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Szemerédi regularity lemma

In mathematics, **the Szemerédi regularity lemma** states that every large enough graph can be divided into subsets of about the same size so that the edges between different subsets behave almost randomly. Szemerédi (1975) introduced a weaker version of this lemma, restricted to bipartite graphs, in order to prove Szemerédi's theorem,[1] and in (Szemerédi 1978) he proved the full lemma.[2] Extensions of the regularity method to hypergraphs were obtained by Rödl and his collaborators[3][4][5] and Gowers.[6][7]

Contents

Statement

The formal statement of Szemerédi's regularity lemma requires some definitions. In what follows G is a graph with vertex set V.

> **Definition 1.** Let X, Y be disjoint subsets of V. The **density** of the pair (X, Y) is defined as:
>
> $$d(X,Y) := \frac{|E(X,Y)|}{|X||Y|}$$
>
> where $E(X, Y)$ denotes the set of edges having one end vertex in X and one in Y.

> **Definition 2.** For $\varepsilon > 0$, a pair of vertex sets X and Y is called **ε-regular**, if for all subsets $A \subseteq X$, $B \subseteq Y$ satisfying $|A| \geq \varepsilon|X|$, $|B| \geq \varepsilon|Y|$, we have
>
> $$|d(X,Y) - d(A,B)| \leq \varepsilon.$$

> **Definition 3.** A partition of V into k sets: $V_1, \ldots, V_k$, is called an **ε-regular partition**, if:
>
> - for all i, j we have: $||V_i| - |V_j|| \leq 1$;
> - all except εk^2 of the pairs V_i, V_j, $i < j$, are ε-regular.

Now we can state the lemma:

> **Regularity Lemma.** For every $\varepsilon > 0$ and positive integer m there exists an integer M such that if G is a graph with at least M vertices, there exists an integer k in the range $m \le k \le M$ and an ε-regular partition of the vertex set of G into k sets.

The bound M for the number of parts in the partition of the graph given by the proofs of Szemeredi's regularity lemma is very large, given by a $O(\varepsilon^{-5})$-level iterated exponential of m. At one time it was hoped that the true bound was much smaller, which would have had several useful applications. However Gowers (1997) found examples of graphs for which M does indeed grow very fast and is at least as large as a $\varepsilon^{-1/16}$-level iterated exponential of m. In particular the best bound has level exactly 4 in the Grzegorczyk hierarchy, and so is not an elementary recursive function.[8]

Extensions

János Komlós, Gábor Sárközy and Endre Szemerédi later (in 1997) proved in the blow-up lemma[9][10] that the regular pairs in Szemerédi regularity lemma behave like complete bipartite graphs under the correct conditions. The lemma allowed for deeper exploration into the nature of embeddings of large sparse graphs into dense graphs.

An inequality of Terence Tao extends the Szemerédi regularity lemma.[11]

It is not possible to prove a variant of the regularity lemma in which all pairs of partition sets are regular. Some graphs, such as the half graphs, require many pairs of partitions (but a small fraction of all pairs) to be irregular.[12]

It is a common variant in the definition of an ϵ-regular partition to require that the vertex sets all have the same size, while collecting the leftover vertices in an "error"-set V_0 whose size is at most an ϵ-fraction of the size of the vertex set of G.

Algorithmic version

An ε-regular partition of a given graph can be found algorithmically. The first constructive version was provided by Alon, Duke, Lefmann, Rödl and Yuster.[13] Subsequently, Frieze and Kannan gave a different version and extended it to hypergraphs.[14]

Here we will briefly describe a different construction due to Alan Frieze and Ravi Kannan that uses singular values of matrices.

The algorithm[15] is based on two crucial lemmas:

Lemma 1:
Fix k and γ and let $G = (V, E)$ be a graph with n vertices. Let P be an equitable partition of V in classes $V_0, V_1, \ldots, V_k$. Assume $|V_1| > 4^{2k}$ and $4^k > 600\gamma^2$. Given proofs that more than γk^2 pairs (V_r, V_s) are not γ-regular, it is possible to find in O(n) time an equitable partition P' (which is a refinement of P) into $1 + k4^k$ classes, with an exceptional class of cardinality at most $|V_0| + n/4^k$ and such that $\operatorname{ind}(P') \ge \operatorname{ind}(P) + \gamma^5/20$

Lemma 2:
Let W be a $R \times C$ matrix with $|R| = p$, $|C| = q$ and $\|W\|_{\inf} \le 1$ and γ be a positive real.
(a) If there exist $S \subseteq R$, $T \subseteq C$ such that $|S| \ge \gamma p$, $|T| \ge \gamma q$ and $|W(S,T)| \ge \gamma |S||T|$ then $\sigma_1(W) \ge \gamma^3 \sqrt{pq}$
(b) If $\sigma_1(W) \ge \gamma\sqrt{pq}$, then there exist $S \subseteq R$, $T \subseteq C$ such that $|S| \ge \gamma' p$, $|T| \ge \gamma' q$ and $W(S,T) \ge \gamma' |S||T|$ where $\gamma' = \gamma^3/108$. Furthermore S, T can be constructed in polynomial time.

These two lemmas are combined in the following algorithmic construction of the Szemerédi regularity lemma.

[Step 1] Arbitrarily divide the vertices of G into an equitable partition P_1 with classes $V_0, V_1, \ldots, V_b$ where $|V_i| = \lfloor n/b \rfloor$ and hence $|V_0| < b$. denote $k_1 = b$.
[Step 2] For every pair (V_r, V_s) of P_i, compute $\sigma_1(W_{r,s})$. If the pair (V_r, V_s) are not ε-regular then by Lemma 2 we obtain a proof that they are not $\gamma = \varepsilon^9/108$-regular.
[Step 3] If there are at most $\varepsilon\binom{k_1}{2}$ pairs that produce proofs of non γ-regularity that halt. P_i is ε-regular.
[Step 4] Apply Lemma 1 where $P = P_i$, $k = k_i$, $\gamma = \varepsilon^9/108$ and obtain P' with $1 + k_i 4^{k_i}$ classes
[Step 5] Let $k_i + 1 = k_i 4^{k_i}$, $P_i + 1 = P'$, $i = i + 1$ and go to Step 2.

The algorithm will terminate with an ε-regular partition in $O(\varepsilon^{-45})$ steps since the improvement at each step is $\gamma^5/20 = O(\varepsilon^{45})$.

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