CHAPTER 5

Recurrent Neural Networks

The multilayer perceptron and the RBF network considered in Chapters 3 and 4 represent two important examples of a class of neural networks known as nonlinear layered feedforward networks. In this chapter, we consider another class of neural networks that have a recurrent structure.

As we have seen from the preceding chapters, time plays a critical role in learning. When time is built into the operation of a neural network through the use of global feedback, which encompasses one or more layers of hidden neurons, or the whole network, it results in the recurrent neural network.

The recurrent neural networks incorporate a static multilayer perceptron or parts thereof, and exploit the nonlinear mapping capability of the multilayer perceptron as well.

5.1 THE HOPFIELD NETWORK

The Hopfield network is a form of recurrent artificial neural network invented by John Hopfield in 1982. It consists of a set of neurons and a corresponding set of unit time delays, formatting a multiple-loop feedback system. A simple example of the architectural graph can be seen in Figure 5.1. The number of feedback loops is equal to the number of neurons. Basically, there is no self-feedback in the model. The output of each neuron is fed back, via a unit time delay element, to each of the other neurons in the network.

The equations that describe the network operation are

$$x(k) = p \tag{5.1}$$

and

$$x(k+1) = \text{satlins}(Wp + b) \tag{5.2}$$

where satlins is the transfer function that is linear in the range $[-1, 1]$, and saturates at 1 for inputs greater than 1 and at -1 for inputs less than -1.

Fundamentals of Computational Intelligence: Neural Networks, Fuzzy Systems, and Evolutionary Computation, First Edition. James M. Keller, Derong Liu, and David B. Fogel.

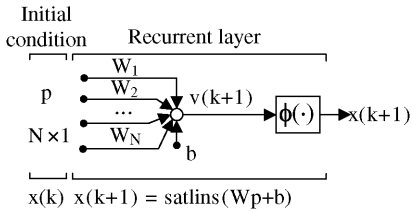


FIGURE 5.1 Architectural graph of the Hopfield network consisting of N neurons.

To illustrate the operation of the network, we have determined a weight matrix and a bias vector that can solve our orange and apple pattern recognition problem. They are given in

$$W = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, \quad b = \begin{bmatrix} 0.9 \\ 0 \\ -0.9 \end{bmatrix} \tag{5.3}$$

Although the procedure for computing the weights and biases for the Hopfield network is beyond the scope of this chapter, we can say a few things about why the parameters in (5.3) work for the apple and orange example.

We want the network output to converge to either the orange pattern, p_1, or the apple pattern, p_2. In both patterns, the first element is 1 and the third element is -1. The difference between the patterns occurs in the second element. Therefore, no matter what pattern is input to the network, we want the first element to converge to -1, and the second element to go to either 1 or -1, whichever is closer to the second element of the input vector.

The equations of operation of the Hopfield network, using the parameters given in (5.3), are

$$\begin{aligned} x_1(k+1) &= \text{satlins}(0.2x_1(k) + 0.9) \\ x_2(k+1) &= \text{satlins}(1.2x_2(k)) \\ x_3(k+1) &= \text{satlins}(0.2x_3(k) - 0.9) \end{aligned} \tag{5.4}$$

Regardless of the initial values, $x_1(k)$, the first element will be increased until it saturates at 1, and the third element will be decreased until it saturates at -1. The second element is multiplied by a number larger than 1. Therefore, if it is initially negative, it will eventually saturate at -1; if it is initially positive, it will saturate at 1.

Let us take our oblong orange to test the Hopfield network. The outputs of the Hopfield network for the first three iterations would be

$$x(k) = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad x(k+1) = \begin{bmatrix} 0.7 \\ -1 \\ -1 \end{bmatrix}, \quad x(k+2) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad x(k+3) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \tag{5.5}$$

The network has converged to the orange pattern.

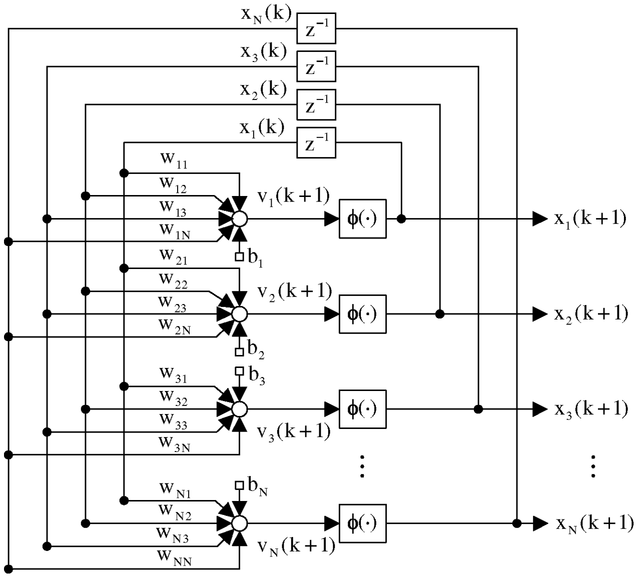


FIGURE 5.2 Discrete model of the Hopfield network consisting of N neurons.

Now, we will introduce a normalized discrete Hopfield network based on the McCulloch–Pitts model, which is shown in Figure 5.2.

The induced local field of neuron j at time step k + 1 is denoted by

$$v_j(k+1) = \sum_{i=1}^{N} w_{ji} x_i(k) + b_j \tag{5.6}$$

where w_{ji} and b_j are the synaptic weight and the bias of neuron j, respectively. Additionally, $w_{ji} = 0$ when $i = j$, which in fact means that the self-feedback does not exist in the model. Then, the output of neuron j by applying the signum function is

$$x_j(k+1) = \phi(v_j(k+1)) = \begin{cases} 1, & \text{if } v_j(k+1) > 0; \\ -1, & \text{if } v_j(k+1) < 0 \end{cases} \tag{5.7}$$

Note that neuron j remains in its previous state if $v_j(k+1)$ is zero.

Next, we consider the circuit model of the continuous Hopfield network depicted in Figure 5.3, where N denotes the number of neurons and $\phi_j(\cdot), j = 1, 2, \ldots, N$, represent the activation functions. The corresponding physical terms in Figure 5.3 are defined as follows:

I_j: external current

w_{ji}: conductance

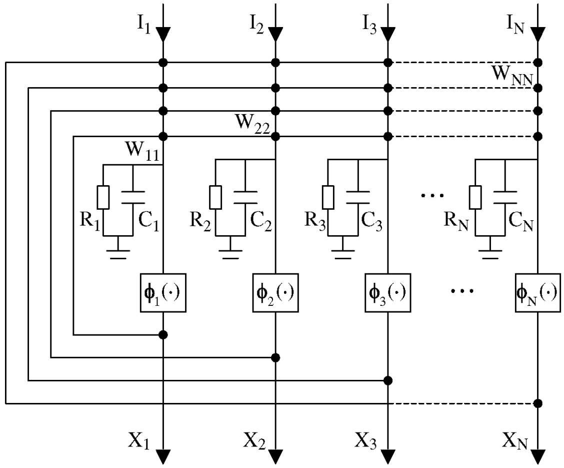


FIGURE 5.3 Circuit model of the continuous Hopfield network consisting of N neurons.

R_j: leakage resistance
C_j: leakage capacitance
x_j: potential

Let the induced local field of neuron j be denoted by

$$v_j(t) = \sum_{i=1}^{N} w_{ji} x_i(t) + I_j \tag{5.8}$$

where $x_i(t)$ is the potential at time t. Also, we can determine the output of neuron j by using the nonlinear relation

$$x_j(t) = \phi_j(v_j(t)) \tag{5.9}$$

Then, according to the Kirchoff's current law, we have

$$C_j(t)\frac{dv_j(t)}{dt} + \frac{v_j(t)}{R_j} = \sum_{i=1}^{N} w_{ji} x_i(t) + I_j, \quad j = 1, 2, \ldots, N \tag{5.10}$$

Hence, the model of the Hopfield network can be given in the following form:

$$C_j(t)\frac{dv_j(t)}{dt} = -\frac{v_j(t)}{R_j} + \sum_{i=1}^{N} w_{ji} \phi_i(v_i(t)) + I_j, \quad j = 1, 2, \ldots, N \tag{5.11}$$

5.2 THE GROSSBERG NETWORK

In this section, we will present the Grossberg network [Grossberg, 1998]. This network was inspired by the operation of the mammalian visual system. Grossberg networks are so heavily influenced by biology that it is difficult to discuss his networks without putting them in their biological context. In this section we want to provide a brief introduction to vision, so that the function of the network will be more understandable.

First, we see Figure 5.4 [Hagan, Demuth, and Beale, 1996]. In part (a), we see an edge as it is originally perceived by the rods and cones, with missing sections. But usually we do not see edges as displayed in part (a). The neural systems in our visual pathway must be performing some operation that compensates for the distortions and completes the image.

Grossberg suggests that there are two primary types of compensatory processing involved. The first, which he calls *emergent segmentation*, completes missing boundaries. The second, which he calls *featural filling-in*, fills in the color and brightness inside the resulting boundaries. Consider, for example, the two figures in Figure 5.5. In part (a) you should be able to see a bright white triangle lying on the top of several other black objects. In fact, no such triangle exists in the figure. It is purely a creation of the emergent segmentation and featural filling-in process of your visual system. The same is true of the bright white circle that appears to lie on the top of the lines in part (b) of the figure.

In addition to emergent segmentation and featural filling-in, there are two other phenomena that give us an indication of what operations are being performed in the

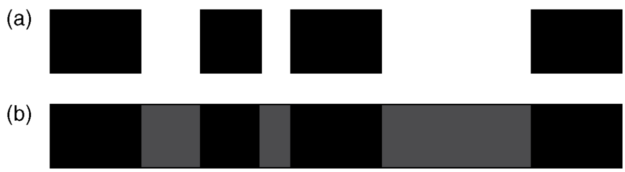


FIGURE 5.4 Compensatory processing.

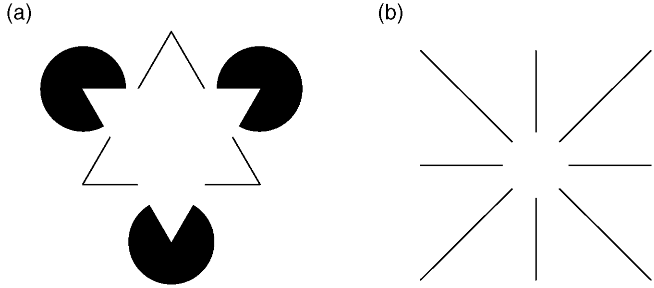


FIGURE 5.5 Emergent segmentation and featural filling-in.

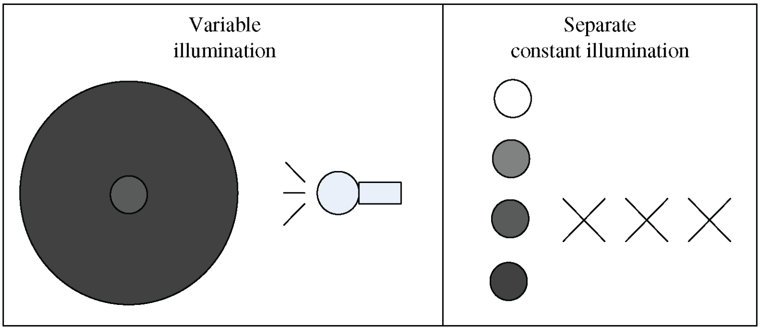


FIGURE 5.6 Test of brightness constancy.

early vision system: *brightness constancy* and *brightness contrast*. The brightness constancy effect is shown as small gray disk inside a darker gray annulus, which is illuminated by white light of a certain intensity. The subject is asked to indicate the brightness of the central disk by looking at a series of gray disks, separately illuminated, and selecting the disk with the same brightness. Next, the brightness of the light illuminating the gray disk and dark annulus is increased, and the subject is again asked to select the disk as matching the original central disk. Even though the total light entering the subject's eye is 10–100 times brighter, it is only the relative brightness that registers (Figure 5.6).

Another phenomenon of the vision system, which is closely related to brightness constancy, is brightness contrast. This effect is illustrated by the two figures in Figure 5.7. At the centers of the two figures, we have two small disks with equivalent gray scale. The small disk in part (a) of the figure is surrounded by a darker annulus, while the small disk in part (b) is surrounded by a lighter annulus. Even though both disks have the same gray scale, the one inside the darker annulus appears brighter. This is because our vision system is sensitive to relative intensities. It would seem that the total activity across the image is held constant.

The properties of the brightness constancy and brightness contrast are very important to our vision system. Since we see things in so many different lighting conditions, if we were not able to compensate for the absolute intensity of a scene, we

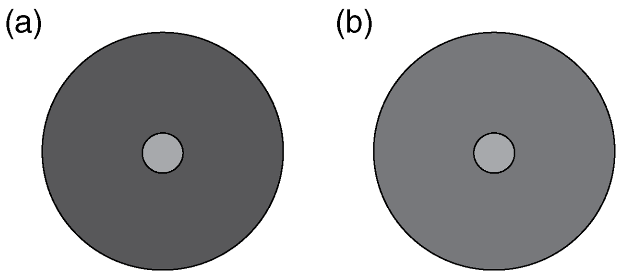


FIGURE 5.7 Test of brightness contrast.

would never learn to recognize things. Grossberg calls this process of normalization "discounting the illuminant."

5.2.1 Basic Nonlinear Model

Before we introduce the Grossberg network, we will begin by looking at some of the building blocks that make up the network [Hagan, Demuth, and Beale, 1996]. The first building block is the "leaky" integrator, which is shown in Figure 5.8.

The response of the leaky integrator to an arbitrary input p(t) is

$$n(t) = e^{-(t/\varepsilon)}n(0) + \frac{1}{\varepsilon}\int_0^t e^{-(t-\tau)/\varepsilon}p(t-\tau)d\tau \tag{5.12}$$

The leaky integrator forms the nucleus of one of Grossberg's fundamental neural models: *the shunting model*, which is shown in Figure 5.9. The equation of operation of this network is

$$\varepsilon\frac{dn}{dt} = -n + (b^+ - n)p^+ - (n + b^-)p^- \tag{5.13}$$

where p^+ is a nonnegative value representing the *excitatory* input to the network, and p^- is a nonnegative value representing the *inhibitory* input. The biases b^+ and b^- are

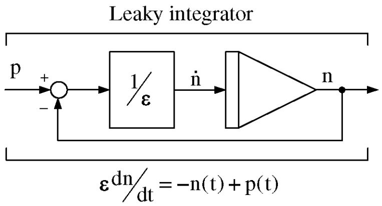


FIGURE 5.8 Leaky integrator.

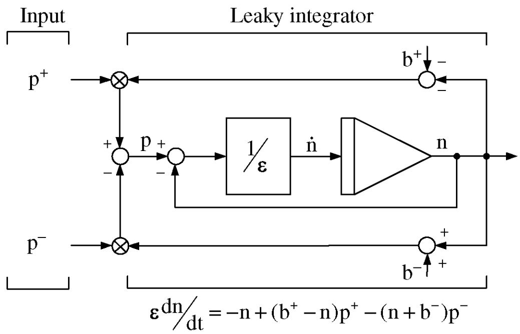


FIGURE 5.9 Shunting model.

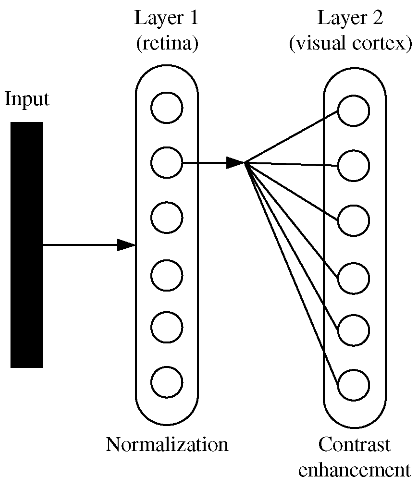


FIGURE 5.10 Grossberg competitive network.

nonnegative constants that determine the upper and lower limits on the neuron response, respectively. From the operation of the shunting model (5.7), we can see that if n(0) falls between b^+ and b^-, then n(t) will remain between these limits.

5.2.2 Two-Layer Competitive Network

We are now ready to present the Grossberg competitive network. There are three components to the Grossberg network: Layer 1, Layer 2, and the adaptive weights. Layer 1 is a rough model of the operation of the retina, while Layer 2 represents the visual cortex. A block diagram of the network is shown in Figure 5.10.

5.2.2.1 Layer 1 Layer 1 of the Grossberg network receives external inputs and normalizes the intensity of the input pattern (Figure 5.11).

The equation of operation of layer 1 is

$$\varepsilon(dn^1/dt) = -n^1 + ({}^+b^+ - n^1)[{}^+W^1]p - (n^1 + {}^-b^1)[{}^-W^1]p \qquad (5.14)$$

where

$${}^+W^1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

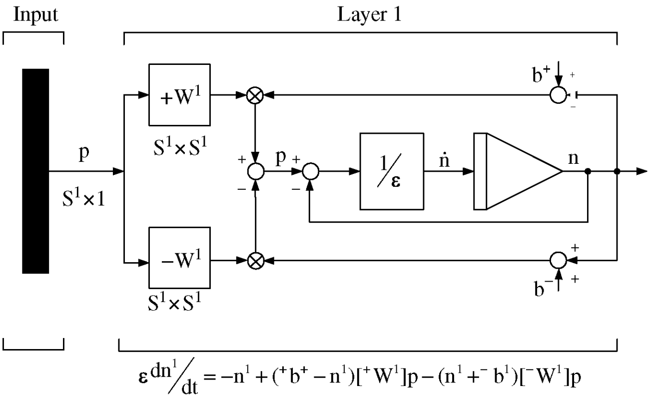


FIGURE 5.11 Layer 1 of the Grossberg network.

Therefore, the excitatory input to neuron i is the ith element of the input vector. The inhibitory input to layer 1 is $[{}^-W^1]p$ where

$$
{}^-W^1 = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}
$$

Thus, the inhibitory input to neuron i is the sum of all elements of the input vector, expect the ith element.

To illustrate the performance of Layer 1, consider the case of two neurons, with the inhibitory bias ${}^-b^1 = 0$, ${}^+b^1 = 1$, $\varepsilon = 0.1$:

$$
(0.1)\frac{dn_1^1(t)}{dt} = -n_1^1(t) + (1 - n_1^1(t))p_1 - n_1^1(t)p_2 \tag{5.15}
$$

$$
(0.1)\frac{dn_2^1(t)}{dt} = -n_2^1(t) + (1 - n_2^1(t))p_1 - n_2^1(t)p_2 s \tag{5.16}
$$

For two inputs $P_1 = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 10 \\ 40 \end{bmatrix}$, we can compute that the response of the network maintains the relative intensities of the inputs, while limiting the total response. The total response $(n_1^1(t) + n_2^1(t))$ will always be less than 1.

5.2.2.2 Layer 2 Layer 2 of the Grossberg network, which is a layer of continuous-time instars, performs several functions. Figure 5.12 is a diagram of layer 2. As with layer 1, the shunting model forms the basis for layer 2. The main difference between layer 2 and layer 1 is that layer 2 uses feedback connections. The feedback enables the

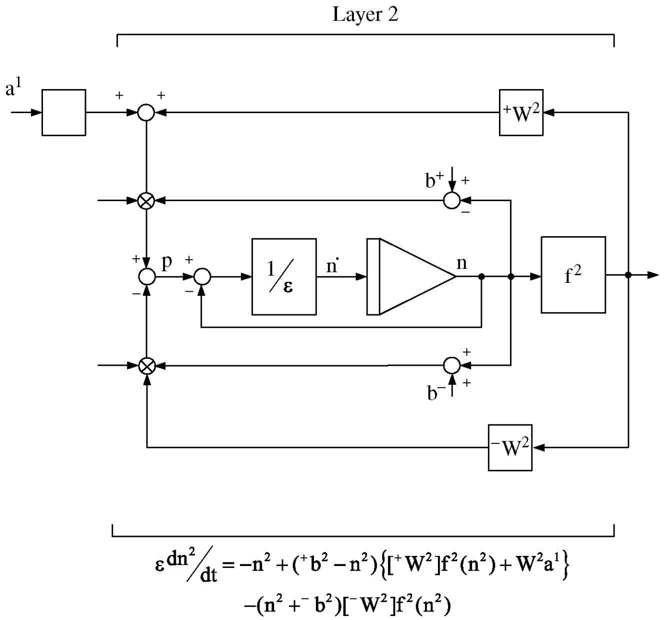


FIGURE 5.12 Layer 2 of the Grossberg network.

network to store a pattern, even after the input has been removed. The feedback also performs the competition that causes the contrast enhancement of pattern.

The equation of operation of layer 2 is

$$\varepsilon\frac{dn^2}{dt} = -n^2 + (^{+}b^2 - n^2)\left\{[^{+}W^2]f^2(n^2) + W^2a^1\right\} \\ -(n^2 + {}^{-}b^2)[^{-}W^2]f^2(n^2) \tag{5.17}$$

To illustrate the performance of layer 2, consider a two-neuron layer with

$$\varepsilon = 0.1, {}^{+}b^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, {}^{-}b^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, W^2 = \begin{bmatrix} ({}_1w^2)^T \\ ({}_2w^2)^T \end{bmatrix} = \begin{bmatrix} 0.9 & 0.45 \\ 0.45 & 0.9 \end{bmatrix} \tag{5.18}$$

and

$$f(n^2) = \frac{10(n)^2}{1 + (n)^2} \tag{5.19}$$

The equations of operation of the layer will be

$$(0.1)\frac{dn_1^2(t)}{dt} = -n_1^2(t) + (1 - n_1^2(t))\left\{f^2(n_1^2(t)) + ({}_1w^2)^T a^1\right\} - n_1^2(t)f^2(n_2^2(t)) \tag{5.20}$$

$$(0.1)\frac{dn_2^2(t)}{dt} = -n_2^2(t) + (1 - n_2^2(t))\left\{f^2(n_2^2(t)) + ({}_2w^2)^T a^1\right\} - n_2^2(t)f^2(n_1^2(t)) \quad (5.21)$$

Define the input vector $a^1 = [0.2, 0.8]^T$. Then the inputs of layer 2 are

$$\begin{aligned} ({}_1w^2)^T a^1 &= \begin{bmatrix} 0.9 & 0.45 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = 0.54 \\ ({}_2w^2)^T a^1 &= \begin{bmatrix} 0.45 & 0.9 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = 0.81 \end{aligned} \quad (5.22)$$

Therefore, the second neuron has 1.5 times as much input as the first neuron. However, after 0.25 s the output of the second neuron is 6.34 times the output of the first neuron. The second characteristic of response is that after the input has been set to zero, the network further enhances the contrast and stores the pattern. After the input is removed, the output of the first neuron decays to zero, while the output of the second reaches a steady state value of 0.79. This output is maintained, even after the input is removed.

5.2.2.3 Learning Law The third component of the Grossberg network is the learning law for the adaptive weights W^2. Grossberg calls these adaptive weights the long-term memory (LTM). This is because the rows of W^2 will represent patterns that have been stored and that the network will be able to recognize. The learning for W^2 is given by

$$\frac{dw_{i,j}^2(t)}{dt} = \alpha\left\{-w_{i,j}^2(t) + n_i^2(t)n_j^1(t)\right\} \quad (5.23)$$

Now we summarize the Grossberg network.

Basic Nonlinear Model: Leaky Integrator

$$\varepsilon\frac{dn}{dt} = -n(t) + p(t)$$

Shunting Model

$$\varepsilon\frac{dn(t)}{dt} = -n(t) + (b^+ - n(t))[{}^+W^1]p - (n(t) + b^1)[{}^-W^1]p$$

Two-Layer Competitive Network

Layer 1

$$\varepsilon(dn^1/dt) = -n^1 + ({}^+b^+ - n^1)[{}^+W^1]p - (n^1 + {}^-b^1)[{}^-W^1]p$$

$${}^+W^1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad {}^-W^1 = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$

Layer 2

$$\varepsilon(dn^2/dt) = -n^2 + ({}^+b^2 - n^2)\left\{[{}^+W^2]f^2(n^2) + W^2a^1\right\} - (n^2 + {}^-b^2)[{}^-W^2]f^2(n^2)$$

Learning Law

$$\frac{dw^2_{i,j}(t)}{dt} = \alpha\left\{-w^2_{i,j}(t) + n^2_i(t)n^1_j(t)\right\}$$

If we substitute the output of the Grossberg network as its input, then the network is recurrent. In 1983, Michael A. Cohen and Stephen Grossberg described a general principle for assessing the stability of a certain class of neural networks, by a system of coupled nonlinear differential equations, given as

$$\frac{du_j}{dt} = a_j(u_j)\left[b_j(u_j) - \sum_{i=1}^{N} c_{ji}\phi_i(u_i)\right], \quad j = 1, 2, \ldots, N \tag{5.24}$$

5.3 CELLULAR NEURAL NETWORKS

In 1988, Leon O. Chua and Lin Yang proposed a novel class of information processing systems called cellular neural networks [Chua and Roska, 2004]. The basic circuit unit of cellular neural networks is called a cell. It contains linear and nonlinear circuit elements, which typically are linear capacitors, linear resistors, linear and nonlinear controlled sources, and independent sources. The structure of cellular neural networks is similar to that found in cellular automata; namely, any cell in a cellular neural network is connected only to its neighbor cells. The adjacent cells can interact directly with each other. Cells not directly connected together may affect each other indirectly because of the propagation effects of the continuous-time dynamics of cellular neural networks. A simple example of a two-dimensional cellular neural network is shown

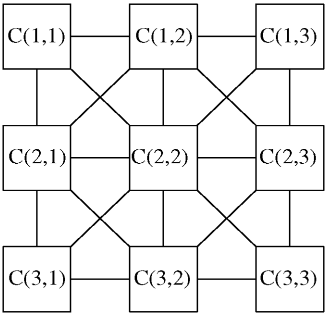


FIGURE 5.13 A two-dimensional cellular neural network. The circuit size is 3×3. The squares are the circuit units called cells.

in Figure 5.13. Actually, we can define a cellular neural network of any dimension. However, we will concentrate on the two-dimensional case in this chapter. The results can be easily generalized to higher dimension cases.

Now, consider an $M \times N$ cellular neural network, having $M \times N$ cells arranged in M rows and N columns. We call the cell on the ith row and jth column cell (i, j), and denote it by $C(i, j)$ as in Figure 5.14.

Ther − neighborhoodof a cell $C(i, j)$ of a cellular neural network is defined by

$$N_r(i, j) = \{C(k, l) : \max\{|k - i|, |l - j|\} \leq r, 1 \leq k \leq M; 1 \leq l \leq N\} \quad (5.25)$$

where r is a positive integer number. The Figure 5.15 shows a neighborhood of the cell located at the center. Usually, we call the r neighborhood a "$(2r + 1) \times (2r + 1)$ neighborhood." Note that the neighborhood defined above exhibits a symmetry property, that is, if $C(i, j) \in N_r(k, l)$, then $C(k, l) \in N_r(i, j)$, for all $C(i, j)$ and $C(k, l)$ in a cellular neural network.

A typical example of a cell $C(i, j)$ of a cellular neural network is shown in Figure 5.16, where the suffixes u, x, and y denote the input, state, and output, respectively. The node voltage v_{xij} of $C(i, j)$ is called the state of the cell and its initial condition is assumed to have a magnitude less than or equal to 1. The node

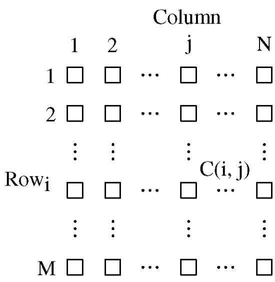


FIGURE 5.14 A two-dimensional cellular neural network. The circuit size is $M \times N$.

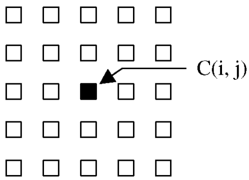


FIGURE 5.15 The neighborhood of cell $C(i, j)$ defined by (5.25) for $r = 2$.

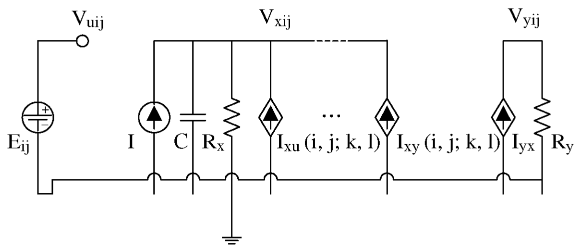
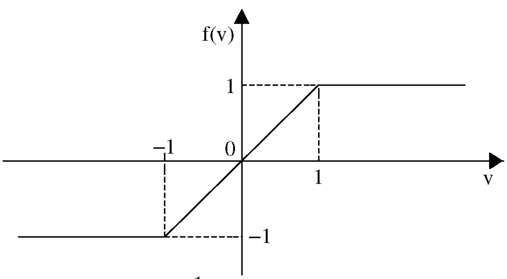


FIGURE 5.16 An example of a cell circuit.

voltage v_{uij} is called the input of $C(i, j)$ and is assumed to be a constant with magnitude less than or equal to 1. The node voltage v_{yij} is called the output.

According to Figure 5.16, each cell $C(i, j)$ contains one independent voltage source E_{ij}, one independent current source I, one linear capacitor C, two linear resistors R_x and R_y, and at most 2 m linear voltage-controlled current sources, which are coupled to its neighbor cells via the controlling input voltage v_{ukl}, and the feedback from the output voltage v_{ykl} of each neighbor cells. In particular, $I_{xy}(i, j; k, l)$ and $I_{xu}(i, j; k, l)$ are linear voltage-controlled current sources with the characteristics $I_{xy}(i, j; k, l) = A(i, j; k, l)v_{ykl}$ and $I_{xu}(i, j; k, l) = B(i, j; k, l)v_{ukl}$ for all $C(i, j) \in N_r(i, j)$. The only nonlinear element in each cell is a piecewise linear voltage-controlled current source $I_{yx} = (1/R_y)f(v_{xij})$ with characteristic $f(\cdot)$ as shown in Figure 5.17.

All of the linear and piecewise line-controlled sources used in the cellular neural network can be easily realized via operational amplifiers (op amps). Applying



$$f(v) = \frac{1}{2}(|v+1| - |v-1|)$$

FIGURE 5.17 The characteristic of the nonlinear controlled source.

Kirchhoff's current law and Kirchhoff's voltage law, the circuit equations of a cell are easily derived as follows:

State equation:

$$C\frac{dv_{xij}(t)}{dt} = -\frac{1}{R_x}v_{xij}(t) + \sum_{C(k,l)\in N_r(i,j)} A(i,j;k,l)v_{ykl}(t) + \sum_{C(k,l)\in N_r(i,j)} B(i,j;k,l)v_{ukl}(t) + I, \quad 1 \le i \le M; \quad 1 \le j \le N \tag{5.26}$$

Output equation:

$$v_{yij}(t) = \frac{1}{2}\left(\left|v_{xij}(t)+1\right| - \left|v_{xij}(t)-1\right|\right), \quad 1 \le i \le M; \quad 1 \le j \le N \tag{5.27}$$

Input equation:

$$v_{uij} = E_{ij}, \quad 1 \le i \le M, \quad 1 \le j \le N \tag{5.28}$$

Constraint conditions:

$$\left|v_{xij}(0)\right| \le 1, \quad 1 \le i \le M, \quad 1 \le j \le N \tag{5.29}$$

$$\left|v_{uij}\right| \le 1, \quad 1 \le i \le M, \quad 1 \le j \le N \tag{5.30}$$

Parameter assumptions:

$$A(i,j;k,l) = A(k,l;i,j), \quad 1 \le i,k \le M, \quad 1 \le j, \quad l \le N \tag{5.31}$$

$$C > 0, \quad R_x > 0 \tag{5.32}$$

The inner cell is the cell that has $(2r+1)^2$ neighbor cells, where r is defined in (5.25). All other cells are called boundary cells. All inner cells of a cellular neural network have the same circuit structures and element values. A cell neural network is completely characterized by the set of all nonlinear differential equations (5.26)–(5.32) associated with the cells in the circuit.

5.4 NEURODYNAMICS AND OPTIMIZATION

First, we describe the mathematical model of a nonlinear dynamic system [Haykin, 2009]. Let $x_1(t)$, $x_2(t)$, ..., $x_N(t)$ denote the state variables of a nonlinear dynamic system, where the continuous time t is the independent variable and N is the order of the system. The dynamics of a large class of nonlinear dynamic systems may be written as the following first-order differential equations:

$$\frac{d}{dt}x_j(t) = F_j(x_j(t)), \quad j = 1, 2, \ldots, N \tag{5.33}$$

where the function $F_j(\cdot)$ is, in general, a nonlinear function of its argument. For convenience of notation, the N state variables are collected into an N-by-1 vector x(t), that is,

$$x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T \tag{5.34}$$

which is called the state vector, or simply state of the system. Then, we can express Eq. 5.33 in a compact form as follows:

$$\frac{d}{dt}x(t) = F(x(t)) \tag{5.35}$$

where the nonlinear function F(x(t)) is

$$F(x(t)) = [F_1(x_1(t)), F_2(x_2(t)), \ldots, F_N(x_N(t))]^T \tag{5.36}$$

A nonlinear dynamic system for which the vector function $F(x(s))$ does not depend explicitly on time t is said to be autonomous. A constant vector $\bar{x}$ is said to be an equilibrium state of system (Eq. 5.35) if it satisfies

$$F(\bar{x}) = 0 \tag{5.37}$$

where 0 is the null vector. Clearly, the constant function $x(t) = \bar{x}$ is a solution of Eq. 5.35 since the velocity vector dx/dt vanishes at $\bar{x}$.

Now, we discuss some of the important issues involved in neurodynamics. Neurodynamics possesses the following general characteristics:

1. *A large number of degrees of freedom:* The human cortex is a highly parallel, distributed system that is estimated to possess about 10 billion neurons, with each neuron modeled by one or more state variables. It is generally believed that both the computational power and the fault-tolerant capability of such a neurodynamic system are the result of the collective dynamics of the system. The system is characterized by a very large number of coupling constants represented by the strengths of the individual synaptic junctions.
2. *Nonlinearity:* A neurodynamic system is inherently nonlinear. In fact, nonlinearity is essential for creating a universal computing machine.
3. *Dissipation:* A neurodynamic system is dissipative. It is therefore characterized by the convergence of the state space volume onto a manifold of lower dimensionality as time goes on.

4. *Noise:* Noise is an intrinsic characteristic of neurodynamic systems. In real-life neurons, membrane noise is generated at synaptic junctions.

5.5 STABILITY ANALYSIS OF RECURRENT NEURAL NETWORKS

For the purpose of studying the stability of recurrent neural networks, it is necessary to be familiar with the definition of stability, in the context of an autonomous nonlinear dynamic system (Eq. 5.35 with equilibrium state $\overline{x}$ [Khalil, 1992].

Definition 5.1 The equilibrium state $\overline{x}$ is said to be uniformly stable if, for any positive constant ε, there exists another positive constant $\delta = \delta(\varepsilon)$ such that the condition

$$\|x(0) - \overline{x}\| < \delta$$

implies that

$$\|x(t) - \overline{x}\| < \varepsilon$$

for all $t > 0$.

Definition 5.2 The equilibrium state $\overline{x}$ is said to be convergent if there exists a positive constant δ such that the condition

$$\|x(0) - \overline{x}\| < \delta$$

implies that

$$x(t) \rightarrow \overline{x} \text{ as } t \rightarrow \infty$$

Definition 5.3 The equilibrium state $\overline{x}$ is said to be asymptotically stable if it is both stable and convergent.

Definition 5.4 The equilibrium state $\overline{x}$ is said to be globally asymptotically stable if it is stable and all trajectories of the system converge to $\overline{x}$ as time t approaches infinity.

By making a comparison among the above definitions, we find that uniform stability means that a trajectory of the system can be made to stay within a small neighborhood of the equilibrium state $\overline{x}$ if the initial state $x(0)$ is close to $\overline{x}$. In addition, the convergence reveals that if the initial state $x(0)$ of a trajectory is close enough to the equilibrium state $\overline{x}$, then the trajectory described by the state vector $x(t)$ will approach $\overline{x}$ as time t approaches infinity. Furthermore, it is only when stability and convergence are both satisfied that we have asymptotic stability. Global asymptotic stability implies that the system will ultimately settle down to a steady state for any choice of initial conditions.

The definition of the positive definite function is also required when doing stability analysis. A function V(x) is called positive definite if it satisfies the following conditions:

1. The function V(x) has continuous partial derivatives with respect to the elements of the state x.
2. $V(\bar{x}) = 0$.
3. $V(x) > 0$ if $x \in \mathfrak{A} - \bar{x}$,, where $\mathfrak{A}$ is a small neighborhood around $\bar{x}$.

Incidentally, the definition of negative definite can be easily derived in the light of the above conditions.

An elegant approach for investigating the stability of dynamic systems is the direct method of Lyapunov, which is based on a continuous scalar function of the state, called a Lyapunov function. The Lyapunov's theorems on the stability analysis of the dynamic system (5.35) are stated as follows [Khalil, 1992]:

Theorem 5.1 The equilibrium state $\bar{x}$ is stable if, in a small neighborhood of $\bar{x}$, there exists a positive definite function V(x) such that its derivative with respect to time is negative semidefinite in the region.

Theorem 5.2 The equilibrium state $\bar{x}$ is asymptotically stable if, in a small neighborhood of $\bar{x}$, there exists a positive definite function V(x) such that its derivative with respect to time is negative definite in the region.

The scalar function V(x) that satisfies the requirements of these two theorems is called a Lyapunov function for the equilibrium state $\bar{x}$.

To summarize, given that V(x) is a Lyapunov function, then according to Theorem 5.1, the equilibrium state $\bar{x}$ is stable if

$$\frac{d}{dt}V(x) \leq 0, \quad \text{for} \quad x \in \mathfrak{A} - \bar{x} \tag{5.38}$$

Similarly, according to Theorem 5.2, the equilibrium state $\bar{x}$ is asymptotically stable if

$$\frac{d}{dt}V(x) < 0, \quad \text{for} \quad x \in \mathfrak{A} - \bar{x} \tag{5.39}$$

It should be noted that global stability of a nonlinear dynamic system generally requires that the condition of radial unboundedness holds, that is,

$$V(x) \to \infty \text{ as } \|x\| \to \infty$$

This condition is usually satisfied according to the Lyapunov functions constructed for neural networks with sigmoid activation functions.

5.5.1 Stability Analysis of the Hopfield Network

To facilitate the stability analysis of Hopfield network (5.28), we make the following three assumptions [Haykin, 2009]:

1. The matrix of synaptic weights is symmetric, that is,

$$w_{ji} = w_{ij}, \quad \text{for all } i \text{ and } j \tag{5.40}$$

2. Each neuron has a nonlinear activation of its own, that is, $\phi_i(\cdot)$.
3. The inverse of the nonlinear activation function exists. According to (5.26), we may write

$$v = \phi_i^{-1}(x) \tag{5.41}$$

Let the sigmoid function $\phi_i(\cdot)$ be defined by the hyperbolic tangent function as

$$x = \phi_i(v) = \tanh\left(\frac{a_i v}{2}\right) = \frac{1 - \exp(-a_i v)}{1 + \exp(-a_i v)} \tag{5.42}$$

which has a slope of $a_i/2$ at the origin, that is,

$$\frac{a_i}{2} = \left.\frac{d\phi_i}{dv}\right|_{v=0} \tag{5.43}$$

Thus, we refer to a_i as the gain of neuron i.

Based on (5.42), the inverse output–input relation of (5.42) can be further written as

$$v = \phi_i^{-1}(x) = -\frac{1}{a_i} \ln\left(\frac{1-x}{1+x}\right) \tag{5.44}$$

Let the standard form of the inverse output–input relation for a neuron of unity gain be denoted as

$$\phi^{-1}(x) = -\ln\left(\frac{1-x}{1+x}\right) \tag{5.45}$$

Then, Eq. 5.44 has an equivalent form presented as follows:

$$\phi_i^{-1}(x) = \frac{1}{a_i}\phi^{-1}(x) \tag{5.46}$$

Define the Lyapunov function of the Hopfield network shown in Figure 5.3 as

$$V = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ji}x_i x_j + \sum_{j=1}^{N}\frac{1}{R_j}\int_0^{x_j}\phi_j^{-1}(x)dx - \sum_{j=1}^{N} I_j x_j \tag{5.47}$$

Differentiating V with respect to time t, we can obtain

$$\begin{aligned}\frac{dV}{dt} &= -\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ji}x_i\frac{dx_j}{dt} + \sum_{j=1}^{N}\frac{1}{R_j}\phi_j^{-1}(x_j)\frac{dx_j}{dt} - \sum_{j=1}^{N} I_j\frac{dx_j}{dt} \\ &= -\sum_{j=1}^{N}\left(\sum_{i=1}^{N} w_{ji}x_i - \frac{1}{R_j}\phi_j^{-1}(x_j) + I_j\right)\frac{dx_j}{dt}\end{aligned} \tag{5.48}$$

From (5.10) and (5.41), we get

$$\frac{dV}{dt} = -\sum_{j=1}^{N}\left(\sum_{i=1}^{N} w_{ji}\phi_i(v_i) - \frac{v_j}{R_j} + I_j\right)\frac{dx_j}{dt} \tag{5.49}$$

Combining (5.12) with (5.49), we can further derive that

$$\frac{dV}{dt} = -\sum_{j=1}^{N} C_j\frac{dv_j}{dt}\frac{dx_j}{dt} \tag{5.50}$$

Next, we apply (5.49) to (5.50) and find that

$$\begin{aligned}\frac{dV}{dt} &= -\sum_{j=1}^{N} C_j\frac{d\phi_j^{-1}(x_j)}{dt}\frac{dx_j}{dt} \\ &= -\sum_{j=1}^{N} C_j\frac{d\phi_j^{-1}(x_j)}{dx_j}\left(\frac{dx_j}{dt}\right)^2\end{aligned} \tag{5.51}$$

In accordance with Figure 2.11, we see that the hyperbolic tangent function is a monotonically increasing function. Thus, the inverse output–input relation $\phi_j^{-1}(x_j)$ is also a monotonically increasing function of the output x_j, which implies that

$$\frac{d\phi_j^{-1}(x_j)}{dx_j} \geq 0, \quad \text{for all } x_j \tag{5.52}$$

Clearly,

$$\left(\frac{dx_j}{dt}\right)^2 \geq 0, \quad \text{for all } x_j \tag{5.53}$$

Therefore, for the energy function V defined in (5.31), we have

$$\frac{dV}{dt} \leq 0, \quad \text{for all } t \tag{5.54}$$

Additionally, from the definition of (5.47), we note that the function V is bounded. In summary, the energy function V is a Lyapunov function of the continuous Hopfield model. Furthermore, the model is stable in accordance with Theorem 5.1.

The time evolution of the continuous Hopfield model described by the system of nonlinear first-order differential equations given in (5.11) represents a trajectory in state space that seeks out the minima of the Lyapunov function V and comes to a stop at such fixed points. From Eq. 5.51, we note that the derivative dV/dt vanishes only if

$$\frac{dx_j}{dt} = 0, \quad \text{for all } j \tag{5.55}$$

Hence, we have

$$\frac{dV}{dt} < 0, \text{ except at a fixed point} \tag{5.56}$$

which reveals that the Lyapunov function V of a Hopfield network is a monotonically decreasing function of time. Consequently, the Hopfield network is asymptotically stable in the Lyapunov sense.

5.5.2 Stability Analysis of the Cohen–Grossberg Network

The Lyapunov function corresponding to Eq. 5.24 is defined as

$$V = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ji}\phi_i(u_i)\phi_j(u_j) - \sum_{j=1}^{N}\int_0^{u_j} b_j(\lambda)\phi_j'(\lambda)d\lambda \tag{5.57}$$

where $\phi_j'(\lambda)$ is the derivative of $\phi_j(\lambda)$ with respect to λ. The following conditions should be held in order to ensure the validity of definition of Eq. 5.41.

1. The synaptic weights of the network are symmetric, that is,

$$c_{ji} = c_{ij} \tag{5.58}$$

2. The function $a_j(u_j)$ satisfies the nonnegativity condition, that is,

$$a_j(u_j) \geq 0 \tag{5.59}$$

3. The nonlinear input–output function $\phi_j(u_j)$satisfies the monotonicity condition, that is,

$$\phi_j'(u_j) = \frac{d\phi_j(u_j)}{du_j} \geq 0 \tag{5.60}$$

With this background, we may now formally state the Cohen–Grossberg theorem.

Theorem 5.3 [Haykin, 2009] Provided that the system of nonlinear differential equations (5.24) satisfies the conditions of symmetry, nonnegativity, and monotonicity, the Lyapunov function V of the system defined by Eq. 5.57 satisfies the condition

$$\frac{dV}{dt} \leq 0 \tag{5.61}$$

Once this basic property of the Lyapunov function *V* is in place, stability of the system follows from Theorem 5.1.

By comparing the general system of Eq. 5.24 with the system of Eq. 5.11 for a continuous Hopfield model, we may make the correspondences between the Hopfield model and the Cohen–Grossberg theorem that are summarized in Table 5.1. Applying the correspondences in Table 5.1 to Eq. 5.57, we can obtain a Lyapunov function for the continuous Hopfield model expressed as

$$V = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ji}\phi_i(v_i)\phi_j(v_j) + \sum_{j=1}^{N}\int_0^{v_j}\left(\frac{v_j}{R_j} - I_j\right)\phi_j'(v)dv \tag{5.62}$$

TABLE 5.1 Correspondences between the Cohen–Grossberg Theorem and the Continuous Hopfield Model

Cohen–Grossberg Theorem	Hopfield Model
u_j	$C_j(v_j)$
$a_j(u_j)$	1
$b_j(u_j)$	$-\frac{v_j}{R_j} + I_j$
c_{ji}	$-w_{ji}$
$\phi_i(u_i)$	$\phi_i(v_i)$

Note that the following equations hold:

1. $\phi_i(v_i) = x_i$.
2. $\int_0^{v_j} \phi_j'(v)dv = \int_0^{x_j} dx = x_j$.
3. $\int_0^{v_j} v\phi_j'(v)dv = \int_0^{x_j} vdx = \int_0^{x_j} \phi_j^{-1}(x)dx$.

By applying them to Eq. 5.62, we can further obtain a Lyapunov function that is identical with Eq. 5.47. This demonstrates that the Hopfield model can be seen as a special case of the Cohen–Grossberg theorem [Haykin, 2009].

The Cohen–Grossberg theorem is a general principle of neurodynamics with a wide range of applications.

EXERCISES

5.1. The Lyapunov function of a Hopfield network is written as

$$V = -\frac{1}{2}\left(7x_1^2 + 12x_1x_2 - 2x_2^2\right)$$

Point out the matrix of synaptic weight of the network.

5.2. Compute the equilibrium state of the following dynamic system:

$$\frac{dx}{dt} = \begin{bmatrix} -x_1 + x_2^2 \\ -x_2(x_1 + 1) \end{bmatrix}$$

Then, confirm the stability of the equilibrium state by choosing the Lyapunov function as

$$V(x) = x^T x$$

5.3. Given a nonlinear dynamic system as

$$\frac{dx}{dt} = \begin{bmatrix} x_2 - 2x_1\left(x_1^2 + x_2^2\right) \\ -x_1 - 2x_2\left(x_1^2 + x_2^2\right) \end{bmatrix}$$

study the stability of the origin, by making use of the following Lyapunov function:

$$V(x) = \alpha x_1^2 + \beta x_2^2$$

5.4. The discrete (time) gamma model of a neurodynamical system is described by the following pair of equations:

$$x_j(k) = \phi\left(\sum_{i<j}\sum_{m} w_{ji}^{(m)} x_i^{(m)}(k)\right) + K_j$$

and

$$x_j^{(m)}(k) = \left(1 - \mu_j\right) x_j^{(m)}(k-1) + \mu_j x_j^{(m-1)}(k-1)$$

where k denotes discrete time, $j = 1, 2, \ldots, N$, and $m = 1, 2, \ldots, M$.

1. Compare the discrete gamma model with the model described in (5.6).
2. Construct a signal flow graph for the recursive part of the gamma model.
3. Find the value of the control parameter μ_j for which the discrete gamma model is stable.

5.5. Consider a Hopfield network made up two neurons. The synaptic weight matrix of the network is

$$W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

The bias applied to each neuron is zero. The following are the four possible states of the network:

$$x^1 = [+1, +1]^T$$
$$x^2 = [-1, +1]^T$$
$$x^3 = [-1, -1]^T$$
$$x^4 = [+1, -1]^T$$

1. Demonstrate that states x^2 and x^4 are stable, whereas states x^1 and x^3 exhibit a limit cycle. Do this demonstration by using the stability condition and energy function, respectively.
2. Confirm the length of the limit cycle characterizing the states x^1 and x^3.

5.6. Define the Lyapunov function related to the cellular neural network described in Section 5.3 as

$$V = -\frac{1}{2}\sum_{(i,j)}\sum_{(k,l)} A(i,j;k,l)v_{yij}(t)v_{ykl}(t) + \frac{1}{2R_x}\sum_{(i,j)} v_{yij}^2(t) - \sum_{(i,j)}\sum_{(k,l)} B(i,j;k,l)v_{yij}(t)v_{ukl} - \sum_{(i,j)} Iv_{yij}(t)$$

Try to perform the stability analysis of the network.