Dynamic Economic Modelling
Tutorial Exercises

David Murakami

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1 Data and the current economy

Download data on UK "Labour Productivity" from the website of the Office for National Statistics (ONS). The correct data should have the name "PRDY".

(a) Using a spreadsheet or other computer program (**Note:** Try and use MATLAB or Python or R if you can!), make a graph of the column labelled "UK Whole Economy: Output per hour worked SA: Index 2018 = 100" over time (code = LZVB). Make sure you only use the quarterly data from 1971Q1 to as close to the present day as possible.

(b) Discuss the shape of the graph after the 2008 financial crisis.

(c) Add a trend line to the graph to more precisely pin down how UK labour productivity behaved after the 2008 financial crisis. What is the big picture? Why does the trend line in your graph look different to that produced by the ONS at http://visual.ons.gov.uk/productivity-puzzle/? Which graph gives the most accurate description of reality, yours or the one by the ONS?

(d) Why might UK labour productivity have behaved as it has since the 2008 financial crisis? Have a look at the Bank of England's Quarterly Bulletin 2014 Q2 and the 2014 speech by Martin Weale "The UK productivity puzzle: an international perspective" for background reading. They are available at: i) "The UK productivity puzzle: an international perspective", ii) Speech by Martin Weale

(e) Examine also the series "Output per Worker: Whole Economy SA: Index 2018=100: UK" (code = A4YM). Are you surprised by the behaviour of the productivity series in 2020 Q2 when the pandemic hit? If so, why? If not, why not?

(f) Can you plot similar data for your home country?

2 Simple representative agent problem

Suppose a representative agent has utility function involving consumption and labour supply of the form

$$U = \ln C - 2L^2,$$

where $\ln C$ is the natural logarithm.

(a) The agent is a "yeoman farmer" that produces their own output with a production function $C = Y = AL^\alpha$, where Y is output per head. Derive the optimal level of labour supply, and comment on what it implies for the relationship between labour supply and productivity growth.

(b) The agent is instead a worker, who receives a real wage w. Their budget constraint is $C = wL + \pi$, where π are profits distributed by firms, and both w and π are assumed to be exogenous by the worker. Derive an expression for optimal labour supply as a function of w and C.

(c) An individual, perfectly competitive firm maximises profits $\pi = AN^\alpha - wN$, where N is the number of workers it employs. Derive the labour demand curve for this firm. What is the relationship between the marginal product of labour and the real wage?

(d) Since each worker works L hours and each firm wishes to employ a total of N hours, equilibrium requires that $L = nN$ where n is the number of firms per worker (or, equivalently, the reciprocal of the number of workers per firm) . As consumption per head is equal to output per head, $C = nY = nAN^\alpha$. Use this, plus the expression for the real wage, in the labour supply equation to derive the equilibrium value for L in terms of model parameters. Compare this to your answer to (a), and comment.

(e) Suppose the number of agents in the economy increases (because of immigration, for example), but the new agents are just like the existing ones. As a result, the number of firms n falls. Show what happens to real wages, output per head and consumption per head.

(f) By deriving an expression for profits per firm in terms of L and n, comment on what might happen in the long run as n changes.

3 Simplified Romer model

Consider a two period version of a Romer model. The technology at the beginning of period 1 is given at a given exogenous level, A_1. In period 1, agents spend proportion $1 - l_1$ of their time in productive activities with production function:

$$Y_1 = A_1(1 - l_1)L,$$

where $L \geq 1$ is the tota labour supply in the economy. Agents spend the remaining proportion l_1 of their time in period 1 producing ideas, according to the production function:

$$A_2 - A_1 = zA_1 l_1 L,$$

with $z \geq 1$ a parameter measuring the productivity of workers producing ideas. In the final period 2, workers spend proportion $1 - l_2$ of their time in productive activities according to:

$$Y_2 = A_2(1 - l_2)L.$$

(a) Assuming that agents only value output produced in periods 1 and 2, what would be the best proportion of time to spend producing ideas in period 2?

(b) Make a plot of the combinations of y_1 and y_2 that are possible when agents choose different proportions of their time to spend on proiduction in period 1. To start you off, what happens to y_1 and y_2 if $l_1 = 0$ so that all time in period 1 is spent on production? What happens if $l_1 = 1$ so that all time in period 1 is spent producing ideas? What about intermediate cases where l_1 is between 0 and 1, for example 0.5?

(c) Agents ideally want to consume production goods in both periods 1 and 2. One way to model this is through a multiplicative utility function:

$$U = y_1 y_2.$$

Use the three production functions given in the introduction to this question to solve out for output per head y_1 and y_2 as functions of l_1 and A_1, z, L alone. What amount of time should be spent producing ideas in period 1 to maximise utility? Add indifference curves to your plot in part (b) to illustrate your answer.

(d) How does the optimal allocation of time in period 1 change as the productivity z of hours spent producing ideas increases?

(e) Without doing any extra calculations, discuss what you would expect to happen if agents start valuing goods produced in period 1 more that those produced in period 2 (why might they do this?). For example, their utility function might be $U = y_1^{\phi} y_2$, where ϕ ¿ 1. Answer intuitively, using a diagram where that would be helpful.

4 The natural rate of unemployment

Consider a bathtub model of unemployment. Let E_t denote the employment level and U_t the unemployment level in period t. Also, let L denote the (constant) labour force. Then, the model consists of the following two equations:

$$\Delta U_{t+1} = sE_t - fU_t,$$
$$E_t + U_t = L,$$

where f denotes the probability with which unemployed people find jobs in a certain period, and s denotes the probability with which employed people lose their jobs in a certain period. In other words, s is the separation rate and f is the job finding rate.

(a) Consider a steady state situation, where neither employment nor unemployment change over time. Describe the unemployment rate in the steady state, as a function of s and f.

Now suppose that the steady state unemployment is described by the formula you provided in part (a), but f and s are not constant. Here is how these variables are determined: The **job finding rate** is given by $f(e) = e$ where e takes values in $[0, 1]$ and denotes the worker's effort. This effort is, in turn, given by $e(b) = 0.5 - 0.05b$, where b denotes the level of unemployment benefits. Assume that b takes values in $[0, 10]$. The **job separation rate** is given by $s(c) = 0.1 - 0.02c$, where c is the fee that a firm has to pay in order to terminate a work relationship (also known as a firing cost). Assume that c takes values in $[0, 5]$.

(b) What is the economic intuition behind the determination of the job finding rate and the job separation rate above?

(c) Write the steady state unemployment rate as a function of b, c.[1] Does steady state unemployment depend positively or negatively on b and c? Discuss.

(d) If you were a policymaker, and your goal was to minimise unemployment, how would you set the policy variables b, c?

(e) If $c = 2.5$, how should the government set b in order to achieve unemployment equal to 10%?

[1] Notice that, by doing so, you have expressed the unemployment rate as a function of two policy variables (that is, two variables that are perfectly controlled by a policymaker).

5 Consumption with different generations

An economy is composed of identical individuals. Each individual lives for 2 periods (you may imagine them as adulthood and old age). Individuals may work during the first period of their life for a proportion L of the day, for an income equal to wL. In the second period they retire and consume their remaining lifetime savings. Their lifetime utility is given by:

$$U = \ln C_1 + \ln C_2 + \ln(1 - L),$$

where C_i is consumption in period i and L is the fraction of time the individual spends working in period 1.

(a) If the rate of interest on savings is R, write down the individual's budget constraints for both periods, and then combine them in a lifetime (intertemporal) budget constraint.

(b) Solve for optimal consumption each period and the optimal work effort. Comment on what you find.

(c) The government introduces a fixed pension paid to individuals in the second period of their lives, funded by a "lump sum tax" paid by those who work.[2] Re-write the intertemporal budget constraint. What will be the impact of the pension on consumption and labour supply decisions of the young? What about the old when the pension is introduced? Give intuition for your answers.

(d) The pension is now funded by an income tax, i.e., a tax equal to τwL where τ is the tax rate. Will the behaviour of the young change in the case where $R = 0$? Interpret this result.

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[2] A lump sum tax is one that is independent of income.

6 Pricing a stock

Consider the arbitrage equation for pricing a risk-free stock:

$$P_t = \frac{D_t + \Delta P_{t+1}}{R},$$

where P_t is the price of the stock in period t, D_t is the dividend the stock pays in period t, and $\Delta P_{t+1} = P_{t+1} - P_t$ is the capital gain the stock will make before the beginning of the next period. R is the interest rate available on an alternative investment in a bank account, assumed to be constant and the same in every period for simplicity – you can also abstract from uncertainty in your answers.[3]

(a) Re-write the arbitrage equation so it reds for P_t on the left hand side (LHS) and R, D_t, and P_{t+1} on the right hand side (RHS).

(b) The arbitrage equation holds for every period, so roll your solution for P_t in part (a) forward one period to read for P_{t+1} on the LHS and R, D_{t+1}, and P_{t+2} on the RHS.

(c) Substitute the solution for P_{t+1} from part (b) into the expression for P_t from part (a) to obtain an expression that has P_t on the LHS and R, D_t, D_{t+1}, and P_{t+2} on the RHS.

(d) Continue rolling the arbitrage equation forward to repeat steps (b) and (c) until you obtain an expression determining the stock price P_t as a function of only the interest rate R and current and all future dividends. Interpret your results.

(e) Alphabet Inc. (the parent company of Google) briefly displaced Apple Inc. as the largest company in the world on Tuesday 2nd February 2016. Its shares traded at $784 each, an implied market capitalisation of $531 billion. Alphabet has never paid any dividends. What does this imply about the calculations you made earlier? Is it possible to rationalise such a high market capitalisation with a lack of dividends?

[3]In other words, do not worry about taking the expectation of future variables with today's information set, $\mathbb{E}_t$.

7 OLG model with population growth

Consider a standard two-period overlapping generations (OLG) model with log utility, $\ln C_t^0 + \ln C_{t+1}^1$, and Cobb-Douglas production technology, $F(K_t, L_t) = K_t^\alpha L_t^{1-\alpha}$. Capital depreciates completely after a period and households only work when young (they inelastically supply labour normalised to 1 when young).

(a) Assume that there is no population growth, so that K_1 is the savings per young $t+1$ household made at time t and the aggregate capital stock per young household at time $t+1$. Write down the optimisation problem of the household when it is young and solve for the optimal consumption and savings decisions it makes, taking the wage rate w_t and the return to capital $R_{t+1} = 1 + r_{t+1}$ as given. Write down the problem of the perfectly competitive firm and show that market clearing implies a law of motion for capital of the form:

$$K_{t+1} = \frac{1}{2}(1-\alpha)K_t^\alpha.$$

Solve for the steady state capital stock. *** **Bonus** *** Log-linearise the law of motion for capital.

(b) Discuss how an increase in α affects the steady state capital stock and its law of motion when there is no population growth.[4]

(c) Now introduce constant population growth so that $L_{t+1} = (1+\eta)L_t$. If K_{t+1}^1 is the savings per young household made at time t then the aggregate capital stock per young household at time $t+1$ will be $(L_t K_{t+1}^1)/L_{t+1}$. Derive the law of motion for capital in the economy with population growth. Solve for steady state and log-linearise the law of motion for capital. How does an increase in the population growth rate η affect these objects?

(d) How would labour and capital income taxes affect the steady state of the economy? To do this, assume that taxes are paid by the household and proportional to labour and capital income, then calculate how the taxes affect first order conditions. You should then be able to identify what changes in steady state. Explain the intuition behind your findings.

(e) Suppose capital saved in period t does not depreciate fully after use in period $t+1$. Instead, assume that $1-\delta$ of the capital stock remains. How does this affect the transition dynamics of the system and how does it affect the steady state? By comparing what happens with depreciation to the effect of a capital tax in part (d), you should be able to answer this part of the question by direct reference to the first order conditions, deriving the implications for transition dynamics and steady state without further calculations.

(f) *** **Bonus** *** Instead of assuming that the utility and technology functions are of the functional forms described above, can you think of any alternative functional forms which would lead to multiple steady states?

[4] You may find it useful to take logs of the steady state equation and then differentiate with respect to α.

8 Simple infinite horizon optimisation problem and the consumption Euler equation

Consider an economy where households make consumption decisions to maximise:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to:

$$c_t + T_t + A_t = Y_t + (1+r)A_{t-1},$$

with A_0 given and $\beta(1+r) = 1$.[5] The instantaneous utility, $u(c_t)$, is not necessarily quadratic, but it is known that income is constant over time, i.e., $Y_t = Y\ \forall t$.

(a) Assume that taxes are set at time 0 and kept constant thereafter, i.e., $T_t = T\ \forall t$. Explain why under these assumptions consumption can be written as a function of financial wealth, A_t, and disposable income. Derive this expression for consumption and compare it to the consumption solution derived under the assumptions of Hall's random walk model (you can state the latter expression without a formal derivation).

(b) Assume the government budget constraint is:

$$G_t + (1+r_b)B_{t-1} = B_t + T_t,$$

where B_0, the initial level of government debt, is given, and government spending is kept constant, i.e., $G_t = G\ \forall t$ (continue to assume $T_t = T, \forall t$). The interest rate for the government, r_b, is not necessarily equal to that for households, r. Explain intuitively and show mathematically how consumption changes with G and B_0.

(c) Assume $r_b < r$, and that the other assumptions in part (b) apply except that the government tax plan is now to set taxes to zero in the first period and to a constant level after the first period, i.e., $T_0 = 0$ and $T_t = T > 0$ for $t \geq 1$, where T is set so that the intertemporal budget constraint is satisfied. Explain intuitively and derive mathematically how, if at all, aggregate demand changes compared with the scenario in part (b).

[5] Alternatively, you can interpret the timing of the state variable, A_t, in start of period notation, i.e., the constraint for the problem is:

$$c_t + T_t + A_{t+1} = Y_t + (1+r)A_t.$$

9 Hall's random walk theory of consumption

Consider the standard intertemporal model of consumption with infinite horizon, rational expectations consumers, and perfect capital markets. Consumption is denoted c, the real interest rate is $R = 1 + r$, and financial assets A.

(a) What further assumptions are needed for Hall's random walk result for consumption to hold? Derive the result mathematically and provide economic intuition for the result (include in your answer economic intuition for the roles played by each of the assumptions that you have identified as necessary for the result).

(b) Assume that in any period income is either y^L or y^H each with probability 0.5, with $y^L < y^H$ and $(y^L + y^H)/2 = y^*$. Explain why **before** y_0 (income in the first period) is known, the intertemporal budget constraint can be written as:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{c_t}{R^t} \right] = RA_0 + \frac{R}{r} y^*.$$

Hint: Use the fact that:

$$\sum_{i=0}^{\infty} ab^i = \frac{a}{1-b}, \quad \text{iff } |b| < 1.$$

(c) Suppose households choose c_0 after learning whether y_0 is y^L or y^H, but still not knowing hwat income is going to be thereafter. How much would households consume if they received $y_0 = y^L$? What if $y_0 = y^H$? In your answers assume that all the further assumptions in part (a) hold, and make use of the random walk result for consumption.

10 The RBC model

Consider a basic RBC model, where the socia planner wants to maximise

$$\mathbb{E}_t\left[\sum_{i=0}^{\infty}\beta^i(U(C_{t+i})-V(L_{t+i}))\right],$$

where C_t is consumption, L_t is hours worked, and β is the representative household's rate of time preference (discount factor). The economy faces constraints described by:

$$\begin{aligned} Y_t &= C_t + I_t, \\ Y_t &= F(K_t, L_t), \\ K_{t+1} &= I_t + (1-\delta)K_t, \end{aligned}$$

where $F(K_t, L_t)$ is the production technology of output, Y_t with constant returns to scale, I_t is investment, and δ is the rate of depreciation of capital. We can simplify the problem by combining the constraints into one equation:

$$F(K_t, L_t) = C_t + K_{t+1} - (1-\delta)K_t.$$

(a) Does the real business cycle (RBC) model predict that real wages should be procyclical or countercyclical? How about employment? Why?

(b) What does the empirical evidence say about the direction and magnitude of the fluctuations in these variables in comparison with the model's predictions?

(c) What are the implications of labour market developments for interpreting the validity of the RBC framework?

11 An analytic RBC model

An economy is populated by an infinitely-lived representative agent with preferences given by:

$$\sum_{t=0}^{\infty} \beta^t \ln C_t,$$

where C_t is consumption and ln is the natural logarithm. β is a discount factor that satisfies $\beta \in (0, 1)$. There is no uncertainty or labour in this economy, and output Y_t is a linear function of capital K_t. Capital can be accumulated through a technology that has a constant elasticity of substitution form in existing capital and investment I_t. The constraints of the economy are therefore:

$$\begin{aligned} C_t + I_t &= Y_t, \\ Y_t &= K_t, \\ K_{t+1} &= K_t^{\alpha} I_t^{\gamma}, \end{aligned}$$

with $\alpha > 0$, $\gamma > 0$, and $\alpha + \gamma < 1$.

(a) Re-write the constraints of the economy as equations for consumption C_t and future capital stock K_t in terms of the current capital stock K_t and the investment rate $s_t = I_t/Y_t$.

(b) Use the constraints expressed in terms of current capital stock and the investment rate to derive the first-order conditions of the social planner's problem. Interpret each condition briefly.

(c) Combine the two first-order conditions and find the steady-state investment rate in this economy. How does it vary with α and γ and why? Hint: You may find it useful to work with a transformation of the investment rate $\nu_t = s_t/[\gamma(1 - s_t)]$.

(d) Derive and discuss the conditions under which there is a unique perfect foresight equilibrium with the investment rate at its steady-state value. Your answer should include a diagram. How do parameter values affect the speed of convergence to the equilibrium?

(e) Returning to the equation for future capital stock derived in part (a), is it possible to have a constant investment rate but ever-increasing capital in the model? Explain your answer and assess the realism of any restrictions applied to parameter values.