DERIVATIVES AND STRUCTURED PRODUCTS

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<< The blue funk option >>

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I. Introduction – the intuition

The history has taught us that the financial agents, being human before all, are most of the time irrational. It is especially the case in times of financial stress when they panic and sometimes act unreasonably to even aggravate the situation. Indeed, due to the high tension on the market, a negative event could induce more assets sold at the same time, which could push the prices downwardly and resulting in even more losses for the market.
The fear (or the lack of confidence) of the investors is then a very important element to take into account, as when it is too high it could imply bad performance.

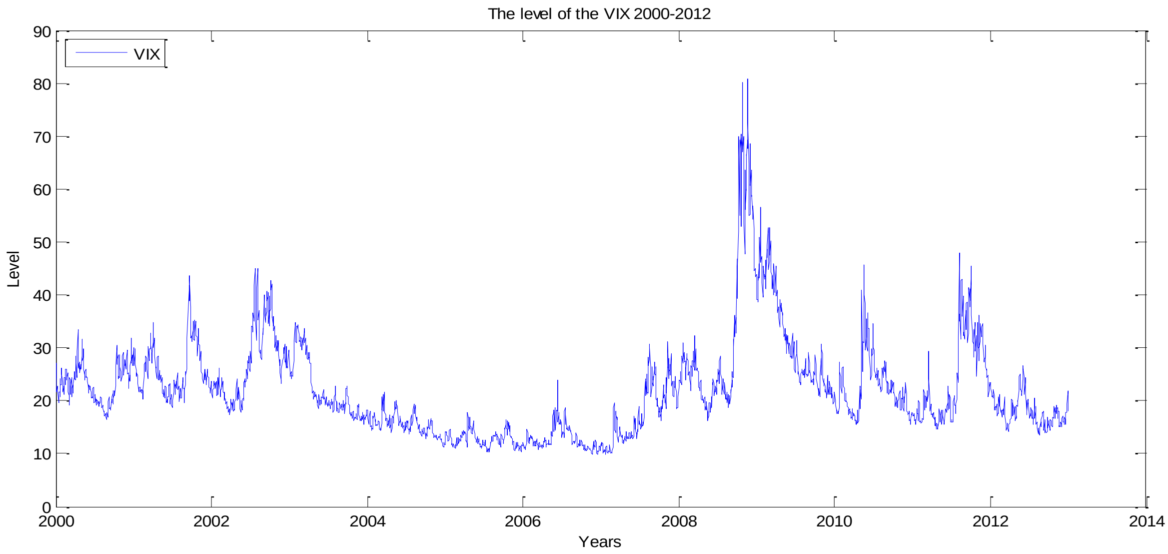
For our project, we wanted to design a product based on the level of fear of the market, inducing different payoffs depending on its level. For the product to be interesting it should generate good performance in critical times, i.e. when the fear is high. Indeed, a high level of fear supposes more risks felt on the market and thus potentially more bad performance generated by the financial assets. If the performance of our derivative is negatively related to the performance of the market, it could allow the investors to actually benefit from a rise in the fear in the market, i.e. gains instead of losses in bad situations. Due to their well known risk aversion, such a product could be very attractive to them, and we could benefit from that character when selling it.

That is the reason why we decided to name our option the **"blue funk" option** or the panic option, relatively to the fact that it does better when the whole market is shivering.

II. The "blue funk" option – the presentation

The underlying

The idea is then to find an index which could be a proxy for the lack of confidence of the market on which we could base our option, i.e. its underlying. For our derivative to be useable, the underlying should be a well-known index recalculated on a regular basis. That is the reason why we decided to choose the **VIX indicator** as the underlying for our product. It is a forward looking indicator, actually also called the "fear index", as it represents the expected volatility or risk felt on the market by the investors. It is calculated regularly using the implied volatilities of at-the-money puts and calls at a one month maturity. The VIX is usually oscillating around 20, higher values representing unusual nervousness felt on the market, and lower values meaning a relative calm and secured market. For example, we see in 2008 a huge peak due to the subprime crisis where the VIX attained its historical highest point of 80.



The payoff structure – an important precision

In the logic of our product, an element that interested us and needs to be presented here is the **"flight-to-quality" behavior** occurring in times of crisis. Basically, it means that when the markets are doing badly, the investors (being more risk averse) tend to cut their risky positions and take refuge in more secured assets like government bonds ("*safe havens*"). The demand for these assets should then increase when the market is in crisis and the VIX is high. Their increasing demand will push their prices upwardly and generate better performance. These types of assets are then interesting to us, as they tend to perform better when the market is down.

That is exactly what we want, and that is the reason we wanted to integrate this effect in some way in our payoff structure. Typically, the short term treasury bills issued by the US government are one of the most common secured investments, as they are often considered as being riskless. This is thus what we will consider in our project as a "safe haven". What is important to remark in this context is that, the asset being a bond, an increase in its demand and in its price (which is good for the holder) induces a *decrease* in the yield of the T-Bill.

The payoff structure – some characteristics

To take into account all elements and design a product which corresponds to our intuition, we decided to implement **two different barriers** in the value of the underlying (inside barriers). These two barriers will define knock-in and knock-out rules for different payoffs, offering then three different payoff zones corresponding to three different scenarios. The two barriers represented by K_1 and K_2 are **fixed** for the whole period of the option and correspond to two different levels of the underlying (the VIX), K_1 being smaller than K_2. These two barriers correspond to different scenarios in our framework. The higher the level of the VIX is the more secured and interesting should be the product.

The payoff zones

1. $\{0\}1_{V_T < K_1 ,\ V_t < K_2}$

2. $a * \{Max(V_t) - K_1\}1_{\ K_1 < V_T < K_2,\ \ K_1 < V_t < K_2}$

3. $b * Max\{Yield_t - Yield_{t+\Delta t}; 0\}1_{\ K_2 \leq V_t}$

$T = maturity\ of\ the\ product$
$V_t = the\ VIX\ at\ time\ t$
$Yield_t = the\ T - Bill\ yield\ at\ time\ t$
$K_i = the\ barrier\ i$
$a\ \&\ b = payoffs\ multiplicator$

The first zone is the negative part for the product owner, as he doesn't get anything. It happens when the VIX ends up under the first barrier at the maturity and it never touched the barrier K_2 . Intuitively, it corresponds to the scenario of the market being very calm at the end of the maturity of the product, and no real alarming event happening during the maturity of the product as it never touched the highest barrier. Indeed, the "blue funk" option being like a bet on an increase of the stress of the market, it seems logical that the investor doesn't get anything when the contingent event (crisis) doesn't happen. Actually, it is a loss for him as he loses the premium paid for the option initially.

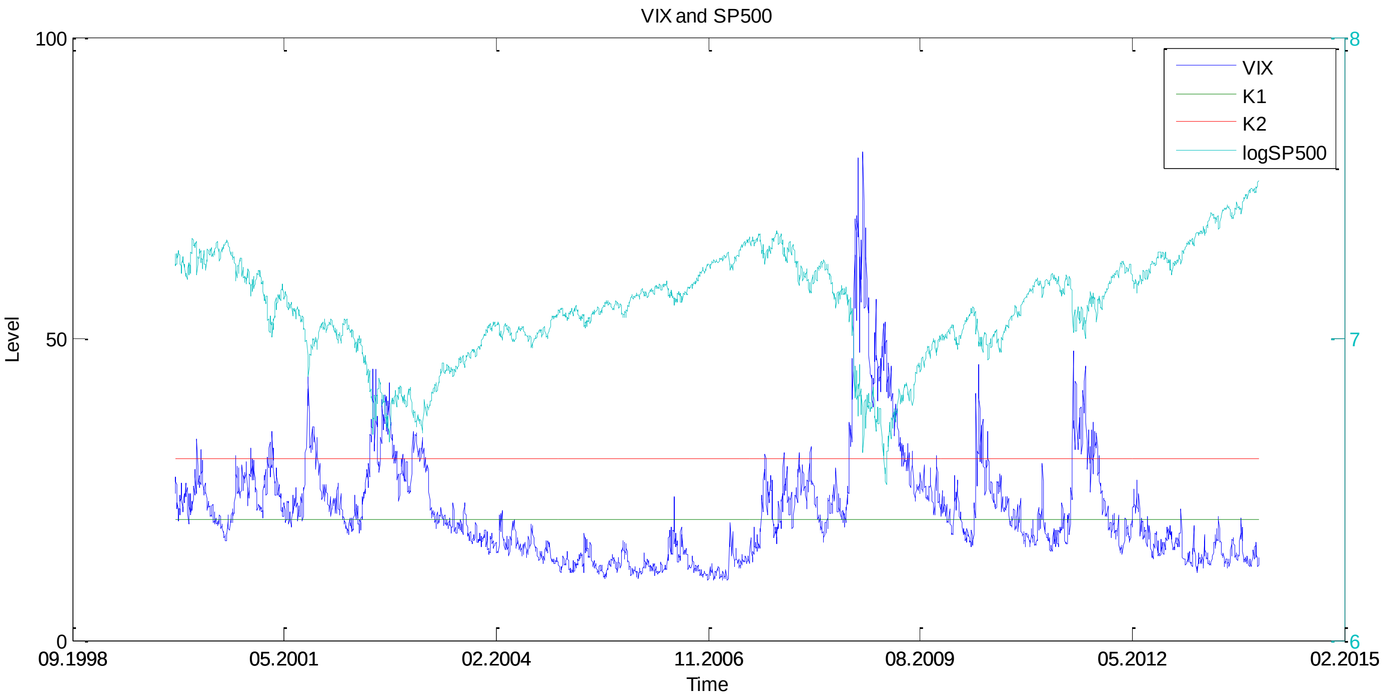
The second zone is like a corridor payoff, and is probably the most profitable part for the investor. If the VIX ends up between K_2 and K_1 and never touched K_2, the owner will receive the maximum value of the VIX minus K_1 times a, on some conditions. One thing important in this payoff is that the maximum value of the VIX is taken within the last values being higher than K_1 until the maturity. The logic behind that being that if the VIX attained high values at the beginning of the period, while crossing down the first barrier afterwards, it should not be taken into account. Every time the underlying goes below the first barrier, the payoff is canceled (*knocking-out* of the payoff). However, when the VIX is below K_1 and crosses it, it corresponds to the activation of the payoff (*knocking-in* of the payoff). Intuitively, it means that if the market suffered from a stressful event at the beginning of the period but calmed rapidly afterwards, it was only a minor event and the investors should not benefit from it. If the market is very calm at the end of the maturity of the option, the market is not in a dangerous state and the investors shouldn't get anything as the market is not in a fearful state. In the opposite, if the market was stressed for the whole period and at the end of the option the market is still stressed, the investors should benefit from that event as it corresponds to a persistent bad situation. The barrier K_2 is only a *knock-out* barrier as it totally blocks this payoff to switch to the third payoff zone.

The last payoff is linked to the concept of "flight-to-quality" behavior explained above. As the VIX attained the barrier K_2 during the maturity of the option, which is an alarming level of stress, it probably means that the market is in a state of crisis. Thus, the "safe haven" represented by the T-Bill should experience an increase in its price and a decrease in the yield. It then gives the owner the right to get the T-Bill yield difference between the day the VIX touched the barrier K_2 and this same yield at the end of the option maturity times a fixed multiplicator b. The more the yield decreases, the more money the investors make. The only condition to receive this payoff is that the VIX touches the barrier K_2 one time during the period. Of course, if the "flight-to-quality" behavior doesn't happen during that period and the yield doesn't decrease, the investors don't get anything neither. Hence, there is still a risk corresponding to this third payoff zone that the investors must be aware of. The "blue funk" option is then also a bet on the decrease of the yields resulting from the stressful time (the VIX touching K_2).

Contract features

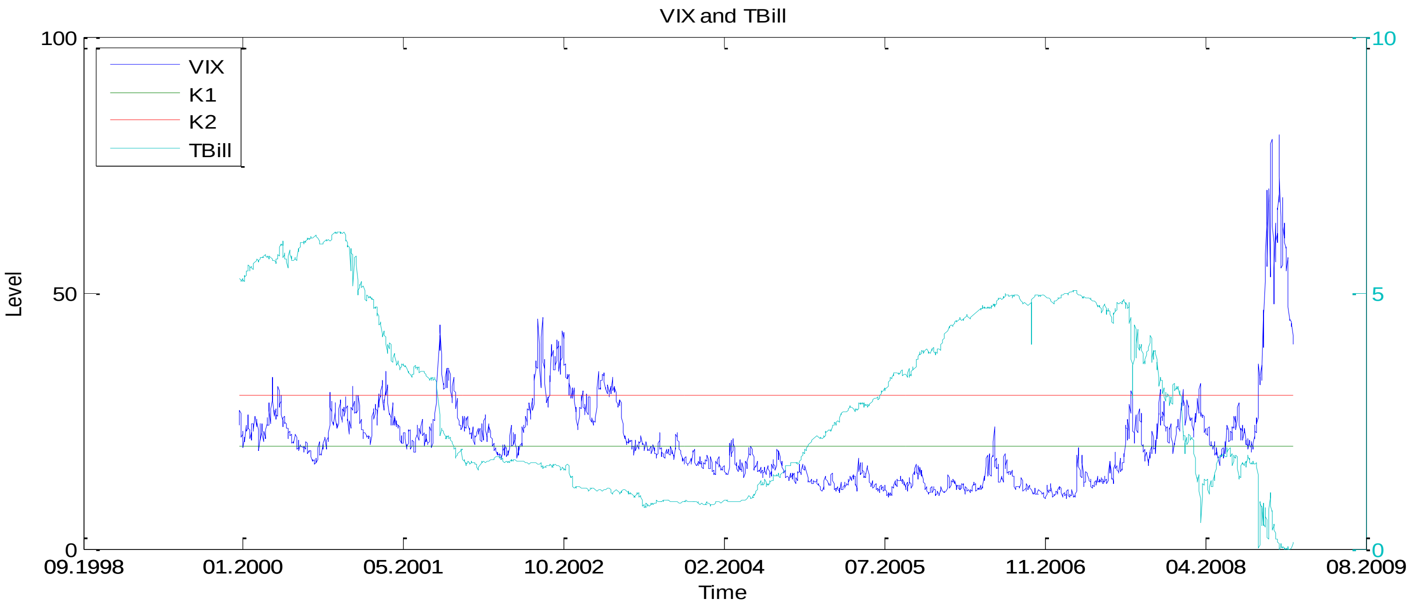
We now have to specify important features to be clear with the details of the product.

The first things we have to fix are the values of the barriers. They are fixed for the whole period and should efficiently correspond to the specific situations that we explained above. We decided to fix the barrier K_1 at 20 and the barrier K_2 at 30.



Indeed, by looking at the figure above, we see that the barriers correspond well to the scenarios we want to happen in both cases.

When the VIX is below 20, it tends to correspond to good performance of the S&P500. That means that our option should not earn anything in this range of values, as it is based on a hypothetic fall of the market. In the opposite, when the VIX is between 20 and 30, it corresponds to more tensed and riskier situations in which the market moves more dangerously. It could be interesting for the investors to get a good payoff in these situations. And when the VIX is above 30, it always corresponds to a fall in the S&P500.



When looking at the VIX and the T-Bill level for these periods, we see that there seems indeed to have a negative relation between them. When the VIX is above 30, The T-Bill tends to fall. It is not systematically the case, but when the event is alarming enough (like a crisis), it is followed by a decrease in the T-Bill yield in the periods following. For example, in the crisis of 2001 or 2008 we see that in both cases the VIX goes above 30 and the T-Bill yield falls afterwards. The risk of the non-decreasing of the T-Bill yield is then linked to the risk of the increase in the VIX to be not significant enough to touch the overall market.

The **multiplicators** of the payoffs (a and b), or the enhancement parameters, can be fixed at will. It must just be kept in mind that the ratio of the gains will always stay the same. In other words, they will boost the gains but also boost the price of the option, keeping the returns unchanged. We could perfectly imagine that they could change depending on the period or on the desire of the issuer. Of course, b should be particularly high, the changes in yield being probably in basis points. We decided to fix a at 10 and b at 100 to take into account the fact that the difference in the yield is in basis points. Plus, we want the third payoff part (when the VIX takes extreme values above 30) to be as profitable as possible.

As for the **maturity of the product**, we decided to fix it at 3 months, i.e. 63 trading days. It should not be too short, as the situation should change for the product to be interesting, but not too long as the investors must still be able to take small bets on the future without being too unsure.

As for the **amount invested**, it is only defined as the price of the option at time zero. The investors will probably have to buy at least a certain number of options to assure a certain initial amount of premiums reached initially. The number of options available depends on the issuer of the option. There is no loss possible for the product, except the loss of the initial price.

Environment profitable with the product

As we have seen before, our product is profitable for the owner in the cases when the financial stress is high, as the payoffs are linked to high level of VIX on the market.

Thus, the context profitable to the investors is a **negative context** for the market (bursting bubbles, bankruptcies). To invest in the product, the investors would then have to take a bet on a maintaining or at least a deterioration of the current situation. If the financial situation improves, the VIX will probably fall during the period, and the result would then be the *knocking-out* of all possible payoffs and imply the loss of the initial premium for the investors.

The most profitable payoff zone is probably the second payoff zone, when the VIX ends up situated between 20 and 30, as the investors simply get the maximum level of the VIX, which is a certain payoff. The third payoff zone, when the VIX touches the 30 barrier, depends on the occurrence of the decrease in the T-Bill yield (the gravity of the event). The more the “flight-to-quality” effect is important, the more the investors will earn money. However, if it does not happen, investors will not get money at all. The third payoff zone corresponds then to potential good payoffs in extreme cases when the whole market does very badly, but there is still a risk of no gains at all.
Of course, it depends on the multiplicators of the payoffs.

Client profile targeted

This product can be sold to every client type like private investors or institutional investors. In fact the price of the product is probably not that high, and the minimum amount of money required is very little for investors as they just need to buy the product. There is no particular client restriction. But as we have seen, the investors who could be interested in such a product are mainly clients who have negatives views about the market. It could be particularly interesting for very risk-averse agents who look for a way to diversify their investments and have an asset which is negatively correlated with the performance of the market. The utility resulting from gains generated in bad situations is indeed much higher than the same gains generated in overall good situations.
Individually, these gains can be risky and costly to replicate for the individual investors. On the other hand, the purchase of the option is very easy. So it could be particularly interesting for basic investors without a lot of time or means to invest in the implementation of a strategy.

Product advantages and defaults

The first advantage is the negative correlation with the market, as we have seen before, i.e. the ability to generate good returns in cases of bad returns by the market.
Another interesting thing is that our product can bring a solution to shorting puzzle problems. For example, there exist some strategies which consist in mostly shorting assets, thus betting on their fall. However, it could be very costly to short, and it is not possible for everybody. Our product could represent a way of betting on the fall of the market without taking the risks related to the actual shorting.
And a good aspect that our product offers is a way to bypass liquidity problems when the market is in a big stress. When the market is much stressed, the VIX is very high. In these contexts we said before that there is a “flight to quality” behavior. Investors sell stocks to rush in government bonds which make yield very profitable. The problem in these situations is to find buyers for the risky positions that investors want to sell. It has to be done at the perfect time, and it is not always possible to sell the positions at the right moment and take directly refuge on more secured assets. It is here that our product can be useful. Investors can profits of these “flight to quality” and they don’t need to rush and take fast decisions before the others. They just need to wait the maturity of the option and take their profits.

To conclude, our product can interest to various investors who own a negative perception of the market. But more precisely, the product can be very useful for investors with shorting restrictions and for illiquidity context

III. The pricing and hedging of the product

Now that we defined its important features and the different situations linked to them, it is necessary to have a good technical approach on the product in order to price it and hedge it correctly. The price must be fair enough for the investors to buy it, while not too low in order for the issuer to make a profit on it.

Models assumptions – the VIX

Our structured product needs first a stochastic model to represent the underlying, i.e. the VIX. We represent its dynamic by the log-normal Ornstein-Uhlenbeck model from *Detemple & Osakwe* (2000). Under the real probability measure the model is:

$$dlogV_t = k[\theta^P - logV_t]dt + \sigma dW_t^P$$

And under the risk-neutral probability measure, the model is:

$$dlogV_t = k[\theta - logV_t]dt + \sigma dW_t^Q$$

Where $\theta = \theta^P - \sigma\delta/k$.

Here θ is the long run mean, k the parameter of speed reversion, σ the volatility and δ the market price of risk.

There are two main reasons of why we chose this model to represent the VIX dynamic. First, by looking at the historical data, we see that the VIX shows a mean-reversion effect. It can increase or decrease but one time or another it will always come back to its long term mean. It is due to the nature of the VIX; it captures the business cycles of the economy, and after a crisis there should always be a period of recovery where the market calms itself. The log- normal Ornstein-Uhlenbeck process takes that effect into account. One another important characteristic of the VIX is that it is always positive. It represents the implied volatility of options, which is always above zero by definition. The logarithm in the dynamic of the log-normal Ornstein-Uhlenbeck guarantees us that the VIX's simulated are always positive.

One drawback resulting from our model is that it doesn't show the persistence that the VIX has in reality. Indeed, when the market is bearish for example, history showed us that the VIX can experience a big increase in a small period and then stay at this level for a certain time. Our model will not take this persistence into account, and will mean-revert almost directly.
Other characteristics that are not captured by our model are the potential jumps, stochastic volatility and central tendencies that the VIX can experience. There exists some models that can take into account this type of behavior and which can thus be maybe more accurate to model the VIX. But with this increase in the realism comes usually an increase in the difficulty of implementation. Indeed,

most of these models require the estimation of many parameters, and we could face several problems like the curse of dimensionality. The log-normal Ornstein-Uhlenbeck delivers a good performance with respect to the constraints that we are facing. Actually, *Javier Mencia and Enrique Sentana* (2012) tested some VIX models in the context of option pricing models and they found out that the log-normal Ornstein Uhlenbeck actually achieves an acceptable performance.
In the context of our project, we preferred opting for a simpler model that we could implement all by ourselves rather than choose a model more complex but impossible to implement without exterior helps.

Model assumptions – the T-Bill yields

An additional difficulty coming from our project is that we also have to model the interest rates, i.e. the T-Bill yields. The model we used for that is the *Vasicek* model which is actually also an Orstein-Uhlenbeck process. The dynamic under the real probability is:

$$dr_t = a(b - r_t)dt + \sigma dW_t^P$$

And under the risk neutral probability measure, the model is:

$$dr_t = a(\bar{r} - r_t)dt + \sigma dW_t^Q$$

Where a is the speed of mean-reversion, $\bar{r}$ is the long run mean and σ is the constant volatility of the interest rates.
This model is rather simple but it is very useful when it comes to interest rates simulations. Indeed, the model takes into account the fact that short term rates can't increase infinitely through the mean reversion behavior. The most useful thing in this model is that it gives a closed-form for interest rates and the bond prices which is an interesting tool in the context of derivatives pricing. However, the main drawback resulting from this model is that there is a positive probability that the interest rates become negative. It could be problematic as we know that it is not the case in reality. Fortunately, it is a very unlikely event and the Vasicek model is still often used for modeling the interest rates, but mathematically it could happen.

Calibration of the models – the estimation of the parameters

Now that we have our models, we need to estimate their parameters. The sample we chose to calibrate the models and estimate the parameters for both variables is the period from 1990 to 1998. The reason of this choice is that we wanted a period without major crises or irregular events which could have induced biases in our estimations. Indeed, if we took the period from 2000 to 2010 for example, it would have implied very high values in the simulations, as the real VIX for this period was abnormally high due to the subprime crisis in 2008. For the T-Bill yields, it would have been also disturbing, as the yields for most of the period are abnormally low.
We used the method of maximum likelihood to estimate the parameters of both processes. We computed the log-likelihood function analytically and used Matlab for the maximization.

For the VIX, the log-likelihood function is given by:

$$\sum_{t=0}^{T} -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1a^2}{2\sigma^2}$$

$a = logV_{t+1} - logV_t - k(\theta - \frac{\sigma\delta}{k} - logV_{t+1})$

For the short rates, the log-likelihood is:

$$\sum_{t=0}^{T} -\frac{1}{2}\log(2\pi v) - \frac{1m^2}{2v}$$

Where v and m are the respectively the conditional variance and mean of the process, whose formulas are given by:

$v = \frac{\sigma^2}{2a^3}(4e^{-a} - e^{-2a} + 2a - 3)$

$m = r_{t+1} - (\bar{r} + \frac{1}{a}(1 - e^{-a})(r_t - \bar{r})$

Calibration of the models – the correlation

As we have two stochastic processes and thus two different Brownian motions, it is necessary to estimate the correlation between the two. In the context of our product, we decided to implement a changing correlation. Indeed, our basic intuition is that the correlation should increase negatively when the VIX hit the second barrier (30). In that case, as the market is probably in a state of crisis, the T-Bill should experience a decrease in its yield due to the “flight-to-quality” behavior and to the potential lowering of the interest rates by the central banks. The yields should then decrease when the VIX is increasing above 30.

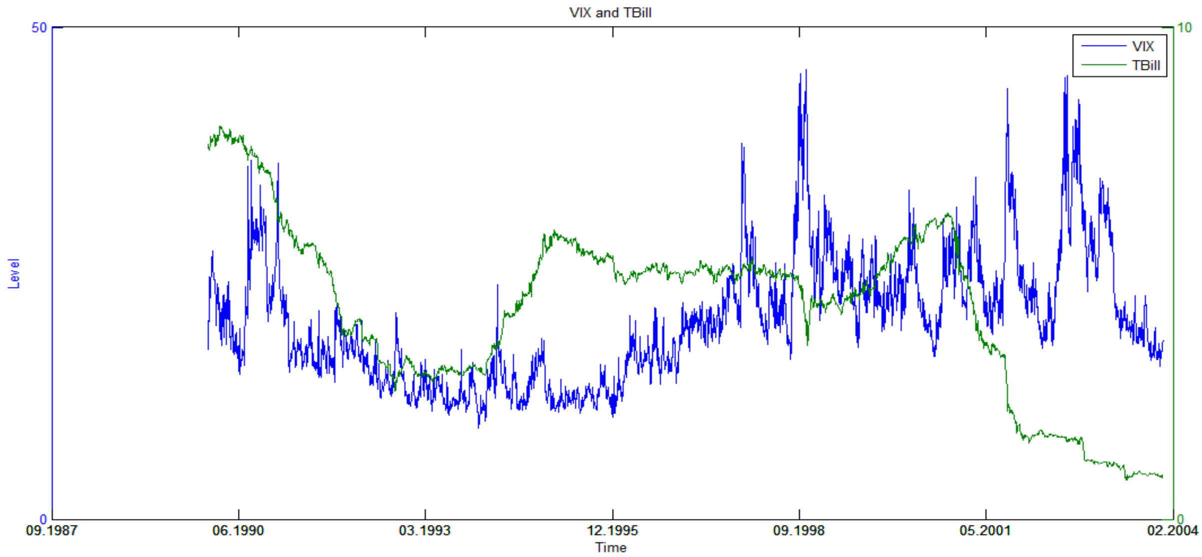
Mathematically, we can represent the correlation between the two Brownian motions in this way:

$dlogV_t = k[\theta - logV_t]dt + \sigma_V dW_t^{1Q}$,

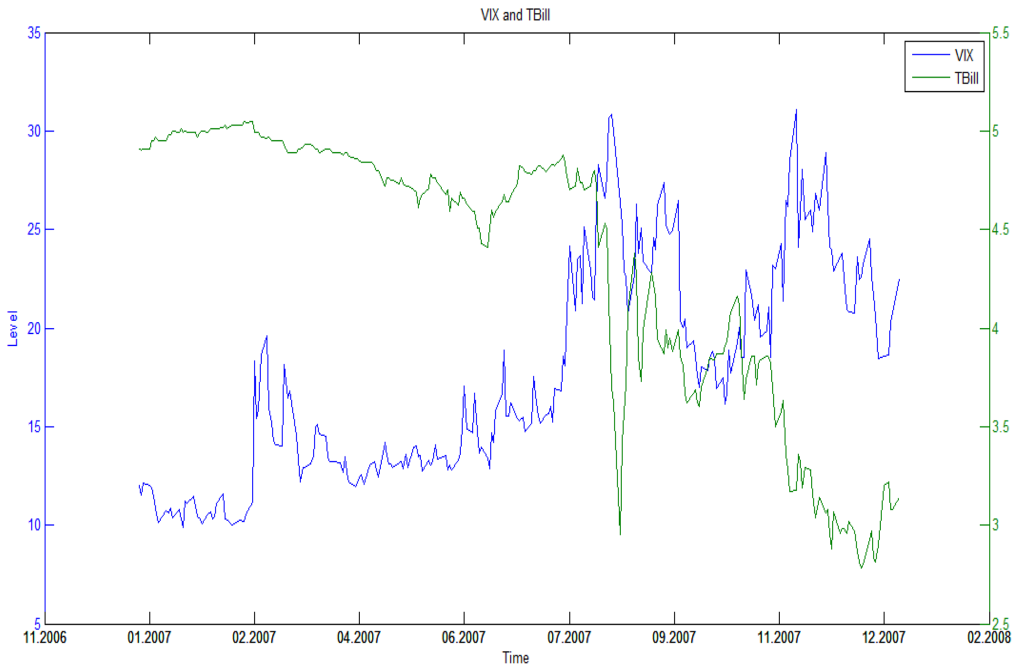
$dr_t = a(\bar{r} - r_t)dt + \sigma_r(\rho_{Vr} dW_t^{1Q} + dW_t^{2Q} + (1 - \rho_{iVr}{}^2)^{\frac{1}{2}})$.

Where $\rho_{iVr}$with i=1,2

There are two different correlations to estimate: the first one when the VIX is below 30 in regular situations, and the second one which is for extreme situations when the VIX is above 30. We could have imposed the correlation coefficients, but we wanted them to be realistic. Hence, we decided to estimate them using two different sample periods. The regular correlation is estimated using a sample from 1990 to 2004 to capture the general correlation between the two variables. Then, for the extreme correlation (when the VIX touches 30), we used the limited sample period of the whole year 2007 to get the behavior in crises. We find that ρ_{Vr1}≈ 0 and ρ_{Vr2}= -0.7387, which corresponds to our intuition and is perfectly consistent with the economic theory.



Above is represented the "normal" market period for the VIX level and the T-Bill yield. It is difficult to find any significant relation between the levels of the two, which implies probably a zero correlation.



In the opposite, in the period of high stress, we observe a radical change in the correlations. As soon as the VIX hit the 30 barrier, almost every time the VIX increases, the T-Bill decreases. There is a strong negative relation between the two.

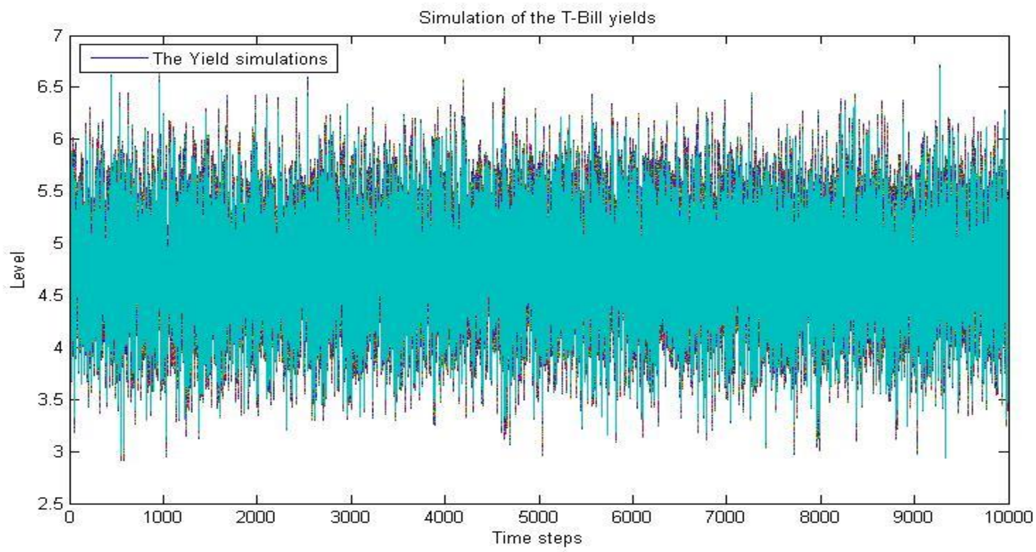
The simulation

Now that we have their parameters, we can start simulating the two processes. It is necessary for the pricing with **Monte Carlo methods**, which are based on the simulation of a very large number of paths for the stochastic processes.

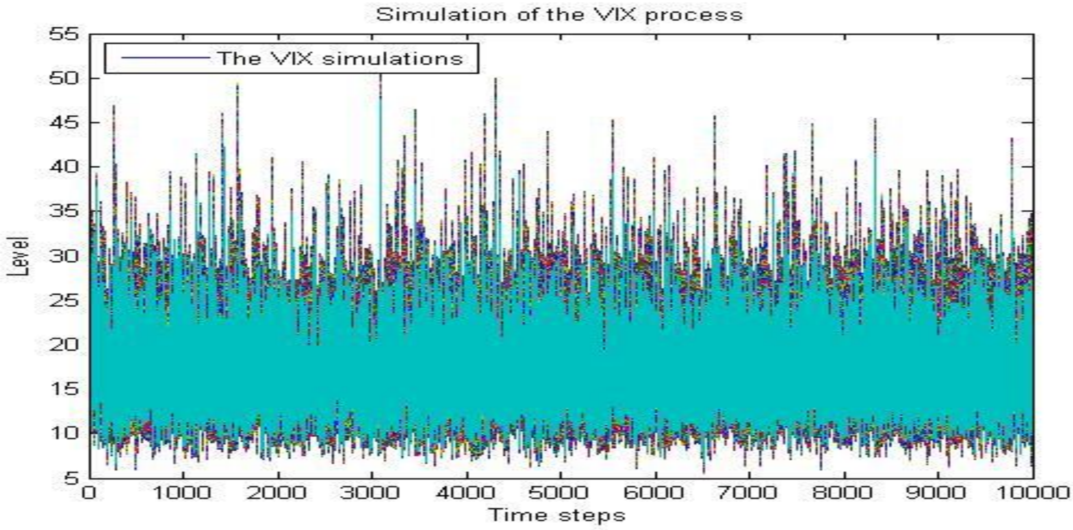
The paths are simulated through their dynamics with the help of a discretization scheme and an initial value to start on. What we used as initial values for both processes (VIX and the T-Bill yield) are the means of both on the sample considered for the estimation. There are different discretization schemes useable but in the context of simulations, it is always important to keep in mind the constant *trade-off* between the efficiency and the rapidity. Indeed, computers and programs have limited abilities when it comes to simulating a huge number of processes. It should be taken into account when deciding how many paths to simulate or what method use for it. Of course, the higher the number of simulations is and the more precise (efficient) will be our estimation. However, if that number is too high, it could be almost impossible to do it numerically. It is the same thing for a method very precise, but which requires multiples steps for the computer. It is especially the case when using matlab which is very slow when it comes to creating big loops.
With that in mind, we opted for the **Euler discretization** as it was proven to provide good results for small time steps.

The first important parameter to fix for the simulation is the number of observations. It is especially critical when pricing a complicate derivative product with multiple payoffs like ours. We decided to generate 10'000 simulations of the paths of both processes, which should be able to generate a satisfactory result. The maturity of the option being 63 days, we decided to divide it by 10'000 giving us time steps of 0.0063 trading days each. One trading day being around 7 hours, one time step corresponds to approximately 3 minutes. It should be small enough to give us good results for the Euler discretization, while still being able to be proceeded through Matlab.
Of course, as our option is complex it should be better to simulate more paths and even smaller time steps to get reliable results for the pricing, but we couldn't do it with our limited means.

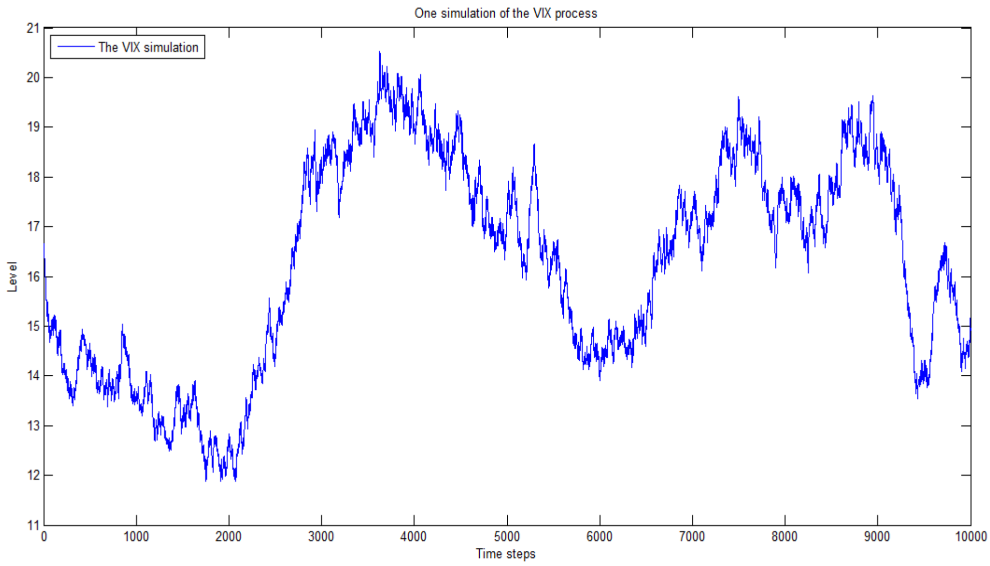
It is of course very important to generate both the paths at the same time, i.e. with the same random Brownian motions. Indeed, the payoff structure of our product depends on both processes at the same time, so it can't be done separately. Of course, as the simulations are random, they must be different every time a simulation is done. Below are examples of simulations for both the yield process and the VIX process with 10'000 observations and 10'000 time steps. The initial value for both processes simulations is the mean of the sample.



We see here that the simulations of the T-Bill yields are within a range of 3% to about 7% annually and centered around the mean of the sample of 4.89%. They are rather high with respect to the rates today which are especially low but we wanted to get a result more general which could be applicable for regular times.



As for the simulations of the VIX paths, we see that they are also centered around their mean of 17.0488, which is also the start of the simulations. They look very volatile, with several values being higher than 30 and some even touching 50. It seems pretty realistic, even though some aspects of the VIX are not taken into account with the process we used, like the persistence. It is necessary to keep in mind that there are 10'000 simulations, so it is not surprising to get some a few very high values, which could perfectly correspond to serious crises.



We also added above one example of a path amongst the 10'000, which can help better visualizing what the process really gives us. Here, we see that the VIX never goes above the first barrier (20), so it could correspond to a very calm period where no stress is felt in the market. In that environment, our option would pay nothing to the investors.

Of course, the simulations themselves being random, they can be very different each time they are simulated. Fortunately, there are several techniques that we can use to reduce the variance of the Monte Carlo methods and improving the accuracy of the estimations. One of them is the antithetic variate, which basically consists in generating for each path another one which is perfectly uncorrelated with it, i.e. its antithetic path. Hence, we have to simulate 10'000 new observations as we have to do it for the whole sample. The perfect negative correlation is implemented by taking the negative of the value of the random terms of the initial paths for the random terms of the antithetic paths. Due to the negative covariance between the two, the estimation variance of a function depending on them (the payoffs in our case) decreases.

The pricing

We have now come to probably the most important part of our project which is the pricing of the option. It is done with the help of all the observations that we simulated and what we call **Monte Carlo methods**.

Basically, the idea is that the price of a claim is simply the expectation of its payoff under Q, discounted on the maturity of the payoff to convert it in today money. Thus, we will first calculate the payoffs related to each simulation and we will discount them on the maturity of the option (three months, so one quarter on an annual basis). Then, we simply take the expectation of all these payoffs discounted (which corresponds to the mean of all payoffs) to get a single price. With enough observations, it should be close to the reality on the hypothesis that we chose the right dynamics for the processes.

In our framework, the T-Bill yields are stochastic and are driven by a *Vasicek* process. As they are often considered as the reference for the riskless rate, we can use them to **discount** the payoffs. The difficulty relative to that is that the level of the rate depends on the time. Thus, we can't simply use a fixed rate in our model; it must be integrated over the maturity of the payoff. However, the advantage for the *Vasicek* model is that there is a closed-form for this integral taking the form of the price of a zero-coupon bond. We can then use this price to discount all the payoffs.

$$E^Q[\exp\left(-\int_0^t r_s ds\right)] = P(r_0, t-0) = \exp(-A_{t-0} - B_{t-0} r_0)$$

$$B_{t-0} = \frac{1}{a}(1 - e^{-a(t-0)})$$

$$A_{t-0} = \big((t-0) - B_{t-0}\big)\bar{r} - \frac{1}{2} v_{t-0}$$

$$v = \frac{\sigma^2}{2a^3}(4e^{-a(t-0)} - e^{-2a(t-0)} + 2a(t-0) - 3)$$

As we have three different payoff zones, it is essential to take into account the indicator functions. When one payoff is realized (the indicator function of the payoff is one), then the other two must not be realized (the indicator function of the payoff is zero) to get the accurate probabilities corresponding to each event. It is very important when calculating the payoffs not to have "crossing" between the payoff zones, as for each simulation the sum of the probabilities linked to each event must be equal to one. In our context, what is important to take into account is that each time the VIX touches 30, the payoff of the option is locked and the indicator functions for the two other payoffs are automatically zero. If it does at one date, what will matter for the payoff is then the simulations of the T-Bill yields between this moment and the maturity of the option as the investor gets the difference (cf. the marketing part). For the second payoff zone, it is also important to take into account only the maximum of the VIX the last time it is above the first barrier, i.e. 20, and not for the whole period. Indeed, if it crosses the barrier there is the *knock-out* of the payoff. We must implement that effect in some way in our loop.

If it is done properly, we can price each payoff zone separately. We have a vector of 10'000 payoffs for each payoff zone, being positive when the indicator condition is satisfied and zero otherwise. We then discount each of these payoffs to get a vector of discounted payoffs, and we calculate the mean of this last vector to get the price for this payoff zone. We go through exactly the same process for the antithetic simulations, and take the mean of the regular price and the antithetic price for the two payoff zones.
Finally, we add up both these prices to get the price of the whole option.

Price Bond feature	Price VIX feature	Total Price
0.1142	2.1858	0.1142+2.1858= 2.3

Here is one example of the price of our option. Of course, every time a new simulation is made, a new price is found. However, with the precautions we took, it should not be too much volatile.
What we see is that the VIX feature (the second payoff zone) is more priced than the Bond feature of our option (the third payoff zone). It is not surprising as the VIX going above 30 is much less frequent than the VIX ending up between 20 and 30. It has an effect on the pricing through the indicator functions of the payoffs. The VIX feature is much more priced, counting for more than 80% of the price of the whole option. It is not surprising either, as if the VIX ends up between the first and second barrier the payoff is much more certain than for the Bond case. This effect is adding up to the fact that it happens more frequently.
Overall, we can say that the option is not very expensive. It is probably related to the fact that in a lot of the cases when the conditions are not satisfied, the option doesn't pay anything to the investors. However, when they are satisfied, the payoff can be very high for the investors. For the VIX feature, the maximum payoff observed with our simulations is 99.8824 for each option, which is far from negligible for an option costing only 43.3077 at time zero. For the bond feature, the maximum observed is 61.5948. It corresponds to a decrease in the yield of 0.60 for the period considered.

The hedging of the product

Our product may be quite tricky to hedge because of the complex payoff structure. In fact we have to find vehicles that are tradable and allow hedging our product.

To hedge against **yield variation**, the firm can buy directly the T-Bill. When the VIX touches the second barrier, we have to buy the T-Bill itself in order to profit from the increase in the price. With the gains on the prices, we will then pay the holders of the options who get the difference in the yields. Of course, the number of T-Bill that we have to buy depends on the multiplicator of the payoff, as we will have to give b times the variation of the yield to the investors.

We also have to find a way to trade on the VIX to hedge against **the VIX variation**. Indeed, the VIX is not tradable. There are several funds which try to replicate the VIX performance like exchanged-traded notes (ETN) and exchanged-traded funds (ETF). We could choose for example the ETN named *iPath S&P500 VIX Short-Term Futures ETN (VXX)* which give exposure on long the first or the second month VIX future contracts. An important thing to understand is that these ETN and ETF on the VIX are only approximations of the VIX performance. They don't replicate directly the VIX level. We have to take into account the "contango" effect which is the difference between future contracts and spot prices, i.e. in our case the VXX and the VIX. As the buyer of the VXX, we pay a premium, so it creates additional costs of hedging. It is necessary to take this fact into account when hedging and maybe transferring this premium to the clients by making them pay a higher premium on the option.

The strategy that could be used by the firm to hedge variations of the yield and the VIX is thus to trade the T-Bill and the VXX by a **delta neutral hedging strategy**. This strategy implies the construction of a portfolio which hedges against the variations of the stochastic parameters (the VIX, the rate). The hedging will be achieved through long and short positions on the VIX and the T-Bill depending on the level of the VIX.

The portfolio could be constructed like that:

$$\pi = -C(logV_t, r_t, t) + \Delta_1 logV_t + \Delta_2 r_t.$$

With π = portfolio

C = the option

Δ_i = the quantities of VIX and T-Bill to buy/short

This gives us the Partial Differential Equation of the portfolio:

$$\begin{aligned}-C_t dt - C_V\big(k[\theta_V - logV_t]dt + \sigma_V dW_t^{1Q}\big) - C_r(a(\bar{r} - r_t)dt + \sigma_r\big(\rho_{Vr} dW_t^{1Q} \ + dW_t^{2Q}\ )) \\ + (1 - \rho_{iVr}{}^2)^{\frac{1}{2}}) - \frac{1}{2} C_{VV}\sigma_V^2 dt - \frac{1}{2} C_{rr}\sigma_r^2 dt - C_{Vr}\sigma_V\sigma_r\rho_{Vr} dt \\ + \Delta_1\big(k[\theta_V - logV_t]dt + \sigma_V dW_t^{1Q}\big) + \Delta_2(a(\bar{r} - r_t)dt + \sigma_r(\rho_{Vr} dW_t^{1Q} \ + dW_t^{2Q} \\ + (1 - \rho_{iVr}{}^2)^{\frac{1}{2}})\end{aligned}$$

To be perfectly hedge $\Delta_1 = C_V \ and \ \Delta_2 = C_r$ which are the deltas of the option with respect to the two variables. If we estimate the derivative, it will give us the positions to take in the goal of hedging the structured product. Ideally, the portfolio should be rebalanced at least daily to capture the risk resulting from the payoff that we designed. However, because of computing limitations, we limited ourselves to a rebalancing every month, and we estimated the delta variation for three dates (t=0, t=21, t=42), using our simulations. We created a function of pricing with several inputs to allow ourselves to calculate the different prices depending on the initial values of the VIX and the yield rate. The deltas are estimated depending on the level of the VIX of the period (smaller than 20, between 20 and 30, and higher than 30). It is important to notice that these are all simulations, so there can be a lot of variations from one simulation to another.

$$\Delta V = \frac{C(V+\varepsilon) - C(V)}{\varepsilon}$$

$$\Delta r = \frac{C(r+\varepsilon) - C(r)}{\varepsilon}$$

ε is +2 for the VIX and +0.5 for the bond.

Delta VIX	V<20	20<V<30	30<V
t=0	0,2032	0.0283	0,0466
t=21	0,7174	-0,3811	0,0589
t=42	2,8418	0,5573	0,0972
Delta Yield	**V<20**	**20<V<30**	**30<V**
t=0	-1,5595	-0,8222	-0,0619
t=21	-1,7666	-1,1750	0,0778
t=42	-2,2621	-2,2375	0,0690

We can see that when the V<20 and 20<V<30, the deltas are very high when time is close to the maturity. In fact a change in the VIX this moment has a lot of chance to give a payoff to the client because we are close to the maturity and the VIX can pass the first barrier. But when 30<V, the deltas are low which is logical because when the VIX crosses the barrier of 30, the payoff doesn't depend anymore of the VIX but of the bond. So the risk for the firm which sells these products has to be very efficient in term of trade when the maturity is close and when the VIX fluctuates under 30. And for the change in the dividend yield, we can see that an increase in the yield one month after has in general a negative impact in the price of the product. It is normal because if the yield increases the client is in the loss.

IV. Conclusion

By looking at the results, we can say that we were not really surprised. Indeed, it is a very particular option, and it pays only in some cases. That induces small prices for the two features, especially for the bond one. But overall, we were pretty satisfied with the results as they correspond to our intuitions. The next step would be to know if the investors would really invest in this type of structured product, but in the context of the project, we were pretty happy to find consistent results. A point on which we were less satisfied is the fact that we couldn't find a closed-form solution for our product. It is mainly due to the fact that the payoff structure is too complex, and even though we maybe had possibilities to do it, there was always something that could not be computed. Given the time restrictions that we had, we had to give up to rather relying on Monte Carlo simulations.

Of course, we are aware of the fact that our method could be improved. For example, the number of observations is a little limited, especially for a complex option like ours. However, the complexity of our option is also what made it difficult to implement numerically. Matlab was quickly reaching its limits when running the loops and it took us some time to obtain the results in some cases. The modeling of the VIX process could have also been improved, with taking into account other characteristics of the VIX, like a stochastic volatility part for example.

Nevertheless, we preferred choosing elements which were within our reach and that we could do fully ourselves from the beginning to the end rather than doing things we didn't really understand.

On a more personal level, we can say that the same reason that made the project difficult made it also interesting. Most of the things we had to do in the context of the project were new to us, and we definitely learned from it on many aspects.