# Derivatives and Structured Products

Session 2

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GEnWeM
GEneva Master in Wealth Management

# Session plan

# One Period Model

- We have shown that a unique arbitrage free price for an option can not be obtained using pure arbitrage arguments only
- We introduce here a model for the evolution of the underlying asset
- Let us first recall the notion of arbitrage which we will use throughout this discussion

**Arbitrage**

An arbitrage strategy requires no (positive) initial investment, never yields a negative terminal value and has a strictly positive expected terminal value

# One Period Model

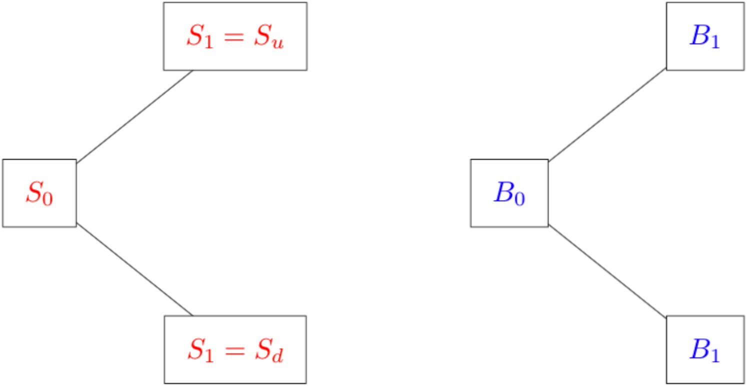
- The binomial market consists initially of two assets

a risky stock

a risk free bond

- Beginning of the period: time $0$, end of the period: time $1$
- The price of the stock at time $0$ is $S_0$
- The price of the bond at time $0$ is $B_0$
- The price of the stock at time $1$ is a random variable $S_1 \in \{S_u, S_d\}$
- The price of the bond at time $1$ is a non random number $B_1$

# One Period Model



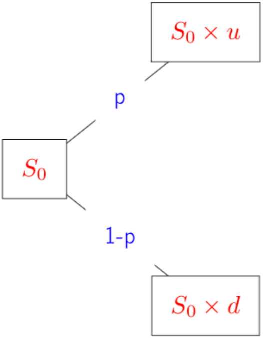
# One Period Model

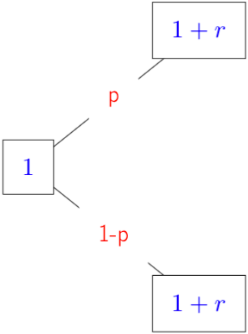
- Define the risk free interest rate as $1 + r = \frac{B_1}{B_0}$
- The risky return on the stock is $u = \frac{S_u}{S_0}$ or $d = \frac{S_d}{S_0}$
- To prevent arbitrage we need to assume

$$\boxed{d < 1 + r < u}$$

- Let us call $p$ the probability of observing $S_1 = S_u$
- $1 - p$ defines the probability of observing $S_1 = S_d$
- We further normalize the model by setting $B_0 = 1$

# One Period Model



# Course plan

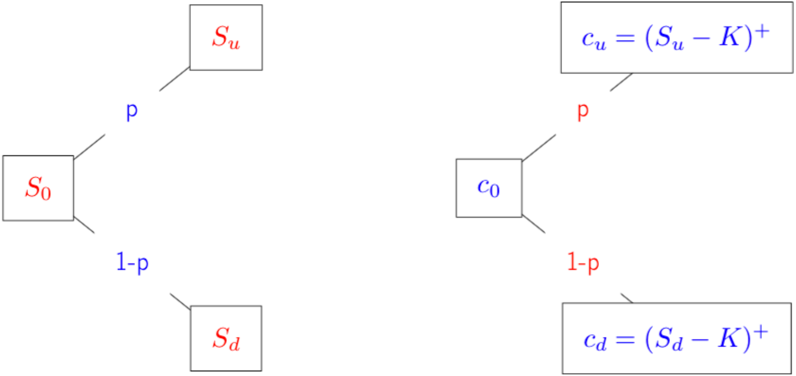
# Replication

- Consider a European Call option written on the underlying asset $S$ with exercise price $K$ and maturity at date $1$
- The payoff of the option is given by

$$\boxed{c_u = (S_u - K)^+} \quad \text{or} \quad \boxed{c_d = (S_d - K)^+}$$

- We write $c_0$ the price of the option at time $0$
- While the payoff is unambiguous, what should be the price at time $0$, such that there are no arbitrage opportunities ?

# Replication



# Replication

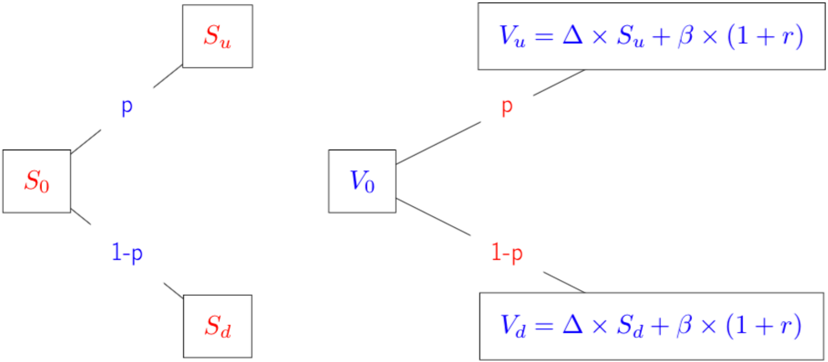
- portfolio $V$ has $\Delta$ shares of the stock and $\beta$ shares of the bond
- $\beta$ shares of the bond = $\beta$ \$ at the risk free rate
- The value of the portfolio at time $0$ is given by

$$\boxed{V_0 = \Delta \times S_0 + \beta}$$

- The value of the portfolio at time $1$ is given by

$$\boxed{V_u = \Delta \times S_u + \beta \times (1 + r)} \qquad \boxed{V_d = \Delta \times S_d + \beta \times (1 + r)}$$

# Replication



# Replication

- Can we choose the portfolio composition such that its value at time 1 is always equal to the payoff of the option ?
- $(\Delta^*, \beta^*)$ must be such that

$$c_u = (S_u - K)^+ = \Delta^* \times S_u + \beta^* \times (1 + r)$$

$$c_d = (S_d - K)^+ = \Delta^* \times S_d + \beta^* \times (1 + r)$$

- The value of this portfolio must be equal to the price of the call to prevent arbitrage and thus

$$\boxed{c_0 = \Delta^* \times S_0 + \beta^*}$$

# Replication

- Solving the previous linear system of equations yields

$$\boxed{\Delta^* = \frac{c_u - c_d}{S_u - S_d}}$$

$$\boxed{\beta^* = \frac{S_u c_d - S_d c_u}{(S_u - S_d)(1+r)}}$$

- The price of the European Call option at time $0$ is given by

$$\boxed{c_0 = \frac{c_u - c_d}{S_u - S_d} S_0 + \frac{S_u c_d - S_d c_u}{(S_u - S_d)(1+r)}}$$

- We can generalize this result to any derivative with payoff $(D_u, D_d)$

$$\boxed{D_0 = \frac{D_u - D_d}{S_u - S_d} S_0 + \frac{S_u D_d - S_d D_u}{(S_u - S_d)(1+r)}}$$

# Course plan

# Risk Neutral Probability

- Collecting terms in $D_u$ and terms in $D_d$

$$D_0 = \frac{1}{1+r}\left[\left(\frac{S_0(1+r)-S_d}{S_u-S_d}\right)D_u + \left(\frac{S_u-S_0(1+r)}{S_u-S_d}\right)D_d\right]$$

- $S_u = S_0 \times u$ and $S_d = S_0 \times d$ imply

$$D_0 = \frac{1}{1+r}\left[\left(\frac{(1+r)-d}{u-d}\right)D_u + \left(\frac{u-(1+r)}{u-d}\right)D_d\right]$$

- $\left(\frac{(1+r)-d}{u-d}\right) > 0$ and $\left(\frac{u-(1+r)}{u-d}\right) > 0$, and $\left(\frac{(1+r)-d}{u-d}\right) + \left(\frac{u-(1+r)}{u-d}\right) = 1$

# Risk Neutral Probability

- Define $q := \frac{(1+r)-d}{u-d}$ , we can then interpret the pricing equation as the expectation of the derivative's payoff, discounted at the risk free rate

$$D_0 = \frac{qD_u+(1-q)D_d}{1+r} := \frac{1}{1+r}E^Q[D_1]$$

- The probability measure $q$ is referred to as a risk neutral measure or martingale measure
- Under this measure the expected return of any asset in the binomial model is the risk free rate

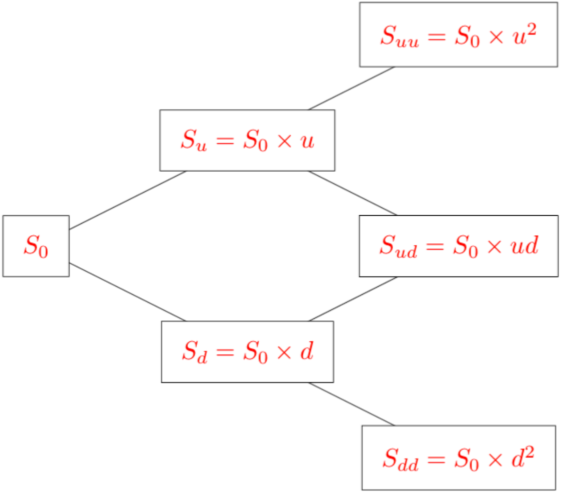
$$S_0 = \frac{1}{1+r}E^Q[S_1]$$

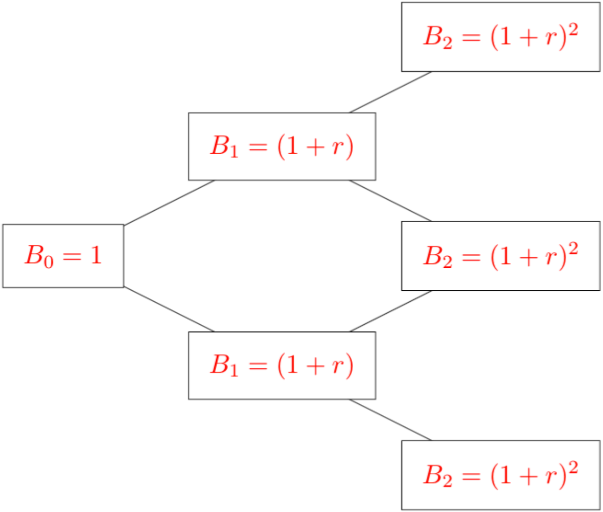
# Course plan

1. One Period Model
   - Replication
   - Risk Neutral Probability
2. Multi-period Model
   - Dynamic Replication
   - Risk Neutral Probability
3. Elements in probability
4. Martingales

# Multi-period Model

- We extend the binomial setting to a finer grid
- The concepts developed in the one-period model still hold
    1. It is possible to replicate the payoff of a derivative with a specific portfolio strategy
    2. The price of a derivative security is the expected value of the payoff discounted at the risk free rate under the risk neutral measure
- We still consider 2 assets, a bond $B$ and a Stock $S$
- The bond yields a risk free return of (1+r) per period
- The evolution of the stock is described by the following tree


$S_{uu} = S_0 \times u^2$
$S_u = S_0 \times u$
$S_0$
$S_{ud} = S_0 \times ud$
$S_d = S_0 \times d$
$S_{dd} = S_0 \times d^2$


$B_2 = (1+r)^2$
$B_1 = (1+r)$
$B_0 = 1$
$B_2 = (1+r)^2$
$B_1 = (1+r)$
$B_2 = (1+r)^2$

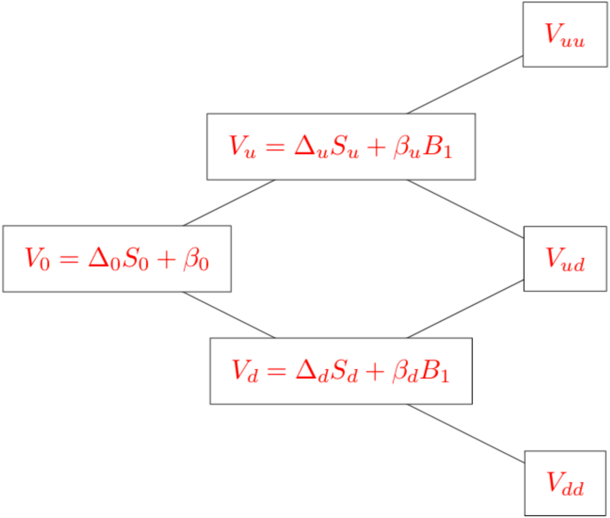
# Course plan

# Dynamic Replication

- A portfolio starting at time zero with $(\Delta_0, \beta_0)$ can be rearranged at any latter point in time
- We focus our attention on portfolios satisfying

$$(\Delta_{t+1} - \Delta_t)S_{t+1} + (\beta_{t+1} - \beta_t)B_{t+1} = 0$$

- We call these portfolio self-financing, as after time $0$ they do not require any further cash inflow
- We solve the replication problem with a self-financing portfolio


$V_{uu}$
$V_u = \Delta_u S_u + \beta_u B_1$
$V_0 = \Delta_0 S_0 + \beta_0$
$V_{ud}$
$V_d = \Delta_d S_d + \beta_d B_1$
$V_{dd}$

# Dynamic Replication

- Self-financing in this case means

$$\Delta_0 S_u + \beta_0 B_1 = \Delta_u S_u + \beta_u B_1$$

$$\Delta_0 S_d + \beta_0 B_1 = \Delta_d S_d + \beta_d B_1$$

- This set of conditions implies

$$(\Delta_u - \Delta_0) S_u + (\beta_u - \beta_0) B_1 = 0$$

$$(\Delta_d - \Delta_0) S_d + (\beta_d - \beta_0) B_1 = 0$$

# Dynamic Replication

- We consider a European derivative with maturity at date $2$
- The payoff is given by some function $D_2(S_2) \in \{D_{uu}, D_{ud}, D_{dd}\}$
- We solve the replication problem backward, starting from the terminal date

$$
\begin{aligned}
D_{uu} &= \Delta_u S_{uu} + \beta_u B_2 \\
D_{ud} &= \Delta_u S_{ud} + \beta_u B_2 = \Delta_d S_{ud} + \beta_d B_2 \\
D_{dd} &= \Delta_d S_{dd} + \beta_d B_2
\end{aligned}
$$

# Dynamic Replication

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\end{aligned}
$$

$$
\begin{tikzcd}[arrows=dash]
 & & \boxed{D_{uu} = \Delta_u S_{uu} + \beta_u B_2} \\
 & \boxed{V_u = \Delta_u S_u + \beta_u B_1} \arrow[ur] \arrow[dr] & \\
\boxed{V_0 = \Delta_0 S_0 + \beta_0} \arrow[ur] \arrow[dr] & & \boxed{D_{ud} = \Delta_u S_{ud} + \beta_u B_2} \\
 & \boxed{V_d = \Delta_d S_d + \beta_d B_1} \arrow[ur] \arrow[dr] & \\
 & & \boxed{V_{dd}}
\end{tikzcd}
$$

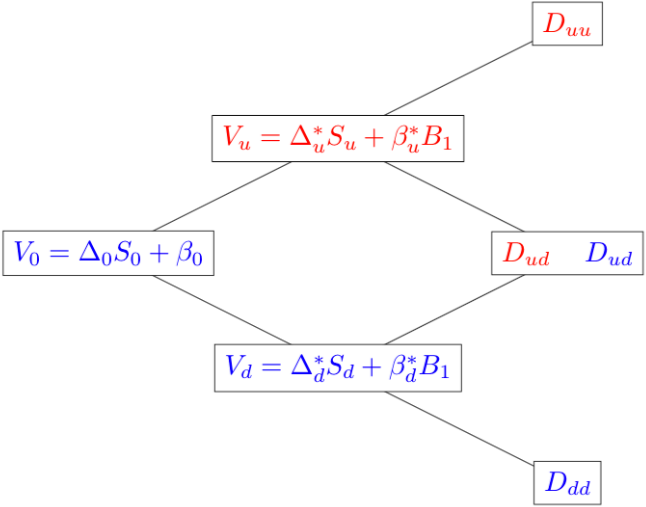
# Dynamic Replication

- We already have the solution to this problem !
- It is identical to the one period replication
- We therefore obtain

$$\boxed{\Delta_u^* = \frac{D_{uu} - D_{ud}}{S_{uu} - S_{ud}}} \qquad \boxed{\beta_u^* = \frac{S_{uu}D_{ud} - S_{ud}D_{uu}}{(S_{uu} - S_{ud})B_2}}$$

- Similarly we can obtain $(\Delta_d^*, \beta_d^*)$ such that

$$D_{ud} = \Delta_d S_{ud} + \beta_d B_2$$
$$D_{dd} = \Delta_d S_{dd} + \beta_d B_2$$


$D_{uu}$
$V_u = \Delta_u^* S_u + \beta_u^* B_1$
$V_0 = \Delta_0 S_0 + \beta_0$
$D_{ud}$ $D_{ud}$
$V_d = \Delta_d^* S_d + \beta_d^* B_1$
$D_{dd}$

# Dynamic Replication

- Obtaining the price of the derivative at time $0$ is similar to the one period problem of replicating a derivative with payoff given by

$$D_u = V_u^* = \Delta_u^* S_u + \beta_u^* B_1$$

$$D_d = V_d^* = \Delta_d^* S_u + \beta_d^* B_1$$

- This in turns implies that

$$\Delta_0^* = \frac{V_u^* - V_d^*}{S_u - S_d}$$

$$\beta_0^* = \frac{S_u V_d^* - S_d V_u^*}{(S_u - S_d) B_1}$$

- This solves the replication problem for any European derivative

$$D_0 = V_0^* = \Delta_0^* S_0 + \beta_0^*$$

# Course plan

# Risk Neutral Probability

- We can also interpret the pricing equation as an expectation
- Consider the value of the replicating at the node $S_u$

$$V_u^* = \Delta_u^* S_u + \beta_u^* * B_1$$

- Given $(\Delta_u^*, \beta_u^*)$ we can define $q := \frac{1+r-d}{u-d}$ (as in the one period model)

$$V_u^* = \frac{1}{1+r}\left[qV_{uu} + (1-q)V_{ud}\right] = \frac{1}{1+r}E^Q\left[V_2 | S_1 = S_u\right]$$

# Risk Neutral Probability

- Similarly we obtain

$$\boxed{V_d^* = \frac{1}{1+r}\left[qV_{ud} + (1-q)V_{dd}\right] = \frac{1}{1+r}E^Q\left[V_2|S_1 = S_d\right]}$$

- Finally at time $0$ we get

$$\boxed{V_0^* = \frac{1}{(1+r)^2}E^Q\left[V_2\right]}$$

- The price of a derivative security is the expected value of the payoff under the risk neutral measure discounted at the risk free rate

# Course plan

# Elements in Probability

- The binomial model is a simple model which will be used as a basis to develop finite probability spaces
- Probability space: A (finite) probability space is defined by a finite sample space $\Omega$ and a probability measure $\mathbb{P}$, i.e. a function assigning a number in $[0,1]$ to each element $\omega \in \Omega$ such that

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$$

- Event: An event is a subset of $\Omega$, the probability of the event $A$ is defined as

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega)$$

# Elements in Probability

- In the context of the binomial model with 3 periods an event is a combination of stock price movements

$$\Omega = \{uuu, uud, udu, udd, duu, dud, ddu, ddd\}$$

- Random Variable: A random variable is an application

$$X(\omega) : \Omega \to \mathbb{R}$$

- Binomial model : $S_2(\omega_1, \omega_2) \in \{S_{uu}, S_{ud}, S_{dd}\}$ is a random variable
- Payoff of a derivative $D(S_T)$ is also a random variable
- A distribution assigns a probability to the occurrence of each of the possible values of the random variable

# Elements in Probability

- Changing the probability measure does not change the random variable, it changes its distribution (c.f. risk neutral)
- Expectation: The expectation of a random variable $X$ is defined as

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega)$$

- The expectation of a random variable depends on the choice of a probability measure

$$\sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega) \neq \sum_{\omega \in \Omega} X(\omega)\widetilde{\mathbb{P}}(\omega)$$

if $\widetilde{\mathbb{P}}(\omega) \neq \mathbb{P}(\omega)$

# Properties of Expectation

- Note that expectation are linear $\mathbb{E}[aX] = a\mathbb{E}[X]$
- Let $f(x) = ax + b$ for some constant $a$ and $b$ then

$$\mathbb{E}[f(X)] = f(\mathbb{E}[X])$$

- For a convex functions $\phi(x)$

$$\mathbb{E}[\phi(X)] \geq \phi(\mathbb{E}[X])$$

- This is Jensen's Inequality

# Elements in Probability

- We can condition an expectation on the information available at some time $n$, we will denote $\mathbb{E}_n[X]$ this conditional expectation
- The following properties are important
  1. Linearity
  $$\mathbb{E}_n[aX] = a\mathbb{E}_n[X]$$
  2. if at time $n$, $X$ is known then
  $$\mathbb{E}_n[XY] = X\mathbb{E}_n[Y]$$
  3. for $m \geq n$ we have
  $$\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X]$$
  4. Jensen's inequality holds for conditional expectation

# Course plan

# Martingales

- Consider the $N$-period binomial model
- A sequences of random variable $M_0, M_1, M_2 ... M_N$ is a stochastic process
- If each $M_n$ depends only on the first $n$ movements, we call this process adapted and it verifies

$$\mathbb{E}_n[M_n] = M_n$$

- If each $M_n$ depends only on the first $n-1$ movements, we call this process predictable and it verifies

$$\mathbb{E}_n[M_{n+1}] = M_{n+1}$$

# Martingales

- If $M_n = \mathbb{E}_n[M_{n+1}]$ then $M_n$ is a martingale
- If $M_n \leq \mathbb{E}_n[M_{n+1}]$ then $M_n$ is a submartingale
- If $M_n \geq \mathbb{E}_n[M_{n+1}]$ then $M_n$ is a supermartingale

# Martingales

- From iterated expectations, we obtain that

$$M_n = \mathbb{E}_n[M_m] \qquad m \geq n$$

- Under the risk neutral measure $q = \frac{1+r-d}{u-d}$,

$$\mathbb{E}^Q\left[\frac{S_m}{(1+r)^m}\right] = \frac{S_n}{(1+r)^n}$$

discounted stock prices are martingales

# Martingales

- A self-financing portfolio $V_n = \Delta_n S_n + \gamma_n B_n$ is such that

$$(\Delta_{n+1} - \Delta_n) S_{n+1} + (\gamma_{n+1} - \gamma_n) B_{n+1} = 0$$

- Under the risk neutral measure $q = \frac{1+r-d}{u-d}$,

$$\boxed{\mathbb{E}^Q\left[\frac{V_m}{(1+r)^m}\right] = \frac{V_n}{(1+r)^n}}$$

self financing portfolios are martingales

# Martingales

- From our previous discussion on replication
- Given a derivative payoff $D_N$, we can choose a self financing portfolio $(\Delta_{N-1}, \gamma_{N-1})$ at date $N-1$ such that $V_N = D_N \quad \mathbb{P}-a.s.$
- Solving the problem backward we obtain a portfolio $V_0 = \Delta_0 S_0 + \gamma_0$ which allows to replicate the payoff of the derivative $D_N$ with certainty
- Under the risk neutral measure all self financing portfolios are **martingales** and we obtain

$$\boxed{\mathbb{E}^Q\left[\frac{V_N}{(1+r)^N}\right] = \mathbb{E}^Q\left[\frac{D_N}{(1+r)^N}\right] = \frac{V_0}{(1+r)^0} = V_0}$$

# Martingales

- This is the **Risk Neutral** valuation formula
- A fundamental result which will apply in much more complicated setting (Continuous time models for instance)
- Note that it implies the following theorem

**Fundamental Theorem of Asset Pricing**

If there exists a risk neutral measure, there can be no arbitrage in the binomial model