Derivatives and Structured Products

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Course outline

Part I: Pricing and hedging

1. A review of discrete time option pricing
2. Martingale pricing - Black and Scholes model
3. Hedging
4. Options on non-traded assets
5. Monte Carlo simulation for option pricing

Part II: Structured equity derivatives

1. Special features
2. Index-linked cash flows
3. Financial architecture and problem solving
4. Applications

Course organisation

- Class notes available on Moodle
- Office hours: by appointment (tony.berrada@unige.ch)
- Office hours: Friday 14-16 : Maxime Auberson (office 311)
- Lecture on monday w/ me
- Exercises on wednesday w/ Maxime

Evaluation

- 40 % : written exam
- 60 % : project and presentation

Session plan

1. Introduction and Definitions
2. Payoff Structure
3. Forward price
4. Rational bounds for option prices

Introduction and Definitions

- Derivative: A derivative is a financial contract which provides its holder with a payoff stream which depends on the past and or current value of one, or many, underlying assets
- Underlying: An underlying asset may actually be anything you want: a stock, an interest rate, a currency, an index, any combination of the previous.
- Forward Contract: A forward contract is an agreement for future delivery of an underlying at a specified date (the maturity), price (the forward price) and location.
- Option: An option is a contract which grants its holder the right, but not the obligation, to buy or sell an underlying, or a function of the current and past value of the underlying, at a specified price (the strike price), on or before a specified date (the maturity).

Introduction and Definitions

- Call An option which grants its holder the right to buy an underlying at a specified price (the strike price), on or before a specified date (the maturity).
- Put An option which grants its holder the right to sell an underlying at a specified price (the strike price), on or before a specified date (the maturity).
- European Option A European option can be exercised only at the maturity date.
- American Option An American option can be exercised at or before the maturity date.
- Bermudean Option An Bermudean option can be exercised at different predetermined dates prior to maturity and at the maturity date.

Introduction and Definitions

- The use of derivatives falls into one of the following two categories

Hedging

- The idea is to reduce (or eliminate) the financial risk associated with a future uncertain situation
    - Protective put to limit downside risk
    - Forward contract to lock in a future sale

Speculation

- Investment with (a priori) positive expected profit

Course plan

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Payoff at maturity

- Forward Contract with delivery price K

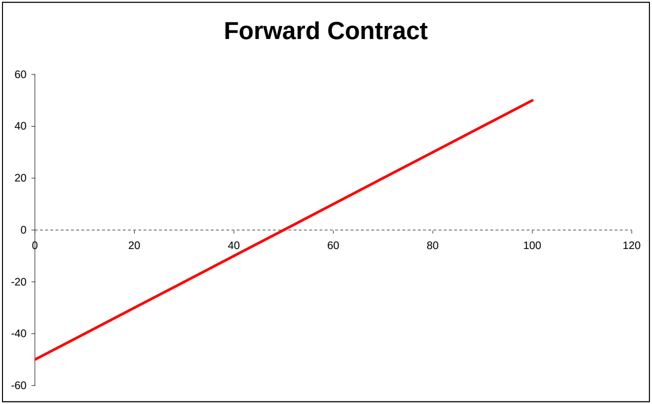
$$\boxed{(S(T) - K)}$$

- Call Option with exercise price K

$$\boxed{(S(T) - K)^+}$$

- Put Option with exercise price K

$$\boxed{(K - S(T))^+}$$


Forward Contract
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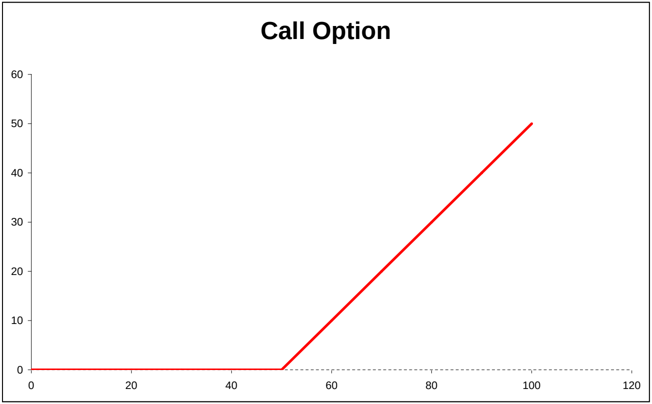
Call Option
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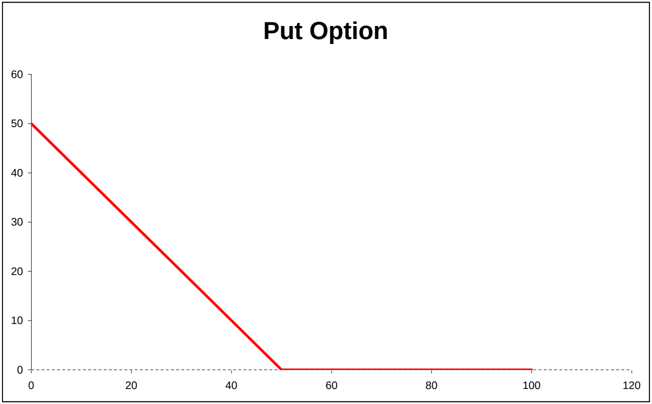
Put Option
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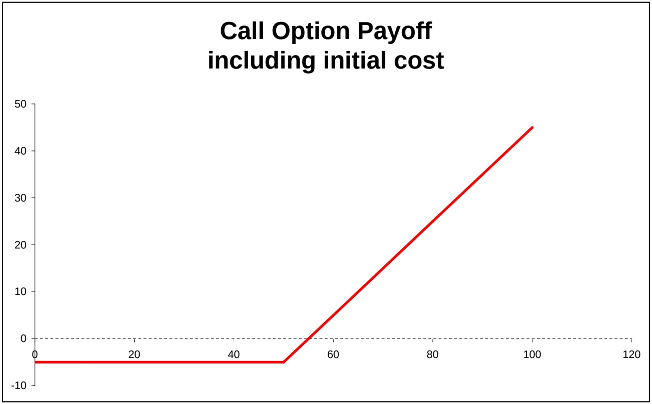
Call Option Payoff
including initial cost
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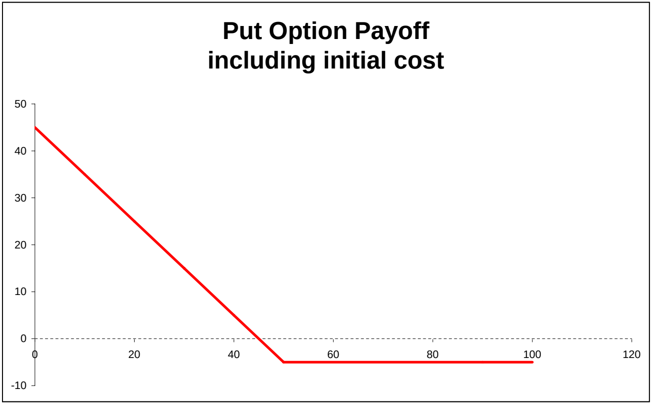
Put Option Payoff
including initial cost
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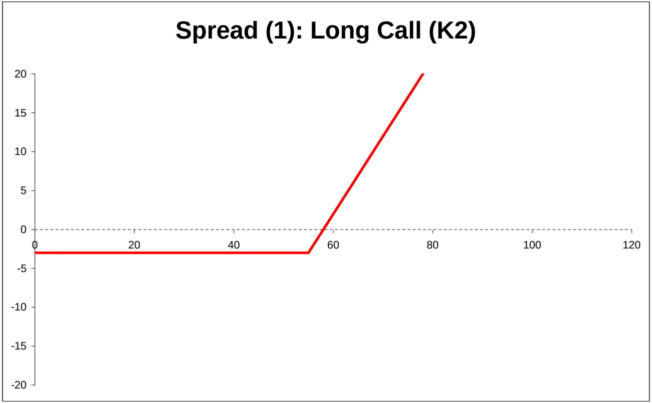
Payoff Structure

- It is possible to combine derivatives to create particular payoffs corresponding to a particular view of the evolution of the market
- Derivatives allow to make 2-dimensional bets, in term of market direction (bullish / bearish) or in terms of market volatility
- Typical examples of option combinations are ($K_1 < K_2 < K_3 < K_4$):

Spread: $[C(K_2) - C(K_1)]$

Straddle: $[C(K_1) + P(K_1)]$

Butterfly Spread: $[C(K_1) + C(K_4) - C(K_2) - C(K_3)]$,



Spread (1): Long Call (K2)
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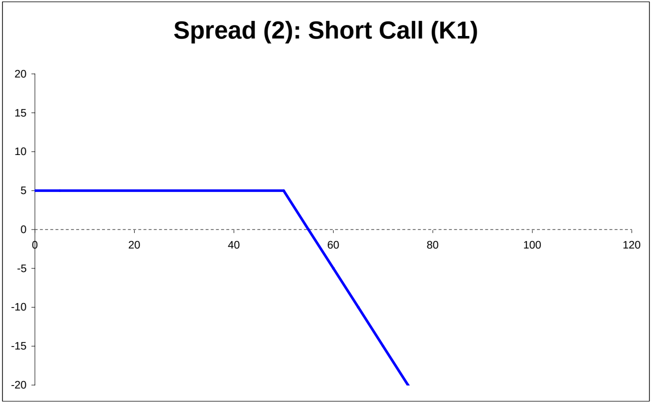
Spread (2): Short Call (K1)
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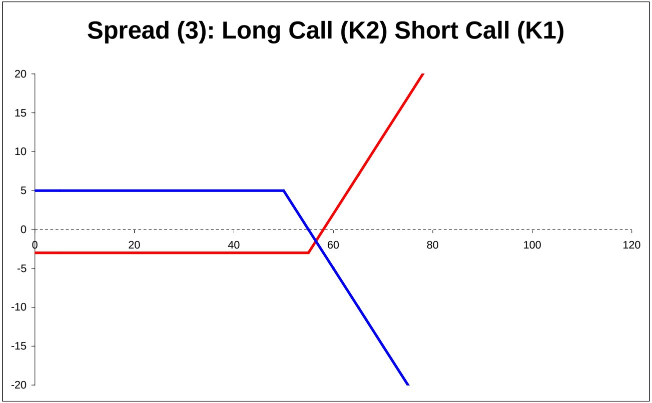
Spread (3): Long Call (K2) Short Call (K1)
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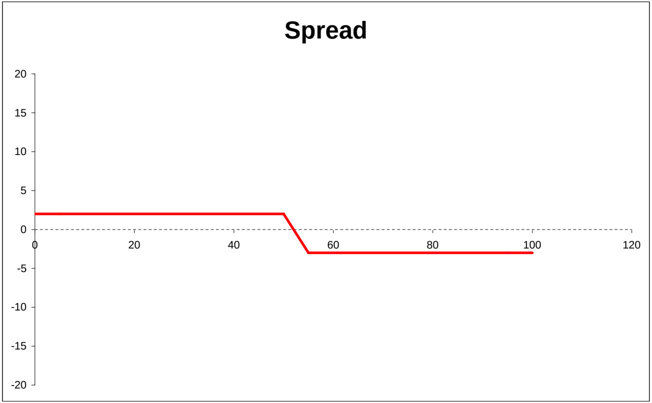
Spread
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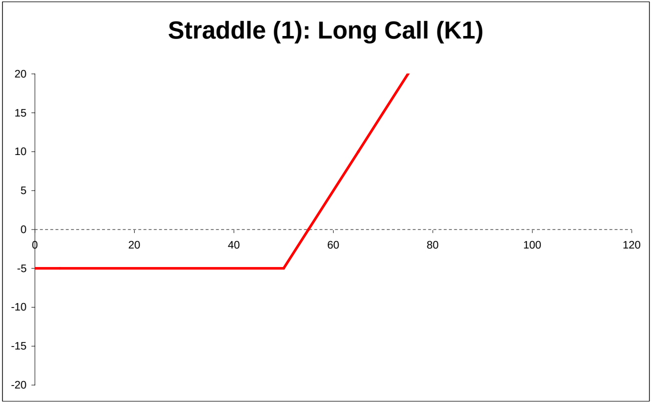
Straddle (1): Long Call (K1)
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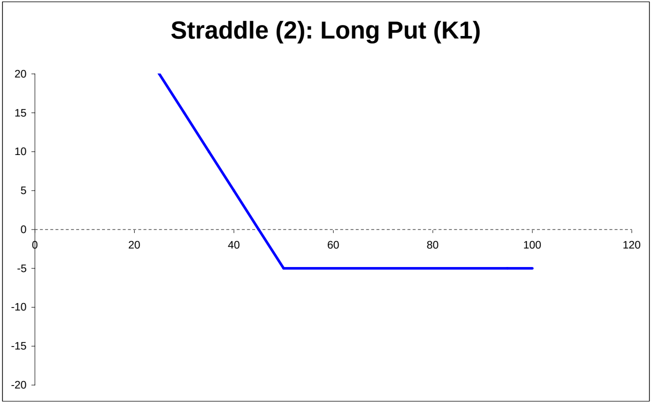
Straddle (2): Long Put (K1)
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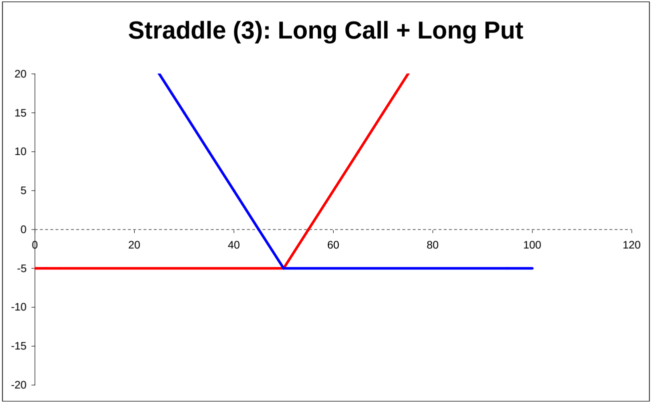

Straddle (3): Long Call + Long Put
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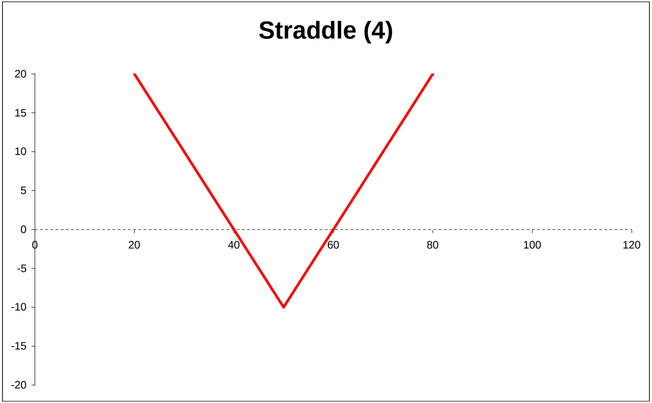

Straddle (4)
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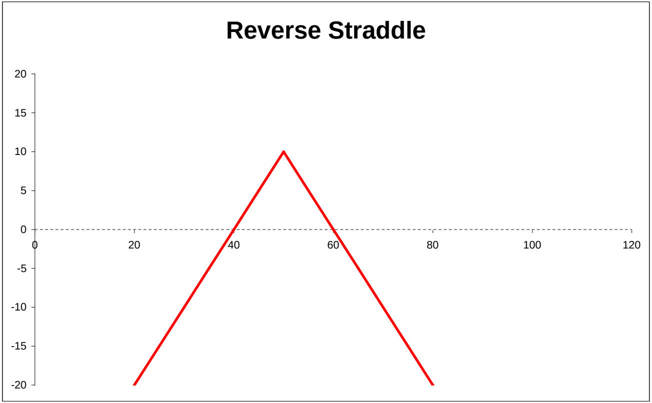

Reverse Straddle
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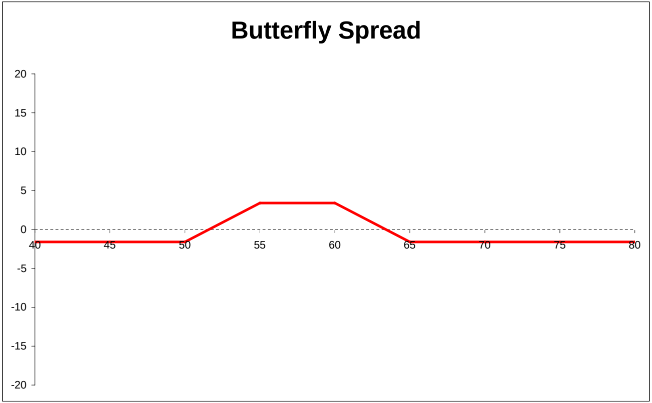
Butterfly Spread
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Course plan

Forward Price

Arbitrage

An arbitrage strategy requires no (positive) initial investment, never yields a negative terminal value and has a strictly positive expected terminal value

In finance, we price derivatives by assuming that there are

NO ARBITRAGE OPPORTUNITIES

- Introducing a new derivative in the market at price P should not give rise to the possibility of an arbitrage

Forward Price

- Let $S(t)$ denote the underlying stock with price at time t
- Let $F(t,T)$ denote the forward price at time t for delivery at time T
- Let r denote the constant, continuously compounded, interest rate
- The arbitrage free forward price is given by the formula

$$\boxed{F(t,T) = S(t)e^{r(T-t)}}$$

Forward Price

- We will verify that this is indeed an arbitrage free price by setting up a strategy that would generate an arbitrage for any other price
- This strategy is called a (Reverse) Cash and Carry
- Given a delivery price K the long position payoff is

$$\boxed{S(T) - K}$$

- the payoff to theshort position is

$$\boxed{K - S(T)}$$

Cash and Carry

time 0	time T
Long $S(0)$	$S(T)$
Short $F(0,T)$	$F(0,T) - S(T)$
Borrow an amount $S(0)$	$-S(0)e^{rT}$
$\boxed{0}$	$\boxed{F(0,T) - S(0)e^{rT}}$

Reverse Cash and Carry

time 0	time T
Short $S(0)$	$-S(T)$
Long $F(0,T)$	$S(T) - F(0,T)$
Lend an amount $S(0)$	$S(0)e^{rT}$
$\boxed{0}$	$\boxed{S(0)e^{rT} - F(0,T)}$

Forward Price

- If $F(0,T) > S(0)e^{rT}$ setting up a Cash and Carry generates a positive payoff without any risk

$$\boxed{\Rightarrow F(0,T) \leq S(0)e^{rT}}$$

- If $F(0,T) < S(0)e^{rT}$ setting up a Reverse Cash and Carry generates a positive payoff without any risk

$$\boxed{\Rightarrow F(0,T) \geq S(0)e^{rT}}$$

Forward Price

- The absence of arbitrage opportunities implies

$$\boxed{F(0,T) = S(0)e^{rT}}$$

- This result is valid provided that there are no short sales restrictions
- Notice that we have not assumed anything about the evolution of the underlying stock price between time 0 and time T
- This is a pure arbitrage result, and it is valid for any evolution of the stock price

Forward Price

- When the underlying security generates an intermediate flow of capital (dividends,...) the forward price formula is modified
- Let us denote by D the certain dividend payment to be received at time t
- The arbitrage free forward price is then given by

$$\boxed{F(0,T) = [S(0) - e^{-rt}D]e^{rT}}$$

- This result is obtained using a slightly modified Cash and Carry strategy

Cash and Carry

time 0	time t	time T
Long $S(0)$	$+D$	$S(T)$
Short $F(0,T)$		$F(0,T) - S(T)$
Borrow $[S(0) - e^{-rt}D]$		$-[S(0) - e^{-rt}D]e^{rT}$
Borrow $e^{-rt}D$	$-D$	
$\boxed{0}$		$\boxed{F(0,T) - [S(0) - e^{-rt}D]e^{rT}}$

Value of a Forward Contract

- Assume that at time t we hold a long position in a forward contract that was entered at time 0, with delivery price $F(0,T)$
- What should the value of this position be at time t ?
- We can guarantee a terminal payoff at time T by taking a short position at time t
- the guaranteed payoff will be

$$\boxed{[S(T) - F(0,T)] + [F(t,T) - [S(T)] = [F(t,T) - F(0,T)]}$$

Value of a Forward Contract

- Since this terminal payoff is riskless, its value at time t is

$$\boxed{e^{-r(T-t)}[F(t,T) - F(0,T)]}$$

- This is the value of the forward contract entered at time 0 which we can further simplify to obtain

$$\boxed{S(t) - e^{-r(T-t)}F(0,T)}$$

Course plan

1. Introduction and Definitions
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4. Rational bounds for option prices

Rational bounds for option prices

- It is possible to obtain forward prices with pure arbitrage arguments
- Option prices can not be obtained similarly
- Assumptions must be made regarding
    1. The statistical properties of the underlying asset
    2. The market's valuation of risk
- We can however provide upper and lower bounds for the price of simple derivatives, irrespective of these assumptions
- Again these bounds will rely on arbitrage arguments

Put-Call Parity

- Consider a european call c and a european put p written on the same underlying S, with the same maturity T, and the same exercise price K
- Consider a portfolio composed of a long position in the call option and a short position in the put option
- The terminal payoff is $(S(T) - K)$ for all possible $S(T)$
- The payoff of this strategy corresponds to a forward contract with delivery price K
- It follows that by absence of arbitrage the following must hold

$$\boxed{c(0) - p(0) = S(0) - Ke^{-rT}}$$

Rational Bounds

- Let $C(t)$ denote the price at time t of an American Call, and $P(t)$ the price of an American put.
- An american option can be exercised at any time prior to maturity, which would generate the following payoff for a Call

$$\boxed{(S(t) - K)^+}$$

- It then follows from the absence of arbitrage that the american option's price is larger than its intrinsic value

$$\boxed{C(t) \geq (S(t) - K)^+)} \qquad \boxed{P(t) \geq (K - S(t))^+)}$$

Rational Bounds

- This is however not the strictest bound
- An american option grants strictly more right than an otherwise identical european option, so is must be the case that

$$\boxed{C(t) \geq c(t)} \qquad \boxed{P(t) \geq p(t)}$$

- Using the Put-Call parity, we obtain

$$\boxed{C(t) \geq c(t) = p(t) + S(t) - Ke^{-r(T-t)}}$$

Rational Bounds

- We also know that an option can only have a positive price and that it is necessarily bounded from above by the underlying price, for a call, and by the strike price for a put so we finally obtain

$$\boxed{S(t) \geq C(t) \geq c(t) \geq \left(S(t) - Ke^{-r(T-t)}\right)^+}$$

$$\boxed{K \geq P(t) \geq p(t) \geq \left(Ke^{-r(T-t)} - S(t)\right)^+}$$