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MEASURES OF MULTIGROUP SEGREGATION

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In this paper we derive and evaluate measures of multigroup segregation. After describing four ways to conceptualize the measurement of multigroup segregation—as the disproportionality in group (e.g., race) proportions across organizational units (e.g., schools or census tracts), as the strength of association between nominal variables indexing group and organizational unit membership, as the ratio of between-unit diversity to total diversity, and as the weighted average of two-group segregation indices—we derive six multigroup segregation indices: a dissimilarity index (D), a Gini index (G), an information theory index (H), a squared coefficient of variation index (C), a relative diversity index (R), and a normalized exposure index (P). We evaluate these six indices against a set of seven desirable properties of segregation indices. We conclude that the information theory index H is the most conceptually and mathematically satisfactory index, since it alone obeys the principle of transfers in the multigroup case. Moreover, H is the only multigroup index that can be decomposed into a sum of between- and within-group components.

This paper was supported by a Spencer Foundation Small Grant to Reardon and NSF grants (NSF SBR-9515153 and SBR-9870870) to Firebaugh. All errors remain the responsibility of the authors. Address all correspondence to Sean F. Reardon, 310F Rackley Building, Pennsylvania State University, University Park, PA 16802 (sean@pop.psu.edu).

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1. INTRODUCTION

The measurement of segregation has a venerable history in U.S. sociology, dating back to the late 1940s and early 1950s, when a variety of segregation indices were proposed and discussed in a series of articles in the *American Sociological Review* (Bell 1954; Cowgill and Cowgill 1951; Duncan and Duncan 1955; Jahn 1950; Jahn, Schmid, and Schrag 1947; Williams 1948). With few exceptions (Theil 1972; Theil and Finezza 1971), however, the major methodological developments in segregation measurement have been limited to measuring segregation between two population groups—between blacks and whites, for example, or men and women (James and Taeuber 1985; Massey and Denton 1988).

As U.S. society becomes more racially diverse, two-group measures of segregation will become increasingly inadequate for describing complex patterns of racial segregation and integration. In this paper we take up the challenge of extending dichotomous indices of segregation to the multigroup case. First, however, we present a brief summary of prior work on segregation measurement.

1.1. *A Brief History of Segregation Measurement*

The first systematic analysis and critique of segregation indices was Duncan and Duncan's seminal 1955 article in the *American Sociological Review*. The problem, as the Duncans saw it, was that segregation measures were often constructed from "naive" notions of segregation rather than being derived from clearly articulated conceptualizations of segregation and its processes:

> [Segregation] is a concept rich in theoretical suggestiveness and of unquestionable heuristic value. Clearly we would not wish to sacrifice the capital of theorization and observation already invested in the concept. Yet this is what is involved in the solution offered by naive operationalism, in more or less arbitrarily matching some convenient numerical procedure with the verbal concept of segregation . . . (1955, p. 217).

Despite the Duncans' call for careful theoretical grounding of segregation measurement, no comprehensive approach to segregation measurement was put forth until the 1980s.[1]

The 1980s saw two important developments in the theory of segregation measurement. First, James and Taeuber (1985)—drawing on Schwartz and Winship's (1980) work on inequality measurement—developed a set of criteria against which segregation measures could be evaluated and used these criteria to demonstrate that indices that are highly correlated in empirical studies may nonetheless behave very differently under certain circumstances, such as when the population shares of groups change. Second, Massey and Denton (1988) used factor analysis to classify segregation indices into five distinct dimensions, which they named *evenness*, *exposure*, *concentration*, *centralization*, and *clustering*. Measures of evenness (e.g., Gini and Dissimilarity) and exposure (P^*) have been most commonly used to measure segregation. These measures describe the distribution of groups among organizational units (e.g., schools, census tracts) without regard for their spatial proximity. The spatial dimensions of segregation are measured by indices of concentration, centralization, and clustering.

The importance of the work of James and Taeuber and of Massey and Denton lies in the clarity they bring to the operationalization of the concept of "segregation." However, neither article discusses measures of multigroup segregation. While several authors have suggested multigroup measures, these indices have been rarely used or discussed. Consequently, the development of multigroup segregation measures lacks the conceptual sophistication of dichotomous measures.

[1]Prior to the mid-1980s, sociologists relied on a variety of indices, most commonly the dissimilarity index (popularized by Taeuber and Taeuber 1965), the P^* exposure indices (Bell 1954; Farley 1984; Farley 1977; Lieberson and Carter 1982a; Lieberson and Carter 1982b), and the variance ratio index (Bell 1954; Coleman, Hoffer, and Kilgore 1982; Duncan and Duncan 1955; Zoloth 1976) to measure residential and educational segregation. A lively debate in the 1970s and 1980s centered on the merits and flaws of these and other segregation indices (Cohen, Falk, and Cortese 1976; Coleman, Hoffer and Kilgore 1982; Cortese, Falk, and Cohen 1976; Falk, Cortese, and Cohen 1978; James and Taeuber 1985; Kestenbaum 1980; Lieberson and Carter 1982b; Morgan 1983; Taeuber and Taeuber 1976; Winship 1977; Winship 1978).

1.2. *Our Objectives*

This article derives and evaluates a set of six multigroup segregation indices. To this point, multigroup segregation indices have been constructed in an ad hoc fashion, without a general set of principles to guide their development.[2] Moreover, there has been no systematic effort to evaluate the properties of these multigroup indices. This article provides such an evaluation.

2. MEASURES OF MULTIGROUP SEGREGATION

2.1. *Notation*

Throughout this paper, we use the following notation: t denotes size and π denotes proportion; subscripts i and j index organizational unit (e.g., school, census tract); and subscripts m and n index group (e.g., racial group). Hence:

$$t_j = \text{number of cases (individuals) in organizational unit } j$$

$$T = \text{total number of cases (note that } \Sigma_j(t_j/T) = 1)$$

$$\pi_m = \text{proportion in group } m \text{ (e.g., proportion black)}$$

$$\pi_{jm} = \text{proportion in group } m\text{, of those in unit } j \text{ (e.g., proportion black in school } j)$$

Because segregation measures are functions of the group proportions (the π_m's and the π_{jm}'s), two measures of the variation in group membership figure prominently in the measurement of segregation—

[2] The first measure to be used as a multigroup segregation index, H—the information theory index, also called the entropy index—was defined by Theil (1972; Theil and Finezza 1971). Derived from a branch of mathematics known as information theory, H has an undeserved reputation for being hard to interpret, and has been rarely used in empirical research (for examples, see Kulis 1997; Miller and Quigley 1990; Reardon and Yun 2001; Reardon, Yun, and Eitle 2000; White 1986). A number of other multigroup indices have been proposed but even less often used. Morgan (1975) and Sakoda (1981) have provided generalizations of the dissimilarity index. Reardon (1998) defined a multigroup generalization of the Gini index. James (1986) defined an exposure-based multigroup generalization of V, the variance ratio index. Carlson (1992) used Goodman and Kruskal's τ_b (Goodman and Kruskal 1954) to measure race/sex occupational segregation; τ_b reduces to V in the two-group case, and so can be seen as a multigroup generalization of V.

Simpson's Interaction Index, denoted I (Lieberson 1969; White 1986) and Theil's Entropy Index, denoted E (Theil 1972):

$$I = \sum_{m=1}^{M} \pi_m (1 - \pi_m) \tag{1}$$

$$E = \sum_{m=1}^{M} \pi_m \ln\left(\frac{1}{\pi_m}\right), \tag{2}$$

where ln is the natural logarithm (here and throughout the article).[3] Both I and E can be seen as measures of the "diversity" of a population since both are equal to zero if and only if all individuals are members of a single group ("no diversity") and both are maximized if and only if individuals are evenly distributed among the M groups ($\pi_m = 1/M$ for all m).

2.2. *Criteria for Evaluating Measures of Multigroup Segregation*

First we state the four James and Taeuber (1985) criteria—organizational equivalence, size invariance, the principle of transfers, and composition invariance—which specify how segregation indices should respond to changes in the distribution of groups among organizational units. Because two-group indices respond the same way to transfers (*one-way* transfers from unit i to unit j) and exchanges (*two-way* transfers between units i and j), James and Taeuber (1985) conflate the two under the rubric "transfers." However, multigroup segregation indices can respond differently to exchanges than to transfers, so we add a fifth criterion, the principle of exchanges.[4]

1. *Organizational equivalence*: If an organizational unit is divided into k units, each with the same group proportions as the original unit, segregation remains unchanged. Likewise, if k organizational units

[3] Note that we define $0 \cdot \ln(1/0) = \lim_{\pi \to 0} (\pi \cdot \ln(1/\pi)) = 0$. Note also that E can be defined using logarithms to any base; we use the natural logarithm for simplicity throughout this article.

[4] In addition to the criteria described here, a multigroup segregation index should (1) be bounded between 0 (no segregation) and 1 (complete segregation); (2) be a continuous function of the within-unit group proportions and the unit sizes; and (3) allow the calculation of segregation among M groups, where $M \geq 2$. The multigroup indices we derive below meet all three criteria.

with identical group proportions are combined into a single unit, segregation is unchanged.

2. *Size invariance*: If the number of persons of each group m in each organizational unit j is multiplied by a constant factor p, segregation is unchanged.
3. *Transfers*: If an individual of group m is moved from organizational unit i to unit j, where the proportion of persons of group m is greater in unit i than in j ($\pi_{im} > \pi_{jm}$), then segregation is reduced.
4. *Exchanges*: If an individual of group m in organizational unit i is exchanged with an individual of group n from organizational unit j, where the proportion of persons of group m is greater in unit i than in j ($\pi_{im} > \pi_{jm}$) and the proportion of persons of group n is greater in unit j than in i ($\pi_{jn} > \pi_{in}$), segregation is reduced.
5. *Composition invariance*: In James and Taeuber's formulation, the composition invariance criterion states that if the number of persons of group m in each unit increases by a constant factor p and the number and distribution of persons of all other groups is unchanged, segregation is unchanged. This definition corresponds to the "margin-free" criterion discussed in the sex segregation literature (e.g., Charles and Grusky 1995; Grusky and Charles 1998; Watts 1992; Watts 1994). Not all scholars of segregation agree, however, that composition invariance is a desirable property for segregation measures (e.g., see Coleman, Hoffer, and Kilgore 1982).

In addition to the James and Taeuber criteria, we define two decomposability properties that are desirable for segregation indices—organizational and group decomposability.

6. *Additive organizational decomposability*: If J organizational units are clustered in K clusters, then a segregation measure should be decomposable into a sum of independent within- and between-cluster components.
7. *Additive Group Decomposability*: If M groups are clustered in N supergroups, then a segregation measure should be decomposable into a sum of independent within- and between-supergroup components.

Although not necessary for defining a meaningful segregation measure, organizational and group decomposability are useful properties for many analyses, since organizational units (schools, census tracts) and/or groups (ethnic groups) are often clustered.

2.3. *Four Approaches to Deriving Multigroup Segregation Measures*

Multigroup segregation indices can be derived from different starting points, reflecting alternative ways to think about segregation. First, segregation can be thought of as a function of the disproportionality in group proportions across organizational units. This conceptualization links the measurement of segregation to the measurement of inequality, since inequality can also be conceptualized as a function of the disproportionality in the distribution of some good (Firebaugh 1998; Firebaugh 1999). Second, segregation can be thought of as association between groups and organizational units, so segregation indices can be related to association measures such as χ^2 and G^2. Third, segregation can be conceptualized as variation in the diversity of units (e.g., variation in the racial diversity of schools). This suggests a variance decomposition approach to the measurement of segregation, in which we first define a measure of the total diversity of a population and then define segregation as the share of this total diversity accounted for by the between-unit (e.g., school-to-school) differences in group proportions. A fourth approach is to construct a multigroup segregation measure as a weighted average of dichotomous segregation indices.

2.3.1. *Segregation as Disproportionality in Group Proportions*

We start with segregation as disproportionality. The disproportionality approach begins with the observation that segregation relates to the ratio of π_{jm} to π_m:

Disproportionality Axiom for Segregation. Segregation is zero when and only when $\pi_{jm}/\pi_m = 1$ for all j and m; otherwise, segregation is greater than zero.

The ratio $r_{jm} = \pi_{jm}/\pi_m$ reflects the extent to which group m is disproportionately represented in unit j ($r_{jm} < 1.0$ indicates group m is underrepresented; $r_{jm} > 1.0$ indicates group m is overrepresented). We can define segregation as the average disproportionality across all units and groups—that is, the average of the deviations of the r_{jm} from 1.0, where "deviation" is measured according to some function f such that $f(1) = 0$. In particular, we can define the weighted average disproportionality, W, as the average value of $f(r_{jm})$ across all units and groups, weighted by unit size and group proportions:

$$W = \sum_{m=1}^{M} \pi_m \sum_{j=1}^{J} \frac{t_j}{T} f(r_{jm}). \tag{3}$$

While there are many possible disproportionality functions *f*, an appropriate function *f* will have the property that *W* obtains its minimum value of zero if and only if $r_{jm} = 1.0$ for all *j* and *m* (if $\pi_{jm} = \pi_m$ for all *j* and *m*). The maximum value of *W* will depend on the specific function *f*. Therefore, to define an index of segregation that is bounded between 0 and 1, we divide *W* by its maximum possible value (the value in the case of complete segregation) to set the upper bound to 1.0:

$$Segregation = \frac{W}{\max(W)} = \frac{W}{W^*}, \tag{4}$$

where W^* is the value of *W* obtained in the case of complete segregation. Equations (3) and (4) define a general class of segregation indices, each member of which is derived from a different disproportionality function *f*.

Firebaugh (1998, 1999) has shown that many inequality indices can be thought of as measures of the average deviation of X_j from μ (the mean of the X_j), where deviation is defined by some disproportionality function *f*. Different inequality indices give different results because they employ different disproportionality functions—the function used to measure distance from 1.0. Columns 1 and 2 of Table 1 list the disproportionality functions for four standard inequality indices. We can derive four segregation indices from these four disproportionality functions (Table 1).[5]

[5] Our approach here to deriving segregation indices from the disproportionality functions used in inequality indices relies on calculating the disproportionality of within-unit group proportions (π_{jm}) relative to overall group proportions (π_m). Earlier discussions define segregation indices in terms of proportions, rather than ratios, or define segregation in terms of the disproportionality of within-unit black/white or male/female ratios, relative to overall ratios (e.g., see Deutsch, Flückiger, and Silber 1994; James and Taeuber 1985). James and Taeuber (1985) note that some black/white racial segregation indices (e.g., the dissimilarity, Gini, and Atkinson indices) can be derived from inequality measures simply by calculating the inequality, across all whites, in the black/white ratios of their schools or tracts. However, this approach has several limitations. First, formulations based on black/white ratios will sometimes encounter the problem of division by zero (James and Taeuber 1985, footnote 7). The r_{jm}-ratio approach avoids this problem since the pertinent ratio is π_{jm}/π_m, and by definition $\pi_m > 0$ for groups in the data. Second, James and Taeuber do not generalize their approach to the case of multiple groups, since there is no clear multigroup analog to the black/white ratio they employ. Finally, while the James and Taeuber approach gives meaningful two-group results for *D*, *G*, and the Atkinson index, our disproportionality approach yields not only multigroup versions of *D* and *G*, but also yields *H* and a multigroup version of *V*.

TABLE 1

Derivation of Disproportionality-Based Multigroup Segregation Indices from Disproportionality Functions

Disproportionality Function $f(r_{jm})$	Corresponding Inequality Index	Maximum[a] of $\sum_{m=1}^{M} \pi_m \sum_{j=1}^{J} \frac{t_j}{T} f(r_{jm})$	Multigroup Segregation Index	Relationship to Association Measure
$\lvert r-1 \rvert /2$	Mean relative deviation	I[b]	$D = \frac{1}{2I} \sum_{m=1}^{M} \pi_m \sum_{j=1}^{J} \frac{t_j}{T} \lvert r_{jm} - 1 \rvert$	n/a
$\lvert r_i - r_j \rvert /2$	Gini coefficient	I[b]	$G = \frac{1}{2I} \sum_{m=1}^{M} \pi_m \sum_{i=1}^{J} \sum_{j=1}^{J} \frac{t_i t_j}{T^2} \lvert r_{im} - r_{jm} \rvert$	n/a
$r \ln r$	Theil	E[c]	$H = \frac{1}{E} \sum_{m=1}^{M} \pi_m \sum_{j=1}^{J} \frac{t_j}{T} r_{jm} \ln r_{jm}$	$H = \frac{G^2}{2TE}$
$(r-1)^2$	Squared coefficient of variation	$(M-1)$	$C = \frac{1}{M-1} \sum_{m=1}^{M} \pi_m \sum_{j=1}^{J} \frac{t_j}{T} (r_{jm} - 1)^2$	$C = \frac{\chi^2}{T(M-1)}$

[a]See Appendix A for proofs.

[b]I denotes the Simpson Interaction Index (Equation [1]).

[c]E denotes Theil's Entropy index (Equation [2]).

Table 1 shows the derivation of four multigroup segregation indices from four disproportionality functions. The mean relative deviation disproportionality function leads to a multigroup measure we call the generalized dissimilarity index (D), since it reduces to the familiar binary dissimilarity index when $M = 2$. This generalized D is equivalent to the generalizations of the dissimilarity index given by others (Morgan 1975; Reardon 1998; Sakoda 1981) and can be interpreted as the percentage of all individuals who would have to transfer among units to equalize the group proportions across units, divided by the percentage who would have to transfer if the system started in a state of complete segregation.

Likewise, the Gini disproportionality function produces a multigroup segregation measure we call the generalized Gini index (G), also defined by Reardon (1998), that reduces to the familiar Gini index of segregation in the two-group case, and can be interpreted as the weighted sum, over M groups, of the weighted average absolute difference in group proportions between all possible pairs of units, divided by the maximum possible value of this sum, obtained if the system were in a state of complete segregation.

The Theil disproportionality function yields the Theil information theory index (H), though Theil derived H using a different approach (Theil 1972; Theil and Finezza 1971). As we shall see below, H can be interpreted as a normalized likelihood-ratio measure of association between two nominal variables indexing group and unit memberships, respectively. It can also be interpreted as one minus the ratio of the average within-unit population diversity to the diversity of the total population.

Finally, the squared coefficient of variation function gives a multigroup segregation measure that we denote by C. This index has not been previously defined in the multigroup case, although it does reduce to the familiar V in the two-group case. It can be interpreted as a measure of the variance of the r_{jm}'s or, as we will see below, as a normalized chi-squared measure of association between groups and units.

2.3.2. *Segregation Indices as Measures of Association*

An alternative approach to deriving segregation indices comes from noting that segregation can be thought of as a measure of association for two-way contingency tables (White 1986, p. 208). Conceptualize the cross-classification of a nominal variable indexing M groups (e.g., racial groups) with a nominal variable indexing J organizational units (e.g., schools). Then define a class of association measures A as

$$A = \sum_{m=1}^{M} \sum_{j=1}^{J} E(t_{jm}) \cdot f\left(\frac{t_{jm}}{E(t_{jm})}\right), \tag{5}$$

where t_{jm} denotes the number of cases in the jm^{th} cell, and $E(t_{jm})$ is the expected count in the jm^{th} cell under the assumption of no association (the expected number of members of group m in unit j in the case of no association or, equivalently, zero segregation). Equation (5) defines a general class of association measures A, where A is the average deviation of the observed cell counts from the expected cell counts, weighted by the expected cell counts, where the deviation is defined by some distance function f. Hence the greater the weighted-average distance of the t_{jm} from the $E(t_{jm})$, the greater is the value of A.

Note that equation (5) subsumes χ^2 and G^2, from which most measures of association are derived. If $f(r)$ measures the squared distance from r to 1.0—that is, if $f(r) = (r-1)^2$—we get $A = \chi^2$, the chi-squared statistic. If $f(r) = r\ln r$, as in the Theil index, we get $A = G^2/2$, or one-half the likelihood ratio chi-squared statistic. Note also that $E(t_{jm}) = \pi_m t_j$ and that $t_{j\,m}/E(t_{jm}) = r_{jm}$. Hence

$$A = \sum_{m=1}^{M} \sum_{j=1}^{J} \pi_m t_j f(r_{jm}) = TW. \tag{6}$$

If we divide an association measure A by its maximum possible value (obtained in the case of complete segregation), we get a class of segregation measures bounded between 0 (in the case of no segregation) and 1 (in the case of complete segregation). Since $max(A) = max(TW) = Tmax(W) = TW^*$, we get:

$$\frac{A}{\max(A)} = \frac{TW}{TW^*} = \frac{W}{W^*}. \tag{7}$$

The association approach to deriving segregation indices, then, yields exactly the same class of segregation indices as does the disproportionality approach. This relationship between the W- and A-class measures makes explicit two potentially useful facts. First, it demonstrates that conventional measures of association (χ^2 and G^2) are functions of the weighted-average distance of the r_{jm} from 1.0, where $r_{jm} = t_{j\,m}/E(t_{j\,m})$. Measures of association, like measures of inequality and segregation, can be based on a general class of disproportionality functions. Second, in the case where

j indexes organizational unit and m indexes group, measures of association (once normalized) are also measures of segregation (see also White 1986; Zoloth 1976).[6] Since considerably more is known about the statistical properties of association measures than of segregation measures, this connection may be useful in applying inferential statistics to segregation measures.

2.3.3. *Segregation Measures as Diversity Ratios*

Our third approach to constructing multigroup segregation measures relies on the calculation and decomposition of diversity. Our approach here is similar in some ways to variance decomposition methods, except that we substitute the idea of "diversity" for variance. In this approach, we define a measure of the "diversity" of a population, and then define segregation as the share of this diversity accounted for by differences in group proportions across units.

We begin by defining more precisely the notion of "diversity." In a population where each individual is a member of exactly one of M mutually exclusive and unordered groups, the diversity $\boldsymbol{d}$ should be defined as a function of the π_m's, the population shares of each of the groups ($\boldsymbol{d} = d(\pi_1, \pi_2, \ldots, \pi_m)$). Moreover, in the case where all individuals are members of the same group ($\pi_n = 1$ for some n and $\pi_m = 0$ for all $m \neq n$), diversity has its minimum value of zero; conversely, diversity is maximized if and only if each group is equally represented in the population ($\boldsymbol{d}_{max} = d(1/M, 1/M, \ldots, 1/M)$). Two such diversity indices are the Simpson Interaction Index (I) and the Entropy (E) (equations [1] and [2] above).

Now suppose we have a population of M groups, with T individuals distributed among J organizational units. Let $\boldsymbol{d}$ be the diversity of the total population, and $\boldsymbol{d}_j$ be the diversity within unit j. From the mean value theorem of calculus, it can be shown that if $\boldsymbol{d}$ is a continuous, differentiable, concave-down function of the π_m's,[7] then

$$0 \leq \sum_{j=1}^{J} \frac{t_j}{T} \boldsymbol{d}_j \leq \boldsymbol{d}, \tag{8}$$

[6] In particular, we can derive the following equalities: $G^2 = 2TEH$ and $\chi^2 = T(M-1)C$.

[7] That is, for all m, the second partial derivative of $\boldsymbol{d}$ with respect to π_m must be negative on $0 < \pi_m < 1$.

with the left-hand equality holding if and only if $\boldsymbol{d}_j = 0$ for all j and the right-hand equality holding if and only if $\pi_{jm} = \pi_m$ for all j and m. The left-hand side of this inequality states that the weighted average of the within-unit diversities will always be greater than or equal to zero, with equality holding only if each unity has no diversity (e.g., each unit is monoracial, in the case of racial diversity). The right-hand inequality states that the weighted average within-unit diversity will always be less than or equal to the total diversity of the population, with equality holding here only if each unit has the same group proportions as the population as a whole. Both I and E are continuous, differentiable, concave-down functions on $0 < \pi_m < 1$, so both satisfy this inequality.

We can use this inequality to construct a general class of segregation indices. Let $\boldsymbol{S}(\boldsymbol{d})$—the segregation measure based on the diversity index $\boldsymbol{d}$—be defined as

$$\boldsymbol{S}(\boldsymbol{d}) = 1 - \frac{\bar{\boldsymbol{d}}_j}{\boldsymbol{d}} = \sum_{j=1}^{J} \frac{t_j}{T\boldsymbol{d}} (\boldsymbol{d} - \boldsymbol{d}_j), \tag{9}$$

where $\bar{\boldsymbol{d}}_j$ is the weighted average within-unit diversity. From equations (8) and (9), it is clear that $\boldsymbol{S}(\boldsymbol{d}) = 0$ if and only if each unit has the same group proportions as the total population, and $\boldsymbol{S}(\boldsymbol{d}) = 1$ if and only if each unit has no diversity. $\boldsymbol{S}(\boldsymbol{d})$ can be interpreted as the average difference between total and within-unit diversity, divided by the total diversity. Since this residual diversity can be attributed only to between-unit differences in group proportions, $\boldsymbol{S}(\boldsymbol{d})$ can also be seen as a measure of the proportion of total diversity attributable to between-unit differences.[8]

Substituting the diversity index I into equation (9), we get

$$\boldsymbol{S}(I) = \sum_{j=1}^{J} \frac{t_j}{TI} (I - I_j). \tag{10}$$

We will refer to $\boldsymbol{S}(I)$ as the *relative diversity index*, and denote it as R. R is equivalent to Goodman and Kruskal's τ_b (Carlson 1992; Goodman and Kruskal 1954), and it can be interpreted as one minus the ratio of the probability that two individuals from the same unit are members of differ-

[8] To see the relationship between the $\boldsymbol{S}$-class measures and variance decomposition methods, note that if we substituted σ^2, the variance of a continuous variable x, for $\boldsymbol{d}$ in equation (9), then $\boldsymbol{S}(\boldsymbol{d})$ would simply be R^2, the between-unit proportion of the variance in x.

ent groups to the probability that any two individuals are members of different groups.

Substituting the entropy index E into equation (9) yields

$$\boldsymbol{S}(E) = \sum_{j=1}^{J} \frac{t_j}{TE} (E - E_j). \tag{11}$$

This is simply Theil's information theory index H (Theil 1972; Theil and Finezza 1971), although written in a different form than shown in Table 1. Although Theil originally derived H using information theory, and we have derived it above from a disproportionality approach, equation (11) shows that H—like R—can also be seen as a measure of the ratio of within-unit diversity to total diversity (see also Zoloth 1976).

2.3.4. *Multigroup Segregation Measures Constructed from Dichotomous Indices*

Each of the three approaches described so far derives multigroup indices of segregation directly from a general mathematical operationalization of the idea of "segregation." A fourth method of constructing a multigroup segregation index derives multigroup indices as weighted averages of dichotomous indices. Suppose we have a population with M groups and S_m is a dichotomous index measuring the segregation of group m from all other groups. We can then define a multigroup segregation index S as a weighted average (where the weights—the W_m's—are positive and sum to 1) of the S_m's:

$$S = \sum_{m=1}^{M} W_m S_m. \tag{12}$$

In equation (12), $S = 0$ only in the case of no segregation and $S = 1$ only in the case of complete segregation. A multigroup index constructed this way necessarily reduces to its dichotomous counterpart when $M = 2$.

We can use equation (12) to derive James's (1986) "generalized segregation index"—a multigroup segregation index that cannot be derived from any of the three approaches described above. James derives this index from the P^*, or exposure, indices (Bell 1954; Farley 1984) by taking a simple weighted average of the normalized exposures of each group to all other groups.[9] Since the normalized exposure index is identical to the variance ratio index V, James's index is simply

$$P = \sum_{m=1}^{M} \pi_m V_m, \tag{13}$$

where V_m is the dichotomous version of V computed between group m and all other groups combined.

P is not, of course, the only multigroup index derivable from equation (12). In principle, we could use equation (12) to generate any number of multigroup indices from existing dichotomous indices, simply by constructing a set of weights (the W_m's) that sum to 1. For example, we could define $W_m = \pi_m$ (to weight groups proportionally to their relative sizes) or $W_m = (1 - \pi_m)/(M - 1)$ (to weight groups proportionally to the relative size of their complement) or $W_m = 1/M$ (to weight all groups equally, regardless of size) or $W_m = \pi_m(1 - \pi_m)/I$ (to weight groups most heavily when $\pi_m = 0.5$ and least heavily when π_m is close to 0 or 1). In fact, in addition to P, four of the five indices derived earlier (all but H) can be written in the form of equation (12):

$$D = \sum_{m=1}^{M} \frac{\pi_m(1 - \pi_m)}{I} D_m; \tag{14}$$

$$G = \sum_{m=1}^{M} \frac{\pi_m(1 - \pi_m)}{I} G_m; \tag{15}$$

$$R = \sum_{m=1}^{M} \frac{\pi_m(1 - \pi_m)}{I} V_m; \tag{16}$$

$$C = \sum_{m=1}^{M} \frac{(1 - \pi_m)}{M - 1} V_m. \tag{17}$$

It is instructive to compare the formulas for C, P, and R. Comparing equations (13), (16), and (17), we see that C, P, and R are each weighted aver-

[9] Although James calls this the "generalized segregation index" and denotes it as *GSI*, we will call it the normalized exposure index and denote it as P to suggest its relationship to the P^* indices.

ages of the V_m's; they differ only in the weight terms.[10] All three indices reduce to V, the variance ratio index, in the two-group case.

Although this approach to deriving multigroup segregation indices as weighted averages of M dichotomous indices appears useful, since it allows us to derive a limitless number of potential indices, including many of those derivable from other approaches, it has serious flaws. The S_m's in equation (12) are not independent of one another—a redistribution of persons that affects S_m necessarily affects at least one S_n, where $n \neq m$. Thus the idea that equation (12) allows us to "weight" the segregation of group m in some way that depends on its relative size is misleading, since the weights do not apply to independent quantities. Put another way, while equation (12) may seem to allow the decomposition of total segregation into M components—each indicating the contribution to total segregation made by the segregation of group m from all other groups—it is meaningless to talk of a single group's contribution to segregation. Segregation is defined by the *relationships* among the groups' distributions across organizational units—not by the distribution across units of each group in isolation.

A second flaw of this approach is that defining multigroup segregation as a weighted average of dichotomous indices may divorce an index from its substantive interpretation. For example, the multigroup dissimilarity index defined earlier based on the disproportionality approach retains the original substantive interpretation of the dichotomous dissimilarity index: it is the proportion of individuals who would have to transfer among units in order to eliminate segregation. A generalized dissimilarity index defined from equation (12) would meet the minimal requirements of a segregation index and would reduce to the dichotomous D when $M = 2$,

[10] Note that C, P, and R can also be written as

$$C = \sum_{m=1}^{M} \sum_{j=1}^{J} \frac{t_j}{T} \frac{(\pi_m - \pi_{jm})^2}{(M-1)\pi_m},$$

$$P = \sum_{m=1}^{M} \sum_{j=1}^{J} \frac{t_j}{T} \frac{(\pi_m - \pi_{jm})^2}{1-\pi_m},$$

$$R = \sum_{m=1}^{M} \sum_{j=1}^{J} \frac{t_j}{TI} (\pi_m - \pi_{jm})^2.$$

These equations indicate that C, P, and R are each measures of the squared difference between the π_{jm}'s and π_m's; they differ only in their denominators.

but it would not retain the substantive meaning of D in the multigroup case (except in the special case where we set $W_m = \pi_m(1 - \pi_m)/I$, as in equation [14]). Likewise, a generalized Gini index derived from equation (12) would lose the substantive interpretation of the generalized Gini index derived from the disproportionality approach.

Because of these flaws, we suggest that multigroup segregation indices not be derived exclusively from equation (12). It may, however, prove computationally useful to write a multigroup index as a weighted sum of dichotomous indices, but this computational convenience should not substitute for substantive meaning.

Table 2 summarizes the six indices we have defined. In addition, we include in Table 2 a formula for each of the indices. In some cases, of course, other expressions of the formulas are possible (e.g., H is often written as in equation [11]).[11]

3. EVALUATION OF THE MULTIGROUP MEASURES

We now turn to evaluating the indices against the seven criteria articulated earlier. Simple algebra shows that all six indices satisfy the organizational equivalence and size invariance criteria. The other criteria, however, require more careful analysis.

Transfers. Of the six indices, only the information theory index H obeys the principle of transfers. For each of the others, it is possible (albeit counterintuitive) to *increase* measured segregation by transferring an individual of group m from organizational unit i to unit j, where the proportion of persons of group m is greater in unit i than in j ($\pi_{im} > \pi_{jm}$) (proofs are shown in Appendix B).

Exchanges. Following James and Taeuber (1985), we evaluate the six indices' compliance with the exchange principle by taking the derivative with respect to an exchange of persons x, where x involves a transfer of persons of group m from unit i to j and a complementary transfer of an equal number of persons of group n from unit j to i. If this derivative is always negative when both $\pi_{im} > \pi_{jm}$ and $\pi_{jn} > \pi_{in}$, the index satisfies

[11] The six multigroup segregation indices defined here can be calculated by a program (-seg-) that runs under the STATA statistical software program and can be downloaded without charge from http://ideas.uqam.ca/ideas/data/bocbocode.html.

TABLE 2
Summary of Six Multigroup Segregation Measures

Multigroup Index	Type	Two-Group Form	Formula	Original Citations for Multigroup Form
Dissimilarity (D)	Disproportionality	D	$D = \sum_{m=1}^{M} \sum_{j=1}^{J} \frac{t_j}{2TI} \lvert \pi_{jm} - \pi_m \rvert$	Morgan 1975; Sakoda 1981
Gini (G)	Disproportionality	G	$G = \sum_{m=1}^{M} \sum_{i=1}^{J} \sum_{j=1}^{J} \frac{t_i t_j}{2T^2 I} \lvert \pi_{im} - \pi_{jm} \rvert$	Reardon 1998
Information Theory (H)	Disproportionality, association, and diversity ratio	H	$H = \sum_{m=1}^{M} \sum_{j=1}^{J} \frac{t_j}{TE} \pi_{jm} \ln \frac{\pi_{jm}}{\pi_m}$	Theil 1972; Theil and Finezza 1971
Squared Coefficient of Variation (C)	Disproportionality association	V	$C = \sum_{m=1}^{M} \sum_{j=1}^{J} \frac{t_j}{T} \frac{(\pi_{jm} - \pi_m)^2}{(M-1)\pi_m}$	New
Relative Diversity (R)	Diversity ratio	V	$R = \sum_{m=1}^{M} \sum_{j=1}^{J} \frac{t_j}{TI} (\pi_{jm} - \pi_m)^2$	Carlson 1992; Goodman and Kruskal 1954; Reardon 1998
Normalized Exposure (P)	Weighted average	V	$P = \sum_{m=1}^{M} \sum_{j=1}^{J} \frac{t_j}{T} \frac{(\pi_{jm} - \pi_m)^2}{(1 - \pi_m)}$	James 1986

the principle of exchanges (see Appendix B for the calculation of the derivatives).

The dissimilarity index fails to satisfy the principle of exchanges when $M > 2$ since the index remains constant in the case of exchanges that move individuals between units where the groups are either over- or underrepresented in both units (see James and Taeuber 1985, for $M = 2$ and Appendix B, below, for $M > 2$).

The derivative of the generalized Gini index with respect to x is

$$\frac{dG}{dx} = \frac{-2}{IT^2}\left[t_i + t_j + \sum_{r=i_m+1}^{j_m-1} t_r + \sum_{s=i_n+1}^{j_n-1} t_s\right], \tag{18}$$

where i_m, j_m, i_n, and j_n are the ranks of schools i and j ranked by decreasing proportions of groups m and n, respectively. Because dG/dx is negative when $i_m < j_m$ and $i_n < j_n$, G satisfies the principle of exchanges (for discussion of the sensitivity of G to exchanges, see James and Taeuber 1985).

Differentiating H with respect to x gives

$$\frac{dH}{dx} = \frac{1}{TE}\left(\ln\frac{\pi_{jm}}{\pi_{im}} + \ln\frac{\pi_{in}}{\pi_{jn}}\right). \tag{19}$$

Clearly $dH/dx < 0$ when $\pi_{im} > \pi_{jm}$ and $\pi_{jn} > \pi_{in}$, so the information theory index complies with the principle of exchanges. The effect of an exchange x on H has two components, one for each of the two groups involved in the exchange. The magnitude of each component is proportional to the difference in logged proportions between the two units. James and Taeuber discuss the sensitivity of H to exchanges in the two-group case (1985, p. 14).

The other three indices—C, P, and R—also obey the principle of exchanges. Differentiating each in turn with respect to x gives

$$\frac{dC}{dx} = \frac{-2}{T(M-1)}\left(\frac{\pi_{im} - \pi_{jm}}{\pi_m} + \frac{\pi_{jn} - \pi_{in}}{\pi_n}\right); \tag{20}$$

$$\frac{dP}{dx} = \frac{-2}{T}\left(\frac{\pi_{im} - \pi_{jm}}{1 - \pi_m} + \frac{\pi_{jn} - \pi_{in}}{1 - \pi_n}\right); \tag{21}$$

$$\frac{dR}{dx} = \frac{-2}{TI}\left[(\pi_{im} - \pi_{jm}) + (\pi_{jn} - \pi_{in})\right]. \tag{22}$$

Again, the effect of an exchange x has two components, one for each of the two groups involved in the exchange. The magnitude of each of these components is proportional to the difference in proportions between the two units. For C, the magnitude of each component is also *inversely* proportional to a group's share of the overall population, while for P the effect is *proportional* to a group's share of the overall population. Thus C is most sensitive to transfers of individuals of small groups while P is most sensitive to transfers of individuals of large groups. Because R does not share this dependence on relative group size, it is preferable to C and P in cases where change in index values should not be sensitive to the relative sizes of the groups involved in the exchanges.

Composition Invariance. James and Taeuber show that the dichotomous H and V fail to satisfy their version of the composition invariance criterion. Consequently, the multigroup H and the three multigroup generalizations of V (C, P, and R) fail to satisfy the criterion. Although the dichotomous D and G do satisfy the James and Taeuber composition invariance condition, simple algebra shows that the multigroup D and G do not. Thus none of the six measures defined here satisfy the composition invariance criterion as defined by James and Taeuber.[12]

Additive Organizational Decomposability. Recall that organizational decomposability refers to the situation where the J organizational units are grouped into K clusters (where $K < J$), such as the grouping of schools within districts, or census tracts within counties or metropolitan areas. A full organizational decomposition should allow us to partition total segregation into $K + 1$ independent additive components—a between-cluster component and K within-cluster components. The portion of total segregation due to segregation within cluster k should be the amount by which total segregation would be reduced if segregation within cluster k were eliminated by rearranging individuals among its units while leaving all

[12]It may seem then that we could easily construct a composition-invariant multigroup index from equation (12) simply by letting S_m be a composition-invariant dichotomous index (such as the dichotomous D or G) and $W_m = 1/M$. However, suppose S_m is a composition-invariant dichotomous index and that the number of members of group 1 is multiplied by a constant p in every unit. Although S_1 (the segregation between group 1 and all other groups combined) will be unchanged, S_2, $S_3, \ldots S_M$ will, in general, change (and thus S will change). So a multigroup index defined as the average of M dichotomous composition-invariant indices will not itself be composition-invariant.

other units unchanged. Evaluating whether an index S can be decomposed, then, requires evaluating whether we can write

$$S = S_K + \sum_{k=1}^{K} g(S_k), \tag{23}$$

where S_K is the between-cluster segregation among the K clusters, S_k is the segregation within cluster k, and g is a strictly increasing function on the interval $[0,1]$ with $g(0) = 0$.

Of the six multigroup indices described here, only two can be decomposed this way. It is straightforward to show that any $\boldsymbol{S}$-class measure (equation [9]) can be written as

$$\boldsymbol{S} = \boldsymbol{S}_K + \sum_{k=1}^{K} \frac{t_k \boldsymbol{d}_k}{T\boldsymbol{d}} \boldsymbol{S}_k, \tag{24}$$

where $\boldsymbol{d}_k$ and $\boldsymbol{d}$ are the diversity within cluster k and in the total population, respectively. From this, we get expressions for the decomposition of H and R:

$$H = H_K + \sum_{k=1}^{K} \frac{t_k E_k}{TE} H_k; \tag{25}$$

$$R = R_K + \sum_{k=1}^{K} \frac{t_k I_k}{TI} R_k. \tag{26}$$

These clearly satisfy the organizational decomposition property, since each of the k within-cluster components is an increasing function of the within-cluster segregation. For both H and R, within-cluster segregation is proportional to the relative size of the cluster (t_k/T), the relative diversity of the cluster $(\mathrm{E_k/E}$ or $I_k/I)$, and the level of within-cluster segregation between units $(H_k$ or $R_k)$. Larger, more diverse, and more segregated clusters contribute more to overall segregation than do smaller, more homogeneous, and less segregated clusters.[13]

The dissimilarity index, because it does not satisfy the principle of exchanges, cannot be appropriately decomposed into within- and between-cluster components—it is possible to reduce within-cluster segregation and leave the total segregation unchanged. Although Rivkin

[13] Reardon, Yun, and Eitle (2000) provide an alternative proof of this decomposition for H.

(1994) has claimed otherwise, the Gini index also cannot be satisfactorily decomposed, since it is possible to reduce within-cluster segregation while increasing the total Gini index, a highly unsatisfactory result. While C and P are decomposable in the dichotomous case—in that case they are equivalent to R, which *is* decomposable—C and P are *not* decomposable for $M > 2$.[14]

Additive Group Decomposability. Recall that group decomposability refers to the situation where M groups are themselves grouped into N mutually exclusive supergroups, where $N < M$. For example, ethnic groups may be grouped into larger groups (Mexicans, Puerto Ricans, Cubans, etc., grouped together into a Hispanic supergroup).

The group decomposability criterion can be formulated analogously to the organizational decomposability criterion. An index S meets the grouping decomposability criterion if we can write

$$S = S_N + \sum_{n=1}^{N} g(S_n), \tag{27}$$

where S_N is the segregation calculated among the N supergroups, S_n is the segregation among the groups making up supergroup n, and g is a strictly increasing function on the interval $[0,1]$ with $g(0) = 0$.[15]

[14] Note that C can be written as

$$C = C_K + \sum_{k=1}^{K} \frac{t_k}{T} \left[\sum_{m=1}^{M} \sum_{j \in k} \frac{t_j \pi_{km}}{t_k \pi_m} \frac{(\pi_{jm}^2 - \pi_{km}^2)}{(M-1)\pi_{km}} \right].$$

The bracketed term is not equal to C_k because of the additional π_{km}/π_m term. Similarly, P can be written as

$$P = P_K + \sum_{k=1}^{K} \frac{t_k}{T} \left[\sum_{m=1}^{M} \sum_{j \in k} \frac{t_j(1-\pi_{km})}{t_k(1-\pi_m)} \frac{(\pi_{jm}^2 - \pi_{km}^2)}{(1-\pi_{km})} \right].$$

Again, the term in the bracket is not equal to P_k because of the additional $(1 - \pi_{km})/(1 - \pi_m)$ term. The multigroup C and P have no organizational decomposition of the form given in equation (23).

[15] It is important to distinguish this type of decomposition from the decomposition of segregation into a weighted sum of dichotomous indices, since the latter "decomposition" contains no independent between- and within-group components.

Of the six indices, apparently only *H* can be decomposed in this manner.[16] Reardon, Yun, and Eitle (2000) show that *H* can be written as

$$H = H_N + \sum_{n=1}^{N} \frac{t_n E_n}{TE} H_n, \tag{28}$$

where H_N is the segregation calculated among the supergroups, T and E are the size and entropy of the population as a whole, and t_n, E_n, and H_n are the size, entropy, and segregation within supergroup n, respectively. The contribution of within-supergroup segregation to the total segregation is proportional to the relative size of the supergroup (t_n/T), the relative diversity of the supergroup (E_n/E), and the level of within-supergroup segregation between groups (H_n). Larger, more diverse, and more segregated supergroups contribute more to overall segregation than do smaller, more homogeneous, and less segregated supergroups.

Table 3 summarizes the compliance of the six indices with the seven criteria. The first four criteria are generally agreed on as essential to any meaningful segregation index. In that light, *H* is the superior index, since it alone satisfies all four criteria for both the two-group and multigroup cases. While all the indices save *D* satisfy the principle of exchanges, only *H* satisfies the stronger principle of transfers in the multigroup case. None of the indices satisfies the composition invariance principle in the multigroup case—which is not necessarily a flaw, as we have noted. Finally, while *R* and *H* satisfy the organizational decomposition property in the multigroup case (and *C* and *P* also satisfy it in the dichotomous case), only *H* satisfies both the organizational and the grouping decomposition properties.

4. DISCUSSION AND CONCLUSION

Taking our cue from scattered work suggesting that some binary indices can be generalized to the multigroup case, we develop methods for deriving and evaluating measures of multigroup segregation. We begin by describing alternative ways to *conceptualize* segregation: as disproportionality in group proportions, which links segregation measures to inequal-

[16] We have no formal proof that a group decomposition of the other indices is impossible, but we are unable to find any such decompositions after considerable algebraic manipulation.

TABLE 3
Properties of Multigroup Segregation Indices

	Dissimilarity (D)	Gini (G)	Information Theory (H)	Squared Coefficient of Variation (C)	Relative Diversity (R)	Normalized Exposure (P)
Organizational equivalence	✓	✓	✓	✓	✓	✓
Size invariance	✓	✓	✓	✓	✓	✓
Transfers						
2-Group	X[a]	✓	✓	✓	✓	✓
M-Group	X	X	✓	X	X	X
Exchanges						
2-Group	X[a]	✓	✓	✓	✓	✓
M-Group	X[a]	✓	✓	✓	✓	✓
Compositional invariance						
2-Group	✓	✓	X	X	X	X
M-Group	X	X	X	X	X	X
Additive organizational decomposability						
2-Group	X	X	✓	✓	✓	✓
M-Group	X	X	✓	X	✓	X
Additive group decomposability	X	X	✓	X	X	X

[a]The dissimilarity index satisfies only a weak form of the principles of transfers and exchanges: transfers and exchanges that move individuals from units of higher to lower proportions may result in no change in D, but they will never result in an increase in D.

ity measures; as association between group and organizational unit, which links segregation measures to χ^2 and G^2; as between-unit diversity relative to total diversity, which links segregation measures to variance decomposition methods; and as a weighted average of two-group segregation, which links multigroup segregation measures to dichotomous measures.

From these approaches we derive a dissimilarity index D, a Gini index G, an information theory index H, a squared coefficient of variation index C, a relative diversity index R, and a normalized exposure index P. D, G, and H are extensions of binary measures bearing the same names, whereas C, P, and R reduce to a single binary measure that is variously known as the variance ratio index (V), eta^2, the normalized exposure index, and Bell's revised index of isolation. Each of these measures captures the dimension of segregation that Massey and Denton (1988) call "evenness"—the extent to which mutually exclusive groups are evenly distributed among organizational units.[17]

Other multigroup measures are possible. In fact, equation (12) suggests that a limitless number of new multigroup indices can be derived from any dichotomous index S_m, though measures derived from the weighted average index approach embodied in equation (12) do not necessarily have any meaningful interpretation. We suggest that proposed indices be evaluated against a conceptually derived set of mathematical criteria, as we have done here. While there may be reasonable disagreement about the desirability of specific criteria or about the completeness of our particular list, it is nonetheless important that segregation indices be defined so that their mathematical behavior matches key conceptual and analytic needs, whatever those may be.

What do our seven criteria show regarding the six indices here? The most important finding is that Theil's information theory index H is the only one of the six that obeys the principle of transfers in the multigroup case. Failure to obey the principle is problematic for a segregation measure, since it means that the measure cannot be trusted to register a decline in segregation when an individual of group m moves from unit i to unit j, where the proportion of persons of group m is higher in unit i than in unit j. All five indices other than H may *increase* when a member of group m moves from a higher-π_m to a lower-π_m unit. H is also the only

[17] The point that segregation measures pertain to mutually exclusive groups is worth noting in light of the recent change in U.S. Census Bureau procedures for classifying race. To measure segregation based on 2000 census data, researchers will need to convert census tabulations into mutually exclusive groupings.

index that permits grouping decomposition, and it is one of only two indices that permits organizational decomposition in the multigroup case. Our overarching recommendation, then, is that researchers use the information theory index H for measuring multigroup segregation, at least when evenness is the conceptual dimension of segregation of interest.

Finally, while H appears superior to the other indices evaluated here, several important methodological issues in the study of multigroup segregation remain. First, we do not know at this point whether the violation of the principle of transfers seriously undermines the non-H indices, or instead is of little practical consequence in most research applications. However, the answer may become clearer in practice, as a new generation of research literature emerges on multigroup segregation. Second, the measures we describe here do not account for spatial dimensions of segregation. Future work should generalize spatial measures of segregation (concentration, clustering, and centralization) to the multigroup case. Third, the link between several of the multigroup measures and familiar measures of statistical association suggests possibilities for applying inferential statistics to segregation measurement, making possible the estimation of segregation levels from sample data rather than population data as well as hypothesis testing regarding differences in segregation levels between populations and over time.

APPENDIX A: MAXIMUM VALUES FOR THE MULTIGROUP SEGREGATION INDICES

The denominators of the disproportionality-based segregation indices have this common form:

$$W^* = \max\left[\sum_{m=1}^{M} \pi_m \sum_{j=1}^{J} \frac{t_j}{T} f(r_{jm})\right]. \tag{A-1}$$

The maximum value of W (for the four functions f that we use) is obtained under complete segregation—when each unit contains members of only a single group (no individual shares a unit with any member of any other group).[18] That is, when

[18] The proofs that, for each of the four disproportionality functions f that we use, W obtains its maximum value only under the condition of complete segregation require relatively straightforward, albeit tedious, applications of calculus. We omit the proofs here in the interest of space. Note that, by our definition, complete segregation is possible only when there are at least as many units as groups. Thus the derivations here assume $J \geq M$.

$$r_{jm} = \frac{\pi_{jm}}{\pi_m} = \begin{cases} \dfrac{1}{\pi_m} & \text{for } \pi_m \text{proportion of the population} \\ 0 & \text{for } (1-\pi_m) \text{ proportion of the population} \end{cases} . \tag{A-2}$$

This yields

$$W^* = \sum_{m=1}^{M} \pi_m \left[\pi_m f\left(\frac{1}{\pi_m}\right) + (1-\pi_m) f(0) \right]. \tag{A-3}$$

We derive the maximum values for each index by substituting the four disproportionality functions into equation (A-3).

Dissimilarity Index (D). If $f(r) = |r-1|/2$, equation (A-3) yields:

$$W^* = \sum_{m=1}^{M} \pi_m \left[\pi_m \frac{\left|\dfrac{1}{\pi_m} - 1\right|}{2} + (1-\pi_m) \frac{|0-1|}{2} \right] = I. \tag{A-4}$$

So the denominator of D is Simpson's Interaction Index I.

Theil's Information Theory Index (H). If $f(r) = r \ln r$, equation (A-3) yields the following (recall we define $0 \cdot \ln 0 = 0$):

$$W^* = \sum_{m=1}^{M} \pi_m \left[\pi_m \left(\frac{1}{\pi_m}\right) \ln\left(\frac{1}{\pi_m}\right) + (1-\pi_m) \cdot 0 \ln 0 \right] = E. \tag{A-5}$$

So the denominator of H is Theil's Entropy Index E.

Squared Coefficient of Variation Index (C). If $f(r) = (r-1)^2$, equation (A-3) yields

$$W^* = \sum_{m=1}^{M} \pi_m \left[\pi_m \left(\frac{1}{\pi_m} - 1\right)^2 + (1-\pi_m)(0-1)^2 \right] = M - 1. \tag{A-6}$$

So the denominator of C is M-1.

Gini Index (G). If $f(r) = |r_{im} - r_{jm}|/2$, the derivation is slightly more complicated. The value of $|r_i - r_j|/2$ depends on the group proportions within both units i and j, so

$$\frac{|r_{im}-r_{jm}|}{2} = \begin{cases} |0-0|/2 = 0 & \text{for } (1-\pi_m)^2 \text{ proportion of the} \\ & \text{population (when } \pi_{im} = \pi_{jm} = 0) \\ |1/\pi_m - 0|/2 = 1/2\pi_m & \text{for } \pi_m\ (1\text{–}\pi_m) \text{ proportion of the} \\ & \text{population (when } \pi_{im} = 1;\ \pi_{jm} = 0) \\ |0 - 1/\pi_m|/2 = 1/2\pi_m & \text{for } \pi_m(1-\pi_m) \text{ proportion of the} \\ & \text{population (when } \pi_{im} = 0; \pi_{jm} = 1) \\ |1/\pi_m - 1/\pi_m|/2 = 0 & \text{for } \pi_m^2 \text{ proportion of the popul-} \\ & \text{tion (when } \pi_{im} = \pi_{jm} = 1) \end{cases} \tag{A-7}$$

Substituting this into a modified version of equation (A-3) and simplifying, we get

$$W^* = \sum_{m=1}^{M} \pi_m \left[(1-\pi_m)^2 \cdot 0 + \pi_m(1-\pi_m) \cdot \frac{1}{2\pi_m} + \pi_m(1-\pi_m) \cdot \frac{1}{2\pi_m} + \pi_m^2 \cdot 0 \right] = I. \tag{A-8}$$

So the denominator of G is Simpson's Interaction Index I.

APPENDIX B: PROOFS OF TRANSFER AND EXCHANGE PROPERTIES

We can evaluate each index's compliance with the principles of transfers and exchanges by taking the derivative of the index with respect to a transfer or exchange x. The four indices derived from the disproportionality approach share the general form below, given in equation (4). (We change the subscripts here since we will use m, n, i, and j to refer to the specific groups and units involved in the transfer or exchange.)

$$S = \frac{1}{W^*} \sum_{r=1}^{M} \sum_{k=1}^{J} \pi_r \frac{t_k}{T} f(r_{kr}). \tag{B-1}$$

Note that any transfer or exchange does not affect the π_r's or T. We can write the derivative of S with respect to a transfer or exchange x as

$$\frac{dS}{dx} = \frac{1}{W^*} \sum_{r=1}^{M} \sum_{k=1}^{J} \left[\pi_r \frac{t_k}{T} \frac{df}{dr_{kr}} \frac{dr_{kr}}{d\pi_{kr}} \frac{d\pi_{kr}}{dx} + \pi_r \frac{1}{T} \frac{dt_k}{dx} f(r_{kr}) \right]. \tag{B-2}$$

Note also that $dr_{kr}/d\pi_{kr} = 1/\pi_r$. In the case where x indicates a transfer of persons of group m from unit i to j, we get the following expressions:

$$\frac{dt_k}{dx} = \begin{cases} -1 & \text{when } k = i \\ 1 & \text{when } k = j \\ 0 & \text{when } k \neq i, j \end{cases} \tag{B-3}$$

$$\frac{d\pi_{kr}}{dx} = \begin{cases} \dfrac{t_{kr}}{t_k^2} & \text{when } k = i \text{ and } r \neq m \\ -\dfrac{t_{kr}}{t_k^2} & \text{when } k = j \text{ and } r \neq m \\ \dfrac{t_{kr} - t_k}{t_k^2} & \text{when } k = i \text{ and } r = m \\ \dfrac{t_k - t_{kr}}{t_k^2} & \text{when } k = j \text{ and } r = m \\ 0 & \text{when } k \neq i, j \end{cases} \tag{B-4}$$

We calculate dS/dx for any of the four segregation indices derived from the disproportionality approach by substituting equations (B-3), (B-4), and the appropriate disproportionality function $f(r)$ into equation (B-2). Using $f(r) = r \ln r$ yields

$$\frac{dH}{dx} = \frac{1}{TE} \ln \frac{\pi_{jm}}{\pi_{im}}. \tag{B-5}$$

When $\pi_{jm} < \pi_{im}$, $dH/dx < 0$, so H satisfies the principle of transfers.

For each of the other five indices, the derivative with respect to a transfer x results in a more complicated function. In each case, the derivative may be positive when $\pi_{jm} < \pi_{im}$, so none of the other functions satisfy the principle of transfers. Rather than compute the derivative of each of these functions, however, we instead provide proofs by counterexample. Two examples suffice to demonstrate that each of the indices other than H does not satisfy the principle of transfers.

Example 1 (Table B1) shows the distribution of members of three groups among three organizational units, and the subsequent distribution following a transfer of a member of group A from unit 2 to unit 1. Since unit 2 begins with a greater proportion of members from group A, this transfer should result in a decrease in the segregation if the principle of transfers is satisfied. However, four of the six indices (D, G, R, and P)

TABLE B1
Failure of Multigroup D, G, R, and P to Obey the Principle of Transfers

	Time 1			
	Unit 1	Unit 2	Unit 3	Total
Group A	6	8	10	24
Group B	181	193	378	752
Group C	12	0	12	24
Total	199	201	400	800

	Time 2			
	Unit 1	Unit 2	Unit 3	Total
Group A	7	7	10	24
Group B	181	193	378	752
Group C	12	0	12	24
Total	200	200	400	800

Index	D	G	H	C	R	P
Time 1	0.1537	0.2085	0.0419	0.0084	0.0073	0.0063
Time 2	0.1636	0.2182	0.0412	0.0082	0.0083	0.0084
Change	+0.0099	+0.0097	−0.0007	−0.0002	+0.0010	+0.0019

show an increase in segregation following the transfer. Clearly then, these indices do not obey the transfer criterion. Example 2, likewise, illustrates that C may increase as a result of a transfer from a unit of higher proportion to one of lower proportion (Table B2).

In the case where x indicates an exchange rather than a transfer, the indices behave differently. Equation (B-2) still applies, but when x is an exchange where an individual of group m in organizational unit i is exchanged with an individual of group n from organizational unit j, we get the following values for dt_k/dx and $d\pi_{kr}/dx$:

$$\frac{dt_k}{dx} = 0, \tag{B-6}$$

$$\frac{d\pi_{kr}}{dx} = \begin{cases} -1/t_k & \text{when } k = i \text{ and } r = m \\ +1/t_k & \text{when } k = j \text{ and } r = m \\ +1/t_k & \text{when } k = i \text{ and } r = n \\ -1/t_k & \text{when } k = j \text{ and } r = n \\ 0 & \text{when } k \neq i, j \text{ or } r \neq m, n \end{cases}. \tag{B-7}$$

TABLE B2
Failure of Multigroup C to Obey the Principle of Transfers

	Time 1			
	Unit 1	Unit 2	Unit 3	Total
Group A	99	101	100	300
Group B	0	100	0	100
Group C	100	0	100	200
Total	199	201	200	600

	Time 2			
	Unit 1	Unit 2	Unit 3	Total
Group A	100	100	100	300
Group B	0	100	0	100
Group C	100	0	100	200
Total	200	200	200	600

Index	D	G	H	C	R	P
Time 1	0.3655	0.3664	0.31468	0.2494	0.18183	0.1501
Time 2	0.3636	0.3636	0.31467	0.2500	0.18182	0.1500
Change	−0.0019	−0.0028	−0.00001	+0.0006	−0.00001	−0.0001

Substituting equations (B-6) and (B-7) into (B-2), we get

$$\frac{dS}{dx} = \frac{1}{TW^*}\left(\frac{df}{dr_{jm}} - \frac{df}{dr_{im}} + \frac{df}{dr_{in}} - \frac{df}{dr_{jn}}\right). \tag{B-8}$$

Again, we calculate dS/dx in the case of an exchange for any of the indices derived from the disproportionality approach used in equation (B-1) by substituting the appropriate disproportionality function $f(r)$ into equation (B-8). Using $f(r) = |r-1|/2$ yields

$$\frac{dD}{dx} = \frac{z_m + z_n}{TI} \tag{B-9}$$

where

$$z_m = \begin{cases} -1 & \text{if } \pi_{im} > \pi_m > \pi_{jm} \\ 1 & \text{if } \pi_{im} < \pi_m < \pi_{jm} \\ 0 & \text{otherwise} \end{cases}$$

$$z_n = \begin{cases} -1 & \text{if } \pi_{jn} > \pi_n > \pi_{in} \\ 1 & \text{if } \pi_{jn} < \pi_n < \pi_{in} \\ 0 & \text{otherwise} \end{cases}$$

Since $dD/dx = 0$ if, for example $\pi_{im} > \pi_{jm} > \pi_m$ and $\pi_{in} < \pi_{jn} < \pi_n$, D does not strictly satisfy the principle of exchanges.

Substituting $f(r) = r \ln r$ into equation (B-8) yields the derivative of H with respect to an exchange x (equation [19]), while substituting $f(r) = (r-1)^2$ into equation (B-8) yields the derivative of C with respect to an exchange x (equation [20]):

$$\frac{dH}{dx} = \frac{1}{TE}\left(\ln\frac{\pi_{jm}}{\pi_{im}} + \ln\frac{\pi_{in}}{\pi_{jn}}\right);$$

$$\frac{dC}{dx} = \frac{-2}{T(M-1)}\left(\frac{\pi_{im} - \pi_{jm}}{\pi_m} + \frac{\pi_{jn} - \pi_{in}}{\pi_n}\right).$$

When $\pi_{jm} < \pi_{im}$ and $\pi_{in} < \pi_{jn}$, $dH/dx < 0$ and $dC/dx < 0$, so both H and C satisfy the principle of exchanges.

The derivative of G with respect to an exchange x is obtained by recalling that we can write G as a weighted sum of dichotomous G_m's (equation [15]). Taking the derivative of G in equation (15), and noting that $dG_r/dx = 0$ for all $r \neq m, n$, we get

$$\frac{dG}{dx} = \frac{\pi_m(1-\pi_m)}{I}\,\frac{dG_m}{dx} + \frac{\pi_n(1-\pi_n)}{I}\,\frac{dG_n}{dx}. \tag{B-10}$$

James and Taeuber (1985) show that the derivative of the dichotomous Gini index with respect to an exchange x is given by

$$\frac{dG_r}{dx} = \frac{-1}{\pi_r(1-\pi_r)T^2}\left[t_i + t_j + 2\sum_{k=i_r+1}^{j_r-1} t_k\right], \tag{B-11}$$

where i_r and j_r are the ranks of schools i and j ranked by decreasing proportions of group r. Substituting this expression into equation (B-10) yields equation (18):

$$\frac{dG}{dx} = \frac{-2}{IT^2}\left[t_i + t_j + \sum_{r=i_m+1}^{j_m-1} t_r + \sum_{s=i_n+1}^{j_n-1} t_s\right]$$

Because dG/dx is negative when $i_m < j_m$ and $i_n < j_n$, G satisfies the principle of exchanges.

Since R is not derived from the disproportionality approach, we cannot obtain its derivative from equation (B-9). Instead, we take its derivative directly from the expression for R in footnote 10:

$$\frac{dR}{dx} = \frac{1}{TI} \sum_{r=1}^{M} \sum_{k=1}^{J} \left[2t_k (\pi_{kr} - \pi_r) \frac{d\pi_{kr}}{dx} \right]. \tag{B-12}$$

Substituting equation (B-7) into this expression gives us equation (22):

$$\frac{dR}{dx} = \frac{-2}{TI} [(\pi_{im} - \pi_{jm}) + (\pi_{jn} - \pi_{in})].$$

When $\pi_{jm} < \pi_{im}$ and $\pi_{in} < \pi_{jn}$, $dR/dx < 0$, so R satisfies the principle of exchanges.

Using a similar approach for P—differentiating P with respect to an exchange x and substituting equation (B-7) into the result—yields equation (21):

$$\frac{dP}{dx} = \frac{-2}{T} \left[\frac{(\pi_{im} - \pi_{jm})}{(1 - \pi_m)} + \frac{(\pi_{jn} - \pi_{in})}{(1 - \pi_n)} \right].$$

When $\pi_{jm} < \pi_{im}$ and $\pi_{in} < \pi_{jn}$, $dR/dx < 0$, so P satisfies the principle of exchanges.

REFERENCES

Bell, W. 1954. "A Probability Model for the Measurement of Ecological Segregation." *Social Forces* 43:357–64.

Carlson, Susan M. 1992. "Trends in Race/Sex Occupational Inequality: Conceptual and Measurement Issues." *Social Problems* 39:269–90.

Charles, Maria, and David B. Grusky. 1995. "Models for Describing the Underlying Structure of Sex Segregation." *American Journal of Sociology* 100:931–71.

Cohen, Jack K., R. Frank Falk, and Charles F. Cortese. 1976. "Reply to Taeuber and Taeuber." *American Sociological Review* 41:889–93.

Coleman, James, Thomas Hoffer, and Sally Kilgore. 1982. "Achievement and Segregation in Secondary Schools: A Further Look at Public and Private School Differences." *Sociology of Education* 55:162–82.

Cortese, Charles F., R. Frank Falk, and Jack K. Cohen. 1976. "Further Considerations on the Methodological Analysis of Segregation Indices." *American Sociological Review* 41:630–37.

Cowgill, Donald O., and Mary S. Cowgill. 1951. "An Index of Segregation Based on Block Statistics." *American Sociological Review* 16:825–31.

Deutsch, Joseph, Yves Flückiger, and Jacques Silber. 1994. "Measuring Occupational Segregation: Summary Statistics and the Impact of Classification Errors and Aggregation." *Journal of Econometrics* 61:133–46.

Duncan, Otis Dudley, and Beverly Duncan. 1955. "A Methodological Analysis of Segregation Indexes." *American Sociological Review* 20:210–17.

Falk, R. Frank, Charles F. Cortese, and Jack Cohen. 1978. "Utilizing Standardized Indices of Residential Segregation: Comment on Winship." *Social Forces* 57:713–16.

Farley, John E. 1984. "P* Segregation Indices: What Can They Tell Us About Housing Segregation in 1980?" *Urban Studies* 21:331–36.

Farley, Reynolds. 1977. "Residential Segregation in Urbanized Areas of the United States in 1970: An Analysis of Social Class and Racial Differences." *Demography* 14:497–518.

Firebaugh, Glenn. 1998. "Measuring Inequality: A Convenient Unifying Framework." Presented at the annual meeting of the Population Association of America, Chicago, IL.

———. 1999. "Empirics of World Income Inequality." *American Journal of Sociology* 104:1597–630.

Goodman, Leo, and William H. Kruskal. 1954. "Measures of Association for Cross-Classifications." *Journal of the American Statistical Association* 49:732–64.

Grusky, David B., and Maria Charles. 1998. "The Past, Present, and Future of Sex Segregation Methodology." *Demography* 35:497–504.

Jahn, Julius A. 1950. "The Measurement of Ecological Segregation: Derivation of an Index Based on the Criterion of Reproducibility." *American Sociological Review* 15:100–104.

Jahn, Julius A., Calvin F. Schmid, and Clarence Schrag. 1947. "The Measurement of Ecological Segregation." *American Sociological Review* 12:293–303.

James, David R., and Karl E. Taeuber. 1985. "Measures of Segregation." *Sociological Methodology* 14:1–32.

James, Franklin J. 1986. "A New Generalized "Exposure-Based" Segregation Index." *Sociological Methods and Research* 14:301–16.

Kestenbaum, Bert. 1980. "Notes on the Index of Dissimilarity: A Research Note." *Social Forces* 59:275–80.

Kulis, Stephen. 1997. "Gender Segregation Among College and University Employees." *Sociology of Education* 70:151–73.

Lieberson, Stanley. 1969. "Measuring Population Diversity." *American Sociological Review* 34:850–62.

Lieberson, Stanley, and Donna K. Carter. 1982a. "A Model for Inferring the Voluntary and Involuntary Causes of Residential Segregation." *Demography* 19:511–26.

———. 1982b. "Temporal Changes and Urban Differences in Residential Segregation: A Reconsideration." *American Journal of Sociology* 88:296–310.

Massey, Douglas S., and Nancy A. Denton. 1988. "The Dimensions of Racial Segregation." *Social Forces* 67:281–315.

Miller, Vincent P., and John M. Quigley. 1990. "Segregation by Racial and Demographic Group: Evidence from the San Francisco Bay Area." *Urban Studies* 27:3–21.

Morgan, Barrie S. 1975. “The Segregation of Socioeconomic Groups in Urban Areas: A Comparative Analysis.” *Urban Studies* 12:47–60.

———. 1983. “A Temporal Perspective on the Properties of the Index of Dissimilarity.” *Environment and Planning A* 15:379–89.

Reardon, Sean F. 1998. “Measures of Racial Diversity and Segregation in Multigroup and Hierarchically Structured Populations.” Presented at the annual meeting of the Eastern Sociological Society, Philadelphia, PA.

Reardon, Sean F., and John T. Yun. 2001. “Suburban Racial Change and Suburban School Segregation, 1987–1995.” *Sociology of Education* 74:79–101.

Reardon, Sean F., John T. Yun, and Tamela McNulty Eitle. 2000. “The Changing Structure of School Segregation: Measurement and Evidence of Multi-racial Metropolitan Area School Segregation, 1989–1995.” *Demography* 37:351–64.

Rivkin, Steven G. 1994. “Residential Segregation and School Integration.” *Sociology of Education* 67:279–92.

Sakoda, J. 1981. “A Generalized Index of Dissimilarity.” *Demography* 18:245–50.

Schwartz, Joseph, and Christopher Winship. 1980. “The Welfare Approach to Measuring Inequality.” *Sociological Methodology* 9:1–36.

Taeuber, Karl E., and Alma F. Taeuber. 1965. *Negroes in Cities: Residential Segregation and Neighborhood Change*. Chicago, IL: Aldine Publishing.

———. 1976. “A Practitioner’s Perspective on the Index of Dissimilarity.” *American Sociological Review* 41:884–89.

Theil, Henri. 1972. *Statistical Decomposition Analysis*. Amsterdam: North-Holland.

Theil, Henri, and Anthony J. Finezza. 1971. “A Note on the Measurement of Racial Integration of Schools by Means of Informational Concepts.” *Journal of Mathematical Sociology* 1:187–94.

Watts, Martin. 1992. “How Should Occupational Sex Segregation be Measured?” *Work, Employment and Society* 6:475–87.

———. 1994. “A Critique of Marginal Matching.” *Work, Employment and Society* 8:421–31.

White, Michael J. 1986. “Segregation and Diversity Measures in Population Distribution.” *Population Index* 52:198–221.

Williams, Josephine J. 1948. “Another Commentary on So-called Segregation Indices.” *American Sociological Review* 13:298–303.

Winship, Christopher. 1977. “A Reevaluation of Indexes of Residential Segregation.” *Social Forces* 55:1058–66.

———. 1978. “The Desirability of Using the Index of Dissimilarity or Any Adjustment of it for Measuring Segregation: Reply to Falk, Cortese, and Cohen.” *Social Forces* 57:717–20.

Zoloth, Barbara S. 1976. “Alternative Measures of School Segregation.” *Land Economics* 52:278–98.