) The distance between two parallel lines

$\vec{r} = \vec{a}_1 + t\vec{u}$; $\vec{r} = \vec{a}_2 + s\vec{u}$ is given by

$$d = \frac{\left|\vec{u} \times \left(\vec{a}_2 - \vec{a}_1\right)\right|}{\left|\vec{u}\right|}$$

(iii) **Skew lines :**

Theorem : (without proof) The distance between the skew lines

$\vec{r} = \vec{a}_1 + t\vec{u}$; $\vec{r} = \vec{a}_2 + s\vec{v}$ is given by

$$d = \frac{\left|\left[\left(\vec{a}_2 - \vec{a}_1\right) \vec{u}\ \vec{v}\right]\right|}{\left|\vec{u} \times \vec{v}\right|}$$

Condition for two lines to intersect :

The shortest distance between the intersecting lines

$\vec{r} = \vec{a}_1 + t\vec{u}$; $\vec{r} = \vec{a}_2 + s\vec{v}$ is 0

The condition for intersecting is $d = 0 \Rightarrow \left[\left(\vec{a}_2 - \vec{a}_1\right)\ \vec{u}\ \vec{v}\right] = 0$ (or)

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \text{ if}$$

$(x_1, y_1, z_1)$and (x_2, y_2, z_2) are the points whose position vectors are $\vec{a}_1$ and $\vec{a}_2$ and l_1, m_1, n_1 ; l_2, m_2, n_2 are the d.rs of the vectors $\vec{u}$ and $\vec{v}$ respectively. ($\vec{u}$ and $\vec{v}$ are not parallel)

Example 2.42 : Find the shortest distance between the parallel lines

$$\vec{r} = \left(\vec{i} - \vec{j}\right) + t\left(2\vec{i} - \vec{j} + \vec{k}\right) \text{ and}$$
$$\vec{r} = \left(2\vec{i} + \vec{j} + \vec{k}\right) + s\left(2\vec{i} - \vec{j} + \vec{k}\right)$$

Solution : Compare the given equations with $\vec{r} = \vec{a}_1 + t\vec{u}$ and $\vec{r} = \vec{a}_2 + s\vec{u}$,

$\vec{a}_1 = \vec{i} - \vec{j}$; $\vec{a}_2 = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{u} = 2\vec{i} - \vec{j} + \vec{k}$

$$\vec{a}_2 - \vec{a}_1 = \vec{i} + 2\vec{j} + \vec{k}$$

$$\vec{u} \times \left(\vec{a}_2 - \vec{a}_1\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -3\vec{i} - \vec{j} + 5\vec{k}$$

$$\left|\vec{u} \times \left(\vec{a}_2 - \vec{a}_1\right)\right| = \sqrt{9 + 1 + 25} = \sqrt{35}$$

$$\left|\vec{u}\right| = \sqrt{4 + 1 + 1} = \sqrt{6}$$

$$\left.\begin{array}{l}\text{The distance between} \\ \text{the parallel lines}\end{array}\right\} = \frac{\left|\vec{u} \times \left(\vec{a}_2 - \vec{a}_1\right)\right|}{\left|\vec{u}\right|} = \frac{\sqrt{35}}{\sqrt{6}}$$

Note : If the equations are in the Cartesian form, write in the vector form and find the distance between them.

Example 2.43 : Show that the two lines $\vec{r} = \left(\vec{i} - \vec{j}\right) + t\left(2\vec{i} + \vec{k}\right)$ and $\vec{r} = \left(2\vec{i} - \vec{j}\right) + s\left(\vec{i} + \vec{j} - \vec{k}\right)$ are skew lines and find the distance between them.

Solution : Compare the given equations with $\vec{r} = \vec{a}_1 + t\vec{u}$ and $\vec{r} = \vec{a}_2 + s\vec{v}$

$\vec{a}_1 = \vec{i} - \vec{j}$; $\vec{a}_2 = 2\vec{i} - \vec{j}$ and $\vec{u} = 2\vec{i} + \vec{k}$; $\vec{v} = \vec{i} + \vec{j} - \vec{k}$

$$\vec{a}_2 - \vec{a}_1 = \vec{i}$$

$$\left[\left(\vec{a}_2 - \vec{a}_1\right)\ \vec{u}\ \vec{v}\right] = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1 \neq 0$$

$\therefore$ They are skew lines.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -\vec{i} + 3\vec{j} + 2\vec{k}$$

$$\left|\vec{u} \times \vec{v}\right| = \sqrt{14}$$

Shortest distance between the lines $= \dfrac{\left|\left[\left(\vec{a}_2 - \vec{a}_1\right)\ \vec{u}\ \vec{v}\right]\right|}{\left|\vec{u} \times \vec{v}\right|}$ … (1)

From (1) shortest distance between them is $\dfrac{1}{\sqrt{14}}$

Example 2.44 : Show that the lines $\dfrac{x-1}{3} = \dfrac{y-1}{-1} = \dfrac{z+1}{0}$ and $\dfrac{x-4}{2} = \dfrac{y}{0} = \dfrac{z+1}{3}$ intersect and hence find the point of intersection.

Solution : The condition for intersecting is

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Compare with $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$, we get

$$(x_1, y_1, z_1) = (1, 1, -1) \;;\; (x_2, y_2, z_2) = (4, 0, -1)$$
$$(l_1, m_1, n_1) = (3, -1, 0) \;;\; (l_2, m_2, n_2) = (2, 0, 3)$$

The determinant becomes

$$\begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 0. \text{ Note that } \vec{u} \text{ and } \vec{v} \text{ are not parallel.}$$

$\therefore$ The lines are intersecting lines.

Point of intersection :

Take $\dfrac{x-1}{3} = \dfrac{y-1}{-1} = \dfrac{z+1}{0} = \lambda$

$\therefore$ Any point on the line is of the form $(3\lambda + 1, -\lambda + 1, -1)$

Take $\dfrac{x-4}{2} = \dfrac{y}{0} = \dfrac{z+1}{3} = \mu$

Any point on this line is of the form $(2\mu + 4,\; 0,\; 3\mu - 1)$

Since they are intersecting, for some λ, μ

$(3\lambda + 1, -\lambda + 1, -1) = (2\mu + 4, 0, 3\mu - 1) \Rightarrow \lambda = 1$ and $\mu = 0$

To find the point of intersection either take $\lambda = 1$ or $\mu = 0$

$\therefore$ The point of intersection is $(4, 0, -1)$.

Note : If the two lines are in the vector form convert into cartesian form and do it.

Example 2.45 : Find the shortest distance between the skew lines

$$\vec{r} = \left(\vec{i} - \vec{j}\right) + \lambda\left(2\vec{i} + \vec{j} + \vec{k}\right) \text{ and}$$
$$\vec{r} = \left(\vec{i} + \vec{j} - \vec{k}\right) + \mu\left(2\vec{i} - \vec{j} - \vec{k}\right)$$

Solution :

Compare the given equation with $\vec{r} = \vec{a}_1 + t\vec{u}$ and $\vec{r} = \vec{a}_2 + s\vec{v}$,

$\vec{a}_1 = \vec{i} - \vec{j}$; $\vec{a}_2 = \vec{i} + \vec{j} - \vec{k}$; $\vec{u} = 2\vec{i} + \vec{j} + \vec{k}$;

$\vec{v} = 2\vec{i} - \vec{j} - \vec{k}$

$$\vec{a}_2 - \vec{a}_1 = 2\vec{j} - \vec{k} \text{ and } \vec{u} \times \vec{v} = 4\vec{j} - 4\vec{k}$$

$$\left[\left(\vec{a}_2 - \vec{a}_1\right)\ \vec{u}\ \ \vec{v}\right] = \begin{vmatrix} 0 & 2 & -1 \\ 2 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix} = 12$$

$$\left|\vec{u} \times \vec{v}\right| = 4\sqrt{2}$$

$$\text{distance} = \frac{\left|\left[\left(\vec{a}_2 - \vec{a}_1\right)\ \vec{u}\ \ \vec{v}\right]\right|}{\left|\vec{u} \times \vec{v}\right|} = \frac{12}{4\sqrt{2}} = \frac{3}{\sqrt{2}}$$

2.6.7 Collinearity of three points :

Theorem (without proof) :

Three points A, B and C with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively are collinear if and only if there exists scalars $\lambda_1, \lambda_2, \lambda_3$, not all zeros such that

$$\lambda_1\vec{a} + \lambda_2\vec{b} + \lambda_3\vec{c} = \vec{o} \text{ and } \lambda_1 + \lambda_2 + \lambda_3 = 0$$

Working rule to find the collinearity :

Write the equation of the line in cartesian form using any two points and verify the third point.

Note : If the three points are collinear then their position vectors are coplanar, but the converse need not be true.

Example 2.46 : Show that the points $(3, -1, -1)$, $(1, 0, -1)$ and $(5, -2, -1)$ are collinear.

Solution :

The equation of the line passing through $(3, -1, -1)$ and $(1, 0, -1)$ is

$$\frac{x-3}{2} = \frac{y+1}{-1} = \frac{z+1}{0} = \lambda \text{ (say)}$$

Any point on the line is of the form $(2\lambda + 3, -\lambda - 1, -1)$

The point $(5, -2, -1)$ is obtained by putting $\lambda = 1$.

$\therefore$ The third point lies on the same line. Hence the three points are collinear.

Note : If the position vectors of the points are given then take the points and do the problem.

Example 2.47 : Find the value of λ if the points (3, 2, – 4), (9, 8, – 10) and (λ, 4, – 6) are collinear.

Solution :

Since the three points are collinear, the position vectors of the points are coplanar.

Let $\vec{a} = 3\vec{i} + 2\vec{j} - 4\vec{k}$; $\vec{b} = 9\vec{i} + 8\vec{j} - 10\vec{k}$; $\vec{c} = \lambda\vec{i} + 4\vec{j} - 6\vec{k}$

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} 3 & 2 & -4 \\ 9 & 8 & -10 \\ \lambda & 4 & -6 \end{vmatrix} = 0$$

$$\Rightarrow 12\lambda = 60 \quad \Rightarrow \quad \lambda = 5$$

EXERCISE 2.7

(1) Find the shortest distance between the parallel lines

(i) $\vec{r} = \left(2\vec{i} - \vec{j} - \vec{k}\right) + t\left(\vec{i} - 2\vec{j} + 3\vec{k}\right)$ and

$\vec{r} = \left(\vec{i} - 2\vec{j} + \vec{k}\right) + s\left(\vec{i} - 2\vec{j} + 3\vec{k}\right)$

(ii) $\dfrac{x-1}{-1} = \dfrac{y}{3} = \dfrac{z+3}{2}$ and $\dfrac{x-3}{-1} = \dfrac{y+1}{3} = \dfrac{z-1}{2}$

(2) Show that the following two lines are skew lines :

$\vec{r} = \left(3\vec{i} + 5\vec{j} + 7\vec{k}\right) + t\left(\vec{i} - 2\vec{j} + \vec{k}\right)$ and

$\vec{r} = \left(\vec{i} + \vec{j} + \vec{k}\right) + s\left(7\vec{i} + 6\vec{j} + 7\vec{k}\right)$

(3) Show that the lines $\dfrac{x-1}{1} = \dfrac{y+1}{-1} = \dfrac{z}{3}$ and $\dfrac{x-2}{1} = \dfrac{y-1}{2} = \dfrac{-z-1}{1}$ intersect and find their point of intersection.

(4) Find the shortest distance between the skew lines $\dfrac{x-6}{3} = \dfrac{y-7}{-1} = \dfrac{z-4}{1}$ and $\dfrac{x}{-3} = \dfrac{y+9}{2} = \dfrac{z-2}{4}$

(5) Show that (2, – 1, 3), (1, – 1, 0) and (3, – 1, 6) are collinear.

(6) If the points (λ, 0, 3), (1, 3, – 1) and (– 5, – 3, 7) are collinear then find λ.

2.7 Planes :

A plane is defined as a surface such that the line joining of any two points on it lies completely on the surface.

Vector and Cartesian Equations of the planes in parametric and non-parametric form :

A plane is determined uniquely in the following cases :

(i) Given a point on the plane and a normal to the plane.

(ii) Given a normal to the plane and distance of the plane from the origin.

(iii) Given a point and two parallel vectors to the plane.

(iv) Given two points on it and a line parallel to the plane.

(v) Given three non-collinear points.

(vi) Equation of a plane that contains two given lines.

(vii) Equation of a plane passing through the line of intersection of two given planes and a given point.

Note : Eventhough the syllabus does not require the derivations (2.7.1 to 2.7.5) and it needs only the results, the equations are derived for better understanding the results.

2.7.1 Equation of a plane passing through a given point and perpendicular to a vector.

Vector Form : Let the plane pass through the point A whose position vector be $\vec{a}$ w.r.to O and perpendicular to the given vector $\vec{n}$.

Let P be any point on the plane and its position vector be $\vec{r}$. Join $\vec{AP}$

Here $\vec{AP}$ is perpendicular to $\vec{n}$

$\therefore\ \vec{AP}.\vec{n} = 0 \Rightarrow \left(\vec{OP} - \vec{OA}\right).\vec{n} = 0$

$\left(\vec{r} - \vec{a}\right).\vec{n} = 0 \Rightarrow \vec{r}.\vec{n} = \vec{a}.\vec{n}$

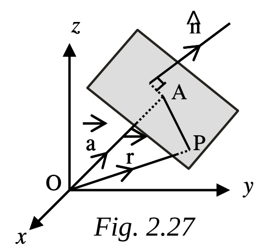


Fig. 2.27

This is the vector equation of the required plane (non parametric form.)

Cartesian Form :

If (x_1, y_1, z_1) are the coordinates of A and a, b, c are the direction ratios of $\vec{n}$

then $\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$; $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$; $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\text{Now, } \left(\vec{r} - \vec{a}\right) . \vec{n} = 0$$

$$\Rightarrow \left[(x - x_1)\vec{i} + (y - y_1)\vec{j} + (z - z_1)\vec{k}\right] . \left(a\vec{i} + b\vec{j} + c\vec{k}\right) = 0$$

$$\Rightarrow \ a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

This is the cartesian equation of the plane (in non-parametric form).

Corollary : The vector equation of the plane passing through the origin and perpendicular to the vector $\vec{n}$ is $\vec{r} . \vec{n} = 0$

2.7.2 Equation of the plane when distance from the origin and unit normal is given :

Let p be the length of the perpendicular ON from the origin O to the given plane. Let $\hat{n}$ be the unit vector normal to the plane in the direction O to N.

Then $\vec{ON} = p\hat{n}$.

Let P be any point on the plane and let its position vector be $\vec{r}$ (i.e.,) $\vec{OP} = \vec{r}$. Join NP.

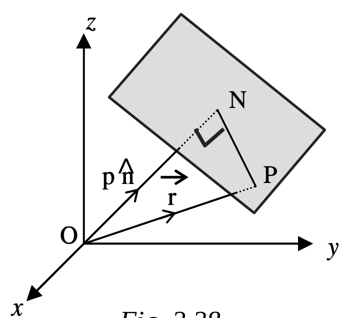


Fig. 2.28

$\vec{NP}$ lies on the plane and $\vec{ON}$ is perpendicular to the plane

$$\Rightarrow \ \vec{NP} . \vec{ON} = 0 \ \Rightarrow \left(\vec{OP} - \vec{ON}\right) . \vec{ON} = 0$$

$$\left(\vec{r} - p\hat{n}\right) . p\hat{n} = 0 \ \Rightarrow \ \vec{r} . \hat{n} - p\hat{n} . \hat{n} = 0$$

$$\text{i.e., } \ \vec{r} . \hat{n} = p \qquad \left(\because \hat{n} . \hat{n} = 1\right)$$

This is the vector equation of the plane (in non-parametric form).

Cartesian form :

If l, m, n are the direction cosines of $\vec{n}$ then $\hat{n} = l\vec{i} + m\vec{j} + n\vec{k}$

$$\vec{r} . \hat{n} = p \Rightarrow \left(x\vec{i} + y\vec{j} + z\vec{k}\right) . \left(l\vec{i} + m\vec{j} + n\vec{k}\right) = p$$

$$lx + my + nz = \text{p}$$

This is the cartesian equation of the plane (in non-parametric form).

Corollary : If $\vec{n}$ is a normal vector but not a unit vector,

$$\text{then} \quad \hat{n} = \frac{\vec{n}}{\left|\vec{n}\right|}$$

$$\vec{r} \cdot \frac{\vec{n}}{\left|\vec{n}\right|} = p \Rightarrow \vec{r} \cdot \vec{n} = p\left|\vec{n}\right| = q \text{ (say)}$$

$$\vec{r} \cdot \vec{n} = q$$

This is the vector equation of the plane perpendicular to the vector $\vec{n}$.

The length of the perpendicular from origin to this plane is $\dfrac{q}{\left|\vec{n}\right|}$

2.7.3 Equation of the plane passing through a given point and parallel to two given vectors :

Let $\vec{a}$ be the position vector of the given point A referred to the origin O. Let $\vec{u}$ and $\vec{v}$ be the given vectors, which are parallel to the plane.

Let P be any point on the plane and let its position vector be $\vec{r}$ (i.e.,) $\overrightarrow{OP} = \vec{r}$.

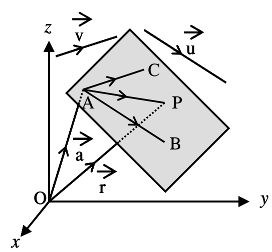


Fig. 2.29

Through A, draw a lines AB and AC parallel to $\vec{u}$ and $\vec{v}$ lying in the planes such that $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AC} = \vec{v}$.

Now $\overrightarrow{AP}$ is coplanar with $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\therefore \overrightarrow{AP} = s\,\overrightarrow{AB} + t\,\overrightarrow{AC} \text{ where } s \text{ and } t \text{ are scalars}$$

$$= s\,\vec{u} + t\,\vec{v}$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$\Rightarrow \vec{r} = a + s\,\vec{u} + t\,\vec{v} \qquad \ldots (1)$$

This is the vector equation of the plane (in parametric form).

Cartesian form :

Let $\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$

$\vec{u} = l_1\vec{i} + m_1\vec{j} + n_1\vec{k}$; $\vec{v} = l_2\vec{i} + m_2\vec{j} + n_2\vec{k}$

From (1) $\vec{r} = \vec{a} + s\vec{u} + t\vec{v}$

$$x\vec{i} + y\vec{j} + z\vec{k} = \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right) + s\left(l_1\vec{i} + m_1\vec{j} + n_1\vec{k}\right) + t\left(l_2\vec{i} + m_2\vec{j} + n_2\vec{k}\right)$$

Equating the coefficients $\vec{i}, \vec{j}, \vec{k}$

$$\left.\begin{aligned} x &= x_1 + sl_1 + tl_2 \\ y &= y_1 + sm_1 + tm_2 \\ z &= z_1 + sn_1 + tn_2 \end{aligned}\right\} \text{These are the parametric equations in cartesian form}$$

$$\Rightarrow \begin{aligned} x - x_1 &= sl_1 + tl_2 \\ y - y_1 &= sm_1 + tm_2 \\ z - z_1 &= sn_1 + tn_2 \end{aligned}$$

Eliminating s and t, we get $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

This is the cartesian equation of the required plane (in non-parametric form).

Non-parametric vector equation

$\vec{AP}$, $\vec{AB}$ and $\vec{AC}$ are coplanar i.e., the vectors $\vec{r} - \vec{a}$, $\vec{u}$, $\vec{v}$ are coplanar

$\therefore \left[\vec{r} - \vec{a},\ \vec{u},\ \vec{v}\right] = 0$ or $\left[\vec{r}\ \vec{u}\ \vec{v}\right] = \left[\vec{a},\ \vec{u},\ \vec{v}\right]$

This is the vector equation of plane in non-parametric form.

2.7.4 Equation of the plane passing through two given points and parallel to a given vector :

Vector Form :

Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A and B (respectively) referred to the origin O. Let $\vec{v}$ be the given vector.

The required plane passes through the points A and B and is parallel to the vector v.

Let P be any point on the plane and let its position vector be $\vec{r}$ (i.e.,) $\overrightarrow{OP} = \vec{r}$.

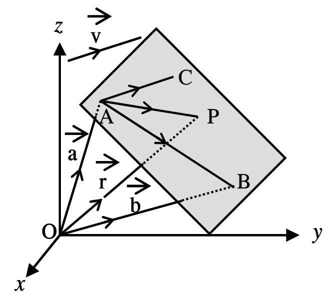


Fig. 2.30

Through A, draw a line AC lying in the plane such that $\overrightarrow{AC} = \vec{v}$.

Now $\overrightarrow{AP}$ is coplanar with $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\therefore \overrightarrow{AP} = s\,\overrightarrow{AB} + t\,\overrightarrow{AC} \text{ where } s \text{ and } t \text{ are scalars}$$

$$= s\left(\overrightarrow{OB} - \overrightarrow{OA}\right) + t\,\vec{v} = s\left(\vec{b} - \vec{a}\right) + t\,\vec{v}$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$\Rightarrow \quad \vec{r} = \vec{a} + s\left(\vec{b} - \vec{a}\right) + t\,\vec{v} \qquad \ldots (1)$$

$$\vec{r} = (1-s)\vec{a} + s\vec{b} + t\,\vec{v}$$

This is the vector equation of the plane (in parametric form).

Non-parametric vector equation

$\overrightarrow{AP}$, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are coplanar i.e., the vectors $\vec{r} - \vec{a}$, $\vec{b} - \vec{a}$ and $\vec{v}$ are coplanar

$$\therefore \left[\vec{r} - \vec{a},\ \vec{b} - \vec{a},\ \vec{v}\right] = 0$$

This is the required vector equation of plane in non-parametric form.

Cartesian form :

Let $\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$; $\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$

$\vec{v} = l\vec{i} + m\vec{j} + n\vec{k}$; $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

From (1)

$$x\vec{i} + y\vec{j} + z\vec{k} = \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right)$$

$$+ s\left[(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}\right] + t\left(l\vec{i} + m\vec{j} + n\vec{k}\right)$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$

$$\left.\begin{aligned} x &= x_1 + s(x_2 - x_1) + tl \\ y &= y_1 + s(y_2 - y_1) + tm \\ z &= z_1 + s(z_2 - z_1) + tn \end{aligned}\right\} \text{These are the parametric equations in cartesian form}$$

$$\Rightarrow \begin{aligned} (x - x_1) &= s(x_2 - x_1) + tl \\ (y - y_1) &= s(y_2 - y_1) + tm \\ (z - z_1) &= s(z_2 - z_1) + tn \end{aligned}$$

Eliminating s and t we get $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0$

This is the (non-parametric) equation of the plane in cartesian form.

2.7.5 Vector and cartesian equations of the plane passing through three given non-collinear points.

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of the points A, B and C referred to the origin O.

The required plane passes through the points A, B and C.

Let P be any point on the plane and let its position vector be $\vec{r}$ (i.e.,) $\vec{OP} = \vec{r}$.

Now join AB, AC and AP.

$\vec{AP}$ is coplanar with $\vec{AB}$ and $\vec{AC}$

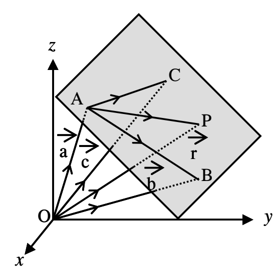


Fig. 2.31

$$\therefore \overrightarrow{AP} = s\,\overrightarrow{AB} + t\,\overrightarrow{AC} \text{ where } s \text{ and } t \text{ are scalars}$$

$$= s\left(\overrightarrow{OB} - \overrightarrow{OA}\right) + t\left(\overrightarrow{OC} - \overrightarrow{OA}\right)$$

$$= s\left(\vec{b} - \vec{a}\right) + t\left(\vec{c} - \vec{a}\right)$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$\Rightarrow \quad \vec{r} = \vec{a} + s\left(\vec{b} - \vec{a}\right) + t\left(\vec{c} - \vec{a}\right) \text{ (or)} \quad \ldots (1)$$

$$\vec{r} = (1 - s - t)\vec{a} + s\vec{b} + t\,\vec{c}$$

This is the vector equation of the plane (in parametric form).

Non-parametric vector equation :

$\overrightarrow{AP}$, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are coplanar.

(i.e.,) $$\left[\overrightarrow{AP},\ \overrightarrow{AB},\ \overrightarrow{AC}\right] = 0$$

$$\therefore \left[\vec{r} - \vec{a},\ \vec{b} - \vec{a},\ \vec{c} - \vec{a}\right] = 0$$

This is the required vector equation of plane in non-parametric form.

Cartesian form :

Let $\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$; $\vec{b} = x_2\vec{i} + y_2\vec{j} + z_3\vec{k}$; $\vec{c} = x_3\vec{i} + y_3\vec{j} + z_3\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

From (1)

$$x\vec{i} + y\vec{j} + z\vec{k} = \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right)$$

$$+ s\left[(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}\right]$$

$$+ t\left[(x_3 - x_1)\vec{i} + (y_3 - y_1)\vec{j} + (z_3 - z_1)\vec{k}\right]$$

Equating the coefficients of $\vec{i}$, $\vec{j}$ and $\vec{k}$, we get

$$\left.\begin{aligned} x &= x_1 + s(x_2 - x_1) + t(x_3 - x_1) \\ y &= y_1 + s(y_2 - y_1) + t(y_3 - y_1) \\ z &= z_1 + s(z_2 - z_1) + t(z_3 - z_1) \end{aligned}\right\} \begin{array}{l}\text{These are the} \\ \text{parametric equations} \\ \text{in cartesian form}\end{array}$$

$$\text{Eliminating } s \text{ and } t \text{ we get } \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

This is the (non-parametric) equation of the plane in cartesian form.

Example 2.48 : Find the vector and cartesian equation of a plane which is at a distance of 8 units from the origin and which is normal to the vector $3\vec{i} + 2\vec{j} - 2\vec{k}$

Solution : Here $p = 8$ and $\vec{n} = 3\vec{i} + 2\vec{j} - 2\vec{k}$

$$\therefore \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\vec{i} + 2\vec{j} - 2\vec{k}}{\sqrt{9+4+4}} = \frac{3\vec{i} + 2\vec{j} - 2\vec{k}}{\sqrt{17}}$$

Hence the required vector equation of the plane is

$$\vec{r} \cdot \hat{n} = p$$

$$\vec{r} \cdot \frac{3\vec{i} + 2\vec{j} - 2\vec{k}}{\sqrt{17}} = 8$$

$$\vec{r} \cdot \left(3\vec{i} + 2\vec{j} - 2\vec{k}\right) = 8\sqrt{17}$$

Cartesian form is $(xi + yj + zk).(3i + 2j - 2k) = 8\sqrt{17}$

$3x + 2y - 2z = 8\sqrt{17}$

Example 2.49 :

The foot of perpendicular drawn from the origin to the plane is (4, – 2, – 5), find the equation of the plane.

Solution : The required plane passes through the point $A(4, -2, -5)$ and is perpendicular to $\overrightarrow{OA}$.

$\therefore \vec{a} = 4\vec{i} - 2\vec{j} - 5\vec{k}$ and $\vec{n} = \overrightarrow{OA} = 4\vec{i} - 2\vec{j} - 5\vec{k}$

$\therefore$ The required equation of the plane is $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$$\vec{r} \cdot \left(4\vec{i} - 2\vec{j} - 5\vec{k}\right) = \left(4\vec{i} - 2\vec{j} - 5\vec{k}\right) \cdot \left(4\vec{i} - 2\vec{j} - 5\vec{k}\right)$$
$$= 16 + 4 + 25$$

$$\vec{r} \cdot \left(4\vec{i} - 2\vec{j} - 5\vec{k}\right) = 45 \qquad \dots (1)$$

Cartesian form :

$$\left(x\vec{i} + y\vec{j} + z\vec{k}\right) \cdot \left(4\vec{i} - 2\vec{j} - 5\vec{k}\right) = 45$$

$$4x - 2y - 5z = 45$$

Example 2.50 : Find the vector and cartesian equations of the plane through the point (2, – 1, – 3) and parallel to the lines

$$\frac{x-2}{3} = \frac{y-1}{2} = \frac{z-3}{-4} \text{ and } \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z-2}{2}.$$

Solution : The required plane passes through $A(2, -1, -3)$ and parallel to $\vec{u} = 3\vec{i} + 2\vec{j} - 4\vec{k}$ and $\vec{v} = 2\vec{i} - 3\vec{j} + 2\vec{k}$

The required equation is $\vec{r} = \vec{a} + s\vec{u} + t\vec{v}$

$$\vec{r} = \left(2\vec{i} - \vec{j} - 3\vec{k}\right) + s\left(3\vec{i} + 2\vec{j} - 4\vec{k}\right) + t\left(2\vec{i} - 3\vec{j} + 2\vec{k}\right)$$

Cartesian form :

(x_1, y_1, z_1) is (2, – 1, –3) ; (l_1, m_1, n_1) is (3, 2, –4) ; (l_2, m_2, n_2) is (2, –3, 2)

The equation of the plane is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

i.e., $\begin{vmatrix} x-2 & y+1 & z+3 \\ 3 & 2 & -4 \\ 2 & -3 & 2 \end{vmatrix} = 0$

$\Rightarrow$ $8x + 14y + 13z + 37 = 0$

This is the required equation in cartesian form.

Example 2.51 : Find the vector and cartesian equations of the plane passing through the points (– 1, 1, 1) and (1, – 1, 1) and perpendicular to the plane $x + 2y + 2z = 5$

Solution : The normal vector to the plane $x + 2y + 2z = 5$ is $\vec{i} + 2\vec{j} + 2\vec{k}$. This vector is parallel to the required plane.

$\therefore$ The required plane passes through the points (– 1, 1, 1) and (1, – 1, 1) and parallel to the vector $\vec{i} + 2\vec{j} + 2\vec{k}$.

Vector equation of the plane :

The vector equation of the plane passing through two given points and parallel to a vector is

$\vec{r} = (1 - s)\vec{a} + s\vec{b} + t\vec{v}$ where s and t are scalars.

Here $\vec{a} = -\vec{i} + \vec{j} + \vec{k}$; $\vec{b} = \vec{i} - \vec{j} + \vec{k}$; $\vec{v} = \vec{i} + 2\vec{j} + 2\vec{k}$

$\therefore \vec{r} = (1 - s)\left(-\vec{i} + \vec{j} + \vec{k}\right) + s\left(\vec{i} - \vec{j} + \vec{k}\right) + t\left(\vec{i} + 2\vec{j} + 2\vec{k}\right)$

This is the required vector equation of the plane.

Cartesian form :

(x_1, y_1, z_1) is $(-1, 1, 1)$; (x_2, y_2, z_2) is $(1, -1, 1)$; (l_1, m_1, n_1) is $(1, 2, 2)$

The equation of the plane is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$

i.e., $\begin{vmatrix} x + 1 & y - 1 & z - 1 \\ 2 & -2 & 0 \\ 1 & 2 & 2 \end{vmatrix} = 0$

$\Rightarrow 2x + 2y - 3z + 3 = 0$

Example 2.52 : Find the vector and cartesian equations of the plane passing through the points $(2, 2, -1)$, $(3, 4, 2)$ and $(7, 0, 6)$

Solution : Vector equation of the plane passing through three given non-collinear points is

$\vec{r} = (1 - s - t)\vec{a} + s\vec{b} + t\vec{c}$ where s and t are scalars.

Here $\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$; $\vec{b} = 3\vec{i} + 4\vec{j} + 2\vec{k}$; $\vec{c} = 7\vec{i} + 6\vec{k}$

$\therefore \vec{r} = (1-s-t)\left(2\vec{i} + 2\vec{j} - \vec{k}\right) + s\left(3\vec{i} + 4\vec{j} + 2\vec{k}\right) + t\left(7\vec{i} + 6\vec{k}\right)$

Cartesian equation of the plane :

Here (x_1, y_1, z_1) is $(2, 2, -1)$; (x_2, y_2, z_2) is $(3, 4, 2)$; (x_3, y_3, z_3) is $(7, 0, 6)$

The equation of the plane is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$

i.e., $\begin{vmatrix} x-2 & y-2 & z+1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} = 0$

$5x + 2y - 3z = 17$

This is the Cartesian equation of the plane.

EXERCISE 2.8

(1) Find the vector and cartesian equations of a plane which is at a distance of 18 units from the origin and which is normal to the vector $2\vec{i} + 7\vec{j} + 8\vec{k}$

(2) Find the unit normal vectors to the plane $2x - y + 2z = 5$.

(3) Find the length of the perpendicular from the origin to the plane $\vec{r} \cdot \left(3\vec{i} + 4\vec{j} + 12\vec{k}\right) = 26$.

(4) The foot of the perpendicular drawn from the origin to a plane is $(8, -4, 3)$. Find the equation of the plane.

(5) Find the equation of the plane through the point whose *p.v.* is $2\vec{i} - \vec{j} + \vec{k}$ and perpendicular to the vector $4\vec{i} + 2\vec{j} - 3\vec{k}$.

(6) Find the vector and cartesian equations of the plane through the point $(2, -1, 4)$ and parallel to the plane $\vec{r} \cdot \left(4\vec{i} - 12\vec{j} - 3\vec{k}\right) = 7$.

(7) Find the vector and cartesian equation of the plane containing the line $\frac{x-2}{2} = \frac{y-2}{3} = \frac{z-1}{3}$ and parallel to the line $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z+1}{1}$.

(8) Find the vector and cartesian equation of the plane through the point $(1, 3, 2)$ and parallel to the lines $\frac{x+1}{2} = \frac{y+2}{-1} = \frac{z+3}{3}$ and $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z+2}{2}$

(9) Find the vector and cartesian equation to the plane through the point $(-1, 3, 2)$ and perpendicular to the planes $x+2y+2z = 5$ and $3x+y+2z = 8$.

(10) Find the vector and cartesian equation of the plane passing through the points $A(1, -2, 3)$ and $B(-1, 2, -1)$ and is parallel to the line $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z-1}{4}$

(11) Find the vector and cartesian equation of the plane through the points $(1, 2, 3)$ and $(2, 3, 1)$ perpendicular to the plane $3x - 2y + 4z - 5 = 0$

(12) Find the vector and cartesian equation of the plane containing the line $\frac{x-2}{2} = \frac{y-2}{3} = \frac{z-1}{-2}$ and passing through the point (– 1, 1, – 1).

(13) Find the vector and cartesian equation of the plane passing through the points with position vectors $3\vec{i} + 4\vec{j} + 2\vec{k}$, $2\vec{i} - 2\vec{j} - \vec{k}$ and $7\vec{i} + \vec{k}$.

(14) Derive the equation of the plane in the intercept form.

(15) Find the cartesian form of the following planes :

(i) $\vec{r} = (s-2t)\vec{i} + (3-t)\vec{j} + (2s+t)\vec{k}$

(ii) $\vec{r} = (1+s+t)\vec{i} + (2-s+t)\vec{j} + (3-2s+2t)\vec{k}$

2.7.6 Equation of a plane passing through the line of intersection of two given planes :

Vector form :

The vector equation of the plane passing through the line of intersection of the planes $\vec{r} \cdot \vec{n}_1 = q_1$ and $\vec{r} \cdot \vec{n}_2 = q_2$ is

$$\left(\vec{r} \cdot \vec{n}_1 - q_1\right) + \lambda\left(\vec{r} \cdot \vec{n}_2 - q_2\right) = 0$$

$$\text{i.e. } \vec{r} \cdot \left(\vec{n}_1 + \lambda\vec{n}_2\right) = q_1 + \lambda q_2$$

Cartesian form :

The cartesian equation of the plane passing through the line of intersection of the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$

Example 2.53 : Find the equation of the plane passing through the line of intersection of the plane $2x - 3y + 4z = 1$ and $x - y = -4$ and passing through the point (1, 1, 1).

Solution :

Any plane through the line of intersection of the given two planes is of the form $(2x - 3y + 4z - 1) + \lambda(x - y + 4) = 0$

But it passes through the point (1, 1, 1). $\therefore \lambda = -\frac{1}{2}$

$\therefore$ The equation of the required plane is $(2x - 3y + 4z - 1) - \frac{1}{2}(x - y + 4) = 0$

i.e., $3x - 5y + 8z - 6 = 0$

Example 2.54 : Find the equation of the plane passing through the intersection of the planes $2x - 8y + 4z = 3$ and $3x - 5y + 4z + 10 = 0$ and perpendicular to the plane $3x - y - 2z - 4 = 0$

Solution :

The equation of the plane passing through the line of intersection of the given two planes is of the form $(2x - 8y + 4z - 3) + \lambda\,(3x - 5y + 4z + 10) = 0$ i.e., $(2 + 3\lambda)\,x + (-8 - 5\lambda)y + (4 + 4\lambda)z + (-3 + 10\lambda) = 0$. But the required plane is perpendicular to the plane $3x - y - 2z - 4 = 0$

$\therefore$ Their normals are perpendicular.

i.e., $(2 + 3\lambda)\,3 + (-8 - 5\lambda)\,(-1) + (4 + 4\lambda)\,(-2) = 0$

$6\lambda + 6 = 0 \quad \Rightarrow \lambda = -1$

$\therefore$ The required equation is $(2x - 8y + 4z - 3) - 1(3x - 5y + 4z + 10) = 0$

$$-x - 3y - 13 = 0$$

$$x + 3y + 13 = 0$$

2.7.7 The distance between a point and a plane :

Let (x_1, y_1, z_1) be a point and $ax + by + cz + d = 0$ be the equation of the plane. The distance between the point and the plane is $\left|\dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$

Corollary (1) :

The distance between the origin and the plane $ax + by + cz + d = 0$ is

$$\left|\frac{d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Corollary (2) :

The distance between the two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $\left|\dfrac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

Note : If the given equation is in vector form, convert into cartesian form and find the distance.

Example 2.55 : Find the distance from the point $(1, -1, 2)$ to the plane $\vec{r} = \left(\vec{i} + \vec{j} + \vec{k}\right) + s\left(\vec{i} - \vec{j}\right) + t\left(\vec{j} - \vec{k}\right)$

Solution :

The given plane is passing through the point (1, 1, 1) and parallel to two vectors $\left(\vec{i} - \vec{j}\right)$ and $\left(\vec{j} - \vec{k}\right)$.

$\therefore$ The corresponding cartesian equation is of the form

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad \begin{cases} (x_1, y_1, z_1) = (1, 1, 1) \\ (l_1, m_1, n_1) = (1 - 1, 0) \\ (l_2, m_2, n_2) = (0, 1, -1) \end{cases}$$

i.e., $\begin{vmatrix} x - 1 & y - 1 & z - 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0$ i.e., $x + y + z - 3 = 0$

Here $(x_1, y_1, z_1) = (1, -1, 2)$

$\therefore$ The distance $= \left|\dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\dfrac{1 - 1 + 2 - 3}{\sqrt{1 + 1 + 1}}\right| = \dfrac{1}{\sqrt{3}}$

Example 2.56 : Find the distance between the parallel planes $\vec{r}.\left(-\vec{i} - \vec{j} + \vec{k}\right) = 3$ and $\vec{r}.\left(\vec{i} + \vec{j} - \vec{k}\right) = 5$

Solution :

The corresponding cartesian equations of the planes are

$-x - y + z - 3 = 0$ and $x + y - z - 5 = 0$

i.e., $x + y - z + 3 = 0$ and $x + y - z - 5 = 0$

distance $= \left|\dfrac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\dfrac{3 + 5}{\sqrt{1 + 1 + 1}}\right| = \dfrac{8}{\sqrt{3}}$

2.7.8 Equation of the plane which contain two given lines (i.e. passing through two given lines)

Let $\vec{r} = \vec{a}_1 + t\vec{u}$ and $\vec{r} = \vec{a}_2 + s\vec{v}$ be the lines, lie on the plane.

Clearly $\vec{r} - \vec{a}_1, \vec{u}, \vec{v}$ are coplanar and $\vec{r} - \vec{a}_2, \vec{u}, \vec{v}$ are also coplanar

Thus $\left[\vec{r} - \vec{a}_1, \vec{u}, \vec{v}\right] = 0$ and $\left[\vec{r} - \vec{a}_2, \vec{u}, \vec{v}\right] = 0$

Note that the above two equations represent the same required plane. The cartesian form is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Where $\vec{a}_1 = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$; $\vec{a}_2 = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$

$\vec{u} = l_1\vec{i} + m_1\vec{j} + n_1\vec{k}$; $\vec{v} = l_2\vec{i} + m_2\vec{j} + n_2\vec{k}$

Note :

(1) If the two lines are parallel then take the two trivial points from the lines and the parallel vector. Now find the equation of the plane passing through two points and parallel to a vector.

(2) Through two skew lines, we can't draw a plane.

Example 2.57 : Find the equation of the plane which contains the two lines

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{1}$

Solution :

Take the trivial point from the first line and the two parallel vectors i.e. $(x_1, y_1, z_1) = (1, 2, 3)$.

$(l_1, m_1, n_1) = (2, 3, 4)$ and $(l_2, m_2, n_2) = (5, 2, 1)$

The required equation is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow 5x - 18y + 11z - 2 = 0$

Example 2.58 : Find the point of intersection of the line passing through the two points $(1, 1, -1)$; $(-1, 0, 1)$ and the xy-plane.

Solution :

The equation of the line passing through $(1, 1, -1)$ and $(-1, 0, 1)$ is

$$\frac{x-1}{2} = \frac{y-1}{1} = \frac{z+1}{-2}$$

It meets the xy-plane i.e. $z = 0$

$$\therefore \frac{x-1}{2} = \frac{y-1}{1} = \frac{1}{-2} \Rightarrow x = 0, \; y = \frac{1}{2}$$

The required point is $\left(0, \frac{1}{2}, 0\right)$

Example 2.59 : Find the co-ordinates of the point where the line

$\vec{r} = \left(\vec{i} + 2\vec{j} - 5\vec{k}\right) + t\left(2\vec{i} - 3\vec{j} + 4\vec{k}\right)$ meets the plane

$\vec{r} \cdot \left(2\vec{i} + 4\vec{j} - \vec{k}\right) = 3$

Solution :

The equation of the straight line in the cartesian form is

$$\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+5}{4} = \lambda \text{ (say)}$$

$\therefore$ Any point on this line is of the form $(2\lambda + 1, -3\lambda + 2, 4\lambda - 5)$.

The cartesian equation of the plane is $2x + 4y - z - 3 = 0$

But the required point lies on this plane.

$\therefore$ $2(2\lambda + 1) + 4(-3\lambda + 2) - (4\lambda - 5) - 3 = 0 \Rightarrow \lambda = 1$

$\therefore$ The required point is $(3, -1, -1)$

EXERCISE 2.9

(1) Find the equation of the plane which contains the two lines $\frac{x+1}{2} = \frac{y-2}{-3} = \frac{z-3}{4}$ and $\frac{x-4}{3} = \frac{y-1}{2} = z - 8$

(2) Can you draw a plane through the given two lines? Justify your answer.

$\vec{r} = \left(\vec{i} + 2\vec{j} - 4\vec{k}\right) + t\left(2\vec{i} + 3\vec{j} + 6\vec{k}\right)$ and

$\vec{r} = \left(3\vec{i} + 3\vec{j} - 5\vec{k}\right) + s\left(-2\vec{i} + 3\vec{j} + 8\vec{k}\right)$

(3) Find the point of intersection of the line

$\vec{r} = \left(\vec{j} - \vec{k}\right) + s\left(2\vec{i} - \vec{j} + \vec{k}\right)$ and xz – plane

(4) Find the meeting point of the line

$\vec{r} = \left(2\vec{i} + \vec{j} - 3\vec{k}\right) + t\left(2\vec{i} - \vec{j} - \vec{k}\right)$ and the plane

$x - 2y + 3z + 7 = 0$

(5) Find the distance from the origin to the plane

$\vec{r} \cdot \left(2\vec{i} - \vec{j} + 5\vec{k}\right) = 7$

(6) Find the distance between the parallel planes

$x - y + 3z + 5 = 0$; $2x - 2y + 6z + 7 = 0$

2.7.9 Angle between two given planes :

The angle between two planes is defined as the angle between their normals.

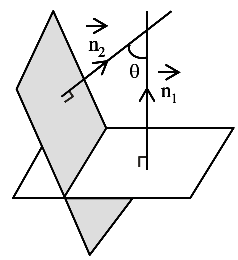


Fig. 2.32

Let the $\vec{r} \cdot \vec{n}_1 = q_1$ and $\vec{r} \cdot \vec{n}_2 = q_2$ the equations of the given two planes (where $\vec{n}_1$ and $\vec{n}_2$ are normals to the planes.)

Now if θ be the angle between the two planes (i.e., between their normals) then

$$\theta = \cos^{-1}\left[\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}\right]$$

Note : (i) If the two planes are perpendicular then $\vec{n}_1 \cdot \vec{n}_2 = 0$

(ii) If the two planes are parallel then $\vec{n}_1 = t\vec{n}_2$ where t is a scalar.

2.7.10 Angle between a line and a plane

The angle between a line and a plane is the complement angle between the line and the normal to the plane.

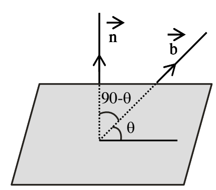


Fig. 2.33

Let $\vec{r} = \vec{a} + t\vec{b}$ be the line and $\vec{r} \cdot \vec{n} = q$ be the plane.

If θ is the angle between the line and the plane then (90 – θ) is the angle between the line and the normal to the plane.

i.e., (90 – θ) is the angle between $\vec{b}$ and $\vec{n}$

$$\therefore \cos(90° - \theta) = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|} \Rightarrow \sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|} \Rightarrow \theta = \sin^{-1}\left(\frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}\right)$$

Note : If the line is parallel to the plane i.e., the normal to the plane is perpendicular to the line then $\vec{b} \cdot \vec{n} = 0$

Example 2.60 : Find the angle between $2x - y + z = 4$ and $x + y + 2z = 4$

Solution : The normals to the given planes are

$\vec{n}_1 = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{n}_2 = \vec{i} + \vec{j} + 2\vec{k}$

Let θ be the angle between the planes then

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Example 2.61 : Find the angle between the line

$\vec{r} = \left(\vec{i} + 2\vec{j} - \vec{k}\right) + \mu\left(2\vec{i} + \vec{j} + 2\vec{k}\right)$ and the plane

$\vec{r} \cdot \left(3\vec{i} - 2\vec{j} + 6\vec{k}\right) = 0$

Solution : Let θ be the angle between the line and the plane.

$$\sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}|\,|\vec{n}|}$$

$$\vec{b} = 2\vec{i} + \vec{j} + 2\vec{k} \; ; \; \vec{n} = 3\vec{i} - 2\vec{j} + 6\vec{k}$$

$$\sin\theta = \frac{16}{3 \times 7} \Rightarrow \theta = \sin^{-1}\left(\frac{16}{21}\right)$$

EXERCISE 2.10

(1) Find the angle between the following planes :

(i) $2x + y - z = 9$ and $x + 2y + z = 7$

(ii) $2x - 3y + 4z = 1$ and $-x + y = 4$

(iii) $\vec{r} \cdot \left(3\vec{i} + \vec{j} - \vec{k}\right) = 7$ and $\vec{r} \cdot \left(\vec{i} + 4\vec{j} - 2\vec{k}\right) = 10$

(2) Show that the following planes are at right angles.

$\vec{r} \cdot \left(2\vec{i} - \vec{j} + \vec{k}\right) = 15$ and $\vec{r} \cdot \left(\vec{i} - \vec{j} - 3\vec{k}\right) = 3$

(3) The planes $\vec{r} \cdot \left(2\vec{i} + \lambda\vec{j} - 3\vec{k}\right) = 10$ and $\vec{r} \cdot \left(\lambda\vec{i} + 3\vec{j} + \vec{k}\right) = 5$ are perpendicular. Find λ.

(4) Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{-2}$ and the plane $3x + 4y + z + 5 = 0$

(5) Find the angle between the line $\vec{r} = \vec{i} + \vec{j} + 3\vec{k} + \lambda\left(2\vec{i} + \vec{j} - \vec{k}\right)$ and the plane $\vec{r}.\left(\vec{i} + \vec{j}\right) = 1$.

2.8 Sphere :

A sphere is the locus of a point which moves in space in such a way that its distance from a fixed point remains constant.

The fixed point is called the centre and the constant distance is called the radius of the sphere.

Note : Eventhough the syllabus does not require the derivations (2.8.1, 2.8.2) and it needs only the results, the equations are derived for better understanding the results.

2.8.1 Vector equation of the sphere whose position vector of centre is $\vec{c}$ and radius is *a*.

Let *O* be the point of reference (origin) and *C* be the centre of the sphere having position vector $\vec{c}$

(i.e.,) $\vec{OC} = \vec{c}$

Let *P* be any point on the sphere whose position vector be $\vec{r}$

(i.e.,) $\vec{OP} = \vec{r}$

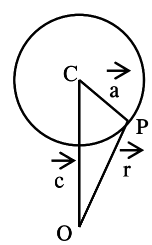


Fig. 2.34

The radius of the sphere is given as *a*. (i.e.,) $\vec{CP} = \vec{a}$

From the figure (2.34) $\vec{OP} = \vec{OC} + \vec{CP}$

$$\vec{r} = \vec{c} + \vec{a}$$

$$\vec{r} - \vec{c} = \vec{a}$$

$$\Rightarrow \left|\vec{r} - \vec{c}\right| = \left|\vec{a}\right| \quad \ldots (1)$$

This is the vector equation of the sphere.

Corollary : Vector equation of a sphere whose centre is origin and radius is *a*.

When O coincides with the centre C then $\vec{c} = \vec{o}$ and the vector equation of the sphere (1) becomes $|\vec{r}| = |\vec{a}|$

Cartesian form :

$$\text{Let} \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

$$\Rightarrow \vec{r} - \vec{c} = (x - c_1)\vec{i} + (y - c_2)\vec{j} + (z - c_3)\vec{k}$$

$$\text{But } |\vec{r} - \vec{c}|^2 = a^2 \qquad \dots (2)$$

From (2) $\quad (x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = a^2 \qquad \dots (3)$

This is the cartesian equation of the sphere whose centre is (c_1, c_2, c_3) and raidus is a.

Corollary : If the centre is at the origin, then the equation (3) takes the form $x^2 + y^2 + z^2 = a^2$.

This is known as the standard form of the equation of the sphere.

Note : General Equation of a Sphere :

The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere with centre $(-u, -v, -w)$ and the radius $= \sqrt{u^2 + v^2 + w^2 - d}$

Note :

(i) the coefficients of x^2, y^2, z^2 are equal.

(ii) The equation does not contain the terms of xy, yz and zx.

2.8.2 Vector and Cartesian equations of the sphere when the extremities of the diameter being given :

Let C be the centre of the sphere. Let A and B be the end points of the diameter AB.

Let $\vec{a}$ be the position vector of the point A and $\vec{b}$ be the position vector of the point B with reference to the origin O.

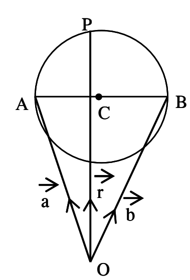


Fig. 2.35

(i.e.,) $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Let P be any point on the surface of the sphere. Let $\vec{r}$ be the position vector of P. (i.e.,) $\overrightarrow{OP} = \vec{r}$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{a}$$

$$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB} = \vec{r} - \vec{b}$$

We know that the diameter AB subtends a right angle at P.

$$\Rightarrow \overrightarrow{AP} \perp \overrightarrow{BP}$$

$$\Rightarrow \overrightarrow{AP} \cdot \overrightarrow{BP} = 0$$

$$\Rightarrow \left(\vec{r} - \vec{a}\right) \cdot \left(\vec{r} - \vec{b}\right) = 0 \qquad \dots (1)$$

which is the required equation of the sphere.

Cartesian Form :

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the end points of the diameter AB.

Let $P\ (x, y, z)$ be any point on the surface of the sphere.

Now $\quad \vec{a} = \overrightarrow{OA} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\ ; \ \vec{b} = \overrightarrow{OB} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$

$$\vec{r} = \overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$$

From (1) $\quad \left(\vec{r} - \vec{a}\right) \cdot \left(\vec{r} - \vec{b}\right) = 0$

$$\left[\left(x\vec{i}+y\vec{j}+z\vec{k}\right)-(x_1\vec{i}+y_1\vec{j}+z_1\vec{k})\right] \cdot \left[\left(x\vec{i}+y\vec{j}+z\vec{k}\right)-(x_2\vec{i}+y_2\vec{j}+z_2\vec{k})\right] = 0$$

$$\left[(x-x_1)\vec{i} + (y-y_1)\vec{j} + (z-z_1)\vec{k}\right] \cdot \left[(x-x_2)\vec{i} + (y-y_2)\vec{j} + (z-z_2)\vec{k}\right] = 0$$

$$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

This is required cartesian form of the equation of the sphere in terms of the end points of the diameter.

Example 2.62 : Find the vector and cartesian equations of the sphere whose centre is $2\vec{i} - \vec{j} + 2\vec{k}$ and radius is 3.

Solution : We know that the vector equation of the sphere with centre and radius is

$$\left|\vec{r} - \vec{c}\right| = a$$

Here $\vec{c} = 2\vec{i} - \vec{j} + 2\vec{k}$ and $a = 3$

$\therefore$ The required vector equation is $\left|\vec{r} - \left(2\vec{i} - \vec{j} + 2\vec{k}\right)\right| = 3$

Cartesian equation :

Putting $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ we get

$$\left[\left(x\vec{i} + y\vec{j} + z\vec{k}\right) - \left(2\vec{i} - \vec{j} + 2\vec{k}\right)\right] = 3$$

$$\left|(x-2)\vec{i} + (y+1)\vec{j} + (z-2)\vec{k}\right| = 3$$

$$\Rightarrow \left|(x-2)\vec{i} + (y+1)\vec{j} + (z-2)\vec{k}\right|^2 = 3^2$$

$$(x-2)^2 + (y+1)^2 + (z-2)^2 = 9$$

$$\Rightarrow x^2 + y^2 + z^2 - 4x + 2y - 4z = 0$$

Example 2.63 : Find the vector and cartesian equation of the sphere whose centre is (1, 2, 3) and which passes through the point (5, 5, 3).

Solution :

$$\text{Radius} = \sqrt{(5-1)^2 + (5-2)^2 + (3-3)^2}$$

$$= \sqrt{16+9} = \sqrt{25} = 5$$

Here $a = 5$ and $\vec{c} = \vec{i} + 2\vec{j} + 3\vec{k}$

$\therefore$ Vector equation of the sphere is

$$\left|\vec{r} - \vec{c}\right| = a$$

$$\left[\vec{r} - \left(\vec{i} + 2\vec{j} + 3\vec{k}\right)\right] = 5 \qquad \ldots (1)$$

Cartesian Equation : Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

From (1)

$$\left|\left(x\vec{i} + y\vec{j} + z\vec{k}\right) - \left(\vec{i} + 2\vec{j} + 3\vec{k}\right)\right| = 5$$

$$\left|(x-1)\vec{i} + (y-2)\vec{j} + (z-3)\vec{k}\right| = 5$$

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 25$$

$$x^2 + y^2 + z^2 - 2x - 4y - 6z - 11 = 0$$

Example 2.64 : Find the equation of the sphere on the join of the points A and B having position vectors $2\vec{i} + 6\vec{j} - 7\vec{k}$ and $2\vec{i} - 4\vec{j} + 3\vec{k}$ respectively as a diameter.

Solution : Vector equation of the sphere is $\left(\vec{r} - \vec{a}\right) . \left(\vec{r} - \vec{b}\right) = 0$

Here $\vec{a} = 2\vec{i} + 6\vec{j} - 7\vec{k}$ and $\vec{b} = 2\vec{i} - 4\vec{j} + 3\vec{k}$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

The required equation is

$$\left[\left(x\vec{i}+y\vec{j}+z\vec{k}\right)-\left(2\vec{i}+6\vec{j}-7\vec{k}\right)\right].\left[\left(x\vec{i}+y\vec{j}+z\vec{k}\right)-\left(2\vec{i}-4\vec{j}+3\vec{k}\right)\right] = 0$$

$$\left[(x-2)\vec{i} + (y-6)\vec{j} + (z+7)\vec{k}\right] . \left[(x-2)\vec{i}+(y+4)\vec{j}+(z-3)\vec{k}\right] = 0 \quad \ldots(1)$$

Cartesian Equation :

From (1)

$(x-2)(x-2) + (y-6)(y+4) + (z+7)(z-3) = 0$

$\Rightarrow x^2 + y^2 + z^2 - 4x - 2y + 4z - 41 = 0$

Example 2.65 : Find the coordinates of the centre and the radius of the sphere whose vector equation is $\vec{r}^{\,2} - \vec{r} . \left(8\vec{i} - 6\vec{j} + 10\vec{k}\right) - 50 = 0$

Solution : Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{r}^{\,2} - \vec{r} . \left(8\vec{i} - 6\vec{j} + 10\vec{k}\right) - 50 = 0$$

$$\vec{r}^{\,2} = x^2 + y^2 + z^2$$

$$\Rightarrow x^2 + y^2 + z^2 - 8x + 6y - 10z - 50 = 0$$

Here $2u$ = coefficient of $x = -8 \Rightarrow u = -4$

$2v$ = coefficient of $y = 6 \Rightarrow v = 3$

$2w$ = coefficient of $z = -10 \Rightarrow w = -5$

Centre : $(-u, -v, -w) = (4, -3, 5)$

Radius : $\sqrt{u^2 + v^2 + w^2 - d} = \sqrt{16 + 9 + 25 + 50} = \sqrt{100} = 10$ uts.

Example 2.66 :

Chord AB is a diameter of the sphere $\left|r - \left(2\vec{i} + \vec{j} - 6\vec{k}\right)\right| = \sqrt{18}$ with coordinate of A as $(3, 2, -2)$ Find the coordinates B.

Solution : The equation of the sphere is $\left| r - \left(2\vec{i} + \vec{j} - 6\vec{k}\right)\right| = \sqrt{18}$

$\Rightarrow$ Centre of the sphere is (2, 1, – 6)

(i.e.,) Position vector of the centre is $2\vec{i} + \vec{j} - 6\vec{k}$

We know that

Centre is the mid point of diameter *AB*

The co-ordinates of *A* are (3, 2, – 2) and let the coordinates of *B* be α, β, γ)

$$\therefore (2, 1, -6) = \left(\frac{\alpha + 3}{2}, \frac{\beta + 2}{2}, \frac{\gamma - 2}{2}\right)$$

$\Rightarrow$ $\alpha = 1,\ \beta = 0,\ \gamma = -10$

$\therefore$ Coordinates of *B* are (1, 0, – 10)

EXERCISE 2.11

(1) Find the vector equation of a sphere with centre having position vector $2\vec{i} - \vec{j} + 3\vec{k}$ and radius 4 units. Also find the equation in cartesian form.

(2) Find the vector and cartesian equation of the sphere on the join of the points *A* and *B* having position vectors $2\vec{i} + 6\vec{j} - 7\vec{k}$ and $-2\vec{i} + 4\vec{j} - 3\vec{k}$ respectively as a diameter. Find also the centre and radius of the sphere.

(3) Obtain the vector and cartesian equation of the sphere whose centre is (1, –1, 1) and radius is the same as that of the sphere $\left|\vec{r} - \left(\vec{i} + \vec{j} + 2\vec{k}\right)\right| = 5$.

(4) If *A* (– 1, 4, – 3) is one end of a diameter *AB* of the sphere $x^2 + y^2 + z^2 - 3x - 2y + 2z - 15 = 0$, then find the coordinates of *B*.

(5) Find the centre and radius of each of the following spheres.

(i) $\left|\vec{r} - \left(2\vec{i} - \vec{j} + 4\vec{k}\right)\right| = 5$

(ii) $\left|2\vec{r} + \left(3\vec{i} - \vec{j} + 4\vec{k}\right)\right| = 4$

(iii) $x^2 + y^2 + z^2 + 4x - 8y + 2z = 5$

(iv) $\vec{r}^{\,2} - \vec{r} \cdot \left(4\vec{i} + 2\vec{j} - 6\vec{k}\right) - 11 = 0$

(6) Show that diameter of a sphere subtends a right angle at a point on the surface.

3. COMPLEX NUMBERS

3.1 Introduction :

The number system that we are aware of today is the gradual development from natural numbers to integers, from integers to rational numbers and from rational numbers to the real numbers.

If we consider the following polynomial equations (i) $x - 1 = 0$, (ii) $x + 1 = 0$, (iii) $x + 1 = 1$, (iv) $2x + 1 = 0$ and (v) $x^2 - 3 = 0$, we see that all of them have solutions in the real number system. However this real number system is not sufficient to solve equations of the form $x^2 + 9 = 0$ i.e., there does not exist any real number which satisfies $x^2 = -9$. The mathematical need to have solutions for equations of the above form led us to extend the real number system to a new kind of number system that allows the square root of negative numbers.

Let us consider solution of a simple quadratic equation $x^2 + 16 = 0$. Its solutions are $x = \pm 4\sqrt{-1}$. We assume that square root of -1 is denoted by the symbol i, called the imaginary unit. Thus for any two real numbers a and b, we can form a new number $a + ib$. This number $a + ib$ is called a complex number. The set of all complex numbers is denoted by $\mathbb{C}$ and the nomenclature of a complex number was introduced by C.F. Gauss, a German mathematician. Hence the extension of the concept of numbers from real numbers enables one to solve any polynomial equation. The symbol i was first introduced in mathematics by the famous Swiss mathematician, Leonhard Euler (1707 – 1783) in 1748. 'i' is the first letter of the Latin word "imaginarius" and it is also referred to as 'iota', a Greek alphabet. Later on the subject was enriched by the original work of A.L. Cauchy, B. Riemann, K. Weierstrass and others.

3.2 The complex number system :

A complex number is of the form $a + ib$ where 'a' and 'b' are real numbers and i is called the imaginary unit, having the property that $i^2 = -1$. If $z = a + ib$ then a is called the real part of z, denoted by $Re(z)$ and b is called the imaginary part of z and is denoted by $Im(z)$.

Some examples of complex numbers are $3 - i2$, $\sqrt{2} + i3$, $-\frac{2}{5} + i$.

Note that 3 is the real part and – 2 is the imaginary part of $3 - i2$ and so on.

Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$. i.e., the corresponding real parts are equal and the corresponding imaginary parts are equal. The real numbers can be considered as a subset of the set of complex numbers with $b = 0$. Hence the complex numbers $0 + i0$ and $-2 + i0$ represents the real numbers 0 and – 2 respectively. If $a = 0$ the complex number $0 + ib$ or ib is called a pure imaginary number.

Negative of a complex number :

If $z = a + ib$ is a complex number then the negative of z is denoted by $-z$ and it is defined as $-z = -a + i(-b)$

Basic Algebraic operations :

Addition : $(a + ib) + (c + id) = (a + c) + i\,(b + d)$

Subtraction : $(a + ib) - (c + id) = (a - c) + i(b - d)$

To perform the operations with complex numbers we can proceed as in the algebra of real numbers replacing i^2 by – 1 whenever it occurs.

Multiplication : $(a + ib)\,(c + id) = ac + iad + ibc + i^2bd$
$$= (ac - bd) + i\,(ad + bc)$$

3.3 Conjugate of a complex number :

If $z = a + ib$, then the conjugate of z is denoted by $\bar{z}$ and is defined by $\bar{z} = a - ib$

Division : $$\frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id}$$

Multiplying the numerator and denominator by the conjugate of the denominator, we get

$$\frac{a + ib}{c + id} = \left[\frac{ac + bd}{c^2 + d^2}\right] + i\left[\frac{bc - ad}{c^2 + d^2}\right]$$

3.3.1 Properties :

(i) $z\bar{z} = (a + ib)\,(a - ib) = a^2 + b^2$ which is a non-negative real number.

(ii) Conjugate of $\bar{z}$ is z i.e., $\bar{\bar{z}} = z$

(iii) If z is real, i.e., $b = 0$ then $z = \bar{z}$. Conversely, if $z = \bar{z}$,
i.e., if $a + ib = a - ib$ then $b = -b \Rightarrow 2b = 0 \Rightarrow b = 0$
($\because 2 \neq 0$ in the real number system). $\therefore\ b = 0 \Rightarrow z$ is real.
Thus z is real $\Leftrightarrow$ the imaginary part is 0

(iv) Let $z = a + ib$. Then $\overline{z} = a - ib$

$\therefore \; z + \overline{z} = (a + ib) + (a - ib) = 2a$

$\Rightarrow a = Re(z) = \dfrac{z + \overline{z}}{2}$

Similarly, $b = Im(z) = \dfrac{z - \overline{z}}{2i}$

(v) The conjugate of the sum of two complex numbers z_1, z_2 is the sum of their conjugates i.e., $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Proof : Let $z_1 = a + ib$ and $z_2 = c + id$

Then
$$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i\,(b + d)$$
$$\overline{z_1 + z_2} = (a + c) - i\,(b + d)$$
$$\overline{z_1} = a - ib, \quad \overline{z_2} = c - id$$
$$\overline{z_1} + \overline{z_2} = (a - ib) + (c - id) = (a + c) - i(b + d)$$
$$= \overline{z_1 + z_2}$$

Similarly it can be proved that the conjugate of the difference of two complex numbers z_1, z_2 is the difference of their conjugates.

i.e., $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(vi) The conjugate of the product of two complex numbers z_1, z_2 is the product of their conjugates.

i.e., $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

Proof : Let $z_1 = a + ib$ and $z_2 = c + id$. Then
$$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$
$$\overline{z_1 z_2} = (ac - bd) - i(ad + bc)$$
$$\overline{z_1} = a - ib, \quad \overline{z_2} = c - id$$
$$\overline{z_1}\,\overline{z_2} = (a - ib)(c - id) = (ac - bd) - i(ad + bc)$$
$$= \overline{z_1 z_2}$$

(vii) The conjugate of the quotient of two complex numbers z_1, z_2, $(z_2 \neq 0)$ is the quotient of their conjugates.

$$\text{i.e.,}\quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2} \quad \text{(without proof)}$$

(viii) $\overline{z^n} = \left(\overline{z}\right)^n$

Example 3.1 : Write the following as complex numbers

(i) $\sqrt{-35}$ (ii) $3 - \sqrt{-7}$

Solution :

(i) $\sqrt{-35} = \sqrt{(-1) \times (35)} = \sqrt{-1}\ .\ \sqrt{35} = i\sqrt{35}$

(ii) $3 - \sqrt{-7} = 3 - \sqrt{(-1) \times 7} = 3 - \sqrt{-1}\sqrt{7} = 3 - i\sqrt{7}$

Example 3.2 : Write the real and imaginary parts of the following numbers :

(i) $4 - i\sqrt{3}$ (ii) $\frac{3}{2}\ i$

Solution :

(i) Let $z = 4 - i\sqrt{3}$; $Re(z) = 4$, $Im(z) = -\sqrt{3}$

(ii) Let $z = \frac{3}{2}\ i$; $Re(z) = 0$, $Im(z) = \frac{3}{2}$

Example 3.3 :

Find the complex conjugate of (i) $2 + i\sqrt{7}$, (ii) $-4 - i9$ (iii) $\sqrt{5}$

Solution :

By definition, the complex conjugate is obtained by reversing the sign of the imaginary part of the complex number. Hence the required conjugates are (i) $2 - i\sqrt{7}$, (ii) $-4 + i9$ and (iii) $\sqrt{5}$ ($\because$ the conjugate of any real number is itself).

Example 3.4 :

Express the following in the standard form of $a + ib$

(i) $(3 + 2i) + (-7 - i)$ (ii) $(8 - 6i) - (2i - 7)$

(iii) $(2 - 3i)(4 + 2i)$ (iv) $\frac{5 + 5i}{3 - 4i}$

Solution :

(i) $(3 + 2i) + (-7 - i) = 3 + 2i - 7 - i = -4 + i$

(ii) $(8 - 6i) - (2i - 7) = 8 - 6i - 2i + 7 = 15 - 8i$

(iii) $(2-3i)(4+2i) = 8+4i-12i-6i^2 = 14-8i$

(iv) $\dfrac{5+5i}{3-4i} = \dfrac{5+5i}{3-4i} \times \dfrac{3+4i}{3+4i} = \dfrac{15+20i+15i+20i^2}{3^2+4^2}$

$$= \frac{-5+35i}{25} = \frac{-1}{5} + \frac{7}{5}i$$

Note :
$$i^4 = 1$$
$$i^3 = -i$$
$$i^2 = -1$$
$$(i)^{4n} = 1$$
$$(i)^{4n-1} = -i$$
$$(i)^{4n-2} = -1 \ ; \ n \in z$$

Example 3.5 : Find the real and imaginary parts of the complex number

$$z = \frac{3i^{20} - i^{19}}{2i-1}$$

Solution :

(i) $$z = \frac{3i^{20} - i^{19}}{2i-1} = \frac{3(i^2)^{10} - (i^2)^9 i}{2i-1}$$

$$= \frac{3(-1)^{10} - (-1)^9 i}{-1+2i}$$

$$= \frac{3+i}{-1+2i}$$

$$= \frac{3+i}{-1+2i} \times \frac{-1-2i}{-1-2i}$$

$$= \frac{-3-6i-i-2i^2}{(-1)^2+2^2}$$

$$= \frac{-1-7i}{5} = \frac{-1}{5} - \frac{7}{5}i$$

$$Re(z) = -\frac{1}{5} \text{ and } Im(z) = \frac{-7}{5}$$

Example 3.6 : If $z_1 = 2+i$, $z_2 = 3-2i$ and $z_3 = \dfrac{-1}{2} + \dfrac{\sqrt{3}}{2}i$

find the conjugate of (i) $z_1 z_2$ (ii) $(z_3)^4$

Solution :

(i) Conjugate of $z_1 z_2$ is $\overline{z_1\, z_2}$

i.e. $\overline{(2 + i)\,(3 - 2i)} = \overline{(2 + i)}\;\overline{(3 - 2i)}$

$= (2 - i)\,(3 + 2i)$

$= (2 - i)\,(3 + 2i)$

$= 6 + 4i - 3i - 2i^2 = 6 + 4i - 3i + 2$

$= 8 + i$

(ii) $$\overline{z_3{}^4} = \left(\overline{z_3}\right)^4 = \left(\overline{-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i}\right)^4$$

$$= \left[\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^2\right]^2 = \left(\frac{1}{4} + \frac{\sqrt{3}}{2}i + \frac{3}{4}\,i^2\right)^2$$

$$= \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)^2 = \frac{1}{4} - \frac{\sqrt{3}}{2}\;i + \frac{3}{4}\,i^2$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$$

EXERCISE 3.1

(1) Express the following in the standard form $a + ib$

(i) $\dfrac{2(i - 3)}{(1 + i)^2}$ (ii) $\dfrac{(1 + i)\,(1 - 2i)}{1 + 3i}$

(iii) $(- 3 + i)\,(4 - 2i)$ (iv) $\dfrac{i^4 + i^9 + i^{16}}{3 - 2i^8 - i^{10} - i^{15}}$

(2) Find the real and imaginary parts of the following complex numbers:

(i) $\dfrac{1}{1 + i}$ (ii) $\dfrac{2 + 5i}{4 - 3i}$ (iii) $(2 + i)\,(3 - 2i)$

(3) Find the least positive integer n such that $\left(\dfrac{1 + i}{1 - i}\right)^n = 1$

(4) Find the real values of x and y for which the following equations are satisfied.

(i) $(1 - i)x + (1 + i)y = 1 - 3i$

(ii) $\dfrac{(1 + i)x - 2i}{3 + i} + \dfrac{(2 - 3i)y + i}{3 - i} = i$

(iii) $\sqrt{x^2 + 3x + 8} + (x + 4)i = y(2 + i)$

(5) For what values of x and y, the numbers $-3 + ix^2y$ and $x^2 + y + 4i$ are complex conjugate of each other?

3.4 Ordered pair Representation :

In view of the representation of complex numbers, it is desirable to define a complex number $a + ib$ as an ordered pair (a, b) of real numbers a and b subject to certain operational definitions. These definitions are as follows:

(i) Equality : $(a, b) = (c, d)$ if and only if $a = c$, $b = d$

(ii) Sum : $(a, b) + (c, d) = (a + c, b + d)$

(iii) Product : $(a, b) . (c, d) = (ac - bd, ad + bc)$

$m(a, b) = (ma, mb)$

Result :

The imaginary unit i is defined as $i = (0, 1)$.

We have $(a, b) = (a, 0) + (0, b) = a(1, 0) + b(0, 1)$

and $(0, 1)\,(0, 1) = (0 - 1, 0 + 0) = (-1, 0)$.

By identifying $(1, 0)$ with 1 and $(0, 1)$ with i we see that $(a, b) = a + ib$.

Thus we associate the complex number $a + ib$ with the ordered pair (a, b). The ordered pair $(0, 0)$ corresponds to the real number 0.

Remark :

Though the set of real numbers is ordered, the set of complex numbers is not ordered. i.e., order relation does not exist in $\mathbb{C}$. Given two complex numbers z_1 and z_2 we cannot say $z_1 < z_2$ or $z_1 > z_2$. We can only say that $z_1 = z_2$ or $z_1 \neq z_2$, since the points are represented in a plane. Thus the order relations 'greater than' and 'less than' are not definied for complex numbers. i.e., the inequalities like 1` $+ i > 3 - 2i$, $i > 0$, $(3 + i) < 2$ etc. are meaningless.

3.5 Modulus of a complex number :

Let $z = a + ib$ be a complex number.

The modulus or absolute value of z denoted by $|z|$ is defined by $|z| = \sqrt{a^2 + b^2}$

From the definition, it is obvious that $|\bar{z}| = |z|$. Since $a^2 + b^2 = z\bar{z}$, $|z| = \sqrt{z\bar{z}}$ (Taking the positive square root)

Result : Let $z = x + iy$

Now, $x < \sqrt{x^2 + y^2}$, then $Re(z) < |z|$ for $y \neq 0$ … (1)

If $y = 0$ then $x = |z|$ then $Re(z) = |z|$ … (2)

Combining (1) and (2) $\boxed{Re(z) \leq |z|}$

Similarly $\boxed{Im(z) \leq |z|}$

Example 3.7 : Find the modulus of the following complex numbers:

(i) $-2+4i$ (ii) $2-3i$ (iii) $-3-2i$ (iv) $4+3i$

Solution :

(i) $|-2+4i| = \sqrt{(-2)^2+4^2} = \sqrt{20} = 2\sqrt{5}$

(ii) $|2-3i| = \sqrt{2^2+(-3)^2} = \sqrt{13}$

(iii) $|-3-2i| = \sqrt{(-3)^2+(-2)^2} = \sqrt{13}$

(iv) $|4+3i| = \sqrt{4^2+3^2} = \sqrt{25} = 5$

3.5.1 Properties :

If $z_1, z_2, \ldots z_m$ are complex numbers, then the following properties hold.

(i) The modulus of a product of two complex numbers is equal to the product of their moduli.

i.e. $|z_1 z_2| = |z_1|\ |z_2|$

Proof : $|z_1 z_2|^2 = (z_1 z_2)\overline{z_1 z_2}$ $[\because z\overline{z} = |z|^2]$

$$= (z_1 z_2)\ \overline{z_1}\ \overline{z_2}$$

$$= \left(z_1 \overline{z_1}\right)\left(z_2 \overline{z_2}\right)$$

$$= |z_1|^2\ |z_2|^2$$

Taking the positive square root on both sides, we get

$$|z_1 z_2| = |z_1|\ |z_2|$$

Note : This result can be extended to any finite number of complex numbers

i.e., $|z_1 z_2 \ldots z_n| = |z_1|\,|z_2| \ldots |z_n|$

(ii) The modulus of a quotient of two complex numbers is equal to the quotient of their moduli.

i.e., $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ where $z_2 \neq 0$.

Proof : Since $z_2 \neq 0$, $z_1 = \left(\frac{z_1}{z_2}\right) . z_2$ and so $| z_1 | = \left|\frac{z_1}{z_2}\right| \, | z_2 |$ (by the previous result)

Therefore $\frac{| z_1 |}{| z_2 |} = \left|\frac{z_1}{z_2}\right|$

Hence $\left|\frac{z_1}{z_2}\right| = \frac{| z_1 |}{| z_2 |}$

(iii) Triangle inequality :

The modules of sum of two complex numbers is always less than or equal to the sum of their moduli.

i.e., $| z_1 + z_2 | \leq | z_1 | + | z_2 |$

Proof : Let z_1 and z_2 be two complex numbers.

We know that $| z_1 + z_2 |^2 = (z_1 + z_2) \overline{(z_1 + z_2)}$ $[\because | z |^2 = z\overline{z}]$

$$= (z_1 + z_2) \left(\overline{z_1} + \overline{z_2}\right)$$

$$= z_1 \overline{z_1} + z_1 \overline{z_2} + z_2 \overline{z_1} + z_2 \overline{z_2}$$

$$= z_1 \overline{z_1} + z_2 \overline{z_2} + z_1 \overline{z_2} + \overline{z_1 \overline{z_2}}$$

$$= | z_1 |^2 + | z_2 |^2 + 2 \, Re \left(z_1 \overline{z_2}\right)$$

$$\leq | z_1 |^2 + | z_2 |^2 + 2 \, | z_1 \overline{z}_2 | \qquad [\because Re\,(z) \leq | z |]$$

$$= | z_1 |^2 + | z_2 |^2 + 2 \, | z_1 | \, | z_2 | \qquad \because | \overline{z} | = | z |$$

$$= [\, | z_1 1 + | z_2 |]^2$$

$$\therefore | z_1 + z_2 |^2 \leq [| z_1 | + | z_2 |]^2$$

Thus taking positive square root we get

$$| z_1 + z_2 | \leq | z_1 | + | z_2 |$$

Note : 1 Writing $- z_2$ for z_2 in this result

We also have $| z_1 - z_2 | \leq | z_1 | + | - z_2 | \Rightarrow | z_1 - z_2 | \leq | z_1 | + | z_2 |$

Note : 2

The above inequality can be immediately extended by induction to any finite number of complex numbers i.e., for any n complex numbers $z_1, z_2, z_3, \ldots, z_n$

$$|z_1 + z_2 + z_3 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$$

(iv) The modulus of the difference of two complex numbers is always greater than or equal to the difference of their moduli.

Proof : Let z_1 and z_2 be two complex numbers.

We know that $|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)}$ $\quad [\because |z|^2 = z\bar{z}]$

$$= (z_1 - z_2)\left(\overline{z_1} - \overline{z_2}\right)$$

$$= z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2}$$

$$= z_1\overline{z_1} + z_2\overline{z_2} - 2\,Re\left(z_1\,\overline{z_2}\right)$$

$$\geq |z_1|^2 + |z_2|^2 - 2|z_1\,\overline{z}_2| \qquad [\because Re(z) \leq |z| \quad - Re(z) \geq -|z|]$$

$$= |z_1|^2 + |z_2|^2 - 2|z_1|\,|\overline{z_2}|$$

$$= |z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2|$$

$$= [\,|z_1| - |z_2|\,]^2$$

$$\therefore |z_1 - z_2|^2 \geq [|z_1| - |z_2|]^2$$

Thus taking positive square root we get

$$|z_1 - z_2| \geq |z_1| - |z_2|$$

Example 3.8 : Find the modulus or the absolute value of $\dfrac{(1+3i)(1-2i)}{(3+4i)}$

Solution :

$$\left|\frac{(1+3i)(1-2i)}{(3+4i)}\right| = \frac{|1+3i|\;|1-2i|}{|3+4i|}$$

$$= \frac{\sqrt{1^2+3^2}\sqrt{1^2+(2)^2}}{\sqrt{3^2+4^2}} = \frac{\sqrt{10}\sqrt{5}}{\sqrt{25}}$$

$$= \frac{\sqrt{10}\sqrt{5}}{5} = \sqrt{2}$$

3.6. Geometrical Representation

3.6.1 Geometrical meaning of a Complex Number

If real scales are chosen on two mutually perpendicular axes $X'OX$ and $Y'OY$ (called the x axis and y axis respectively), We can locate any point in the plane determined by these lines, by the ordered pair of real numbers (a, b) called rectangular co-ordinates of the point.

Since every complex number $a + ib$ can be considered as an ordered pair (a, b) of real numbers, we can represent such number by a point P in the xy plane, called the complex plane. Such a representation is also known as the Argand diagram. The complex number represented by P can therefore be read as either (a, b) or $a + ib$.

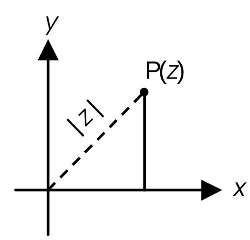


Fig. 3.1

With this representation the modulus of the complex number $z = a + ib$ represents the distance between z and the origin i.e., $| z | = \sqrt{a^2 + b^2}$. The complex number $z = a + ib$ can also be represented by the vector $\overrightarrow{OP}$ (*Fig. 3.1)* where $P = (a, b)$ and pictured as an arrow from the origin to the point (a, b). To each complex number there corresponds one and only one point in the plane, and conversely to each point in the plane there corresponds one and only one complex number. Because of this we often refer to the complex number z as the point z.

Clearly, the set of real numbers $(x, 0)$ corresponds to the x-axis called real axis. The set of all purely imaginary number $(0, y)$ corresponds to the y-axis called the imaginary number axis. The origin identifies the complex number $0 = 0 + i0$.

3.6.2 Polar form of a Complex Number :

Let (r, θ) be the polar co-ordinates of the point

$P = P(x, y)$ in the complex plane corresponding to the complex number

$z = x + iy$.

Then we get from the figure (*Fig. 3.2)*,

$\cos\theta = \dfrac{OM}{OP} = \dfrac{x}{r}$ and $\sin\theta = \dfrac{PM}{OP} = \dfrac{y}{r}$

$x = r\cos\theta$; $y = r\sin\theta$

where $r = \sqrt{x^2 + y^2} = | x + iy |$ is called the modulus or the absolute value of $z = x + iy$ denoted by mod z or $| z |$

(i.e., the distance from the origin to the point z)

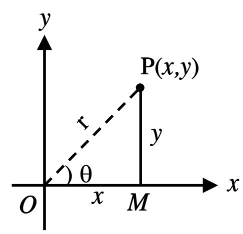


Fig. 3.2

$\tan\theta = \dfrac{y}{x}$, $\therefore\ \theta = \tan^{-1}\dfrac{y}{x}$ is called the amplitude or argument of $z = x + iy$ denoted by amp z or arg z and is measured as the angle which the line OP makes with the positive x-axis (in the anti clockwise sense).

Thus $z = x + iy = r(\cos\theta + i\sin\theta)$ is called the polar form or the modulus amplitude form of the complex number. It is sometimes convenient to use the abbreviation cis θ for $\cos\theta + i\sin\theta$.

$\theta = \tan^{-1}\dfrac{y}{x}$ is applicable only for first quadrant numbers i.e., x & y are positive.

3.6.3 Principal Value :

The argument of z is not unique. Any two distinct arguments of z differ from each other by an integral multiple of 2π. In order to specify a unique value of arg z, we may restrict its value to some interval of length 2π. For this purpose, we introduce the concept of "principal value" for arg z as follows :

For an arbitrary $z \neq 0$ the principal value of arg z is defined to be the unique value of z that satisfies $-\pi < \arg z \leq \pi$.

Note : For $z = 0$, the argument is indeterminate.

Results :

(1) For any two complex numbers z_1 and z_2

(i) $| z_1 z_2 | = | z_1 | . | z_2 |$ (ii) $\arg (z_1 . z_2) = \arg z_1 + \arg z_2$

Proof :

Let $z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$

then $| z_1 | = r_1$, $\arg z_1 = \theta_1$; $| z_2 | = r_2$, $\arg z_2 = \theta_2$

$$\begin{aligned} z_1.z_2 &= r_1r_2 (\cos\theta_1 + i\sin\theta_1) . (\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2 [(\cos\theta_1 . \cos\theta_2 - \sin\theta_1 . \sin\theta_2) + \\ &\qquad i (\sin\theta_1 . \cos\theta_2 + \cos\theta_1 . \sin\theta_2)] \end{aligned}$$

$$= r_1r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$\therefore \ |z_1 z_2| = r_1r_2 = |z_1| . |z_2| \text{ and}$$

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

Note :

This result can be extended to any finite number of complex numbers i.e.,

(i) $|z_1 . z_2 \ \dots \ z_n| = |z_1| . |z_2| \ \dots \ |z_n|$

(ii) $\arg(z_1 z_2 \dots z_n) = \arg z_1 + \arg z_2 + \dots + \arg z_n$

(2) For any two complex numbers z_1 and z_2

(i) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $(z_2 \neq 0)$ (ii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Proof :

Let $z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i \sin\theta_2)$

Then $|z_1| = r_1$, $\arg z_1 = \theta_1$ and $|z_2| = r_2$, $\arg z_2 = \theta_2$

Now
$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i \sin\theta_1)}{r_2(\cos\theta_2 + i \sin\theta_2)}$$

$$= \frac{r_1}{r_2} \frac{(\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 - i \sin\theta_2)}{(\cos\theta_2 + i \sin\theta_2)(\cos\theta_2 - i \sin\theta_2)}$$

$$= \frac{r_1}{r_2} \frac{(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

$$\therefore \ \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \text{ and}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$$

Exponential form of a Complex Number :

The symbol $e^{i\theta}$ or $\exp(i\theta)$ (called exponential of $i\theta$) is defined by

$e^{i\theta} = \cos\theta + i \sin\theta$

This relation is known as **Euler's formula.**

If $z \neq 0$ then $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. This is called the exponential form of the complex number z. By straight forward multiplication of $e^{i\theta_1} = (\cos\theta_1 + i\sin\theta_1)$ and $e^{i\theta_2} = \cos\theta_2 + i\sin\theta_2$

we have $e^{i\theta_1}.e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$

Remarks :

(1) If $\theta_1 = \theta$ and $\theta_2 = -\theta$ in the above definition

then we have $e^{i\theta} . e^{i(-\theta)} = e^{i(\theta - \theta)} = e^{i0} = 1$

$\Rightarrow e^{i(-\theta)} = \dfrac{1}{e^{i\theta}}$. Thus writing $e^{i(-\theta)}$ as $e^{-i\theta}$

we observe that $e^{-i\theta} = \dfrac{1}{e^{i\theta}}$

(2) If $\theta_1 = \theta_2 = \theta$ then $(e^{i\theta})^2 = e^{2i\theta}$. By mathematical induction it can be shown that $(e^{i\theta})^n = e^{in\theta}$ where $n = 0, 1, 2 \ldots$

(3) Since $\overline{e^{i\theta}} = e^{-i\theta}$, we see that, if $z = re^{i\theta}$ then $\overline{z} = re^{-i\theta}$

(4) Two complex numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ are equal if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2n\pi$, $n \in Z$ (the set of all integers)

General rule for determining the argument θ.

Let $z = x + iy$

where $x, y \in R$

$\theta = \pi - \alpha$	$\theta = \alpha$
$\theta = -\pi + \alpha$	$\theta = -\alpha$

Take $\alpha = \tan^{-1}\dfrac{|y|}{|x|}$

(i)	Both $\cos\theta$ and $\sin\theta$ are $+ve$. z lies in the first quadrant.	$\theta = \alpha$
(ii)	$\sin\theta$ is $+ve$, $\cos\theta$ is $-ve$ z lies in the second quadrant.	$\theta = \pi - \alpha$
(iii)	Both $\sin\theta$ and $\cos\theta$ are $-ve$ z lies in the third quadrant.	$\theta = -\pi + \alpha$
(iv)	$\sin\theta$ is $-ve$ and $\cos\theta$ is $+ve$, z lies in the fourth quadrant.	$\theta = -\alpha$

Example 3.9 : Find the modulus and argument of the following complex numbers :

(i) $-\sqrt{2} + i\sqrt{2}$ (ii) $1 + i\sqrt{3}$ (iii) $-1 - i\sqrt{3}$

Solution :

(i) Let $-\sqrt{2} + i\sqrt{2} = r(\cos\theta + i\sin\theta)$

Equating the real and imaginary parts separately

$$\begin{array}{r|l} r\cos\theta = -\sqrt{2} & r\sin\theta = \sqrt{2} \\ r^2\cos^2\theta = 2 & r^2\sin^2\theta = 2 \end{array}$$

$$r^2(\cos^2\theta + \sin^2\theta) = 4$$

$$r = \sqrt{4} = 2$$

$$\left.\begin{aligned} \cos\theta &= \frac{-\sqrt{2}}{2} = \frac{-1}{\sqrt{2}} \\ \sin\theta &= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \end{aligned}\right\} \Rightarrow \theta \text{ in the } 2^{nd} \text{ quadrant}$$

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

modulus $r = 2$, argument $\theta = \frac{3\pi}{4}$

Hence $-\sqrt{2} + i\sqrt{2} = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(ii) Let $1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

Equating the real and imaginary parts separately

$$\begin{array}{r|l} r\cos\theta = 1 & r\sin\theta = \sqrt{3} \\ r^2\cos^2\theta = 1 & r^2\sin^2\theta = 3 \end{array}$$

$$r^2(\cos^2\theta + \sin^2\theta) = 4 \Rightarrow r = 2$$

$$\left.\begin{aligned} \cos\theta &= \frac{1}{2} \\ \sin\theta &= \frac{\sqrt{3}}{2} \end{aligned}\right\} \Rightarrow \theta \text{ in the } 1^{st} \text{ quadrant}$$

$$\theta = \frac{\pi}{3} \quad \left[\because \theta = \tan^{-1}\frac{y}{x}\right]$$

modulus $r = 2$, argument $\theta = \frac{\pi}{3}$

Hence $1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(iii) Let $\qquad -1 - i\sqrt{3} = r(\cos\theta + i\sin\theta)$

Equating the real and imaginary parts separately

$$\begin{array}{r|l} r\cos\theta = -1 & r\sin\theta = -\sqrt{3} \\ r^2\cos^2\theta = 1 & r^2\sin^2\theta = 3 \end{array}$$

$$r^2(\cos^2\theta + \sin^2\theta) = 4$$

$$\Rightarrow \quad r = 2$$

$$\left.\begin{array}{l} \cos\theta = \dfrac{-1}{2} \\ \sin\theta = \dfrac{-\sqrt{3}}{2} \end{array}\right\} \Rightarrow \theta \text{ in the 3}^{\text{rd}} \text{ quadrant}$$

$$\theta = -\pi + \frac{\pi}{3} = \frac{-2\pi}{3}$$

modulus $r = 2$, argument $\theta = \frac{-2\pi}{3}$

Hence $-1 - i\sqrt{3} = 2\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right] = 2\left[\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right]$

Example 3.10 : If $(a_1 + ib_1)(a_2 + ib_2) \dots (a_n + ib_n) = A + iB$,

prove that (i) $(a_1^2 + b_1^2)(a_2^2 + b_2^2) \dots (a_n^2 + b_n^2) = A^2 + B^2$

(ii) $\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \dots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = k\pi + \tan^{-1}\left(\frac{B}{A}\right), k \in Z$

Solution :

Given $(a_1 + ib_1)(a_2 + ib_2) \dots (a_n + ib_n) = A + iB$

$|(a_1 + ib_1)(a_2 + ib_2) \dots (a_n + ib_n)| = |A + iB|$

$|(a_1 + ib_1)|\ |(a_2 + ib_2)| \dots |(a_n + ib_n)| = |A + iB|$

$\sqrt{a_1^2 + b_1^2}\ \sqrt{a_2^2 + b_2^2} \dots \sqrt{a_n^2 + b_n^2} = \sqrt{A^2 + B^2}$

Squaring on both sides

$(a_1^2 + b_1^2)(a_2^2 + b_2^2) \dots (a_n^2 + b_n^2) = A^2 + B^2$

Also

$\arg[(a_1 + ib_1)(a_2 + ib_2) \dots (a_n + ib_n)] = \arg(A + iB)$

$$\arg(a_1 + ib_1) + \arg(a_2 + ib_2) \ldots + \arg(a_n + ib_n) = \arg(A + iB) \qquad \ldots (1)$$

Now $\arg\ (a_i + ib_i) = \tan^{-1}\left(\dfrac{b_i}{a_i}\right)$

Hence (1) becomes

$$\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$$

By taking the general value,

$$\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = k\pi + \tan^{-1}\left(\frac{B}{A}\right)$$

Example 3.11 :

P represents the variable complex number z, find the locus of P if

(i) $Re\left(\dfrac{z+1}{z+i}\right) = 1$ (ii) $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{3}$

Solution :

Let $z = x + iy$ then

(i)

$$\frac{z+1}{z+i} = \frac{x+iy+1}{x+iy+i} = \frac{(x+1)+iy}{x+i(y+1)}$$

$$= \frac{[(x+1)+iy]}{x+i(y+1)} \times \frac{[x-i(y+1)]}{[x-i(y+1)]}$$

$$= \frac{x(x+1)+y(y+1)+i(yx-xy-x-y-1)}{x^2+(y+1)^2}$$

$$= \frac{x(x+1)+y(y+1)+i(-x-y-1)}{x^2+(y+1)^2}$$

Given that $Re\left(\dfrac{z+1}{z+i}\right) = 1$

$$\therefore \frac{x(x+1)+y(y+1)}{x^2+(y+1)^2} = 1$$

$$\Rightarrow \quad x^2+y^2+x+y = x^2+y^2+2y+1$$

$\Rightarrow$ $x - y = 1$ which is a straight line.

$\therefore$ The locus of P is a straight line.

(ii) $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{3}$

$$\therefore \arg(z-1) - \arg(z+1) = \frac{\pi}{3}$$

$$\arg(x+iy-1) - \arg(x+iy+1) = \frac{\pi}{3}$$

$$\arg[(x-1)+iy] - \arg[(x+1)+iy] = \frac{\pi}{3}$$

$$\tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} = \frac{\pi}{3}$$

$$\tan^{-1}\left[\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1+\left(\frac{y}{x-1}\right)\left(\frac{y}{x+1}\right)}\right] = \frac{\pi}{3}$$

$$\Rightarrow \qquad \frac{2y}{x^2-1+y^2} = \tan\frac{\pi}{3}$$

$$\frac{2y}{x^2+y^2-1} = \sqrt{3}$$

$$2y = \sqrt{3}x^2 + \sqrt{3}y^2 - \sqrt{3}$$

$$\therefore \sqrt{3}x^2 + \sqrt{3}y^2 - 2y - \sqrt{3} = 0 \text{ is the required locus.}$$

Result : (without proof) :

If $|z - z_1| = |z - z_2|$ then the locus of z is the perpendicular bisector of the line joining the two points z_1 and z_2.

3.6.4 Geometrical meaning of conjugate of a complex number :

Let $z = x + iy$ be a complex number represented by P in the Argand diagram. Then we know that its conjugate $\overline{z}$ is given by $\overline{z} = x - iy$.

i.e., $z = (x, y) \Rightarrow \overline{z} = (x, -y)$

$\therefore$ If Q represents the conjugate $\overline{z}$, then conjugate of z is obviously the mirror image of the complex point z on the real axis (*Fig. 3.3*). This clearly indicates that

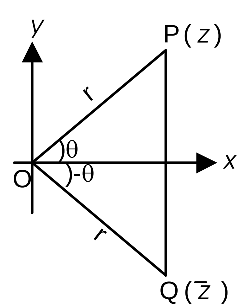


Fig. 3.3

$z = \overline{z} \Leftrightarrow z$ is purely a real number. Also $\overline{\overline{z}} = z$.

In polar coordinates let

$$z = r(\cos\theta + i\sin\theta) \text{ then}$$
$$\bar{z} = r(\cos(-\theta) + i\sin(-\theta)) \quad \text{so if}$$
$$z = (r, \theta) \text{ then } \bar{z} = (r, -\theta)$$

Thus the moduli of both z and $\bar{z}$ are same i.e., $r = \sqrt{x^2 + y^2}$ But the amplitude of z is θ and that of $\bar{z}$ is $-\theta$. Hence $|\bar{z}| = |z|$ and amp $\bar{z} = -\text{amp } z$.

Fig. 3.4 gives the simple geometric relations among the complex number z, its negation $-z$ and its conjugate $\bar{z}$

$-z = (-x, -y)$. It is the point symmetrical to z about the origin.

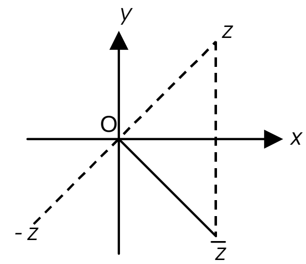


Fig. 3.4

3.6.5 Geometrical representation of sum of two complex numbers

Let A and B represent the two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the Argand diagram. Complete the parallelogram $OACB$. Then C represents the complex number $z_1 + z_2$.

Proof :

Since $OACB$ is a parallelogram, diagonals OC and AB bisect at M.

$\therefore$ From *Fig. 3.5*, the midpoint M of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad \ldots (1)$$

If C is (h, k) then midpoint M of OC is also given by $\left(\frac{0 + h}{2}, \frac{0 + k}{2}\right)$

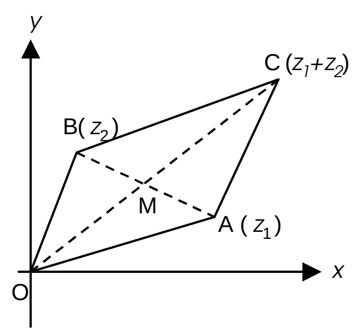


Fig. 3.5

i.e., M is $\left(\frac{h}{2}, \frac{k}{2}\right) \qquad \ldots (2)$

$\therefore$ From (1) and (2)

$$\frac{h}{2} = \frac{x_1 + x_2}{2} \; ; \; \frac{k}{2} = \frac{y_1 + y_2}{2}$$

$$\Rightarrow h = x_1 + x_2 \; ; \; k = y_1 + y_2$$

$\therefore$ C is $(x_1 + x_2,\ \ y_1 + y_2)$

Hence C represents the complex number, $z_1 + z_2$

Note : $OA = |z_1|$, $OB = |z_2|$ and $OC = |z_1 + z_2|$

In any triangle the sum of two sides is greater than the third side.

$\therefore$ from ΔOAC we have $OA + AC > OC$ or $OC < OA + AC$

$$|z_1 + z_2| < |z_1| + |z_2| \qquad \dots (1)$$

Further, if the points are collinear

$$|z_1 + z_2| = |z_1| + |z_2| \qquad \dots (2)$$

Combining (1) and (2) we get

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

This is the reason why this inequality is called the triangle inequality.

3.6.6 Vector interpretation of complex numbers :

Let A and B represent the two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the Argand diagram. Complete the parallelogram $OACB$. Then C represents the complex number $z_1 + z_2$.

A complex number $z = x + iy$ can be considered as a vector OP whose initial point is the origin O and whose terminal point is $P = P(x, y)$. We sometimes call $OP = x + iy$ the position vector of P. Two vectors having the same length or magnitude and the same direction but different initial points such as OP and AB are considered equal. *(Fig. 3.6)* $\therefore$ $OP = AB = x + iy$

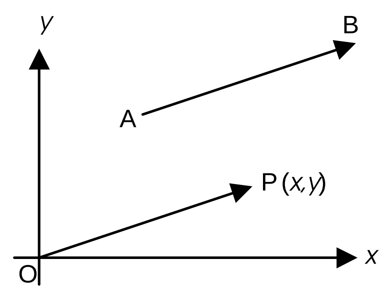


Fig. 3.6

Based on the above interpretation of complex numbers as vectors, addition of complex numbers corresponds to the "parallelogram law" for addition of vectors. Thus to add the complex numbers z_1 and z_2, we complete the parallelogram $OABC$ whose sides OA and OC correspond to z_1 and z_2. The diagonal OB of this parallelogram corresponds to $z_1 + z_2$. *(Fig. 3.7)*

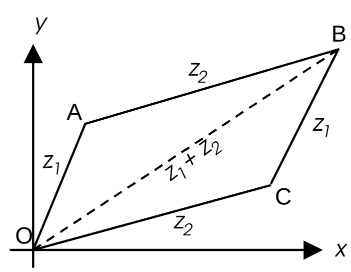


Fig. 3.7

3.6.7. Geometrical representation of difference of two complex numbers

Let A and B represent the two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ respectively in the Argand diagram. Produce BO to B' such that $OB' = OB$. Then B' represents the complex number $-z_2$. Complete the parallelogram $OADB'$. Then D represents the sum of the complex numbers z_1 and $-z_2$ or $z_1 - z_2$ i.e., D represents the difference of complex numbers z_1 and z_2. (*Fig. 3.8*)

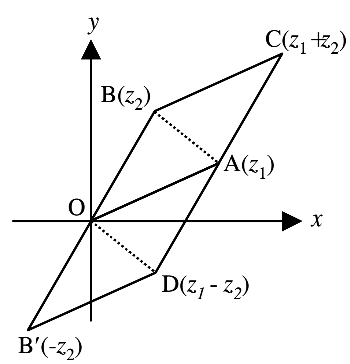


Fig. 3.8

Result :

From the diagram $OD = AB$. But $OD = |\ z_1 - z_2\ |$. AB is the distance between z_1 and z_2. Thus, distance between two complex numbers z_1 and z_2 is $|\ z_1 - z_2\ |$.

Note :

Complete the parallelogram $OACB$. Then C represents the complex number $z_1 + z_2$.

3.6.8 Geometrical representation of product of two complex numbers:

Let A and B represent the two complex numbers z_1 and z_2 respectively in the Argand diagram. Let $z_1 = r_1$ ($\cos\theta_1 + i\sin\theta_1$) and $z_2 = r_2$ ($\cos\theta_2 + i\sin\theta_2$)

Then $OA = r_1$, $\underline{|XOA} = \theta_1$

$OB = r_2$, $\underline{|XOB} = \theta_2$.

Take a point L on OX such that $OL = 1$ unit. Draw the triangle OBC directly similar to $\Delta\ OLA$. (*Fig. 3.9*)

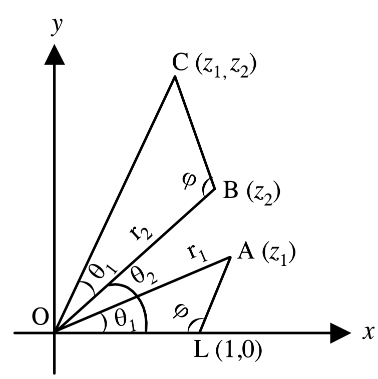


Fig. 3.9

then $\dfrac{OB}{OL} = \dfrac{OC}{OA}$ i.e., $\dfrac{r_2}{1} = \dfrac{OC}{r_1}$

$\therefore OC = r_1 r_2$

Also $\underline{|XOC} = \underline{|XOB} + \underline{|BOC}$

$= \underline{|XOB} + \underline{|XOA}$

$= \theta_2 + \theta_1$ or $\theta_1 + \theta_2$ $\left(\because \underline{|XOA} = \underline{|BOC}\right)$

$\therefore$ The point C represents the complex number $z_1 z_2$ with polar coordinates $(r_1 r_2, \ \theta_1 + \theta_2)$

Note :

If P represents the complex number

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

then the effect of multiplication by $(\cos\alpha + i\sin\alpha) = e^{i\alpha}$ is the rotation of $P(z)$ counter clockwise about O through an angle α.

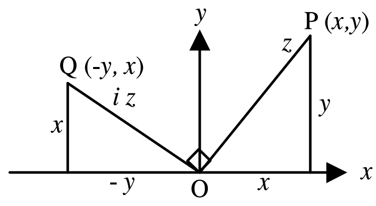


Fig. 3.10

In particular, since $i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = e^{\frac{i\pi}{2}}$, the effect of multiplication of any complex numbers $P(z)$ by i is the rotation of P counter clockwise about the origin through an angle 90°. *(Fig. 3.10)*

3.6.9 Geometrical representation of the quotient of two complex numbers

Let A and B represent two complex numbers z_1 and z_2 in the Argand diagram.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$; $(z_2 \neq 0)$

Then $OA = r_1$, $\underline{|XOA} = \theta_1$

$OB = r_2$, $\underline{|XOB} = \theta_2$.

Take a point L on OX such that $OL = 1$ unit. Draw the triangle OAC directly similar to $\Delta\, OBL$. *(Fig. 3.11)*

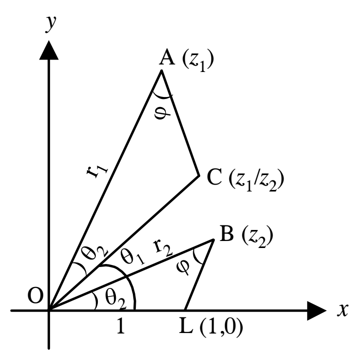


Fig. 3.11

then $\qquad \dfrac{OA}{OB} = \dfrac{OC}{OL} \qquad$ i.e., $\dfrac{r_1}{r_2} = \dfrac{OC}{1}$

$$\therefore\ OC = \frac{r_1}{r_2}$$

$$\underline{|XOC} = \underline{|XOA} - \underline{|COA} = \theta_1 - \theta_2$$

$\therefore$ C is the point whose polar coordinates are $\left(\dfrac{r_1}{r_2},\ \theta_1 - \theta_2\right)$. Hence C represents the complex number $\dfrac{z_1}{z_2}$

Example 3.12 : Graphically prove that $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$

Solution :

By triangle inequality in ΔOAB,

$|z_1 + z_2| \le |z_1| + |z_2| \qquad \ldots(1)$

By triangle inequality in ΔOBP,

$|z_1 + z_2 + z_3| \le |z_1 + z_2| + |z_3|$

$\le |z_1| + |z_2| + |z_3|$ from (1)

$\therefore |z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$

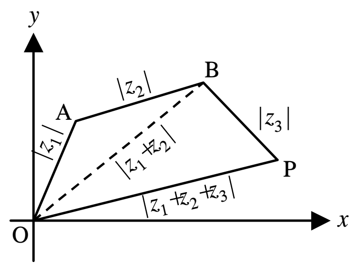


Fig. 3.12

Example 3.13 : Prove that the complex numbers $3 + 3i$, $-3 - 3i$, $-3\sqrt{3} + 3\sqrt{3}\,i$ are the vertices of an equilateral triangle in the complex plane.

Solution :

Let A, B and C represent the complex numbers $(3 + 3i)$, $(-3 - 3i)$ and $(-3\sqrt{3} + i3\sqrt{3})$ in the Argand diagram.

$AB = |(3 + 3i) - (-3 - 3i)|$

$\quad = |6 + 6i| = \sqrt{72}$

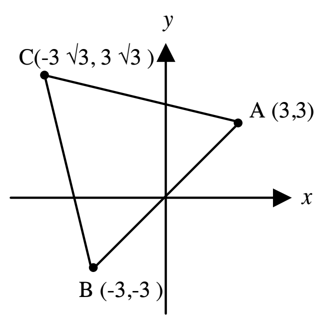


Fig. 3.13

$$BC = \left|(-3 - 3i) - \left(-3\sqrt{3} + 3\sqrt{3}i\right)\right|$$

$$= \left|\left(-3 + 3\sqrt{3}\right) + i\left(-3 - 3\sqrt{3}\right)\right| = \sqrt{72}$$

$$CA = \left|\left(-3\sqrt{3} + 3\sqrt{3}i\right) - (3 + 3i)\right|$$
$$= \left|\left(-3\sqrt{3} - 3\right) + i\left(3\sqrt{3} - 3\right)\right| = \sqrt{72}$$
$$AB = BC = CA$$

$\therefore$ $\Delta\, ABC$ is an equilateral triangle.

Example 3.14 : Prove that the points representing the complex numbers $2i$, $1 + i$, $4 + 4i$ and $3 + 5i$ on the Argand plane are the vertices of a rectangle.

Solution :

Let A, B, C and D represent the complex numbers $2i$, $(1 + i)$, $(4 + 4i)$ and $(3 + 5i)$ in the Argand diagram respectively.

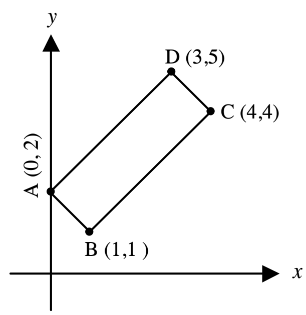


Fig. 3.14

$$AB = |\,2i - (1 + i)\,|$$
$$= |-1 + i\,| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$
$$BC = |\,(1 + i) - (4 + 4i)\,|$$
$$= |-3 - 3i\,|$$
$$= \sqrt{(-3)^2 + (-3)^2} = \sqrt{9 + 9} = \sqrt{18}$$
$$CD = |\,(4 + 4i) - (3 + 5i)\,|$$
$$= |\,1 - i\,| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$DA = |\,(3 + 5i) - 2i\,| = |\,3 + 3i\,| = \sqrt{3^2 + 3^2} = \sqrt{18}$$

$\therefore$ $AB = CD$ and $BC = DA$

$$AC = |\,(0 + 2i) - (4 - 4i)\,|$$
$$= |-4 - 2i\,|$$
$$= \sqrt{(-4)^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20}$$
$$AB^2 + BC^2 = 2 + 18 = 20$$
$$AC^2 = 20$$

Hence $\quad AB^2 + BC^2 = AC^2$

As pairs of opposite sides are equal and $\lfloor B = 90°$, $\therefore$ $ABCD$ is a rectangle.

Example 3.15 : Show that the points representing the complex numbers $7 + 9i$, $-3 + 7i$, $3 + 3i$ form a right angled triangle on the Argand diagram.

Solution :

Let A, B and C represent the complex numbers

$7 + 9i$, $-3 + 7i$ and $3 + 3i$ in the Argand diagram respectively.

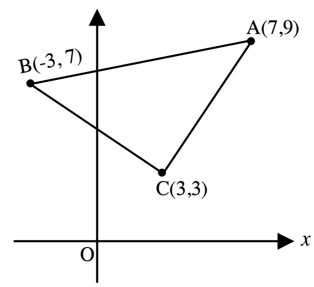


Fig. 3.15

$$AB = |(7 + 9i) - (-3 + 7i)|$$
$$= |10 + 2i| = \sqrt{10^2 + 2^2} = \sqrt{104}$$
$$BC = |(-3 + 7i) - (3 + 3i)|$$
$$= |-6 + 4i|$$
$$= \sqrt{(-6)^2 + 4^2} = \sqrt{36 + 16} = \sqrt{52}$$
$$CA = |(3 + 3i) - (7 + 9i)|$$
$$= |-4 - 6i|$$
$$= \sqrt{(-4)^2 + (-6)^2} = \sqrt{16 + 36} = \sqrt{52}$$
$$\Rightarrow AB^2 = BC^2 + CA^2$$
$$\Rightarrow \underline{|BCA} = 90°$$

Hence ΔABC is a right angled isosceles triangle.

Example 3.16 : Find the square root of $(-7 + 24i)$

Solution :

Let $\sqrt{-7 + 24i} = x + iy$

On squaring,

$$-7 + 24i = (x^2 - y^2) + 2ixy$$

Equating the real and imaginary parts

$$x^2 - y^2 = -7 \text{ and } 2xy = 24$$
$$x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + 4x^2y^2}$$
$$= \sqrt{(-7)^2 + (24)^2} = 25$$

Solving, $\quad x^2 - y^2 = -7$ and $x^2 + y^2 = 25$

we get $\quad x^2 = 9$ and $y^2 = 16$

$$\therefore x = \pm 3 \text{ and } y = \pm 4$$

Since xy is positive, x and y have the same sign.

$\therefore$ $(x = 3, y = 4)$ or $(x = -3, y = -4)$

$$\therefore \sqrt{-7 + 24i} = (3 + 4i) \text{ or } (-3 - 4i)$$

EXERCISE 3.2

(1) If $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = x + iy$
show that $2.5.10 \ldots (1 + n^2) = x^2 + y^2$

(2) Find the square root of $(-8 - 6i)$

(3) If $z^2 = (0, 1)$ find z.

(4) Prove that the triangle formed by the points representing the complex numbers $(10 + 8i)$, $(-2 + 4i)$ and $(-11 + 31i)$ on the Argand plane is right angled.

(5) Prove that the points representing the complex numbers $(7 + 5i)$, $(5 + 2i)$, $(4 + 7i)$ and $(2 + 4i)$ form a parallelogram. (Plot the points and use midpoint formula).

(6) Express the following complex numbers in polar form.
(i) $2 + 2\sqrt{3}\, i$ (ii) $-1 + i\sqrt{3}$ (iii) $-1 - i$ (iv) $1 - i$

(7) If $\arg (z - 1) = \frac{\pi}{6}$ and $\arg (z + 1) = 2\frac{\pi}{3}$ then prove that $|z| = 1$

(8) P represents the variable complex number z. Find the locus of P, if
(i) $\text{Im}\left[\frac{2z + 1}{iz + 1}\right] = -2$ (ii) $|z - 5i| = |z + 5i|$
(iii) $Re\left(\frac{z - 1}{z + i}\right) = 1$ (iv) $|2z - 3| = 2$ (v) $\arg\left(\frac{z - 1}{z + 3}\right) = \frac{\pi}{2}$

3.7 Solutions of polynomial equations :

Consider the quadratic equation $x^2 - 4x + 7 = 0$

Its discriminant is $b^2 - 4ac = (-4)^2 - (4)(7)(1)$
$= 16 - 28 = -12$ which is negative.

$\therefore$ The roots of this quadratic equation are not real. The roots are given by

$$\frac{-(-4) \pm \sqrt{-12}}{2} = \frac{4 \pm \sqrt{-12}}{2} = 2 \pm i\sqrt{3}$$

Thus we see that the roots $2 + i\sqrt{3}$ and $2 - i\sqrt{3}$ are conjugate to each other.

We shall now consider the cube roots of unity.

Let x be the cube root of unity then

$$x = (1)^{\frac{1}{3}}$$

$$\Rightarrow x^3 = 1$$

$$\Rightarrow (x - 1)(x^2 + x + 1) = 0$$

$$\therefore\ x - 1 = 0 \quad ; \quad x^2 + x + 1 = 0$$

Hence $x = 1$ and $$x = \frac{-1 \pm \sqrt{1 - (4)(1)(11)}}{2}$$

$\therefore$ Cube roots of unity are 1, $\dfrac{-1 + \sqrt{3}\, i}{2}$, $\dfrac{-1 - \sqrt{3} i}{2}$

Here again the two complex roots $\dfrac{-1 + \sqrt{3}\, i}{2}$ and $\dfrac{-1 - \sqrt{3} i}{2}$ are conjugate to each other.

From the above two examples one can infer that in an equation with real coefficients, imaginary roots occur in pairs (i.e., one root is the conjugate of the other). This paved the way for the following theorem.

Theorem :

For any polynomial equation $P(x) = 0$ with real coefficients, imaginary (complex) roots occur in conjugate pairs.

Proof :

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$ be a polynomial equation of degree n with real coefficients.

Let z be a root of $P(x) = 0$. We show that $\bar{z}$ is also a root of $P(x) = 0$.

Since z is a root of $P(x) = 0$

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0 \qquad \ldots (1)$$

Taking the conjugate on both sides

$$\overline{P(z)} = \overline{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} = \overline{0}$$

Using the idea that the conjugate of the sum of the complex numbers is equal to the sum of their conjugates,

$$\overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \ldots + \overline{a_1 z} + \overline{a_0} = 0$$

i.e. $\overline{a_n}\ \overline{z^n} + \overline{a_{n-1}}\ \overline{z^{n-1}} + \ldots + \overline{a_1}\ \overline{z} + \overline{a_0} = 0$

Since $\overline{z^n} = \left(\overline{z}\right)^n$ and

$a_0, a_1, a_2 \ldots a_n$ are real numbers, each of them is its own conjugate and hence we get

$$a_n \overline{z^n} + a_{n-1} \overline{z^{n-1}} + \ldots + a_1 \overline{z} + a_0 = 0$$

which is same as $P(\overline{z}) = 0$

This means $\overline{z}$ is also a root of $P(x) = 0$.

Hence the result.

Example 3.17 : Solve the equation $x^4 - 4x^2 + 8x + 35 = 0$, if one of its roots is $2 + \sqrt{3}\, i$

Solution :

Since $2 + i\sqrt{3}$ is a root, $2 - i\sqrt{3}$ is also a root.
Sum of the roots = 4

Product of the roots $\left(2 + i\sqrt{3}\right)\left(2 - i\sqrt{3}\right) = 4 + 3 = 7$

$\therefore$ The corresponding factor is $x^2 - 4x + 7$

$\therefore\ x^4 - 4x^2 + 8x + 35 = (x^2 - 4x + 7)(x^2 + px + 5)$

Equating x term, we get $8 = 7p - 20 \Rightarrow p = 4$

$\therefore$ Other factor is $(x^2 + 4x + 5)$

$\therefore\ x^2 + 4x + 5 = 0 \ \Rightarrow\ x = -2 \pm i$

Thus the roots are $2 \pm i\sqrt{3}$ and $-2 \pm i$

EXERCISE 3.3

(1) Solve the equation $x^4 - 8x^3 + 24x^2 - 32x + 20 = 0$ if $3 + i$ is a root.

(2) Solve the equation $x^4 - 4x^3 + 11x^2 - 14x + 10 = 0$ if one root is $1 + 2i$

(3) Solve : $6x^4 - 25x^3 + 32x^2 + 3x - 10 = 0$ given that one of the roots is $2 - i$

3.8 De Moivre's Theorem and its applications :

Theorem :

For any rational number n, $\cos n\theta + i \sin n\theta$ is the value or one of the values of $(\cos\theta + i\sin\theta)^n$

Proof :

Case I : Let n be a positive integer.

By simple multiplication we have

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

Similarly $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$$= \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$$

By extending it to the product of n complex number we have

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots (\cos\theta_n + i\sin\theta_n)$

$$= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$$

In this expression put $\theta_1 = \theta_2 \dots = \theta_n = \theta$, *then* we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Case II : Let n be a negative integer and equal to $-m$; (m is a +ve integer)

$$\begin{aligned}
\therefore\ (\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m} \\
&= \frac{1}{(\cos\theta + i\sin\theta)^m} \\
&= \frac{1}{\cos m\theta + i\sin m\theta} \qquad \text{by case I} \\
&= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)} \\
&= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \\
&= \cos m\theta - i\sin m\theta \\
&= \cos(-m)\theta + i\sin(-m)\theta \\
&= \cos n\theta + i\sin n\theta
\end{aligned}$$

Case III : Let n be a fraction and equal to $\frac{p}{q}$, where q is a positive integer and p is any integer.

Consider $\left[\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right]^q = \cos\theta + i\sin\theta$

Therefore $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is such that its qth power is $\cos\theta + i\sin\theta$.

Hence $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$.

Raise each of these quantities to the pth power.

$\therefore \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^P$ is one of the values of $\left[(\cos\theta + i\sin\theta)^{\frac{1}{q}}\right]^p$

i.e., $\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$

ie., $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Note : De Moivre's theorem holds good for irrational values also but the proof is beyond the scope of this book.

Properties :

(i) $$(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i\sin n\theta$$

(ii) $$(\cos\theta - i\sin\theta)^{n} = \{\cos(-\theta) + i\sin(-\theta)\}^{n}$$
$$= \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i\sin n\theta$$

(iii) $$(\sin\theta + i\cos\theta)^{n} = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^{n}$$
$$= \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$$

Example 3.18 : Simplify : $\dfrac{(\cos 2\theta + i\sin 2\theta)^3\,(\cos 3\theta - i\sin 3\theta)^{-3}}{(\cos 4\theta + i\sin 4\theta)^{-6}\,(\cos\theta + i\sin\theta)^8}$

Solution :

$$\text{The given expression} = \frac{(\cos 2\theta + i\sin 2\theta)^3\,(\cos 3\theta - i\sin 3\theta)^{-3}}{(\cos 4\theta + i\sin 4\theta)^{-6}\,(\cos\theta + i\sin\theta)^8}$$
$$= \frac{(e^{i2\theta})^3 . (e^{-i3\theta})^{-3}}{(e^{i4\theta})^{-6}\,(e^{i\theta})^8} = \frac{e^{i6\theta}\;\; e^{i9\theta}}{e^{-i24\theta} . e^{i8\theta}}$$
$$= e^{i15\theta} . e^{i16\theta}$$
$$= e^{i31\theta} = \cos 31\theta + i\sin 31\theta$$

Alternative method :

$$\text{The given expression} = \frac{(\cos\theta + i\sin\theta)^6\,(\cos\theta + i\sin\theta)^9}{(\cos\theta + i\sin\theta)^{-24}\,(\cos\theta + i\sin\theta)^8}$$
$$= (\cos\theta + i\sin\theta)^{6 + 9 + 24 - 8}$$
$$= (\cos\theta + i\sin\theta)^{31}$$
$$= \cos 31\theta + i\sin 31\theta$$

Example 3.19 : Simplify : $\dfrac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5}$

Solution :

$$\frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} = \frac{(\cos\theta + i\sin\theta)^4}{\left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^5}$$

$$= \cos\left[4\theta - 5\left(\frac{\pi}{2} - \theta\right)\right] + i\sin\left[4\theta - 5\left(\frac{\pi}{2} - \theta\right)\right]$$

$$= \cos\left(9\theta - \frac{5\pi}{2}\right) + i\sin\left(9\theta - \frac{5\pi}{2}\right)$$

$$= \cos\left(\frac{5\pi}{2} - 9\theta\right) - i\sin\left(\frac{5\pi}{2} - 9\theta\right)$$

$$= \cos\left(\frac{\pi}{2} - 9\theta\right) - i\sin\left(\frac{\pi}{2} - 9\theta\right)$$

$$= \sin 9\theta - i\cos 9\theta$$

Alternative method :

$$\frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} = \frac{1}{i^5}\left[\frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^5}\right]$$

$$= -i(\cos 4\theta + i\sin 4\theta)(\cos 5\theta + i\sin 5\theta)$$

$$= -i[\cos 9\theta + i\sin 9\theta]$$

$$= \sin 9\theta - i\cos 9\theta$$

Result : $|z| = 1 \Leftrightarrow \overline{z} = \frac{1}{z}$

Example 3.20 : If n is a positive integer, prove that

$$\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$$

Solution :

Let $z = \sin\theta + i\cos\theta$

$\therefore \frac{1}{z} = \sin\theta - i\cos\theta$

$$\therefore \left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n = \left(\frac{1 + z}{1 + \frac{1}{z}}\right)^n = z^n = (\sin\theta + i\cos\theta)^n$$

$$= \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n$$

$$= \left[\cos n\left(\frac{\pi}{2} - \theta\right) + i \sin n\left(\frac{\pi}{2} - \theta\right)\right]$$

Example 3.21 : If n is a positive integer, prove that

$$\left(\sqrt{3} + i\right)^n + \left(\sqrt{3} - i\right)^n = 2^{n+1} \cos\frac{n\pi}{6}$$

Solution :

Let $\left(\sqrt{3} + i\right) = r(\cos\theta + i \sin\theta)$

Equating real and imaginary parts separately, we have

$r \cos\theta = \sqrt{3}$ and $r \sin\theta = 1$

$$\therefore\ r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{4} = 2$$

$$\cos\theta = \frac{\sqrt{3}}{2},\ \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

Hence $$\left(\sqrt{3} + i\right) = 2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$$

$$\left(\sqrt{3} + i\right)^n = \left[2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)\right]^n = 2^n\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)^n$$

$$= 2^n\left(\cos\frac{n\pi}{6} + i \sin\frac{n\pi}{6}\right) \qquad \ldots (1)$$

To determine $\sqrt{3} - i$, we replace i in the above result by $-\,i$ we get

$$\left(\sqrt{3} - i\right) = 2\left(\cos\frac{\pi}{6} - i \sin\frac{\pi}{6}\right)$$

$$\therefore\ \left(\sqrt{3} - i\right)^n = 2^n\left(\cos\frac{n\pi}{6} - i \sin\frac{n\pi}{6}\right) \qquad \ldots (2)$$

Adding (1) and (2) we have

$$\left(\sqrt{3} + i\right)^n + \left(\sqrt{3} - i\right)^n = 2^n\left(2 \cos\frac{n\pi}{6}\right)$$

$$= 2^{n+1} \cdot \cos\frac{n\pi}{6}$$

Example 3.22 : If α and β are the roots of $x^2 - 2x + 2 = 0$ and $\cot\theta = y + 1$,

show that $\dfrac{(y+\alpha)^n - (y+\beta)^n}{\alpha - \beta} = \dfrac{\sin n\theta}{\sin^n\theta}$

Solution :

The roots of the equation $x^2 - 2x + 2 = 0$ are $1 \pm i$.

Let $\alpha = 1 + i$ and $\beta = 1 - i$

Then $(y + \alpha)^n = [(\cot\theta - 1) + (1 + i)]^n$

$$= (\cot\theta + i)^n$$

$$= \frac{1}{\sin^n\theta}\ [\cos\theta + i \sin\theta]^n$$

$$= \frac{1}{\sin^n\theta}\ [\cos n\theta + i \sin n\theta]$$

Similarly $(y + \beta)^n = \frac{1}{\sin^n\theta}\ [\cos n\theta - i \sin n\theta]$

$$(y + \alpha)^n - (y + \beta)^n = \frac{2i \sin n\theta}{\sin^n\theta}$$

Further $\alpha - \beta = (1 + i) - (1 - i) = 2i$

$$\frac{(y + \alpha)^n - (y + \beta)^n}{\alpha - \beta} = \frac{2i \sin n\theta}{2i \sin^n\theta} = \frac{\sin n\theta}{\sin^n\theta}$$

EXERCISE 3.4

(1) Simplify : $\dfrac{(\cos 2\theta - i \sin 2\theta)^7\ (\cos 3\theta + i \sin 3\theta)^{-5}}{(\cos 4\theta + i \sin 4\theta)^{12}\ (\cos 5\theta - i \sin 5\theta)^{-6}}$

(2) Simplify : $\dfrac{(\cos \alpha + i \sin \alpha)^3}{(\sin \beta + i \cos \beta)^4}$

(3) If $\cos \alpha + \cos \beta + \cos \gamma = 0 = \sin \alpha + \sin \beta + \sin \gamma$, prove that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\alpha + \beta + \gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin (\alpha + \beta + \gamma)$

(iii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

(iv) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(Hints : Take $a = \text{cis } \alpha$, $b = \text{cis } \beta$, $c = \text{cis } \gamma$

$$a + b + c = 0 \Rightarrow a^3 + b^3 + b^3 = 3abc$$

$$1/a + 1/b + 1/c = 0 \Rightarrow a^2 + b^2 + c^2 = 0)$$

(v) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \dfrac{3}{2}$

For problems 4 to 9, $m, n \in N$

(4) Prove that

(i) $(1 + i)^n + (1 - i)^n = 2^{\frac{n+2}{2}} \cos \dfrac{n\pi}{4}$

(ii) $(1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n = 2^{n+1} \cos \dfrac{n\pi}{3}$

(iii) $(1 + \cos\theta + i \sin\theta)^n + (1+\cos\theta - i \sin\theta)^n = 2^{n+1} \cos^n(\theta/2) \cos \dfrac{n\theta}{2}$

(iv) $(1 + i)^{4n}$ and $(1 + i)^{4n+2}$ are real and purely imaginary respectively

(5) If α and β are the roots of the equation $x^2 - 2px + (p^2 + q^2) = 0$ and $\tan\theta = \dfrac{q}{y+p}$ show that $\dfrac{(y+\alpha)^n - (y+\beta)^n}{\alpha - \beta} = q^{n-1} \dfrac{\sin n\theta}{\sin^n\theta}$

(6) If α and β are the roots of $x^2 - 2x + 4 = 0$

Prove that $\alpha^n - \beta^n = i2^{n+1} \sin \dfrac{n\pi}{3}$ and deduct $\alpha^9 - \beta^9$

(7) If $x + \dfrac{1}{x} = 2\cos\theta$ prove that

(i) $x^n + \dfrac{1}{x^n} = 2\cos n\theta$ (ii) $x^n - \dfrac{1}{x^n} = 2i \sin n\theta$

(8) If $x + \dfrac{1}{x} = 2\cos\theta$ and $y + \dfrac{1}{y} = 2\cos\phi$ show that

(i) $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m} = 2\cos(m\theta - n\phi)$ (ii) $\dfrac{x^m}{y^n} - \dfrac{y^n}{x^m} = 2i\sin(m\theta - n\phi)$

(9) If $x = \cos\alpha + i\sin\alpha$; $y = \cos\beta + i\sin\beta$

prove that $x^m y^n + \dfrac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$

(10) If $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$ and $c = \cos 2\gamma + i\sin 2\gamma$ prove that

(i) $\sqrt{abc} + \dfrac{1}{\sqrt{abc}} = 2\cos(\alpha + \beta + \gamma)$

(ii) $\dfrac{a^2\, b^2 + c^2}{abc} = 2\cos 2(\alpha + \beta - \gamma)$

3.9. Roots of a complex number

Definition :

A number ω is called an nth root of a complex number z, if $\omega^n = z$ and we write $\omega = z^{\frac{1}{n}}$

Working rule to find the *n*th roots of a complex number :

Step 1 : Write the given number in polar form.

Step 2 : Add $2k\pi$ to the argument

Step 3 : apply De Moivre's theorem (bring the power to inside)

Step 4 : Put $k = 0, 1 \ldots$ upto $n - 1$

Illustration :

$$\text{Let} \quad z = r(\cos\theta + i\sin\theta)$$

$$= r\{\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\}, \; k \text{ is an integer.}$$

$$\therefore z^{\frac{1}{n}} = [r\{\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right]$$

where $k = 0, \; 1, 2 \ldots (n - 1)$

Only these values of k will give n different values of $z^{\frac{1}{n}}$ provided $z \neq 0$

Note :

(1) The number of nth roots of a non-zero complex number is n.

(2) The moduli of these roots is the same non negative real number.

(3) The argument of these n roots are equally spaced. That is if θ is the principal value of arg z i.e., $-\pi \leq \theta \leq \pi$ then the arguments of other roots of z are obtained by adding respectively $\frac{2\pi}{n}, \frac{4\pi}{n}, \ldots$ to $\frac{\theta}{n}$

(4) If k be given integral values greater than or equal to n, these n values are repeated and no fresh root is obtained.

3.9.1 The *n*th roots of unity

$$1 = (\cos 0 + i\sin 0) = \cos 2k\pi + i\sin 2k\pi$$

$$n\text{th roots of unity} = 1^{\frac{1}{n}} = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{n}}$$

$$= \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) \text{ where } k = 0, 1, 2, \ldots n - 1$$

$\therefore$ The nth roots of unity are $\cos 0 + i\sin 0$, $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$,

$\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}$, $\cos\frac{6\pi}{n} + i\sin\frac{6\pi}{n}$, ……, $\cos(n-1)\frac{2\pi}{n} + i\sin(n-1)\frac{2\pi}{n}$

$$\text{Let} \quad \omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{i\frac{2\pi}{n}}$$

Then the nth roots of unity are

$e^0, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, e^{i\frac{6\pi}{n}}, \ldots e^{i\frac{2(n-1)}{n}\pi}$ become $1, \omega, \omega^2 \ldots \omega^{n-1}$.

It is clear that the roots are in geometric progression.

Results :

(1) $\omega^n = 1$

$$\omega^n = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = \cos 2\pi + i\sin 2\pi = 1$$

(2) Sum of the roots is 0

i.e., $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$

$\because$ LHS $= 1 + \omega + \omega^2 + \ldots + \omega^{n-1}$ is a G.P. with n terms.

$$= \frac{1.(1-\omega^n)}{1-\omega} \qquad \because \left[1 + r + r^2 + \ldots + r^{n-1} = \frac{1-r^n}{1-r}\right]$$

$= 0 =$ R.H.S.

(3) The roots are in G.P with common ratio ω

(4) The arguments are in A.P with common difference $\frac{2\pi}{n}$

(5) Product of the roots $= (-1)^{n+1}$

3.9.2. Cube roots of unity : $(1)^{\frac{1}{3}}$

Let $x = (1)^{\frac{1}{3}}$

$\therefore x = (\cos 0 + i\sin 0)^{\frac{1}{3}}$

$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{3}}$, where k is an integer.

$$x = \left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right), \quad \text{where } k = 0, 1, 2$$

The three roots are

$\cos 0 + i\sin 0,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

i.e., $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\therefore$ The roots are $1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Result :

The modulus of each root of $(1)^{\frac{1}{3}}$ is 1

$\therefore$ All these roots lie on the circumference of the unit circle. Let *A*, *B* and *C* be points represented by the three roots $1, -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}$ in ordered pair form. The angles between *OA* and *OB*, *OB* and *OC*, *OC* and *OA* are each $\frac{2\pi}{3}$ radians or 120°. Hence when these points are joined by straight lines they will form the vertices of an equilateral triangle.

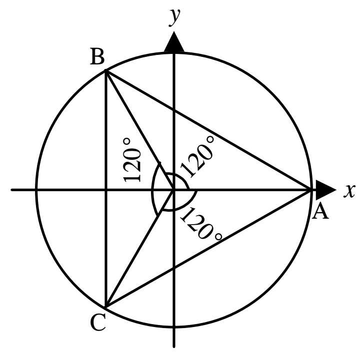


Fig. 3.16

If we denote the second root $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ by ω then the other root $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2$ becomes ω^2.

Hence we observe that the cube roots of unity, namely $1, \omega, \omega^2$ are in G.P.

Note :

(i) Even if $\frac{-1-i\sqrt{3}}{2}$ is taken as ω it can be proved that $\frac{-1+i\sqrt{3}}{2} = \omega^2$

(ii) $1 + \omega + \omega^2 = 0$ (by actual addition) i.e., the sum of the cube roots of unity is zero.

(iii) Since ω is a root of the equation $x^3 = 1$, we see that $\omega^3 = 1$.

Fourth roots of unity :

Let x be a fourth root of unity. Then $x = (1)^{\frac{1}{4}}$

$\therefore\ x^4 = 1 = (\cos 2k\pi + i\sin 2k\pi)$ where k is an integer.

$$\begin{aligned} x &= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{4}} \\ &= \left(\cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4}\right) \\ &= \left(\cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}\right) \quad \text{where } k = 0, 1, 2, 3 \end{aligned}$$

The four roots are

$\cos 0 + i \sin 0, \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}, \ \cos \pi + i \sin \pi, \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$

i.e., $1, i, -1(= i^2), -i\ (= i^3)$. Let us denote $\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ by ω. Then the four roots of unity are $1, \omega, \omega^2, \omega^3$.

The fourth roots of unity form the vertices of a square all lying on the unit circle.

We observe that the sum of the fourth roots of unity is zero.

i.e., $1 + \omega + \omega^2 + \omega^3 = 0$

and $\omega^4 = 1$

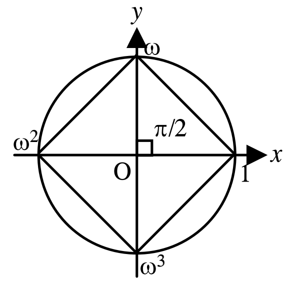


Fig. 3.17

Note : The values of ω used in cube roots of unity and in fourth roots of unity are different.

Sixth roots of unity : Let x be a sixth root of unity. Then $x = (1)^{1/6}$

$$\therefore \ 1 = \cos 0 + i \sin 0$$

$$(1)^{1/6} = (\cos 2k\pi + i \sin 2k\pi)^{1/6}$$

where k is an integer.

By De Moivre's theorem

$$x = (1)^{\frac{1}{6}} = \left(\cos \frac{2k\pi}{6} + i \sin \frac{2k\pi}{6}\right), \text{ where } k = 0, 1, 2, 3, 4, 5$$

The six roots are

$\cos 0 + i \sin 0 = 1$

$\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$

$\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$

$\cos \frac{3\pi}{3} + i \sin \frac{3\pi}{3}$

$\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$

$\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}$

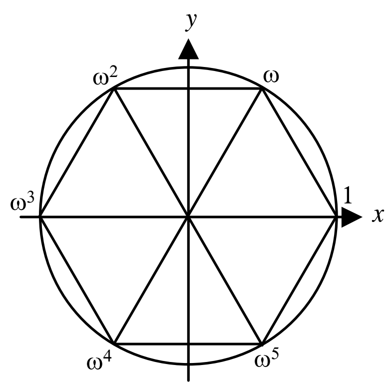


Fig. 3.18

Then the six, sixth roots of unity are $1, \omega, \omega^2, \omega^3, \omega^4, \omega^5$

From the above figure it can be noted that the six roots of unity form the vertices of a hexagon all lying on the unit circle (*Fig. 3.18).* Thus it can be seen that the n, nth roots of unity form the vertices of n sided regular polygon all lying on the unit circle *(Fig. 3.19)*.

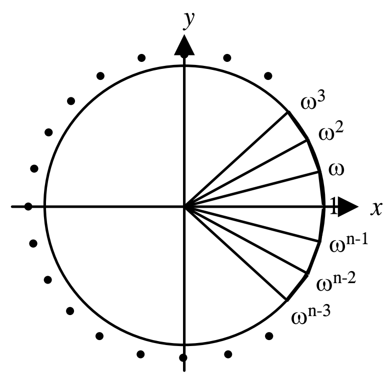


Fig. 3.19

Example 3.23 : Solve the equation $x^9 + x^5 - x^4 - 1 = 0$

Solution :

$$x^9 + x^5 - x^4 - 1 = 0 \Rightarrow x^5 (x^4 + 1) - 1 (x^4 + 1) = 0$$
$$\Rightarrow (x^5 - 1)(x^4 + 1) = 0$$
$$\Rightarrow x^5 - 1 = 0 \ ; \ x^4 + 1 = 0$$
$$x = (1)^{\frac{1}{5}} ; (-1)^{\frac{1}{4}}$$

(i) $x = (1)^{\frac{1}{5}} = (\cos 0 + i \sin 0)^{\frac{1}{5}}$

$$= (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{5}}$$
$$= \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}, \ k = 0, 1, 2, 3, 4$$

(ii) $(-1)^{\frac{1}{4}} = (\cos \pi + i \sin \pi)^{\frac{1}{4}}$

$$= \{\cos (2k + 1)\pi + i \sin (2k + 1)\pi\}^{\frac{1}{4}}$$
$$= \cos \frac{(2k + 1)\pi}{4} + i \sin \frac{(2k + 1)\pi}{4} \quad k = 0, 1, 2, 3$$

Thus we have 9 roots.

Example 3.24 : Solve the equation $x^7 + x^4 + x^3 + 1 = 0$

Solution :

$$x^7 + x^4 + x^3 + 1 = 0 \Rightarrow x^4 (x^3 + 1) + 1 (x^3 + 1) = 0$$
$$\Rightarrow (x^4 + 1)(x^3 + 1) = 0$$

$$x^4 = -1 \quad ; x^3 = -1$$

(i) $x = (-1)^{\frac{1}{4}}$

$= (\cos\pi + i\sin\pi)^{\frac{1}{4}}$

i.e., $= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{4}}$

$= \left[\cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}\right] \; ; \; k = 0, 1, 2, 3$

(ii) $x^3 = -1 \;\Rightarrow\; x = (-1)^{\frac{1}{3}}$

$= (\cos\pi + i\sin\pi)^{\frac{1}{3}}$

$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{3}}$

$= \cos(2k+1)\frac{\pi}{3} + i\sin(2k+1)\frac{\pi}{3}, \; k = 0, 1, 2$

Note :

The roots are $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$; $\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$; $\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$ and $\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$, $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $(\cos\pi + i\sin\pi)$ and $\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

i.e., $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \; -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \; -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}, \; \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$\frac{1}{2} + i\frac{\sqrt{3}}{2}, \; -1, \; \frac{1}{2} - i\frac{\sqrt{3}}{2}$

Example 3.25 : Find all the values of $(\sqrt{3} + i)^{\frac{2}{3}}$

Solution :

Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$\Rightarrow r\cos\theta = \sqrt{3}, \; r\sin\theta = 1$

$\Rightarrow r = \sqrt{(\sqrt{3})^2 + 1} = 2$

$\text{Cos}\,\theta = \frac{\sqrt{3}}{2}, \;\; \sin\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{6}$

$\therefore (\sqrt{3} + i)^{\frac{2}{3}} = 2^{\frac{2}{3}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{\frac{2}{3}}$

$$= 2^{\frac{2}{3}}\left[\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^2\right]^{\frac{1}{3}}$$

$$= 2^{\frac{2}{3}}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{1}{3}}$$

$$= 2^{\frac{2}{3}}\left[\cos\left(2k\pi + \frac{\pi}{3}\right) + i\sin\left(2k\pi + \frac{\pi}{3}\right)\right]^{\frac{1}{3}}$$

$$= 2^{\frac{2}{3}}\left[\cos(6k+1)\frac{\pi}{9} + i\sin(6k+1)\frac{\pi}{9}\right] \text{ where } k = 0, 1, 2$$

Note: The values are

$$2^{\frac{2}{3}}\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right),\ 2^{\frac{2}{3}}\left(\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right),\ 2^{\frac{2}{3}}\left(\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right)$$

Aliter :

$$(\sqrt{3} + i)^{\frac{2}{3}} = 2^{\frac{2}{3}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{\frac{2}{3}}$$

$$= 2^{\frac{2}{3}}\left(\cos\left(2k\pi + \frac{\pi}{6}\right) + i\sin\left(2k\pi + \frac{\pi}{6}\right)\right)^{\frac{2}{3}}$$

$$= 2^{\frac{2}{3}}\left[\cos(12k+1)\frac{\pi}{6} + i\sin(12k+1)\frac{\pi}{6}\right]^{\frac{2}{3}}$$

$$= 2^{\frac{2}{3}}\left[\cos(12k+1)\frac{\pi}{9} + i\sin(12k+1)\frac{\pi}{9}\right] \text{ where } k = 0, 1, 2$$

The different values are $2^{\frac{2}{3}}\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$, $2^{\frac{2}{3}}\left(\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right)$, $2^{\frac{2}{3}}\left(\cos\frac{25\pi}{9} + i\sin\frac{25\pi}{9}\right)$

i.e., $2^{\frac{2}{3}}\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$, $2^{\frac{2}{3}}\left(\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right)$, and $2^{\frac{2}{3}}\left(\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right)$ since $\cos\frac{25\pi}{9} + i\sin\frac{25\pi}{9} = \cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}$

Thus we have obtained the same values in this case also.

Note : If we add $2k\pi$ before taking the power 2 inside, we will get the same answer.

EXERCISE 3.5

(1) Find all the values of the following :

(i) $(i)^{\frac{1}{3}}$ (ii) $(8i))^{\frac{1}{3}}$ (iii) $\left(-\sqrt{3}-i\right)^{\frac{2}{3}}$

(2) If $x = a + b$, $y = a\omega + b\omega^2$, $z = aw^2 + b\omega$ show that

(i) $xyz = a^3 + b^3$

(ii) $x^3 + y^3 + z^3 = 3\,(a^3 + b^3)$ where ω is the complex cube root of unity.

(3) Prove that if $\omega^3 = 1$, then

(i) $(a + b + c)\ (a + b\omega + c\omega^2)\ (a + b\omega^2 + c\omega) = a^3 + b^3 + c^3 - 3abc$

(ii) $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^5 + \left(\dfrac{-1-i\sqrt{3}}{2}\right)^5 = -1$

(iii) $\dfrac{1}{1+2\omega} - \dfrac{1}{1+\omega} + \dfrac{1}{2+\omega} = 0$

(4) Solve :

(i) $x^4 + 4 = 0$ (ii) $x^4 - x^3 + x^2 - x + 1 = 0$

(5) Find all the values of $\left(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$ and hence prove that the product of the values is 1.

4. ANALYTICAL GEOMETRY

4.1 Introduction :

Tracing the history of Mathematics, around 430 B.C., study of conic sections or conics, i.e., study of plane sections of a right circular cone began. The study included degenerate or singular conics (comprising single point, pair of distinct lines, two coincident lines etc., which were already dealt with in detail in lower classes) and non-degenerate or non-singular conics (comprising of circles, parabolas, ellipses and hyperbolas).

The study of conic sections from Greek Geometry, developed by Apollonius, is described today as graphs of quadratic equations in the co-ordinate plane. The Greek mathematicians of Plato's time (429 − 347 B.C.) described these curves as the curves formed by slicing a double cone (right circular cone of two nappes) with a plane and hence the name conic sections.

Analytic Geometry grew out of need for establishing uniform techniques for solving geometrical problems, the aim being to apply them to the study of curves, which are of particular importance in practical problems.

The aim was achieved in the co-ordinate method viz., cartesian, polar, bi-polar (where calculations are fundamental and constructions play a subordinate role). Thus solving problems by the method of Analytical Geometry requires less inventiveness. This method of the ancient Greek origin ($\approx$ 1 – 2 B.C.) was systematically developed in the first half of the 17^{th} century by great mathematicians Fermat, Descartes, Kepler, Newton, Euler, Leibnitz, L'Hôpital, Clairaut, Cramer and the Jacobis.

A major breakthrough in the study occurred with the development of the hypothesis of Planetary Phenomena by the German mathematician cum physicist Johannes Kepler. He stated that all the planets in the solar system including the earth are moving in elliptical orbits with sun at one of a foci, governed by inverse square law. This led to the development of Newton's gravitation theory.

Euler applied the co-ordinate method in a systematic study of space curves and surfaces, which was further developed by Albert Einstein in his theory of relativity.

Needless to say that today the development in this area has conquered industry, medicine and scientific research. And we shall cite a few of them before getting into the depth of actual study of conics.

4.1.1 Geometry and Practical applications of a parabola :

- A parabola is a conic section obtained on slicing a right circular cone by a plane parallel to the line joining vertex and any other point of the cone *(Fig.4.1)*

Fig. 4. 1

- If P is any point on the parabola with focus F and vertex V, the angles subtended by FP and PX with the tangent at P are equal where PX is parallel to the axis VFA of the parabola. *(Fig. 4.2)*

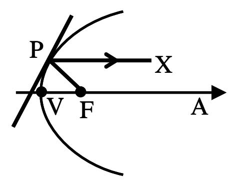


Fig. 4. 2

This property is made use of in parabolic reflectors (surface obtained by revolving the parabola about its axis and coated with silver paint) of sound, light and radio waves when the respective source is placed at the focus S as given in *(Fig. 4.3)*. Light (or sound or radio waves) from S falls on the reflecting surface gets reflected parallel to the axis of parabola. For example, Flash light, head light of motor vehicles, parabolic mirrors, spot light reflectors, selective microphone sounding boards etc.

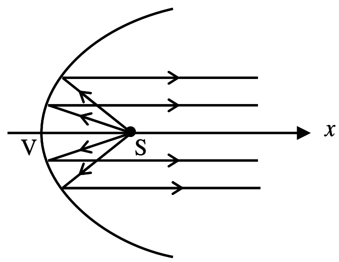


Fig. 4. 3

The same reflectors can be employed in intensifying signals. Electromagnetic waves arriving parallel to the axis of the parabolic reflector will be focussed at the focus where a suitable receiver 'R' could be placed. *(Fig. 4.4)*

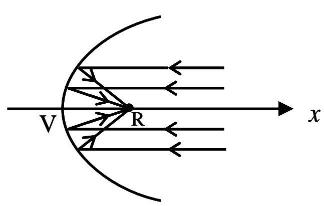


Fig. 4. 4

For example, Radio telescope, television satellite dishes, solar heaters, radar antenna's etc.

- The strongest simple arch is parabolic in shape.

- The supporting cable of a uniformly loaded bridge is parabolic in shape (weight of cable neglected in comparison with weight of the bridge).
- The path of an object thrown or projected obliquely upwards is a parabola. Also bombs dropped from a moving war plane or food packets dropped from helicopters during cyclone time to people in need (not moving vertically upwards or downwards) traces a parabola.
- Some comets have parabolic path with sun at the focus.

4.1.2 Geometry and Practical Applications of an ellipse :

- An ellipse is a conic obtained on slicing across obliquely one nappe of a cone. *(Fig. 4.5)*
- If P is any point on the ellipse and $F_1$and F_2 its foci, the angle subtended by F_1P and F_2P with the tangent at P are equal and if a source of light or sound is placed at one focus of an ellipsoidal reflector (surface generated by revolving an ellipse about its major axis) all the waves will be reflected so as to pass through the other focus *(Fig. 4.6)*

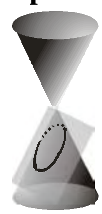


Fig. 4. 5

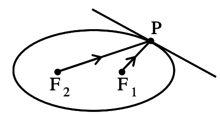


Fig. 4. 6

This property is also used in "Wispering Gallery", the roof or walls of which are shaped like an ellipsoidal reflector.

- The ellipsoidal reflectors are designed for Nd : YAG (ND^{3+} Neodymium ions; YAG – Yttrium Aluminium Garnetz) laser that is widely used in medicine, industry and scientific research.

A light reflector in the form of a tube whose cross section is an ellipse has Nd : YAG rod and a linear flash lamp placed at the foci of the ellipse. (Fig. 4.7). Here light emitted from the lamp is effectively coupled to the Nd : YAG rod to produce laser beam.

- In Bohr-Sommerfeld theory of the atom electron orbit can be circular or elliptical.

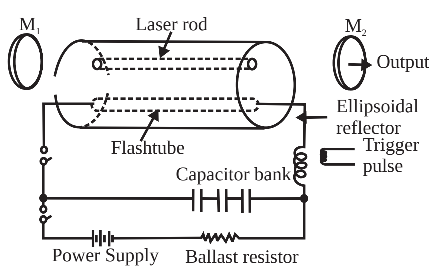


Fig. 4. 7

- The orbits of our planet earth and all other planets and planetoids in our solar system are elliptical with sun situated at one of the foci. Also all the satellites, either natural or artificial to all the planets in the solar system have elliptic orbits (with the force binding them following inverse square law). *[Fig.4.8(a)]*

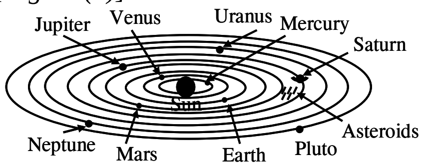


Fig. 4.8(a)

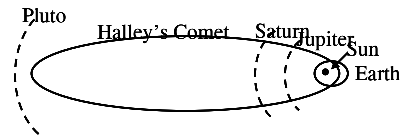


Fig. 4.8(b)

Path of Halley's Comet (which returns after every 75 years) is an ellipse with e $\approx$ 0.97 and the sun at a focus *[Fig. 4.8 (b)], e* being the eccentricity.

- Elliptical arches are often used for their beauty.

- Steam boilers are believed to have greatest strength when heads are made elliptical with major and minor axes in the ratio 2 : 1.

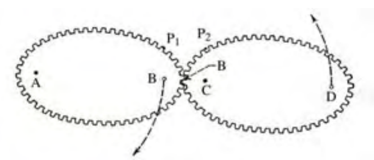


Fig. 4. 9

⋆ Gears are sometimes (for particular need) made elliptical in shape *(Fig. 4.9)*

Fig. 4. 10

- The orbit of Comet Kohoutek is an ellipse with e $\approx$ 0.9999 (*Fig. 4.10)*.

- The shape of our mother earth is an oblate spheroid i.e., the solid of revolution of an ellipse about its minor axis, bulged along equatorial region and flat along the polar region.

- The area of action of an airplane which leaves a moving carrier and returns in a given time (with no wind) is an ellipse with the take off and landing positions of the carrier as foci.

- The track of a plane making an On-pylon turn in a wind of constant velocity is an ellipse with one focus directly over the pylon *(Fig. 4.11)*.

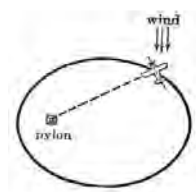


Fig. 4. 11

4.1.3 Geometry and Practical Applications of a Hyperbola :

- A hyperbola is a conic obtained on slicing a double napped cone by a plane parallel to the axis of the cone (*Fig.4.12*)

Fig. 4. 12

- The lines from the foci to any point on a hyperbola make equal angles with the tangent at that point. Hence if the surface of a reflector is generated by revolving a hyperbola about its transverse axis, all rays of light converging on one focus are reflected to the other *(Fig.4.13)*

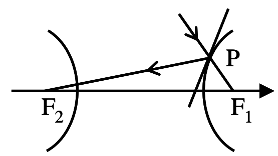


Fig. 4. 13

- This property is made use of in some telescopes in conjunction with a parabolic reflector. American space research foundation NASA's Hubble space telescope uses hyperbolic reflectors in conjunction with parabolic reflectors *(Fig. 4.14)*.

- Hyperbolas are useful in range-finding. (The difference in the times at which a sound is heard at two listening posts is proportional to the difference of the distances from the posts to the point of emanation of sound. A third listening post serves to give another hyperbola and the point of emanation is at the point of intersection of the two curves).

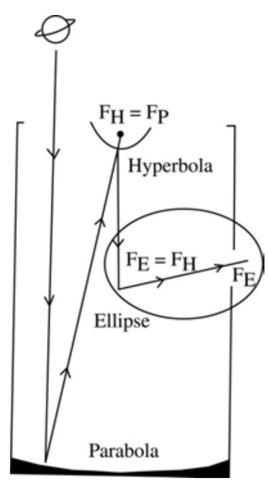


Fig. 4. 14

- Boyle's law pv = constant is hyperbolic in relationship. The same is true of relationship of any two quantities, which are inversely proportional.

- Hyperbolic paths arise in Einstein's theory of relativity and form a basis for LORAN (Long Range Navigation) radio navigation system.

4.2 Definition of a Conic :

Consider a circle C. Let A be the line through the centre of C and perpendicular to the plane of C and let V be a point on A not in the plane of C. Let P be a point on C and draw the infinite straight lines through P that also passes through V. As P moves around C, what sweeps out is called a right circular cone with the axis A and vertex V. Each of the lines

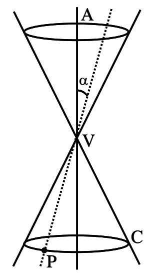


Fig. 4. 15

PV is called a generator of the cone, and the angle α between the axis and the generator is called a vertex angle (semi-vertical angle). The upper and lower portions of the cone that meet at the vertex are called nappes of the cone *(Fig. 4.15).*

The curves obtained by slicing the cone with a plane not passing through the vertex are called conic sections or simply conics.

A conic is the locus of a point which moves in a plane, so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line. *(Fig.4.16)*

The fixed point is called focus, the fixed straight line is called directrix, and

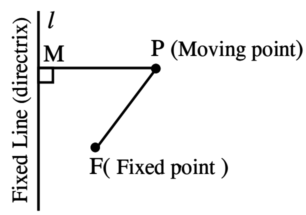


Fig. 4. 16

the constant ratio is called eccentricity, which is denoted by 'e'.

From the figure we have $\dfrac{FP}{PM}$ = constant = e

4.2.1 General equation of a Conic :

Let $F(x_1, y_1)$ be the focus, $lx + my + n = 0$, the equation of the directrix 'l' and 'e' the eccentricity of the conic.

Let $P(x, y)$ be any point on the conic. Drop a perpendicular from P to 'l'.

$$FP = \sqrt{(x-x_1)^2 + (y-y_1)^2}$$

PM = Perpendicular distance from $P(x, y)$ to the line $lx + my + n = 0$

$$= \pm \frac{lx + my + n}{\sqrt{l^2 + m^2}}$$

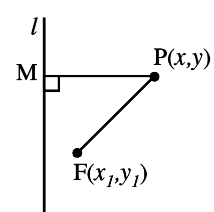


Fig. 4. 17

By the definition of a conic, $\frac{FP}{PM} = e$

$$\therefore FP^2 = e^2 PM^2$$

$$\therefore (x - x_1)^2 + (y - y_1)^2 = e^2 \left[\pm \frac{lx + my + n}{\sqrt{l^2 + m^2}} \right]^2$$

Simplifying this we get an equation of second degree in x and y of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

4.2.2 Classification with respect to the general equation of a conic :

The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ represents either a (non–degenerate) conic or a degenerate conic. If it is a conic, then it is

(i) a parabola if $B^2 - 4AC = 0$ (ii) an ellipse if $B^2 - 4AC < 0$

(iii) a hyperbola if $B^2 - 4AC > 0$

4.2.3. Classification of conics with respect to eccentricity :

1. If $e < 1$, then the conic is an ellipse.

   From the figure 4.18 we observe that F_2P_i is always less than P_iM_i.

   i.e., $\frac{F_2P_i}{P_iM_i} = e < 1, (i = 1, 2, 3\ldots)$

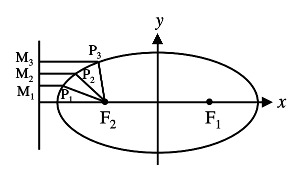


Fig. 4. 18

2. If $e = 1$, then the conic is a parabola.

   From the figure 4.19 we observe that FP_i is always equal to P_iM_i.

   i.e., $\frac{FP_i}{P_iM_i} = e = 1, (i = 1, 2, 3\ldots)$

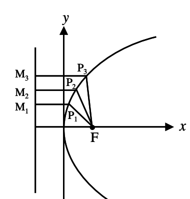


Fig. 4. 19

3. If $e > 1$, then the conic is a hyperbola.

   From the figure 4.20 we observe that F_1P_i is always greater than P_iM_i.

   i.e., $\frac{F_1P_i}{P_iM_i} = e > 1, (i = 1, 2, 3\ldots)$

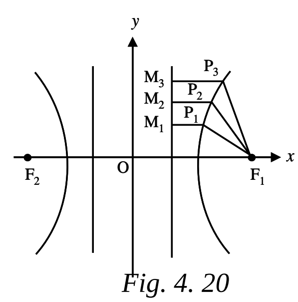


Fig. 4. 20

4.3 Parabola :

The locus of a point whose distance from a fixed point is equal to its distance from a fixed line is called a parabola. That is a parabola is a conic whose eccentricity is 1.

Note : Eventhough the syllabus does not require the derivation of standard equation and the tracing of parabola (4.3.1, 4.3.2) and it needs only the standard equation and the diagram, the equation is derived and the curve is traced for better understanding.

Now we derive and trace the standard equation of a parabola.

4.3.1. Standard equation of a parabola :

Given :

★ Fixed point (F)

★ Fixed line (l)

★ Eccentricity ($e = 1$)

★ Moving point $P(x, y)$

Construction :

★ Plot the fixed point F and draw the fixed line 'l'.

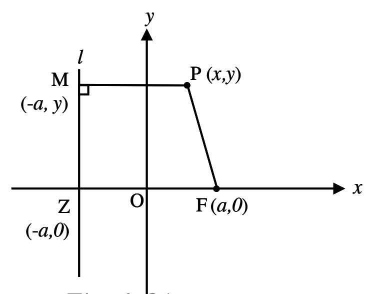


Fig. 4. 21

★ Drop a perpendicular (FZ) from F to l.

★ Take $FZ = 2a$ and treat it as x-axis.

★ Draw a perpendicular bisector to FZ and treat it as y-axis.

★ Let $V(0, 0)$ be the origin.

★ Drop a perpendicular (PM) from P to l.

★ The known points are $F(a, 0)$, $Z(-a, 0)$ and hence M is $(-a, y)$.

By the definition of a conic, $\dfrac{FP}{PM} = e = 1 \Rightarrow FP^2 = \text{PM}^2$

$(x-a)^2 + (y-0)^2 = (x+a)^2 + (y-y)^2$

$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$ which simplifies to $y^2 = 4ax$.

This is the standard equation of the parabola.

To trace a curve, we shall use the tools dealt in detail in chapter 6.

4.3.2.Tracing of the parabola $y^2 = 4ax$:

(i) Symmetry property :

It is symmetrical about x-axis.

i.e., x-axis divides the curve into two symmetrical parts.

(ii) Special points :

The parabola passes through the origin since (0, 0) satisfies the equation $y^2 = 4ax$.

To find the points on x-axis, put $y = 0$. We get $x = 0$ only.

$\therefore$ the parabola cuts the x-axis only at the origin (0, 0).

To find the points on y-axis, put $x = 0$. We get $y = 0$ only.

$\therefore$ the parabola cuts the y-axis only at the origin (0, 0).

(iii) Existence of the curve :

For $x < 0$, y^2 becomes negative. i.e., y is imaginary. Therefore the curve does not exist for negative values of x. i.e., the curve exists only for non-negative values of x.

(iv) The curve at infinity :

As x increases, y^2 also increases.

i.e., as $x \to \infty$, $y^2 \to \infty$

i.e., as $x \to \infty$, $y \to \pm\infty$

$\therefore$ the curve is open rightward. *[Fig. 4.22]*

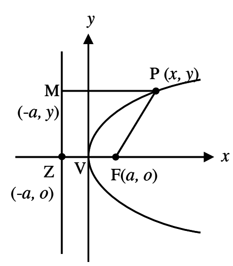


Fig. 4. 22

4.3.3. Important definitions regarding a parabola :

Focus : The fixed point used to draw the parabola is called the focus (F). Here, the focus is $F(a, 0)$.

Directrix : The fixed line used to draw a parabola is called the directrix of the parabola. Here, the equation of the directrix is $x = -a$.

Axis : The axis of the parabola is the axis of symmetry. The curve $y^2 = 4ax$ is symmetrical about x-axis and hence x-axis or $y = 0$ is the axis of the parabola $y^2 = 4ax$. Note that the axis of the parabola passes through the focus and perpendicular to the directrix.

Vertex : The point of intersection of the parabola and its axis is called its vertex. Here, the vertex is $V(0, 0)$.

Focal distance : The focal distance is the distance between a point on the parabola and its focus.

Focal chord : A chord which passes through the focus of the parabola is called the focal chord of the parabola.

Latus Rectum : It is a focal chord perpendicular to the axis of the parabola. Here, the equation of the latus rectum is $x = a$.

End points of latus rectum and length of latus rectum :

To find the end points, solve the equation of latus rectum $x = a$ and $y^2 = 4ax$.

Using $\quad x = a$ in $y^2 = 4ax$

we get $\quad y^2 = (4a)a = 4a^2$

$\therefore y = \pm 2a$

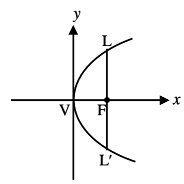


Fig. 4. 23

If L and L' are the end points of latus rectum then L is $(a, 2a)$ and L' is $(a, -2a)$. The length of latus rectum = $LL' = 4a$. Length of semi-latus rectum = $FL = FL' = 2a$. So far we have discussed standard equation of a parabola which is open rightward. But we have parabolas which are open leftward, open upward and open downward.

4.3.4. Other standard parabolas :

1. **Open leftward :**

   $\boldsymbol{y^2 = -4ax \; [a > 0]}$

   If $x > 0$, then y becomes imaginary. i.e., the curve does not exist for $x > 0$ i.e., the curve exist for $x \leq 0$.

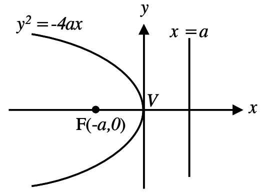


Fig. 4. 24

2. **Open upward :**

   $\boldsymbol{x^2 = 4ay \; [a > 0]}$

   If $y < 0$, then y becomes imaginary. i.e., the curve does not exist for $y < 0$ i.e., the curve exist for $y \geq 0$.

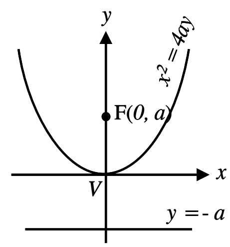


Fig. 4. 25

3. **Open downward :**

   $\boldsymbol{x^2 = -4ay \; [a > 0]}$

   If $y > 0$, then y becomes imaginary. i.e., the curve does not exist for $y > 0$ i.e., the curve exist for $y \leq 0$.

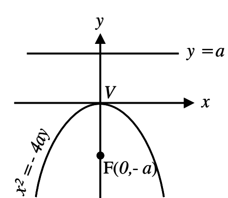


Fig. 4. 26

Remark : So far we have discussed four standard types of parabolas. There are plenty of parabolas which cannot be classified under these standard types. For example, consider the following parabolas.

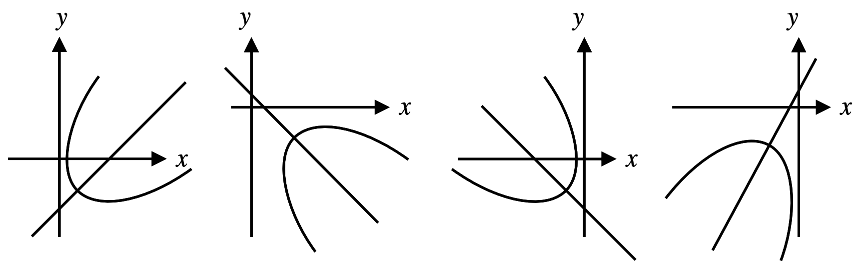


Fig. 4. 27

For the above parabolas, the axes are neither parallel to x-axis nor parallel to y-axis. In such cases the equation of the parabolas include xy term, which is beyond the scope of this book, eventhough we will find the equation of the parabolas which are not in standard form. Note that for the standard types the axis is either parallel to x-axis or parallel to y-axis. We will study only these four types.

All the parabolas discussed so far have vertex at the origin. In general the vertex need not be at the origin for any parabola. Hence we need the concept of shifting the origin or translation of the axes.

4.3.5 The process of shifting the origin or translation of axes :

Consider the xoy system. Draw a line parallel to x-axis (say X-axis) and draw a line parallel to y-axis (say Y-axis). Let $P(x, y)$ be a point with respect to xoy system and $P(X, Y)$ be the same point with respect to XOY system.

Let the co-ordinates of O′ with respect to xoy system be (h, k)

The co-ordinate of P with respect to xoy system :

$OL = OM + ML = h + X$

i.e., $x = X + h$

Similarly $y = Y + k$

$\therefore$ The new co-ordinates of P with respect to XOY system

$X = x - h$

$Y = y - k$

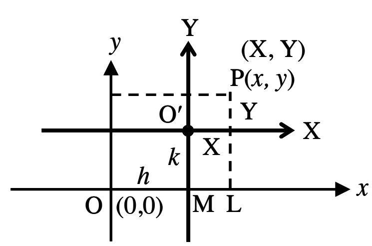


Fig. 4. 28

4.3.6 General form of the standard equation of a parabola, which is open rightward (i.e., the vertex other than origin) :

Consider a parabola with vertex V whose co-ordinates with respect to XOY system is (0, 0) and with respect to xoy system is (h, k).

Since it is open rightward, the equation of the parabola w.r.t. XOY system is $Y^2 = 4aX$.

By shifting the origin $X = x - h$ and $Y = y - k$, the equation of the parabola with respect to old xoy system is $(y - k)^2 = 4a(x - h)$.

This is the general form of the standard equation of the parabola, which is open rightward. Similarly the other general forms are

$(y - k)^2 = -4a(x - h)$ (open leftwards)

$(x - h)^2 = 4a(y - k)$ (open upwards)

$(x - h)^2 = -4a(y - k)$ (open downwards)

Note : To find the general form, replace x by $x - h$ and y by $y - k$ if the vertex is (h, k)

Remark : The above form of equations do not have xy term.

Example 4.1: Find the equation of the following parabola with indicated focus and directrix.

(i) $(a, 0)$; $x = -a$ $a > 0$

(ii) $(-1, -2)$; $x - 2y + 3 = 0$

(iii) $(2, -3)$; $y - 2 = 0$

Solution: (i) Let $P(x, y)$ be any point on the parabola. If PM is drawn perpendicular to the directrix,

$$\frac{FP}{PM} = e = 1$$

$$\Rightarrow \quad FP^2 = PM^2$$

$$(x - a)^2 + (y - 0)^2 = \left(\pm \frac{x + a}{\sqrt{1^2}}\right)^2$$

$$\Rightarrow \quad (x - a)^2 + y^2 = (x + a)^2$$

$$\Rightarrow \quad y^2 = 4ax$$

This is the required equation.

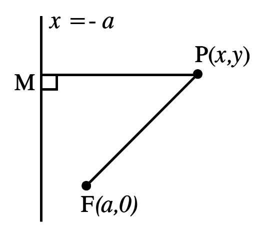


Fig. 4. 29

Alternative method :

From the given data, the parabola is open rightward.

$\therefore$ The equation of the parabola is of the form $(y - k)^2 = 4a\,(x - h)$

We know that the vertex is the midpoint of Z(– a, 0) and focus F(a, 0), where Z is the point of intersection of the directrix and the x-axis.

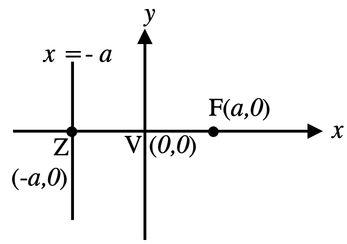


Fig. 4. 30

$$\therefore \text{ Vertex is } \left(\frac{-a+a}{2}, \quad \frac{0+0}{2}\right) = (0, 0) = (h, k)$$

Again the distance between the vertex and the focus $VF = a$

$\therefore$ The required equation is $(y - 0)^2 = 4a\,(x - 0)$ i.e., $y^2 = 4ax$

(ii) Let $P(x, y)$ be any point on the parabola. If PM is drawn perpendicular to the directrix,

$$\frac{FP}{PM} = e = 1$$

$$\Rightarrow \quad FP^2 = PM^2$$

$$(x+1)^2 + (y+2)^2 = \left(\pm\frac{x - 2y + 3}{\sqrt{1^2+2^2}}\right)^2$$

$$\Rightarrow 4x^2 + 4xy + y^2 + 4x + 32y + 16 = 0$$

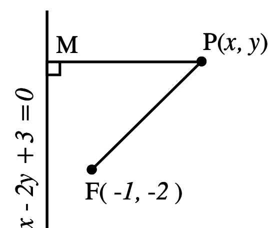


Fig. 4. 31

Note : Here the directrix is parallel to neither x-axis nor y-axis. This type is not standard type. Therefore we can't do this problem as in the alternative method of previous problem.

(iii) Let $P(x, y)$ be any point on the parabola. If PM is drawn perpendicular to the directrix

$$\frac{FP}{PM} = e = 1$$

$$\Rightarrow \quad FP^2 = PM^2$$

$$\text{i.e., } (x-2)^2 + (y+3)^2 = \left(\pm\frac{y-2}{\sqrt{1}}\right)^2$$

$$(x-2)^2 + (y+3)^2 = (y-2)^2$$

$$\Rightarrow \quad x^2 - 4x + 10y + 9 = 0$$

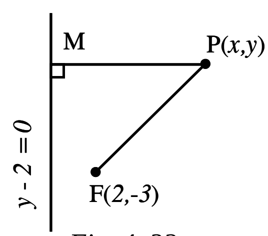


Fig. 4. 32

Note : Since the directrix $y = 2$ is parallel to x-axis, the type is standard and hence this problem can be solved by alternative method of 4.1(i).

Example 4.2 : Find the equation of the parabola if

(i) the vertex is (0, 0) and the focus is $(-a, 0)$, $a > 0$

(ii) the vertex is (4, 1) and the focus is (4, – 3)

Solution: (i) From the given data the parabola is open leftward

The equation of the parabola is of the form

$$(y - k)^2 = -4a(x - h)$$

Here, the vertex (h, k) is (0, 0) and $VF = a$

$\therefore$ The required equation is

$$(y - 0)^2 = -4a\,(x - 0)$$

$$y^2 = -4ax$$

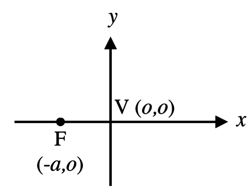


Fig. 4. 33

(ii) From the given data the parabola is open downward.

$\therefore$ The equation is of the form

$$(x - h)^2 = -4a\,(y - k)$$

Here, the vertex (h, k) is (4, 1) and the distance between the vertex and the focus

$VF = a$

$\Rightarrow \sqrt{(4-4)^2 + (1+3)^2} = 4 = a$

$\therefore$ the required equation is

$(x - 4)^2 = -4(4)\,(y - 1)$

$(x - 4)^2 = -16(y - 1)$

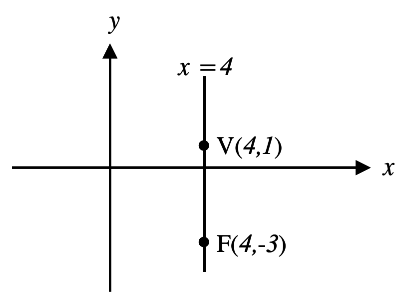


Fig. 4. 34

Example 4.3: Find the equation of the parabola whose vertex is (1, 2) and the equation of the latus rectum is $x = 3$.

Solution: From the given data the parabola is open rightward.

$\therefore$ The equation is of the form

$$(y - k)^2 = 4a(x - h)$$

Here, the vertex $V\,(h, k)$ is (1, 2)

Draw a perpendicular from V to the latus rectum.

It passes through the focus.

$\therefore$ F is (3, 2)

Again $VF = a = 2$

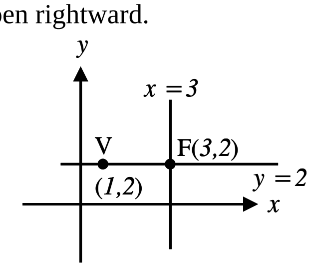


Fig. 4. 35

$\therefore$ The required equation is

$$(y-2)^2 = 4(2)(x-1)$$
$$(y-2)^2 = 8(x-1)$$

Example 4.4: Find the equation of the parabola if the curve is open rightward, vertex is (2, 1) and passing through point (6, 5).

Solution: Since it is open rightward, the equation of the parabola is of the form

$$(y-k)^2 = 4a(x-h)$$

The vertex $V(h, k)$ is (2, 1)

$$\therefore \quad (y-1)^2 = 4a(x-2)$$

But it passes through (6, 5)

$$\therefore \quad 4^2 = 4a(6-2) \quad \Rightarrow \quad a = 1$$

$\therefore$ The required equation is $(y-1)^2 = 4(x-2)$

Example 4.5 : Find the equation of the parabola if the curve is open upward, vertex is (– 1, – 2) and the length of the latus rectum is 4.

Solution: Since it is open upward, the equation is of the form

$$(x-h)^2 = 4a(y-k)$$

Length of the latus rectum $= 4a = 4$ and this gives $a = 1$

The vertex $V(h, k)$ is (– 1, – 2)

Thus the required equation becomes $\quad (x+1)^2 = 4(y+2)$

Example 4.6 : Find the equation of the parabola if the curve is open leftward, vertex is (2, 0) and the distance between the latus rectum and directrix is 2.

Solution: Since it is open leftward, the equation is of the form

$$(y-k)^2 = -4a(x-h)$$

The vertex $V(h, k)$ is (2, 0)

The distance between latus rectum and directrix $= 2a = 2$ giving $a = 1$ and the equation of the parabola is

$$(y-0)^2 = -4(1)(x-2)$$
$$\text{or} \quad y^2 = -4(x-2)$$

Example 4.7 : Find the axis, vertex, focus, directrix, equation of the latus rectum, length of the latus rectum for the following parabolas and hence draw their graphs.

(i) $y^2 = 4x$ (ii) $x^2 = -4y$ (iii) $(y+2)^2 = -8(x+1)$

(iv) $y^2 - 8x + 6y + 9 = 0$ (v) $x^2 - 2x + 8y + 17 = 0$

Solution:

(i) $y^2 = 4x$

$(y - 0)^2 = 4(1)(x - 0)$

Here (h, k) is $(0, 0)$ and $a = 1$

Axis : The axis of symmetry is x-axis.

Vertex : The vertex $V(h, k)$ is $(0, 0)$

Focus : The focus $F(a, 0)$ is $(1, 0)$

Directrix : The equation of the directrix is $x = -a$ i.e. $x = -1$

Latus Rectum : The equation of the latus rectum is $x = a$ i.e. $x = 1$

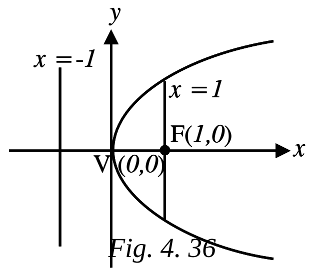


Fig. 4. 36

and its length is $4a = 4(1) = 4$ $\therefore$ the graph of the parabola looks as in *Fig. 4.36*.

(ii) $x^2 = -4y$

$(x - 0)^2 = -4(1)(y - 0)$

Here (h, k) is $(0, 0)$ and $a = 1$

Axis : y-axis or $x = 0$

Vertex : $V(0, 0)$

Focus : $F(0, -a)$ i.e. $F(0, -1)$

Directrix : $y = a$ i.e. $y = 1$

Latus rectum : $y = -a$ i.e. $y = -1$

: length = 4

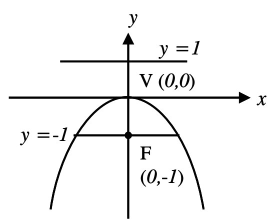


Fig. 4. 37

$\therefore$ the graph looks as in *Fig. 4.37*

(iii) $(y + 2)^2 = -8(x + 1)$

$Y^2 = -8X$ where $X = x + 1$

$Y^2 = -4(2)X$ $Y = y + 2$ $a = 2$

The type is open leftward.

	Referred to X, Y	Referred to x, y $X = x + 1$, $Y = y + 2$
Axis	$Y = 0$	$Y = 0 \Rightarrow y + 2 = 0$
Vertex	$(0, 0)$	$X = 0$; $Y = 0$ $\Rightarrow x + 1 = 0$; $y + 2 = 0$ $x = -1$, $y = -2$ $\therefore V(-1, -2)$

Focus	$(-a, 0)$ i.e. $(-2, 0)$	$X = -2$; $Y = 0$ $\Rightarrow x + 1 = -2, y + 2 = 0$ $x = -3, y = -2$ $F(-3, -2)$
Directrix	$X = a$ i.e. $X = 2$	$X = 2 \Rightarrow x + 1 = 2$ $\Rightarrow x = 1$
Latus rectum	$X = -a$ i.e. $X = -2$	$X = -2 \Rightarrow x + 1 = -2$ $\Rightarrow x = -3$
Length of Latus rectum	$4a = 8$	8

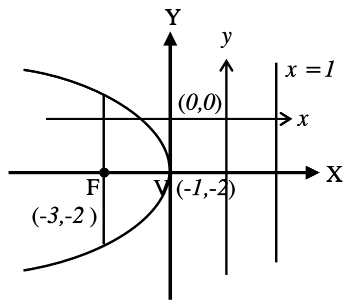


Fig. 4. 38

(iv)
$$\begin{aligned} y^2 - 8x + 6y + 9 &= 0 \\ y^2 + 6y &= +8x - 9 \\ (y+3)^2 - 9 &= +8x - 9 \\ (y+3)^2 &= 8x \\ Y^2 &= 8X \qquad \text{where } X = x \\ Y^2 &= 4(2)X \qquad\quad Y = y + 3 \\ a &= 2 \end{aligned}$$

The type is open rightward

	Referred to X, Y	Referred to x, y $X = x$, $Y = y + 3$
Axis	$Y = 0$	$Y = 0 \Rightarrow y + 3 = 0$
Vertex	$(0, 0)$	$X = 0$; $Y = 0$ $\Rightarrow x = 0$; $y + 3 = 0$ $\therefore V(0, -3)$

Focus	$(a, 0)$ i.e. $(2, 0)$	$X = +2$; $Y = 0$ $\Rightarrow x = 2, y + 3 = 0$ $F(2, -3)$
Directrix	$X = -a$ i.e. $X = -2$	$X = -2 \Rightarrow x = -2$
Latus rectum	$X = a$ i.e. $X = 2$	$X = 2 \Rightarrow x = 2$
Length	$4a = 8$	8

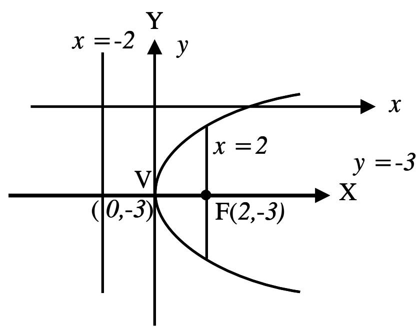


Fig. 4. 39

(v)
$$x^2 - 2x + 8y + 17 = 0$$
$$x^2 - 2x = -8y - 17$$
$$(x-1)^2 - 1 = -8y - 17$$
$$(x-1)^2 = -8y - 16$$
$$(x-1)^2 = -8(y+2)$$
$$X^2 = -8Y \qquad \text{where } X = x - 1$$
$$X^2 = -4(2)Y \qquad Y = y + 2$$
$$a = 2$$

The type is open downward

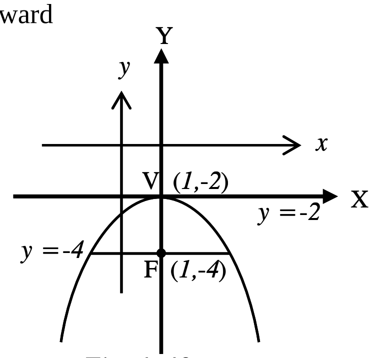


Fig. 4. 40

	Referred to X, Y	Referred to x, y $X = x - 1$, $Y = y + 2$
Axis	$X = 0$	$X = 0 \;\Rightarrow\; x - 1 = 0$ $\Rightarrow\; x = 1$
Vertex	(0, 0)	$X = 0$; $Y = 0$ $\Rightarrow\; x - 1 = 0,\; y + 2 = 0$ $\therefore\; V(1, -2)$
Focus	$(0, -a)$ i.e. $(0, -2)$	$X = 0$; $Y = -2$ $\Rightarrow\; x - 1 = 0, y + 2 = -2$ $F(1, -4)$
Directrix	$Y = a$ i.e. $Y = 2$	$Y = 2 \;\Rightarrow\; y + 2 = 2$ $\Rightarrow\; y = 0$
Latus rectum	$Y = -a$ i.e. $Y = -2$	$Y = -2 \;\Rightarrow\; y + 2 = -2$ $y = -4$
Length	$4a = 8$	8

4.3.7 Some practical problems :

Example 4.8 :

The girder of a railway bridge is in the parabolic form with span 100 ft. and the highest point on the arch is 10 ft. above the bridge. Find the height of the bridge at 10 ft. to the left or right from the midpoint of the bridge.

Solution:

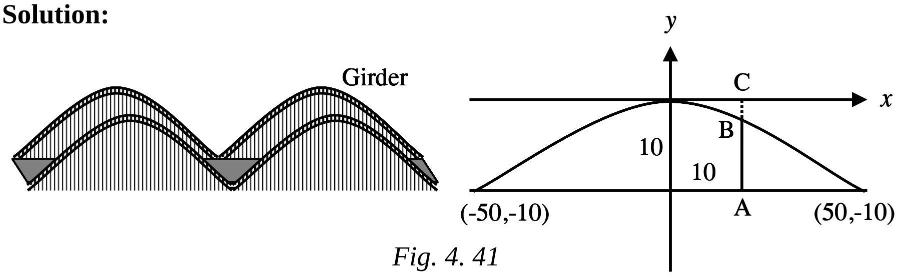


Fig. 4. 41

Consider the parabolic girder as open downwards

i.e., $$x^2 = -4ay$$

It passes through $(50, -10)$

$$\therefore\quad 50 \times 50 = -4a(-10)$$

$$\Rightarrow \quad a = \frac{250}{4}$$

$$\therefore \quad x^2 = -4\left(\frac{250}{4}\right)y$$

$$x^2 = -250y$$

Let $B(10, y_1)$ be a point on the parabola.

$$\therefore \; 100 = -250y_1$$

$$y_1 = -\frac{100}{250} = -\frac{2}{5}$$

Let AB be the height of the bridge at 10 ft to the right from the mid point

$AC = 10$ and $BC = \frac{2}{5}$

$AB = 10 - \frac{2}{5} = 9\frac{3}{5}$ ft

i.e. the height of the bridge at the required place = $9\frac{3}{5}$ ft.

Example 4.9 :

The headlight of a motor vehicle is a parabolic reflector of diameter 12cm and depth 4cm. Find the position of bulb on the axis of the reflector for effective functioning of the headlight.

Solution:

By the property of parabolic reflector the position of the bulb should be placed at the focus.

By taking the vertex at the origin, the equation of the reflector is $y^2 = 4ax$

Let PQ be the diameter of the reflector

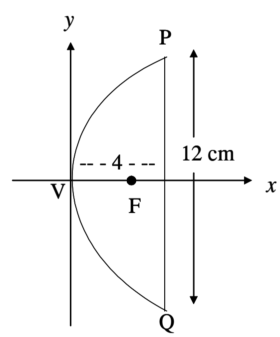


Fig. 4. 42

$\therefore$ P is (4, 6) and since $P(4, 6)$ lies on the parabola, $36 = 4a \times 4 \Rightarrow a = 2.25$

The focus is at a distance of 2.25cm from the vertex on the x-axis.

∴ The bulb has to be placed at a distance of 2.25 cms from the centre of the mirror.

Example 4.10 :

On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4mts when it is 6 mts away from the point of projection. Finally it reaches the ground 12 mts away from the starting point. Find the angle of projection.

Solution:

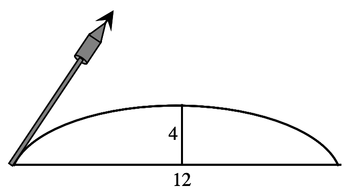


Fig. 4. 43

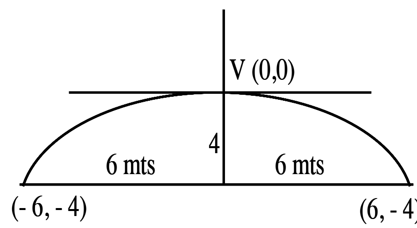


Fig. 4. 44

The equation of the parabola is of the form $x^2 = -4ay$ (by taking the vertex at the origin). It passes through (6, – 4)

$$\therefore\ 36 = 16a \quad \Rightarrow \quad a = \frac{9}{4}$$

The equation is $\qquad x^2 = -9y \qquad \ldots(1)$

Find the slope at (– 6, – 4)

Differentiating (1) with respect to x, we get

$$2x = -9\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2}{9}\ x$$

$$\text{At } (-6, -4), \frac{dy}{dx} = -\frac{2}{9} \times -6 = \frac{4}{3} \quad \text{i.e. } \tan\theta = \frac{4}{3}$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

∴ The angle of projection is $\tan^{-1}\left(\frac{4}{3}\right)$

Example 4.11 :

A reflecting telescope has a parabolic mirror for which the distance from the vertex to the focus is 9mts. If the distance across (diameter) the top of the mirror is 160cm, how deep is the mirror at the middle?

Solution:

Let the vertex be at the origin.

$$VF = a = 900$$

The equation of the parabola is

$$y^2 = 4 \times 900 \times x$$

Let x_1 be the depth of the mirror at the middle

Since $(x_1, 80)$ lies on the parabola

$80^2 = 4 \times 900 \times x_1 \Rightarrow x_1 = \frac{16}{9}$

$\therefore$ depth of the mirror $= \frac{16}{9}$ cm

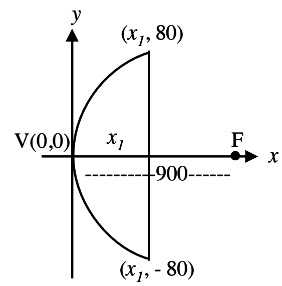


Fig. 4. 45

Example 4.12 :

Assume that water issuing from the end of a horizontal pipe, 7.5m above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

Solution:

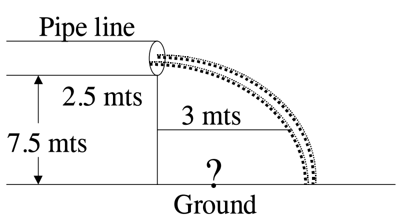


Fig. 4. 46

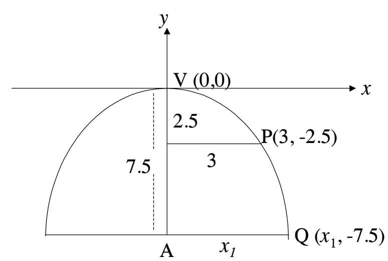


Fig. 4. 47

As per the given information, we can take the parabola as open downwards i.e. $x^2 = -4ay$

Let P be the point on the flow path, 2.5m below the line of the pipe and 3m beyond the vertical line through the end of the pipe.

$\therefore$ P is $(3, -2.5)$

Thus $\quad 9 = -4a(-2.5)$

$$\Rightarrow a = \frac{9}{10}$$

$\therefore$ The equation of the parabola is $x^2 = -4 \times \frac{9}{10}\, y$

Let x_1 be the distance between the bottom of the vertical line on the ground from the pipe end and the point on which the water touches the ground. But the height of the pipe from the ground is 7.5 *m*

The point $(x_1, -7.5)$ lies on the parabola

$$x_1^{\,2} = -4 \times \frac{9}{10} \times (-7.5) = 27$$

$$x_1 = 3\sqrt{3}$$

$\therefore$ The water strikes the ground $3\sqrt{3}$ m beyond the vertical line.

Example 4.13 :

A comet is moving in a parabolic orbit around the sun which is at the focus of a parabola. When the comet is 80 million kms from the sun, the line segment from the sun to the comet makes an angle of $\frac{\pi}{3}$ radians with the axis of the orbit. find (i) the equation of the comet's orbit (ii) how close does the comet come nearer to the sun? (Take the orbit as open rightward).

Solution:

Take the parabolic orbit as open rightward and the vertex at the origin.

Let *P* be the position of the comet in which *FP* = 80 million kms.

Draw a perpendicular *PQ* from *P* to the axis of the parabola.

Let $FQ = x_1$

From the triangle *FQP*,

$$PQ = FP \,.\, \sin\frac{\pi}{3}$$

$$= 80 \times \frac{\sqrt{3}}{2} = 40\sqrt{3}$$

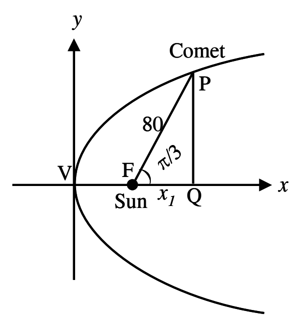


Fig. 4. 48

$$\text{Thus } FQ = x_1 = FP \,.\, \cos\frac{\pi}{3} = 80 \times \frac{1}{2} = 40$$

$$\therefore VQ = a + 40 \text{ if } VF = a\,; \quad P \text{ is } (VQ, PQ) = \left(a + 40, 40\sqrt{3}\right)$$

Since *P* lies on the parabola $y^2 = 4ax$

$$\left(40\sqrt{3}\right)^2 = 4a(a + 40)$$

$$\Rightarrow a = -60 \text{ or } 20$$

$$a = -60 \text{ is not acceptable.}$$

$\therefore$ The equation of the orbit is

$$y^2 = 4 \times 20 \times x$$

$$y^2 = 80x$$

The shortest distance between the Sun and the Comet is VF i.e. a

$\therefore$ The shortest distance is 20 million kms.

Example 4.14 :

A cable of a suspension bridge hangs in the form of a parabola when the load is uniformly distributed horizontally. The distance between two towers is 1500 ft, the points of support of the cable on the towers are 200ft above the road way and the lowest point on the cable is 70ft above the roadway. Find the vertical distance to the cable (parallel to the roadway) from a pole whose height is 122 ft.

Solution :

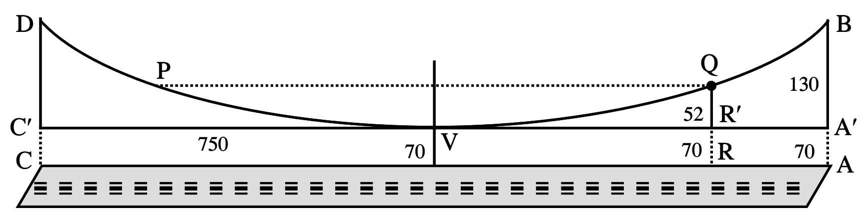


Fig. 4. 49

Take the lowest point on the cable as the vertex and take it as origin. Let AB and CD be the towers. Since the distance between the two towers is 1500 ft.

$$VA' = 750 \text{ ft} \; ; \; AB = 200 \text{ ft}$$

$$\therefore A'B = 200 - 70 = 130 \text{ ft}$$

Thus the point B is (750, 130)

The equation of the parabola is $x^2 = 4ay$

Since B is a point on $x^2 = 4ay$

$$(750)^2 = 4a(130)$$

$$\Rightarrow 4a = \frac{75 \times 750}{13}$$

$\therefore$ The equation is $x^2 = \dfrac{75 \times 750}{13} y$

Let PQ be the vertical distance to the cable from the pole RQ.

$RQ = 122, \quad RR' = 70 \quad \Rightarrow \quad R'Q = 52$

Let VR' be $x_1 \therefore Q$ is $(x_1, 52)$

Q is a point on parabola

$$x_1^2 = \frac{75 \times 750}{13} \times 52$$

$$x_1 = 150\sqrt{10}$$

$$PQ = 2x_1 = 300\sqrt{10} \text{ ft.}$$

EXERCISE 4.1

(1) Find the equation of the parabola if

(i) Focus : $(2, -3)$; directrix : $2y - 3 = 0$

(ii) Focus : $(-1, 3)$; directrix : $2x + 3y = 3$

(iii) Vertex : $(0, 0)$; focus : $(0, -4)$

(iv) Vertex : $(1, 4)$; focus : $(-2, 4)$

(v) Vertex : $(1, 2)$; latus rectum : $y = 5$

(vi) Vertex : $(1, 4)$; open leftward and passing through the point $(-2, 10)$

(vii) Vertex : $(3, -2)$; open downward and the length of the latus rectum is 8.

(viii) Vertex : $(3, -1)$; open rightward ; the distance between the latus rectum and the directrix is 4.

(ix) Vertex : $(2, 3)$; open upward ; and passing through the point $(6, 4)$.

(2) Find the axis, vertex, focus, equation of directrix, latus rectum, length of the latus rectum for the following parabolas and hence sketch their graphs.

(i) $y^2 = -8x$ (ii) $x^2 = 20y$

(iii) $(x - 4)^2 = 4(y + 2)$ (iv) $y^2 + 8x - 6y + 1 = 0$

(v) $x^2 - 6x - 12y - 3 = 0$

(3) If a parabolic reflector is 20cm in diameter and 5cm deep, find the distance of the focus from the centre of the reflector.

(4) The focus of a parabolic mirror is at a distance of 8cm from its centre (vertex). If the mirror is 25cm deep, find the diameter of the mirror.

(5) A cable of a suspension bridge is in the form of a parabola whose span is 40 mts. The road way is 5 mts below the lowest point of the cable. If an extra support is provided across the cable 30 mts above the ground level, find the length of the support if the height of the pillars are 55 mts.

4.4 Ellipse :

Definition : The locus of a point in a plane whose distance from a fixed point bears a constant ratio, less than one to its distance from a fixed line is called ellipse.

Note : Eventhough the syllabus does not require the derivation of standard equation and the tracing of ellipse (4.4.1, 4.4.2) and it needs only the standard equation and the diagram, the equation is derived and the curve is traced for better understanding.

Now, we will derive the standard equation of an ellipse.

4.4.1 Standard equation of the ellipse :

Given :

★ Fixed point F

★ Fixed line l

★ Eccentricity e ($e < 1$)

★ Moving point P (x, y)

Construction :

★ Plot the fixed point F and draw the fixed line l

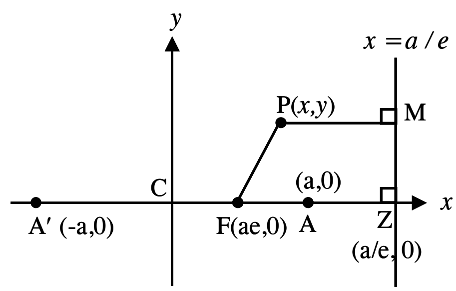


Fig. 4. 50

★ Drop a perpendicular (FZ) from F to l

★ Drop a perpendicular (PM)

★ Plot the points A, A' which divides FZ internally and externally in the ratio $e : 1$

★ Take $AA' = 2a$ and treat it as x-axis.

★ Draw a perpendicular bisector to AA' and treat it as y-axis.

★ Let C be the origin.

★ The known points are the origin $C(0, 0)$, $A(a, 0)$, $A'(-a, 0)$

★ To find the co-ordinates of F and M, do the following :

Since A, A' divides FZ internally and externally in the ratio $e : 1$ respectively,

$$\frac{FA}{AZ} = \frac{e}{1}$$
$$\therefore FA = e\, AZ$$
i.e., $CA - CF = e\,(CZ - CA)$
$$\therefore a - CF = e\,(CZ - a) \quad \ldots(1)$$

$$\frac{FA'}{A'Z} = \frac{e}{1}$$
$$\therefore FA' = e\, A'Z$$
i.e. $A'C + CF = e\,(A'C + CZ)$
$$\therefore a + CF = e(a + CZ) \quad \ldots(2)$$

$$(2) + (1) \Rightarrow 2a = e\,[2CZ] \Rightarrow CZ = \frac{a}{e}$$
$$(2) - (1) \Rightarrow 2CF = e(2a) \Rightarrow CF = ae$$

$\therefore M$ is $\left(\frac{a}{e},\ y\right)$ and F is $(ae, 0)$

To obtain the equation of the ellipse, do the following :

Since P is any point on the ellipse
$$\frac{FP}{PM} = e \Rightarrow FP^2 = e^2PM^2$$
$$\text{i.e. } (x - ae)^2 + (y - 0)^2 = e^2\left[\left(x - \frac{a}{e}\right)^2 + (y - y)^2\right]$$
$$\Rightarrow x^2 - e^2x^2 + y^2 = a^2 - a^2e^2$$
$$(1 - e^2)\, x^2 + y^2 = a^2\,(1 - e^2)$$

Dividing by $a^2\,(1 - e^2)$, we get
$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$
i.e.
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2\,(1 - e^2)$$

This is known as the standard equation of an ellipse.

4.4.2 Tracing of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$

(i) Symmetry property :

It is symmetrical about x-axis and y-axis simultaneously and hence about the origin.

(ii) Special points :

It does not pass through the origin.

To find the points on x-axis, put $y = 0$, we get $x = \pm a$. Therefore the curve meets the x-axis at $A(a, 0)$ and $A'(-a, 0)$.

To find the points on y-axis, put $x = 0$, we get $y = \pm b$. Therefore the curve meets the y-axis at $B(0, b)$ and $B'(0, -b)$

(iii) Existence of the curve :

Write the equation of the ellipse as $y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$. y is real only if $a^2 - x^2 \geq 0$. i.e., the curve does not exist for $a^2 - x^2 < 0$ or $x^2 - a^2 > 0$

Equivalently the curve does not exist for $x > a$ and $x < -a$. Thus the curve exists only when $-a \leq x \leq a$.

Write the equation of the ellipse as $x = \pm \frac{a}{b} \sqrt{b^2 - y^2}$. x is real only if $b^2 - y^2 \geq 0$. The curve does not exist for $b^2 - y^2 < 0$ i.e., $y^2 - b^2 > 0$ i.e., the curve does not exist for $y > b$ and $y < -b$. The curve exist only when $-b \leq y \leq b$. $\therefore$ Ellipse is a closed curve bounded by the lines $x = \pm a$ and $y = \pm b$. Thus the curve is

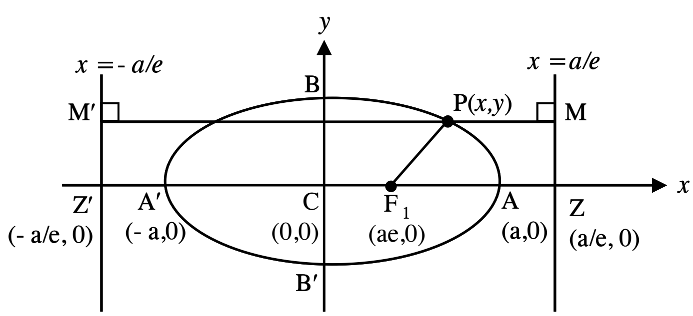


Fig. 4. 51

4.4.3 Important definitions regarding ellipse :

Focus : The fixed point is called focus, denoted as F_1 $(ae, 0)$.

Directrix : The fixed line is called directrix l of the ellipse and its equation is $x = \frac{a}{e}$.

Major axis : The line segment AA' is called the major axis and the length of the major axis is $2a$. The equation of the major axis is $y = 0$.

Minor axis : The line segment BB' is called the minor axis and the length of minor axis is $2b$. Equation of the minor axis is $x = 0$. Note that the length of major axis is always greater than minor axis.

Centre : The point of intersection of the major axis and minor axis of the ellipse is called the centre of the ellipse. Here $C(0, 0)$ is the centre of the ellipse. Note that the centre need not be the origin of the ellipse always.

End points of latus rectum and length of latus rectum :

To find the end points, solve $x = ae$ … (1) and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ … (2)

Using (2) in (1) we get

$$\frac{a^2 e^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \frac{y^2}{b^2} = 1 - e^2$$

$$\therefore y^2 = b^2 (1 - e^2)$$

$$= b^2 \left(\frac{b^2}{a^2}\right) \quad \left\{ \begin{array}{l} \because b^2 = a^2 (1 - e^2) \\ \text{or } \frac{b^2}{a^2} = 1 - e^2 \end{array} \right.$$

$$\therefore y = \pm \frac{b^2}{a}$$

If L_1 and L_1' are the end points of the latus rectum then L_1 is $\left(ae, \frac{b^2}{a}\right)$ and L_1' is $\left(ae, -\frac{b^2}{a}\right)$.

The end points of the other latus rectum are $\left(-ae, \pm \frac{b^2}{a}\right)$.

The length of the latus rectum is $\frac{2b^2}{a}$.

For the above discussed ellipse, the major axis is along x-axis. There is another standard ellipse in which the major axis is along the y-axis.

Vertices : The points of intersection of the ellipse and its major axis are called its vertices. Here the vertices of the ellipse are $A(a, 0)$ and $A'(-a, 0)$.

Focal distance : The focal distance with respect to any point P on the ellipse is the distance of P from the referred focus.

Focal chord : A chord which passes through the focus of the ellipse is called the focal chord of the ellipse.

A special property : Thanks to the symmetry about the origin, it permits

(i) the second focus F_2 $(-ae, 0)$

(ii) the second directrix $x = -\frac{a}{e}$

Latus rectum : It is a focal chord perpendicular to the major axis of the ellipse. The equations of latus rectum are $x = ae$, $x = -ae$.

Eccentricity

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

Remark :

In the case of an ellipse $0 < e < 1$. As $e \to 0$, $\frac{b}{a} \to 1$ i.e., $b \to a$ or the length of the minor and major axes are close in size. i.e., the ellipse is close to being a circle.

As $e \to 1$, $\frac{b}{a} \to 0$ and the ellipse degenerates into a line segment (degenerate conic) i.e., the ellipse is flat.

4.4.4 The other standard form of the ellipse :

If the major axis of the ellipse is along the y-axis, then the equation of the ellipse takes the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, for $a > b$.

For this type of ellipse, we have the following as explained in the earlier ellipse.

Centre : C $(0, 0)$

Vertices : A $(0, a)$, A' $(0, -a)$

Foci : F_1 $(0, ae)$, F_2 $(0, -ae)$

Equation of major axis is $x = 0$

Equation of minor axis is $y = 0$

End points of minor axis : B $(b, 0)$, B' $(-b, 0)$

Equation of directrices : $y = \pm a/e$

End points of latus rectums : $\left(\pm\dfrac{b^2}{a},\ ae\right)$, $\left(\pm\dfrac{b^2}{a},\ -ae\right)$

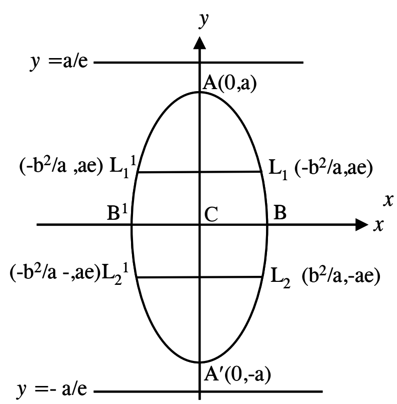


Fig. 4. 52

4.4.5 General forms of standard ellipses :

To obtain the general forms of standard ellipses, replace x by $x - h$ and y by $(y - k)$ if the centre is $C(h, k)$.

Thus the general forms of standard ellipses are $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1,\ a > b$

Focal property of an ellipse :

The sum of the focal distances of any point on an ellipse is constant and is equal to the length of the major axis.

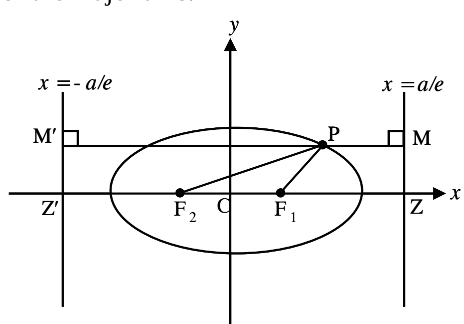


Fig. 4. 53

Proof :

To prove : $F_1P + F_2P = 2a$

Let P be a point on the ellipse. Drop the perpendiculars PM and PM' to the directrices $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively .

We know that $\frac{F_1P}{PM} = e, \quad \frac{F_2P}{PM'} = e$

$$\therefore F_1P = ePM, \quad F_2P = e\,PM'$$

$$\begin{aligned} \therefore \; F_1P + F_2P &= e(PM + PM') \\ &= e(MM') \\ &= e \,.\, \frac{2a}{e} \\ &= 2a \\ &= \text{length of the major axis} \end{aligned}$$

Example 4.15 : Find the equation of the ellipse whose foci are (1, 0) and (– 1, 0) and eccentricity is $\frac{1}{2}$.

Solution:

The centre of the ellipse is the midpoint of FF' where F is (1, 0) and F' is (– 1, 0).

$\therefore$ Centre C is $\left(\frac{1-1}{2}, \frac{0+0}{2}\right) = (0, 0)$

But $F_1F_2 = 2ae = 2$ and $e = \frac{1}{2}$

$2a \times \frac{1}{2} = 2$

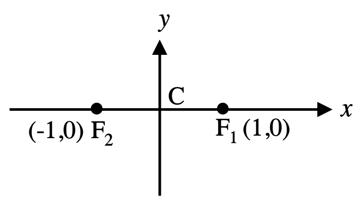


Fig. 4. 54

$$a = 2$$

$$b^2 = a^2(1 - e^2) = 4\left(1 - \frac{1}{4}\right) = 3$$

From the given data the major axis is along x-axis.

$\therefore$ the equation of the ellipse is of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{3} = 1$$

Example 4.16 : Find the equation of the ellipse whose one of the foci is (2, 0) and the corresponding directrix is $x = 8$ and eccentricity is $\frac{1}{2}$

Solution:

Let $P(x, y)$ be a moving point. By definition

$$\frac{FP}{PM} = e$$

$$\therefore \quad FP^2 = e^2\, PM^2$$

$$(x-2)^2 + (y-0)^2 = \frac{1}{4}\left(\pm\frac{x-8}{\sqrt{1}}\right)^2$$

$$(x-2)^2 + y^2 = \frac{1}{4}\,(x-8)^2$$

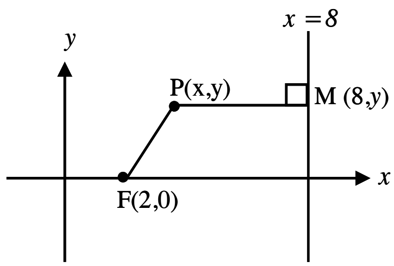


Fig. 4. 55

$$4\left[(x-2)^2 + y^2\right] = (x-8)^2$$

$$3x^2 + 4y^2 = 48$$

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

Aliter :

From the given data, the major axis is along the x-axis and the equation of the ellipse may be taken as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

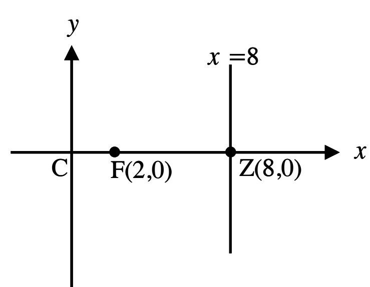


Fig. 4. 56

$$FZ = \frac{a}{e} - ae = 6$$

$$\text{But } e = \frac{1}{2} \Rightarrow 2a - \frac{1}{2}\,a = 6$$

$$\Rightarrow \frac{3}{2}a = 6 \Rightarrow a = 4$$

$$b^2 = a^2(1-e^2) = 16\left(1-\frac{1}{4}\right) = 16 \times \frac{3}{4} = 12$$

$\therefore$ The required equation is $\frac{x^2}{16} + \frac{y^2}{12} = 1$

Example 4.17 : Find the equation of the ellipse with focus (– 1, – 3), directrix $x - 2y = 0$ and eccentricity $\frac{4}{5}$

Solution:

Let $P(x, y)$ be a moving point. By definition

$$\frac{FP}{PM} = e$$

$$\therefore \quad FP^2 = e^2\, PM^2$$

$$(x+1)^2 + (y+3)^2 = \frac{16}{25}\left[\pm\frac{x-2y}{\sqrt{1+4}}\right]^2$$

$$125\,[(x+1)^2 + (y+3)^2] = 16\,(x-2y)^2$$

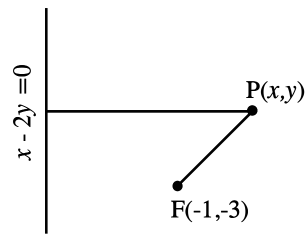


Fig. 4. 57

$$\Rightarrow 109x^2 + 64xy + 61y^2 + 250x + 750y + 1250 = 0$$

Example 4.18 : Find the equation of the ellipse with foci (± 4, 0) and vertices (± 5, 0)

Solution:

Let the foci be $F_1(4, 0)$ and F_2 (– 4, 0), vertices be $A(5, 0)$ and $A'(-5, 0)$. The centre is the midpoint of AA'

i.e., C is $\left(\frac{-5+5}{2}, \frac{0+0}{2}\right)$

$= (0, 0)$

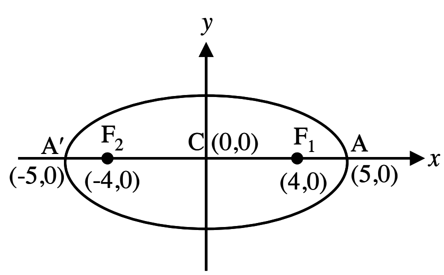


Fig. 4. 58

From the given data, the major axis is along the x-axis and the equation of the ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Here $CA = a = 5$

$CF = ae = 4$ since $e = \frac{4}{5}$

$b^2 = a^2(1 - e^2) = 25 - 16 = 9$ and the required equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Example 4.19 : The centre of the ellipse is (2, 3). One of the foci is (3, 3). Find the other focus.

Solution:

From the given data the major axis is parallel to the x axis. Let F_1 be (3, 3)

Let F_2 be the point (x, y). Since C (2, 3) is the midpoint of F_1 and F_2 on the major axis $y = 3$

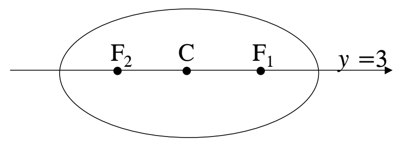


Fig. 4. 59

$$\frac{x+3}{2} = 2 \text{ and } \frac{y+3}{2} = 3$$

This gives $x = 1$ and $y = 3$. Thus the other focus is (1, 3).

Example 4.20 : Find the equation of the ellipse whose centre is (1, 2), one of the foci is (1, 3) and eccentricity is $\frac{1}{2}$

Solution:

The major axis is parallel to y-axis.

$\therefore$ The equation is of the form

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$CF_1 = ae = 1$$

But $e = \frac{1}{2} \Rightarrow a = 2,\ a^2 = 4$

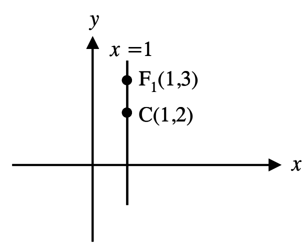


Fig. 4. 60

$$b^2 = a^2(1-e^2) = 4\left(1-\frac{1}{4}\right) = 3\ ; \quad C(h, k) = (1, 2)$$

$\therefore$ The required equation is $\frac{(x-1)^2}{3} + \frac{(y-2)^2}{4} = 1$

Example 4.21 : Find the equation of the ellipse whose major axis is along x-axis, centre at the origin, passes through the point (2, 1) and eccentricity $\frac{1}{2}$

Solution:

Since the major axis is along the x-axis and the centre is at the origin, the equation of the ellipse is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

It passes through the point (2, 1). $\therefore \frac{4}{a^2} + \frac{1}{b^2} = 1$ … (1)

$$e = \frac{1}{2}$$

$$b^2 = a^2 (1 - e^2) \Rightarrow b^2 = a^2 \left(1 - \frac{1}{4}\right)$$

$$\therefore 4b^2 = 3a^2 \quad \ldots (2)$$

Solving (1) and (2) we get $a^2 = \frac{16}{3}$, $b^2 = 4$

$\therefore$ The required equation is $\frac{x^2}{16/3} + \frac{y^2}{4} = 1$

Example 4.22 : Find the equation of the ellipse if the major axis is parallel to y-axis, semi-major axis is 12, length of the latus rectum is 6 and the centre is (1, 12)

Solution:

Since the major axis is parallel to y-axis the equation of the ellipse is of the form

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

The centre C (h, k) is (1, 12)

Semi major axis $a = 12 \Rightarrow a^2 = 144$

Length of the latus rectum $\frac{2b^2}{a} = 6 \Rightarrow \frac{2b^2}{12} = 6$

$\therefore$ $b^2 = 36$ and the required equation is $\frac{(x-1)^2}{36} + \frac{(y-12)^2}{144} = 1$

Example 4.23 : Find the equation of the ellipse given that the centre is (4, – 1), focus is (1, – 1) and passing through (8, 0).

Solution :

From the given data since the major axis is parallel to the x axis, the equation is of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The centre $C(h, k)$ is (4, – 1)

$$\frac{(x-4)^2}{a^2} + \frac{(y+1)^2}{b^2} = 1$$

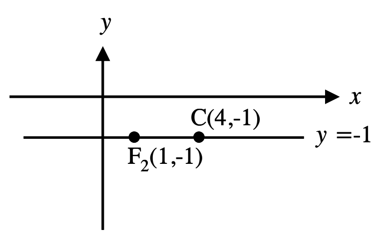


Fig. 4. 61

It passes through (8, 0) $\therefore \frac{16}{a^2} + \frac{1}{b^2} = 1$ … (1)

But $CF_1 = ae = 3$

$$b^2 = a^2(1 - e^2) = a^2 - a^2e^2 = a^2 - 9$$

(1) $\Rightarrow$ $\frac{16}{a^2} + \frac{1}{a^2 - 9} = 1$

$\Rightarrow$ $16a^2 - 144 + a^2 = a^4 - 9a^2$

$\Rightarrow$ $a^4 - 26a^2 + 144 = 0$

$\Rightarrow$ $a^2 = 18$ or 8

Case (i) : $a^2 = 18$

$b^2 = a^2 - 9 = 18 - 9 = 9$

Case (ii) : $a^2 = 8$

$b^2 = 8 - 9 = -1$ which is not possible

$\therefore$ $a^2 = 18, \quad b^2 = 9$

Thus the equation is $\frac{(x-4)^2}{18} + \frac{(y+1)^2}{9} = 1$

Example 4.24 : Find the equation of the ellipse whose foci are (2, 1), (– 2, 1) and length of the latus rectum is 6.

Solution :

From the given data the major axis is parallel to the x axis.

$\therefore$ The equation is of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

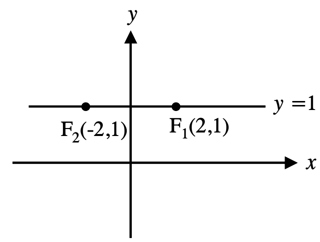


Fig. 4. 62

Since the centre is the midpoint of F_1F_2

C is $\left(\frac{-2+2}{2}, \frac{1+1}{2}\right) = (0, 1)$

and the equation becomes

$$\frac{x^2}{a^2} + \frac{(y-1)^2}{b^2} = 1$$

$$F_1 F_2 = 2ae = 4 \Rightarrow a^2 e^2 = 4$$

$$a^2 e^2 = a^2 - b^2$$

$$\therefore a^2 - b^2 = 4 \qquad \ldots (1)$$

The length of the latus rectum $\dfrac{2b^2}{a} = 6 \quad b^2 = 3a \qquad \ldots (2)$

(1) $\Rightarrow$ $a^2 - 3a - 4 = 0$ (by (2))

$\Rightarrow$ $a = 4$ or -1

$a = -1$ is absurd

$\therefore a = 4$

$b^2 = 3a = 12$

Thus the equation is $\dfrac{x^2}{16} + \dfrac{(y-1)^2}{12} = 1$

Example 4.25 : Find the equation of the ellipse whose vertices are (– 1, 4) and (– 7, 4) and eccentricity is $\dfrac{1}{3}$.

Solution :

From the given data the major axis is parallel to x axis.

$\therefore$ The equation is of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

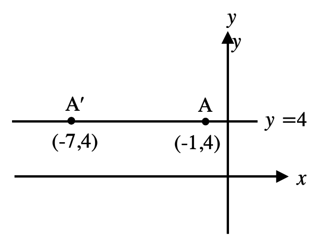


Fig. 4. 63

The centre is the midpoint of AA'

$\therefore C$ is $\left(\dfrac{-1-7}{2}, \dfrac{4+4}{2}\right) = (-4, 4)$

Thus the equation becomes

$$\frac{(x+4)^2}{a^2} + \frac{(y-4)^2}{b^2} = 1$$

We know that $AA' = 2a = 6 \Rightarrow a = 3$

$$b^2 = a^2(1 - e^2) = 9\left(1 - \frac{1}{9}\right) = 8$$

The required equation is $\dfrac{(x+4)^2}{9} + \dfrac{(y-4)^2}{8} = 1$

Example 4.26 : Find the equation of the ellipse whose foci are (1, 3) and (1, 9) and eccentricity is $\frac{1}{2}$

Solution :

From the given data the major axis is parallel to *y* axis.

$\therefore$ The equation is of the form

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

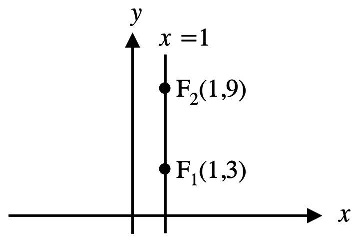


Fig. 4. 64

The centre of the ellipse is the midpoint of $F_1 F_2$

$\therefore$ C is $\left(\frac{1+1}{2}, \frac{3+9}{2}\right) = (1, 6)$

$$F_1 F_2 = 2ae = 6$$

$$ae = 3$$

$$\text{But} \quad e = \frac{1}{2} \qquad \therefore \quad a = 6$$

$$b^2 = a^2(1-e^2) = 36\left(1-\frac{1}{4}\right) = 27$$

Thus the required equation is

$$\frac{(x-1)^2}{27} + \frac{(y-6)^2}{36} = 1$$

Property (without proof) :

A point moves such that the sum of its distances from two fixed points in a plane is a constant. The locus of this point is an ellipse.

Example 4.27 : Find the equation of a point which moves so that the sum of its distances from (– 4, 0) and (4, 0) is 10.

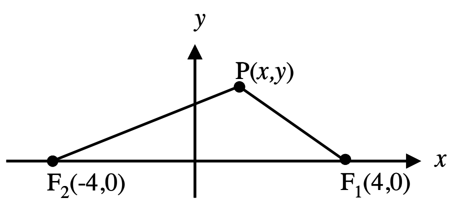


Fig. 4. 65

Solution :

Let F_1 and F_2 be the fixed points (4, 0) and (– 4, 0) respectively and $P(x_1, y_1)$ be the moving point.

It is given that $F_1P + F_2P = 10$

i.e., $\sqrt{(x_1 - 4)^2 + (y_1 - 0)^2} + \sqrt{(x_1 + 4)^2 + (y_1 - 0)^2} = 10$

Simplifying we get

$9x_1^2 + 25y_1^2 = 225$. $\therefore$ The locus of (x_1, y_1) is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Example 4. 28 : Find the equations and lengths of major and minor axes of

(i) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ (ii) $4x^2 + 3y^2 = 12$ (iii) $\frac{(x-1)^2}{9} + \frac{(y+1)^2}{16} = 1$

Solution :

(i) The major axis is along x-axis and the minor axis is along y-axis. This gives the equation of major axis as $y = 0$ and the equation of minor axis as $x = 0$. We have $a^2 = 9$; $b^2 = 4$ $\Rightarrow$ $a = 3$, $b = 2$

$\therefore$ The length of major axis is $2a = 6$ and the length of minor axis is $2b = 4$

(ii) $\frac{x^2}{3} + \frac{y^2}{4} = 1$

The major axis is along y-axis and the minor axis is along x-axis.

$\therefore$ The equation of major axis is $x = 0$ and the equation of minor axis is $y = 0$. Here $a^2 = 4$; $b^2 = 3$ $\Rightarrow$ $a = 2$, $b = \sqrt{3}$

$\therefore$ The length of major axis $(2a) = 4$

The length of minor axis $(2b) = 2\sqrt{3}$

(iii) Let $x - 1 = X$ and $y + 1 = Y$

$\therefore$ The given equation becomes $\frac{X^2}{9} + \frac{Y^2}{16} = 1$ Clearly the major axis is along Y-axis and the minor axis is along X-axis.

$\therefore$ The equation of major axis is $X = 0$ and the equation of minor axis is $Y = 0$

i.e., the equation of major axis is $x - 1 = 0$ and the equation of minor axis is $y + 1 = 0$

Here $a^2 = 16$, $b^2 = 9$

$\Rightarrow$ $a = 4$, $b = 3$

$\therefore$ Length of major axis $(2a) = 8$

$\therefore$ Length of minor axis $(2b) = 6$

Example 4. 29 : Find the equations of axes and length of axes of the ellipse $6x^2 + 9y^2 + 12x - 36y - 12 = 0$

Solution :

$$\begin{aligned} 6x^2 + 9y^2 + 12x - 36y - 12 &= 0 \\ (6x^2 + 12x) + (9y^2 - 36y) &= 12 \\ 6(x^2 + 2x) + 9(y^2 - 4y) &= 12 \\ 6\{(x + 1)^2 - 1\} + 9\ \{(y - 2)^2 - 4\} &= 12 \\ 6(x + 1)^2 + 9\ (y - 2)^2 &= 12 + 6 + 36 \\ 6(x + 1)^2 + 9(y - 2)^2 &= 54 \\ \frac{(x + 1)^2}{9} + \frac{(y - 2)^2}{6} &= 1 \end{aligned}$$

Let $X = x + 1$; $Y = y - 2$

$\therefore$ The equation becomes $\frac{X^2}{9} + \frac{Y^2}{6} = 1$

Clearly the major axis is along X-axis and the minor axis is along Y-axis.

$\therefore$ The equation of the major axis is $Y = 0$ and the equation of the minor axis is $X = 0$.

The equation of the major axis is $y - 2 = 0$ and of minor axis is $x + 1 = 0$

i.e., the equation of the major axis is $y - 2 = 0$

Here $a^2 = 9$, $b^2 = 6$ $\Rightarrow$ $a = 3$, $b = \sqrt{6}$

$\therefore$ The length of major axis $(2a) = 6$

The length of minor axis $(2b) = 2\sqrt{6}$

Example 4.30 : Find the equations of directrices, latus rectum and length of latus rectums of the following ellipses.

(i) $\frac{x^2}{16} + \frac{y^2}{9} = 1$ (ii) $25x^2 + 9y^2 = 225$ (iii) $4x^2 + 3y^2 + 8x + 12y + 4 = 0$

Solution :

(i) The major axis is along x-axis

Here $a^2 = 16$, $b^2 = 9$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

Equations of directrices are $x = \pm \frac{a}{e}$

$$x = \frac{\pm 16}{\sqrt{7}}$$

Equations of the latus rectums are $x = \pm ae$

$$x = \pm \sqrt{7}$$

Length of the latus rectum $\frac{2b^2}{a} = \frac{2 \times 9}{4} = \frac{9}{2}$

(ii) $25x^2 + 9y^2 = 225 \quad \therefore \frac{x^2}{9} + \frac{y^2}{25} = 1$

Here $a^2 = 25, \quad b^2 = 9$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

The equations of the directrices are $y = \pm \frac{a}{e}$

$$y = \frac{\pm 25}{4}$$

Equations of the latus rectum are $y = \pm ae$

$$y = \pm 4$$

Length of the latus rectum is $\frac{2b^2}{a} = \frac{2 \times 9}{5} = \frac{18}{5}$

(iii)

$$\begin{aligned}
4x^2 + 3y^2 + 8x + 12y + 4 &= 0 \\
(4x^2 + 8x) + (3y^2 + 12y) + 4 &= 0 \\
4(x^2 + 2x) + 3(y^2 + 4y) &= -4 \\
4\{(x+1)^2 - 1\} + 3\{(y+2)^2 - 4\} &= -4 \\
4(x+1)^2 + 3(y+2)^2 &= 12 \\
\frac{(x+1)^2}{3} + \frac{(y+2)^2}{4} &= 1
\end{aligned}$$

$$\frac{X^2}{3} + \frac{Y^2}{4} = 1 \text{ where } X = x + 1 \ , \ Y = y + 2$$

The major axis is along Y axis. Here $a^2 = 4$, $b^2 = 3$ and $e = \frac{1}{2}$

Equations of the directrices are $Y = \pm \frac{a}{e}$ i.e. $Y = \pm \frac{2}{(1/2)}$

$Y = \pm 4$

(i) $Y = 4 \Rightarrow y + 2 = 4 \Rightarrow y = 2$

(ii) $Y = -4 \Rightarrow y + 2 = -4 \Rightarrow y = -6$

The directrices are $y = 2$ and $y = -6$

Equations of the latus rectum are $Y = \pm ae$ i.e. $Y = \pm 2\left(\frac{1}{2}\right)$

$Y = \pm 1$

(i) $Y = 1 \Rightarrow y + 2 = 1$

$\Rightarrow y = -1$

(ii) $Y = -1 \Rightarrow y + 2 = 1$

$\Rightarrow y = -3$

$\therefore$ Equation of the latus rectum are $y = -1$ and $y = -3$

Length of the latus rectum is $\frac{2b^2}{a} = \frac{2 \times 3}{2} = 3$

Example 4.31 : Find the eccentricity, centre, foci, vertices of the following ellipses : (i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (ii) $\frac{x^2}{4} + \frac{y^2}{9} = 1$

(iii) $\frac{(x+3)^2}{6} + \frac{(y-5)^2}{4} = 1$ (iv) $36x^2 + 4y^2 - 72x + 32y - 44 = 0$

Solution : (i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

The major axis is along x-axis $a^2 = 25$, $b^2 = 9$

$e = \frac{4}{5}$ and $ae = 4$

Clearly centre C is (0, 0),

Foci are $(\pm ae, 0) = (\pm 4, 0)$

Vertices are $(\pm a, 0) = (\pm 5, 0)$

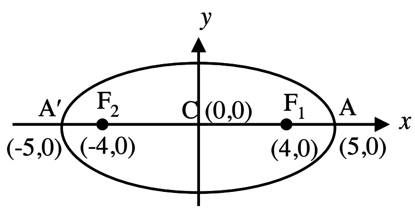


Fig. 4. 66

(ii) The major axis is along y-axis $a^2 = 9, b^2 = 4$

$e = \frac{\sqrt{5}}{3}$ and $ae = \sqrt{5}$

Clearly centre C is $(0, 0)$

Foci are $(0, \pm ae) = (0, \pm\sqrt{5})$

Vertices are $(0, \pm a) = (0, \pm 3)$

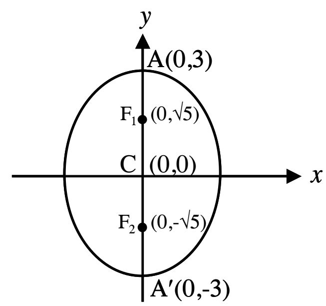


Fig. 4. 67

(iii) Let $x + 3 = X$, $y - 5 = Y$

$\therefore$ The equation becomes $\frac{X^2}{6} + \frac{Y^2}{4} = 1$

The major axis is along X-axis

$a^2 = 6$, $b^2 = 4$

$e = \frac{1}{\sqrt{3}}$ and $ae = \sqrt{2}$

	Referred to X, Y	Referred to x, y $X = x + 3$; $Y = y - 5$
Centre	$(0, 0)$	$X = 0$; $Y = 0$ $\Rightarrow$ $x + 3 = 0$, $y - 5 = 0$ $x = -3$, $y = 5$ Centre $C(-3, 5)$
Vertices	$(\pm a, 0)$ i.e. $(\pm\sqrt{6}, 0)$ (i) $(\sqrt{6}, 0)$	(i) $X = \sqrt{6}$, $Y = 0$ $x + 3 = \sqrt{6}$, $y - 5 = 0$ $x = \sqrt{6} - 3$, $y = 5$ $A(-3 + \sqrt{6}, 5)$
	(ii) $(-\sqrt{6}, 0)$	(ii) $X = -\sqrt{6}$, $Y = 0$ $x + 3 = -\sqrt{6}$, $y - 5 = 0$ $x = -3 - \sqrt{6}$, $y = 5$ $A'(-3 - \sqrt{6}, 5)$

foci	$(\pm ae, 0)$ i.e. $\left(\pm\sqrt{2}\,, 0\right)$ (i) $\left(\sqrt{2}\,, 0\right)$	(i) $X = \sqrt{2}\,,\ Y = 0$ $x + 3 = \sqrt{2}\,,\ y - 5 = 0$ $x = -3 + \sqrt{2}\,, y = 5$ $F_1\left(-3 + \sqrt{2}\,, 5\right)$
	(ii) $\left(-\sqrt{2}, 0\right)$	(ii) $X = -\sqrt{2}\,,\ Y = 0$ $x + 3 = -\sqrt{2}\,, y - 5 = 0$ $x = -3 - \sqrt{2}\,, y = 5$ $F_2\left(-3 - \sqrt{2}\,, 5\right)$

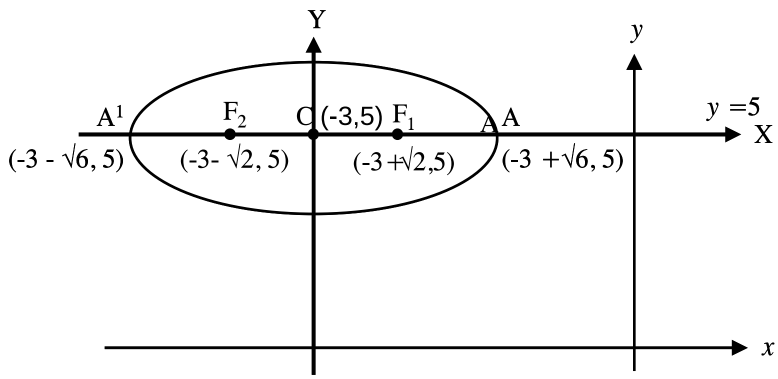


Fig. 4. 68

(iv) $36x^2 + 4y^2 - 72x + 32y - 44 = 0$

$$36\,(x^2 - 2x) + 4\,(y^2 + 8y) = 44$$

$$36\,\{(x-1)^2 - 1\} + 4\,\{(y+4)^2 - 16\} = 44$$

$$36(x-1)^2 + 4\,(y+4)^2 = 144$$

$$\frac{(x-1)^2}{4} + \frac{(y+4)^2}{36} = 1$$

i.e., $\dfrac{X^2}{4} + \dfrac{Y^2}{36} = 1$ where $X = x - 1\,,\ Y = y + 4$

The major axis is along Y-axis.

$a^2 = 36,\quad b^2 = 4$

$e = \dfrac{2\sqrt{2}}{3}$ and $ae = 4\sqrt{2}$

	Referred to X, Y	Referred to x, y $X = x - 1$; $Y = y + 4$
Centre	(0, 0)	$X = 0$; $Y = 0$ $\Rightarrow$ $x - 1 = 0$, $y + 4 = 0$ $x = 1, y = -4$ Centre $C(1, -4)$
Vertices	$(0, \pm a)$ i.e. $(0, \pm 6)$ (i) (0, 6)	(i) $X = 0$, $Y = 6$ $x - 1 = 0$, $y + 4 = 6$ $x = 1$, $y = 2$ $A(1, 2)$
	(ii) $(0, -6)$	(ii) $X = 0, Y = -6$ $x - 1 = 0, y + 4 = -6$ $x - 1 = 0$, $y + 4 = -6$ $x = 1$, $y = -10$ $A'(1, -10)$
Foci	$(0, \pm ae)$ i.e. $\left(0, \pm 4\sqrt{2}\right)$ (i) $\left(0, 4\sqrt{2}\right)$	(i) $X = 0$; $Y = 4\sqrt{2}$ $x - 1 = 0$, $y + 4 = 4\sqrt{2}$ $x = 1, y = 4\sqrt{2} - 4$ $F_1\left(1, 4\sqrt{2} - 4\right)$
	(ii) $\left(0, -4\sqrt{2}\right)$	(ii) $X = 0$, $Y = -4\sqrt{2}$ $x - 1 = 0$; $y + 4 = -4\sqrt{2}$ $x = 1$, $y = -4 - 4\sqrt{2}$ $F_2\left(1, -4 - 4\sqrt{2}\right)$

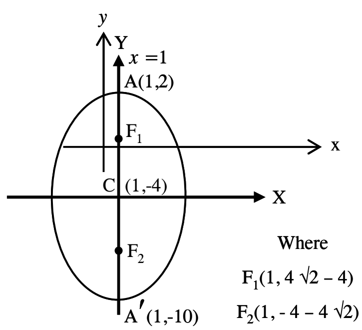


Fig. 4. 69

4.4.6 Some practical problems :

Example 4.32 : An arch is in the form of a semi-ellipse whose span is 48 feet wide. The height of the arch is 20 feet. How wide is the arch at a height of 10 feet above the base?

Solution :

Take the mid point of the base as the centre C (0, 0)

Since the base wide is 48 feet, the vertices A and A' are (24, 0) and (– 24, 0) respectively.

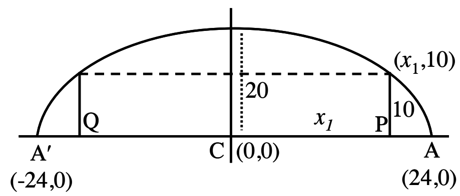


Fig. 4. 70

Clearly $2a = 48$ and $b = 20$.

The corresponding equation is $\dfrac{x^2}{24^2} + \dfrac{y^2}{20^2} = 1$ … (1)

Let x_1 be the distance between the pole whose height is 10m and the centre.

Then $(x_1, 10)$ satisfies the equation (1)

$$\therefore \frac{x_1^{\,2}}{24^2} + \frac{10^2}{20^2} = 1 \quad \Rightarrow \; x_1 = 12\sqrt{3}$$

Clearly the width of the arch at a height of 10 feet is $2x_1 = 24\sqrt{3}$

Thus the required width of arch is $24\sqrt{3}$ feet.

Example 4.33 : The ceiling in a hallway 20ft wide is in the shape of a semi ellipse and 18 ft high at the centre. Find the height of the ceiling 4 feet from either wall if the height of the side walls is 12ft.

Solution :

Let PQR be the height of the ceiling which is 4 feet from the wall.

From the diagram PQ = 12 ft

To find the height QR

Since the width is 20ft, take A, A' as vertices with A as (10, 0) and A' as (– 10, 0). Take the midpoint of AA' as the centre which is (0, 0)

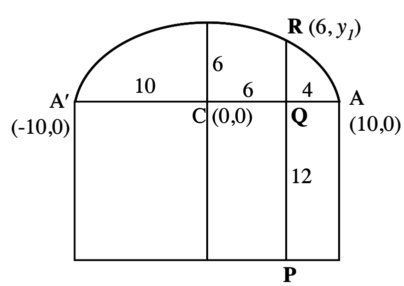


Fig. 4. 71

From the diagram $AA' = 2a = 20 \Rightarrow a = 10$

and $b = 18 - 12 = 6$

$$\therefore \frac{x^2}{100} + \frac{y^2}{36} = 1$$

Let QR be y_1 then R is $(6, y_1)$

Since R lies on the ellipse,

$$\frac{36}{100} + \frac{y_1^2}{36} = 1 \Rightarrow y_1 = 4.8$$

$$\therefore PQ + QR = 12 + 4.8$$

$\therefore$ The required height of the ceiling is 16.8 feet.

Example 4.34 : The orbit of the earth around the sun is elliptical in shape with sun at a focus. The semi major axis is of length 92.9 million miles and eccentricity is 0.017. Find how close the earth gets to sun and the greatest possible distance between the earth and the sun.

Solution :

Semi-major axis CA is

$a = 92.9$ million miles

Given $e = 0.017$

The closest distance of the earth from the sun = FA

and farthest distance of the earth from the sun = FA'

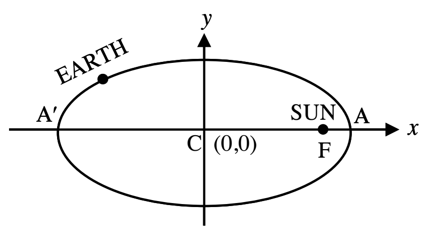


Fig. 4. 72

$$CF = ae = 92.9 \times 0.017$$

$$\begin{aligned} FA = CA - CF &= 92.9 - 92.9 \times 0.017 \\ &= 92.9\,[1 - 0.017] \\ &= 92.9 \times 0.983 = 91.3207 \text{ million miles} \end{aligned}$$

$$\begin{aligned} FA' = CA' + CF &= 92.9 + 92.9 \times 0.017 \\ &= 92.9\,(1 + 0.017) \\ &= 92.9 \times 1.017 = 94.4793 \text{ million miles} \end{aligned}$$

Example 4.35 : A ladder of length 15m moves with its ends always touching the vertical wall and the horizontal floor. Determine the equation of the locus of a point P on the ladder, which is 6m from the end of the ladder in contact with the floor.

Solution :

Let AB be the ladder and $P(x_1, y_1)$ be a point on the ladder such that $AP = 6$m.

Draw PD perpendicular to x-axis and PC perpendicular to y-axis.

Clearly the triangles ADP and PCB are similar.

$$\therefore \frac{PC}{DA} = \frac{PB}{AP} = \frac{BC}{PD}$$

i.e., $$\frac{x_1}{DA} = \frac{9}{6} = \frac{BC}{y_1}$$

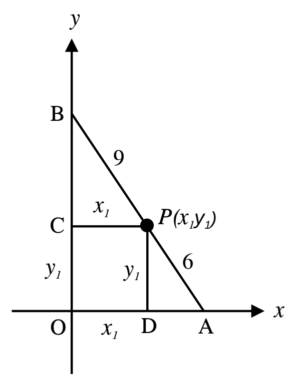


Fig. 4. 73

$$\Rightarrow \quad DA = \frac{6x_1}{9} = \frac{2x_1}{3} \quad ; \quad BC = \frac{9y_1}{6} = \frac{3}{2}y_1$$

$$OA = OD + DA = x_1 + \frac{2x_1}{3} = \frac{5}{3}x_1 \; ; \; OB = OC + BC = y_1 + \frac{3y_1}{2} = \frac{5}{2}y_1$$

But $OA^2 + OB^2 = AB^2 \Rightarrow \frac{25}{9}x_1^2 + \frac{25}{4}y_1^2 = 225$

$$\frac{x_1^2}{9} + \frac{y_1^2}{4} = 9$$

$\therefore$ The locus of (x_1, y_1) is $\frac{x^2}{81} + \frac{y^2}{36} = 1$, which is an ellipse.

EXERCISE 4.2

(1) Find the equation of the ellipse if

(i) one of the foci is $(0, -1)$, the corresponding directrix is $3x + 16 = 0$ and $e = \frac{3}{5}$

(ii) the foci are $(2, -1)$, $(0, -1)$ and $e = \frac{1}{2}$

(iii) the foci are $(\pm 3, 0)$ and the vertices are $(\pm 5, 0)$

(iv) the centre is $(3, -4)$, one of the foci is $\left(3 + \sqrt{3}, -4\right)$ and $e = \frac{\sqrt{3}}{2}$

(v) the centre at the origin, the major axis is along x-axis, $e = \frac{2}{3}$ and passes through the point $\left(2, \frac{-5}{3}\right)$

(vi) the length of the semi major axis, and the latus rectum are 7 and $\frac{80}{7}$ respectively, the centre is (2, 5) and the major axis is parallel to y-axis.

(vii) the centre is (3, – 1), one of the foci is (6, – 1) and passing through the point (8, – 1).

(viii) the foci are (± 3, 0), and the length of the latus rectum is $\frac{32}{5}$.

(ix) the vertices are (± 4, 0) and $e = \frac{\sqrt{3}}{2}$

(2) If the centre of the ellipse is (4, – 2) and one of the focus is (4, 2), find the other focus?

(3) Find the locus of a point which moves so that the sum of its distances from (3, 0) and (– 3, 0) is 9

(4) Find the equations and length of major and minor axes of

(i) $9x^2 + 25y^2 = 225$ (iii) $9x^2 + 4y^2 = 20$

(ii) $5x^2 + 9y^2 + 10x - 36y - 4 = 0$ (iv) $16x^2 + 9y^2 + 32x - 36y - 92 = 0$

(5) Find the equations of directrices, latus rectum and lengths of latus rectums of the following ellipses :

(i) $25x^2 + 169y^2 = 4225$ (ii) $9x^2 + 16y^2 = 144$

(iii) $x^2 + 4y^2 - 8x - 16y - 68 = 0$ (iv) $3x^2 + 2y^2 - 30x - 4y + 23 = 0$

(6) Find the eccentricity, centre, foci, vertices of the following ellipses and draw the diagram :

(i) $16x^2 + 25y^2 = 400$ (ii) $x^2 + 4y^2 - 8x - 16y - 68 = 0$

(iii) $9x^2 + 4y^2 = 36$ (iv) $16x^2 + 9y^2 + 32x - 36y = 92$

(7) A kho-kho player in a practice session while running realises that the sum of the distances from the two kho-kho poles from him is always 8m. Find the equation of the path traced by him if the distance between the poles is 6m.

(8) A satellite is travelling around the earth in an elliptical orbit having the earth at a focus and of eccentricity 1/2 . The shortest distance that the satellite gets to the earth is 400 kms. Find the longest distance that the satellite gets from the earth.

(9) The orbit of the planet mercury around the sun is in elliptical shape with sun at a focus. The semi-major axis is of length 36 million miles and the eccentricity of the orbit is 0.206. Find (i) how close the mercury gets to sun? (ii) the greatest possible distance between mercury and sun.

(10) The arch of a bridge is in the shape of a semi-ellipse having a horizontal span of 40ft and 16ft high at the centre. How high is the arch, 9ft from the right or left of the centre.

4.5 Hyperbola :

Definition: The locus of a point whose distance from a fixed point bears a constant ratio, greater than one to its distance from a fixed line is called a hyperbola.

Note : Eventhough the syllabus does not require the derivation of standard equation and the tracing of hyperbola (4.5.1, 4.5.2) and it needs only the standard equation and the diagram, the equation is derived and the curve is traced for better understanding.

We shall now derive the standard equation of the hyperbola.

4.5.1. Standard equation of the hyperbola :

Given :

★ Fixed point (F)

★ Fixed line (l)

★ Eccentricity e, ($e > 1$)

★ Moving point $P(x, y)$

Construction

★ Plot the fixed point F and draw the fixed line 'l'.

★ Drop a perpendicular (FZ) from F to l.

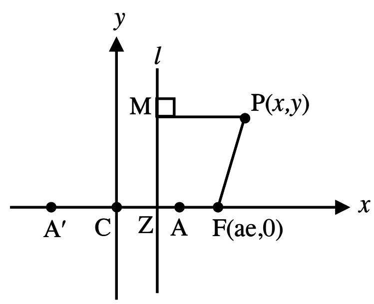


Fig. 4. 74

★ Drop a perpendicular (PM) from P to l.

★ Plot the points A, A' which divides FZ internally and externally in the ratio $e : 1$ respectively.

★ Take $AA' = 2a$ and treat it as x-axis.

★ Draw a perpendicular bisector of AA' and treat it as y-axis.

Let C be the origin. The known points are $C(0, 0)$, $A(a, 0)$, $A'(-a, 0)$.

To find the co-ordinates of F and M do the following :

Since A, A' divides FZ internally and externally in the ratio $e : 1$ respectively,

$$\frac{FA}{AZ} = \frac{e}{1}$$
$$\therefore FA = e\,AZ$$
i.e. $CF - CA = e\,(CA - CZ)$
$$\therefore CF - a = e\,(a - CZ) \qquad \ldots(1)$$

$$\frac{FA'}{A'Z} = \frac{e}{1}$$
$$\therefore FA' = e\,A'Z$$
i.e. $A'C + CF = e\,(A'C + CZ)$
$$\therefore a + CF = e(a + CZ) \qquad \ldots(2)$$

$$(2) - (1) \quad \Rightarrow \quad 2a = e\,[2CZ] \Rightarrow CZ = \frac{a}{e}$$
$$(2) + (1) \quad \Rightarrow \quad 2CF = e(2a) \quad \Rightarrow \quad CF = ae$$

$\therefore M$ is $\left(\frac{a}{e}, y\right)$ and F is $(ae, 0)$

To obtain the equation of the hyperbola we do the following:

Since P is a point on the hyperbola,

We have $\quad \frac{FP}{PM} = e \Rightarrow FP^2 = e^2PM^2$

$$\therefore \quad (x - ae)^2 + (y - 0)^2 = e^2\left[\left(x - \frac{a}{e}\right)^2 + (y - y)^2\right]$$
$$x^2 - 2aex + a^2e^2 + y^2 = e^2\,\frac{[e^2x^2 - 2aex + a^2]}{e^2}$$
$$x^2 - e^2x^2 + y^2 = a^2 - a^2e^2$$
$$(e^2 - 1)x^2 - y^2 = a^2(e^2 - 1)$$
$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2\,(e^2 - 1)$ is a positive quantity.

This is the required standard equation of the hyperbola.

4.5.2 Tracing of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(i) Symmetry :

The hyperbola is symmetric about x-axis, y-axis and hence the hyperbola is symmetric about the origin.

(ii) Special points :

The hyperbola does not pass through the origin.

To find the points on x-axis, put $y = 0$, we get $x = \pm\, a$. Therefore the curve meets the x-axis at $A(a, 0)$ and $A'(-\,a, 0)$.

To find the points on y-axis, put $x = 0$, we get $y^2 = -b^2$. i.e., y is imaginary. Therefore the curve does not meet the y-axis.

(iii) Existence of the curve :

Write the equation of the hyperbola as $y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$. If $x^2 - a^2 < 0$ i.e., $-a < x < a$, y is imaginary. i.e., the curve does not exist for $-a < x < a$. Therefore the curve exists for $x \leq -a$ and $x \geq a$. Note that for all y, the curve exists.

(iv) The curve at infinity :

As x increases y^2 also increases i.e., as $x \to \infty$, $y^2 \to \infty$. as $x \to \infty$, $y \to \pm\infty$.

Thus the curve branches out to infinity on either side.

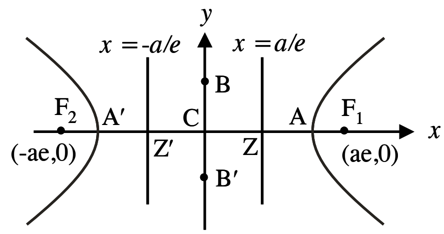


Fig. 4. 75

4.5.3 Important definitions regarding hyperbola :

Focus : The fixed point is called a focus F_1 $(ae, 0)$ of the hyperbola.

Directrix : The fixed line is called the directrix of the hyperbola and its equation is $x = \frac{a}{e}$.

Transverse axis : The line segment AA' joining the vertices is called the transverse axis and the length of the transverse axis is $2a$. The equation of transverse axis is $y = 0$. Note that the transverse axes cut both the branches of the curve.

Conjugate axis : The line segment joining the points $B(0, b)$ and $B'(0, -b)$ is called the conjugate axis. The length of the conjugate axis is $2b$. The equation of the conjugate axis is $x = 0$

Centre : The point of intersection of the transverse and conjugate axes of the hyperbola is called the centre of the hyperbola. Here $C(0, 0)$ is called the centre of the hyperbola.

Vertices : The points of intersection of the hyperbola and its transverse axis is called its vertices. The vertices of the hyperbola are $A(a, 0)$ and $A'(-a, 0)$.

As in the case of ellipse, hyperbola also has the special property of the second focus $F_2(-ae, 0)$ and the second directrix $x = -\frac{a}{e}$.

Eccentricity : $e = \sqrt{1 + \dfrac{b^2}{a^2}}$

Remark :

In the case of a hyperbola $e > 1$. As $e \to 1$, $\dfrac{b}{a} \to 0$ i.e., as $e \to 1$, b is very small related to a and the hyperbola becomes a pointed nose. As $e \to \infty$, b is very large related to a and the hyperbola becomes flat.

Latus rectum : It is a focal chord perpendicular to the transverse axis of the hyperbola. The equations of the latus rectum are $x = \pm\, ae$.

End points of latus rectum and length of latus rectum :

To find the end points, solve $x = ae$ … (1) and $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ … (2)

Using (1) in (2) we get

$$\frac{a^2 e^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore \frac{y^2}{b^2} = e^2 - 1$$

$$\therefore y^2 = b^2 (e^2 - 1)$$

$$= b^2 . \left(\frac{b^2}{a^2}\right) \quad (\because b^2 = a^2 (e^2 - 1))$$

$$\therefore y = \pm \frac{b^2}{a}$$

If L_1 and L_1' are the end points of one latus rectum then L_1 is $\left(ae, \dfrac{b^2}{a}\right)$ and L_1' is $\left(ae, -\dfrac{b^2}{a}\right)$.

Similarly the end points of the other latus rectum are $\left(-ae, \pm \dfrac{b^2}{a}\right)$ and the length of the latus rectum is $\dfrac{2b^2}{a}$.

For the above discussed hyperbola, the transverse axis is along x-axis. There is another standard hyperbola in which the transverse axis is along y-axis.

4.5.4 The other form of the hyperbola:

If the transverse axis is along y-axis and the conjugate axis is along x-axis, then the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

For this type of hyperbola, we have the following as explained in the earlier hyperbola

Centre	:	$C\,(0, 0)$
Vertices	:	$A\,(0, a),\ A'\,(0, -a)$
Foci	:	$F_1\,(0, ae), F_2\,(0, -ae)$
Equation of transverse axis is	:	$x = 0$
Equation of conjugate axis is	:	$y = 0$
End points of conjugate axis	:	$(b, 0),\ (-b, 0)$
Equations of latus rectum	:	$y = \pm ae$
Equations of directrices	:	$y = \pm \frac{a}{e}$
End points of latus rectum	:	$\left(\pm \frac{b^2}{a},\ ae\right), \left(\pm \frac{b^2}{a},\ -ae\right)$

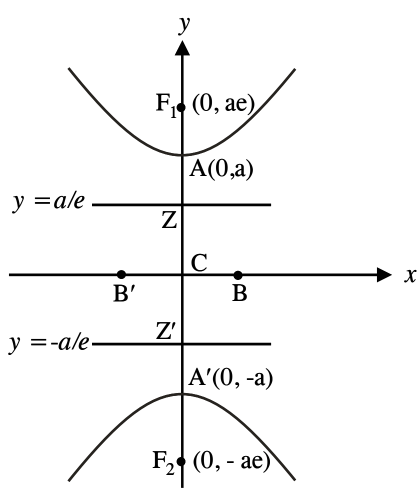


Fig. 4. 76

Example 4. 36 : Find the equation of hyperbola whose directrix is $2x + y = 1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

Solution:

Let $P(x, y)$ be any point on the hyperbola. Draw PM perpendicular to the directrix.

By definition, $\frac{\text{FP}}{\text{PM}} = e \Rightarrow \therefore FP^2 = e^2 . PM^2$

i.e., $(x-1)^2 + (y-2)^2 = 3\left(\frac{2x+y-1}{\sqrt{4+1}}\right)^2$

$(x-1)^2 + (y-2)^2 = \frac{3}{5}(2x+y-1)^2$

i.e., $7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0$

This is the required equation of the hyperbola.

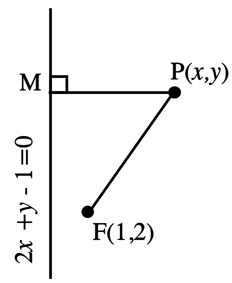


Fig. 4. 77

Example 4.37 : Find the equation of the hyperbola whose transverse axis is along x-axis. The centre is $(0, 0)$ length of semi-transverse axis is 6 and eccentricity is 3.

Solution:

Since the transverse axis is along x-axis and the centre is $(0, 0)$, the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Given that semi-transverse axis $a = 6$, eccentricity $e = 3$

We know that $b^2 = a^2 (e^2 - 1)$

$$\therefore\ b^2 = 36(8) = 288$$

$\therefore$ The equation of the hyperbola is

$$\frac{x^2}{36} - \frac{y^2}{288} = 1$$

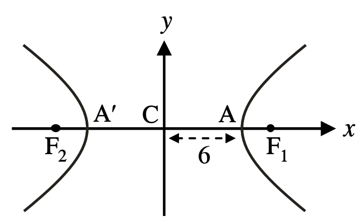


Fig. 4. 78

Example 4.38 : Find the equation of the hyperbola whose transverse axis is parallel to x-axis, centre is $(1, 2)$, length of the conjugate axis is 4 and eccentricity $e = 2$.

Solution:

Since the transverse axis is parallel to x-axis, the equation is of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Here centre $C(h, k)$ is $(1, 2)$.

The length of conjugate axis $2b = 4$ and $e = 2$

$$b^2 = a^2(e^2 - 1)$$

$$4 = a^2(4 - 1)$$

$$\Rightarrow a^2 = \frac{4}{3}$$

$\therefore$ The required equation is $\dfrac{(x-1)^2}{4/3} - \dfrac{(y-2)^2}{4} = 1$

Example 4.39 : Find the equation of the hyperbola whose centre is (1, 2). The distance between the directrices is $\frac{20}{3}$, the distance between the foci is 30 and the transverse axis is parallel to y-axis.

Solution:

Since the transverse axis is parallel to y-axis, the equation is of the form

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Here centre $C(h, k)$ is $(1, 2)$

The distance between the directrices $\dfrac{2a}{e} = \dfrac{20}{3} \Rightarrow \dfrac{a}{e} = \dfrac{10}{3}$

The distance between the foci, $2ae = 30 \Rightarrow ae = 15$

$$\frac{a}{e}(ae) = \frac{10}{3} \times 15 \Rightarrow a^2 = 50$$

Also $\dfrac{ae}{a/e} \Rightarrow e^2 = \dfrac{9}{2}$

$$b^2 = a^2(e^2 - 1) \Rightarrow b^2 = 50\left(\frac{9}{2} - 1\right) = 175$$

The required equation is $\dfrac{(y-2)^2}{50} - \dfrac{(x-1)^2}{175} = 1$

Example 4.40 : Find the equation of the hyperbola whose transverse axis is parallel to y-axis, centre (0, 0), length of semi-conjugate axis is 4 and eccentricity is 2.

Solution:

From the given data the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Given that semi-conjugate axis $b = 4$ and $e = 2$,

$$b^2 = a^2 (e^2 - 1)$$
$$4^2 = a^2 (2^2 - 1)$$
$$\therefore \quad a^2 = \frac{16}{3}$$

Hence the equation of the hyperbola is $\frac{y^2}{16/3} - \frac{x^2}{16} = 1$

$$\text{or} \qquad 3y^2 - x^2 = 16$$

Example 4.41 : Find the equation of the hyperbola whose foci are $(\pm 6, 0)$ and length of the transverse axis is 8.

Solution:

From the given data the transverse axis is along x-axis.

$\therefore$ The equation is of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

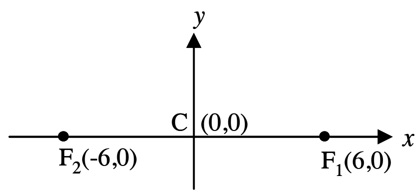


Fig. 4. 79

The centre is the midpoint of F_1 and F_2

i.e., C is $\left(\frac{-6+6}{2}, \frac{0+0}{2}\right) = (0, 0)$

The length of the transverse axis $2a = 8$, $\Rightarrow$ $a = 4$

$$F_1F_2 = 2ae = 12 \quad ae = 6$$
$$\therefore 4e = 6$$
$$e = \frac{6}{4} = \frac{3}{2}$$
$$b^2 = a^2 (e^2 - 1) = 16\left(\frac{9}{4} - 1\right) = \frac{16 \times 5}{4} = 20$$

$\therefore$ The required equation is $\frac{x^2}{16} - \frac{y^2}{20} = 1$

Example 4.42 : Find the equation of the hyperbola whose foci are $(5, \pm 4)$ and eccentricity is $\frac{3}{2}$.

Solution:

From the given data the transverse axis is parallel to y-axis and hence the equation of the hyperbola is of the form

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

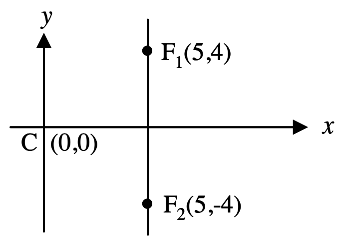


Fig. 4. 80

The centre $C\ (h, k)$ is the midpoint of F_1 and F_2

i.e., C is $\left(\frac{5+5}{2}, \frac{4-4}{2}\right) = (5, 0)$

$$F_1F_2 = 2ae = \sqrt{(5-5)^2 + (4+4)^2} = 8$$

$$ae = 4$$

$$\text{But } e = \frac{3}{2} \quad \therefore a = \frac{8}{3}$$

$$b^2 = a^2(e^2 - 1) = \frac{64}{9}\left(\frac{9}{4} - 1\right)$$

$$= \frac{80}{9}$$

$\therefore$ The required equation is

$$\frac{(y-0)^2}{64/9} - \frac{(x-5)^2}{80/9} = 1 \quad \text{or} \quad \frac{9y^2}{64} - \frac{9(x-5)^2}{80} = 1$$

Example 4.43 : Find the equation of the hyperbola whose centre is (2, 1), one of the foci is (8, 1) and the corresponding directrix is $x = 4$.

Solution:

From the given data the equation is of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

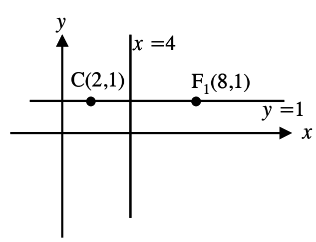


Fig. 4. 81

Centre $C\ (h, k)$ is (2, 1)

$CF_1 = ae = 6$

(Draw CZ perpendicular to $x = 4$)

The distance between the centre and directrix

$$CZ = \frac{a}{e} = 2$$

$$\therefore\ ae \cdot \frac{a}{e} = 6 \times 2 \Rightarrow a^2 = 12$$

$$\frac{ae}{a/e} = \frac{6}{2} \Rightarrow e^2 = 3$$

$$b^2 = a^2 (e^2 - 1) \quad \therefore \quad b^2 = 12(3 - 1) = 24$$

$\therefore$ The required equation is

$$\frac{(x-2)^2}{12} - \frac{(y-1)^2}{24} = 1$$

Example 4.44 : Find the equation of the hyperbola whose foci are (0, ± 5) and the length of the transverse axis is 6.

Solution:

From the given data the transverse axis is along y-axis and hence the equation is of the form

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

The centre C (h, k) is the midpoint of F_1 and F_2

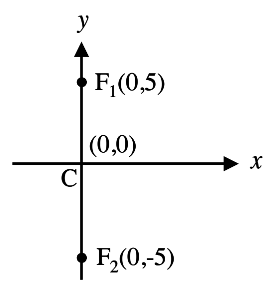


Fig. 4. 82

i.e. C is $\left(\frac{0+0}{2}, \frac{5-5}{2}\right) = (0, 0)$

$$F_1F_2 = 2ae = 10$$

The length of the transverse axis $= 2a = 6$

$$\Rightarrow \quad a = 3 \quad \text{and } e = \frac{5}{3}$$

$$b^2 = a^2 (e^2 - 1)$$

$$= 9\left(\frac{25}{9} - 1\right)$$

$$= 16$$

$\therefore$ The required equation is $\frac{y^2}{9} - \frac{x^2}{16} = 1$

Example 4.45 : Find the equation of the hyperbola whose foci are $\left(0, \pm\sqrt{10}\right)$ and passing through (2, 3).

Solution:

From the data, the transverse axis is along the y-axis. $\therefore$ it is of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Given that the foci are $(0, \pm ae) = \left(0, \pm\sqrt{10}\right)$

$$\Rightarrow \quad ae = \sqrt{10}$$

$$\text{Also } b^2 = a^2(e^2 - 1) = a^2e^2 - a^2$$

$$b^2 = 10 - a^2$$

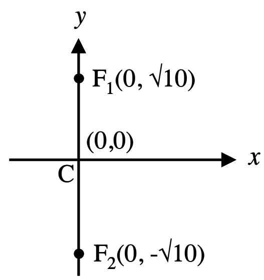


Fig. 4. 83

$\therefore$ Equation of the hyperbola is $\dfrac{y^2}{a^2} - \dfrac{x^2}{10 - a^2} = 1$

It passes through (2, 3),

$$\frac{9}{a^2} - \frac{4}{10 - a^2} = 1$$

$$\frac{9(10 - a^2) - 4a^2}{a^2(10 - a^2)} = 1$$

$$90 - 9a^2 - 4a^2 = 10a^2 - a^4$$

$$\text{or } a^4 - 23a^2 + 90 = 0$$

$$(a^2 - 18)(a^2 - 5) = 0$$

$$a^2 = 18 \text{ or } 5$$

If $a^2 = 18$, $b^2 = 10 - 18 = -8$ which is impossible.

If $a^2 = 5$, $b^2 = 10 - 5 = 5$

$\therefore$ Equation of the hyperbola is $\dfrac{y^2}{5} - \dfrac{x^2}{5} = 1$ or $y^2 - x^2 = 5$

Example 4.46 : Find the equations and length of transverse and conjugate axes of the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

Solution:

The centre is at the origin, the transverse axis is along x-axis and the conjugate axis is along the y-axis. i.e., transverse axis is x-axis i.e., $y = 0$ and the conjugate axis y-axis i.e., $x = 0$.

Hence $a^2 = 9$, $b^2 = 4$ $\Rightarrow$ $a = 3$, $b = 2$

$\therefore$ Length of transverse axis $= 2a = 6$

Length of conjugate axis $= 2b = 4$

Example 4.47 : Find the equations and length of transverse and conjugate axes of the hyperbola $16y^2 - 9x^2 = 144$

Solution: $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$

The centre is at the origin, the transverse axis is along y-axis, and the conjugate axis is along x-axis.

$\therefore$ The transverse axis is y-axis, i.e. $x = 0$

The conjugate axis is x-axis i.e. $y = 0$.

Here $a^2 = 9$, $b^2 = 16$ $\Rightarrow$ $a = 3$, $b = 4$

$\therefore$ The length of transverse axis $= 2a = 6$

The length of conjugate axis $= 2b = 8$

Example 4.48 : Find the equations and length of transverse and conjugate axes of the hyperbola $9x^2 - 36x - 4y^2 - 16y + 56 = 0$

Solution:

$$9(x^2 - 4x) - 4(y^2 + 4y) = -56$$

$$9\{(x-2)^2 - 4\} - 4\{(y+2)^2 - 4\} = -56$$

$$9(x-2)^2 - 4(y+2)^2 = 36 - 16 - 56$$

$$9(x-2)^2 - 4(y+2)^2 = -36$$

$$4(y+2)^2 - 9(x-2)^2 = 36$$

$$\frac{(y+2)^2}{9} - \frac{(x-2)^2}{4} = 1$$

$$\frac{Y^2}{9} - \frac{X^2}{4} = 1 \text{ where } \begin{cases} X = x - 2 \\ Y = y + 2 \end{cases}$$

Clearly the transverse axis is along y-axis and the conjugate axis is along x-axis. i.e. transverse axis is y-axis or $X = 0$ i.e., $x - 2 = 0$

The conjugate axis is X-axis or $Y = 0$ i.e., $y + 2 = 0$

Here $a^2 = 9$, $b^2 = 4$ $\Rightarrow$ $a = 3$, $b = 2$

$\therefore$ The length of transverse axis $= 2a = 6$

The length of conjugate axis $= 2b = 4$

Example 4.49 : Find the equations of directrices, latus rectum and length of latus rectum of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$

Solution:

The centre is at the origin and the transverse axis is along x-axis.

The equations of the directrices are $x = \pm \frac{a}{e}$

The equations of the latus rectum are $x = \pm\, ae$

Length of the latus rectum $= \frac{2b^2}{a}$

Here $a^2 = 9$, $b^2 = 4$

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{4}{9}} = \frac{\sqrt{13}}{3}$$

$\therefore$ The equations of the directrices are

$$x = \pm \frac{3}{\sqrt{13}/3} \text{ i.e. } x = \pm \frac{9}{\sqrt{13}}$$

The equation of the latus rectum are $x = \pm \sqrt{13}$

Length of the latus rectum is $\frac{2b^2}{a} = \frac{8}{3}$

Example 4. 50 : Find the equations of directrices, latus rectum and length of latus rectum of the hyperbola $16y^2 - 9x^2 = 144$

Solution: $\frac{y^2}{9} - \frac{x^2}{16} = 1$

Here $a^2 = 9$, $b^2 = 16$ $\quad e = \frac{5}{3}$

The transverse axis is along the y-axis.

$\therefore$ The equations of the directrices are $y = \pm \frac{a}{e}$ i.e., $y = \pm \frac{9}{5}$

The equation of the latus rectum are $y = \pm\, ae$ i.e., $y = \pm 5$

Length of the latus rectum is $\frac{2b^2}{a} = \frac{32}{3}$

Example 4.51 : Find the equations of directrices, latus rectum and length of latus rectum of the hyperbola $9x^2 - 36x - 4y^2 - 16y + 56 = 0$

Solution: By simplifying we get $\frac{Y^2}{9} - \frac{X^2}{4} = 1$ where $\begin{cases} Y = y + 2 \\ X = x - 2 \end{cases}$

Here $a^2 = 9$, $b^2 = 4$

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \frac{\sqrt{13}}{3}$$

$$ae = \sqrt{13}, \quad \frac{a}{e} = \frac{9}{\sqrt{13}}$$

The transverse axis is along Y-axis.

$\therefore$ The equations of the directrices are $Y = \pm \frac{a}{e}$ i.e. $Y = \pm \frac{9}{\sqrt{13}}$

(i) $Y = \frac{9}{\sqrt{13}} \Rightarrow y + 2 = \frac{9}{\sqrt{13}} \Rightarrow y = \frac{9}{\sqrt{13}} - 2$

(ii) $Y = -\frac{9}{\sqrt{13}} \Rightarrow y + 2 = \frac{-9}{\sqrt{13}} \Rightarrow y = \frac{-9}{\sqrt{13}} - 2$

The equations of the latus rectum are $Y = \pm ae$ i.e. $Y = \pm \sqrt{13}$

(i) $Y = \sqrt{13} \Rightarrow y + 2 = \sqrt{13} \Rightarrow y = \sqrt{13} - 2$

(ii) $Y = -\sqrt{13} \Rightarrow y + 2 = -\sqrt{13} \Rightarrow y = -\sqrt{13} - 2$

Length of the latus rectum is $\frac{2b^2}{a} = \frac{8}{3}$

Example 4.52 : The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Determine the equation of the hyperbola if its eccentricity is 2.

Solution :

The equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$\Rightarrow a^2 = 25$, $b^2 = 9$, $\quad e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

$$\therefore \quad ae = 4$$

The foci of the ellipse are $(\pm ae, 0) = (\pm 4, 0)$

Given that the foci of the hyperbola coincide with the foci of the ellipse, foci of the hyperbola are $(\pm ae, 0) = (\pm 4, 0)$

$$\therefore \ ae = 4$$

Given that the eccentricity of the hyperbola is 2

$$a(2) = 4 \quad \Rightarrow \ a = 2$$

For a hyperbola $\quad b^2 = a^2 (e^2 - 1)$

$$= a^2 e^2 - a^2$$

$$= 16 - 4 = 12$$

$\therefore$ The equation of the hyperbola is $\dfrac{x^2}{4} - \dfrac{y^2}{12} = 1$

Property (without proof) :

A point moves such that the difference of its distances from two fixed points in a plane is a constant. The locus of this point is a hyperbola and this difference is equal to the length of the transverse axis.

Example 4.53 : Find the equation of the locus of all points such that the differences of their distances from (4, 0) and (– 4, 0) is always equal to 2.

Solution :

By the property, the locus is a hyperbola. Take the fixed points as foci.

$\therefore$ F_1 is (4, 0) and F_2 is (– 4, 0)

Let $P(x, y)$ be a point on the hyperbola.

$F_1P \sim F_2P$ = length of transverse axis = $2a = 2$

$\therefore a = 1$

Centre is the midpoint of $F_1F_2 = (0, 0)$

Hence from the given data the hyperbola is of the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$$F_1F_2 = 2ae = 8$$

$$ae = 4 \Rightarrow e = 4$$

$$b^2 = a^2 (e^2 - 1)$$

$$= 1(16 - 1) = 15$$

$\therefore$ The equation is $\dfrac{x^2}{1} - \dfrac{y^2}{15} = 1$

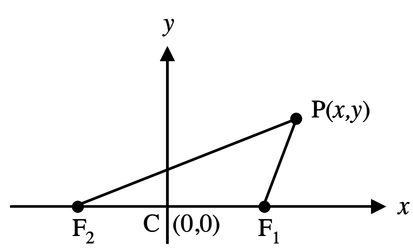


Fig. 4. 84

Alternate method:

Let $P(x, y)$ be a point on the hyperbola and let F_1 and F_2 be the fixed points (4, 0) and (– 4, 0).

It is given that $F_1P \sim F_2P = 2$

$$\sqrt{(x-4)^2+(y-0)^2} \sim \sqrt{(x+4)^2+(y-0)^2} = 2$$

Simplifying, we get $\dfrac{x^2}{1} - \dfrac{y^2}{15} = 1$

Example 4.54 : Find the eccentricity, centre, foci and vertices of the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$ and also trace the curve

Solution :

$a^2 = 4, \quad b^2 = 5$

$\Rightarrow \quad e = \sqrt{1 + \dfrac{b^2}{a^2}} = \dfrac{3}{2}$

$\therefore \quad ae = 2 \times \dfrac{3}{2} = 3.$

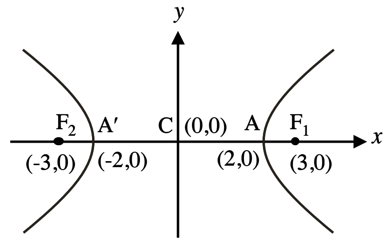


Fig. 4. 85

The transverse axis is along the x-axis

Centre : (0, 0)

Foci : $(\pm ae, 0) = (\pm 3, 0)$

vertices : $(\pm a, 0) = (\pm 2, 0)$

Example 4.55 : Find the eccentricity, centre, foci and vertices of the hyperbola $\dfrac{y^2}{6} - \dfrac{x^2}{18} = 1$ and also trace the curve.

Solution :

$a^2 = 6 \quad b^2 = 18$

$\Rightarrow \quad e = \sqrt{1 + \dfrac{b^2}{a^2}} = \sqrt{\dfrac{24}{6}} = 2$

$\therefore \; ae = 2\sqrt{6}$

The transverse axis is along the y-axis

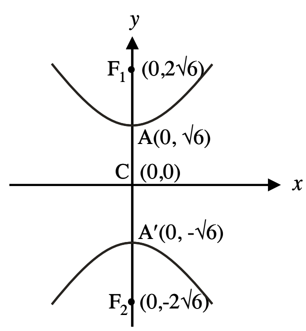


Fig. 4. 86

Centre : (0, 0)

Foci are : $(0, \pm ae) = \left(0, \pm 2\sqrt{6}\right)$

vertices are : $(0, \pm a) = \left(0, \pm\sqrt{6}\right)$

Example 4.56 : Find the eccentricity, centre, foci and vertices of the hyperbola $9x^2 - 16y^2 - 18x - 64y - 199 = 0$ and also trace the curve.

Solution:

$$9(x^2 - 2x) - 16(y^2 + 4y) = 199$$

$$9\{(x-1)^2 - 1\} - 16\{(y+2)^2 - 4\} = 199$$

$$9(x-1)^2 - 16(y+2)^2 = 199 + 9 - 64$$

$$9(x-1)^2 - 16(y+2)^2 = 144$$

$$\frac{(x-1)^2}{16} - \frac{(y+2)^2}{9} = 1$$

$$\text{i.e., } \frac{X^2}{16} - \frac{Y^2}{9} = 1 \text{ where } \begin{cases} X = x - 1 \\ Y = y + 2 \end{cases}$$

$$a^2 = 16, \quad b^2 = 9 \quad \Rightarrow \quad e = \sqrt{1 + \frac{b^2}{a^2}} = \frac{5}{4}$$

$$ae = 4 \times \frac{5}{4} = 5$$

The transverse axis is parallel to X-axis.

	Referred to X, Y	Referred to x, y $X = x - 1, Y = y + 2$
Centre	(0, 0)	$X = 0$; $Y = 0$ $x - 1 = 0$; $y + 2 = 0$ $x = 1$; $y = -2$ $\therefore C(1, -2)$
Foci	$(\pm ae, 0)$ is $(\pm 5, 0)$ (i) (5, 0)	(i) $X = 5$; $Y = 0$ $x - 1 = 5$; $y + 2 = 0$ $x = 6$; $y = -2$ $\therefore F_1(6, -2)$
	(ii) $(-5, 0)$	(ii) $X = -5$; $Y = 0$ $x - 1 = -5$; $y + 2 = 0$ $\therefore F_2(-4, -2)$

Vertices	$(\pm a, 0)$ i.e. $(\pm 4, 0)$ (i) (4, 0)	(i) $X = 4$; $Y = 0$ $x - 1 = 4$; $y + 2 = 0$ $\therefore$ A (5, – 2)
	(ii) (– 4, 0)	(ii) $X = -4$; $Y = 0$ $x - 1 = -4$; $y + 2 = 0$ $\therefore$ A′ (– 3, – 2)

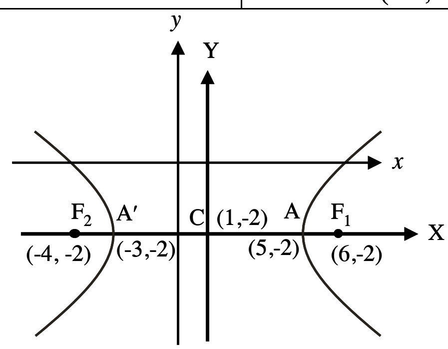


Fig. 4. 87

Example 4.57 : Find the eccentricity, centre, foci and vertices of the following hyperbola and draw the diagram : $9x^2 - 16y^2 + 36x + 32y + 164 = 0$

Solution:

$$9(x^2 + 4x) - 16(y^2 - 2y) = -164$$

$$9\{(x+2)^2 - 4\} - 16\{(y-1)^2 - 1\} = -164$$

$$9(x+2)^2 - 16(y-1)^2 = -164 + 36 - 16$$

$$16(y-1)^2 - 9(x+2)^2 = 144$$

$$\frac{(y-1)^2}{9} - \frac{(x+2)^2}{16} = 1$$

$$\frac{Y^2}{9} - \frac{X^2}{16} = 1 \text{ where } \begin{cases} X = x + 2 \\ Y = y - 1 \end{cases}$$

$$a^2 = 9, \quad b^2 = 16 \quad \Rightarrow \quad e = \sqrt{1 + \frac{b^2}{a^2}} = \frac{5}{3}$$

$$ae = 5$$

The transverse axis is parallel to Y-axis.

	Referred to X, Y	Referred to x, y $X = x + 2$, $Y = y - 1$
Centre	(0, 0)	$X = 0$; $Y = 0$ $x + 2 = 0$; $y - 1 = 0$ $x = -2$; $y = 1$ $\therefore$ $C(-2, 1)$
Foci	$(0, \pm ae)$ i.e., $(0, \pm 5)$ (i) (0, 5)	(i) $X = 0$; $Y = 5$ $x + 2 = 0$; $y - 1 = 5$ $x = -2$; $y = 6$ $\therefore$ $F_1(-2, 6)$
	(ii) $(0, -5)$	(ii) $X = 0$; $Y = -5$ $x + 2 = 0$; $y - 1 = -5$ $x = -2$; $y = -4$ $\therefore$ $F_2(-2, -4)$
Vertices	$(0, \pm a)$ (i) (0, 3)	(i) $X = 0$; $Y = 3$ $x + 2 = 0$; $y - 1 = 3$ $\therefore$ $A(-2, 4)$
	(ii) $(0, -3)$	(ii) $X = 0$; $Y = -3$ $x + 2 = 0$; $y - 1 = -3$ $x = -2$; $y = -2$ $\therefore$ $A'(-2, -2)$

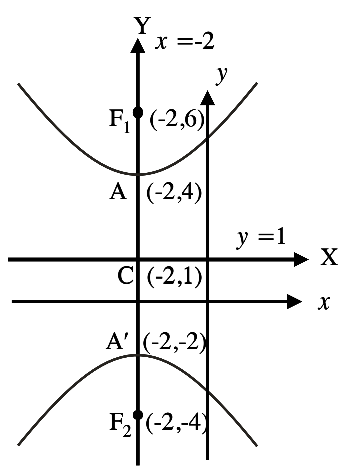


Fig. 4. 88

Example 4.58 :

Points A and B are 10 km apart and it is determined from the sound of an explosion heard at those points at different times that the location of the explosion is 6 km closer to A than B. Show that the location of the explosion is restricted to a particular curve and find an equation of it.

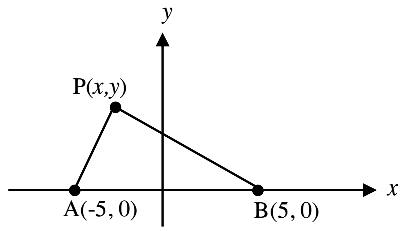


Fig. 4. 89

Given :

$$PB - PA = 6$$

i.e., $$\sqrt{(x-5)^2 + y^2} - \sqrt{(x+5)^2 + y^2} = 6$$

Simplifying we get $$-9y^2 + 16x^2 = 144$$

$$\frac{-y^2}{16} + \frac{x^2}{9} = 1 \text{ i.e., } \frac{x^2}{9} - \frac{y^2}{16} = 1 \text{ which is a hyperbola.}$$

EXERCISE 4.3

(1) Find the equation of the hyperbola if

(i) focus : (2, 3) ; corresponding directrix : $x + 2y = 5$, $e = 2$

(ii) centre : (0, 0) ; length of the semi-transverse axis is 5 ; $e = \frac{7}{5}$ and the conjugate axis is along x-axis.

(iii) centre : (0, 0) ; length of semi-transverse axis is 6 ; $e = 3$, and the transverse axis is parallel to y-axis.

(iv) centre : (1, – 2) ; length of the transverse axis is 8 ; $e = \frac{5}{4}$ and the transverse axis is parallel to x-axis.

(v) centre : (2, 5) ; the distance between the directrices is 15, the distance between the foci is 20 and the transverse axis is parallel to y-axis.

(vi) foci : $(0, \pm 8)$; length of transverse axis is 12

(vii) foci : $(\pm 3, 5)$; $e = 3$

(viii) centre : $(1, 4)$; one of the foci $(6, 4)$ and the corresponding directrix is $x = \frac{9}{4}$.

(ix) foci : $(6, -1)$ and $(-4, -1)$ and passing through the point $(4, -1)$

(2) Find the equations and length of transverse and conjugate axes of the following hyperbolas :

(i) $144x^2 - 25y^2 = 3600$ (ii) $8y^2 - 2x^2 = 16$

(iii) $16x^2 - 9y^2 + 96x + 36y - 36 = 0$

(3) Find the equations of directrices, latus rectums and length of latus rectum of the following hyperbolas :

(i) $4x^2 - 9y^2 = 576$ (ii) $9x^2 - 4y^2 - 36x + 32y + 8 = 0$

(4) Show that the locus of a point which moves so that the difference of its distances from the points $(5, 0)$ and $(-5, 0)$ is 8 is $9x^2 - 16y^2 = 144$.

(5) Find the eccentricity, centre, foci and vertices of the following hyperbolas and draw their diagrams.

(i) $25x^2 - 16y^2 = 400$ (ii) $\frac{y^2}{9} - \frac{x^2}{25} = 1$

(iii) $x^2 - 4y^2 + 6x + 16y - 11 = 0$ (iv) $x^2 - 3y^2 + 6x + 6y + 18 = 0$

4.6 Parametric form of Conics:

Conic	Parametric equations	Parameter	Range of parameter	Any point on the conic
Parabola	$x = at^2$ $y = 2at$	t	$-\infty < t < \infty$	't' or $(at^2, 2at)$
Ellipse	$x = a\cos\theta$ $y = b\sin\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\cos\theta, b\sin\theta)$
Hyperbola	$x = a\sec\theta$ $y = b\tan\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\sec\theta, b\tan\theta)$

Note: For ellipse, we have another parametric form of equations $x = \frac{a(1-t^2)}{1+t^2}$, $y = \frac{b.2t}{1+t^2}$, $-\infty < t < \infty$. This result will be obtained by putting $\tan\frac{\theta}{2} = t$ in the parametric equations $x = a\cos\theta$ and $y = b\sin\theta$.

Thus we have two forms of representations of conics i.e., cartesian form and parametric form. Now we will derive the equations of chord, tangent and normal to the conics.

4.7 Chords, tangents and normals

We derive these equations using both forms of conics.

4.7.1 Cartesian form

(i) Parabola

Equation of the chord joining $A(x_1, y_1)$ and $B(x_2, y_2)$ on the parabola $y^2 = 4ax$

Since (x_1, y_1) and (x_2, y_2) lie on the parabola,

$$y_1^2 = 4ax_1, y_2^2 = 4ax_2$$

$$y_1^2 - y_2^2 = 4a(x_1 - x_2)$$

$$\Rightarrow \quad \frac{y_1 - y_2}{x_1 - x_2} = \frac{4a}{y_1 + y_2}$$

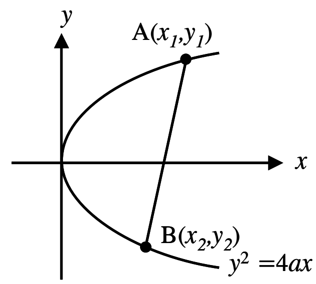


Fig. 4. 90

i.e., the slope (m) of the chord $AB = \dfrac{4a}{y_1 + y_2}$

The equation of the chord, using slope (m) and point (x_1, y_1) is

$$(y - y_1) = \frac{4a}{y_1 + y_2}(x - x_1)$$

If the point (x_2, y_2) coincides with (x_1, y_1) then the chord becomes the tangent at (x_1, y_1). Therefore, to obtain tangent at (x_1, y_1), put $x_2 = x_1$ and $y_2 = y_1$ in the equation of the chord. $\therefore$ the equation of the tangent is

$$(y - y_1) = \frac{4a}{y_1 + y_1}(x - x_1)$$

$$\Rightarrow \quad yy_1 = 2a(x + x_1)$$

$$(\text{use } y_1^2 = 4ax_1)$$

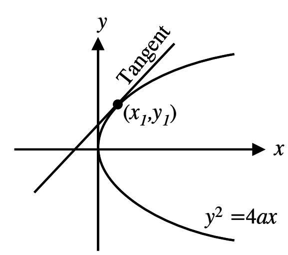


Fig. 4. 91

Thus the equation of the tangent at (x_1, y_1) to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

To find the equation of the normal using perpendicularity.

Equation of the tangent is

$$2ax - y_1 y + 2ax_1 = 0$$

$\therefore$ the normal is of the form

$$y_1 x + 2ay = k$$

But it passes through (x_1, y_1)

$\therefore\ k = x_1 y_1 + 2ay_1$

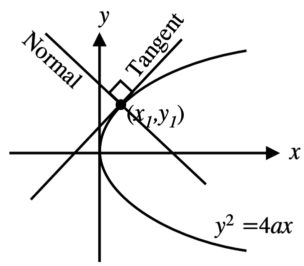


Fig. 4. 92

Thus the equation of the normal at (x_1, y_1) to the parabola is $y_1 x + 2ay = x_1 y_1 + 2ay_1$

(ii) Ellipse

Equation of the chord joining $A(x_1, y_1)$ and $B(x_2, y_2)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since (x_1, y_1) and (x_2, y_2) lie on the ellipse,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \quad \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1$$

By simplification, the slope

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-b^2 (x_1 + x_2)}{a^2 (y_1 + y_2)}$$

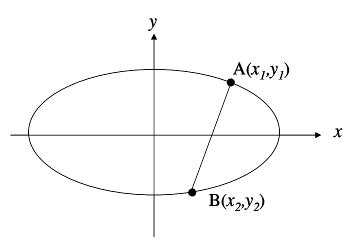


Fig. 4. 93

$\therefore$ the equation of the chord is

$$(y - y_1) = \frac{-b^2 (x_1 + x_2)}{a^2 (y_1 + y_2)} (x - x_1)$$

To get the equation of the tangent at (x_1, y_1) put $x_2 = x_1$ and $y_2 = y_1$ in the equation of the chord.

$\therefore$ The equation of the tangent at (x_1, y_1) is

$$(y - y_1) = \frac{-b^2 (x_1 + x_1)}{a^2 (y_1 + y_1)} (x - x_1)$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

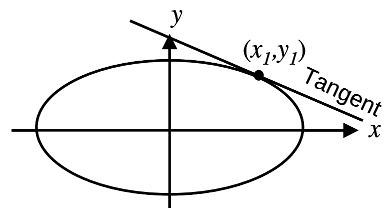


Fig. 4. 94

To get the equation of the normal, use the perpendicularity property to a straight line.

$\therefore$ The equation of the tangent is

$x_1b^2x + y_1a^2y - a^2b^2 = 0$

$\therefore$ The equation of the normal is of the form $y_1a^2x - x_1b^2y = k$

But it passes through (x_1, y_1)

$\therefore\ k = (a^2 - b^2)\, x_1\, y_1$

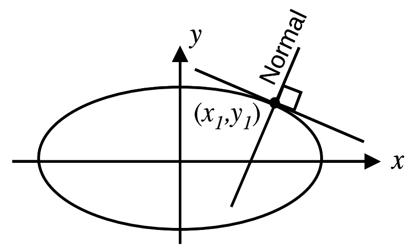


Fig. 4. 95

$\therefore$ The required equation is

$$y_1a^2x - x_1b^2y = (a^2 - b^2)\, x_1y_1 \text{ or } \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

(iii) Hyperbola

Following the same procedure as in the case of ellipse we get the equation of the chord as

$$y - y_1 = \frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)} (x - x_1)$$

The equation of the tangent at (x_1, y_1) as $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

and the normal at (x_1, y_1) as $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

Note : To get the results for the hyperbola replace b^2 as $-b^2$ in the results of ellipse.

4.7.2 Parametric form :

To get the parametric forms of equations of chord, tangent and normal to conics, replace (x_1, y_1), by the corresponding 'any point' in the parametric form.

(i) Parabola :

The equation of the chord joining (x_1, y_1) and (x_2, y_2) on the parabola is

$$y - y_1 = \frac{4a}{y_1 + y_2}(x - x_1)$$

$\therefore$ The equation of the chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ or 't_1' and 't_2' on the parabola is

$$y - 2at_1 = \frac{4a}{2at_1 + 2at_2}(x - at_1^2)$$

$$\text{i.e.}\quad y(t_1 + t_2) = 2x + 2a\,t_1t_2$$

To find the equation of the tangent at 't' put $t_1 = t_2 = t$ in the equation of the chord. We get

$$y(2t) = 2x + 2at^2$$

$$\text{i.e.}\ \ yt = x + at^2$$

Another method:

The tangent at (x_1, y_1) to $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

$\therefore$ The tangent at $(at^2, 2at)$ is

$$y(2at) = 2a\,(x + at^2)$$

$$\text{i.e.,}\quad yt = x + at^2$$

Applying the perpendicularity, we get the equation of the normal at 't' as $y + tx = 2at + at^3$

Similarly we can derive the equation of chord, tangent and normal for ellipse and hyperbola.

Note : The equation of tangent at (x_1, y_1) is obtained from the equation of the curve by replacing x^2 by xx_1, y^2 by yy_1, xy by $\frac{1}{2}(xy_1 + x_1y)$, x by $\frac{1}{2}(x + x_1)$ and y by $\frac{1}{2}(y + y_1)$

To find the condition that *y* = *mx* + *c* may be a tangent to the conics

(1) Parabola :

Let $y = mx + c$ be a tangent to the parabola $y^2 = 4ax$ at (x_1, y_1).

We know that at (x_1, y_1), the equation of the tangent is $yy_1 = 2a(x + x_1)$

$\therefore$ The above two equations represent the same tangent and hence their corresponding coefficients are proportional

$$\therefore \quad 2ax - y_1y + 2ax_1 = 0$$

$$mx - y + c = 0$$

$$\Rightarrow \quad \frac{2a}{m} = \frac{-y_1}{-1} = \frac{2ax_1}{c}$$

$$\Rightarrow \quad x_1 = \frac{c}{m}\ , \quad y_1 = \frac{2a}{m}$$

Since (x_1, y_1) lies on the parabola, $y_1^2 = 4ax_1$, $\frac{4a^2}{m^2} = 4a \,.\, \frac{c}{m}$

$$\text{i.e. }, \quad c = \frac{a}{m}$$

Thus we have three results to the parabola $y^2 = 4ax$.

(1) The condition for the tangency is $c = \frac{a}{m}$

(2) The point of contact is $\left(\frac{c}{m}\ ,\ \frac{2a}{m}\right)$ i.e., $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

(3) The equation of any tangent is of the form $y = mx + \frac{a}{m}$

Note : Instead of taking the equation of the tangent in the cartesian form, we can prove the same result by taking the tangent in the parametric form.

Similarly, we can derive the results for other conics also.

Results connected with ellipse :

(i) The condition that $y = mx + c$ may be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$

(ii) The point of contact is $\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$ where $c^2 = a^2m^2 + b^2$

(iii) The equation of any tangent is of the form $y = mx \pm \sqrt{a^2m^2 + b^2}$

Note : In $y = mx \pm \sqrt{a^2m^2 + b^2}$, either $y = mx + \sqrt{a^2m^2 + b^2}$ holds or $y = mx - \sqrt{a^2m^2 + b^2}$ holds

Results connected with hyperbola :

(i) The condition that $y = mx + c$ may be a tangent to the hyperbola is $c^2 = a^2m^2 - b^2$

(ii) The point of contact is $\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$ where $c^2 = a^2m^2 - b^2$

(iii) The equation of any tangent is of the form $y = mx \pm \sqrt{a^2m^2 - b^2}$

Note : In $y = mx \pm \sqrt{a^2m^2 - b^2}$, either $y = mx + \sqrt{a^2m^2 - b^2}$ or $y = mx - \sqrt{a^2m^2 - b^2}$ is correct but not both.

4.7.3 Equation of chord of contact of tangents from a point (x_1, y_1)

to the (i) Parabola $y^2 = 4ax$ (ii) ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (iii) hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution :

The equation of tangent at $Q(x_2, y_2)$ is $yy_2 = 2a(x + x_2)$

It passes through the point $P(x_1, y_1)$

$y_1y_2 = 2a\,(x_1 + x_2)$ … (1)

The equation of tangent at $R(x_3, y_3)$ is $yy_3 = 2a(x + x_3)$

It passes through the point $P(x_1, y_1)$

$\therefore\ y_1y_3 = 2a(x_1 + x_3)$ … (2)

The result (1) and (2) show that $Q(x_2, y_2)$ and $R(x_3, y_3)$ lie on the straight line $yy_1 = 2a(x + x_1)$.

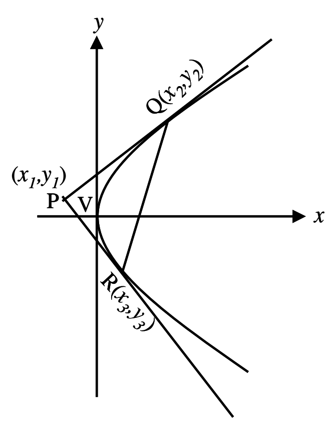


Fig. 4. 96

$\therefore$ Equation of QR, the chord of contact of tangents is $yy_1 = 2a(x + x_1)$

Similarly we can find the required equations of the chord of contact for ellipse as $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ and for the hyperbola as $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$

Example 4.59 : Find the equations of the tangents to the parabola $y^2 = 5x$ from the point (5, 13). Also find the points of contact.

Solution:

The equation of the parabola is $y^2 = 5x$ Here $4a = 5 \Rightarrow a = \dfrac{5}{4}$

Let the equation of the tangent be $y = mx + \dfrac{a}{m}$ i.e., $y = mx + \dfrac{5}{4m}$ … (1)

Since it passes through (5, 13) we have

$$13 = 5m + \frac{5}{4m}$$

$$\therefore 20m^2 - 52m + 5 = 0$$

$$(10m - 1)\ (2m - 5) = 0$$

$$\therefore m = \frac{1}{10} \text{ or } m = \frac{5}{2}$$

Using the values of m, we get the equations of tangents are $2y = 5x + 1$, $10y = x + 125$.

The points of contact are given by $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$, where $a = \dfrac{5}{4}$ $m = \dfrac{5}{2}$, $\dfrac{1}{10}$

$\therefore$ the points of contact are $\left(\dfrac{1}{5}, 1\right)$, (125, 25)

Example 4.60 : Find the equation of the tangent at $t = 1$ to the parabola $y^2 = 12x$

Solution: Equation of the parabola is $y^2 = 12x$.

Here $4a = 12$, $a = 3$

't' represents the point $(at^2, 2at)$. $\therefore$ $t = 1$ represents the point = (3, 6)

Equation of tangent at (x_1, y_1) to the parabola $y^2 = 12x$ is $yy_1 = 12\,\dfrac{(x + x_1)}{2}$

$\therefore$ Equation of tangent at (3, 6) is $y(6) = \dfrac{12\,(x + 3)}{2}$ i.e., $x - y + 3 = 0$

Alternative form :

The equation of the tangent at 't' is $yt = x + at^2$

$$\text{Here} \quad 4a = 12 \Rightarrow a = 3$$

$$\text{Also} \quad t = 1$$

$\therefore$ The equation of the tangent is $y = x + 3$

$$x - y + 3 = 0$$

Example 4.61 : Find the equation of the tangent and normal to the parabola $x^2 + x - 2y + 2 = 0$ at (1, 2)

Solution:

The equation of the parabola is $x^2 + x - 2y + 2 = 0$

Equation of the tangent at (x_1, y_1) to the given parabola is

$$xx_1 + \frac{x + x_1}{2} - 2\frac{(y + y_1)}{2} + 2 = 0 \text{ i.e., } x(1) + \frac{x+1}{2} - 2\frac{(y+2)}{2} + 2 = 0$$

On simplification we get $3x - 2y + 1 = 0$

Equation of the normal is of the form $2x + 3y + k = 0$

This normal passes through (1, 2)

$$\therefore\ 2 + 6 + k = 0 \quad \therefore\ k = -8$$

$\therefore$ Equation of the normal is $2x + 3y - 8 = 0$

Example 4.62 : Find the equations of the two tangents that can be drawn from the point (5, 2) to the ellipse $2x^2 + 7y^2 = 14$

Solution:

Equation of the ellipse is

$$2x^2 + 7y^2 = 14$$

$$\text{i.e., } \frac{x^2}{7} + \frac{y^2}{2} = 1$$

Here $a^2 = 7$, $b^2 = 2$

Let the equation of the tangent be $y = mx + \sqrt{a^2m^2 + b^2}$

$$\therefore\ y = mx + \sqrt{7m^2 + 2}$$

Since this line passes through the point (5, 2) we get

$$2 = 5m + \sqrt{7m^2 + 2}$$

$$\text{i.e.}\quad 2 - 5m = \sqrt{7m^2 + 2}$$

$$\therefore\ (2 - 5m)^2 = 7m^2 + 2$$

$$4 + 25m^2 - 20m = 7m^2 + 2$$

$$18m^2 - 20m + 2 = 0$$

$$9m^2 - 10m + 1 = 0$$

$$\therefore \; (9m - 1)\;(m - 1) = 0$$

$$\therefore \; m = 1 \quad \text{or} \quad m = \frac{1}{9}$$

To find the equations of the tangents, use slope-point form

(i) $m = 1$,

The equation is $y - 2 = 1(x - 5)$ i.e., $x - y - 3 = 0$

(ii) $m = 1/9$

The equation is $y - 2 = \frac{1}{9}(x - 5)$, i.e., $x - 9y + 13 = 0$.

Thus the equations of the tangents are $x - y - 3 = 0$, $x - 9y + 13 = 0$

Example 4.63 : Find the equation of chord of contact of tangents from the point (2, 4) to the ellipse $2x^2 + 5y^2 = 20$

Solution:

The equation of chord of contact of tangents

from (x_1, y_1) to $2x^2 + 5y^2 - 20 = 0$ is $2xx_1 + 5yy_1 - 20 = 0$

$\therefore$ the required equation from (2, 4) is $2x(2) + 5y(4) - 20 = 0$

i.e. $x + 5y - 5 = 0$

EXERCISE 4.4

(1) Find the equations of the tangent and normal

(i) to the parabola $y^2 = 12x$ at $(3, -6)$

(ii) to the parabola $x^2 = 9y$ at $(-3, 1)$

(iii) to the parabola $x^2 + 2x - 4y + 4 = 0$ at $(0, 1)$

(iv) to the ellipse $2x^2 + 3y^2 = 6$ at $\left(\sqrt{3}, 0\right)$

(v) to the hyperbola $9x^2 - 5y^2 = 31$ at $(2, -1)$

(2) Find the equations of the tangent and normal

(i) to the parabola $y^2 = 8x$ at $t = \frac{1}{2}$

(ii) to the ellipse $x^2 + 4y^2 = 32$ at $\theta = \frac{\pi}{4}$

(iii) to the ellipse $16x^2 + 25y^2 = 400$ at $t = \frac{1}{\sqrt{3}}$

(iv) to the hyperbola $\frac{x^2}{9} - \frac{y^2}{12} = 1$ at $\theta = \frac{\pi}{6}$

(3) Find the equations of the tangents

(i) to the parabola $y^2 = 6x$, parallel to $3x - 2y + 5 = 0$

(ii) to the parabola $y^2 = 16x$, perpendicular to the line $3x - y + 8 = 0$

(iii) to the ellipse $\frac{x^2}{20} + \frac{y^2}{5} = 1$, which are perpendicular to $x + y + 2 = 0$

(iv) to the hyperbola $4x^2 - y^2 = 64$, which are parallel to $10x - 3y + 9 = 0$

(4) Find the equation of the two tangents that can be drawn

(i) from the point $(2, -3)$ to the parabola $y^2 = 4x$

(ii) from the point $(1, 3)$ to the ellipse $4x^2 + 9y^2 = 36$

(iii) from the point $(1, 2)$ to the hyperbola $2x^2 - 3y^2 = 6$.

(5) Prove that the line $5x + 12y = 9$ touches the hyperbola $x^2 - 9y^2 = 9$ and find its point of contact.

(6) Show that the line $x - y + 4 = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 12$. Find the co-ordinates of the point of contact.

(7) Find the equation to the chord of contact of tangents from the point

(i) $(-3, 1)$ to the parabola $y^2 = 8x$

(ii) $(2, 4)$ to the ellipse $2x^2 + 5y^2 = 20$

(iii) $(5, 3)$ to the hyperbola $4x^2 - 6y^2 = 24$

Results without Proof :

(1) Two tangents can be drawn to (i) a parabola (ii) an ellipse and (iii) a hyperbola, from any point on the plane.

(2) (a) Three normals can be drawn to a parabola

(b) Four normals can be drawn to (i) an ellipse and (ii) a hyperbola from any point on the plane.

(3) The equation of chord of contact of tangents from a point (x_1, y_1)

(i) a parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

(ii) an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

(iii) a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

(4) The chord of contact of tangents from any point on the directrix (i) of a parabola passes through its focus (ii) passes through the corresponding focus for ellipse and hyperbola

(5) The condition that $lx + my + n = 0$ may be a tangent to

(i) the parabola $y^2 = 4ax$ is $am^2 = ln$

(ii) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $a^2l^2 + b^2m^2 = n^2$

(iii) the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $a^2l^2 - b^2m^2 = n^2$

(6) The condition that $lx + my + n = 0$ may be a normal to

(i) the parabola $y^2 = 4ax$ is $al^3 + 2alm^2 + m^2n = 0$

(ii) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$

(iii) the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$

(7) The locus of the foot of the perpendicular from a focus to a tangent to

(i) the parabola $y^2 = 4ax$ is $x = 0$

(ii) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the circle $x^2 + y^2 = a^2$

(iii) the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the circle $x^2 + y^2 = a^2$

(This circle is also called auxiliary circle)

(8) The locus of the point of intersection of perpendicular tangents to

(i) the parabola $y^2 = 4ax$ is $x = -a$ (the directrx)

(ii) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 + b^2$ (This circle is called director circle)

(iii) an hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$ (This circle is also called director circle)

(9) The point of intersection of the tangents at 't_1' and 't_2' to the parabola $y^2 = 4ax$ is $[at_1t_2, a(t_1 + t_2)]$

(10) The normal at the point 't_1' on the parabola $y^2 = 4ax$ meets the parabola again at the point 't_2', then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$

(11) If 't_1' and 't_2' are the extremities of any focal chord of the parabola $y^2 = 4ax$, then $t_1 t_2 = -1$

Note : For the proof of above results one may refer the Solution Book.

4.8. Asymptotes

Consider the graph of a function $y = f(x)$. As a point P on the curve moves farther and farther away from the origin, it may happen that the distance between P and some fixed line tends to zero. This fixed line is called an asymptote.

Note that it is possible only when the curve is open. Since hyperbola is open and $y \to \pm \infty$ as $x \to +\infty$ and $x \to -\infty$ hyperbola have asymptotes.

Definition :

An asymptote to a curve is the tangent to the curve such that the point of contact is at infinity. In particular the asymptote touches the curve at $+\infty$ and $-\infty$.

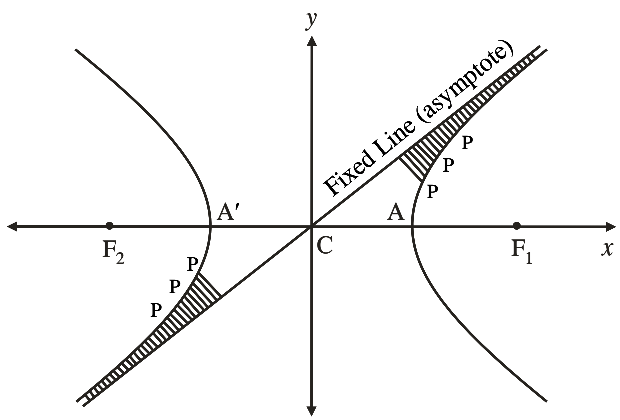


Fig. 4. 97

The equations of the asymptotes to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Assume that the equation of an asymptote is of the form $y = mx + c$.

To find the points of intersection of the hyperbola and the asymptote, solve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $y = mx + c$.

$$\therefore \frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

$$\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right)x^2 - \frac{2mc}{b^2}x - \left(\frac{c^2}{b^2} + 1\right) = 0$$

The points of contact are at infinity. i.e., the roots of the equations are infinite. Since the roots are infinite, the coefficients of x^2 and x must be zero.

$$\therefore \frac{1}{a^2} - \frac{m^2}{b^2} = 0 \quad \text{and} \quad \frac{-2mc}{b^2} = 0$$

$$\text{i.e.,} \quad m = \pm\frac{b}{a} \text{ and } c = 0$$

$$\text{Then} \quad y = \pm\frac{b}{a}x$$

$\therefore$ there are two asymptotes to the hyperbola whose equations are

$$y = \frac{b}{a}x \quad \text{and } y = \frac{-b}{a}x$$

i.e. $\frac{x}{a} - \frac{y}{b} = 0$ and $\frac{x}{a} + \frac{y}{b} = 0$

The combined equation of asymptotes is

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0 \quad \text{i.e.} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

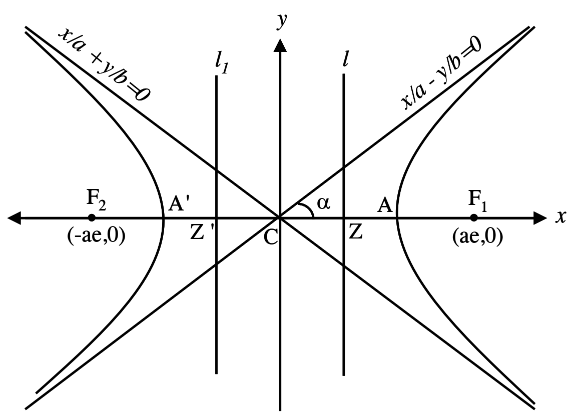


Fig. 4. 98

Results regarding asymptotes :

(1) The asymptotes pass through the centre $C(0, 0)$ of the hyperbola.

(2) The slopes of asymptotes are $\frac{b}{a}$ and $-\frac{b}{a}$ i.e., the transverse axis and conjugate axis bisect angles between the asymptotes.

(3) If 2α is the angle between the asymptotes then the slope of $\frac{x}{a} - \frac{y}{b} = 0$ is $\tan\alpha = \frac{b}{a}$.

$\therefore$ angle between the asymptotes is $2\alpha = 2\tan^{-1}\frac{b}{a}$

(4) We know that $\sec^2\alpha = 1 + \tan^2\alpha$

$$\sec^2\alpha = 1 + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2} = e^2$$

$$\Rightarrow \sec\alpha = e \Rightarrow \alpha = \sec^{-1}e$$

$\therefore$ angle between the asymptotes $2\alpha = 2\sec^{-1}e$

Important Note :

Eventhough the asymptotes are straight lines, if the angle between the asymptotes is obtuse, take obtuse angle as the angle between them and not the corresponding acute angle.

(5) The standard equation of hyperbola and combined equation of asymptotes differs only by a constant.

(6) If $l_1 = 0$ and $l_2 = 0$ are the separate equations of asymptotes, then the combined equation of the asymptotes is $l_1 l_2 = 0$.

$\therefore$ the equation of the corresponding hyperbola is of the form $l_1 l_2 = k$, where k is a constant. To find this k, we need a point on the hyperbola.

Example 4.64 : Find the separate equations of the asymptotes of the hyperbola $3x^2 - 5xy - 2y^2 + 17x + y + 14 = 0$

Solution: The combined equation of the asymptotes differs from the hyperbola by a constant only.

$\therefore$ the combined equation of the asymptotes is

$$3x^2 - 5xy - 2y^2 + 17x + y + k = 0$$

$$\begin{aligned} \text{Consider } 3x^2 - 5xy - 2y^2 &= 3x^2 - 6xy + xy - 2y^2 \\ &= 3x(x - 2y) + y(x - 2y) \\ &= (3x + y)(x - 2y) \end{aligned}$$

$\therefore$ The separate equations are $3x + y + l = 0$, $x - 2y + m = 0$

$$\therefore \quad (3x + y + l)\ (x - 2y + m) = 3x^2 - 5xy - 2y^2 + 17x + y + k$$

Equating the coefficients of x, y terms and constant term, we get

$$l + 3m = 17 \qquad \ldots (1)$$

$$-2l + m = 1 \qquad \ldots (2)$$

$$l\,m = k$$

Solving (1) and (2) we get $l = 2$, $m = 5$ and $k = 10$

Hence separate equations of asymptotes are $3x + y + 2 = 0$, $x - 2y + 5 = 0$

The combined equation of asymptotes is

$$3x^2 - 5xy - 2y^2 + 17x + y + 10 = 0$$

Note : The hyperbola, discussed above is not a standard hyperbola.

Example 4.65 : Find the equation of the hyperbola which passes through the point (2, 3) and has the asymptotes $4x + 3y - 7 = 0$ and $x - 2y = 1$.

Solution:

The separate equations of the asymptotes are $4x + 3y - 7 = 0$, $x - 2y - 1 = 0$

$\therefore$ combined equation of asymptotes is $(4x + 3y - 7)\ (x - 2y - 1) = 0$

The equation of the hyperbola differs from this combined equation of asymptotes by a constant only.

$\therefore$ the equation of the hyperbola is of the form

$$(4x + 3y - 7)\ (x - 2y - 1) + k = 0$$

But this passes through (2, 3)

$$(8 + 9 - 7)\ (2 - 6 - 1) + k = 0 \qquad \therefore\ k = 50$$

$\therefore$ The equation of the corresponding hyperbola is

$$(4x + 3y - 7)\ (x - 2y - 1) + 50 = 0$$

i.e., $\quad 4x^2 - 5xy - 6y^2 - 11x + 11y + 57 = 0$

Example 4.66 : Find the angle between the asymptotes of the hyperbola $3x^2 - y^2 - 12x - 6y - 9 = 0$

Solution:

$$3x^2 - y^2 - 12x - 6y - 9 = 0$$

$$3(x^2 - 4x) - (y^2 + 6y) = 9$$

$$3\,\{(x - 2)^2 - 4\} - \{(y + 3)^2 - 9\} = 9$$

$$3(x - 2)^2 - (y + 3)^2 = 12$$

$$\frac{(x - 2)^2}{4} - \frac{(y + 3)^2}{12} = 1$$

Here $a = 2$, $b = \sqrt{12} = 2\sqrt{3}$

The angle between the asymptotes is

$$2\alpha = 2\tan^{-1}\frac{b}{a} = 2\tan^{-1}\frac{2\sqrt{3}}{2} = 2\tan^{-1}\sqrt{3} = 2\times\frac{\pi}{3} = \frac{2\pi}{3}$$

Another method : $a^2 = 4$, $b^2 = 12$

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{12}{4}} = 2$$

The angle between the asymptotes is

$$2\alpha = 2\sec^{-1}2 = 2\times\frac{\pi}{3} = \frac{2\pi}{3}$$

Example 4.67 : Find the angle between the asymptotes to the hyperbola $3x^2 - 5xy - 2y^2 + 17x + y + 14 = 0$

Solution: Combined equation of the asymptotes differs from that of the hyperbola by a constant only.

$\therefore$ Combined equation of asymptotes is $3x^2 - 5xy - 2y^2 + 17x + y + k = 0$

$$\begin{aligned} 3x^2 - 5xy - 2y^2 &= 3x^2 - 6xy + xy - 2y^2 \\ &= 3x(x-2y) + y(x-2y) \\ &= (x-2y)(3x+y) \end{aligned}$$

$\therefore$ Separate equations are $x - 2y + l = 0$, $3x + y + m = 0$

Let m_1 and m_2 be the slopes of these lines, then $m_1 = \frac{1}{2}$, $m_2 = -3$

$\therefore$ angle between the lines is $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{1/2-(-3)}{1+1/2(-3)}\right| = 7$

$$\theta = \tan^{-1}(7)$$

Alternative method : Combined equation of asymptotes is nothing but pair of straight lines. Hence the angle between the asymptotes is

$$\tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$$

Comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

We have $a = 3$, $b = -2$, $2h = -5$

$$\begin{aligned} \tan\theta &= \left|\frac{2\sqrt{\frac{25}{4}+6}}{3-2}\right| \\ &= \left|\frac{2\times 7}{2}\right| = 7 \end{aligned}$$

$$\theta = \tan^{-1}(7)$$

Note : Since the above hyperbola is not in the standard form, it is difficult to identify whether the angle between the asymptotes is obtuse or acute. According to the above method we will get only the acute angle as the angle between the asymptotes.

Therefore if the hyperbola in the standard form, use either $2\tan^{-1}\frac{b}{a}$ or $2\sec^{-1}e$ to find the angle between the asymptotes and take the angle as it is.

Example 4.68 : Prove that the product of perpendiculars from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is constant and the value is $\frac{a^2b^2}{a^2+b^2}$

Solution:

Let $P(x_1, y_1)$ be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \therefore \frac{{x_1}^2}{a^2} - \frac{{y_1}^2}{b^2} = 1$ … (1)

The perpendicular distance from (x_1, y_1) to the asymptote

$\frac{x}{a} - \frac{y}{b} = 0$ is $\dfrac{\frac{x_1}{a} - \frac{y_1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$ and to $\frac{x}{a} + \frac{y}{b} = 0$ is $\dfrac{\frac{x_1}{a} + \frac{y_1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$

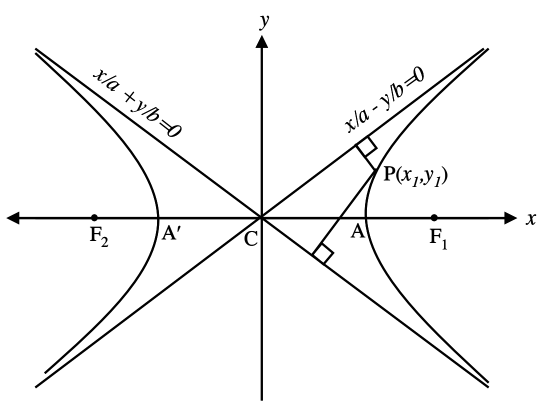


Fig. 4. 99

$$\therefore \text{ Product of perpendicular distances} = \dfrac{\frac{x_1}{a} + \frac{y_1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} \cdot \dfrac{\frac{x_1}{a} - \frac{y_1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$$

$$= \frac{\dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2}}{\dfrac{1}{a^2} + \dfrac{1}{b^2}} = \frac{1}{\dfrac{b^2 + a^2}{a^2\, b^2}} \quad \text{(by (1))}$$

$$= \frac{a^2\, b^2}{a^2 + b^2}, \text{ which is a constant.}$$

EXERCISE 4.5

(1) Find the equation of the asymptotes to the hyperbola

(i) $36x^2 - 25y^2 = 900$ (ii) $8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0$

(2) Find the equation of the hyperbola if

(i) the asymptotes are $2x + 3y - 8 = 0$ and $3x - 2y + 1 = 0$ and (5, 3) is a point on the hyperbola

(ii) its asymptotes are parallel to $x + 2y - 12 = 0$ and $x - 2y + 8 = 0$, (2, 4) is the centre of the hyperbola and it passes through (2, 0).

(3) Find the angle between the asymptotes of the hyperbola

(i) $24x^2 - 8y^2 = 27$ (ii) $9(x - 2)^2 - 4(y + 3)^2 = 36$

(iii) $4x^2 - 5y^2 - 16x + 10y + 31 = 0$

4.9 Rectangular hyperbola

Definition:

A hyperbola is said to be a rectangular hyperbola if its asymptotes are at right angles.

The angle between the asymptotes is given by $2\tan^{-1}\dfrac{b}{a}$. But angle between the asymptotes of the rectangular hyperbola is 90° .

$\therefore\ 2\tan^{-1}\left(\dfrac{b}{a}\right) = 90°\ \therefore\ \dfrac{b}{a} = \tan 45° \ \Rightarrow\ a = b.$

Using $a = b$ in the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, we get the equation of rectangular hyperbola as $x^2 - y^2 = a^2$. Hence the combined equation of the asymptotes is $x^2 - y^2 = 0$. The separate equations are $x - y = 0$ and $x + y = 0$. i.e., $x = y$ and $x = -y$. The transverse axis is $y = 0$, conjugate axis is $x = 0$.

All the results corresponding to the rectangular hyperbola of the form $x^2 - y^2 = a^2$ are obtained simply by putting $a = b$ in the corresponding results of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

This type of rectangular hyperbola is not a standard one. For standard type, the asymptotes are the co-ordinate axes.

The standard rectangular hyperbola $xy = c^2$ is obtained by rotating the rectangular hyperbola $x^2 - y^2 = a^2$ through an angle 45° about the origin in the anticlockwise direction.

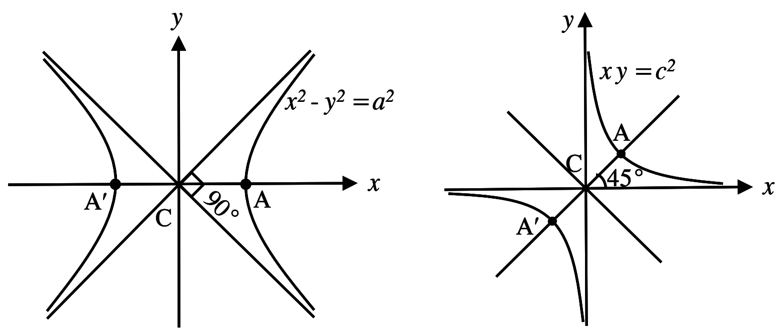


Fig. 4. 100

4.9.1 Standard equation of a rectangular hyperbola :

For a standard rectangular hyperbola the asymptotes are co-ordinate axes. Since the axes are the asymptotes, the equations of the asymptotes are $x = 0$ and $y = 0$. The combined equation of the asymptotes is $xy = 0$. Therefore the equation of the standard rectangular hyperbola is of the form $xy = k$. To find k, we need a point on the rectangular hyperbola.

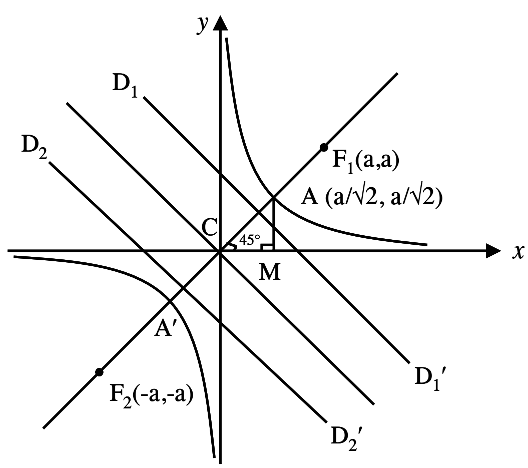


Fig. 4. 101

Let the asymptotes meet at C. Let $AA' = 2a$ be the length of the transverse axis. Draw AM perpendicular to x-axis. Since the asymptotes bisect the angle between the axes, $\underline{|ACM} = 45°$. $CM = a\cos 45° = \frac{a}{\sqrt{2}}$, $AM = a\sin 45° = \frac{a}{\sqrt{2}}$

$\therefore$ co-ordinates of A are $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$. This point lies on the rectangular hyperbola $xy = k$. $\therefore$ $k = \frac{a}{\sqrt{2}} \cdot \frac{a}{\sqrt{2}}$ or $k = \frac{a^2}{2}$ and

the equation of the rectangular hyperbola is $xy = \frac{a^2}{2}$ or

$$xy = c^2 \qquad \text{where } c^2 = \frac{a^2}{2}.$$

Eccentricity of the hyperbola is given by $b^2 = a^2(e^2 - 1)$. Since $a = b$ in a rectangular hyperbola, $a^2 = a^2(e^2 - 1)$

Eccentricity of the rectangular hyperbola is $e = \sqrt{2}$.

Also the vertices of the rectangular hyperbola are $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$, $\left(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$ and foci are (a, a), $(-a, -a)$.

The equation of transverse axis is $y = x$ and the conjugate axis is $y = -x$.

If the centre of the rectangular hyperbola is at (h, k) and the asymptotes are parallel to x and y-axis, the general form of standard rectangular hyperbola is $(x - h)(y - k) = c^2$.

The parametric equation of the rectangular hyperbola $xy = c^2$ are

$x = ct$, $y = \frac{c}{t}$ where 't' is the parameter and 't' is any non-zero real number.

Any point on the rectangular hyperbola is $\left(ct\ ,\ \frac{c}{t}\right)$. This point is often referred to as the point 't'.

Results :

★ Equation of the tangent at (x_1, y_1) to the rectangular hyperbola $xy = c^2$ is $xy_1 + yx_1 = 2c^2$

★ Equation of the tangent at 't' is $x + yt^2 = 2ct$.

★ Equation of normal at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.

★ Equation of normal at 't' is $y - xt^2 = \frac{c}{t} - ct^3$

★ Two tangents and four normals can be drawn from a point to a rectangular hyperbola.

Example 4.69 : Find the equation of the standard rectangular hyperbola whose centre is $\left(-2, \frac{-3}{2}\right)$ and which passes through the point $\left(1, \frac{-2}{3}\right)$

Solution:

The equation of the standard rectangular hyperbola with centre at (h, k) is $(x - h)(y - k) = c^2$

The centre is $\left(-2, \frac{-3}{2}\right)$.

$\therefore$ the equation of the standard rectangular hyperbola is $(x+2)\left(y + \frac{3}{2}\right) = c^2$

It passes through $\left(1, \frac{-2}{3}\right)$ $\therefore$ $(1 + 2)\left(\frac{-2}{3} + \frac{3}{2}\right) = c^2 \Rightarrow c^2 = \frac{5}{2}$

Hence the required equation is $(x + 2)\left(y + \frac{3}{2}\right) = \frac{5}{2}$ or

$2xy + 3x + 4y + 1 = 0$

Example 4.70 : The tangent at any point of the rectangular hyperbola $xy = c^2$ makes intercepts a, b and the normal at the point makes intercepts p, q on the axes. Prove that $ap + bq = 0$

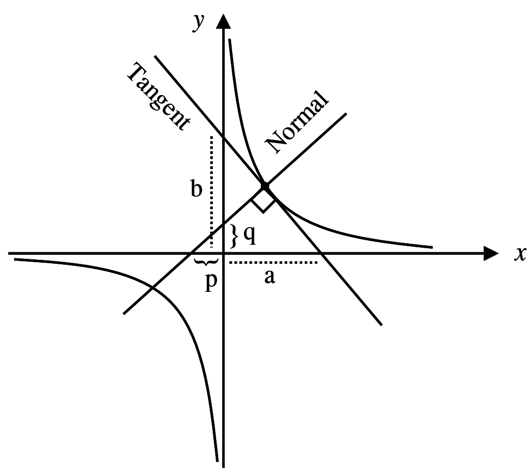


Fig. 4. 102

Solution: Equation of tangent at any point 't' on $xy = c^2$ is $x + yt^2 = 2ct$

or $\dfrac{x}{2ct} + \dfrac{y}{2c/t} = 1$

$\therefore$ Intercept on the axes are $a = 2ct$, $b = \dfrac{2c}{t}$.

Equation of normal at 't' on $xy = c^2$ is $y - xt^2 = \dfrac{c}{t} - ct^3$

$$\frac{x}{\left(\dfrac{\dfrac{c}{t} - ct^3}{-t^2}\right)} + \frac{y}{\left(\dfrac{c}{t} - ct^3\right)} = 1$$

$\therefore$ intercept on axes are $\quad p = \dfrac{-1}{t^2}\left(\dfrac{c}{t} - ct^3\right)$, $\quad q = \dfrac{c}{t} - ct^3$

$$\begin{aligned}\therefore\ ap + bq &= 2ct\left(\frac{-1}{t^2}\right)\left(\frac{c}{t} - ct^3\right) + \frac{2c}{t}\left(\frac{c}{t} - ct^3\right)\\ &= -\frac{2c}{t}\left(\frac{c}{t} - ct^3\right) + \frac{2c}{t}\left(\frac{c}{t} - ct^3\right)\\ &= 0\end{aligned}$$

Example 4.71 : Show that the tangent to a rectangular hyperbola terminated by its asymptotes is bisected at the point of contact.

Solution:

The equation of tangent at $P\left(ct, \dfrac{c}{t}\right)$ is $x + yt^2 = 2ct$

Putting $y = 0$ in this equation we get the co-ordinates of A as $(2ct,\ 0)$. Putting $x = 0$ we get the co-ordinates of B as $\left(0, \dfrac{2c}{t}\right)$

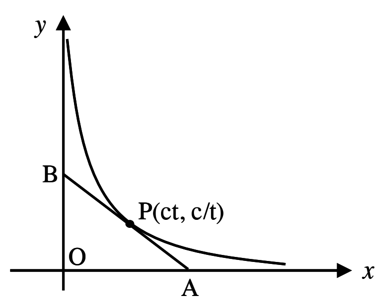


Fig. 4. 103

The mid-point of AB is $\left(\dfrac{2ct + 0}{2}, \dfrac{0 + \dfrac{2c}{t}}{2}\right) = \left(ct, \dfrac{c}{t}\right)$

which is the point P. This shows that the tangent is bisected at the point of contact.

EXERCISE 4.6

(1) Find the equation of the standard rectangular hyperbola whose centre is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and which passes through the point $\left(1, \frac{1}{4}\right)$.

(2) Find the equation of the tangent and normal (i) at (3, 4) to the rectangular hyperbolas $xy = 12$ (ii) at $\left(-2, \frac{1}{4}\right)$ to the rectangular hyperbola $2xy - 2x - 8y - 1 = 0$

(3) Find the equation of the rectangular hyperbola which has for one of its asymptotes the line $x + 2y - 5 = 0$ and passes through the points (6, 0) and (– 3, 0).

(4) A standard rectangular hyperbola has its vertices at (5, 7) and (– 3, –1). Find its equation and asymptotes.

(5) Find the equation of the rectangular hyperbola which has its centre at (2, 1), one of its asymptotes $3x - y - 5 = 0$ and which passes through the point (1, – 1).

(6) Find the equations of the asymptotes of the following rectangular hyperbolas.

(i) $xy - kx - hy = 0$ (ii) $2xy + 3x + 4y + 1 = 0$

(iii) $6x^2 + 5xy - 6y^2 + 12x + 5y + 3 = 0$

(7) Prove that the tangent at any point to the rectangular hyperbola forms with the asymptotes a triangle of constant area.

Results without proof :

(1) The foot of the perpendicular from a focus of a hyperbola on an asymptote lies on the corresponding directrix.

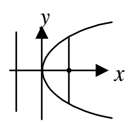
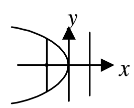
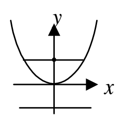
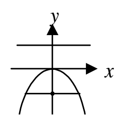
(2) (i) Two tangents (ii) four normals can be drawn from a point to the rectangular hyperbola $xy = c^2$.

(3) The condition that the line $lx + my + n = 0$ may be a tangent to the rectangular hyperbola $xy = c^2$ is $4c^2lm = n^2$

(4) If the normal to the rectangular hyperbola $xy = c^2$ at 't_1' meets the curve again at 't_2' prove that $t_1^3 t_2 = -1$.

Note : For the proof of above results one may refer the Solution Book.

Now, we summarise the results of the four standard types of parabolas.

Type	Equation	Diagram	Focus	Equation of Directrix	Axis	Vertex	Equation of Latus Rectum	Length of Latus Rectum
Open rightwards	$y^2 = 4ax$		$(a, 0)$	$x = -a$	$y = 0$	$(0, 0)$	$x = a$	$4a$
Open leftwards	$y^2 = -4ax$		$(-a, 0)$	$x = a$	$y = 0$	$(0, 0)$	$x = -a$	$4a$
Open upwards	$x^2 = 4ay$		$(0, a)$	$y = -a$	$x = 0$	$(0, 0)$	$y = a$	$4a$
Open downwards	$x^2 = -4ay$		$(0, -a)$	$y = a$	$x = 0$	$(0, 0)$	$y = -a$	$4a$

Thus we get the following :

Cartesian form :	**Parabola**	**Ellipse**	**Hyperbola**
Equation of chord joining (x_1, y_1) and (x_2, y_2)	$y - y_1 = \frac{4a}{y_1 + y_2}(x - x_1)$	$y - y_1 = -\frac{b^2(x_1+x_2)}{a^2(y_1+y_2)}(x - x_1)$	$y - y_1 = \frac{b^2(x_1+x_2)}{a^2(y_1+y_2)}(x - x_1)$
Equation of tangent at (x_1, y_1)	$yy_1 = 2a(x + x_1)$	$xx_1 / a^2 + yy_1/b^2 = 1$	$xx_1 / a^2 - yy_1/b^2 = 1$
Equation of normal at (x_1, y_1)	$xy_1 + 2ay = x_1y_1 + 2ay_1$	$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$	$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

Parametric form :	**Parabola**	**Ellipse**	**Hyperbola**
Equation of chord	Chord joining the points 't_1' and 't_2' is $y(t_1 + t_2) = 2x + 2at_1t_2$	Chord joning the points 'θ'$_1$ and 'θ'$_2$ is $\frac{x}{a}\cos\frac{(\theta_1+\theta_2)}{2} + \frac{y}{b}\sin\frac{(\theta_1+\theta_2)}{2} = \cos\frac{(\theta_1-\theta_2)}{2}$	Chord joning the points 'θ'$_1$ and 'θ'$_2$ is $\frac{x}{a}\cos\frac{(\theta_1-\theta_2)}{2} - \frac{y}{b}\sin\frac{(\theta_1+\theta_2)}{2} = \cos\frac{(\theta_1+\theta_2)}{2}$
Equation of tangent	at 't' is $yt = x + at^2$	at 'θ' is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$	at 'θ' is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$
Equation of normal	at 't' is $tx + y = 2at + at^3$	$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$	$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

OBJECTIVE TYPE QUESTIONS

Choose the correct or most suitable answer :

(1) The rank of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 4 & -4 & 8 \end{bmatrix}$ is

(1) 1(2) 2 (3) 3 (4) 4

(2) The rank of the diagonal matrix $\begin{bmatrix} -1 & & & & \\ & 2 & & & \\ & & 0 & & \\ & & & -4 & \\ & & & & 0 \end{bmatrix}$

(1) 0(2) 2 (3) 3 (4) 5

(3) If $A = [2 \ \ 0 \ \ 1]$, then rank of AA^T is

(1) 1(2) 2 (3) 3 (4) 0

(4) If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then the rank of AA^T is

(1) 3(2) 0 (3) 1 (4) 2

(5) If the rank of the matrix $\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{bmatrix}$ is 2, then λ is

(1) 1(2) 2 (3) 3 (4) any real number

(6) If A is a scalar matrix with scalar $k \neq 0$, of order 3, then A^{-1} is

(1) $\frac{1}{k^2}$ I (2) $\frac{1}{k^3}$ I (3) $\frac{1}{k}$ I (4) kI

(7) If the matrix $\begin{bmatrix} -1 & 3 & 2 \\ 1 & k & -3 \\ 1 & 4 & 5 \end{bmatrix}$ has an inverse then the values of k

(1) k is any real number (2) $k = -4$ (3) $k \neq -4$ (4) $k \neq 4$

(8) If $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, then (adj A) A =

(1) $\begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (3) $\begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$ (4) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

(9) If A is a square matrix of order n then $|\operatorname{adj} A|$ is

(1) $|A|^2$ (2) $|A|^n$ (3) $|A|^{n-1}$ (4) $|A|$

(10) The inverse of the matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is

(1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (3) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (4) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(11) If A is a matrix of order 3, then $\det(kA)$

(1) $k^3\det(A)$ (2) $k^2\det(A)$ (3) $k\det(A)$ (4) $\det(A)$

(12) If I is the unit matrix of order n, where $k \neq 0$ is a constant, then $\operatorname{adj}(kI) =$

(1) $k^n(\operatorname{adj} I)$ (2) $k(\operatorname{adj} I)$ (3) $k^2(\operatorname{adj}(I))$ (4) $k^{n-1}(\operatorname{adj} I)$

(13) If A and B are any two matrices such that $AB = O$ and A is non-singular, then

(1) $B = O$ (2) B is singular (3) B is non-singular (4) $B = A$

(14) If $A = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$, then A^{12} is

(1) $\begin{bmatrix} 0 & 0 \\ 0 & 60 \end{bmatrix}$ (2) $\begin{bmatrix} 0 & 0 \\ 0 & 5^{12} \end{bmatrix}$ (3) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(15) Inverse of $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ is

(1) $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ (2) $\begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ (3) $\begin{bmatrix} 3 & -1 \\ -5 & -3 \end{bmatrix}$ (4) $\begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$

(16) In a system of 3 linear non-homogeneous equation with three unknowns, if $\Delta = 0$ and $\Delta_x = 0$, $\Delta_y \neq 0$ and $\Delta_z = 0$ then the system has

(1) unique solution (2) two solutions

(3) infinitely many solutions (4) no solutions

(17) The system of equations $ax + y + z = 0$; $x + by + z = 0$; $x + y + cz = 0$ has a non-trivial solution then $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} =$

(1) 1(2) 2 (3) -1 (4) 0

(18) If $ae^x + be^y = c$; $pe^x + qe^y = d$ and $\Delta_1 = \begin{vmatrix} a & b \\ p & q \end{vmatrix}$; $\Delta_2 = \begin{vmatrix} c & b \\ d & q \end{vmatrix}$,

$\Delta_3 = \begin{vmatrix} a & c \\ p & d \end{vmatrix}$ then the value of (x, y) is

(1) $\left(\frac{\Delta_2}{\Delta_1}, \frac{\Delta_3}{\Delta_1}\right)$ (2) $\left(\log \frac{\Delta_2}{\Delta_1}, \log \frac{\Delta_3}{\Delta_1}\right)$

(3) $\left(\log \frac{\Delta_1}{\Delta_3}, \log \frac{\Delta_1}{\Delta_2}\right)$ (4) $\left(\log \frac{\Delta_1}{\Delta_2}, \log \frac{\Delta_1}{\Delta_3}\right)$

(19) If the equation
$$-2x + y + z = l$$
$$x - 2y + z = m$$
$$x + y - 2z = n$$
such that $l + m + n = 0$, then the system has

(1) a non-zero unique solution (2) trivial solution

(3) Infinitely many solution (4) No Solution

(20) If $\vec{a}$ is a non-zero vector and m is a non-zero scalar then $m\vec{a}$ is a unit vector if

(1) $m = \pm 1$ (2) $a = | m |$ (3) $a = \frac{1}{| m |}$ (4) $a = 1$

(21) If $\vec{a}$ and $\vec{b}$ are two unit vectors and θ is the angle between them, then $\left(\vec{a} + \vec{b}\right)$ is a unit vector if

(1) $\theta = \frac{\pi}{3}$ (2) $\theta = \frac{\pi}{4}$ (3) $\theta = \frac{\pi}{2}$ (4) $\theta = \frac{2\pi}{3}$

(22) If $\vec{a}$ and $\vec{b}$ include an angle 120° and their magnitude are 2 and $\sqrt{3}$ then $\vec{a} \cdot \vec{b}$ is equal to

(1) $\sqrt{3}$ (2) $-\sqrt{3}$ (3) 2 (4) $-\frac{\sqrt{3}}{2}$

(23) If $\vec{u} = \vec{a} \times \left(\vec{b} \times \vec{c}\right) + \vec{b} \times \left(\vec{c} \times \vec{a}\right) + \vec{c} \times \left(\vec{a} \times \vec{b}\right)$, then

(1) u is a unit vector (2) $\vec{u} = \vec{a} + \vec{b} + \vec{c}$

(3) $\vec{u} = \vec{0}$ (4) $\vec{u} \neq \vec{0}$

(24) If $\vec{a} + \vec{b} + \vec{c} = 0$, $\left|\vec{a}\right| = 3$, $\left|\vec{b}\right| = 4$, $\left|\vec{c}\right| = 5$ then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{2\pi}{3}$ (3) $\frac{5\pi}{3}$ (4) $\frac{\pi}{2}$

(25) The vectors $2\vec{i} + 3\vec{j} + 4\vec{k}$ and $a\vec{i} + b\vec{j} + c\vec{k}$ are perpendicular when

(1) $a = 2,\ b = 3,\ c = -4$ (2) $a = 4,\ b = 4,\ c = 5$
(3) $a = 4,\ b = 4,\ c = -5$ (4) $a = -2,\ b = 3,\ c = 4$

(26) The area of the parallelogram having a diagonal $3\vec{i} + \vec{j} - \vec{k}$ and a side $\vec{i} - 3\vec{j} + 4\vec{k}$ is

(1) $10\sqrt{3}$ (2) $6\sqrt{30}$ (3) $\frac{3}{2}\sqrt{30}$ (4) $3\sqrt{30}$

(27) If $\left|\vec{a} + \vec{b}\right| = \left|\vec{a} - \vec{b}\right|$ then

(1) $\vec{a}$ is parallel to $\vec{b}$

(2) $\vec{a}$ is perpendicular to $\vec{b}$

(3) $\left|\vec{a}\right| = \left|\vec{b}\right|$

(4) $\vec{a}$ and $\vec{b}$ are unit vectors

(28) If $\vec{p}$, $\vec{q}$ and $\vec{p} + \vec{q}$ are vectors of magnitude λ then the magnitude of $\left|\vec{p} - \vec{q}\right|$ is

(1) 2λ (2) $\sqrt{3}\lambda$ (3) $\sqrt{2}\lambda$ (4) 1

(29) If $\vec{a} \times \left(\vec{b} \times \vec{c}\right) + \vec{b} \times \left(\vec{c} \times \vec{a}\right) + \vec{c} \times \left(\vec{a} \times \vec{b}\right) = \vec{x} \times \vec{y}$ then

(1) $\vec{x} = \vec{0}$ (2) $\vec{y} = \vec{0}$

(3) $\vec{x}$ and $\vec{y}$ are parallel (4) $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$ or $\vec{x}$ and $\vec{y}$ are parallel

(30) If $\vec{PR} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{QS} = -\vec{i} + 3\vec{j} + 2\vec{k}$ then the area of the quadrilateral $PQRS$ is

(1) $5\sqrt{3}$ (2) $10\sqrt{3}$ (3) $\frac{5\sqrt{3}}{2}$ (4) $\frac{3}{2}$

(31) The projection of $\vec{OP}$ on a unit vector $\vec{OQ}$ equals thrice the area of parallelogram $OPRQ$. Then $\underline{|POQ}$ is

(1) $\tan^{-1}\frac{1}{3}$ (2) $\cos^{-1}\left(\frac{3}{10}\right)$ (3) $\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ (4) $\sin^{-1}\left(\frac{1}{3}\right)$

(32) If the projection of $\vec{a}$ on $\vec{b}$ and projection of $\vec{b}$ on $\vec{a}$ are equal then the angle between $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{4}$ (4) $\frac{2\pi}{3}$

(33) If $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) \times \vec{c}$ for non-coplanar vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ then

(1) $\vec{a}$ parallel to $\vec{b}$ (2) $\vec{b}$ parallel to $\vec{c}$

(3) $\vec{c}$ parallel to $\vec{a}$ (4) $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

(34) If a line makes 45°, 60° with positive direction of axes x and y then the angle it makes with the z axis is

(1) 30° (2) 90′ (3) 45° (4) 60°

(35) If $\left[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}\right] = 64$ then $\left[\vec{a},\ \vec{b},\ \vec{c}\right]$ is

(1) 32 (2) 8 (3) 128 (4) 0

(36) If $\left[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}\right] = 8$ then $\left[\vec{a},\ \vec{b},\ \vec{c}\right]$ is

(1) 4 (2) 16 (3) 32 (4) – 4

(37) The value of $\left[\vec{i}+\vec{j},\ \vec{j}+\vec{k},\ \vec{k}+\vec{i}\right]$ is equal to

(1) 0 (2) 1 (3) 2 (4) 4

(38) The shortest distance of the point (2, 10, 1) from the plane

$\vec{r}\,.\left(3\vec{i}-\vec{j}+4\vec{k}\right) = 2\sqrt{26}$ is

(1) $2\sqrt{26}$ (2) $\sqrt{26}$ (3) 2 (4) $\frac{1}{\sqrt{26}}$

(39) The vector $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right)$ is

(1) perpendicular to $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$

(2) parallel to the vectors $\left(\vec{a} \times \vec{b}\right)$ and $\left(\vec{c} \times \vec{d}\right)$

(3) parallel to the line of intersection of the plane containing $\vec{a}$ and $\vec{b}$ and the plane containing $\vec{c}$ and $\vec{d}$

(4) perpendicular to the line of intersection of the plane containing $\vec{a}$ and $\vec{b}$ and the plane containing $\vec{c}$ and $\vec{d}$

(40) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are a right handed triad of mutually perpendicular vectors of magnitude *a, b, c* then the value of $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ is

(1) $a^2 b^2 c^2$ (2) 0 (3) $\frac{1}{2}abc$ (4) abc

(41) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar and

$\left[\vec{a} \times \vec{b},\ \vec{b} \times \vec{c},\ \vec{c} \times \vec{a}\right] = \left[\vec{a} + \vec{b},\ \vec{b} + \vec{c},\ \vec{c} + \vec{a}\right]$ then $\left[\vec{a},\ \vec{b},\ \vec{c}\right]$ is

(1) 2(2) 3 (3) 1 (4) 0

(42) $\vec{r} = s\vec{i} + t\vec{j}$ is the equation of

(1) a straight line joining the points $\vec{i}$ and $\vec{j}$

(2) *xoy* plane (3) *yoz* plane (4) *zox* plane

(43) If the magnitude of moment about the point $\vec{j} + \vec{k}$ of a force $\vec{i} + a\vec{j} - \vec{k}$ acting through the point $\vec{i} + \vec{j}$ is $\sqrt{8}$ then the value of a is

(1) 1(2) 2 (3) 3 (4) 4

(44) The equation of the line parallel to $\frac{x-3}{1} = \frac{y+3}{5} = \frac{2z-5}{3}$ and passing through the point (1, 3, 5) in vector form is

(1) $\vec{r} = \left(\vec{i} + 5\vec{j} + 3\vec{k}\right) + t\left(\vec{i} + 3\vec{j} + 5\vec{k}\right)$

(2) $\vec{r} = \vec{i} + 3\vec{j} + 5\vec{k} + t\left(\vec{i} + 5\vec{j} + 3\vec{k}\right)$

(3) $\vec{r} = \left(\vec{i} + 5\vec{j} + \frac{3}{2}\vec{k}\right) + t\left(\vec{i} + 3\vec{j} + 5\vec{k}\right)$

(4) $\vec{r} = \vec{i} + 3\vec{j} + 5\vec{k} + t\left(\vec{i} + 5\vec{j} + \frac{3}{2}\vec{k}\right)$

(45) The point of intersection of the line $\vec{r} = \left(\vec{i} - \vec{k}\right) + t\left(3\vec{i} + 2\vec{j} + 7\vec{k}\right)$ and the plane $\vec{r} \cdot \left(\vec{i} + \vec{j} - \vec{k}\right) = 8$ is

(1) (8, 6, 22) (2) (– 8, – 6, – 22) (3) (4, 3, 11) (4) (– 4, – 3, – 11)

(46) The equation of the plane passing through the point (2, 1, – 1) and the line of intersection of the planes $\vec{r} \cdot \left(\vec{i} + 3\vec{j} - \vec{k}\right) = 0$ and $\vec{r} \cdot \left(\vec{j} + 2\vec{k}\right) = 0$ is

(1) $x + 4y - z = 0$ (2) $x + 9y + 11z = 0$

(3) $2x + y - z + 5 = 0$ (4) $2x - y + z = 0$

(47) The work done by the force $\vec{F} = \vec{i} + \vec{j} + \vec{k}$ acting on a particle, if the particle is displaced from $A(3, 3, 3)$ to the point $B(4, 4, 4)$ is

(1) 2 units (2) 3 units (3) 4 units (4) 7 units

(48) If $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{b} = 3\vec{i} + \vec{j} + 2\vec{k}$ then a unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$ (2) $\frac{\vec{i} - \vec{j} + \vec{k}}{\sqrt{3}}$

(3) $\frac{-\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{3}}$ (4) $\frac{\vec{i} - \vec{j} - \vec{k}}{\sqrt{3}}$

(49) The point of intersection of the lines $\frac{x-6}{-6} = \frac{y+4}{4} = \frac{z-4}{-8}$ and $\frac{x+1}{2} = \frac{y+2}{4} = \frac{z+3}{-2}$ is

(1) (0, 0, – 4) (2) (1, 0, 0) (3) (0, 2, 0) (4) (1, 2, 0)

(50) The point of intersection of the lines

$\vec{r} = \left(-\vec{i} + 2\vec{j} + 3\vec{k}\right) + t\left(-2\vec{i} + \vec{j} + \vec{k}\right)$ and

$\vec{r} = \left(2\vec{i} + 3\vec{j} + 5\vec{k}\right) + s\left(\vec{i} + 2\vec{j} + 3\vec{k}\right)$ is

(1) (2, 1, 1) (2) (1, 2, 1) (3) (1, 1, 2) (4) (1, 1, 1)

(51) The shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ is

(1) $\frac{2}{\sqrt{3}}$ (2) $\frac{1}{\sqrt{6}}$ (3) $\frac{2}{3}$ (4) $\frac{1}{2\sqrt{6}}$

(52) The shortest distance between the parallel lines

$\frac{x-3}{4} = \frac{y-1}{2} = \frac{z-5}{-3}$ and $\frac{x-1}{4} = \frac{y-2}{2} = \frac{z-3}{3}$ is

(1) 3(2) 2 (3) 1 (4) 0

(53) The following two lines are $\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z-1}{2}$

(1) parallel (2) intersecting (3) skew (4) perpendicular

(54) The centre and radius of the sphere given by

$x^2 + y^2 + z^2 - 6x + 8y - 10z + 1 = 0$ is

(1) (– 3, 4, – 5), 49(2) (– 6, 8, – 10), 1

(3) (3, – 4, 5), 7 (4) (6, – 8, 10), 7

(55) The value of $\left[\frac{-1 + i\sqrt{3}}{2}\right]^{100} + \left[\frac{-1 - i\sqrt{3}}{2}\right]^{100}$ is

(1) 2(2) 0 (3) – 1 (4) 1

(56) The modulus and amplititude of the complex number $\left[e^{3 - i\pi/4}\right]^3$ are respectively

(1) $e^9, \frac{\pi}{2}$ (2) $e^9, \frac{-\pi}{2}$ (3) $e^6, \frac{-3\pi}{4}$ (4) $e^9, \frac{-3\pi}{4}$

(57) If $(m - 5) + i(n + 4)$ is the complex conjugate of $(2m + 3) + i(3n - 2)$ then (n, m) are

(1) $\left(-\frac{1}{2} - 8\right)$ (2) $\left(-\frac{1}{2}, 8\right)$ (3) $\left(\frac{1}{2}, -8\right)$ (4) $\left(\frac{1}{2}, 8\right)$

(58) If $x^2 + y^2 = 1$ then the value of $\frac{1 + x + iy}{1 + x - iy}$ is

(1) $x - iy$ (2) $2x$ (3) $-2iy$ (4) $x + iy$

(59) The modulus of the complex number $2 + i\sqrt{3}$ is

(1) $\sqrt{3}$ (2) $\sqrt{13}$ (3) $\sqrt{7}$ (4) 7

(60) If $A + i$B $= (a_1 + ib_1)(a_2 + ib_2)(a_3 + ib_3)$ then $A^2 + B^2$ is

(1) $a_1^{\,2} + b_1^{\,2} + a_2^{\,2} + b_2^{\,2} + a_3^{\,2} + b_3^{\,2}$

(2) $(a_1 + a_2 + a_3)^2 + (b_1 + b_2 + b_3)^2$

(3) $(a_1^{\,2} + b_1^{\,2})(a_2^{\,2} + b_2^{\,2})(a_3^{\,2} + b_3^{\,2})$

(4) $(a_1^{\,2} + a_2^{\,2} + a_3^{\,2})(b_1^{\,2} + b_2^{\,2} + b_3^{\,2})$

(61) If $a = 3 + i$ and $z = 2 - 3i$ then the points on the Argand diagram representing az, $3az$ and $-az$ are

(1) Vertices of a right angled triangle

(2) Vertices of an equilateral triangle

(3) Vertices of an isosceles triangle

(4) Collinear

(62) The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if

(1) $z_1 + z_4 = z_2 + z_3$ (2) $z_1 + z_3 = z_2 + z_4$

(3) $z_1 + z_2 = z_3 + z_4$ (iv) $z_1 - z_2 = z_3 - z_4$

(63) If z represents a complex number then arg (z) + arg $\left(\overline{z}\right)$ is

(1) $\pi/4$ (2) $\pi/2$ (3) 0 (4) $\pi/4$

(64) If the amplitude of a complex number is $\pi/2$ then the number is

(1) purely imaginary (2) purely real

(3) 0 (4) neither real nor imaginary

(65) If the point represented by the complex number iz is rotated about the origin through the angle $\frac{\pi}{2}$ in the counter clockwise direction then the complex number representing the new position is

(1) iz (2) $-iz$ (3) $-z$ (4) z

(66) The polar form of the complex number $(i^{25})^3$ is

(1) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ (2) $\cos\pi + i\sin\pi$

(3) $\cos\pi - i\sin\pi$ (4) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

(67) If P represents the variable complex number z and if $|2z - 1| = 2|z|$ then the locus of P is

(1) the straight line $x = \frac{1}{4}$ (2) the straight line $y = \frac{1}{4}$

(3) the straight line $z = \frac{1}{2}$ (4) the circle $x^2 + y^2 - 4x - 1 = 0$

(68) $\frac{1 + e^{-i\theta}}{1 + e^{i\theta}} =$

(1) $\cos\theta + i\sin\theta$ (2) $\cos\theta - i\sin\theta$

(3) $\sin\theta - i\cos\theta$ (4) $\sin\theta + i\cos\theta$

(69) If $z_n = \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}$ then $z_1 z_2 \ldots z_6$ is

(1) 1 (2) -1 (3) i (4) $-i$

(70) If $-\overline{z}$ lies in the third quadrant then z lies in the

(1) first quadrant (2) second quadrant

(3) third quadrant (4) fourth quadrant

(71) If $x = \cos\theta + i\sin\theta$ the value of $x^n + \frac{1}{x^n}$ is

(1) $2\cos n\theta$ (2) $2i\sin n\theta$ (3) $2\sin n\theta$ (4) $2i\cos n\theta$

(72) If $a = \cos\alpha - i\sin\alpha$, $b = \cos\beta - i\sin\beta$

$c = \cos\gamma - i\sin\gamma$ then $(a^2c^2 - b^2)/abc$ is

(1) $\cos 2(\alpha - \beta + \gamma) + i\sin 2(\alpha - \beta + \gamma)$

(2) $-2\cos(\alpha - \beta + \gamma)$

(3) $-2i\sin(\alpha - \beta + \gamma)$

(4) $2\cos(\alpha - \beta + \gamma)$

(73) $z_1 = 4 + 5i$, $z_2 = -3 + 2i$ then $\frac{z_1}{z_2}$ is

(1) $\frac{2}{13} - \frac{22}{13}i$ (2) $-\frac{2}{13} + \frac{22}{13}i$

(3) $\frac{-2}{13} - \frac{23}{13}i$ (4) $\frac{2}{13} + \frac{22}{13}i$

(74) The value of $i + i^{22} + i^{23} + i^{24} + i^{25}$ is

(1) i (2) $-i$ (3) 1 (4) -1

(75) The conjugate of $i^{13} + i^{14} + i^{15} + i^{16}$ is

(1) 1(2) -1 (3) 0 (4) $-i$

(76) If $-i+2$ is one root of the equation $ax^2 - bx + c = 0$, then the other root is

(1) $-i-2$ (2) $i-2$ (3) $2+i$ (4) $2i+i$

(77) The quadratic equation whose roots are $\pm i\sqrt{7}$ is

(1) $x^2 + 7 = 0$ (2) $x^2 - 7 = 0$

(3) $x^2 + x + 7 = 0$ (4) $x^2 - x - 7 = 0$

(78) The equation having $4 - 3i$ and $4 + 3i$ as roots is

(1) $x^2 + 8x + 25 = 0$ (2) $x^2 + 8x - 25 = 0$

(3) $x^2 - 8x + 25 = 0$ (4) $x^2 - 8x - 25 = 0$

(79) If $\frac{1-i}{1+i}$ is a root of the equation $ax^2 + bx + 1 = 0$, where a, b are real then (a, b) is

(1) (1, 1) (2) (1, – 1) (3) (0, 1) (4) (1, 0)

(80) If $-i+3$ is a root of $x^2 - 6x + k = 0$ then the value of k is

(1) 5 (2) $\sqrt{5}$ (3) $\sqrt{10}$ (4) 10

(81) If ω is a cube root of unity then the value of

$(1-\omega+\omega^2)^4 + (1+\omega-\omega^2)^4$ is

(1) 0 (2) 32 (3) – 16 (4) – 32

(82) If ω is the nth root of unity then

(1) $1 + \omega^2 + \omega^4 + \ldots = \omega + \omega^3 + \omega^5 + \ldots$

(2) $\omega^n = 0$ (3) $\omega^n = 1$ (4) $\omega = \omega^{n-1}$

(83) If ω is the cube root of unity then the value of

$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$ is

(1) 9 (2) – 9 (3) 16 (4) 32

(84) The axis of the parabola $y^2 - 2y + 8x - 23 = 0$ is

(1) $y = -1$ (2) $x = -3$ (3) $x = 3$ (4) $y = 1$

(85) $16x^2 - 3y^2 - 32x - 12y - 44 = 0$ represents

(1) an ellipse (2) a circle (3) a parabola (4) a hyperbola

(86) The line $4x + 2y = c$ is a tangent to the parabola $y^2 = 16x$ then c is

(1) – 1 (2) – 2 (3) 4 (4) – 4

(87) The point of intersection of the tangents at $t_1 = t$ and $t_2 = 3t$ to the parabola $y^2 = 8x$ is

(1) $(6t^2, 8t)$ (2) $(8t, 6t^2)$ (3) $(t^2, 4t)$ (4) $(4t, t^2)$

(88) The length of the latus rectum of the parabola $y^2 - 4x + 4y + 8 = 0$ is

(1) 8 (2) 6 (3) 4 (4) 2

(89) The diretrix of the parabola $y^2 = x + 4$ is

(1) $x = \frac{15}{4}$ (2) $x = -\frac{15}{4}$ (3) $x = -\frac{17}{4}$ (4) $x = \frac{17}{4}$

(90) The length of the latus rectum of the parabola whose vertex is (2, – 3) and the directrix $x = 4$ is

(1) 2 (2) 4 (3) 6 (4) 8

(91) The focus of the parabola $x^2 = 16y$ is

(1) (4, 0) (2) (0, 4) (3) (– 4, 0) (4) (0, – 4)

(92) The vertex of the parabola $x^2 = 8y - 1$ is

(1) $\left(-\frac{1}{8}, 0\right)$ (2) $\left(\frac{1}{8}, 0\right)$ (3) $\left(0, \frac{1}{8}\right)$ (4) $\left(0, -\frac{1}{8}\right)$

(93) The line $2x + 3y + 9 = 0$ touches the parabola $y^2 = 8x$ at the point

(1) (0, – 3) (2) (2, 4) (3) $\left(-6, \frac{9}{2}\right)$ (4) $\left(\frac{9}{2}, -6\right)$

(94) The tangents at the end of any focal chord to the parabola $y^2 = 12x$ intersect on the line

(1) $x - 3 = 0$ (2) $x + 3 = 0$ (3) $y + 3 = 0$ (4) $y - 3 = 0$

(95) The angle between the two tangents drawn from the point (– 4, 4) to $y^2 = 16x$ is

(1) 45° (2) 30° (3) 60° (4) 90°

(96) The eccentricity of the conic $9x^2 + 5y^2 - 54x - 40y + 116 = 0$ is

(1) $\frac{1}{3}$ (2) $\frac{2}{3}$ (3) $\frac{4}{9}$ (4) $\frac{2}{\sqrt{5}}$

(97) The length of the semi-major and the length of semi minor axis of the ellipse $\frac{x^2}{144} + \frac{y^2}{169} = 1$ are

(1) 26, 12 (2) 13, 24 (3) 12, 26 (4) 13, 12

(98) The distance between the foci of the ellipse $9x^2 + 5y^2 = 180$ is

(1) 4 (2) 6 (3) 8 (4) 2

(99) If the length of major and semi-minor axes of an ellipse are 8, 2 and their corresponding equations are $y - 6 = 0$ and $x + 4 = 0$ then the equations of the ellipse is

(1) $\frac{(x+4)^2}{4} + \frac{(y-6)^2}{16} = 1$ (2) $\frac{(x+4)^2}{16} + \frac{(y-6)^2}{4} = 1$

(3) $\frac{(x+4)^2}{16} - \frac{(y-6)^2}{4} = 1$ (4) $\frac{(x+4)^2}{4} - \frac{(y-6)^2}{16} = 1$

(100) The straight line $2x - y + c = 0$ is a tangent to the ellipse $4x^2 + 8y^2 = 32$ if c is

(1) $\pm 2\sqrt{3}$ (2) ± 6 (3) 36 (4) ± 4

(101) The sum of the distance of any point on the ellipse $4x^2 + 9y^2 = 36$ from $(\sqrt{5},\ 0)$and $(-\sqrt{5},\ 0)$is

(1) 4 (2) 8 (3) 6 (4) 18

(102) The radius of the director circle of the conic $9x^2 + 16y^2 = 144$ is

(1) $\sqrt{7}$ (2) 4 (3) 3 (4) 5

(103) The locus of foot of perpendicular from the focus to a tangent of the curve $16x^2 + 25y^2 = 400$ is

(1) $x^2 + y^2 = 4$ (2) $x^2 + y^2 = 25$ (3) $x^2 + y^2 = 16$ (4) $x^2 + y^2 = 9$

(104) The eccentricity of the hyperbola $12y^2 - 4x^2 - 24x + 48y - 127 = 0$ is

(1) 4 (2) 3 (3) 2 (4) 6

(105) The eccentricity of the hyperbola whose latus rectum is equal to half of its conjugate axis is

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{5}{3}$ (3) $\frac{3}{2}$ (4) $\frac{\sqrt{5}}{2}$

(106) The difference between the focal distances of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 24 and the eccentricity is 2. Then the equation of the hyperbola is

(1) $\frac{x^2}{144} - \frac{y^2}{432} = 1$ (2) $\frac{x^2}{432} - \frac{y^2}{144} = 1$

(3) $\frac{x^2}{12} - \frac{y^2}{12\sqrt{3}} = 1$ (4) $\frac{x^2}{12\sqrt{3}} - \frac{y^2}{12} = 1$

(107) The directrices of the hyperbola $x^2 - 4(y-3)^2 = 16$ are

(1) $y = \pm \frac{8}{\sqrt{5}}$ (2) $x = \pm \frac{8}{\sqrt{5}}$ (3) $y = \pm \frac{\sqrt{5}}{8}$ (4) $x = \pm \frac{\sqrt{5}}{8}$

(108) The line $5x - 2y + 4k = 0$ is a tangent to $4x^2 - y^2 = 36$ then k is

(1) $\frac{4}{9}$ (2) $\frac{2}{3}$ (3) $\frac{9}{4}$ (4) $\frac{81}{16}$

(109) The equation of the chord of contact of tangents from (2, 1) to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is

(1) $9x - 8y - 72 = 0$ (2) $9x + 8y + 72 = 0$
(3) $8x - 9y - 72 = 0$ (4) $8x + 9y + 72 = 0$

(110) The angle between the asymptotes to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is

(1) $\pi - 2\tan^{-1}\left(\frac{3}{4}\right)$ (2) $\pi - 2\tan^{-1}\left(\frac{4}{3}\right)$

(3) $2\tan^{-1}\frac{3}{4}$ (4) $2\tan^{-1}\left(\frac{4}{3}\right)$

(111) The asymptotes of the hyperbola $36y^2 - 25x^2 + 900 = 0$ are

(1) $y = \pm\frac{6}{5}x$ (2) $y = \pm\frac{5}{6}x$ (3) $y = \pm\frac{36}{25}x$ (4) $y = \pm\frac{25}{36}x$

(112) The product of the perpendiculars drawn from the point (8, 0) on the hyperbola to its asymptotes is $\frac{x^2}{64} - \frac{y^2}{36} = 1$ is

(1) $\frac{25}{576}$ (2) $\frac{576}{25}$ (3) $\frac{6}{25}$ (4) $\frac{25}{6}$

(113) The locus of the point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is

(1) $x^2 + y^2 = 25$ (2) $x^2 + y^2 = 4$ (3) $x^2 + y^2 = 3$ (4) $x^2 + y^2 = 7$

(114) The eccentricity of the hyperbola with asymptotes $x + 2y - 5 = 0$, $2x - y + 5 = 0$ is

(1) 3 (2) $\sqrt{2}$ (3) $\sqrt{3}$ (4) 2

(115) Length of the semi-transverse axis of the rectangular hyperbola $xy = 8$ is

(1) 2 (2) 4 (3) 16 (4) 8

(116) The asymptotes of the rectangular hyperbola $xy = c^2$ are

(1) $x = c, y = c$ (2) $x = 0, y = c$ (3) $x = c, y = 0$ (4) $x = 0, y = 0$

(117) The co-ordinate of the vertices of the rectangular hyperbola $xy = 16$ are

(1) (4, 4), (– 4, – 4) (2) (2, 8), (– 2, – 8)

(3) (4, 0), (– 4, 0) (4) (8, 0), (– 8, 0)

(118) One of the foci of the rectangular hyperbola $xy = 18$ is

(1) (6, 6) (2) (3, 3) (3) (4, 4) (4) (5, 5)

(119) The length of the latus rectum of the rectangular hyperbola $xy = 32$ is

(1) $8\sqrt{2}$ (2) 32 (3) 8 (4) 16

(120) The area of the triangle formed by the tangent at any point on the rectangular hyperbola $xy = 72$ and its asymptotes is

(1) 36 (2) 18 (3) 72 (4) 144

(121) The normal to the rectangular hyperbola $xy = 9$ at $\left(6, \frac{3}{2}\right)$ meets the curve again at

(1) $\left(\frac{3}{8}, 24\right)$ (2) $\left(-24, \frac{-3}{8}\right)$ (3) $\left(\frac{-3}{8}, -24\right)$ (4) $\left(24, \frac{3}{8}\right)$

ANSWERS

EXERCISE 1.1

(1) (i) $\begin{bmatrix} -4 & 1 \\ -2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$

(2) $\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

(4) (i) $\frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$ (ii) $\frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ (v) $\frac{1}{5}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}$ (6) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

EXERCISE 1.2

(1) $x = 3, y = -1$ (2) $x = -1, y = 2$ (3) $x = 1, y = 3, z = 5$

(4) $x = 4, y = 1, z = 0$ (5) $x = 1, y = 1, z = 1$

EXERCISE 1.3

(1) 2 (2) 1 (3) 2 (4) 3 (5) 2 (6) 2

EXERCISE 1.4

(1) (1, 1) (2) No solution (3) $\left(\frac{1}{4}(9 - 5k), k\right)$; $k \in R$

(4) (1, 1, 1) (5) $(4 - k, 3k - 4, k)$; $k \in R$ (6) (1, 2, 3)

(7) $\left(\frac{1}{3}(5k - 12), \frac{1}{3}(15 - 4k), k\right)$; $k \in R$ (8) $\left(\frac{1}{2}(2 + s - t, s, t)\right)$; s, $t \in R$

(9) (1, 2, 1) (10) $(50 + 2k, 50 - 3k, k)$; $k = 0, 1, 2, \ldots 16$

EXERCISE 1.5

(1) (i) Consistent : $x = 4, y = -1, z = 2$

(ii) Consistent :$x = 2k - 1$, $y = 3 - 2k$, $z = k$ infinitely many solutions.

(iii) Inconsistent

(iv) Inconsistent

(v) Consistent : $x = 1 - k_1 + k_2$, $y = k_1$, $z = k_2$, infinitely many solutions.

(2) If $\lambda \neq 0$ the system has a unique solution.

If $\lambda = 0$, the system has infinitely many solutions.

(3) When $k \neq 1, k \neq -2$ the system has a unique solution.
When $k = 1$, the system is consistent and has infinitely many solutions.
When $k = -2$ the system is inconsistent and has no solution.

EXERCISE 2.1

(1) 4 (2) -15 (3) $\frac{3}{2}$ (4) (i) $m = -15$ (ii) $m = \frac{2}{3}$

(5) $\left(\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}\right)$ (10) 22 (11) -25

(14) (i) 0 (ii) $\frac{-10}{\sqrt{30}}$ (iii) $\frac{9}{\sqrt{21}}$

EXERCISE 2.2

(5) 7 (6) $\frac{50}{3}$ (7) 17 (8) $\frac{124}{7}$

EXERCISE 2.3

(1) $\sqrt{6}$ (2) $3\sqrt{7}$ (3) $\pm\frac{-\vec{i} - \vec{j} + 3\vec{k}}{\sqrt{11}}$ (4) $\pm\frac{10\vec{i} - 10\vec{j} + 5\vec{k}}{3}$ (5) $\frac{\pi}{4}$ (6) $\frac{\pi}{6}$

EXERCISE 2.4

(1) $6\sqrt{59}$ (2) $\frac{49}{2}$ (3) $6\sqrt{5}$ (4) $\frac{1}{2}\sqrt{165}$

(8) $-24\vec{i} + 13\vec{j} + 4\vec{k}$ (10) $7\sqrt{10}$, $\left(\frac{3}{\sqrt{10}}, 0 \frac{-1}{\sqrt{10}}\right)$

EXERCISE 2.5

(2) -3 (11) -4

EXERCISE 2.6

(1) $\left(\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}\right)$ (2) (i) not possible (ii) yes (3) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(4) The d.c's are $\left(\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}\right)$ and $\vec{r} = 7\left(3\vec{i} + 4\vec{j} + 5\vec{k}\right)$

(5) $\pm\left(-\frac{1}{\sqrt{26}}, \frac{-4}{\sqrt{26}}, \frac{3}{\sqrt{26}}\right)$

(6) $\vec{r} = \left(3\vec{i} - 4\vec{j} - 2\vec{k}\right) + t\left(9\vec{i} + 6\vec{j} + 2\vec{k}\right)$; $\frac{x-3}{9} = \frac{y+4}{6} = \frac{z+2}{2}$

(7) $\vec{r} = \left(\vec{i} - 2\vec{j} + \vec{k}\right) + t\left(-\vec{i} + 2\vec{k}\right)$; $\frac{x-1}{-1} = \frac{y+2}{0} = \frac{z-1}{2}$

(8) $\cos^{-1}\left(\frac{20}{21}\right)$ (9) $\cos^{-1}\left(\frac{1}{\sqrt{21}}\right)$

EXERCISE 2.7

(1) (i) $\sqrt{\frac{5}{2}}$ (ii) $\sqrt{\frac{285}{14}}$ (3) (1, – 1, 0) (4) $3\sqrt{30}$ (6) – 2

EXERCISE 2.8

(1) $\frac{\vec{r}\cdot\left(2\vec{i} + 7\vec{j} + 8\vec{k}\right)}{\sqrt{117}} = 18$; $2x + 7y + 8z = 54\sqrt{13}$

(2) $\pm\frac{(2\vec{i} - \vec{j} + 2\vec{k})}{3}$ (3) 2 units (4) $8x - 4y + 3z = 89$ (5) $4x + 2y - 3z = 3$

(6) $\left[(x-2)\vec{i} + (y+1)\vec{j} + (z-4)\vec{k}\right]\cdot\left(4\vec{i} - 12\vec{j} - 3\vec{k}\right) = 0$
$4x - 12y - 3z - 8 = 0$

(7) $\vec{r}\left(2\vec{i} + 2\vec{j} + \vec{k}\right) + s\left(2\vec{i} + 3\vec{j} + 3\vec{k}\right) + t\left(3\vec{i} + 2\vec{j} + \vec{k}\right)$
$3x - 7y + 5z + 3 = 0$

(8) $\vec{r} = \left(\vec{i} + 3\vec{j} + 2\vec{k}\right) + s\left(2\vec{i} - \vec{j} + 3\vec{k}\right) + t\left(\vec{i} + 2\vec{j} + 2\vec{k}\right)$
$8x + y - 5z - 1 = 0$

(9) $\vec{r} = \left(-\vec{i} + 3\vec{j} + 2\vec{k}\right) + s\left(\vec{i} + 2\vec{j} + 2\vec{k}\right) + t\left(3\vec{i} + \vec{j} + 2\vec{k}\right)$
and $2x + 4y - 5z = 0$

(10) $\vec{r} = \left(\vec{i} - 2\vec{j} + 3\vec{k}\right) + s\left(-2\vec{i} + 4\vec{j} - 4\vec{k}\right) + t\left(2\vec{i} + 3\vec{j} + 4\vec{k}\right)$
and $2x - z + 1 = 0$

(11) $\vec{r} = \left(\vec{i} + 2\vec{j} + 3\vec{k}\right) + s\left(\vec{i} + \vec{j} - 2\vec{k}\right) + t\left(3\vec{i} - 2\vec{j} + 4\vec{k}\right)$
$2y + z - 7 = 0$

(12) $\vec{r} = \left(-\vec{i} + \vec{j} - \vec{k}\right) + s\left(3\vec{i} + \vec{j} + 2\vec{k}\right) + t\left(2\vec{i} + 3\vec{j} - 2\vec{k}\right)$
and $8x - 10y - 7z + 11 = 0$

(13) $\vec{r} = \left(3\vec{i} + 4\vec{j} + 2\vec{k}\right) + s\left(-\vec{i} + 6\vec{j} - 3\vec{k}\right) + t\left(4\vec{i} - 4\vec{j} - \vec{k}\right)$
and $6x + 13y - 28z - 14 = 0$

(15) (i) $2x - 5y - z + 15 = 0$ (ii) $2y - z - 1 = 0$

EXERCISE 2.9

(1) $11x - 10y - 13z + 70 = 0$ (2) No. Because of the lines are skew lines

(3) (2, 0, 0) (4) (6, – 1, – 5) (5) $\frac{7}{\sqrt{30}}$ (6) $\frac{3}{2\sqrt{11}}$

EXERCISE 2.10

(1) (i) $\frac{\pi}{3}$ (ii) $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$ (iii) $\cos^{-1}\left(\frac{9}{\sqrt{231}}\right)$ (3) $\frac{3}{5}$ (4) $\sin^{-1}\left(\frac{3}{2\sqrt{91}}\right)$ (5) $\frac{\pi}{3}$

EXERCISE 2.11

(1) $\left|\vec{r} - \left(2\vec{i} - \vec{j} + 3\vec{k}\right)\right| = 4$ and $x^2 + y^2 + z^2 - 4x + 2y - 6z - 2 = 0$

(2) $\left[\vec{r} - \left(2\vec{i} + 6\vec{j} - 7\vec{k}\right)\right] \cdot \left[\vec{r} - \left(-2\vec{i} + 4\vec{j} - 3\vec{k}\right)\right] = 0$ and
$x^2 + y^2 + z^2 - 10y + 10z + 41 = 0$

Centre is (0, 5, – 5) and radius is 3 units.

(3) $\left|\vec{r} - \left(\vec{i} - \vec{j} + \vec{k}\right)\right| = 5$; $x^2 + y^2 + z^2 - 2x + 2y - 2z - 22 = 0$

(4) B (4, – 2, 1)

(5) (i) Centre (2, – 1, 4) ; $r = 5$ units

(ii) Centre $\left(-\frac{3}{2}, \frac{1}{2}, -2\right)$, $r = 2$ units

(iii) Centre (– 2, 4, – 1), $r = \sqrt{26}$ units

(iv) Centre (2, 1, – 3), $r = 5$ units

EXERCISE 3.1

(1) (i) $1 + 3i$ (ii) $-i$ (iii) $-10 + 10i$ (iv) 1

(2)

	R.P.	I.P.
(i)	$\frac{1}{2}$	$\frac{-1}{2}$
(ii)	$\frac{-7}{25}$	$\frac{26}{25}$
(iii)	8	-1

(3) $n = 4$

(4) (i) $x = 2,\ y = -1$

(ii) $x = 3,\ y = -1$

(iii) $x = -7,\ y = -3$ and $x = \frac{-8}{3}\ ,\ y = \frac{4}{3}$

(5) $x = \pm 1,\ y = -4$ and $x = \pm 2i,\ y = 1$

EXERCISE 3.2

(2) $1 - 3i$ and $-1 + 3i$

(3) $\left(\frac{1}{\sqrt{2}} + i\ \frac{1}{\sqrt{2}}\right)$; $\left(-\frac{1}{\sqrt{2}} - i\ \frac{1}{\sqrt{2}}\right)$

(6) (i) $4 \text{ cis } \frac{\pi}{3}$ (ii) $2 \text{ cis } \frac{2\pi}{3}$ (iii) $\sqrt{2} \text{ cis } \left(-\frac{3\pi}{4}\right)$ (iv) $\sqrt{2} \text{ cis } \left(-\frac{\pi}{4}\right)$

(8) (i) $x + 2y = 2$ (ii) $y = 0$ (iii) $x + y + 1 = 0$

(iv) $4x^2 + 4y^2 - 12x + 5 = 0$ (v) $x^2 + y^2 + 2x - 3 = 0$

EXERCISE 3.3

(1) $3 \pm i\ ,\ 1 \pm i$ (2) $1 \pm 2i,\ 1 \pm i$ (3) $2 \pm i,\ \frac{2}{3}, \frac{-1}{2}$

EXERCISE 3.4

(1) $\text{cis } (-107\theta)$ (2) $\text{cis } (3\alpha + 4\beta)$

EXERCISE 3.5

(1) (i) $\text{cis } \frac{\pi}{6},\ \text{cis } \frac{5\pi}{6},\ \text{cis } \frac{9\pi}{6}$ (ii) $2 \text{ cis } \frac{\pi}{6},\ 2\text{cis } \frac{5\pi}{6},\ 2\text{cis } \frac{9\pi}{6}$

(iii) $2^{2/3} \text{ cis } \left(\frac{-5\pi}{9}\right),\ 2^{2/3} \text{ cis } \frac{\pi}{9},\ 2^{2/3} \text{ cis } \left(\frac{7\pi}{9}\right)$

(4) (i) $\sqrt{2}$ cis $\frac{\pi}{4}$, $\sqrt{2}$ cis $\frac{3\pi}{4}$, $\sqrt{2}$ cis $\frac{5\pi}{4}$ and $\sqrt{2}$ cis $\frac{7\pi}{4}$

(ii) cis $\frac{\pi}{5}$, cis $\frac{3\pi}{5}$, cis $\frac{7\pi}{5}$, cis $\frac{9\pi}{5}$

(5) cis $(2k-1)\frac{\pi}{4}$, k = 0, 1, 2, 3

EXERCISE 4.1

(1) (i) $4x^2 - 16x + 36y + 43 = 0$ (ii) $9x^2 - 12xy + 4y^2 + 38x - 60y + 121 = 0$

(iii) $x^2 = -16y$ (iv) $(y-4)^2 = -12(x-1)$

(v) $(x-1)^2 = 12(y-2)$ (vi) $(y-4)^2 = -12(x-1)$

(vii) $(x-3)^2 = -8(y+2)$ (viii) $(y+1)^2 = 8(x-3)$

(ix) $(x-2)^2 = 16(y-3)$

2)

Q. No.	Axis	Vertex	Focus	Equation of directrix	Equation of Latus rectum	Length of the Latus rectum
(i)	$y = 0$	(0, 0)	(– 2, 0)	$x - 2 = 0$	$x + 2 = 0$	8
(ii)	$x = 0$	(0, 0)	(0, 5)	$y + 5 = 0$	$y - 5 = 0$	20
(iii)	$x - 4 = 0$	(4, – 2)	(4, – 1)	$y + 3 = 0$	$y + 1 = 0$	4
(iv)	$y - 3 = 0$	(1, 3)	(– 1, 3)	$x - 3 = 0$	$x + 1 = 0$	8
(v)	$x - 3 = 0$	(3, – 1)	(3, 2)	$y + 4 = 0$	$y - 2 = 0$	12

(3) Distance = 5 cm.

(4) Diameter = $40\sqrt{2}$ cm (5) $20\sqrt{2}$ mts

EXERCISE 4.2

(1) (i) $16x^2 + 25y^2 - 96x + 50y - 231 = 0$

(ii) $\frac{(x-1)^2}{4} + \frac{(y+1)^2}{3} = 1$ (iii) $\frac{x^2}{25} + \frac{y^2}{16} = 1$

(iv) $\frac{(x-3)^2}{4} + \frac{(y+4)^2}{1} = 1$ (v) $\frac{x^2}{9} + \frac{y^2}{5} = 1$

(vi) $\frac{(x-2)^2}{40} + \frac{(y-5)^2}{49} = 1$ (vii) $\frac{(x-3)^2}{25} + \frac{(y+1)^2}{16} = 1$

(viii) $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (ix) $\frac{x^2}{16} + \frac{y^2}{4} = 1$

(2) (4, – 6)

(3) $\frac{x^2}{81/4} + \frac{y^2}{45/4} = 1$

(4)

No.	Equation of major axis	Equation of minor axis	Length of major axis	Length of minor axis
(i)	$y = 0$	$x = 0$	10	6
(ii)	$y - 2 = 0$	$x + 1 = 0$	6	$2\sqrt{5}$
(iii)	$x = 0$	$y = 0$	$2\sqrt{5}$	$\frac{4\sqrt{5}}{3}$
(iv)	$x + 1 = 0$	$y - 2 = 0$	8	6

(5)

No.	Equation of directrices	Equation of the latus rectums	Length of the latus rectum
(i)	$x = \pm \frac{169}{12}$	$x = \pm 12$	$\frac{50}{13}$
(ii)	$x = \pm \frac{16}{\sqrt{7}}$	$x = \pm \sqrt{7}$	$\frac{9}{2}$
(iii)	$x = 4 \pm \frac{20}{\sqrt{3}}$	$x = 4 \pm 5\sqrt{3}$	5
(iv)	$y = 10; y = -8$	$y = 4 ; y = -2$	$4\sqrt{3}$

(6)

No.	e	Centre	Foci	Vertices
(i)	$\frac{3}{5}$	(0, 0)	(± 3, 0)	(± 5, 0)
(ii)	$\frac{\sqrt{3}}{2}$	(4, 2)	$\left(4 \pm 5\sqrt{3}, 2\right)$	(14, 2) ; (– 6, 2)
(iii)	$\frac{\sqrt{5}}{3}$	(0, 0)	$\left(0, \pm \sqrt{5}\right)$	(0, ± 3)
(iv)	$\frac{\sqrt{7}}{4}$	(– 1, 2)	$\left(-1, 2 \pm \sqrt{7}\right)$	(– 1, 6), (– 1, – 2)

(7) $\frac{x^2}{16} + \frac{y^2}{7} = 1$ (8) 1200 km

(9) (i) 28.584 million miles (ii) 43.416 million miles

(10) $\frac{4}{5}\sqrt{319}$ feet

EXERCISE 4.3

(1) (i) $x^2 - 16xy - 11y^2 + 20x + 50y - 35 = 0$ (ii) $\frac{y^2}{25} - \frac{x^2}{24} = 1$

(iii) $\frac{y^2}{36} - \frac{x^2}{288} = 1$ (iv) $\frac{(x-1)^2}{16} - \frac{(y+2)^2}{9} = 1$

(v) $\frac{(y-5)^2}{75} - \frac{(x-2)^2}{25} = 1$ (vi) $\frac{y^2}{36} - \frac{x^2}{28} = 1$

(vii) $\frac{x^2}{1} - \frac{(y-5)^2}{8} = 1$ (viii) $\frac{(x-1)^2}{25/4} - \frac{(y-4)^2}{75/4} = 1$

(ix) $\frac{(x-1)^2}{9} - \frac{(y+1)^2}{16} = 1$

(2)

No.	Equation of transverse axis	Equation of Conjugate axis	Length of Transverse axis	Length of Conjugate axis
(i)	$y = 0$	$x = 0$	10	24
(ii)	$x = 0$	$y = 0$	$2\sqrt{2}$	$4\sqrt{2}$
(iii)	$y - 2 = 0$	$x + 3 = 0$	6	8

(3)

No.	Equations of Directrices	Equation of Latus rectums	Length of latus rectum
(i)	$x = \pm\frac{36}{\sqrt{13}}$	$x = \pm 4\sqrt{13}$	$\frac{32}{3}$
(ii)	$y = 4 \pm \frac{9}{\sqrt{13}}$	$y = 4 \pm \sqrt{13}$	$\frac{8}{3}$

(5)

No.	Eccentricity	Centre	Foci	Vertices
(i)	$e = \frac{\sqrt{41}}{4}$	(0, 0)	$\left(\pm\sqrt{41}, 0\right)$	$(\pm 4, 0)$
(ii)	$e = \frac{\sqrt{34}}{3}$	(0, 0)	$\left(0, \pm\sqrt{34}\right)$	$(0, \pm 3)$
(iii)	$e = \frac{\sqrt{5}}{2}$	(– 3, 2)	$\left(-3 \pm \sqrt{5}, 2\right)$	(– 1, 2), (– 5, 2)
(iv)	$e = 2$	(– 3, 1)	(– 3, 5) (– 3, – 3)	(– 3, 3) (– 3, – 1)

EXERCISE 4.4

(1) (i) $x + y + 3 = 0$; $x - y - 9 = 0$

(ii) $2x + 3y + 3 = 0$; $3x - 2y + 11 = 0$

(iii) $x - 2y + 2 = 0$; $2x + y - 1 = 0$

(iv) $x = \sqrt{3}$; $y = 0$

(v) $18x + 5y = 31$; $5x - 18y - 28 = 0$

(2) (i)$2x - y + 1 = 0$; $2x + 4y - 9 = 0$

(ii) $x + 2y - 8 = 0$; $2x - y - 6 = 0$

(iii) $4x + 5\sqrt{3}\,y = 40$; $10\sqrt{3}\,x - 8y - 9\sqrt{3} = 0$

(iv) $4\sqrt{3}\,x - 3y = 18$; $3x + 4\sqrt{3}\,y - 14\sqrt{3} = 0$

(3) (i) $3x - 2y + 2 = 0$ (ii) $x + 3y + 36 = 0$

(iii) $y = x \pm 5$ (iv) $10x - 3y \pm 32 = 0$

(4) (i) $x + 2y + 4 = 0$; $x + y + 1 = 0$

(ii) $x - 2y + 5 = 0$; $5x + 4y - 17 = 0$

(iii) $3x + y - 5 = 0$; $x - y + 1 = 0$

(5) $\left(5, \frac{-4}{3}\right)$ (6) (– 3, 1)

(7) (i) $4x - y - 12 = 0$ (ii) $x + 5y - 5 = 0$ (iii) $10x - 9y - 12 = 0$

EXERCISE 4.5

(1) (i) $\left(\frac{x}{5} - \frac{y}{6}\right) = 0$ and $\left(\frac{x}{5} + \frac{y}{6}\right) = 0$

(ii) $4x - y + 1 = 0$ and $2x + 3y - 1 = 0$

(2) (i) $(2x + 3y - 8)\ (3x - 2y + 1) = 110$

(ii) $(x + 2y - 10)\ (x - 2y + 6) + 64 = 0$

(3) (i) $\frac{2\pi}{3}$ (ii) $2\tan^{-1}\frac{3}{2}$ (iii) $2\tan^{-1}\frac{\sqrt{5}}{2}$

EXERCISE 4.6

(1) $\left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = \frac{9}{8}$

(2) (i) $4x + 3y - 24 = 0$; $3x - 4y + 7 = 0$

(ii) $x + 8y = 0$; $32x - 4y + 65 = 0$

(3) $(x + 2y - 5)\ (2x - y + 4) = 16$

(4) $(x - 1)\ (y - 3) = 16$

$x - 1 = 0$ and $y - 3 = 0$

(5) $(3x - y - 5)\ (x + 3y - 5) - 7 = 0$

(6) (i) $x - h = 0$ and $y - k = 0$

(ii) $(x + 2) = 0$; $\left(y + \frac{3}{2}\right) = 0$

(iii) $3x - 2y + 3 = 0$

$2x + 3y + 2 = 0$

KEY TO OBJECTIVE TYPE QUESTIONS

Q.No	Key	Q.No	Key	Q.No	Key	Q.No	Key	Q.No	Key
1	1	26	4	51	2	76	3	101	3
2	3	27	2	52	1	77	1	102	4
3	1	28	2	53	3	78	3	103	2
4	3	29	4	54	3	79	4	104	3
5	1	30	3	55	3	80	4	105	4
6	3	31	1	56	4	81	3	106	1
7	3	32	1	57	1	82	3	107	2
8	4	33	3	58	4	83	1	108	3
9	3	34	4	59	3	84	4	109	1
10	3	35	2	60	3	85	4	110	3
11	1	36	1	61	4	86	4	111	2
12	4	37	3	62	2	87	1	112	2
13	1	38	3	63	3	88	3	113	1
14	2	39	3	64	1	89	3	114	2
15	1	40	4	65	3	90	4	115	2
16	4	41	1	66	4	91	2	116	4
17	1	42	2	67	1	92	3	117	1
18	2	43	2	68	2	93	4	118	1
19	3	44	4	69	2	94	2	119	4
20	3	45	2	70	4	95	4	120	4
21	4	46	2	71	1	96	2	121	3
22	2	47	2	72	3	97	4		
23	3	48	4	73	3	98	3		
24	4	49	1	74	1	99	2		
25	3	50	3	75	3	100	2		