Inverse problems in nonlinear wave propagation

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Abstract and goal

The ability to control a desired dynamics or pattern has attracted considerable attention over the last decades. Often, the applied control signals are treated as weak perturbations and an equation of motion for perturbed solution in response to the applied controls is derived.
Recently [1-3], we've pose the inverse question of how to design controls that lead to a propagation of the solution following a desired velocity protocol?

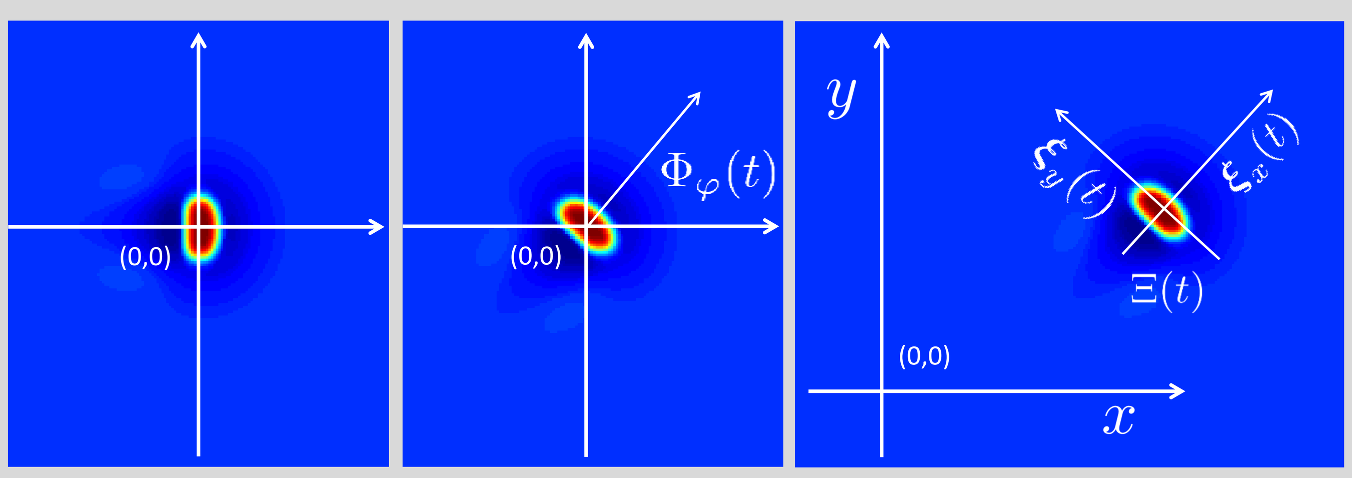
Thereby, our goal is to derive analytical expressions for a minimal invasive, open loop control to adjust the position and orientation of wave patterns according to a ***prescibed protocol of motion***.

Goldstone-mode control

- n-component $\mathbf{u}(\mathbf{r},t) = (\mathbf{u}_1(\mathbf{r},t),\ldots,\mathbf{u}_n(\mathbf{r},t))^T$ system

$$\partial_t \boldsymbol{u}(\mathbf{r},t) - \boldsymbol{\mathcal{L}}\boldsymbol{u}(\mathbf{r},t) - \boldsymbol{\mathcal{N}}(\boldsymbol{u}(\mathbf{r},t)) = \boldsymbol{\mathcal{B}}\boldsymbol{f}(\mathbf{r},t), \quad \mathbf{r}\in\mathbb{R}^2 \tag{1}$$

  linear $\boldsymbol{\mathcal{L}}$ and nonlinear $\boldsymbol{\mathcal{N}}$ operator , vector of controls $\boldsymbol{f}(\mathbf{r},t) = (f_1,\ldots,f_m)^T$, nxm coupling matrix $\boldsymbol{\mathcal{B}}$
- Eq. (1) possesses a TW solution $\mathbf{U}_c(\boldsymbol{\xi}) = \mathbf{U}_c(\mathcal{A}(-\omega t)\,(\mathbf{r}-\mathbf{v}^0 t))$ moving either with constant velocity $\mathbf{v^0} = (v_x^0, v_y^0)^T$ or $\omega \neq 0$
- aim to control the position of a pattern $\mathbf{U}_c(\mathbf{r},t)$ according to a prescribed ***protocol of motion*** $\Xi(t) = (\boldsymbol{\Phi},\Phi_\varphi)^T \equiv (\Phi_x,\Phi_y,\Phi_\varphi)^T$



- for desired trajectory $\boldsymbol{u}_d(\mathbf{r},t) = \boldsymbol{U}_c(\mathcal{A}(-\Phi_\varphi(t))\,(\mathbf{r}-\boldsymbol{\Phi}(t)))$

  control is proportional to Goldstone modes of the pattern [2]

$$\boldsymbol{f}_{\rm Gold}(\mathbf{r},t) = \boldsymbol{\mathcal{B}}^{-1}\left(\mathcal{A}^z(-\omega\, t)\begin{pmatrix} v_x^0 \\ v_y^0 \\ \omega\end{pmatrix} - \mathcal{A}^z(-\Phi_\varphi(t))\begin{pmatrix}\dot\Phi_x(t)\\ \dot\Phi_y(t)\\ \dot\Phi_\varphi(t)\end{pmatrix}\right) \times \begin{pmatrix} \partial_x \mathbf{U}_c\left(\mathcal{A}\left(-\Phi_\varphi(t)\right)\cdot(\mathbf{r}-\mathbf{\Phi}(t))\right) \\ \partial_y \mathbf{U}_c\left(\mathcal{A}\left(-\Phi_\varphi(t)\right)\cdot(\mathbf{r}-\mathbf{\Phi}(t))\right) \\ \partial_\phi \mathbf{U}_c\left(\mathcal{A}\left(-\Phi_\varphi(t)\right)\cdot(\mathbf{r}-\mathbf{\Phi}(t))\right)\end{pmatrix}$$

- ❑ no detailed knownledge of underlying system dynamics ($\boldsymbol{\mathcal{L}}$, $\boldsymbol{\mathcal{N}}$) needed
- ❑ only stimulates translation modes → no deformations
- ❑ open-loop control realized by spatio-temporal forcing
- ❑ signals are localized at controlled pattern
- ❑ Goldstone-mode control is solution to unregularized optimal control problem and remarkably close to regularized one [2-4]
- ❑ $\mathbf{U}_c(\mathbf{r},t), \mathbf{v}^0$, and ω have to be measured with sufficient accuracy **only once**

References

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Neural field equations

- neural field systems exhibit self-organized, spatio-temporal structures like traveling fronts and pulses, spiral-waves, and localized bump solutions
- make them a convenient tool to describe various neural processes, such as working memory, motion perception, visual hallucinations etc.

- two-component neural field equations, with local activity $u(\mathbf{r},t)$, linear adaptation variable $v(\mathbf{r},t)$, coupling kernel $\omega(|\mathbf{r}|)$, firing function $\mathcal{N}$, and external input $I(\mathbf{r},t)$ [5]

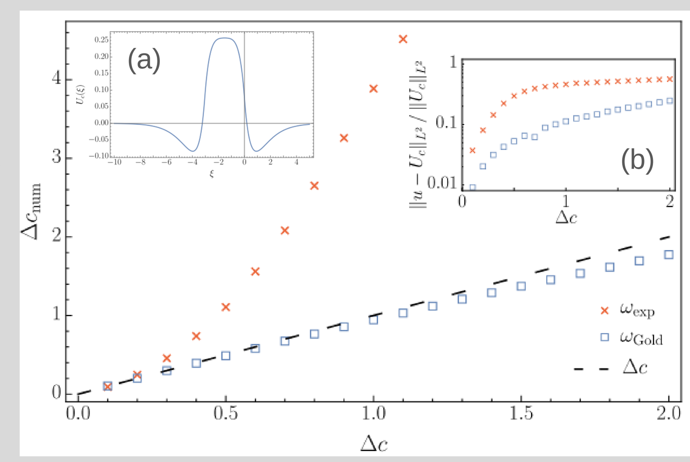
$$\begin{aligned}\partial_t u(\mathbf{r},t) &= -\,u + \int_\Omega \mathrm{d}\mathbf{r}' w\left(|\mathbf{r}-\mathbf{r}'|\right)\mathcal{N}\left[u\left(\mathbf{r}',t\right)\right] - \beta\, v\left(\mathbf{r},t\right) + I\left(\mathbf{r},t\right), \\ \partial_t v(\mathbf{r},t) &= \varepsilon\left(u(\mathbf{r},t) - v(\mathbf{r},t)\right)\end{aligned} \tag{2}$$

Position control by kernel modulation

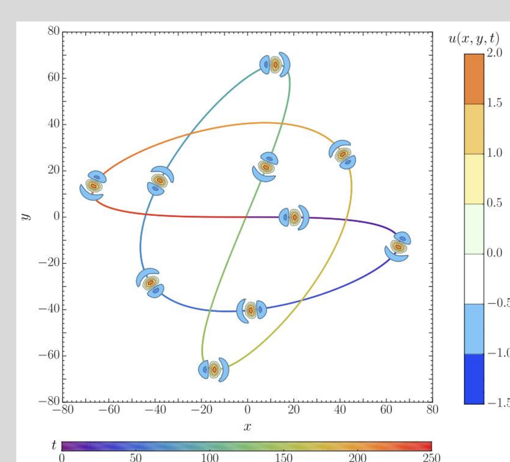
- additive kernel modulation $\omega \to \omega_0 + \omega_{\rm Gold}$ has to equal Goldstone-mode control
- for scalar (β = 0) NFE in 1D [5]

$$\omega_{\rm Gold}(x,t) = \frac{v_x^0 - \dot\phi(t)}{v_x^0}\int_x^\infty \mathrm{d}x' e^{\frac{x-x'}{v_x^0}}\frac{\mathrm{d}\,\omega_0(x')}{\mathrm{d}\,x'} \stackrel{v_x^0\to 0}{=} -\dot\phi(t)\frac{\mathrm{d}\,\omega_0(x)}{\mathrm{d}\,x} \tag{3}$$

- allows for explicit position control without excitation of deformation modes



Velocity changes Δc_{num} of immobile bump (a) controlled by ω_{Gold} and exponentially decaying ω_{exp} kernel modulations. Inset (b): Ratio between L^2-shape deviations of Goldstone kernel controlled solution to unperturbed bump.
$\beta = 0, I = 0, \omega_0 = \exp(-1.8|x|) - 0.5\exp(-|x|)$,
$\mathcal{N}[u] = \Theta(u-\vartheta), \vartheta = 0.077$

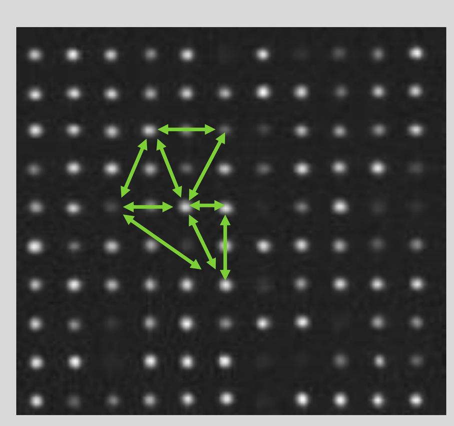


First component u of an orientation controlled bump solution at different times t, indicated by color gradient.

Orientation control by additive Inputs in 2D

- in order to control the bump's orientation, the angular derivative of the solution is excited
- compared to position control, lower control amplitudes are often sufficient as the scheme makes use of the "natural" movement of bump solution in direction of its symmetry axis
- but orientation control becomes unstable if amplitudes are too small and numerical perturbations are dominant

Network of chemical oscillators



Chemical oscillators are monitored with camera. Based on measured state of the oscillators (M_{red}) they are illuminated individually with light from a spatial light modulator.

- network build by porous cation-exchange beads that are loaded with the catalyst in oscillating BZ reaction
- modelled by three component Oregonator model

$$\begin{aligned}\dot u_i &= \left[u_i - u_i^2 - w_i(u_i-\mu)\right]/\epsilon_u \\ \dot v_i &= u_i - v_i \\ \dot w_i &= \left[f\, v_i + \phi_i - w_i(u_i+\mu)\right]/\epsilon_w\end{aligned} \tag{4}$$

- *u*, *v*, *w* stand for dimensionless concentrations of $HBrO_2$, M_{ox}, and Br^-

- elements are either excitable or oscillatory
- based on measured state

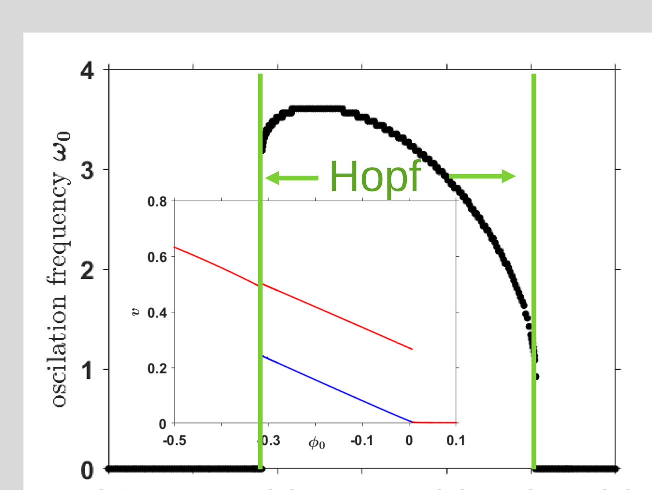
  $$[M_{\rm red}] \propto v_i^{\rm norm}(\phi_i) = \frac{v_{\max}(\phi_i) - v_i}{v_{\max}(\phi_i) - v_{\min}(\phi_i)}$$

  the activity of each node is calcultated via

  $$\dot I_i = \left[\Theta\left(v_i^{\rm norm} - \theta_{\rm tresh}\right) - I_i\right]/\epsilon_I$$
- nodes are coupled to a ring network

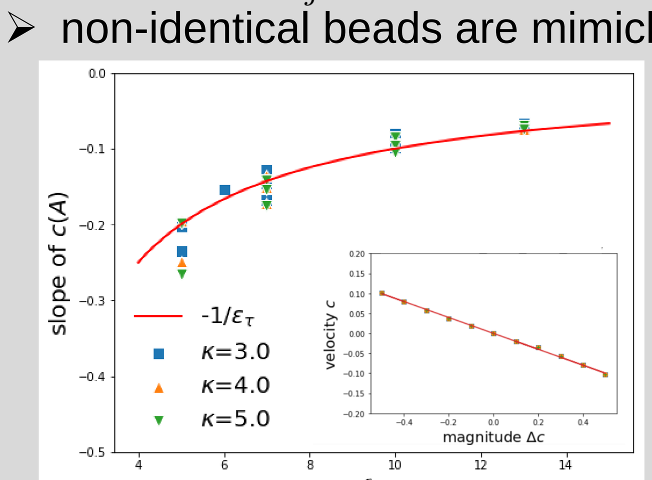
  $$G_{i,j} = \kappa\left(1 - A\,\cos\left(2\pi\, n_{\rm head}(i-j)/N\right)\right)$$
- light intensity at *i*-th node is determined by

  $$\phi_i = \phi_{0,i} + \sum_j G_{i,j} I_j$$

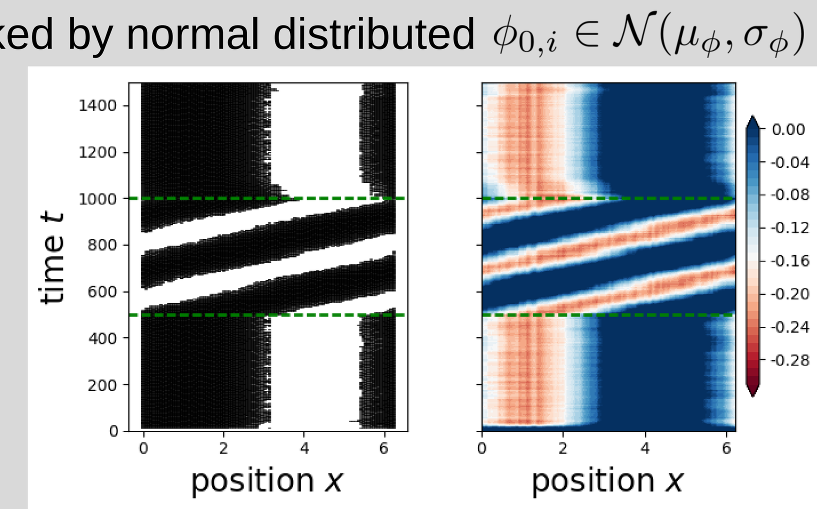


Impact of light flux on oscillation frequency in Eq.(4).
Inset: Maximum and minimum value of v on limit cycle.
$\epsilon_u = 40.0, \epsilon_w = 3600.0, \mu = 0.0020, f = 1.160$

- non-identical beads are mimicked by normal distributed $\phi_{0,i} \in \mathcal{N}(\mu_\phi, \sigma_\phi)$



Position control of bump solution in 3k Oreg. network via control approach Eq. (3).
Inset: velocity vs. control magnitude Δc.
Main: Impact of ϵ_I on slope for various combinations of κ and A.



Position control of bump solution in ring network. Modulated kernel with $\Delta c = -0.3$ is switched on at t=500 and off at t=1000.
Left: Spike events (black dashes). Right: Space-time dynamics of the non-local light flux applied to individual elements.
$\epsilon_u = 40, \epsilon_w = 3600, \epsilon_I = 10, \mu = 0.002, f = 1.16, \kappa = 3.0, A = 1.5,$
$\theta_{\rm thresh} = 0.9, \mu_\phi = -0.4, \sigma_\phi = 0.01.$