Multivariate → data $(X_1, X_2 \cdots X_p)$, we consider live in (it's a vector) a multivariate space, that is $p>2$

→ this variables ($X_1 \cdots X_p$) are linked / dependent

analysis → MA is a collection of methods based on probabilistic models to understand, analyze and interpret the data

→ data reduction
discrimination
sorting
groups
dependence among variables

different goal

The model depend on the type of variables, the type of the dependence, and the type of the problems.

社会学 P_1 : this class —> unsupervised learning

include a lot of

classic statistics

you have data $(X_1 \cdots X_p) \in R^{p \times p}$,

you want fulfill some goals.

no label

---

you have data $(X_1 \cdots X_p) \in R^{p \times 1}$

and have some response variable

$(Y_1 \cdots Y_n) \in R$ or category

we want do $Y = f(X)$

have label

usually USL 在 SL 之前做.下

$\Rightarrow$ supervised learning

content P1

Background on multivariate distribution

① the data: sample mean, sample covariance

② the model: population versions, multivariate families of distributions.

③ densities, cumulative distribution functions

content P2

dependence modeling

① How we can describe the dependence between 2 variables

② Copulas: separating the marginals from the dependence

content P3

graphic model

① what's the structure of multivariate obs.

② graphs as a description of conditional independence relations.

③ High-dimensional data sets and regularization

content P4

Unsupervised learning

① Dimsion reduction data compression.

② Visualization of high-dimensional data

③ principal components analysis

④ clustering method

⑤ extend PCA to non-linear transformations (Autoders)

⑥ sample from the distribution of images)

Variational Autoencoder and GAN

VAE

Model and data perspective

Model perspective:

① the model is a random vector $X = (X_1 \cdots X_p)^T \in R^p$ following a certain distribution F_X describing the entire (infinite) population

② we can compute the statistics such as the population mean or population variance for X

Data perspective

① the data/sample $X_1 \cdots X_n \in R^p$ are thought of finite number n of observations from X

questions? range $n \in [0, p]$ right?

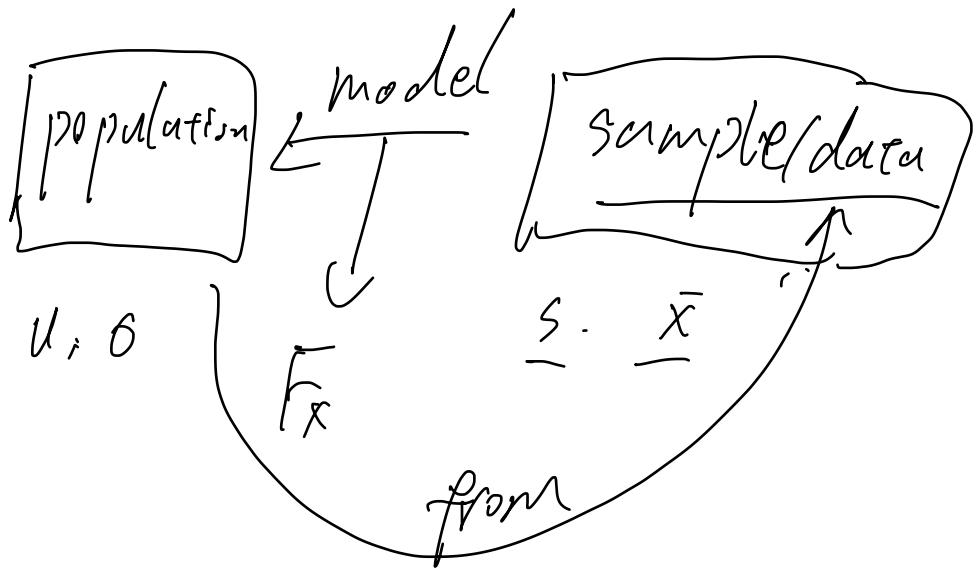
1-n 为 feature p 为维度 → nxp

unknown

this distribution is

② based on data, we can compute sample version of the mean or the variance. they are approximations of the population version.

model { vector distribution infinite population



the model: $p=2$, 2-dimensional

Let's $X = (X_1, X_2)^T$ be a bivariate normal distribution which is $N_2(\mu, \Sigma)$, with mean $(2,5)$ and covariance matrix: $\Sigma = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}$ ↳ model(的角度

the data:

we sumlate $n=50$ samples $x_1 \cdots x_n \in R^2$ from the normal distribution

一个x的取值. p.

we get $\bar{x} = (2.06, 5.07)$ $S = \begin{pmatrix} 0.98 & 0.71 \\ 0.71 & 1.11 \end{pmatrix}$

从样本观察数据的特征

model(simulation → data

inference, but never know it exactly.

Some notation: data 个体 variable 变量

for a vector $x \in \mathbb{R}^p$ we write

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \qquad x^T = (x_1 \cdots x_p)$$

$u = \bar{x} = \sum_{i=1}^{n} x_i / n$ unbiased

=> if we get more data, $\bar{x} \to u$

mode 众数

sample 有限 ⇒

if $x, y \in \mathbb{R}^p$, $x^T y \in \mathbb{R}$ and $x y^T \in \mathbb{R}^{p \times p}$

$A \in \mathbb{R}^{m \times m}$ in boldface the diagonal matrix a's

$$\mathrm{diag}(a_1 \cdots a_m) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_m \end{pmatrix} \in \mathbb{R}^{m \times m}$$

for n data $x_1 \cdots x_n \in \mathbb{R}^p$, we write

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & & & \vdots \\ \vdots & & & \\ x_{n1} & \cdots & & x_{np} \end{pmatrix} \in \mathbb{R}^{n \times p} = \begin{bmatrix} x_1^T \\ \vdots \\ x_p^T \end{bmatrix}$$

obs n 数行数

→ feature p 数列数

population

Let $X=(X_1 \cdots X_p)^T \sim F$ be a random p-vetor with join distribution function:

$$F_X(y) = F(y) = F(y_1, y_2 \cdots y_p) = P(X_1 \le y_1, \cdots X_p \le y_p) \in R^p$$

the univariate marginal distribution of X_j, $j=1 \cdots p$ is

$$F_j(y_j) = P(\infty \cdots X_j \le y_j, \cdots \infty)$$

similarly, we get higher dimensional marginals by

$$F_{jk}(y_j, y_k) = P(X_j \le y_j, X_k \le y_k)$$

F describes the margins and the dependence structure of the population

if F is differentable (we assume), the join density is the derivative:

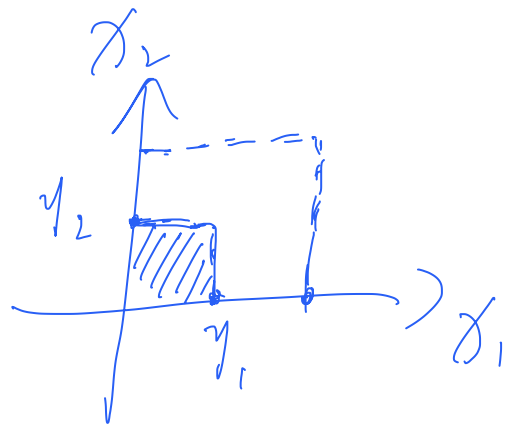
$$f(y) = f(y_1 \cdots y_p) = \frac{\partial^p F(y)}{\partial y_1 \cdots \partial y_p}$$

or equivalently:

$$F(y) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_p} f(u_1 \cdots u_p)\, d_1 \cdots d u_p$$

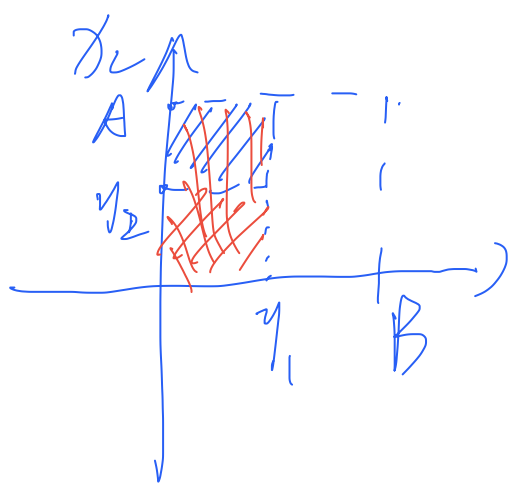
本质上：F[上页画0处]作为 distribution，可以替换为任意 distribution：

举例：uniform distribution



$$F_\gamma(y_1, y_2) = P(\gamma_1 \le y_1, \gamma_2 \le y_2)$$

"///" is the P



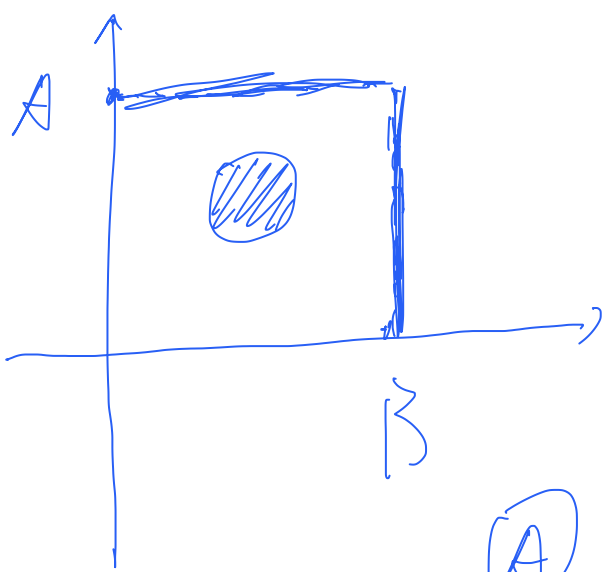
从图可推导得 ⟹ $P(\gamma_1 \le y_1, y_2 \le \gamma_2)$

A is the range $[c, A]$

⇓ 得 F

$$F(y_1) - F(y_1, y_2)$$

"///" — "▨"



⟹ if we want to know the $P(\gamma)$ which is not limit to rectangles

Ⓐ one way we try to is 用 rectangles 逼近.

the joint density existence ⟺ the marginal densities

$$f_j(x_j) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_1 \cdots u_{j-1}\, u_{j+1} \cdots u_p)\, du_1 \cdots du_{j-1} du_{j+1} \cdots du_p$$

the density function integrates to 1

$$F(\infty, \infty) = 1$$

① PS

Sample quantity: we don't know distribution we need to estimate it

Let $X_1 \cdots X_n \in \mathbb{R}^p$ be the n samples of X

You should see $X_1 \cdots X_n$ as random realizations of X

the emprical distribution function

③ $\hat{F}(y) = \frac{1}{n} \sum_{i=1}^{n} 1\{X_{i1} \le y_1, \cdots X_{ip} \le y_p\}$ ② PS

is a non-parametric estimtor of F — empirical distribution

there is no "empirical" sample-equivalent for the density function without further assumptions. why? ④

example:

Let $X = (X_1, X_2)^T$ be a bivariate random variable with density function: $f(y_1, y_2) = \begin{cases} r(y_1^{\alpha} + y_2^{\beta}), & 0 < y_1, y_2 < 1 \\ 0, & \text{otherwise} \end{cases}$ $\alpha, \beta > 0$

$\int_0^1 \int_0^1 r(y_1^{\alpha} + y_2^{\beta})\, dy_1 dy_2 = 1$ ⟹ get the r

$0 < y_1, y_2 < 1$

$= \int_0^1 \int_0^1 r y_1^{\alpha} dy_1 dy_2 + \int_0^1 \int_0^1 r y_2^{\alpha} dy_1 dy_2$

we get: $r = \left[(\alpha+1)^{-1} + (\beta+1)^{-1} \right]^{-1}$

$= \int_0^1 r y_1^{\alpha} dy_1 \int_0^1 dy_2 + \int_0^1 r y_2^{\beta} dy_2 \int_0^1 dy_1$

$= \left[\frac{r}{\alpha+1} y_1^{\alpha+1} \right]_0^1 \left[y_2 \right]_0^1 + \left[\frac{r}{\beta+1} y_2^{\beta+1} \right]_0^1 \left[y_1 \right]_0^1$

$= \frac{r}{\alpha+1} + \frac{r}{\beta+1} = 1$

只要满足即可估计 F

$\alpha = 1, \beta = 2, r = 6/5$

$F = \int_0^{y_1} \int_0^{y_2} \frac{6}{5} (u_1 + u_2^2)\, du_1 du_2$

$F = \frac{6}{5} y_1 y_2 \left[\frac{1}{2} y_1 + \frac{1}{3} y_2^2 \right]$

$F(X_1 < \frac{1}{2}, X_2 > \frac{1}{2}) = \frac{1}{4}$ (求 Probabilities 的流程)

for $\hat{F}$ 类似的:

$\hat{P}(X_1 < \frac{1}{2}, X_2 > \frac{1}{2}) = \hat{F}(\frac{1}{2}, 1) - \hat{F}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{n} \sum_{i=1}^{n} 1\{X_{i1} \le \frac{1}{2}, X_{i2} > \frac{1}{2}\}$

population quantity:

Let $X = (X_1 \cdots X_p)^T$ be a random vector

we can define the expectation of $g(x)$, where g $\mathbb{R}^p \to \mathbb{R}^m$ is some function

$E(g(x)) = \int_{\mathbb{R}^p} g(u)\, dF(u) = \int_{\mathbb{R}^p} g(u) f(u)\, du \in \mathbb{R}^m$

? 这里被写成 density vector

sample quantity

for n samples $x_1 \cdots x_n \in \mathbb{R}^p$ of X, we define the sample version of $E(g(x))$ as

recall PS ② ⟹

$\hat{E}(g(x)) = \int_{\mathbb{R}^p} g(u)\, d\hat{F}(u) = \frac{1}{n} \sum_{i=1}^{n} g(x_{i1} \cdots x_{ip})$ ⟹ empirical

样本期望估计 $\int_{\Omega} f(x)\, dx$

For a random vector $X \in \mathbb{R}^p$ and $Y \in \mathbb{R}^q$ matrix $A \in \mathbb{R}^{m \times p}$ $B \in \mathbb{R}^{m \times q}$ and constant vector $b \in \mathbb{R}^m$

$$Z = \underset{m \times p \cdot p \times 1}{AX} + \underset{m \times q \cdot q \times 1}{BY} + \underset{m \times 1}{b} \in \mathbb{R}^m$$

$E(Z) = A \cdot E(X) + B \cdot E(Y) + b$ ⟹ 线性性质.

the trace is an operator on square matrices that returns the sum of the diagnoal elements. 对角元素和

Let $A = \{a_{kl}\}_{k,l=1}^{m}$ be $m \times m$ matrix, then

$$Tr(A) = \sum_{i=1}^{m} a_{ii}$$

the trace is linear and circular (with compatible matrix)

可乘的前提 AB BA 行=列

$Tr(A+B) = Tr(A) + Tr(B)$, $Tr(ABC) = Tr(CAB)$

For a random vector $X \in \mathbb{R}^p$ and a determinstic matrix $A \in \mathbb{R}^{p \times p}$ 特别或矩阵

$$E(\underset{1\times p}{X^T} \underset{p\times p}{A} \underset{p\times 1}{X}) = E[Tr(X^T A X)] = E[Tr(A X X^T)] = Tr[A \cdot E(X X^T)]$$

circular; $1\times p$; 线性性质

how can I prove it ?

$X^T A X$ is a scalar, $Tr(\text{scalar}) = \text{scalar}$

by linearity $E[Tr(C)] = Tr[E(C)]$

population quantity

the variance-covariance matrix of X is the matrix of variances and covariances between the p components

$$\Sigma = cov(X) = \{\sigma_{kl}\}_{k,l=1}^{p} = E[(X - E(X))(X - E(X))^T]$$

$$\sigma_{kl} = E[(X_k - E(X_k)) \cdot (X_l - E(X_l))], \quad \sigma_k^2 = \sigma_{kk}$$

the correlation matrix is the matrix of correlations obtained through normalization.

$$\rho_{kl} = \frac{\sigma_{kl}}{\sigma_k \cdot \sigma_l}, \quad \rho_{kk} = 1$$

gama

$T = cor(X) = \{\rho_{kl}\}_{k,l=1}^{p} = \Delta^{-1/2} \Sigma \Delta^{-1/2}$, with $diag(\sigma_1^2 \cdots \sigma_p^2)$

recall by cauchy-schwarz inequality we can get $\rho_{kl} \in [-1, 1]$

$|\langle x, y \rangle|^2 \le \langle x, x \rangle \cdot \langle y, y \rangle$

$|\langle x, y \rangle| \le \|x\| \cdot \|y\|$

sample quantity.

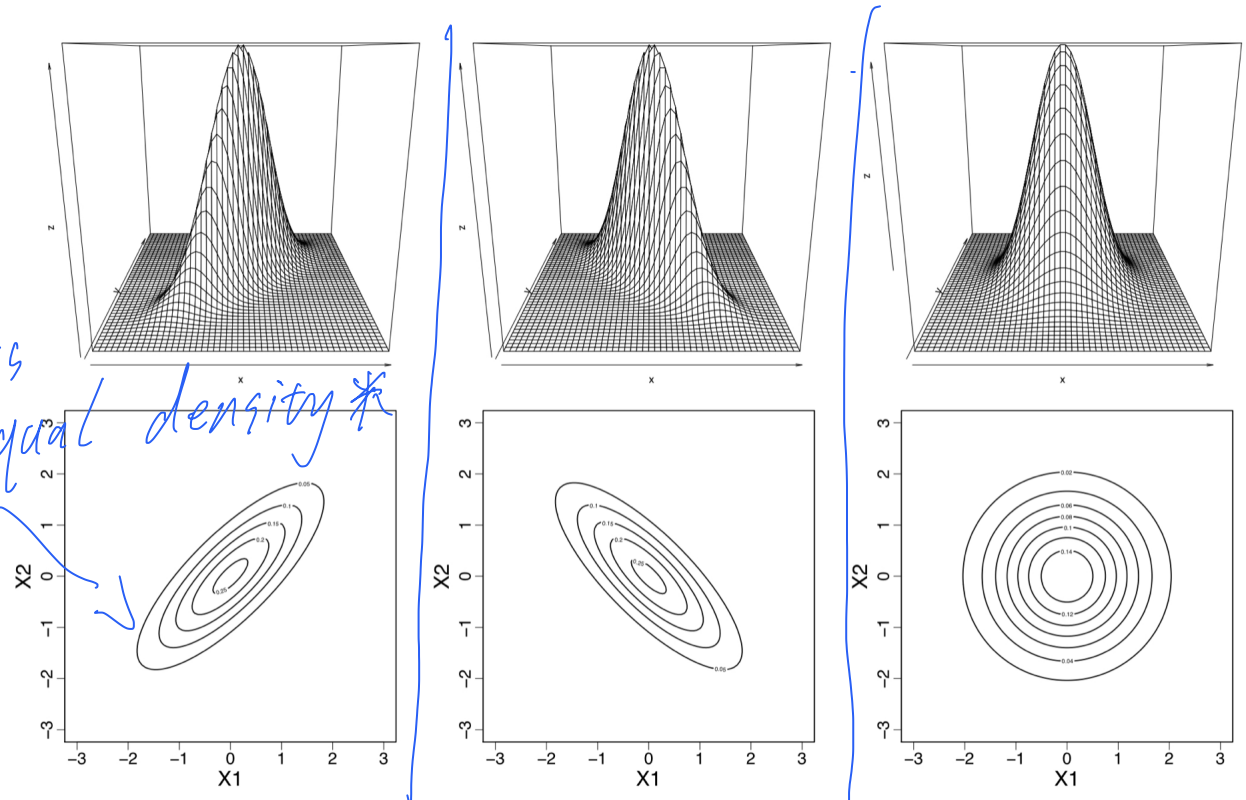
$$S = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^T \in \mathbb{R}^{p \times p}$$

$r_{kl} = \frac{s_{kl}}{s_k s_l}$

why not n, because it's an unbiased estimator of Σ bessel's correction

$R = D^{-1/2} S D^{-1/2}$ with $D = diag(s_1^2 \cdots s_p^2)$

PS:① bivariate normal distribution



points is equal density 来

这是 different density function

上面的图可以理解为: $z = f_{x,y}(x, y)$

z 为 probability

下图为其在 x-y 平面上的投影

其 $U = \begin{pmatrix} a \\ b \end{pmatrix}$ 改变的是下图的位置

$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ σ_{11} σ_{12} 改变在 x or y 的跨度

σ_{21} σ_{12} 是相关小量

来怎么理解: $z = 0 = f_{(x,y)}(x, y) = \dfrac{1}{(2\pi)^{n/2} \cdot |\Sigma|^{1/2}} e^{[-\frac{1}{2}(x-u)\Sigma^{-1}(x-u)]}$

其 $n = len(x)$

$x = (x_1, x_2, x_3 \cdots)$

= 当 z=0

covariance = V = density value

(2) PS: $F(\gamma) = P(\chi_1 \le \gamma_1, \cdots \chi_p \le \gamma_p)$

$= E[\text{Indicator function}]$

$= E[I(\chi_1 \le \gamma_1 \cdots \chi_p \le \gamma_p)]$

cove: prob is expectation of indicator function

$I\{A\} = \begin{cases} 1 & \text{if it's true} \\ 0 & \text{if it's false} \end{cases}$

∴ if we want estimate the $F(\gamma)$.

we need to estimate the expectation of the indicator function.

So how to estimate the expectation of the indicator function? => use monte-carlo

$$\hat{F}(\gamma) = \frac{1}{n} \cdot \sum_{i=1}^{n} 1 \cdot \{\chi_{i1} \le \gamma_1, \cdots \chi_{ip} \le \gamma_p\} \quad (3)$$

expectation: [prob x value] 对应 Indicator function.

PS: it's non-parametric estimator

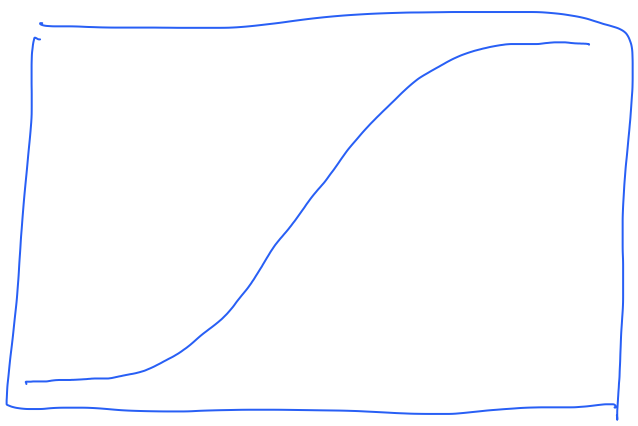
good estimator:

it means it coverage to the true CDF

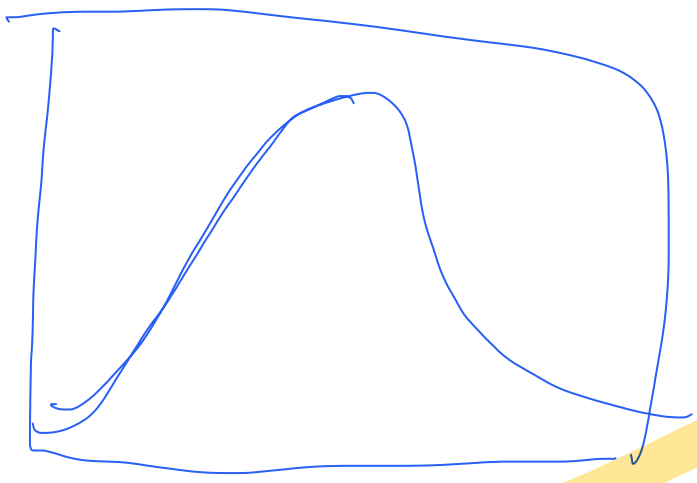
$\hat{F}(x) \to F(x)$

move data

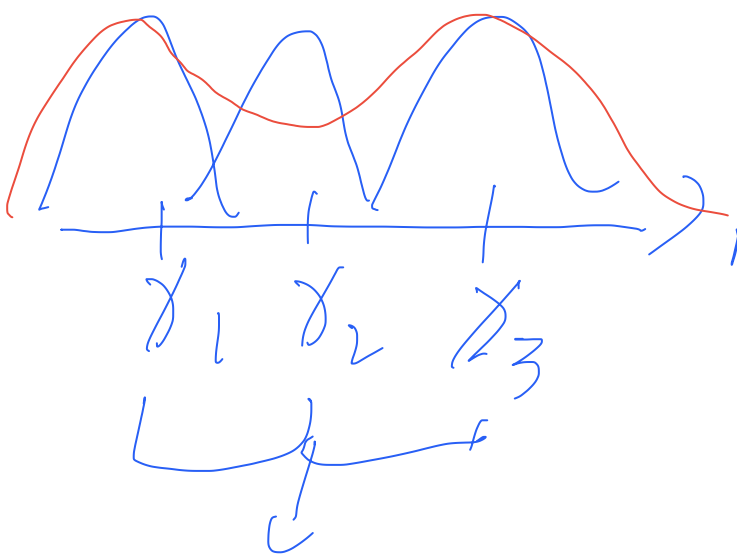
CDF of normal distribution



density function of normal distribution



这里只是估计了 CDF
没有估计 density
如何应对? way1: 连续下、求导.
way2: kernel function estimate



need some assumption

how to choose

red: kernel density

non-parameter

but it not empirical

$\gamma_1 \cdots \sim$ normal distribution

下文图：本质 $cov(b+Ax) \to AZA^T$

$\Rightarrow m \to p$

是转化了 dimension

a covariance matrix $\Sigma = cov(X)$ is symmetric and positive-semidefinite, that is,

$$x^T \Sigma x \geq 0, \ \text{for all } x \in R^p$$

if the above holds strictly with >0, then Σ is positive definite

if Σ is positive definite and Σ^{-1} with invertible positive-definited

we can have a decomposition $\Sigma^{1/2}$ satisfying

$$\Sigma^{1/2}(\Sigma^{1/2})^T = \Sigma$$ choleske-decomposition

for a matrix $A \in R^{m \times p}$, a vector $b \in R^m$ (a constant vector), and a random vector $X \in R^p$

⑤ $cov(b+AX) = E[A(X-E(X))(X-E(X))^T A^T] = A\Sigma A^T \in R^{m \times m}$

the covariance notion also generalizes to two vectors

Let $X \in R^p$ and $Y \in R^q$ be random vectors, then their covariance matrix is $p \times q$.

$cov(X;Y) = E[(X-E(X))(Y-E(Y))^T] = \{cov(X_i, Y_j)\}_{i,j=1}^{p,q}$

note that $\Sigma_{XY} = \Sigma_{YX}^T$

population quantity

Let $X \in R^p$ be a random vector with $\Sigma = cov(X)$ and $\Delta = diag(\sigma_1^2, \cdots \sigma_p^2)$ and correlation matrix:

$$cov(X) = \Gamma = \Delta^{-1/2}\Sigma\Delta^{-1/2}$$

then the random vector: satisfies:

$\frac{x-u}{\sigma}$ 标准 $\Rightarrow Y = \Delta^{-1/2}(X-E(X)) \Rightarrow E(Y)=0$, $cov(Y) = \Gamma$ (i)

for fun standardization, consider:

$$Y = \Sigma^{-1/2}(X-E(X))$$

then $E(Y)=0$ and $cov(Y) = I_p = diag(1, \cdots 1)$ (ii)

for (i) $cov(Y) = cov(\Delta^{-1/2}X - \Delta^{-1/2}E(X)) = cov(\Delta^{-1/2}X)$

$= E[\Delta^{-1/2}(X-E(X))(X-E(X))^T(\Delta^{-1/2})^T]$ transpose 交换顺序

$= \Delta^{-1/2}\cdot\Sigma\cdot(\Delta^{-1/2})^T = \Delta^{-1/2}\cdot\Sigma\cdot\Delta^{-1/2} = \Gamma$

$\Delta = diag(\sigma_1^2 \cdots \sigma_p^2) = \begin{bmatrix}\sigma_1^2 & & \\ & \ddots & \\ & & \sigma_p^2\end{bmatrix} \Rightarrow (\Delta^{-1/2}) = (\Delta^{-1/2})^T$

for (ii): $cov(Y) = cov(\Sigma^{-1/2}X - \Sigma^{-1/2}E(X)) = cov(\Sigma^{-1/2}\cdot X)$

$= E[\Sigma^{-1/2}\cdot(X-E(X))(X-E(X))^T(\Sigma^{-1/2})^T]$

$= \Sigma^{-1/2}\cdot\Sigma\cdot(\Sigma^{-1/2})^T = \Sigma^{-1}\cdot\Sigma = I$

sample quantity:

the analogous sample standardization for the data $X_1 \cdots X_n$ are

$y_i = D^{-1/2}(x_i - \bar{x})$ $y_2 = S^{-1/2}(x_i - \bar{x})$

S is the sample covariance

D is the $diag(s_1^2 \cdots s_p^2)$

Independence:

By definite: the p components $X_1 \cdots X_p$ of $X \sim F$ are mutually independent

$$F(y) = \prod_{j=1}^{p} F_j(y_j), \ \forall y \in R^p$$ ⇐ CDF

⇕

$$f(y) = \prod_{j=1}^{p} f_j(y_j), \ \forall y \in R^p$$ ⇐ PDF

if two random variables are independent, then the $E(xy)$

$E(xy) = E(x)\cdot E(y)$

consequently, if $X = (X_1 \cdots X_p)^T$ mutually independent,

$\sigma_{kt} = cov(X_k; X_t) = E[(X_k - E(X_k))(X_t \ldots$

$= 0$ 独立

$cov(X) = diag(\sigma_1^2 \cdots \sigma_p^2)$

Let $X=(X_1,X_2)^T$ be a bivariate normal random variables with mean $=0$ and covariance $\Sigma=\begin{pmatrix}1&\rho\\ \rho&1\end{pmatrix}$

the density function:

$$f(y_1,y_2)=\frac{1}{2\pi\sqrt{1-\rho^2}}e^{\left\{-\frac{1}{2(1-\rho^2)}(y_1^2-2\rho y_1y_2+y_2^2)\right\}}$$

原形式：（补充）

$$f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y}\cdot e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2-2\rho\cdot\frac{x-\mu_x}{\sigma_x}\cdot\frac{y-\mu_y}{\sigma_y}+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$\mu=\begin{bmatrix}\mu_x\\ \mu_y\end{bmatrix}\quad\Sigma=\begin{pmatrix}\sigma_x^2&\sigma_x\sigma_y\cdot\rho\\ \sigma_x\sigma_y\rho&\sigma_y^2\end{pmatrix}$$

$$f_1(y_1)=\int_{-\infty}^{+\infty}f(y_1,y_2)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y_1^2}{2}}\quad\text{same to } y_2$$

if $\rho=0\Rightarrow X_1,X_2$ independent?

L)

$$f_1(y_1)f_2(y_2)=\frac{1}{2\pi}e^{-\frac{1}{2}(y_1^2+y_2^2)}=I.$$

if $I=f(y_1,y_2)\not\Rightarrow\rho=0$.

summary:

$$\begin{cases}X_1,X_2\sim N\\ \rho=0\end{cases}\Rightarrow X_1,X_2\text{ independent}$$

X_1,X_2 indep $\Rightarrow\rho=0$

conditional distribution:

Let $X\in R^p$, $Y\in R^q$ be random vectors, and Let $(X^T,Y^T)^T$ be the joint vector with density $f(x,y)$. Then the conditional distribution of X given the $Y=y$

$$f_{X|Y=y(x)}=\frac{f_{(x,y)}(x,y)}{f_y(y)}$$

if X,Y independent, $f_X(x)=f_Y(y)$

for the bivariate normal distribution our numerical example with $X=X_1$, $Y=X_2$.

$$f_{X|Y=y(x)}=\frac{1}{2\pi\sqrt{1-\rho^2}}\cdot e^{\left\{-\frac{(x-\rho y)}{2(1-\rho^2)}\right\}}$$

therefore $(X|Y=y)\sim N(\rho y,1-\rho^2)$

we also can define the $E(X|Y=y)$:

$$E(X|Y=y)=\int_x x\cdot f_{X|Y}(x|Y=y)\,dx$$

more generally: $E(g(x)|Y=y)=\int_x g(x)\cdot f_{X|y}(x|Y=y)\,dx$

the conditional expectation can be seen as a random variable: $E(X|Y)$

with the important property

$$E(x)=E[E(x|Y)]$$

similarly:

$$\mathrm{cov}(X|Y=y)=E[(x-E(x|Y))(x-E(x|Y))^T|y]$$

if X and Y ~~indepent~~ independent.

$E(X|Y)=E(X)$

$\mathrm{cov}(X|Y)=\mathrm{cov}(X)$ → CDF

$F_{X|Y}(x|Y)=F_X(x)$, $f_{X|Y}(x|Y)=f_X(x)$ $\Rightarrow$ both is sufficient condition

PDF

连续(但只有在normal distribution)

$$f_{x,y_1}(x_1,y_1)=f_X(x)\cdot f_Y(y)$$

实在不理解就：

$$E(x)=\int_{-\infty}^{+\infty}x\cdot f(x)\,dx$$

$$F(x)=\int_{-\infty}^{x}f(t)\,dt\Rightarrow\text{CDF}$$

Module 02:
multivariate normal and other distributions

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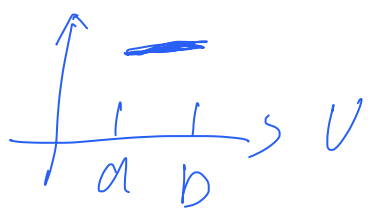
the univariate uniform distribution: 均匀分布(别看到1个distribution就是正态分布)

Let $S=(a,b)$ be interval, then the uniform distribution $U \sim U(a,b)$ is a uniformly distributed random variable if $f(u)=\frac{1}{b-a}$, for $u \in (a,b)$



prob density function

a important example: $S=(0,1) \Rightarrow$ standard uniform distribution

recall that any random variable X with continous cumulative distribution function F satisfies:

$F(x) \sim U(0,1)$, since:

$$P(F(x) \le u) = P(x \le F^{-1}(u)) = F(F^{-1}(u)) = u, \forall u \in (0,1)$$

(b)

we can thus generate the samples of X using a standard uniform random variable $U \sim U(0,1)$, by setting $X = F^{-1}(U)$

$$P(X \le x) = P(F^{-1}(U) \le x) = F(x)$$

$X \sim F$; $P(U \le F(x))$ (a)

PS: ⑥ $P(X \le x) = F(x)$ 解释了第二个文号内容.

F是 CDF

因为是 uniform distribution ∴ $F(x) = x$ 解释三个文号

$F^{-1}(u)$ 是反函数 => 解释一个文号

⑦: 问题在于求 inverse function. $X \sim F$ 是知道的 CDF了

but we don't know 是否可以求反

for example: $F_\lambda(x) = 1 - \exp(-\lambda \cdot x) = y$ 若CDF是这个

反 $\rightarrow F_\lambda^{-1}(y) = \frac{\log(1-y)}{-\lambda}$ => exponential function

若令: $x = \frac{\log(1-y)}{-\lambda} \sim \exp(\lambda) = F^{-1}(v)$ 证明.

$v \sim$ uniform distribution

$P(X \le x) = P(F^{-1}(v) \le x) = P(v \le F(x)) = F(x)$

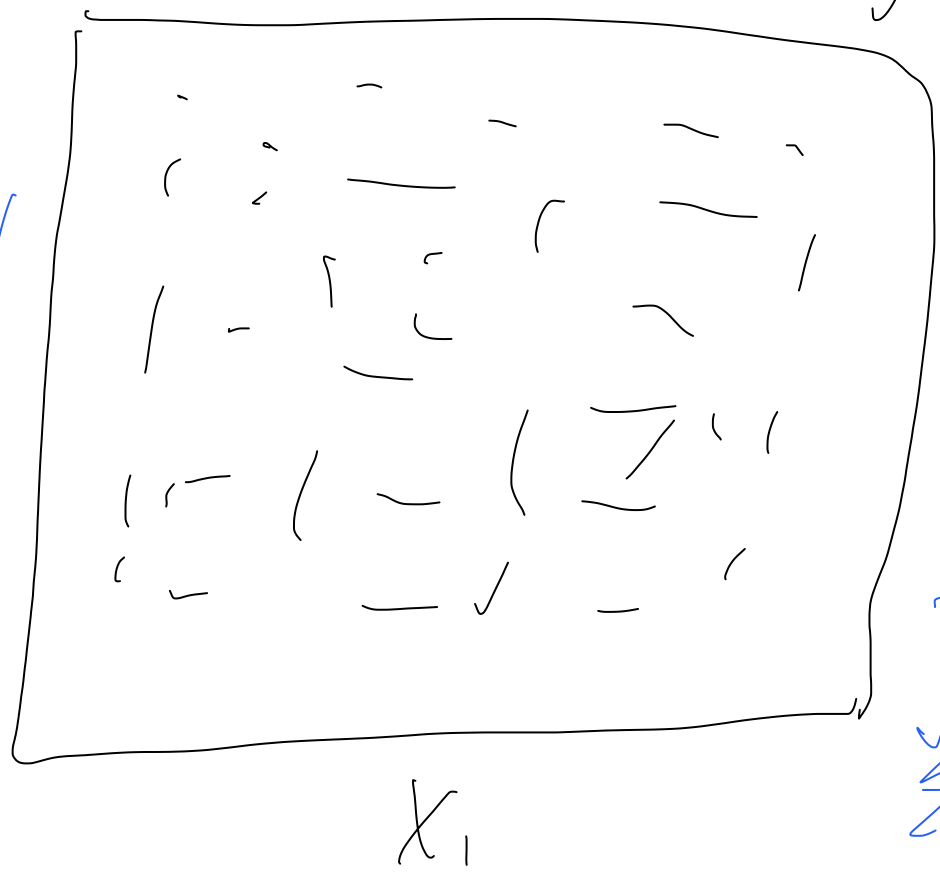
In general, we can construct a multivariate distribution for $X = (X_1 \cdots X_p)^T$ by letting $X_1 \cdots X_p$ be independent univariate random variables. The density is:

$$f(x_1, \cdots x_p) = f_{x_1}(x_1) \cdot f_{x_2}(x_2) \cdots f_{x_p}(x_p)$$

Let X_j, $j=1 \cdots p$ be random variables uniformly distributed on interval $S_j = (a_j, b_j)$, $j=1 \cdots p$. Then $X = (X_1 \cdots X_p)^T$ has a multivariate uniform distribution with density

$$f(x_1 \cdots x_p) = \frac{1}{\prod_{j=i}^{p} (b_j - a_j)}, \quad x_j \in S_j$$

by definition the marginals X_j are mutually independent and all uniformly distributed



多元均匀分布有很强的限制

⇓

独立性

X_1 之间必须独立

⇐ 所有的点均匀

多元均匀分布以二元为例，在 XOY 面上投影下来应为均匀的

当它们是独立的时候

the multivariate normal distribution

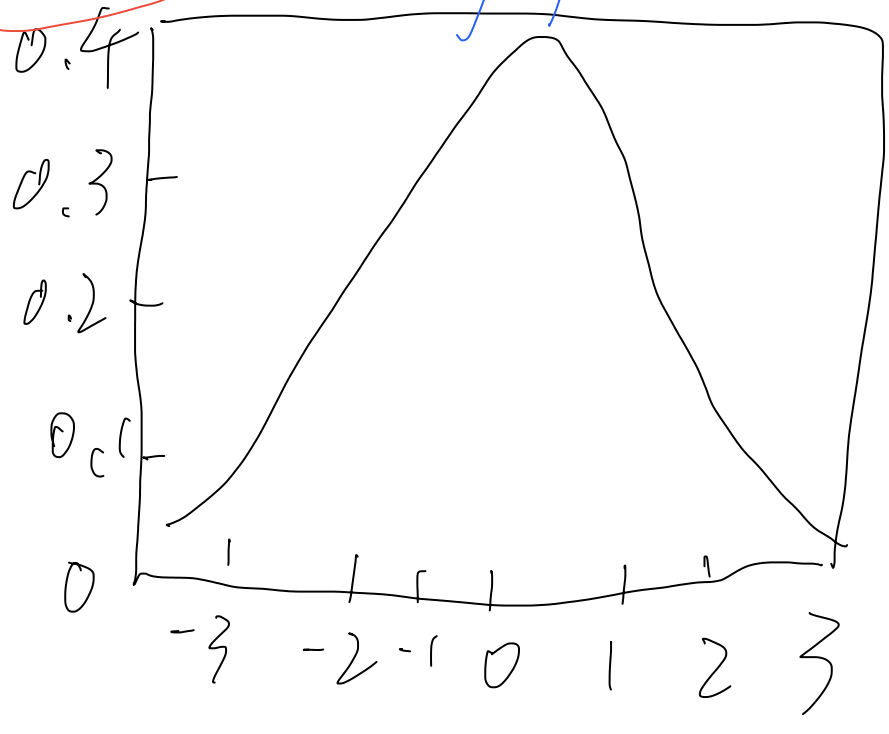
recall the $X \sim N(u, \sigma^2)$ has a univariate normal distribution if its density is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-u)^2}{2\sigma^2}\right\} \qquad u = E(x), \quad \sigma^2 = Var(x)$$

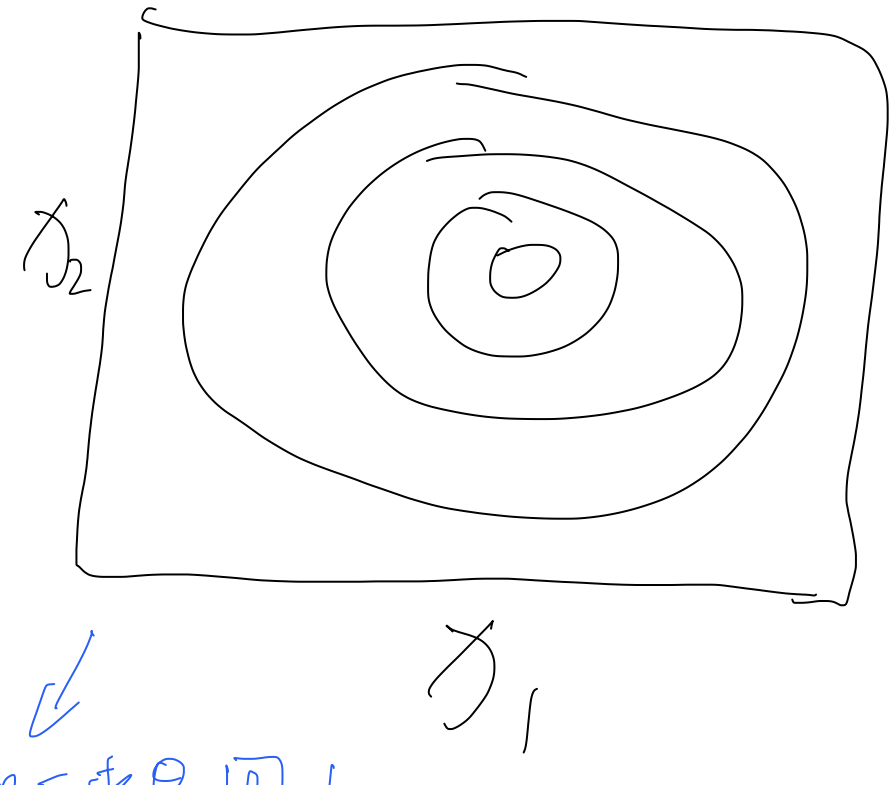
the standard multivariate normal distribution follows by the same "independent construction" as before, namely let $X_1 \cdots X_p$ be standard normal $N(0,1)$, then $X = (X_1 \cdots X_p)^T$ has density

$$f(x) = f_{x_1}(x_1) \cdots f_{x_p}(x_p) = \frac{1}{\sqrt{(2\pi)^p}} \exp\left\{-\frac{1}{2} x^T x\right\}, \quad x \in \mathbb{R}^p$$

dnorm(x) => density function



某个截面. X



投下来是圆!

当 x_1 x_2 独立.

Density of transformed random vectors.

Let X be a p random vector and $Y = H(X)$ for $H: R^p \to R^p$ a bijective (i.e: invertible) transformation (8)

双射的 → (可逆的)

if f_X is the density of X, then Y has the density $f_Y(y) = f_X(H^{-1}(y)) \left| \det(J_{H^{-1}}(y)) \right|$

要搞懂

逆函数 雅可比矩阵

where for a mapping $T = (T_1 \cdots T_p): R^p \to R^p$

$J_T(y)$: is the Jacobian matrix at point $y \in R^p$

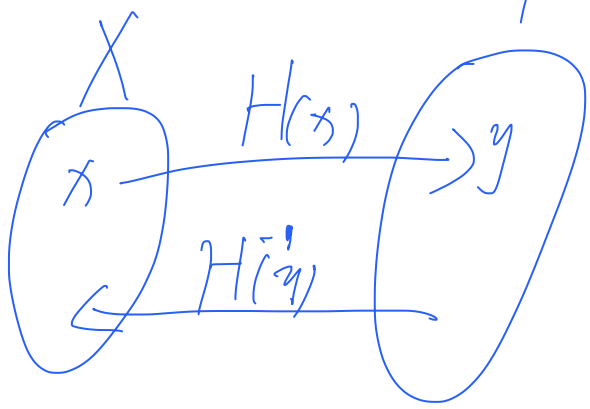
=> high dimension derivative

$$J_T(y) = \begin{pmatrix} \frac{\partial T_1}{\partial y_1} & \cdots & \frac{\partial T_1}{\partial y_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial T_p}{\partial y_1} & \cdots & \frac{\partial T_p}{\partial y_p} \end{pmatrix} \in R^{p \times p}$$

the matrix J_T is the best linear approximation of T at y

how to prove (9)

8:



know $X \sim F$

to get $Y \sim ?$

$$P(Y \le y) = P(H(x) \le y)$$

$$F_Y(y) = P(X \le H^{-1}(y))$$

$$F_Y(y) = F_X(H^{-1}(y)) \Leftarrow \text{CDF}$$

$$f_Y(y) = f_X(H^{-1}(y)) \left(\frac{d(H^{-1}(y))}{dy}\right) \Leftarrow \text{chain rules}$$

对其求导 derivate.

这是一维的，高维如正文.

9

你认为T是一个未知的函数，比如二次函数，然后你想让计算机在某一点上，然后你试着在这一点上找到最佳超平面，它在这一点上最接近函数，所以它是在点Y上函数T的最佳线性近似

10

$$H(x) = Ax$$

$$H: R^p \to R^p$$

$$H(x) = \begin{pmatrix} H_1(x) \\ \vdots \\ H_p(x) \end{pmatrix}$$

$$H_i(x) = A_i \cdot x = \sum a_{ij} \cdot x_j$$

$$J_H(x) = \begin{bmatrix} \frac{\partial H_1(x)}{\partial x_1} & \\ \vdots & \\ & \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pp} \end{bmatrix} = A$$

example: matrix $A \in \mathbb{R}^{p\times p}$, $b \in \mathbb{R}^p$

the mapping $H(x) = Ax + b$ is affine

$\Rightarrow$ $J_H(x) = A$ 未导出已，具体 ? (10)

scaling and rotation: 改变形状

further construction priciples for multivariate distributions are

scaling: Let $\Lambda = diag(\lambda_1 \cdots \lambda_p)$ be a scaling matrix and define a new random vector $\Lambda^{1/2}X$ (11)

$x \in \mathbb{R}^p$ $f_{\Lambda^{1/2}X}(x) = \frac{1}{\sqrt{(2\pi)^p det(\Lambda)}} exp\{-\frac{1}{2}x^T\Lambda^{-1}x\}$ (12)

PS $y = H(x) = \Lambda^{1/2}x$

$f_y(x) = f_x(H^{-1}(y))\,|det(J_{H^{-1}}(y))|$ ↘ why? ✓

Rotation: let V be a rotation matrix and define a new random vector $V\Lambda^{1/2}X$. (13)

$f_{V\Lambda^{1/2}X}(x) = \frac{1}{\sqrt{(2\pi)^p det(V\Lambda V^T)}} \cdot exp\{-\frac{1}{2}x^T(V\Lambda V^T)^{-1}x\}$

(11) $\Lambda^{1/2} x = y$

$H(x) = \Lambda^{1/2} x \Rightarrow \Lambda^{-1/2}\Lambda^{1/2} x = \Lambda^{-1/2} y$

$H^{-1}(y) = \longleftarrow \quad x = \Lambda^{-1/2} y$

$$\det\left(J_{H^{-1}(y)}\right) = \prod_{i=1}^{p} \Lambda_i^{-1/2} \Rightarrow \frac{1}{\sqrt{\det(\Lambda_i)}} \Rightarrow$$ (12)

$$-\frac{1}{2} x^T \Lambda^{-1} x = \left(x \cdot \Lambda^{-1/2}\right)^T \cdot x \cdot \Lambda^{-1/2}$$

$$-\frac{1}{2} x^T \Lambda^{-1} x = x^T \Lambda^{-1/2} \Lambda^{-1/2} \cdot x$$

(13) $V \cdot V^T = I, \; y = V \Lambda^{1/2} x_i$

$y = \Lambda^{1/2} x$

$z = H(y) = V \cdot y$

$V^T = V^{-1} \longrightarrow H^{-1}(z) = V^{-1} \cdot z = V^T z \Rightarrow J_{H^{-1}(z)} = V^T$

$\det(V \Lambda V^T) = \det(V) \cdot \det(\Lambda) \cdot \det(V^T)$

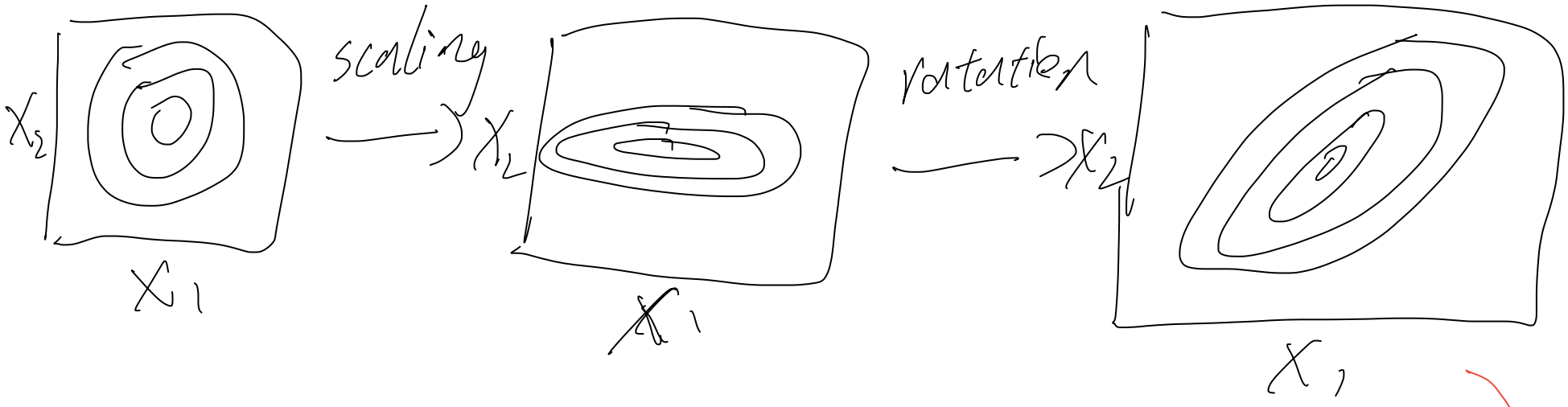
$\det(A) \cdot \det(B) = \det(A \cdot B)$

$x \sim N(0, I)$

$cov(V \Lambda^{1/2} \cdot X) = V \cdot \Lambda^{1/2} \cdot cov(X) \cdot (V \cdot \Lambda^{1/2})^T$

$= V \cdot \Lambda^{1/2} \, I \cdot (V \cdot \Lambda^{1/2})^T$

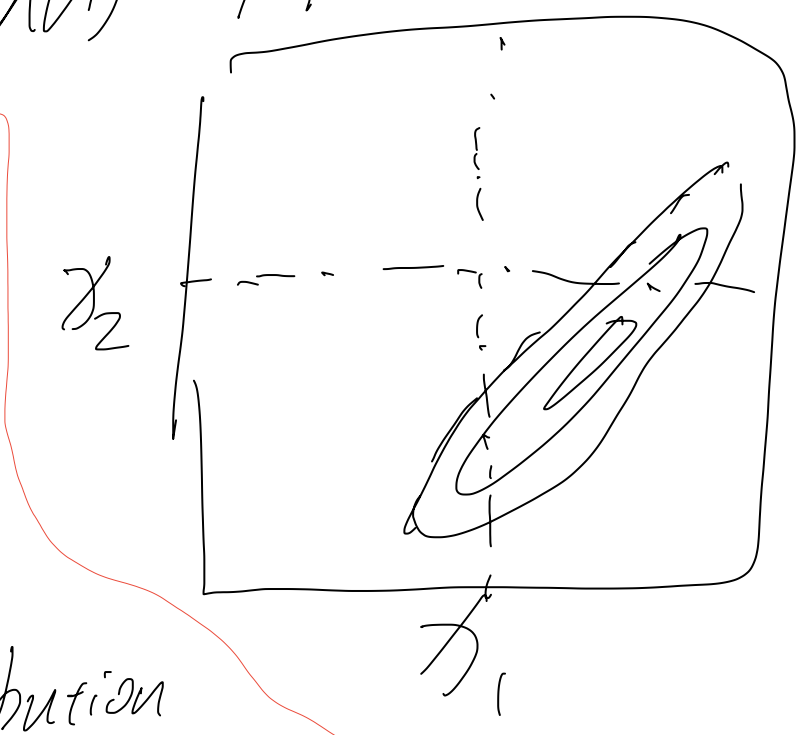
$= V \Lambda V^T$



location shifting: for some $u \in \mathbb{R}^p$, let $H(x) = x + u$

均值移位置

$$f_{UN^{1/2}X}(x) = \frac{1}{\sqrt{(2\pi)^p \det(UNU^T)}} \exp\left\{-\frac{1}{2}(x-u)^T (UNU^T)^{-1}(x-u)\right\}$$



let Σ be a positive-definite $p \times p$-matrix and $u \in \mathbb{R}^p$. Then $X = (X_1 \cdots X_p)^T$ has a multivariate normal distribution $N_p(u, \Sigma)$ if its density satisfies

$$f(x) = \frac{1}{\sqrt{(2\pi)^p \det(\Sigma)}} \exp\left\{-\frac{1}{2}(x-u)^T \Sigma^{-1}(x-u)\right\}, \quad x \in \mathbb{R}^p$$

every positive-definite matrix Σ has eigendecomposition

$\Sigma = V \Lambda V^T$ where $\Lambda = diag(\lambda_1 \cdots \lambda_p)$, λ_i is the eigenvalues of Σ, and V is a rotation matrix. Therefore, we get this represetation

$$X \sim u + V \Lambda^{1/2} N_p(0, I_p)$$

任意 V

by the rules for expectation and covariance this gives: $E[X] = u$, $cov(X) = \Sigma$

Let $X \sim N_p(u, \Sigma)$

① the marginal X_j has a univariate normal distribution $N(u_j, \sigma_j^2)$

② Let A be a $m \times p$ matrix, $b \in R^m$, then $b + AX$ is multivariate normal $N_m(b + Au, A\Sigma A^T)$

in particular $\Sigma^{-1/2}(X - u) \sim N_p(0, I_p)$

$\frac{X-u}{\sigma}$ 标准

where $\Sigma^{-1/2}$ is the choleski decomposition of Σ^{-1} satisfying $\Sigma^{-1/2} \cdot \Sigma^{-1/2} = \Sigma^{-1}$

3) X_j and X_k are independent if their covariance $\sigma_{jk} = 0$. That is the components of X are mutually independent if and only if

$$\Sigma = \text{diag}(\sigma_1^2 \cdots \sigma_p^2)$$

this equivalence between "independent" and "uncorrelated" if specific to the normal distribution.

the normal distribution is completely characterized by its mean vector μ and covariance matrix Σ