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Polyhedral Dynamics and the Controllability of Dynamical Systems

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1. INTRODUCTION

A recurring theme in contemporary system-theoretic work is the employment of mathematical tools from algebra and geometry to analyze the structure of dynamical processes. The motivation for the shift in emphasis from analysis to algebra has been to provide a language suitable for describing the *global* structure of dynamical systems, which at the same time unifies numerous subclasses of problems, e.g. discrete/continuous-time, input-output/state variable form, finite-state/infinite-dimensional, etc. Reasonable summaries of much of this work can be found in the works [1–3].

In the present work, the above algebraic trend is extended in a new direction to analyze the controllability structure of systems. The methodology employed, termed "polyhedral dynamics", (or "q-analysis" in [4]), has its basis in combinatorial topology and proceeds by mapping the given process into a simplicial complex and then employing both standard and non-standard algebraic tools to study the connective structure of the complex. Since the mapping of the system into a complex may be carried out in a number of non-equivalent ways, the first basic problem in utilizing the methodology is to find a mapping which leads to interesting statements about the process under investigation. We shall present just such a mapping suitable for studying the controllability structure of linear dynamical processes. For the most part, in this preliminary report attention is restricted to the linear case in order to make the underlying ideas as transparent as possible, as well as to facilitate comparison of the results obtained with more traditional approaches [5, 6].

The principle results of this paper show that, from the viewpoint of algebraic topology, single-input controllable linear systems have a trivial topological structure. Furthermore, we show how the standard arithmetic invariants of simplicial homology theory can be interpreted as statements about such systems.

In addition to classical homological concepts, the polyhedral dynamics methodology also suggests introduction of a new topological concept, *q-connec-*

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tion, which characterizes the dimensional "nearness" of pairs of simplices in a complex. Utilizing this concept, we present an alternate characterization of controllable linear systems, one which is based upon the manner in which the controllable subspaces are connected.

The paper concludes with a discussion of approaches to treat nonlinear processes, as well as indications of how to employ more advanced tools from algebraic topology to study the case of multi-input linear problems.

2. Linear Systems and Controllability

We study the single-input, constant linear system

$$x_{k+1} = Fx_k + gu_k, \qquad x_0 = c, \tag{Σ}$$

where $x_k \in R^n$, $u_k \in R$ and F and g are real constant matrices of appropriate sizes.

The set of states which may be transferred to the origin by application of a suitable input sequence $\{u_k\}$ is characterized by the *controllability* matrix [5]

$$\mathscr{C} = [g \mid Fg \mid F^2g \mid \cdots \mid F^{n-1}g],$$

i.e. the linearly independent colums of $\mathscr{C}$ span the controllable subspace of Σ. The system is said to be completely controllable if all initial states are controllable, i.e. if (and only if) rank $\mathscr{C} = n$. We note, in passing, that much of the modern work in algebraic system theory cited above was motivated by the fact that from the viewpoint of controllability, all of the dynamical properties of Σ are contained in the *algebraic* structure of $\mathscr{C}$. This observation will also be central to our treatment of Σ by algebro-topological tools.

3. Polyhedral Dynamics

As noted, the conceptual idea underlying polyhedral dynamics is to regard each system as being represented by an appropriate simplicial complex and to study the connective structure of the system by appeal to the many tools of algebraic topology, which have been refined over the past few decades [7, 8].

To fix the basic idea, let $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_m\}$ be two finite sets of elements and let $\lambda \subset Y \times X$ be a binary relation. Here "$\times$" denotes the usual Cartesian product of two sets. We may geometrically represent the relation λ by a simplicial complex in the following manner: identify the elements of X with the vertices of the complex and let the elements of Y represent the simplices. The relation λ then defines those vertices which form each simplex in the abstract complex $K_Y(X; \lambda)$, i.e. vertex x_j belongs to simplex y_i if and only if

$(y_i, x_j) \in \lambda$. Algebraically, we may represent the complex $K_Y(X; \lambda)$ by the $m \times n$ incidence matrix $\Lambda = [\Lambda_{ij}]$, where

$$\Lambda_{ij} = 1, \qquad (y_i, x_j) \in \lambda$$
$$= 0, \qquad \text{otherwise.}$$

Interchanging the roles of the sets X and Y generates the conjugate complex $K_X(Y; \lambda^*)$, whose incidence matrix is obtained by transposing Λ, i.e., $\Lambda^* = \Lambda'$.

In addition to the traditional concepts from algebraic topology such as homology, cohomology, homotopy, etc. [8], it turns out to be useful to define a new relation, termed a "chain of q-connection", for studying the structure of a simplicial complex. Formally, we have

DEFINITION 1. Given two simplices σ_p and σ_r in the complex $K_Y(X; \lambda)$, we say they are joined by a *chain of connection* if there exists a finite sequence of simplices $\sigma_{\alpha_1}, \sigma_{\alpha_2}, \ldots, \sigma_{\alpha_s}$ such that

(1) σ_{α_1} is a face of σ_p,

(2) σ_{α_s} is a face of σ_r,

(3) σ_{α_i} and $\sigma_{\alpha_{i+1}}$ have a common face, say σ_{ρ_i}, $i = 1, 2, \ldots, s-1$.

We shall say that such a chain is of *length* $(s - 1)$ and the chain is a *q-connection* if

$$\min\{\alpha_1, \rho_1, \rho_2, \ldots, \rho_{s-1}, \alpha_s\} = q.$$

Remark. Here we employ the standard convention that the subscript on a simplex denotes its geometric dimension, e.g. dim $\sigma_i = i$, where the geometric dimension equals the number of vertices comprising the simplex minus one.

The importance of the relation of q-connection is that for each value of q, it defines an equivalence relation on $K_Y(X; \lambda)$. Thus, we can study the equivalence classes of the complex as one measure of the connective structure.

With each value of $q = 0, 1, 2, \ldots, N$ ($N =$ dim of highest dimensional simplex in $K_Y(X; \lambda)$), we associate a positive integer Q, the number of distinct equivalence classes in K at dimension level q. The vector

$$Q = (Q_N, Q_{N-1}, \ldots, Q_1, Q_0),$$

is termed the *structure vector* of $K_Y(X; \lambda)$, where $N = \dim K$. Determination of the vector Q is sometimes termed performing a q-analysis on the complex [4].

The structure vector Q (and the vector Q^* formed from the conjugate complex $K_X(Y; \lambda^*)$) gives useful insight into the global connectivity pattern among the simplices of K. However, it does not provide too much useful information about the degree to which any individual simplex is integrated into K. Thus, we introduce another notion, *eccentricity*, to account for this aspect of the system structure.

DEFINITION 2 [4]. Given a simplex $\sigma \in K_Y(X; \lambda)$, we define the *eccentricity* of σ to be

$$\operatorname{ecc} \sigma = \frac{\hat{q} - \check{q}}{\check{q} + 1},$$

where $\hat{q} = \dim$ of σ, $\check{q} =$ largest q-value at which σ is connected to another distinct simplex in K.

From Definition 2, we note the satisfying properties that $\operatorname{ecc} \sigma = +\infty$ if σ is totally disconnected from the remainder of the complex and $\operatorname{ecc} \sigma = 0$ if σ is totally imbedded within another simplex in K.

There are many other mathematical considerations surrounding the setup we have just described, such as graded patterms, t-forces and so on, which are covered in detail elsewhere [4, 9]. Since these notions are not needed for the material of this report, we shall restrict our main attention to the connectivity concepts introduced above.

Before proceeding to our main results, we note that the abstract simplicial complex generated by the binary relation λ allows us to utilize all of the standard tools of algebraic topology for the study of such objects. In particular, we shall make use of the homological structure of $K_Y(X; \lambda)$ for some of our results and refer the reader to [7, 8] or any of the other standard treatises cited therein for the appropriate background.

4. THE COMPLEX OF Σ

Since the abstract complex $K_Y(X; \lambda)$ requires only the finite sets X and Y and the relation λ for its definition, associating a simplicial complex to the system Σ is clearly equivalent to specification of the above elements using the basic problem data. As our aims are to study the controllable structure of the system, the well known algebraic procedure for generating an exterior algebra from a finite-dimensional vector space [10] provides an appropriate key for specifying a useful complex, recalling the fact that the controllable states for Σ form a subspace of R^n, the subspace generated by the linearly independent columns of the matrix $\mathscr{C}$.

For the sake of definiteness, we label the n column vectors of $\mathscr{C}$ as $x_1, x_2, \ldots, x_n$. Thus, $g \leftrightarrow x_1$, $Fg \leftrightarrow x_2$, and so on and Identify the elements of the set X with the vectors of $\mathscr{C}$, i.e. the vertex set

$$X = \{x_1, x_2, \ldots, x_n\}.$$

The p-simplices of the complex will consist of those collections of $p + 1$ elements of X which are linearly independent, $p = 0, 1, \ldots, n - 1$. If we introduce the standard "wedge" product operation "$\wedge$" from exterior algebra, then the elements of the set Y, the simplex set, are the basis elements of the exterior product space $\Lambda^{p+1}X$, while the *p-chains* of the complex are the members of $\Lambda^{p+1}X$. The binary relation $\lambda \subset Y \times X$ is easily seen to be "$(y_i, x_j) \in \lambda$ if and only if the vector x_j belongs to the simplex y_i."

As a simple illustration of the above definition, consider the system Σ in control canonical form with

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_2 & -\alpha_1 & -\alpha_0 \end{bmatrix}, \qquad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We have

$$\mathscr{C} = \begin{bmatrix} 0 & 0 & 1 & \\ 0 & 1 & -\alpha_0 & \\ 1 & -\alpha_0 & -\alpha_1 & +\alpha_0^2 \end{bmatrix} = [x_1 \mid x_2 \mid x_3].$$

Hence, the sets X and Y are

$$\begin{aligned} X &= \{x_1, x_2, x_3\}, \\ Y &= \{x_1, x_2, x_3, x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_3, x_1 \wedge x_2 \wedge x_3\}. \end{aligned}$$

We shall return to this example later.

5. Main Results

The definitions of the previous sections allow us to associate a simplicial complex with any finite-dimensional constant, single-input linear system. We now exploit these definitions to study the relationship between the connective structure of the complex and the standard notion of controllability from linear system theory. Our first result in this direction establishes the connection between the homology groups $\{H_i\}$ of the complex and the concept of controllability.

THEOREM 1. *The system Σ is completely controllable if and only if the associated complex has a trivial homology, i.e. $H_i \cong 0$, $0 < i \leqslant n - 1$.*

Proof. Assume the homology is trivial. In particular, we then have $H_{n-2} \cong 0$. Let

$$z = \sum_k \sum_{i_1 < i_2 < \cdots < i_{n-1}} \alpha_{i_1 i_2 \cdots i_k \cdots i_{n-1}} \langle x_{i_1} x_{i_2} \cdots \hat{x}_{i_k} \cdots x_{i_{n-1}} \rangle$$

be an $n - 2$ cycle, i.e. $\partial z = 0$. But, since $H_{n-2} \cong 0$, there exists an $n - 1$ chain $y \in K$ such that $\partial y = z$, i.e. z is a bounding cycle.

By definition of the boundary operator ∂, we have

$$\partial z = \sum_l \sum_k \sum_{i_1 < i_2 < \cdots < i_{n-1}} (-1)^l \langle x_{i_1} x_{i_2} \cdots \hat{x}_{i_k} \hat{x}_{i_l} \cdots x_{i_n} \rangle \alpha_{i_1 \cdots i_n} = 0.$$

Since this equation must hold identically in the symbols $x_{i_1}, x_{i_2}, \ldots, x_{i_n}, \alpha_{i_1 \cdots i_n} \equiv 0$. Thus, we have a linear relation between the coefficients $\alpha_{i_1 \cdots i_n}$, which insures that there exists a scalar β and a simplex $\langle x_{i_1} \cdots x_{i_n} \rangle \in K$ such that

$$\partial(\beta \langle x_{i_1} \cdots x_{i_n} \rangle) = z,$$

i.e. we choose $y = \beta \langle x_{i_1} \cdots x_{i_n} \rangle$. But, since $\langle x_{i_1} \cdots x_{i_n} \rangle \in K$, this implies that $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ are linearly independent and, hence, Σ is controllable.

Now let Σ be controllable. Then the simplex $\langle x_1 x_2 \cdots x_n \rangle \in K$. Furthermore, each nonempty subset of $\{x_1, x_2, \ldots, x_n\}$ is also linearly independent and, consequently, each simplex associated with each such subset is also contained in K. Hence, all homology groups $H_i \cong 0$, $0 < i \leqslant n - 1$.

The intuitive content of Theorem 1 is that the abstract "space" associated with a controllable system Σ has no "holes." Thus, the appearance of an uncontrollable part in a system is geometrically equivalent to "punching" a hole in the underlying abstract space. We shall return to this interpretation later.

Since the notion of q-connectivity also relates to the manner in which the simplices of the complex are joined, it is not surprising that a result analogous to Theorem 1 can be obtained relating the structure vector Q and the concept of controllability.

THEOREM 2. *The system Σ is controllable if and only if the structure vector* $Q = (\overset{n-1}{1}1 \cdots \overset{0}{1})$.

Proof. If Σ is controllable, the complex K contains the single $(n-1)$-simplex $\sigma_{n-1} = \langle x_1 x_2 \cdots x_n \rangle$. Further, each face of σ_{n-1} is also in K. Hence, at the level $q = n - 1$ we have a single component σ_{n-1}. At level $q = n - 2$, we have the simplex σ_{n-1}, which is $(n-2)$-connected to each of its $n - 2$ faces and, as there are no other $n - 2$ simplices in K, we have only one component at the level $q = n - 2$. A similar argument holds for the remaining dimension levels $q = n - 3, n - 4, \ldots, 0$.

Assume $Q = (\overset{n-1}{1}1 \cdots \overset{0}{1})$. Thus, we have one component at the level $q = n - 1$, which must be the simplex $\langle x_1 x_2 \cdots x_n \rangle$, since this is the only $(n-1)$-simplex candidate for K. But, $\langle x_1, x_2 \cdots x_n \rangle \in K$ implies the set $\{x_1, x_2, \ldots, x_n\}$ is linearly independent, i.e. Σ is controllable.

It is of some interest to see the corresponding versions of the above result for the case of the conjugate complex $K_X(Y; \lambda^*)$. The general result is expressed by the next theorem.

THEOREM 3. *The system Σ is controllable if and only if the structure vector Q^* of the conjugate complex has the form*

$$Q^* = (\overset{n-1}{2^n} - 1, 2^n - 1, \ldots, 2^n - 1, \overset{\beta}{1}, 1, \ldots, 1, \overset{0}{1}),$$

where

$$\beta = \sum_{k=0}^{n-1} \binom{n-2}{k-1}.$$

Proof. At level k, the total number of simplices which can contain a given vertex is $\sum_{k=0}^{n-1} \binom{n-1}{k} = 2^n - 1$. Thus, given two vertices x_i, x_j, they may appear in β different simplices. Upon computation of the matrix $\Lambda'\Lambda - \Omega$, where Λ is the incidence matrix of the complex and Ω is the matrix all of whose entries equal one, we see that all diagonal entries are $2^n - 1$, while all off-diagonal elements are β. Since the (i, j) element of $\Lambda'\Lambda - \Omega$ denotes the dimension of the face shared by the ith and jth simplex in the conjugate complex, the result follows immediately.

It is of interest to note that the preceding result does not relate to the dual complex in the classical sense, i.e. it does not address the cohomological structure of the complex of K. However, since Theorem 1 shows that a controllable system Σ is homeomorphic to an n-dimensional manifold, i.e. to R^n, the Poincaré Duality Theorem [11] may be invoked to show that a controllable Σ also implies a trivial co-homological structure. The cohomology problem is intimately related to the problem of minimal "energy" required to reach a given state from the origin. Since the details are messy, we shall not consider the question here, but defer its consideration to a future paper.

6. INVARIANTS OF Σ

The preceding results enable us to make contact with another important algebraic aspect of system theory: the theory of invariants. As is well known in algebraic topology, the Betti numbers of a simplicial complex are (arithmetic) invariants with respect to homeomorphisms of the underlying space [11]. (Recall: the ith Betti number β_i is the number of free generators of the ith homology group H_i).

COROLLARY 1. *The system Σ is controllable if and only if the Betti numbers $\{\beta_i\}$ of the associated complex are $\beta_0 = 1$, $\beta_i = 0$, $1 \leqslant i \leqslant n - 2$. Furthermore, these numbers form a set of arithmetic invariants for Σ with respect to continuous coordinate changes in the state and control spaces.*

If we let α_i represent the number of i-dimensional simplices in the complex K, we may define the Euler characteristic χ of K as

$$\chi(K) = \sum_{i=0}^{n} (-1)^i \alpha_i, \qquad N = \dim K.$$

Since the Betti numbers and the Euler characteristic are related as

$$\sum_{i=0}^{N} (-1)^i \beta_i = \sum_{i-0}^{N} (-1)^i \alpha_i = \chi(K),$$

we immediately obtain a second corrollary of Theorem 1.

COROLLARY 2. *The system Σ is controllable if and only if the Euler characteristic $\chi(K) = 1$. In addition, the number $\chi(K)$ is a topological invariant of Σ.*

7. EXAMPLES

To illustrate the above results, as well as to provide a background for the much more complicated multi-input case, we present some numerical examples of controllable and uncontrollable systems.

EXAMPLE 1 (continusd). Here we consider the three-dimensional problem in control canonical form which was introduced in Section 4. The incidence matrix for this problem is

$$\begin{array}{c} \\ \\ \Lambda = (Y) \end{array}
\begin{array}{c} (X) \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array},$$

where the sets X and Y are taken in the order given earlier.

Computing $\Lambda\Lambda' - \Omega$, it is easy to see that the q-analysis provides

$$\begin{aligned} \text{at level } q = 2:&\quad Q_2 = 1 \quad \{\sigma_2{}^1\} \\ q = 1:&\quad Q_1 = 1 \quad \{\sigma_2{}^1\sigma_1{}^3\sigma_1{}^2\sigma_1{}^1\} \\ q = 0:&\quad Q_0 = 1 \quad \{\text{all}\}. \end{aligned}$$

Thus, the structure vector is $Q = (1\ \ 1\ \ 1)$, as predicted by Theorem 2.

Using the incidence matrix Λ, it is easy to see that the homological structure is trivial since all possible simplices of all dimensions are present in the complex. Thus, in confirmation of Theorem 1, we have $H_0 \cong J$, $H_1 \cong 0$, $H_2 \cong 0$ (here J is the set of integers).

EXAMPLE 2. Now consider the elementary uncontrollable system Σ defined by

$$F = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3), \qquad g = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

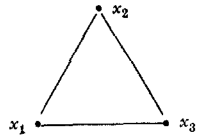
The controllability matrix is

$$\mathscr{C} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{bmatrix} = [x_1 \mid x_2 \mid x_3].$$

Thus, the complex of Σ is determined by the vertex set $X = \{x_1, x_2, x_3\}$. The simplices in K are

$$\begin{array}{lll} \sigma_0^1 = x_1, & \sigma_0^2 = x_2, & \sigma_0^3 = x_3, \\ \sigma_1^1 = x_1 \wedge x_2, & \sigma_1^2 = x_1 \wedge x_3, & \sigma_1^3 = x_2 \wedge x_3. \end{array}$$

Hence, there is no 2-dimensional simplex in K. Geometrically, we may depict the complex as



It is an easy exercise to verify that

$$\sigma_1^3 + \sigma_1^1 - \sigma_1^2$$

is a nontrivial 1-cycle in K; hence, there is a nontrivial homology at the 1-level. In this case, we see that the above 1-cycle is the sole generator of the homology group H_1. Consequently, the homological structure of K is

$$H_0 \cong J, \qquad H_1 \cong J$$

and we confirm Theorem 1, that the uncontrollable system Σ has a nontrivial homology.

Using the incidence matrix

$$\Lambda = Y \begin{array}{c|ccc|} & & X & \\ & x_1 & x_2 & x_3 \\ \sigma_0^1 & 1 & 0 & 0 \\ \sigma_0^2 & 0 & 1 & 0 \\ \sigma_0^3 & 0 & 0 & 1 \\ \sigma_1^1 & 1 & 1 & 0 \\ \sigma_1^2 & 1 & 0 & 1 \\ \sigma_1^3 & 0 & 1 & 1 \end{array},$$

we readily compute the q-structure as

$$\begin{aligned} \text{at level } q = 1:&\quad Q_1 = 3 \quad \{\sigma_1^3\},\quad \{\sigma_1^2\},\quad \{\sigma_1^3\}, \\ q = 0:&\quad Q_0 = 1 \quad \text{(all}\}. \end{aligned}$$

Thus, the structure vector is $Q = (3\ \ 1)$ in confirmation of Theorem 2.

Unfortunately, the simple classification scheme presented here for single-input systems cannot be extended to the multi-input case without further refinement, as the following examples show.

EXAMPLE 3.

$$F = \begin{bmatrix} 2 & 4 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Forming the controllability matrix, we find

$$\mathscr{C} = \begin{bmatrix} 0 & 1 & 4 & 6 & 4 & 8 & 12 & 20 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, Σ is not controllable.

Employing the same identifications as before, some elementary calculations show that the complex of Σ contains no simplices of dimension greater than $n = 2$. The homological structure can be calculated as

$$H_0 \cong J, \qquad H_1 \cong \underbrace{J \oplus J \oplus \cdots \oplus J}_{21\ \text{times}}.$$

EXAMPLE 4.

$$F = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The controllability matrix is

$$\mathscr{C} = \begin{bmatrix} 1 & 0 & 3 & 0 & 10 & 1 \\ 0 & 1 & 2 & 1 & 8 & 1 \\ 0 & 0 & 1 & 1 & 6 & 2 \end{bmatrix}.$$

Hence, Σ is controllable. The homological structure can be seen to be

$$H_0 \cong J, \qquad H_1 \cong 0, \qquad H_2 \cong \underbrace{J \oplus J \oplus \cdots \oplus J}_{10 \text{ times}}.$$

Thus, we have a nontrivial homology at the level $n - 1 = 2$, in contrast to the single-input result of Theorem 1.

EXAMPLE 5.

$$F = \text{same as in Example 3}, \qquad G = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here

$$\mathscr{C} = \begin{bmatrix} 0 & 1 & 5 & 6 & 3 & 8 & 19 & 20 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 0 & -3 & 0 & 9 & 0 & -27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and Σ is not controllable. After some algebra, it is found that Σ has the homological structure

$$H_0 \cong J, \qquad H_1 \cong 0, \qquad H_2 \cong \underbrace{J \oplus J \oplus \cdots \oplus J}_{31 \text{ times}}, \qquad H_3 \cong 0.$$

Thus, comparison of Examples 3–5 indicates that a more detailed analysis of the interrelation between the homological structure, the dimension of the state space and the number of system inputs will be needed to produce the multi-input results analogous to Theorem 1.

8. DISCUSSION

A number of interesting points for future investigation are raised by the preceding results. In this section we shall summarize some of the more pressing questions.

(a) *Multi-input linear problems.* A noted above, a more careful consideration of the complex of Σ will be required to precisely characterize the relationship between the controllable systems and the topological structure of the associated complex. It would seem that the notion of ***relative*** homology, using the subcomplexes generated by each input would provide a suitable starting point for such a study. The Mayer–Vietoris sequence [7] would presumably provide much of the information needed to isolate the relevant homological structure.

(b) *Nonlinear problems.* Recent work in geometric system theory [12] has shown that for important classes of nonlinear problems, it is possible to characterize the controllable manifold in an algebraic manner similar to the linear case. For instance, the system

$$\dot{x} = f(x, u), \qquad x(0) = x_0,$$

where f is an analytic function of its arguments, can be studied for controllability by examining the Lie algebra of vector fields generated by ***constant*** controls u. If we denote $f^k = f(x, u^k)$, where u^k is a constant control. it is shown in [12] that the controllable structure is algebraically characterized by linear combinations of elements of the form

$$[f^1[f^2[\cdots[f^i, f^{i-1}]\cdots]],$$

where [,] denotes the Lie bracket operation. Using the elements $\{f^i\}$ as as vertices, and objects of the above type as simplices, it is possible to construct a simplicial complex for Σ and to study its topological structure as above.

(c) *Duality.* An important aspect of classical linear system theory is the vector-space duality between controllability and observability. On the other hand, an equally significant duality exists in algebraic topology between homology and cohomology. It would be of some interest to look into how either of these duality theories interfaces with the conjugacy concept introduced by the relation λ^* discussed above.

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