Astro 272: Stellar Interiors
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Equations of Stellar Structure

a) Continuity / Mass Conservation

$$\boxed{\frac{dM}{dr} = 4\pi r^2 \rho}$$

b) Hydrostatic Equilibrium

$$\boxed{\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho}$$

c) Energy Generation Equation

$$\boxed{\frac{dL}{dr} = 4\pi r^2 \rho \epsilon}$$

$\rightarrow$ power generation per unit mass (sources + sinks)

d) Energy Transport

radiative/photon diffusion $\quad \vec{J} = -D \nabla n$

particle flux $\quad$ diffusion coefficient $\sim \frac{1}{3} v \lambda_{MFP}$ $\quad$ particle density

$\lambda \sim \frac{1}{\kappa\rho} \qquad v \to c \qquad u = aT^4$

$$F = -\frac{1}{3}\frac{c}{\kappa\rho} 4aT^3 \frac{dT}{dr} = \frac{L}{4\pi r^2}$$

$$\boxed{\frac{dT}{dr} = -\frac{3}{16\pi}\frac{\kappa\rho}{ac}\frac{L}{r^2 T^3}}$$

Convection $\quad \frac{d\ln T}{d\ln P} = \nabla_{ad} = \frac{2}{5}$ for ideal gas

Basic Timescales

a) Dynamical Timescale

sound speed $\quad c_s^2 = \left(\frac{dP}{d\rho}\right)_{ad} = \Gamma_1 \frac{P}{\rho}$ $\quad$ 1^{st} adiabatic exponent

$$t_{dyn} \sim \frac{R}{c_s} \sim \frac{R}{(\Gamma P/\rho)^{1/2}} \qquad P \sim \frac{GM\rho}{R} \qquad \frac{P}{\rho} \sim \frac{GM}{R}$$

$$\therefore \quad t_{dyn} \sim \left(\frac{R^3}{GM}\right)^{1/2} \propto \frac{1}{\sqrt{G\rho}} \sim 1600\,s \left(\frac{R}{R_\odot}\right)^{3/2}\left(\frac{M}{M_\odot}\right)^{-1/2}$$

$\longrightarrow$ changes in pressure

b) Thermal Timescale (Kelvin-Helmholtz)

$\rightarrow$ changes in temperature

$$t_{KH} \sim \frac{GM}{R^2 L} \sim 2\times10^7 \text{ yrs} \left(\frac{M}{M_\odot}\right)\left(\frac{L}{L_\odot}\right)^{-1}\left(\frac{R}{R_\odot}\right)^{-2}$$

c) Nuclear Timescale

$$t_{nuc} \sim \frac{0.1\, Mc^2\,(0.007)}{L} \sim 10^{10} \text{ yrs} \left(\frac{M}{M_\odot}\right)\left(\frac{L}{L_\odot}\right)^{-1}\left(\frac{\epsilon}{0.007}\right)$$

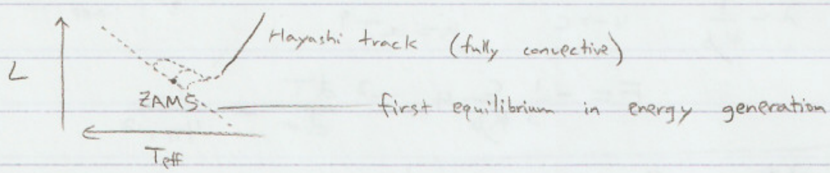
Useful Numbers:

$M_\odot = 2\times10^{33}$ gm $\qquad L_\odot = 3.86\times10^{33}$ erg/s

$\sigma = 5.67\times10^{-5}$ erg cm^{-3} s^{-1} K^{-4} $= \frac{4ac}{3}$ (?)

Overview of Stellar Evolution

Star Formation



Main Sequence

Low mass stars: fully convective $\lesssim 0.3\, M_\odot$ M

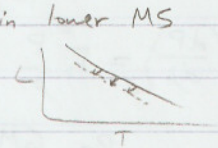
Intermediate : radiative core, convective envelope $0.3 - 1.5$

upper MS : convective core, radiative envelope >1.5 A

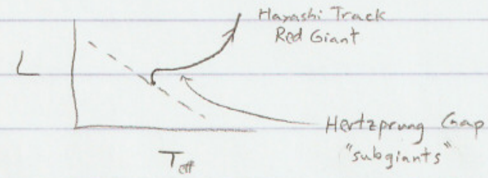
Lower limit $\lesssim 0.1\, M_\odot$ $\qquad$ Upper limit $\gtrsim 40\, M_\odot$ $\qquad$ $L \propto M^3$

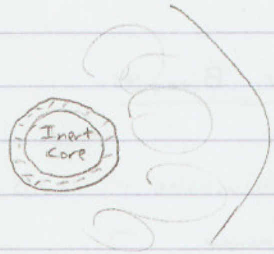
Metallicity Dependance: $L \propto Z^{0.35}$ in lower MS

subdwarfs (lower metallicity)

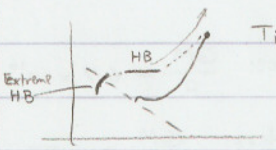


After Main Sequence





10% of mass fused in core → Schönberg-Chandrasekhar Limit



Tip of RGB → Helium ignition $\quad$ $0.45 - 0.47\, M_\odot$ of He core

Lower MS stars – degenerate ignition (He flash)

Upper MS stars – non-degenerate ignition

Asymptotic Giant Branch

mass loss – stellar winds
– stellar pulsations

Eventually lose the envelope to become a white dwarf

$T_{eff} > 10^5\,K$ @ tip of WD sequence

"Planetary Nebulae"

More massive stars

$C \rightarrow Si \rightarrow \ldots \rightarrow Fe$

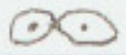
Photodissociation leads to core collapse → Supernova
→ Black Hole ?

Binary Evolution

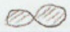
Roche Potential

in rotating ref. frame
(centrifugal term needs to be included)

$$\frac{R_{L_1}}{a} \sim \frac{0.49\, q^{2/3}}{0.6\, q^{2/3} + \ln(1+q^{1/3})} \qquad q = \frac{M_1}{M_2}$$

Detached binary 

Semi-detached binary  $\quad$ mass transfer can be stable or unstable

Contact binary  Algol $\quad$ with unstable can get Common Envelope Evolution

Mass Transfer Binaries

1) Cataclysmic Variables : WD + MS

dwarf novae
intermittent novae
novae

} thermonuclear runaway (H) on a white dwarf surface

2) X-ray Bursters : NS + MS

$0.007\,c^2 = 6.3 \times 10^{18}$ erg/gm

Grav potential on WD surface: $\frac{GM}{R} = 6.7 \times 10^{16}$ erg/gm

WD: lot of energy used to eject material

NS surface: $\frac{GM}{R} = 1.87 \times 10^{20}$ erg/gm

NS: no material ejected

Supernovae
- Type I (no H)
  - Ia WD collapse ($M > 1.4 M_\odot$)
  - Ib
  - Ic
- Type II (H) – core collapse

(Ib, Ic and Type II: core collapse)

Equations of State

Microphysical description → distribution function f (6-D)

$$f(\vec{x},\vec{p}) = \frac{dN}{d^3x\, d^3p} = \frac{1}{h^3} \sum_j \frac{g_j}{\exp\left[\left(-\mu + \epsilon_j + \epsilon_{(p)}\right)/kT\right] \pm 1}$$

g_j – spin degeneracy; μ – chemical potential; ϵ_j – internal energy of particles (excited states); $\epsilon_{(p)}$ – kinetic energy

$$\mu = \left(\frac{\partial E}{\partial N_i}\right)_{S,V}$$

$$n(\vec{x}) = \int_p f(\vec{x},\vec{p})\, 4\pi p^2\, dp \qquad E = \int f(\vec{x},\vec{p})\, \epsilon(p)\, 4\pi p^2\, dp \qquad \epsilon(p) = \left(p^2c^2 + m^2c^4\right)^{1/2} - mc^2$$

$$P = \frac{1}{3} \int_p f(\vec{x},\vec{p})\, p \cdot v\, 4\pi p^2\, dp$$

Blackbody Radiation

photons : bosons of spin 1 $\quad g=2$ (2 polarization state) $\quad \mu_\gamma = 0 \quad \epsilon_j = 0$

$$n_\gamma = \frac{1}{h^3} \int \frac{2}{\exp(\epsilon/kT) - 1}\, 4\pi p^2\, dp \qquad \epsilon = pc$$

$$n_\gamma = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2\, dp}{e^{\frac{pc}{kT}} - 1} = \frac{8\pi}{h^3}\left(\frac{kT}{c}\right)^3 \int_0^\infty \frac{x^2\, dx}{e^x - 1} \qquad P_{rad} = \frac{8\pi^5 k^4}{15 c^3 h^3} \frac{T^4}{3} = \frac{a}{3} T^4$$

$$E_{rad} = aT^4 = 3 P_{rad}$$

Ideal Gas

$\frac{\mu}{kT} << -1$ (non-interacting gas)

very large negative number — no internal states $\epsilon = \epsilon_0$

nonrelativistic $\epsilon = \frac{p^2}{2m}$ — (ignore ± 1)

$$n = \frac{4\pi}{h^3} g \int_0^\infty p^2 e^{\frac{\mu - \epsilon_0 - p^2/2m}{kT}} dp \qquad e^{\frac{\mu}{kT}} = \frac{n h^3}{g(2\pi m kT)^{3/2}} e^{\frac{\epsilon_0}{kT}} << 1$$

$$P = g \frac{4\pi}{h^3} \frac{\pi^{1/2}}{8m} (2mkT) e^{-\epsilon_0/kT} e^{\mu/kT}$$

$$\Longrightarrow P = nkT \qquad E = \frac{3}{2} nkT$$

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The Saha Equation

$H^+ + e^- \leftrightarrow H^0$ $\quad \chi_H$ – ionization potential

Thermodynamic equilibrium $\quad \sum_i \mu_i dN_i = 0$

Stochiometric coefficients

$\boxed{\mu_+ + \mu_- = \mu_0}$ — chemical potential

Ionization Balance: (1) Assume Boltzmann limit

(2) $\epsilon_0 = 0$ for e^- & H^+

$\epsilon_0 = -\chi_H$ for H^0 (13.6 eV)

(3) $g_0 = 2$ for H^0 $\quad g_+ = 1 \quad g_- = 2$ (or $g_+ = 2 \quad g_- = 1$)

degeneracy — spins aligned or anti aligned

$$n_e = 2 \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{\frac{\mu_-}{kT}} \qquad n_+ = \frac{(2\pi m kT)^{3/2}}{h^3} e^{\frac{\mu_+}{kT}}$$

$$n_0 = 2 \frac{(2\pi (m_e + m) kT)^{3/2}}{h^3} e^{\frac{\mu_0}{kT}} e^{\frac{\chi_H}{kT}}$$

$$\frac{n_+ n_e}{n_0} = \underbrace{\frac{(2\pi kT)^{3/2}}{h^3} \left(\frac{m_e m}{m_e + m}\right)^{3/2}}_{2.415 \times 10^{15}\ \text{cm}^{-3}\ T^{3/2}} \cancel{e^{\frac{\mu_- + \mu_+ - \mu_0}{kT}}}^{\,1} e^{-\frac{\chi_H}{kT}}$$

$n_+ + n_0 = n$ (Baryon conservation) $\qquad n_e = n_+$ (Charge Conservation)

$$y = \frac{n_+}{n}$$

$$\frac{y^2}{1-y} = \frac{1}{n}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\frac{\chi_H}{kT}}$$

Ionization (50%) $\rightarrow \frac{\chi_H}{kT} \sim 10 \implies T \sim 10^4 K$

Fermi-Dirac Equation of State

$$n = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{\exp\{(-\mu + mc^2 + \epsilon(p))/kT\} + 1} \qquad \epsilon(p) = mc^2\left(\sqrt{1+\left(\frac{p}{mc}\right)^2} - 1\right)$$

Completely degenerate $\rightarrow T = 0$

$\rightarrow$ exponential term is step function @ $E = \mu - mc^2 = \epsilon_F$

$$E_F = mc^2\left(\sqrt{1 + x_F^2} - 1\right) \qquad x_F = \frac{p_F}{mc} \qquad \text{Fermi energy}$$

$$n = \frac{8\pi}{h^3}\int_0^{p_F} p^2\,dp = \frac{8\pi}{3}\left(\frac{mc}{h}\right)^3 x_F^3$$

$x_F \ll 1$ non relativistic

$x_F \gg 1$ relativistic

mean molecular weight $\frac{\rho}{\mu_e} \sim 10^6\ \text{g/cm}^3$ for electrons

$$P_e = \frac{8\pi}{3}\frac{m_e^4 c^5}{h^3}\int_0^{x_F}\frac{x^4\,dx}{\sqrt{1+x^2}} \qquad E_e = 8\pi\left(\frac{m_e c}{h}\right)^3 m_e c^2 \int_0^{x_F} x\left(\sqrt{1+x^2} - 1\right) dx$$

$P \sim \rho^{5/3}$ non rel $\qquad P \sim \rho^{4/3}$ rel $\qquad E_e \sim \rho^{5/3} \qquad E_e \sim \rho^{4/3}$

$$\frac{\partial M}{\partial r} = 4\pi r^2 \rho \qquad \frac{dP}{dr} = -\rho g \qquad \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Polytrope $P = k\rho^\Gamma \longrightarrow \boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n}$ Lane-Ender Eqn

$\Gamma = 1 + \frac{1}{n}$

where $\rho = \lambda\theta^n \qquad r = \alpha\xi \qquad \alpha = \left(\frac{n+1}{4\pi G}k\lambda^{\frac{1}{n-1}}\right)^{1/2}$

$$M = \int_0^R 4\pi r^2\rho\,dr = 4\pi\alpha^3\lambda\int_0^{\xi_1}\xi^2\theta^n\,d\xi = 4\pi\alpha^3\lambda\int_0^{\xi_1}\frac{d}{d\xi}\left(\xi\frac{d\theta}{d\xi}\right)d\xi$$

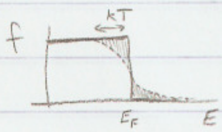
$$= 4\pi\alpha^3\lambda\left(-\xi^2\frac{d\theta}{d\xi}\right)\Big|_{\xi_1}$$

$M \sim \rho_c^{\frac{3-n}{2n}} \qquad R \sim \rho_c^{\frac{1-n}{2n}}$

Non-relativistic degeneracy $\Gamma = 5/3 = 1 + \frac{1}{n} \Rightarrow n = 3/2 \qquad R \propto M^{-1/3}$

Relativistic degenerate $\Gamma = \frac{4}{3} = 1 + \frac{1}{n} \rightarrow n = 3$

Chandrasekhar Mass — max mass supported by degeneracy



$E_F \gtrsim kT \Rightarrow$ degeneracy

Non relativistic case: $E_F \sim \frac{mc^2}{2} x_F^2 \rightarrow kT \sim \frac{mc^2}{2} x_F^2$

$\frac{\rho}{\mu_e} = 6 \times 10^{-9} \; g/cm^3 \; T^{3/2}$

Relativistic: $\frac{\rho}{\mu_e} = 4.6 \times 10^{-24} \; T^3$

First Order Corrections

- Coulomb corrections: $E_{coul} \sim \frac{Z^2 e^2}{a}$ — avg. separation $\qquad \Gamma_c = \frac{Z^2 e^2}{a k T}$ Coulomb coupling parameter

$\frac{4\pi a^3}{3} \sim \frac{1}{n}$

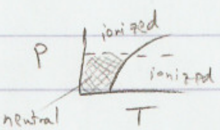
$\Gamma_c \sim 1 \Rightarrow \rho \sim 8.5 \times 10^{-17} \; T^3 \; g/cm^3$
$Z \sim 1$

$\Gamma_c \sim 170 \Rightarrow$ crystallization — old WD & NS crusts

- Pressure Ionization

atoms too close → separation close to orbital radii of e^- → not really neutral

$a_0 \sim 0.5 \times 10^{-8} \; cm \sim a \sim \left(\frac{3}{4\pi n}\right)^{1/3} \Rightarrow \rho \sim 3 \; g/cm^3$



- Radiation Pressure

$\frac{aT^4}{3} \sim \rho \frac{N_A k T}{\mu} \qquad \rho \sim 1.5 \times 10^{-23} \; T^3$

Radiative Transfer

Specific Intensity $I(\theta)\,d\Omega$ = energy flux passing through surface in direction θ within a solid angle $d\Omega$

energy density $u(\theta)\,d\Omega = \frac{I(\theta)}{c}\,d\Omega$

total energy density $u = \int_{4\pi} u(\theta)\,d\Omega = \frac{2\pi}{c}\int_{-1}^{1} I(\mu)\,d\mu \qquad \mu = \cos\theta$

flux $\quad F = \int I(\theta)\cos\theta\,d\Omega = 2\pi\int_{-1}^{1} I(\mu)\,\mu\,d\mu$

if I = constant $\quad \int_{-1}^{1}\mu\,d\mu = 0 \quad$ zero net flux $\qquad$ net flux requires anisotropic intensity

$I(\mu) \rightarrow$ sources & sinks

emission $\quad$ absorption

— scattering —

$dI = j\rho\,ds - \kappa\rho I\,ds$

emissivity $\quad$ opacity

$$\boxed{\frac{1}{\rho}\frac{dI}{ds} = j - \kappa I}$$ Equation of radiative transfer

in isotropic equilibrium $\frac{dI}{ds} = 0 \Rightarrow I = j/\kappa$

$u = \frac{4\pi}{c} I = aT^4$ in LTE $\quad I = B(T)$

$S_\nu = \frac{j_\nu}{\kappa_\nu}$ source function $\qquad d\tau_\nu = -\kappa_\nu\rho\,dz \qquad dz = \cos\theta\,ds = \mu\,ds$

$$\boxed{\mu\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu}$$

$$I(\tau,\mu) = I(\tau_0,\mu)\,e^{-\frac{\tau_0-\tau}{\mu}} + \int_\tau^{\tau_0} e^{-\frac{t-\tau}{\mu}}\frac{S(t)}{\mu}\,dt$$

(outward) if $\mu > 0 \quad \tau_0 \to \infty$

$$I(\tau,\mu \geq 0) = \int_\tau^\infty e^{\frac{-(t-\tau)}{\mu}}\frac{S(t)}{\mu}\,dt$$

if $\mu < 0$ inwardly directed

$\tau_0 = 0$ & $I(\tau_0,\mu) = 0 \qquad I(\tau,\mu<0) = \int_\tau^0 e^{-\frac{(t-\tau)}{\mu}}\frac{S(t)}{\mu}\,dt$

LTE: $S_\nu = B_\nu$

$$S(t) = B(\tau) + (t-\tau)\frac{\partial B}{\partial \tau}$$

$$I(\tau, \mu \geq 0) = B(\tau) + \mu\frac{\partial B}{\partial \tau}$$ outward

$$I(\tau, \mu < 0) = B(\tau)(1 - e^{\tau/\mu}) + \mu\frac{\partial B}{\partial \tau}\left(e^{\tau/\mu}\left(\frac{\tau}{\mu} - 1\right) + 1\right)$$ inward

$$\Longrightarrow B(\tau) + \mu\frac{\partial B}{\partial \tau} \quad \text{for large } \tau$$

$$F(\tau) = 2\pi\int_{-1}^{1}\left(B(\tau) + \mu\frac{\partial B}{\partial \tau}\right)\mu\, d\mu = \frac{4\pi}{3}\frac{\partial B}{\partial \tau}$$

Flux depends on gradient (temperature gradient)

$$\frac{\partial B}{\partial \tau}\Big/B = \frac{\frac{3}{4\pi}F}{\sigma T^4} \sim \frac{L}{4\pi r^2}\frac{3}{4\pi}\frac{1}{\sigma T^4} \sim \frac{3}{16\pi^2}\frac{L}{\sigma r^2 T^4} \sim 10^{-13}$$

(solar values put in)

↳ small anisotropy leads to transport

$$F_\nu = \frac{4\pi}{3}\frac{\partial B_\nu}{\partial \tau_\nu} = -\frac{4\pi}{3}\frac{1}{\kappa_\nu \rho}\frac{\partial B_\nu}{\partial r} = -\frac{4\pi}{3}\frac{1}{\kappa_\nu \rho}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial r}$$

$$F = \int F_\nu\, d\nu = -\frac{4\pi}{3}\frac{1}{\rho}\frac{dT}{dr}\underbrace{\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T}\, d\nu}$$

$$\equiv \frac{1}{\langle\kappa\rangle}\int_0^\infty \frac{\partial B_\nu}{\partial T}\, d\nu$$ Rosseland mean opacity

$$\frac{\partial}{\partial T}\int_0^\infty B_\nu\, d\nu = \frac{ac}{\pi}T^3$$

$$\boxed{F = -\frac{4ac}{3}\frac{T^3}{\langle\kappa\rangle\rho}\frac{dT}{dr}}$$

$$\nabla = \frac{d\ln T}{d\ln P} = \frac{3}{16\pi}\frac{1}{acG}\frac{P\kappa}{T^4}\frac{1}{M}$$

Radiative Opacity

photon beam

$$\sigma = \frac{\#\text{ events per unit time per target}}{\#\text{ of photons}}$$

$$I\sigma n_p\, ds = I\,\kappa\rho\, ds \Longrightarrow \kappa = \frac{\sigma n}{\rho}$$

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Nonrelativistic Thomson scattering by electrons

$$\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 0.6652 \times 10^{-24}\ \mathrm{cm}^2 \sim 2/3\ \mathrm{barn}$$

Fully ionized case $\quad n_e = \underbrace{\frac{X\rho}{m_p}}_{\#\, e^- \text{ from Hydrogen}} + \underbrace{\frac{1}{2}(1-X)\frac{\rho}{m_p}}_{\#\, e^- \text{ from He}} = \frac{\rho}{2m_p}(1+X)$

$$\kappa = \frac{\sigma n_e}{\rho} = \frac{\sigma}{\rho}\frac{\rho}{2m_p}(1+X) = 0.2(1+X)\ \mathrm{cm^2/g}$$

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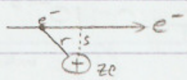
Free-free Emission

inverse to Bremsstrahlung



$$P(t) = \frac{2}{3}\frac{e^2}{c^3}a(t)$$ a^2 Larmor formula

$$a = \frac{F}{m} = \frac{Ze^2}{mr^2}$$

$$E_s = \frac{Z^2 e^6 \pi}{3c^3 m_e^2}\frac{1}{v\,s^3}$$ energy radiated

(s: impact parameter)

characteristic frequency $\omega \sim \frac{v}{s}$

convert E_s to function of ω: $\quad E_\omega d\omega = -\underbrace{2\pi s\, ds}_{\text{area of annular target}} E_s \qquad ds>0$ implies $d\omega<0$

$$E_\omega = \frac{2Z^2 e^6 \pi^2}{3c^3 m_e^2}\frac{1}{v^2}$$ independent of ω

$$n_e(v)dv = 4\pi n_e\left(\frac{m_e}{2kT\pi}\right)^{3/2} e^{-\frac{m_e v^2}{2kT}} v^2\, dv$$ Maxwell-Boltzmann dist'n

flux of e^- per unit velocity $\quad n_e v\, dv$

$E_\omega n_e v\, dv =$ energy output per target ion per unit freq.

$$4\pi j_\omega \rho = n_I \int E_\omega n_e(v)\, v\, dv$$ total power emitted per unit freq. and volume

(j_ω: emission coefficient)

$\frac{1}{2}m_e v_m^2 = \hbar\omega \quad$ minimum velocity to produce photon

$$4\pi j_\omega \rho\, d\omega = \frac{2\pi}{3}\frac{Z^2 e^6}{m_e c^3}\left(\frac{2\pi}{m_e kT}\right)^{1/2} n_e n_I e^{-\frac{\hbar\omega}{kT}} d\omega$$

Integrate over freq.:

$$4\pi j_\rho = \frac{2\pi}{3}\frac{z^2 e^6}{m_e c^3 \hbar}\left(\frac{2\pi kT}{m_e}\right)^{1/2} n_e n_I \sim 10^{-27} Z^2 n_I n_e T^{1/2} \text{ erg cm}^{-3}\text{ s}^{-1}$$

$$j/\kappa = B(T) \Rightarrow \kappa = j/B(T) \qquad \boxed{\kappa_{ff} \simeq 4\times10^{-24}\frac{Z^2 n_e n_I T^{-3.5}}{\rho}\text{ cm}^2\text{ g}^{-1}} \propto \rho T^{-3.5}$$

$(n \propto \rho)$

Kramer's Opacity

10^{-25} if doing full Q.M.

4.4.3 Bound-Free Opacity: $\kappa_{bf} \simeq 4\times10^{25} Z(1+X)\rho T^{-3.5}$ same form as above

4.4.4 H^- Opacity

0.75 eV to remove this extra e^-

$2500\,K < T < 10^4\,K$

$$\kappa_{H^-} \simeq 2.5\times10^{-31}\left(\frac{Z}{0.02}\right)\rho^{1/2} T^9 \text{ cm}^2\text{ g}^{-1}$$

4.5 Conduction

in dense medium $\rightarrow$ e^- conduction

$$F_{cond} = -D_e \frac{dT}{dr}$$

e^- diffusion coefficient $\qquad D_e \sim \frac{1}{3} C_V v_e \lambda$

specific heat @ constant V

Mean free path

conductive opacity $\quad \kappa_{cond} = \frac{4acT^3}{3D_e\rho}$

$$F = F_{rad} + F_{cond} \qquad \frac{1}{\kappa_{tot}} = \frac{1}{\kappa_{rad}} + \frac{1}{\kappa_{cond}}$$

Degenerate e^- gas: $\frac{3}{2}k \times \#\, e^-$ that can carry energy $= C_V$

$$\sim \frac{3}{2} k\, n_e \underbrace{\frac{kT}{E_F}}$$

fraction available to transport energy

$$C_V \approx \frac{3}{2}k\,\frac{8\pi}{3}\left(\frac{m_e c}{h}\right)^3 x_F^3 \frac{kT}{\frac{1}{2}m_e c^2 x_F^2} = \frac{8\pi m_e^2 c}{h^3} k^2 T x_F \rightarrow C_V \sim \rho^{1/3} T$$

$$p_F \sim m_e v_F \sim x \sim \rho^{1/3} \qquad \lambda = \frac{1}{\sigma n}$$

$$\frac{Ze^2}{s} \sim m_e v_F^2 \qquad s \sim \frac{1}{v^2} \sim \rho^{-2/3} \qquad \sigma = \pi s^2$$

$$\therefore \lambda \sim \frac{\rho^{4/3}}{n} \sim \rho^{1/3}$$

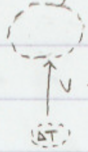
$$D_e \sim \rho^{1/3} T \rho^{1/3} \rho^{1/3} \sim \rho T$$

$$K_{cond} \approx 4 \times 10^{-8} \frac{\mu_e^2}{\mu_I} Z^2 \left(\frac{T}{\rho}\right)^2 \ cm^2\ g^{-1}$$

Chpt 5

Convection

Mixing Length Theory (Benard Problem)



$\longrightarrow$ slow enough to maintain pressure equilibrium $\Delta P = 0$

Flux of energy $F_{conv} = \rho c_p \Delta T\, v$

shares excess with surroundings after traveling a distance ℓ ← mixing length

$\ell_m = \ell$

$$\frac{\Delta T}{T} = \frac{1}{T}\frac{\partial}{\partial r}(\Delta T)\frac{\ell_m}{2} = \frac{1}{T}\frac{\ell_m}{2}\frac{\partial}{\partial r}(T_e - T)$$

$$= \frac{1}{2}\frac{\ell}{P}\frac{dP}{dr}(\nabla - \nabla_e) = \frac{1}{2}\frac{\ell}{H_p}(\nabla - \nabla_e)$$

$$\nabla = \frac{\partial \ln T}{\partial \ln P}$$

$$\frac{\Delta \rho}{\rho} = -\delta \frac{\Delta T}{T} \qquad K_r = -g\frac{\Delta \rho}{\rho} \quad \text{radial bouyancy force}$$

$$v^2 = \frac{1}{2} K_r \frac{\ell}{2} = g\,\delta(\nabla - \nabla_e)\frac{\ell^2}{8H_p} \quad \text{velocity of blob}$$

Flux:

$$F_{conv} = \rho c_p (\nabla - \nabla_e)\frac{1}{2}\frac{\ell}{H_p} T \left(g(\nabla - \nabla_e)\,\delta \frac{\ell^2}{8H_p}\right)^{1/2}$$

$$= \rho c_p T \sqrt{g\delta}\ \frac{\ell^2}{4\sqrt{2}} H_p^{-3/2} (\nabla - \nabla_e)^{3/2}$$

Radiative flux out of element $f = \dfrac{4acT^3}{3\kappa\rho}\dfrac{\partial T}{\partial s}$

luminosity of blob $\sim 4\pi s^2 f \sim 4\pi s^2 \dfrac{4acT^3}{3\kappa\rho}\dfrac{\Delta T}{s} \sim \dfrac{4acT^3}{3\kappa\rho}\Delta T(4\pi s) = \lambda$

temp decrease per unit length $\left(\dfrac{dT}{dr}\right)_e = \left(\dfrac{dT}{dr}\right)_{ad} - \dfrac{\lambda}{\rho V c_p v}$

in blob (?) — volume — velocity

$$\nabla_e - \nabla_{ad} = \frac{\lambda H_p}{\rho \mathcal{V} c_p v T} = \frac{4acT^3}{3\kappa\rho^2 c_p v} \frac{(\nabla - \nabla_e)}{2} \frac{3\ell}{s^2}$$

$s \sim \frac{\ell}{2}$

$$\frac{\nabla_e - \nabla_{ad}}{\nabla - \nabla_e} = \frac{4acT^3}{\kappa\rho^2 c_p \ell v} \qquad v \sim (\nabla - \nabla_e)^{1/3}$$

$$\nabla_e - \nabla_{ad} = 2U(\nabla - \nabla_e)^{1/2} \quad \text{with} \quad U = \frac{2acT^3}{c_p\rho^2\kappa\ell^2}\sqrt{\frac{8H_p}{g\delta}}$$

$$F_{tot} = F_{rad} + F_{conv} = \frac{4ac}{3}\frac{GT^4 m}{\kappa \underbrace{\rho}_{\text{or } P?} r^2}\nabla$$

$$\underbrace{\phantom{F_{tot}}}_{\frac{4acG}{3}\frac{T^4 m}{\kappa\rho r^2}\nabla_{rad}}$$

$$\longrightarrow \quad (\nabla - \nabla_e)^{3/2} = \frac{8}{9}U(\nabla_{rad} - \nabla)$$

$$\nabla_e - \nabla_{ad} = \nabla_e - \nabla + \nabla - \nabla_{ad} = 2U(\nabla - \nabla_e)^{1/2}$$

$$(\nabla - \nabla_e)^{1/2} = -U + \xi \qquad \xi^2 = \nabla - \nabla_{ad} + U^2$$

$$(\xi - U)^3 = \frac{8}{9}U\left(\nabla_{rad} - (\xi^2 + \nabla_{ad} - U^2)\right) \qquad W = \nabla_{rad} - \nabla_{ad}$$

$$(\xi - U)^3 + \frac{8}{9}U(\xi^2 - U^2 - W) = 0$$

cubic for ξ to determine ∇ once we have U & W

$U \to 0 \quad \xi \to U \Rightarrow \nabla \to \nabla_{ad}$ gradient close to adiabatic gradient

only small change needed for convection

$\longrightarrow$ efficient convection

$U \to \infty \quad \nabla \to \nabla_{rad}$ inefficient convection $\longrightarrow$ most transport by radiation

Chpt 6 1/28/2008

Energy Sources

$$\frac{\partial L}{\partial r} = 4\pi \rho r^2 \epsilon \qquad \frac{\partial L}{\partial M} = \epsilon$$

$$\frac{\partial Q}{\partial t} = \underbrace{\underbrace{\frac{\partial u}{\partial t}}_{\text{internal energy}} + P \underbrace{\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)}_{\text{grav. pot energy}}}_{\text{gravitational energy} = -\epsilon_{grav}} = \epsilon - \frac{\partial L}{\partial M}$$

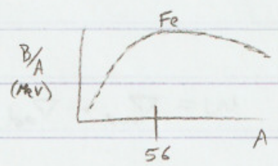
$$\therefore \quad \frac{\partial L}{\partial M} = \epsilon + \epsilon_{grav}$$

Nuclear Energy Sources

$$\alpha + X \to Z^* \to Y + \beta$$

$B = (\text{mass of constituent nucleons} - \text{mass of bound nucleus})\, c^2$

Average binding energy per nucleus is B/A — mass #



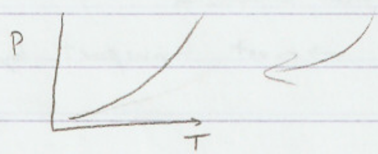
$$E_{coul} \sim \frac{Z_\alpha Z_x e^2}{R} \sim \text{few MeV}$$

R – minimum distance of approach $\sim 10^{-13}$ cm

$KT \sim \text{MeV} \Rightarrow 10^{10}$ K required

6.2.4 Barrier penetration probability $P \sim e^{-2\pi\eta}$ $\quad \eta \sim \dfrac{Z_\alpha Z_x e^2}{\hbar v} \sim 0.16\, Z_\alpha Z_x \left(\dfrac{\mu}{E}\right)^{1/2}$

usually becomes significant when $KT \sim 100$ keV



Hydrogen Burning

6.3 PP chain

PP I : ${}^1H + {}^1H \rightarrow {}^2H + e^+ + \nu_e$ ${}^2H + {}^1H \rightarrow {}^3He + \gamma$ ${}^3He + {}^3He \rightarrow {}^4He + 2\,{}^1H$

PP II: ${}^3He + {}^4He \rightarrow {}^7Be + \gamma$ ${}^7Be + e^- \rightarrow {}^7Li + \nu_e + \gamma$ ${}^7Li + {}^1H \rightarrow 2\,{}^4He$

PP III: ${}^7Be + {}^1H \rightarrow {}^8B + \gamma$ ${}^8B \rightarrow {}^8Be + e^+ + \nu_e$ ${}^8Be \rightarrow 2\,{}^4He$

(6.76) $$\epsilon_{pp} = 2.4 \times 10^4 \frac{\rho X^2}{T_9^{2/3}} e^{-3.380/T_9^{1/3}} \text{ erg g}^{-1} \text{ s}^{-1}$$

6.3.1 D + Li Burning

1) "Deuterium Main Sequence" → short period when pre-MS stars burn deuterium before reaching MS

2) "Li test" → Li burnt in centers of MS stars ⇒ presence of Li in spectrum is a sign of young age or low mass

6.4 CNO cycle

catalyzed H burning pg 304

$C \rightarrow N \rightarrow C \rightarrow N \rightarrow O \rightarrow N \rightarrow C$ $$\epsilon_{CNO} = 4.4 \times 10^{25} \frac{\rho X Z}{T_9^{2/3}} e^{-15.228/T_9^{1/3}} \text{ erg g}^{-1} \text{ s}^{-1}$$

low mass stars – pp higher mass stars – CNO

transition ~ solar mass

6.5 Helium Burning

${}^4He + {}^4He \leftrightarrow {}^8Be$ ${}^4He + {}^8Be \rightarrow {}^{12}C^* \rightarrow {}^{12}C + \gamma$

C^*: excited state → resonant

Triple-α process

(6.80) $$\epsilon_{3\alpha} = 5.1 \times 10^8 \frac{\rho^2 Y^3}{T_9^3} e^{\frac{-4.4027}{T_9}} \text{ erg g}^{-1} \text{ s}^{-1}$$

Carbon Burning

$${}^{12}C + {}^{12}C \rightarrow \begin{cases} {}^{24}Mg + \gamma \\ {}^{23}Mg + n \\ {}^{23}Na + p \\ {}^{20}Ne + \alpha \\ {}^{16}O + 2d \end{cases}$$

Photondisintegration: ${}^{20}Ne + \gamma \rightarrow {}^{16}O + \alpha$ photons break up nuclei

6.8 Neutrinos $\sigma_\nu \sim 10^{-44} \left(\frac{\epsilon_\nu}{1\,MeV}\right)^2$ cm^2 $\lambda = \frac{1}{n\sigma_\nu} \sim 10^{20} \frac{\epsilon_\nu^{-2}}{\rho}$ cm

Neutron star $\rho \sim \frac{1\ m_p}{(1\ fm)^3} \sim 10^{15}\ g/cm^3$ (nuclear densities)

$\lambda \sim 10^5\ \epsilon_\nu^{-2}$ cm no longer optically thin

Neutrino Production Mechanisms

1) Pair annihilation $e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e$

At high T, detailed balance $\gamma + \gamma \Leftrightarrow e^- + e^+$

$KT \sim 2m_ec^2 \implies T \sim 10^{10}\ K$

2) Photoneutrinos or Bremsstrahlung Neutrinos

$e^- + \gamma \rightarrow e^- + \nu_e + \bar{\nu}_e$ hot white dwarfs

3) Plasma Neutrinos

E + B oscillations in plasmas $\rightarrow$ plasmons (collective motion of plasma)

plasmons can decay to produce $\nu_e - \bar{\nu}_e$

6.2.6 Nucleosynthesis

$\rightarrow$ above iron peak elements are made by absorbing neutrons not protons

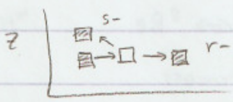
$n + (A,Z) \rightarrow (A+1, Z) \nearrow$ stable

$\searrow$ unstable $\rightarrow (A+1, Z+1) + e^-$ β-decay (slow)

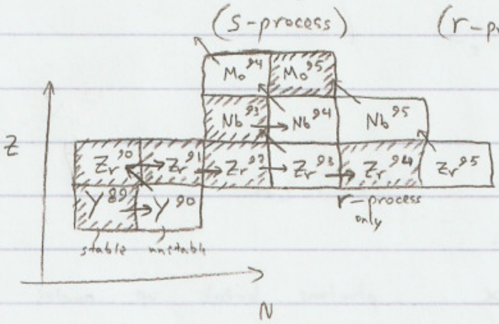
can add neutrons slowly or rapidly

slowly $\downarrow$ unstable products decay (s-process)

rapidly $\downarrow$ no time for decay (r-process)



N = # neutrons



Chpt 7

Stellar Structure Eqns

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho \qquad \frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \qquad \frac{\partial T}{\partial P} = \frac{T}{P}\nabla$$

$$P = P(\rho, T, \underbrace{\vec{x}}_{\text{composition}}) \qquad \kappa = \kappa(\rho, T, \vec{x}) \qquad \epsilon = \epsilon(\rho, T, \vec{x})$$

4^{th} order Diffn eqn $\Rightarrow$ 4 boundary conditions

@ center $r=0$ $M_r = 0$ $L_r = 0$

@ surface ρ & $T = 0$ (simplest) more generally match P&T from atmosphere models

7.2 **Simple Models**

Polytrope $P_e = 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3}$ dyne cm^{-2} eg nonrelativistic degeneracy

In convection zone, $\nabla = \nabla_{ad} = 1 - \frac{1}{\Gamma_2} \Rightarrow \underbrace{P \propto T^{\frac{\Gamma_2}{\Gamma_2 - 1}} \quad T \propto \frac{P}{\rho}}_{P \propto \rho^{\Gamma_2}}$

In general for a polytrope: $\boxed{P = k\rho^{1+\frac{1}{n}}}$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho$$

$$\rho = \rho_c \theta^n \quad P = k\rho_c^{1+\frac{1}{n}}\theta^{n+1} = P_c\theta^{1+n} \qquad r = r_n \xi \qquad r_n^2 = \frac{n+1}{4\pi G} k \rho_c^{\frac{1}{n}-1}$$

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n} \quad \text{Lane-Emden Eqn}$$

@ $\xi = 0$ $\theta = 1$ $\frac{dP}{dr} \to 0$ as $r \to 0 \Rightarrow \frac{d\theta}{d\xi} = 0$ @ $\xi = 0$

Integrate outwards + determine where $\theta = 0$

pg 335 3 cases with analytic solutions:

1) $n=0$ $\theta = 1 - \frac{\xi^2}{6}$

$n = 3/2$ for $P \propto \rho^{5/3}$

2) $n=1$ $\theta = \frac{\sin \xi}{\xi}$

3) $n=5$ $\theta = \left(1 + \frac{\xi^2}{3}\right)^{-1/2}$ infinite in size but finite mass

$n > 5 \to$ infinite in extent and mass

7.2.2 Numerical Solutions

1) Shooting Method – start at one end and work out/in hoping to reach boundary cond.

$x = \xi \quad y = \theta \quad z = \frac{d\theta}{d\xi} = \frac{dy}{dx}$

$y' = \frac{dy}{dx} = z \qquad z' = \frac{dz}{dx} = -y^n - \frac{2}{x} z$

$y_{n+1} = y_n + h\left(\frac{dy}{dx}\right)_n = y_n + h\, f(x_n, y_n)$

4th order Runge-Kutta $\qquad y_{i+1} = y_i + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}$ etc. (pg 339)

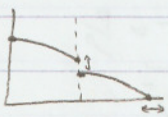
↳ taking intermediate steps

$\theta_n(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \frac{n(8n-5)}{15120}\xi^6 + \ldots$

since $\frac{d^2\theta}{d\xi^2} = -\theta^n - \frac{2}{\xi}\frac{d\theta}{d\xi}$

$\theta = 1 + 0 + \frac{1}{2}\xi^2\left(\frac{d^2\theta}{d\xi^2}\right)_0 \qquad \frac{d\theta}{d\xi} = \xi\left(\frac{d^2\theta}{d\xi^2}\right)_0$

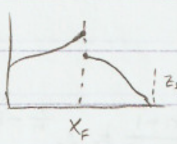
$\Rightarrow \frac{d^2\theta}{d\xi^2} = -\frac{1}{3}$



Both sides, match at some point

Fitting method

1/30/2008



y_o, z_o

y_i, z_i from interior

all at x_F

$Y(x_s, z_s) = y_i - y_o \qquad Z(x_s, z_s) = z_i - z_o \qquad y_s = 0$ (surface)

want $Y = Z = 0$

Taylor expand

$Y(x_s + \Delta x_s, z_s + \Delta z_s) = Y(x_s, z_s) + \frac{\partial Y}{\partial x_s}\Delta x_s + \frac{\partial Y}{\partial z_s}\Delta z_s$ 'Newton-Raphson method'

7.2.4 "Relaxation Methods" → Henyey method

$\frac{dy}{dx} = f(x, y, z) \qquad \frac{dz}{dx} = g(x, y, z)$

Boundary conditions

$b_1(x, y, z) = 0 \qquad b_N(x_N, y_N, z_N)$

entire star is fitted as whole (not just outer/inner)

$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{1}{2}\left(f_{i+1} + f_i\right)$ for $i = 1$ to $N-1$

$2N-2$ equations $+ 2$ BCs $\Rightarrow 2N$ constraints for $2N$ variables

Add perturbations $y_i \to y_i + \Delta y_i$

$$y_{i+1} - y_i - \frac{\Delta x}{2}(f_i + f_{i+1}) = \left(1 + \frac{\Delta x}{2}\left(\frac{\partial f}{\partial y}\right)_i\right)\Delta y_i + \left(\frac{\Delta x}{2}\left(\frac{\partial f}{\partial y}\right)_{i+1} - 1\right)\Delta y_{i+1}$$
$$+ \left(\frac{\Delta x}{2}\left(\frac{\partial f}{\partial z}\right)_i\right)\Delta z_i + \left(\frac{\Delta x}{2}\left(\frac{\partial f}{\partial z}\right)_{i+1}\right)\Delta z_{i+1}$$

want this to be zero for solution

get matrix $\vec{M} \cdot \vec{u} = \vec{R}$ $\longrightarrow$ invert to solve for $\vec{u}$

M: N×N coefficient matrix; u: vector of perturbations; R: degree to which eqns are not satisfied

matrix M is band-diagonal $\begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$ can invert on timescales $\sim N$ (not N^3 as general matrices)

$\rightarrow$ most often used since more robust

7.2.7 <u>Eddington Standard Model</u>

$$\nabla = \frac{P}{T}\frac{dT/dr}{dP/dr} \qquad P_{rad} = aT^4$$

$$\nabla = \frac{1}{4}\frac{P}{P_{rad}}\frac{dP_{rad}}{dP} = \frac{3}{16\pi ac}\frac{P\kappa}{T^4}\frac{L_r}{GM_r}$$

$$\frac{dP_{rad}}{dP} = \frac{L\kappa}{4\pi c}\frac{L_r/L}{GM\,M_r/M} \qquad \langle \epsilon(r) \rangle = \frac{\int_0^r \epsilon\, dM_r}{\int_0^r dM_r} = \frac{L_r}{M_r} \qquad \langle \epsilon(R) \rangle = \frac{L}{M}$$

$$\eta(r) = \frac{\langle \epsilon(r) \rangle}{\langle \epsilon(R) \rangle} \qquad \boxed{\frac{dP_{rad}}{dP} = \frac{L}{4\pi c GM}\eta(r)\kappa(r)}$$

$$P_{rad} = \frac{L}{4\pi c GM}\langle \kappa\eta \rangle P \qquad \langle \kappa\eta \rangle = \frac{1}{P(r)}\int_0^r \kappa\eta\, dP$$

β = ratio of gas pressure over total pressure $\quad 1 - \beta = \frac{P_{rad}}{P}$

$$\boxed{1 - \beta = \frac{L}{4\pi c GM}\langle \kappa\eta \rangle}$$

Eddington's assumption $\longrightarrow$ $\kappa\eta$ = constant since η peaks at center + drops, κ increases outward

$\therefore$ β = constant

since β is constant can get T and P

$$T(r) = \left(\frac{3N_A k}{a\mu}\ \frac{1-\beta}{\beta}\right)^{1/3} \rho_{(r)}^{1/3} \longrightarrow P(r) = \left(\left(\frac{N_A k}{\mu}\right)^4 \frac{3}{a}\ \frac{1-\beta}{\beta^4}\right)^{1/3} \rho^{4/3}(r)$$

$$\Longrightarrow \frac{1-\beta}{\beta^4} = 0.003\,\mu^4 \left(\frac{M}{M_\odot}\right)^2$$

ex: $\mu^4\left(\frac{M}{M_\odot}\right)^2 \sim 1$ $\quad \beta \sim 0.997$
→ nearly all pressure is from gas

~ 50 $\quad \beta \sim 0.5$ → half pressure from radiation

7.3 Boundary Conditions

$$M_r \to \frac{4\pi}{3}\rho_c r^3 \qquad P_r \to P_c - \frac{2\pi}{3} G\rho_c^2 r^2$$

$$L_r \to \frac{4\pi}{3}\rho_c \epsilon_c r^3 \qquad T_r \to T_c - \frac{1}{8ac}\frac{\kappa_c \rho_c^2}{T_c^3}\epsilon_c r^2$$

@ center

@ surface → conditions derived from stellar atmosphere model
(radiative transfer in optically thin limit)

$$\frac{dT}{dr} = -\frac{3}{16\pi ac}\frac{\kappa\rho L}{T^3 r^2} \qquad \frac{dP_{rad}}{d\tau} = \frac{L}{4\pi r^2 c}$$

$$P_{rad} = \int_0^\tau \frac{L}{4\pi R^2 c}\,d\tau = \frac{L}{4\pi R^2 c}\tau + P_{rad}(\tau=0) = \frac{\sigma T_{eff}^4}{c}\tau + P_{rad}(\tau=0)$$

$$P_{rad} = \frac{2\pi}{c}\int_0^\pi I(\theta)\cos^2(\theta)\sin(\theta)\,d\theta$$

Eddington assumption: $I = \frac{\sigma T^4}{\pi}$ out going $\quad I = 0$ incoming

$$P_{rad}^{(\tau=0)} = \frac{2\pi}{3c} I(\tau=0)$$

$$L = 4\pi R^2\, 2\pi \int_0^{\pi/2} I\cos\theta\sin\theta\,d\theta = 4\pi R^2 \pi\, I(\tau=0)$$

$$P_{rad}(\tau=0) = \frac{2}{3c}\sigma T_{eff}^4 \qquad P_{rad}(\tau) = \frac{\sigma}{c}\left(\tau + \frac{2}{3}\right)T_{eff}^4$$

$$\longrightarrow \boxed{T^4(\tau) = \frac{1}{2}T_{eff}^4\left(1 + \frac{3}{2}\tau\right)}$$

$$\frac{dP}{dr} = -\rho g \to \frac{dP}{d\tau} = \frac{g}{\kappa} \qquad P(\tau) = g_s\int_0^\tau \frac{d\tau}{\kappa} = \frac{2\,g_s}{3\,\kappa_p} + P_{rad}(\tau=0)$$

$$\boxed{P(\tau_p) = \frac{2}{3}\frac{g_s}{\kappa_p}\left(1 + \frac{\kappa_p L}{4\pi c GM}\right)} \qquad \frac{1}{\kappa_p} = \int_0^{2/3}\frac{d\tau}{\kappa} \quad \text{photosphere opacity}$$

→ important when near 1

$$\frac{\kappa_p L}{4\pi c GM} = 1 \Longrightarrow L = \frac{4\pi c GM}{\kappa_p}$$

Eddington luminosity $= 3.5\times10^4\left(\frac{M}{M_\odot}\right)L_\odot$ using electron scattering

7.3.2 Boundary Conditions : Radiative Envelope

$$\nabla = \frac{3}{16\pi ac} \frac{P\kappa}{T^4} \frac{L}{M} \qquad \kappa = \kappa_0 \rho^n T^{-s} \qquad P = \frac{\rho}{\mu m_p} kT$$

$$\kappa = \kappa_g P^n T^{-(n+s)}$$

$$P^n dP = \frac{16\pi ac\, GM}{3\kappa_g L} T^{n+s+3} dT$$

$$P^{n+1} = \frac{n+1}{n+s+4} \frac{16\pi ac}{3\kappa_g} \frac{GM}{L} T^{n+s+4} \left(\frac{1-(T_0/T)^{n+s+4}}{1-(P_0/P)^{n+1}} \right)$$

Kramer's Opacity $n=1$ $s=3.5$

$n+1=2$ $\quad n+s+4=8.5$ $\quad$ interior of star insensitive to P_0, T_0 for Kramer's κ

$\rightarrow$ rapidly converges regardless of B.C.

H^- opacity $n=1/2$ $\quad s=-9$ $\quad n+s+4=-4.5$ $\quad \rightarrow$ very sensitive to B.C.

$\rightarrow$ gradient (∇) quickly increases $\Longrightarrow$ convection

$$\nabla \rightarrow \frac{n+1}{n+s+4} = \frac{1}{1+n_{eff}} \qquad P \propto \rho^{1+\frac{1}{n_{eff}}}$$

$\hookrightarrow$ effective polytropic index $\quad n_{eff} = \frac{s+3}{n+1}$

7.3.3 Convective Envelope

$$\nabla = \frac{1}{1+n_{eff}} + \left(\frac{T_{eff}}{T}\right)^{n+s+4} \left[\nabla_p - \frac{1}{1+n_{eff}} \right]$$

$$\hookrightarrow \nabla_p = \frac{3\kappa_0 L}{16\pi ac\, GM} \left(\frac{\mu m_p}{k}\right)^n \frac{P_p^{n+1}}{T_{eff}^{n+s+4}}$$

$P_p = \frac{2}{3} \frac{g_s}{\kappa_p}$ @ photosphere

$$= \frac{3L}{16\pi ac\, GM} \frac{2g}{3\kappa} \frac{\kappa}{T_{eff}^4} = \frac{1}{8} = \nabla_p$$

$$\nabla(r) = -\frac{1}{3} + \frac{11}{24} \left(\frac{T_{eff}}{T}\right)^{-9/2}$$

$\nabla = 0.4$ (from ideal gas $(\frac{2}{5})$) $\quad T/T_{eff} \sim 1.11 \quad \rightarrow$ convective

$$P = 2^{2/3} P_p$$

7.3.3 Fully Convective Stars

$$P = K' T^{5/2} \qquad K' = \frac{0.016}{\mu^{2/5}} \left(\frac{M}{M_\odot}\right)^{-1/2} \left(\frac{R}{R_\odot}\right)^{-3/2} \qquad \nabla = \nabla_{ad} = \frac{2}{5} = \frac{d\ln T}{d\ln P}$$

$\rightarrow$ 3/2 polytrope

$$T_{eff} \simeq 2600\, \mu^{13/51} \left(\frac{M}{M_\odot}\right)^{7/51} \left(\frac{L}{L_\odot}\right)^{1/102} [K]$$ for H^- opacity

(~ 4000K in reality)

$\longrightarrow$ this describes Hayashi track

Entropy

$$T\,dS = dE + P\,d\left(\frac{1}{\rho}\right) \qquad \frac{dS}{dr} = C_p(\nabla - \nabla_{ad})\frac{d\ln P}{dr} \qquad (7.144)$$

$\longrightarrow$ determines run of entropy in the star based on $\nabla - \nabla_{ad}$

$$S = \frac{3}{2}\frac{k}{\mu m_p}\ln\left(\frac{P}{\rho^{5/3}}\right)$$

2/4/8

Star Formation & Protostellar Collapse

Collapse of gas cloud: $t_{ff} \sim \frac{1}{\sqrt{\rho G}}$ $\qquad t_s \sim \frac{R}{c_s} \sim \frac{R}{\left(\frac{kT}{\mu m_p}\right)^{1/2}}$ pressure waves

c_s — sound speed

collapse if $t_{ff} < t_s$ $\qquad R \sim \left(\frac{M}{\rho}\right)^{1/3}$

$$\hookrightarrow M > \boxed{M_J \simeq \left(\frac{kT}{G\mu m_p}\right)^{3/2}\rho^{-1/2} \sim 1.7\times 10^4\, M_\odot \left(\frac{T}{100\text{k}}\right)^{3/2}\left(\frac{n}{1\,\text{cm}^{-3}}\right)^{-1/2}}$$

$M_J \gg M_\odot \longrightarrow$ stars tend to form in clusters

at first: density is low so collapsing gas can cool efficiently

Fragmentation $\Rightarrow$ opacity limited

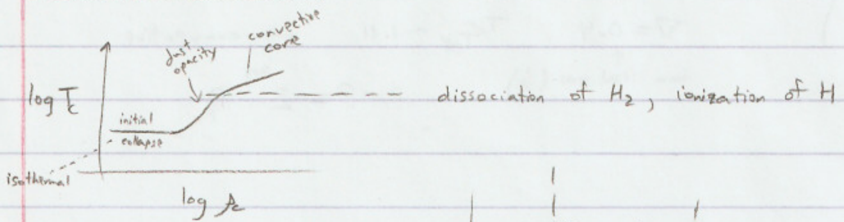
$\longrightarrow$ rise in T offsets rise in ρ & M_J reaches a minimum

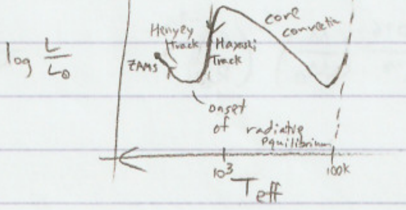
$$L \sim \frac{GM^2}{Rt_{ff}} \sim \left(\frac{3}{4\pi}\right)^{1/2}\frac{G^{3/2}M^{5/2}}{R^{5/2}} \sim 4\pi R^2 \sigma T^4$$

$$\Rightarrow M^5 = \frac{64\pi^3}{3}\frac{\sigma^2 T^8 R^9}{G^3}$$

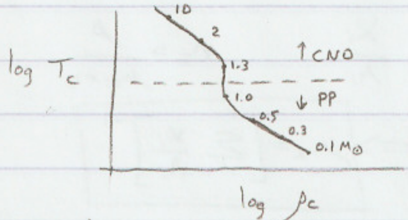
$$\boxed{M_J \sim 0.02\, M_\odot \frac{T^{1/4}}{\mu^{9/4}}}$$

limited by whether the collapsing gas can cool





Main Sequence

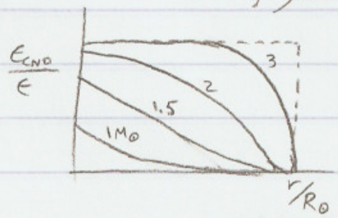


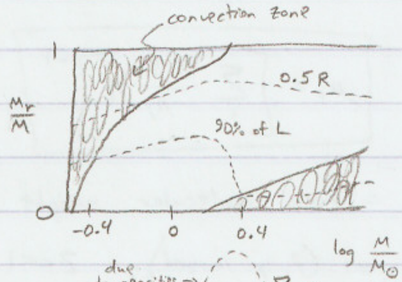
$R \sim M^{0.8}$ (upper)

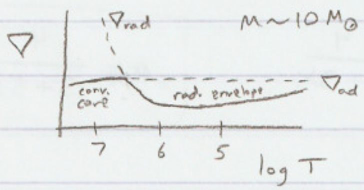
$\sim M^{0.57}$ (lower)

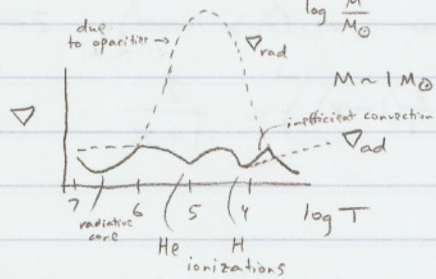
$L \sim M^3$

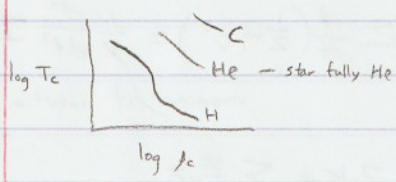
(Hayashi track pg 155 The Origin of Stars)

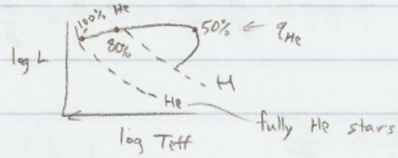


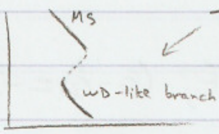












T kept low as density increases

"pyconuclear" burning — density high enough to overcome Coulomb barrier

Chpt 9 ## The Sun

9.1 $\frac{Z}{X} = 0.0275$ $\quad Y \simeq 0.25$

$X = 0.709$

$Y = 0.271$

$Z = 0.020$

Solar Lithium problem

Sun has less Li than expected

9.2 on ZAMS $L = 0.725 L_\odot$ $\quad R = 0.886 R_\odot$

mass loss $\dot{M} \sim 10^{-14} M_\odot \, yr^{-1}$ so not important in MS

L dominated by pp process (CNO ~ 1%)

inner 73% of radius : radiative outer 27% of radius contains 3% of mass

Change in Composition

$$n_{I,i} = \frac{\rho X_i}{A_i m_p} \qquad n_I = \sum_i n_{I,i} = \frac{\rho}{m_p}\sum_i \frac{X_i}{A_i} \qquad n_I = \frac{\rho}{\mu_I m_p}$$

$$\Rightarrow \boxed{\mu_I = \left[\sum_i \frac{X_i}{A_i}\right]^{-1}}$$

ionized fraction

$$n_{e_i} = y_i Z_i n_{I,i} \qquad \boxed{\mu_e = \left[\sum_i \frac{Z_i X_i}{A_i} y_i\right]^{-1}}$$

$$\mu = \left[\frac{1}{\mu_I} + \frac{1}{\mu_e}\right]^{-1} = \text{mean molecular weight}$$

stellar interior $y_i = 1$ (fully ionized) $Z << 1$ $Y = 1 - X$

$$\mu_e = \left(X + \frac{2}{4}(1-X)\right)^{-1} = \frac{2}{1+X} \qquad \mu_I = \frac{4}{1+3X} \qquad \boxed{\mu = \frac{4}{3+5X}}$$

Chpt 1

Virial Theorem

$$\sum_i \vec{p}_i \cdot \vec{r}_i \qquad \frac{d}{dt}\sum_i \vec{p}_i \cdot \vec{r}_i = \frac{d}{dt}\sum_i m_i \dot{\vec{r}}_i \cdot \vec{r}_i = \frac{d}{dt}\sum \frac{d}{dt}\left(\frac{1}{2} m_i r_i^2\right) = \frac{1}{2}\frac{d^2}{dt^2} I$$

moment of inertia

$$\sum_i \underbrace{\frac{d\vec{p}_i}{dt}}_{\vec{F}_i} \cdot \vec{r}_i + \underbrace{\sum_i \vec{p}_i \cdot \frac{d\vec{r}_i}{dt}}_{\sum m_i v_i^2} \qquad \frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \sum_i \vec{F}_i \cdot \vec{r}_i$$

Using gravity $\sum_i \vec{F}_i \cdot \vec{r}_i = \sum_{i<j}\left(\vec{F}_{ij}\cdot\vec{r}_i + \vec{F}_{ji}\cdot\vec{r}_j\right) = \sum_{i<j} \vec{F}_{ij}\cdot(\vec{r}_i - \vec{r}_j) = -\sum_{i,j} \frac{G m_i m_j}{r_{ij}} = \Omega$

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \Omega}$$ Virial Equilibrium $\frac{d^2 I}{dt^2} = 0$ $\boxed{2K + \Omega = 0}$

9.2.2 Internal energy $K = \frac{3}{2} kT \frac{M}{\mu m_p}$ Binding energy $\Omega \propto -\frac{GM^2}{R}$

$$\frac{3}{2} kT \frac{M}{\mu m_p} = \frac{1}{2}\frac{GM^2}{R} \qquad T = \frac{1}{3}\frac{GM\mu m_p}{Rk} \qquad \boxed{T \sim \mu M^{2/3} \rho^{1/3}}$$

Virial temperature

Radiative transport $L \sim \frac{r^2}{\kappa\rho}\frac{\partial T^4}{\partial r} \sim \frac{RT^4}{\kappa\rho}$ $\kappa \sim \kappa_0 \rho T^{-3.5}$

$$L \sim \frac{M^{5.33}\rho^{0.117}}{\kappa_0}\mu^{7.5} \qquad L(t) \sim \mu^{7.5} \qquad \mu = \frac{4}{3+5X}$$

$$\frac{\partial X}{\partial t} = -\frac{L(t)}{MQ} \qquad Q = 6\times10^{18} \text{ ergs g}^{-1}$$

(9.10) $\Rightarrow \frac{L(t)}{L(0)} = \left(1 - 0.3\frac{L(0)}{L_\odot}\frac{t}{t_\odot}\right)^{-\frac{15}{17}}$ $\mu(0) \sim 0.6$ $t_\odot \sim 4.5\times10^9$ yrs

9.3 Solar Neutrino "Problem"

PP | CNO

$^1H + {}^1H \rightarrow {}^2H + e^+ + \nu_e$ — continuum up to 0.42 MeV | $^{13}N \rightarrow {}^{13}C + e^+ + \nu_e$

$^7Be + e^- \rightarrow {}^7Li + \nu_e + \gamma$ — 0.384 MeV or 0.862 MeV (dominant) | $^{15}O \rightarrow {}^{15}N + e^+ + \nu_e$

$^8B \rightarrow {}^8Be + e^+ + \nu_e$ — continuum up to 15 MeV | $^{17}F \rightarrow {}^{17}O + e^+ + \nu_e$

Detection

$\nu_e + {}^{37}Cl \rightarrow {}^{37}Ar + e^-$ threshold of 0.81 MeV

└ radioactive

Flux in ^7Be ν is 10^{10} $cm^{-2}\, s^{-1}$ $\sigma_\nu \sim 10^{-44}$ cm^2 $\Rightarrow$ 10^{-35} s^{-1} per target nucleus

Solar Neutrino Unit (SNU) = 10^{-36} captures s^{-1} target^{-1}

predicted: 7.9 ± 2.4 SNU   Homestake: 2.07 ± 0.3 SNU

Neutrino oscillations

$|\nu_e\rangle = |\nu_1\rangle \cos\theta + |\nu_2\rangle \sin\theta$ $|\nu_\mu\rangle = |\nu_1\rangle \sin\theta + |\nu_2\rangle \cos\theta$

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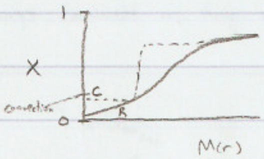
pg 137
Ex. 2.22

Schonberg – Chandrasekhar Limit



$T_c \sim$ isothermal
└ core

$\frac{dT}{dr} \propto L$ L small in core
H shell burning

Virial Theorem in the core $2K + \Omega = 0$

$$K = \frac{3}{2} kT \frac{M_c}{\mu m_H} = \frac{3}{2} M \frac{P}{\rho}$$

$$2\int_0^{M_c} \frac{3}{2}\frac{P}{\rho}\, dm = \int_0^M \frac{Gm}{r}\, dm + 4\pi R^3 P_0$$

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho \qquad \frac{dM}{dr} = 4\pi r^2 \rho \quad \Rightarrow \quad \frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$

$$\int_0^M 4\pi r^3 \frac{dP}{dm}\, dm = \int_0^M \cancel{4\pi r^3} \frac{(-GM)}{\cancel{4\pi r^4}}\, dm = -\int_0^M \frac{GM}{r}\, dm$$

$$= \left[4\pi r^3 P\right]\Big|_0^M - \int_0^M 12\pi r^2 \frac{dr}{dm} P\, dm = (4\pi r^3 P)\Big|_0^M - 3\int_0^M \frac{P}{\rho}\, dm$$

$$\boxed{\int_0^M \frac{GM}{r}\, dm = 3\int_0^M \frac{P}{\rho}\, dm - 4\pi R^3 P_0}$$

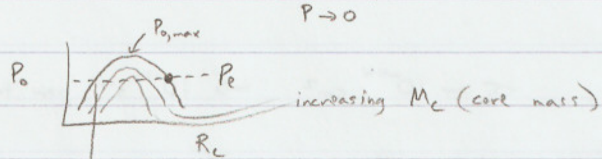
Virial theorem with external pressure

$$\overset{\text{internal}}{3E_i} + \overset{\text{gravity}}{E_g} = 4\pi R^3 P_0 \qquad P_0 = C_1 \frac{M_c T_0}{R_c^3} - C_2 \frac{M_c^3}{R_c^4}$$

$$\frac{dP}{dR_c} = 0 \implies P_{0,max} \sim M_c^{-2}$$ maximum pressure of isothermal core

$$\frac{dP}{dR} \sim \frac{M}{R^2}\cdot\frac{M}{R^3} \qquad P_e \sim \frac{M^2}{R^4}$$ pressure of envelope

$$\frac{MT}{R^3} \sim \frac{M^2}{R^4}$$ (from virial theorem in envelope $P \to 0$) $\therefore\ T \sim M/R \qquad P_e \sim \frac{T_0^4}{M^2}$



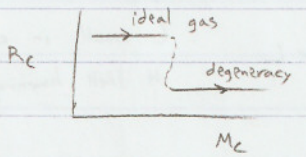
thermally unstable (expand core → more P_0 → more expansion)

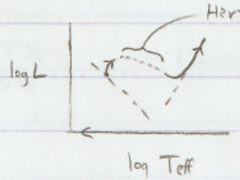
At critical core mass, $P_{0,max} = P_e$

core mass fraction $\boxed{q_{sc} = 0.37\left(\frac{\mu_{env}}{\mu_{core}}\right)^2}$

typically 10% of total mass

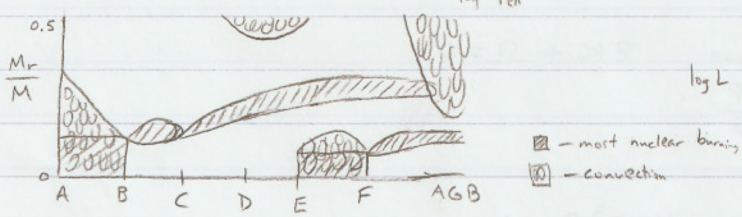
Post Main Sequence Evolution: Massive Stars

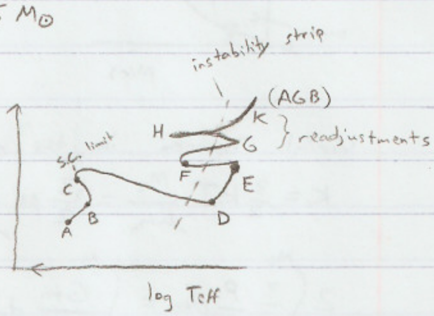




Hertzsprung Gap – stars not in thermal equilibrium

$5 M_\odot$





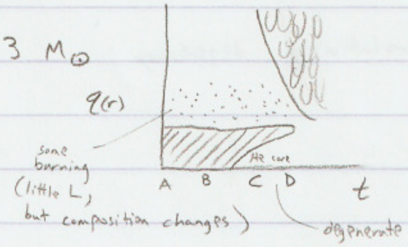
Helium ignition — non-degenerate in massive stars

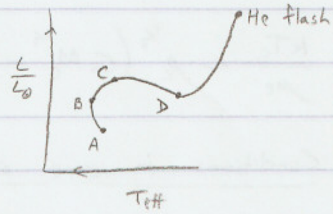
$3\alpha \to {}^{12}C$

${}^{12}C + \alpha \to {}^{16}O$

Cepheid pulsations $t \sim \frac{1}{\sqrt{\rho}} \sim \left(\frac{R^3}{M}\right)^{1/2}$ → leads to period-luminosity relation

1.3 $M_\odot$





Helium ignition occurs under degenerate conditions.

→ temperature rises until degeneracy is lifted $(T \sim 10^9 K)$

Luminosity – Core Mass Relation (in RGB)

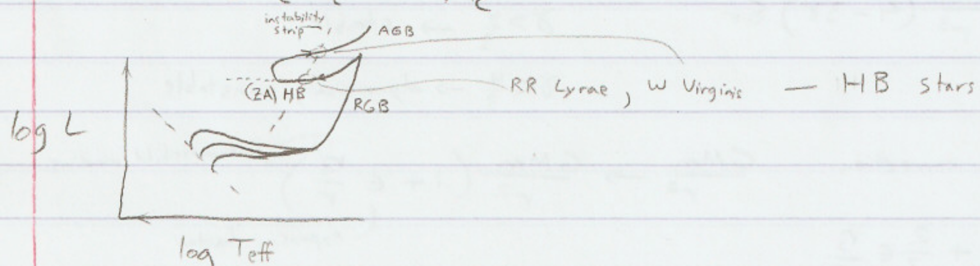
$L \sim 4\pi r^2 \Delta r \, \epsilon_0 \rho^2 T^n$ $\qquad \rho_c \sim \frac{M_c}{R_c^3} \sim M_c^2$ since $R_c \sim M_c^{-1/3}$ degeneracy

e^- cant carry energy/heat in degenerate matter (no free states)

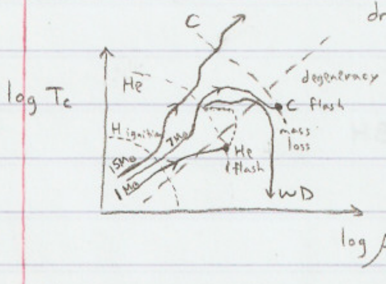
→ heat capacity dominated by ions

$T_c \sim \frac{M_c}{R_c}$ $\left(\begin{array}{l}\text{ideal ion gas} \\ + \text{virial theorem}\end{array}\right) \Rightarrow T_c \sim M_c^{4/3}$

$\therefore \quad L \sim R_c^2 M_c^4 M_c^{\frac{4n}{3}} \sim M_c^{\frac{10+4n}{3}}$ $\qquad$ pp $n \sim 4: L \sim M_c^7$



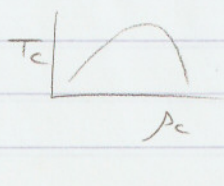
Thermal pulses (AGB) — convection zone gets deep enough and resupplies H to exhausted layers that can burn it drives convection zone out which stops mixing (and repeat)



Evolution of C–O core

$P \sim \frac{\rho}{\mu m_H} kT + k\left(\frac{\rho}{\mu_e}\right)^\gamma$ $\qquad$ Hydro eqns $P_c \sim \frac{G M_c \rho_c}{R_c} \sim G M_c^{2/3} \rho_c^{4/3}$ if $R_c = \left(\frac{M_c}{\rho_c}\right)^{1/3}$

$\frac{kT_c}{\mu e} = G M_c^{2/3} \rho_c^{1/3} - k \rho_c^{\gamma-1} \mu_e^{-\gamma}$



$$\frac{kT_c}{\mu_e} \sim \rho_c^{1/3}\left(GM_c^{2/3} - k\mu_e^{-4/3}\right)$$ limit of relativistic degeneracy

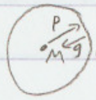
Conditions in core of massive star

Pressure is dominated by degenerate e^-

$T \sim 10^{10}\,K \Rightarrow e^-$ are relativistic

photodisintegration of heavy nuclei is important energy sink

GR effects also reduce stability



$$m\ddot{r} = 4\pi r^2 P - \frac{GMm}{r^2} = 0$$ in equilibrium

Adiabatic perturbation $\frac{dP}{P} = \gamma \frac{d\rho}{\rho} = -3\gamma \frac{dr}{r}$

$$m\delta\ddot{r} = 8\pi r\,\delta r\,P + 4\pi r^2 P\left(-3\gamma\frac{\delta r}{r}\right) + \frac{2GMm}{r^3}\delta r$$

$$= \frac{GMm}{r^3}(4-3\gamma)\,\delta r$$

$\gamma > \frac{4}{3} \rightarrow$ stable

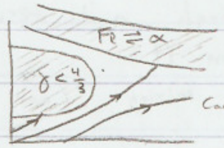
$\gamma < \frac{4}{3} \rightarrow$ dynamically unstable

Simple GR correction $\frac{GMm}{r^2} \rightarrow \frac{GMm}{r^2}\left(1 + \epsilon\frac{r_s}{r}\right)$

Schwarschild radius

expansion factor

$$\gamma_{crit} = \frac{4}{3} + \frac{2}{3}\epsilon\frac{r_s}{r}$$



4-8 $M_\odot$ stars may ignite C under degenerate conditions

$M < 8\,M_\odot \rightarrow$ WD

$M > 8\,M_\odot \rightarrow$ NS / BH

2/13/2008

Stellar Explosions

Combustion-driven explosions
↓
nuclear ignition burning front propagates through the star

detonation
↓
when compression & heating of material by the schock is sufficient to ignite nuclear burning

deflagration
↓
material is only ignited on a longer timescale of energy transport within the star (conduction, convection)

This also happens in accreting white dwarfs.

a) low accretion rates
 → accreted material is degenerate
 ↳ helium flash → double detonation wave
 ↳ shocks propagate both inwards + outwards

b) high accretion rates
 → helium non-degenerate when it ignites
 → C grows until central density + temperatures are high enough for Carbon flash ($M \sim M_{ch}$)
 → Type Ia Supernova

Collapse

At end of massive star's life, iron core becomes dynamically unstable
 → free-fall collapse initially
 → collapse is only halted when $\rho \sim 10^{14}$ g cm^{-3} (neutron degeneracy pressure)
 → equation of state "stiffens" → generates core bound

$E_{bind} \sim \frac{GM_c^2}{R_c} \sim 3 \times 10^{53}$ ergs for $1.4 M_\odot$ core

$E_{bind, envelope} \sim \frac{GM^2}{R_{wd}} \sim 3 \times 10^{52}$ ergs for $10 M_\odot$ star

But low efficiency of energy conversion

1) most of energy is emitted in neutrinos

2) shock wave breaks up Si nuclei + others in overlying layers

$$E \sim \frac{M}{m_p} Q \sim 2\times10^{51}\ \text{ergs}\left(\frac{M}{1M_\odot}\right)\left(\frac{Q}{1\ \text{MeV/nucleon}}\right)$$

→ not just photons, but particles as well

r-process can take place here

Neutrinos

$p + e^- \to n + \nu_e$ "inverse β-decay"

Typical ν energies ~ Fermi energy of e^-

$$\frac{E_\nu}{m_e c^2} \sim \frac{E_{F,e^-}}{m_e c^2} \simeq 10^{-2}\left(\frac{\rho}{\mu_e}\right)^{1/3}$$ for relativistic e^-

Principal opacity → "coherent" scattering off nuclei

$$\nu + (Z,A) \to \nu + (Z,A)$$

$$\sigma_\nu \sim 10^{-45}\ \text{cm}^2\ A^2\left(\frac{E_\nu}{m_e c^2}\right)^2 \sim 10^{-49}\ \text{cm}^2\ A^2\left(\frac{\rho}{\mu_e}\right)^{2/3}$$

$$l_\nu \sim \frac{1}{n\sigma_\nu} \sim \frac{1.7\times10^{25}\ \text{cm}}{\mu_e A}\left(\frac{\rho}{\mu_e}\right)^{-5/3}$$

$\mu_e = 2$ $A \sim 100$

$\rho \sim 3.6\times10^9\ \text{g cm}^{-3}$ $\Rightarrow l_\nu \sim 10^7\ \text{cm}$

Neutrinos trapped if $\rho > 3\times10^{11}\ \text{g cm}^{-3}$ (diffusion time < collapse time)

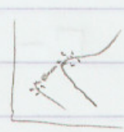
Spread of neutrinos from SN1987A due to diffusion through neutron star

neutrino driven convection may revive stalled supernova shocks

Compact Objects

1) White Dwarfs

$M \lesssim 8-10\, M_\odot$ on M.S.
(5?)

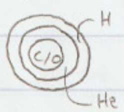


relating mass of WD and mass of M.S.

$T_{eff} \sim 10^5 K \rightarrow 4000K$ (maybe 3000K)
(planetary nebulae nuclei : PNN)

Core composition $\rightarrow$ Carbon / Oxygen more massive $\rightarrow$ Oxygen / Neon / Magnesium

Atmosphere composition $\rightarrow$ DA — see Balmer lines, no He or metals - majority



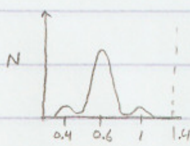
DB — see He I lines, no H or metals

DC — no lines, continuum spectrum

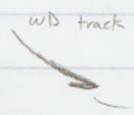
DO — He II lines, hot WD, some He I or H

DZ — show metal lines

DQ — molecular carbon possible dredge up of C



binary companion disrupted evolution



Simple Cooling Curve

core - isothermal, energy transport of e^- very efficient

Radiative transport + Kramer's opacity in envelope

$$p^2 = \frac{2}{8.5}\,\frac{16}{3}\,\pi\,\frac{ac\,GM}{\kappa_0 L}\left(\frac{k_b}{\mu m_p}\right)T^{8.5}$$

Transition to degenerate region $P \sim \frac{\rho k T}{\mu m_p} \sim 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3}$

$$T_{core} \sim 6.4 \times 10^7 K \left(\frac{L}{L_\odot}\frac{M_\odot}{M}\right)^{2/7} \frac{\mu_e^{5/2}\, Z}{\mu^2}$$

$$L = \frac{dE}{dt} = \frac{d}{dt}\left(\frac{3}{2} k T_{core} \frac{M}{A m_p}\right) \Rightarrow L = 4.9 \times 10^{-4} L_\odot \left(\frac{M}{M_\odot}\right)\left(\frac{t}{10^9 yrs}\right)^{-7/5}\left(\frac{A}{12}\,\frac{\mu^2}{\mu_e^{5/3} Z}\right)^{-7/5}$$

Core Crystallization

$$\Gamma \sim \frac{Ze^2}{r}\frac{1}{kT}$$

(r: mean interparticle separation)

$\Gamma \sim 170 \rightarrow$ crystallization

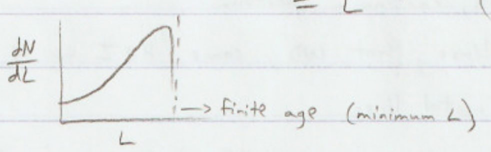
→ latent heat

→ change in heat capacity because fewer lattice vibrations excited

Luminosity Function

$$\frac{dN}{dL} = \frac{dN}{dt}\frac{dt}{dL} = \frac{dt}{dL} \quad \left(\frac{dN}{dt} \sim \text{constant for galactic disk}\right)$$

$$= L^{-12/7} \quad \left(t \propto L^{-5/7}\right) \rightarrow \text{LF increases as } L \text{ drops}$$



narrow peak for cluster $\left(\frac{dN}{dt} \neq \text{constant}\right)$

2) Neutron Stars

$$p + e^- \rightleftharpoons n + \nu_e$$

beginning of collapse → large, neutron-rich nuclei + e^-

$\rho \gtrsim 10^{11}\ g\ cm^{-3}$ → "neutron drip" i.e. free neutrons in statistical equilibrium with nuclei + e^-

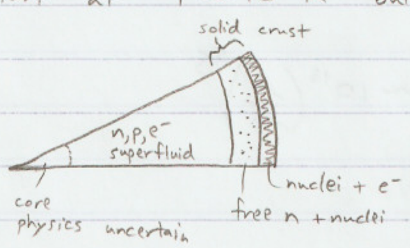
$\rho \gtrsim 10^{14}\ g\ cm^{-3}$ → equilibrium with n, p, e^-

$n_n : n_p : n_e = 8:1:1$ neutron star still has p, e^-

start at $T \sim 10^{10}\ K$ but cool rapidly

→ first by ν then X-rays

→ observable in X-rays for 10^8 yrs



rate of cooling decreases with time

2/20/2008

Pulsars

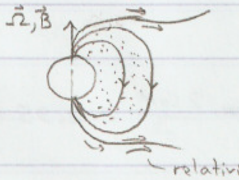
periodic radio souces
- period: milliseconds — tens of seconds

$$t_p \sim \frac{P}{\dot{P}} \sim 10^7 \text{ yrs} \quad (P \sim 1s)$$
$$\sim 10^{10} \text{ yrs} \quad (P \sim ms)$$

Braking index $\quad n = \frac{\Omega \ddot{\Omega}}{(\dot{\Omega})^2} \quad$ or $\quad 2-n = \frac{P\ddot{P}}{(\dot{P})^2}$

Magnetic Dipole Moment

Vacuum Aligned Rotator: (not a pulsar)



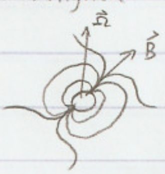
In frame of star: frame is non-inertial

$$\vec{E} = \frac{(\vec{r} \times \vec{\Omega}) \times \vec{B}}{c}$$

populate charge among field $\quad \rho = \frac{\nabla \cdot \vec{E}}{4\pi} = -\frac{\vec{\Omega} \cdot \vec{B}}{2\pi c} \quad$ Goldreich – Julian density

"Light Cylinder" $\quad R_L = \frac{c}{\Omega} \quad$ maximum distance a plasma can be in corotation with field lines (otherwise faster than light)

Non-Aligned Rotator:



Dipole moment gives you $n = 3$

Actual observed: $n \sim 2-3$

Light comes from dipole moment radiation (pulsar) and relativistic wind

→ need changing magnetic field for radiation

particle flux carries most of energy

$$\dot{P} = \frac{B^2 R^6}{c^3 I} \frac{(2\pi)^2}{P} \implies B = 10^{12} \left(P \cdot \frac{\dot{P}}{10^{-15}} \right)^{1/2} G$$

(assuming all energy lost via magnetic radiation)

moment of inertia $\sim MR^2$

$$\underbrace{\tau = \frac{P}{2\dot{P}} = 10^7 - 10^8 \text{ yrs}}_{\text{young pulsars}} \quad \text{spin-down period}$$

Millisecond Pulsars

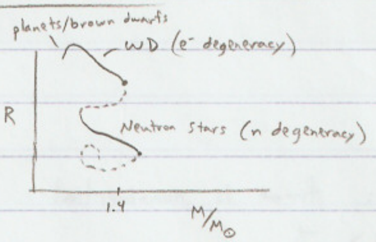
$P \sim 1-10\,ms$ $B \sim 10^9\,G$ $\tau \gtrsim 10^{10}\,yrs$ (sub millisecond might break itself apart)

Most are found in binaries

→ as they accrete material they spin up and somehow decrease B

→ pulsar planets around millisecond pulsars → selection effects since need accuracy of $10^{-6}\,s$

Black Holes



Upper limit on neutron star mass $\sim 3-5\,M_\odot$

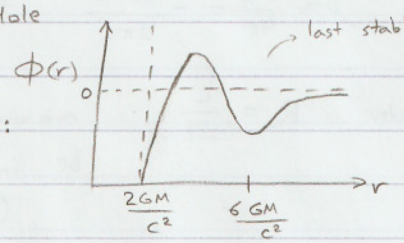
More? → BH

Schwarzschild radius $r_s = \frac{2GM}{c^2} = 2.95\,km\left(\frac{M}{M_\odot}\right)$

(nonrotating BH)

Rotating (Kerr) Black Hole

For equatorial region:



Binary stars → K star around unseen $10\,M_\odot$ object

Supermassive Black Holes

- centers of galaxies $\sim 10^6 - 10^9\,M_\odot$

Quasars ⇒ $L \sim 10^{44}\,ergs\,s^{-1}$

H fusion $\epsilon \sim 0.007 \sim \frac{7\,MeV}{m_p c^2}$ $L \sim \epsilon \dot{M} c^2 \Rightarrow \dot{M} \sim 0.25\,M_\odot\,yr^{-1}$

too much for stars to account for $\sim \frac{2.5 \times 10^6\,M_\odot}{10^7\,yr}\left(\frac{L}{10^{44}\,erg/s}\right)$

Asteroseismology

Chpt 8 ↳ Helioseismology   WD, Cepheids, COROT satellite

8.1 Adiabatic Radial Pulsations

↓ dynamical times much less than thermal times

↘ motion of spherical shells

$$\frac{\partial M_r}{\partial r} = 4\pi r^2 \rho \qquad \ddot{r} = -4\pi r^2 \frac{\partial P}{\partial M_r} - \frac{GM_r}{r^2} \qquad (\text{Force})$$

Perturbation theory $\quad r(t, M_r) = r_0(M_r)\left(1 + \frac{\delta r}{r_0}\right) \qquad \rho(r, M_r) = \rho_0(M_r)\left(1 + \frac{\delta\rho}{\rho_0}\right)$

small perturbations

$$\frac{\partial M_r}{\partial\left(r_0\left(1+\frac{\delta r}{r_0}\right)\right)} = 4\pi\left(r_0\left(1+\frac{\delta r}{r_0}\right)\right)^2 \rho_0\left(1+\frac{\delta\rho}{\rho_0}\right)$$

$$\frac{\partial M_r}{\partial r_0}\left(1 - \frac{\delta r}{r_0} - r_0\frac{\partial}{\partial r_0}\left(\frac{\delta r}{r_0}\right)\right) = 4\pi r_0^2 \rho_0\left(1 + 2\frac{\delta r}{r_0} + \frac{\delta\rho}{\rho_0}\right)$$

zeroth order: $\quad \frac{\partial M_r}{\partial r_0} = 4\pi r_0^2 \rho_0$

1st order: $\quad \frac{\delta\rho}{\rho_0} = -3\frac{\delta r}{r_0} - r_0\frac{\partial}{\partial r_0}\left(\frac{\delta r}{r_0}\right) \qquad (8.6)$

Force Equation: $\quad \rho_0 r_0 \frac{d^2}{dt^2}\left(\frac{\delta r}{r_0}\right) = -\left(4\frac{\delta r}{r_0} + \frac{\delta P}{P_0}\right)\frac{\partial P_0}{\partial r_0} - P_0\frac{\partial}{\partial r_0}\left(\frac{\delta P}{P_0}\right)$

From adiabatic approx: $\quad \Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{ad.} \Rightarrow \frac{\delta P}{P_0} = \Gamma_1 \frac{\delta\rho}{\rho_0}$

2 eqns in δr & $\delta\rho$ → decompose into normal modes $\quad \frac{\delta r}{r_0} = \zeta(r_0) e^{i\sigma t}$

$$\frac{d\zeta}{dr_0} = -\frac{1}{r_0}\left(3\zeta + \frac{1}{\Gamma_1}\frac{\delta P}{P}\right) \qquad \frac{d}{dr_0}\left(\frac{\delta P}{P}\right) = -\frac{d\ln P}{dr}\left(4\zeta + \frac{\sigma^2 r^3}{GM_r}\zeta + \frac{\delta P}{P}\right)$$

@ $r=0$: $\quad 3\zeta + \frac{1}{\Gamma_1}\frac{\delta P}{P} = 0 \quad$ boundary condition

Additional amplitude constraint → determine eigenvalues & eigenstates

8.1.1 Sturm – Liouville

↳ all σ^2 are real < pure oscillation / pure decaying/growing mode

also $\exists$ a minimum σ^2

8.2 Non-Adiabatic Modes

$$\frac{\partial L_r}{\partial M_r} = \varepsilon - \frac{P}{\rho(\Gamma_3 - 1)}\left(\frac{\partial \ln P}{\partial t} - \Gamma_1 \frac{\partial \ln \rho}{\partial t}\right)$$

$$\delta\varepsilon - \frac{\partial \delta L_r}{\partial M_r} = i\sigma C_v T\left(\frac{\delta T}{T} - (\Gamma_3 - 1)\frac{\delta\rho}{\rho}\right) \quad (8.39)$$

↑ pulsations can grow/decay

From radiative diffusion eqn.

$$\frac{\delta L_r}{L_r} = 4\zeta - \frac{\delta\kappa}{\kappa} + 4\frac{\delta T}{T} + \frac{1}{\partial T/\partial r}\frac{d}{dr}\left(\frac{\delta T}{T}\right) \quad (8.58)$$

8.2.1 Quasi-adiabatic

→ assume adiabatic relation between $\delta\rho$ & δT + assume $\kappa \sim \rho^n T^{-s}$

$$\frac{\delta L_r}{L_r} = -\frac{\left(\frac{4}{3} + n\right)}{\Gamma_3 - 1}\frac{\delta T}{T} + (s+4)\frac{\delta T}{T} \quad (8.59)$$

Kramer's opacity $n=1$, $s=3.5$ & assume $\Gamma_3 = 5/3 \Rightarrow \frac{\delta L_r}{L_r} = 4\frac{\delta T}{T}$

In ionization zones, $\Gamma_3 < 5/3$ opacity more complicated

$\frac{\delta L_r}{L_r} = -(\dots)\frac{\delta T}{T}$ increase in T, decrease L ← κ-effect

opacity changes drive oscillation

γ-effect → equation of state drives pulsation ($\to \Gamma_3$)

Non-adiabatic effects important when driving occurs @ surface (Cepheids)

because $t_{th} \sim \frac{C_v T \Delta M}{L}$ & $\Delta M =$ surface layer mass

$$2\,t_{th} \sim t_{dyn}$$

8.3 **Adiabatic Non-radial**

1st order perturbation : $\vec{\xi}$, δP

$$\sigma^2 \xi_r = \frac{\partial}{\partial r}\left(\frac{\delta P}{\rho}\right) - A \frac{\Gamma_1 P}{\rho} \nabla \cdot \vec{\xi} \quad (8.89) \qquad A = \frac{d \ln \rho}{dr} - \frac{1}{\Gamma_1} \frac{d \ln P}{dr}$$

$$\sigma^2 \xi_\theta = \frac{\partial}{\partial \theta}\left(\frac{1}{r} \frac{\delta P}{\rho}\right)$$

$$\sigma^2 \xi_\varphi = \frac{1}{\sin\theta} \frac{\partial}{\partial \varphi}\left(\frac{1}{r} \frac{\delta P}{\rho}\right)$$

decompose into spherical harmonics $Y_{\ell m}(\theta, \varphi)$

(8.102) $r \dfrac{d \xi_r}{dr}$

(8.103) $r \dfrac{d \xi_t}{dr}$

(8.99) $$\boxed{N^2 = -Ag = -g\left(\frac{d \ln \rho}{dr} - \frac{1}{\Gamma_1} \frac{d \ln P}{dr}\right)}$$ Brunt-Väisälä frequency (radial)

Lamb frequency (azimuthal) $$\boxed{S_\ell^2 = \frac{\ell(\ell+1)}{r^2} \frac{\Gamma_1 P}{\rho} = \frac{\ell(\ell+1)}{r^2} c_s^2} \quad (8.100)$$

$K_t^2 = \dfrac{\ell(\ell+1)}{r^2} = \dfrac{S_\ell^2}{c_s^2}$ transverse wave number

8.3.3 (?) using $e^{ik_r r}$, we get dispersion relation

$$\boxed{k_r^2 = \frac{k_t^2}{\sigma^2 S_\ell^2}\left(\sigma^2 - N^2\right)\left(\sigma^2 - S_\ell^2\right)} \quad (8.108)$$

if $\sigma^2 < N^2, S_\ell^2$ or $\sigma^2 > N^2, S_\ell^2 \rightarrow k_r^2 > 0 \Rightarrow$ standing wave eigenfunction

if $S_\ell^2 < \sigma^2 < N^2 \rightarrow k_r^2 < 0 \Rightarrow$ evanescent

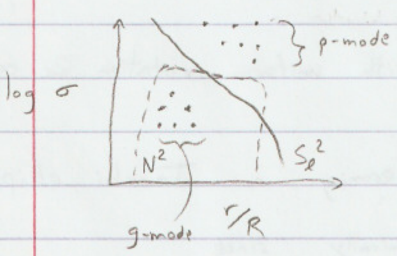


Fig 8.3

$\sigma^2 \gg N^2, S_\ell^2 \rightarrow \sigma^2 \sim (k_r^2 + k_t^2)\, c_s^2$ p-mode

sound speed

(8.109, 110) pressure/acoustic mode

$\sigma^2 \ll N^2, S_\ell^2 \rightarrow \sigma^2 \sim \dfrac{k_t^2}{k_r^2 + k_t^2} N^2$ g-mode

gravity

5 minute oscillations on Sun $\rightarrow$ non-radial p-modes

ZZ Ceti stars (WD) $\rightarrow$ g-modes ($P \sim$ few hundred seconds)

<u>Binary Stars</u>

2/25/2008

<u>Formation:</u>

1) Hierarchical fragmentation

Heggie (1975) MNRAS 173, 729

$\rightarrow$ formation of single stars uncorrelated in space

$\rightarrow$ sometimes bound to other stars

$$f(x,e) \sim \frac{e}{x^{5/2}} \quad \left(x = \frac{1}{\underset{\text{separation}}{a}} = \text{binding energy} \qquad e = \text{eccentricity} \right)$$

thermal eccentricity distribution $p(e)\,de \sim e\,de$

Note $f(x) \rightarrow \infty$ as $x \rightarrow 0$

"hard" & "soft" binary

$\underset{\text{velocity dispersion}}{\sigma^2} \sim x$

hard: $|x| > \sigma^2$ strongly bound compared to σ^2

soft: perturbations in cluster make them less bound and can disrupt them

hard binaries get harder
soft binaries get softer

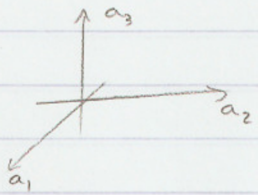
2) Fission — rapidly rotating cores can fission to form two bodies

$\rightarrow$ close binary

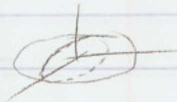
Maclaurin Spheroid — incompressible fluid with uniform rotation & self-gravitation

If $\dfrac{\overset{\text{rotation energy}}{E_R}}{\underset{\text{gravity energy}}{|E_G|}} > 0.14 \Longrightarrow$ secularly unstable to becoming a Jacobi ellipsoid

(takes longer than dynamically since you have to dissipate energy



not necessarily on evolutionary cycle

Maclaurin Spheroid: $a_1 \sim a_2 > a_3$ flattened sphere 

Jacobi Ellipsoid: $a_1 > a_2 \sim a_3$ cigar / football shape 

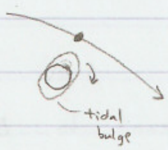
$$\frac{E_R}{|E_G|} > 0.1628 \Rightarrow \text{Jacobi is secularly unstable to becoming pear shape}$$

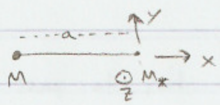
$$\frac{E_R}{|E_G|} > 0.2738 \Rightarrow \text{Both Maclaurin \& Jacobi systems are dynamically unstable}$$

$\rightarrow$ non-axisymmetric configuration

Compressibility & Differential Rotation affect these scenarios

3) Tidal Capture





$$\Phi(x,y,z) = \frac{-GM}{((a+x)^2 + y^2 + z^2)^{1/2}}$$

$$\Phi(x,y,z) \simeq \underbrace{-\frac{GM}{a}}_{\substack{\text{monopole} \\ \text{(arb. constant)}}} + \underbrace{\frac{GM}{a^2}x}_{\substack{\text{dipole} \\ \text{(point mass)}}} - \underbrace{\frac{GM}{2a^3}(2x^2 - y^2 - z^2)}_{\substack{\text{quadropole} \\ \text{(tidal force)}}}$$

$$\vec{g}_T = \frac{GM}{a^3}\left(2x\hat{\imath} - y\hat{\imath} - z\hat{k}\right) \qquad \vec{g} = -\nabla\Phi$$

$$\frac{GM_*}{(R_* + \underbrace{\epsilon}_{\text{size of tidal bulge}})^2} \sim \frac{GM_*}{R_*^2} - \frac{2GM}{a^3}(R_* + \epsilon)$$

$$\frac{GM_*}{R_*^2}\left(1-\frac{2\epsilon}{R_*}\right) \sim \frac{GM_*}{R_*^2} - \frac{2GM}{a^3}(R_* + \epsilon)$$

$$-\frac{2GM}{a^3}(R_* + \epsilon) \sim -\frac{2GM_*}{R_*^3}\epsilon$$

$$\boxed{\frac{\epsilon}{R_*} \sim \frac{M}{M_*}\left(\frac{R_*}{a}\right)^3}$$

Change in potential energy

$$\Delta E \sim \underbrace{\frac{GM_*}{R_*^2}\epsilon}_{\Delta\phi} \cdot \underbrace{\frac{\epsilon}{R_*} M_*}_{\text{mass in bulge}} \sim \frac{GM_*^2}{R_*}\left(\frac{\epsilon}{R_*}\right)^2 \sim \frac{GM^2}{R_*^2}\left(\frac{R_*}{a}\right)^6$$

$\rightarrow$ need to dissipate energy to have a capture (fluid motions $\rightarrow$ heat $\rightarrow$ radiation, for example)

Need efficient dissipation mechanism with rate: $\frac{\Delta E}{t_{orb}}$

$\Delta E_{perturb} \sim \frac{1}{2} m v^2$ set equal to ΔE and get:

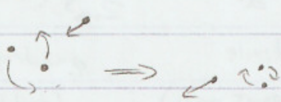
$$\frac{a}{R_*} \lesssim \left(\frac{GM}{R_* v^2}\right)^{1/6}$$

perturber has to pass within a of the star to be captured

$M \sim 1\, M_\odot$, $R \sim 1\, R_\odot$, $v \sim 10$ km/s $\Rightarrow a \sim 3.5\, R_\odot$

4) Dynamical Capture

— change properties of binary



lightest one gets thrown out

final binary will contain the two most massive of the three bodies concerned

Observations

1) ~70% of MS $_{\wedge}$ solar-type stars are in binaries

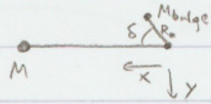
Duquennay & Mayor (1991) A&A 248, 485

2) Period distribution is uniform in log P for $0 < \log \frac{P}{\text{days}} < 9$

3) Mass distribution roughly consistent with random pairing from a normal IMF

Binary Evolution

Tidal Interactions



$\longrightarrow$ torque

$$\vec{N} = \vec{r} \times \vec{F} \qquad x F_y \Longrightarrow N \sim (R_* \cos\delta) \frac{GM}{a^3} M_{bulge} R_* \sin\delta$$

↳ from previous derivation

$$\sim \frac{GM^2}{R_*} \left(\frac{R_*}{a}\right)^6 \delta$$

Describe perturbed star as a forced, damped oscillator with natural frequency $\omega_0 \sim \left(\frac{GM_*}{R_*^3}\right)^{1/2}$

$\gamma = \frac{1}{t_F}$ inverse of damping time scale

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = F(t)$$

$$F = F_0 e^{i \Delta\omega t} \qquad \Delta\omega = \Omega - \Omega_*$$

if $x = A e^{i\Delta\omega t}$ $\Longrightarrow -(\Delta\omega)^2 A + i\Delta\omega\gamma A + \omega_0^2 A = F_0$

$$\frac{A}{F_0} = \frac{\omega_0^2 - \Delta\omega^2}{(\omega_0 - \Delta\omega)^2 + \gamma^2 \Delta\omega^2} - \frac{i\gamma\Delta\omega}{(\omega_0 - \Delta\omega)^2 + \gamma^2\Delta\omega^2} = R e^{i\phi}$$

Fast dissipation (high γ) larger phase lag (?)

$\Longrightarrow$ imaginary part $\Longrightarrow$ oscillation has a phase lag with respect to the perturber

To synchronize: $t_{sync} \sim \frac{I_* \Delta\omega}{N}$

$\left(e^{i\phi} = \cos\phi + i\sin\phi \sim 1 + i\phi \right)$

limit of small phase lag $\Longrightarrow \phi \sim \frac{\gamma\Delta\omega}{\omega_0^2} (\sim \delta)$

$$t_{sync} \sim I_* \frac{R_*}{GM^2}\left(\frac{a}{R_*}\right)^6 \frac{\Delta\omega}{\delta} \sim \frac{I_* R_*}{GM^2}\left(\frac{a}{R_*}\right)^6 \frac{\omega_0^2}{\gamma}$$

$$\sim I_* \frac{\cancel{R_*}}{\cancel{G}M^2}\left(\frac{a}{R_*}\right)^6 \frac{\cancel{G}M_*}{R_*^{\cancel{3}\,2}} t_F \sim \frac{I_*}{M_* R_*^2} \frac{M_* \cancel{R_*^2}}{M} \frac{M_*}{\cancel{R_*^2}} \left(\frac{a}{R_*}\right)^6 t_F$$

$$\boxed{t_{sync} \sim t_F \frac{I_*}{M_* R_*^2}\left(\frac{M_*}{M}\right)^2 \left(\frac{a}{R_*}\right)^6}$$

$$t_{sync} \sim 4\times10^4 \text{ years} \left(\frac{1+q}{q}\right)^2 \left(\frac{M}{M_\odot}\right) P_{days}^4 \qquad q = \frac{M_*}{M}$$

Circularizing the orbit

$$t_{circ} \sim \frac{4 t_F}{63} \left(\frac{M_*}{M}\right)\left(\frac{M_*}{M+M_*}\right)\left(\frac{a}{R_*}\right)^8$$

$$\sim 3\times10^5 \text{ yrs} \frac{(1+q)^{5/3}}{q} \left(\frac{M}{M_*}\right)^{8/3} P_{days}^{16/3}$$

M_* probably $M_\odot$

in Hubble time: $P \lesssim 15.8$ days $\rightarrow$ circularized

$\lesssim 5.7$ days should be circularized

2/27/2008

Magnetic Braking

$$\dot{M} = 4\pi r^2 \rho_w \left(\frac{2GM}{R}\right)^{1/2} \qquad B(r) = B_S\left(\frac{R}{r}\right)^2$$

v

$$\rho_w v_w^2 \sim \frac{B^2}{8\pi} \qquad \text{(Alfvén radius)}$$

Ram Pressure ~ Magnetic P.

$$\frac{\dot{M}}{4\pi r^2}\left(\frac{2GM}{R}\right)^{1/2} \sim \frac{B_s^2}{8\pi}\left(\frac{R}{r}\right)^4$$

$$\Rightarrow r_A \sim 5.5\times10^{11}\text{ cm} \left(\frac{B_s}{1G}\right)\left(\frac{R}{R_\odot}\right)^{9/4}\left(\frac{M}{M_\odot}\right)^{-1/4}\left(\frac{\dot{M}}{10^{-14}\, M_\odot/yr}\right)^{-1/2}$$

$$\dot{J} = \dot{M}_w \Omega r_A^2 \sim 1.4\times10^{31}\text{ g cm}^2\text{ s}^{-1}\; B_s^2 \left(\frac{R}{R_\odot}\right)^{9/2}\left(\frac{M}{M_\odot}\right)^{-1/2}\left(\frac{P}{day}\right)^{-1}$$

Tidal synchronization $\Rightarrow$ removes J from orbit which is then lost through wind

$\Rightarrow$ binary spirals together

$P_{orb} \leq$ few days

Gravitational Radiation

$$L_{GW} = \frac{32}{5} \frac{G^4}{c^5} \frac{M^2 \mu^2}{a^5} f(e) \qquad f(e) = \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right)\left(1 - e^2\right)^{-7/2}$$

$M = M_1 + M_2 \quad \mu \equiv$ reduced mass

$$t_{merge} = 4.7 \times 10^{10} \text{ yrs } \frac{P_{days}^{8/3}}{\left(\frac{M}{M_\odot}\right)^{2/3} \left(\frac{\mu}{M_\odot}\right)}$$ for circular orbits

---

$M_1 \quad M_2$

$$\underbrace{\vec{v}}_{\text{inertial frame velocity}} = \underbrace{\dot{\vec{r}}}_{\text{velocity in rotating frame}} + \vec{r} \times \vec{\Omega} \implies \vec{a} = \dot{\vec{v}} + \vec{\Omega} \times \vec{v} = \ddot{\vec{r}} + \underbrace{2 \vec{\Omega} \times \dot{\vec{r}}}_{\text{Coriolis}} + \underbrace{\vec{\Omega} \times (\vec{\Omega} \times \vec{r})}_{\text{Centrifugal}}$$

Extra potential

$$\phi_\Omega = -\frac{1}{2} \Omega^2 (x^2 + y^2)$$

Hydrostatic Eq:

$$\frac{1}{\rho} \nabla P = -\nabla \phi \implies$$

satellites in L_2 more stable
$\rightarrow$ based on Earth-Moon

Detached | Semi-detached | Contact Binaries

Roche Lobe:

$$\frac{R_L}{a} = \frac{0.49\, q^{2/3}}{0.6\, q^{2/3} + \ln(1 + q^{1/3})} \qquad q = \frac{M_2}{M_1}$$

fitting formula for "radius" of Roche Lobe of star 1

Total angular momentum

$$J_{tot} = \underbrace{M_1 a_1^2 \Omega + M_2 a_2^2 \Omega}_{\text{orbital}} + \underbrace{I_1 \Omega + I_2 \Omega}_{\text{spin}} \qquad \frac{a_1}{a} = \frac{M_2}{M_1 + M_2}$$

$$J_{tot} = \frac{M_1 M_2}{M_1 + M_2} a^2 \Omega + \text{spin (small)} \qquad \frac{a_2}{a} = \frac{M_1}{M_1 + M_2}$$

1) Conservative Mass Transfer

$\dot{M}_{tot} = 0 \qquad \dot{J}_{orb} = 0$

$\dot{M}_1 = -\dot{M}_2 \qquad \dot{J}_{orb} = 0 = \frac{\dot{M}_1}{M} J + \frac{\dot{M}_2}{M} J + \frac{1}{2} J \frac{\dot{a}}{a} - \frac{1}{2} \cancel{(\dot{M}_1 + \dot{M}_2)}^{\,0} \frac{J}{M_1 + M_2}$ (?)

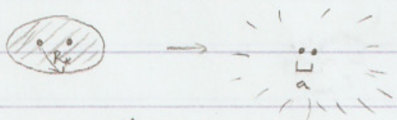
$$\boxed{\frac{\dot{a}}{a} = \frac{2 \dot{M}_1 (M_1 - M_2)}{M_1 + M_2}}$$

$M_1 > M_2 \Rightarrow \dot{M}_1 < 0 \Rightarrow \dot{a} < 0$

If more massive star is donor → orbit shrinks

If less " " " " → orbit expands

usually unstable
(as mass decreases so does Roche lobe)

instability → common envelope  friction dissipates energy and cores spiral in

$$\frac{G M_c M_{env}}{R_*} \sim \frac{G M_c M_2}{a} \Rightarrow a \sim \frac{M_2}{M_{env}} R_*$$

Types

RS Canum Venaticorum Stars / "RS CanVan" stars (BY Draconis)

eclipsing, enhanced chromospheric activity (UV/X-rays)

close, tidally synchronised stars → rapid rotation (detached)

Algols

MS primary (A,B type) + G,K giant/subgiant filling Roche Lobe (semi-detached)

(gained mass through transfer of mass

W Ursa Majoris (W Uma)

short period, eclipsing systems $P \sim 5-18$ hours

that are contact binaries 

Cataclysmic Variables

WD + MS, stable (less massive MS transfers mass to WD)

X-ray Binaries

NS + companion

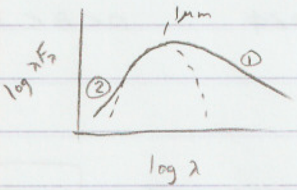
- High Mass XRB — O,B star wind  (not by Roche lobe overflow); X-ray pulsars
- Low Mass XRB — lower MS companion overflows Roche lobe

3/3/2008

Protoplanetary Disks

YSO = Young Stellar Objects

classified by SED



① IR excess – hot dust

② UV excess – ongoing accretion
(hot region of star + inner disk)

$$\alpha_{IR} = \frac{d \log \lambda F_\lambda}{d \log \lambda}$$
slope

K band – N band
(2μm) (10μm)

Class 0: SED peaks in far-IR ($\lambda \sim 100\mu m$)

Class I: flat or rising SED into mid-IR ($\alpha_{IR} > 0$)

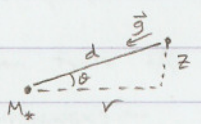
Class II: falling SED into mid-IR ($-1.5 < \alpha_{IR} < 0$) "Classical T Tauri stars"

Class III: small-to-negligible IR excess but still signs of accretion
→ "weak-line T Tauri stars"

Accretion Disk Structure

viscous accretion disk ↙ ↘ passively illuminated disk

vertical structure



$$\frac{dP}{dz} = -\rho g_z$$

$v \propto \sqrt{\frac{GM}{r}}$ $v = \Omega r$

$$g_z = \frac{GM_*}{d^2}\sin\theta = \frac{GM_*}{d^3} z$$ assuming $M_* >> M_{disk}$

orbital frequency $\Omega = \sqrt{\frac{GM}{r^3}}$

$$\simeq \frac{GM_*}{r^3} z \quad (z << r) \qquad g_z \simeq \Omega^2 z \qquad \frac{dP}{dz} = -\Omega^2 z \rho$$

Isothermal disk (at a given radius)

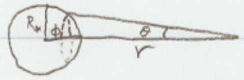
↳ $P = \rho c_s^2$

$$c^2 \frac{d\rho}{dz} = -\Omega^2 \rho z \Rightarrow \rho = \rho(0) e^{-\frac{z^2}{h^2}} \qquad \boxed{h = \sqrt{2}\frac{c_s}{\Omega}}$$ scale height

$$\frac{h}{r} \simeq \frac{c_s}{v_\phi}$$ thickness determined by c_s ...

$$c_s \sim r^{-\beta} \Rightarrow \boxed{\frac{h}{r} \propto r^{-\beta + 1/2}}$$ can lead to flared disks

Radial Temperature Profile : Passively Illuminated Disk (Flat)



$F = \int I_* \sin\theta \cos\phi \, d\Omega$ optically thick: $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$

$\quad \sin\theta \, d\theta \, d\phi$ $0 < \theta < \sin^{-1}\left(\frac{R_*}{r}\right)$

$$F = I_* \left[\sin^{-1}\left(\frac{R_*}{r}\right) - \left(\frac{R_*}{r}\right)\sqrt{1-\left(\frac{R_*}{r}\right)^2} \right]$$

$\frac{1}{\pi}\sigma T_*^4$ $F = \sigma T_{disk}^4 \Rightarrow \left(\frac{T_{disk}}{T_*}\right)^4 = \frac{1}{\pi}\left[\sin^{-1}\left(\frac{R_*}{r}\right) - \left(\frac{R_*}{r}\right)\sqrt{1-\left(\frac{R_*}{r}\right)^2} \right]$

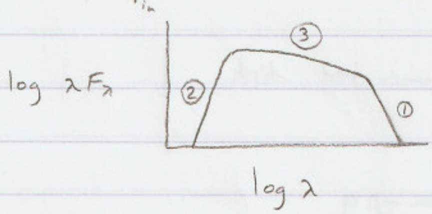
$$F_{disk} = \underset{\text{top + bottom}}{2} \int_0^\infty 2\pi r \sigma T_{disk}^4 \, dr = \boxed{\frac{1}{4} F_* = F_{disk}}$$

By Taylor expanding : $\boxed{T_{disk} \propto r^{-3/4}} \Rightarrow c_s \sim r^{-3/8} \Rightarrow \frac{h}{r} \propto r^{1/8}$

flaring disk (but weak so ~OK)

Spectral Energy Distribution

$$F_\lambda \propto \int_{r_{in}}^{r_{out}} 2\pi r \, B_\lambda(T(r)) \, dr \qquad B_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$



① At long wavelength $\lambda \gg \frac{hc}{kT(r_{out})}$

Rayleigh-Jeans Tail : $\lambda F_\lambda \sim \lambda^{-3}$

② At short wavelength $\lambda \ll \frac{hc}{kT(r_{in})}$

$\rightarrow$ exponential cutoff

③ $x = \frac{hc}{\lambda k T(r_{in})}\left(\frac{r}{r_{in}}\right)^{3/4}$

$$\Rightarrow F_\lambda \sim \lambda^{-7/3} \int \frac{x^{5/3}}{e^x - 1} dx \Rightarrow \boxed{\lambda F_\lambda \sim \lambda^{-4/3}} \sim \text{flat} \quad (\text{Class II})$$

F_nu ~ nu^(1/3)

Flared model (self-consistently) $\Rightarrow T \sim r^{-1/2}$

Dust is the dominant absorber + emitter

efficient absorber of $\lambda < 2\pi a$ light   a ~ size of dust particle

inefficient emitter of $\lambda > 2\pi a$ light

$\Rightarrow$ 2 layers   ① hot surface layer   ② cooler dust interior   } 50% of processing each

Actively Accreting Disks

$$\frac{v_\phi^2}{r} = \frac{GM_*}{r^2} + \frac{1}{\rho}\frac{dP}{dr}$$

centrifugal   gravity   pressure

$$-\frac{1}{\rho}\frac{dP}{dr} = -\frac{1}{\rho}\frac{\rho c_s^2}{r} \sim -\frac{GM_*}{r^2}\left(\frac{h}{r}\right)^2$$

→ can be described as Keplerian

Continuity eqn: $r\frac{d\Sigma}{dt} + \frac{d}{dr}\left(r\Sigma v_r\right) = 0$

Σ = surface density v_r – radial velocity

Conservation of angular momentum: $r\frac{d}{dt}\left(\Sigma r^2\Omega\right) + \frac{d}{dr}\left(r\Sigma v_r r^2\Omega\right) = \frac{1}{2\pi}\frac{dG}{dr}$

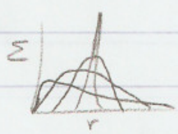
torques on gas

viscous forces ⇒ $G = 2\pi r \, \nu \, \Sigma r \frac{d\Omega}{dr} \, r$

circumference   kinematic viscosity   lever arm

viscous force / length → $r \times F$ = torque

Combine both: $$\boxed{\frac{d\Sigma}{dt} = \frac{3}{r}\frac{d}{dr}\left[r^{1/2}\frac{d}{dr}\left(\nu\Sigma r^{1/2}\right)\right]}$$ Diffusion equation

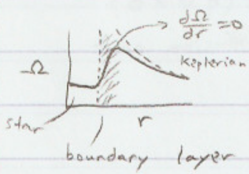


mass goes in, angular momentum goes out

Steady state solution $\left(\frac{d}{dt} = 0\right)$

$$\Sigma r^3 \Omega v_r = \nu \Sigma r^3 \frac{d\Omega}{dr} + \text{constant} \qquad \dot{M} = -2\pi r \Sigma v_r$$

$$-\frac{\dot{M}}{2\pi} r^2 \Omega = \nu \Sigma r^3 \frac{d\Omega}{dr} + \text{constant}$$



@ boundary layer   constant $\simeq \frac{\dot{M}}{2\pi} r_*^2 \sqrt{\frac{GM_*}{r_*^3}}$ zero torque boundary condition

$$\Rightarrow \nu\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{r_*}{r}}\right)$$

$$T_{disk} \sim r^{-3/4}$$

3/5/2008

Origin of Viscosity

molecular viscosity $\nu \sim \underbrace{\lambda}_{\text{mean free path}} c_s \sim \frac{c_s}{n\sigma}$

@ 1 AU $\Sigma \sim 10^3 \text{ g cm}^{-2}$ $\frac{h}{r} \sim 0.05$ thin disk $\Rightarrow n \sim 4 \times 10^{14} \text{ cm}^{-3}$ from Σ, h (?)

$c_s \sim 1.5 \text{ km s}^{-1}$ $\frac{h}{r} \simeq \frac{c_s}{v_\phi}$

$\sigma \sim 10^{-15} \text{ cm}^2$ (H atoms (?)) $\lambda \sim 2.5 \text{cm}$ $\therefore$ $\nu \sim 4 \times 10^5 \text{ cm}^2 \text{ s}^{-1}$

$t \sim \frac{R^2}{\nu} \sim 10^{13}$ yrs ... too long

Shakura-Sunyaev $\nu \sim \alpha c_s h \Rightarrow t = \left(\frac{h}{r}\right)^{-2} \frac{1}{\alpha \Omega}$ need $\alpha \sim 10^{-2}$ to match observations

turbulence/viscosity on larger scales

Reynold's number $\mathbb{R}_e = \frac{uL}{\nu}$ large $\mathbb{R}_e \rightarrow$ turbulent

but velocity shear stabilizes motions

① Self-gravity — can give effective viscosity on global scales

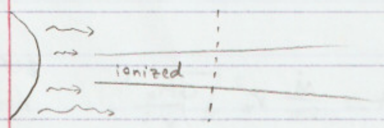
$M_{disk} \gtrsim 0.1 M_*$ $\rightarrow$ global modes ($m=1$, for example) can transfer angular momentum

spiral density waves

② Magneto Rotational Instability (MRI)

ionized disks provide $\alpha \sim 10^{-2}$ by amplifying fields in disk | $\vec{B}$ fields coupled to gas are amplified due to shear motions

MRI & Protoplanetary Disks



$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \underbrace{\nabla \times (\overset{\text{magnetic diffusivity}}{\eta} \nabla \times \vec{B})}_{\text{diffusion part}}$$

$\eta = 6.5 \times 10^{-3} x^{-1} \text{ cm}^2 \text{ s}^{-1}$ $\left(x = \frac{n_e}{n_H}\right)$

$\tau_{MRI} < \tau_{Damp}$

$\frac{h}{v_A} < \frac{h^2}{\eta}$

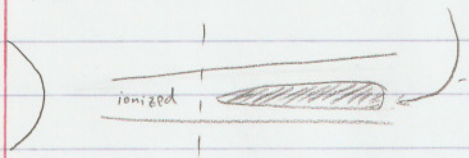
$\Rightarrow \eta < h v_A$

timescale for MRI $\tau \sim \frac{h}{v_A}$ $v_A = \sqrt{\frac{B^2}{4\pi\rho}}$ Alfvén speed

damping timescale $\tau \sim \frac{h^2}{\eta}$ $\Longrightarrow \eta < h v_A$ to cutoff MRI

$v_A \lesssim c_s$ @ 1 AU: $x > 10^{-13}$ small, but hard to get at that distance

Might have dead zone inside accretion disk



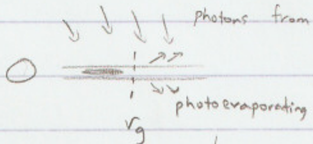
---- does nothing

but outer parts ionized / partially ionized and flow inwards (accretion)

FU Orionis Events – dump dead zone rapidly on star due to self-gravity

Disk Dispersal

proplyds in Orion 



ionized by O/B stars → $T \sim 10^4$ K

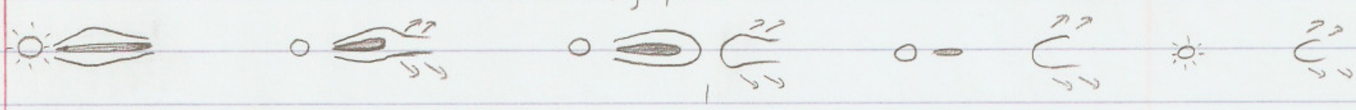
$$v_g = \frac{GM_*}{c_s^2}$$

sound speed enough to escape gravity

Photoevaporation can also happen due to x-rays/uv from star

$$\dot{M} = 4 \times 10^{-10} \, M_\odot \, yr^{-1} \left(\frac{\Phi}{10^{41} s^{-1}}\right)\left(\frac{M_*}{M_\odot}\right)^{1/2}$$

flux of ionizing photons



Condensation

Vertical temperature structure $\quad \tau = \frac{1}{2} \kappa \Sigma$

$$F(z) = -\frac{16 \sigma T^3}{3 \kappa \rho} \frac{\partial T}{\partial z} \qquad \frac{-16\sigma}{3\kappa} \int_{T_c}^{T_{disk}} T^3 \, dT = \sigma T_{disk}^4 \int_0^z \rho \, dz$$

$$\boxed{\frac{T_c^4}{T_{disk}^4} \simeq \frac{3}{4} \tau}$$

1680 K Al_2O_3 $\quad$ 180 K H_2O ice "ice line"

1400 K $MgAl_2O_4$ $\quad$ 130 K NH_3

Planet Formation

3/10/2008

Dust   Rocks   Planetesimals   Earth Mass   Planetary Cores

Dust: sub-μm $\leftrightarrow$ cm scales   coupled to gas, slow drifts

Rocks: $\sim$meter scales   weakly coupled to gas

Planetesimals: ~ 10 km   decoupled to gas, Keplerian dynamics   lots of collisions

Earth Masses: waves in gas / gravitational coupling

Planetary Cores: $\gtrsim 10\, M_\oplus$ gravitationally bound gas envelopes

Dust Dynamics

$$F_{drag} = \frac{1}{2} \underbrace{C_D}_{\text{drag coeficient}} \underbrace{\pi a^2}_{\text{geometric cross-section}} \underbrace{\rho \bar{v}^2}_{\text{ram pressure}}$$

$$Re = \frac{2av}{\underbrace{\nu}_{\text{viscosity}}}$$

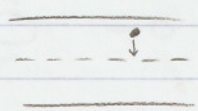
small — viscous

large — turbulent

$$C_D = \begin{cases} 24\, Re^{-1} & Re < 1 \quad \longrightarrow C_D \sim \frac{c_s}{v} \\ 24\, Re^{-0.6} & 1 \leq Re < 800 \\ 0.44 & Re > 800 \end{cases}$$

$$\bar{v} = \left(\frac{8}{\pi}\right)^{1/2} c_s$$

frictional time $t_{fric} = \dfrac{mv}{|F_{drag}|}$ usually short (well-coupled to gas)



$F_{grav} = m \underbrace{\Omega_k^2}_{\text{keplerian frequency}} z$ $F_{drag} = \frac{4}{3}\pi a^2 \bar{v} \rho v$ using $Re < 1$

$$\longrightarrow v_{settle} = \frac{\Omega_k^2}{\bar{v}} \frac{\rho_d}{\rho} a z$$

settling time scale $t_{settle} \sim 2 \times 10^5$ yrs @ 1 AU

$\rightarrow$ quite fast

Radial drifts:

small dust ($a < 1$ cm) $\rightarrow$ orbit with gas velocity but drift inwards

rocks ($a > 1$ m) $\rightarrow$ orbit with Kepler velocity $\Longrightarrow$ experience drag from slower moving gas

Radial force balance: $\dfrac{v_{\phi,gas}^2}{r} = \dfrac{GM_*}{r^2} + \dfrac{1}{\rho}\dfrac{dP}{dr}$

$P = \underset{\rho_0 c_s^2}{P_0} \left(\frac{r}{r_0}\right)^{-n}$ $v_{\phi,gas} = v_k (1 - \eta)^{1/2}$ $\eta = n \dfrac{c_s^2}{v_k^2}$

Momentum Equations

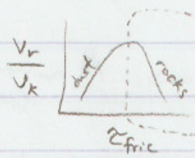
$$\frac{dv_r}{dt} = \frac{v_\phi^2}{r} - \Omega_k^2 r - \frac{1}{t_{fric}}\left(v_r - v_{r,gas}\right) \qquad \frac{d}{dt}\left(r v_\phi\right) = -\frac{r}{t_{fric}}\left(v_\phi - v_{\phi,gas}\right)$$

→ radial drifts

→ wants to bring both components into corotation

$$\frac{v_r}{v_k} = \frac{-\eta}{\tau_{fric} + \tau_{fric}^{-1}} \qquad \tau_{fric} = t_{fric}\,\Omega_k$$

→ terminal radial velocity



Goldreich – Ward Mechanism

Dust settles into midplane and can become gravitationally unstable.

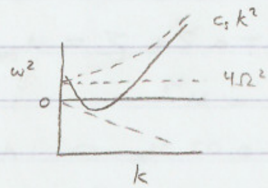
$$\omega^2 = c_s k^2 - 2\pi G \Sigma_0 |k| + 4\Omega^2$$

$\omega^2 > 0 \rightarrow$ stable | $\omega^2 < 0 \rightarrow$ unstable

→ stabilizing effect of pressure

gravitational effects → instability

→ stabilizing effect of rotation



$$k_{crit} = \frac{\pi G \Sigma_0}{c_s^2} \qquad \omega^2 = 0 \ @ \ k = k_{crit} \qquad \frac{c_s \Omega}{G\Sigma_0} = \frac{\pi}{2}$$

Toomre Q $\boxed{Q = \frac{c_s \Omega}{\pi G \Sigma}}$ $Q < 1$ gravitationally unstable → form clumps

$Q \gg 1$ stable

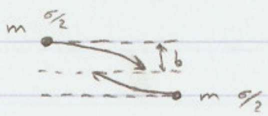
$Q \sim 1$ waves

$$\lambda_{crit} = \frac{2\pi}{k_{crit}} = \frac{2c_s^2}{G\Sigma_0} \qquad m \sim \pi \Sigma_{dust} \lambda_{crit}^2 \sim 3\times 10^{18}\, g \ (r \sim 6\, km)$$

→ skips small rocks / large dust

but we've neglected turbulence → puffs up midplane, increases Q

Planetesimals



$$\frac{1}{4} m\sigma^2 = m V_{max}^2 - \frac{Gm^2}{R_c} \qquad \text{Energy balance}$$

far away closest approach

$$V_{max} = \frac{1}{2}\frac{b\sigma}{R_c}$$

Collision $\Rightarrow R_c \leq R$ (size of body)

$$\Rightarrow b^2 = \underbrace{R^2}_{\sim\text{head on collision}} + \underbrace{\frac{4GmR}{\sigma^2}}_{\text{gravitational focusing}} \Longrightarrow \boxed{b^2 = R^2\left(1 + \frac{V_{esc}^2}{\sigma^2}\right)}$$

his notes

$$V_{esc}^2 = \frac{4GM}{R}$$

(gravitational focusing factor

Safronov number

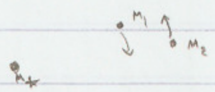
Growth of planetesimals

$$\frac{dM}{dt} = \underbrace{\rho_{pl}}_{\text{density of planetesimals}} \sigma \pi b^2 = \rho_{pl} \sigma \pi R^2 \left(1 + \frac{v_{esc}^2}{\sigma^2}\right) = \boxed{\Sigma \Omega \pi R^2 \left(1 + \frac{v_{esc}^2}{\sigma^2}\right) = \frac{dM}{dt}}$$

$\frac{dM}{dt} \propto M^{2/3} \Rightarrow R \sim t$ with little grav. focusing

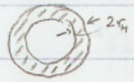
with grav focusing $\frac{dM}{dt} \propto MR \Rightarrow M \sim \frac{1}{(M_0^{-1/3} - kt)^3}$ $\Rightarrow$ runaway accretion

Isolation Mass



Hill radius $r_H \sim \left(\frac{M_p}{3M_*}\right)^{1/3} a$

$r < r_H \rightarrow$ planetesimal gravity dominates



$2\pi a \, 2r_H \, \Sigma = \underbrace{M_{zone}}_{\text{mass of feeding zone}}$

$\propto M^{1/3}$

maximum mass (?) for accreting object

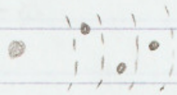
$M_{iso} \sim 0.07 \, M_\oplus$ @ 1 AU

$\sim 9 \, M_\oplus$ for Jupiter distance

3/12/2008

Planetesimals - Oligarchic Growth

big ones get bigger



Solar System disk gets dominated by a few large bodies

Get larger with collisions

Gas Giants:

1) Core Accretion or 2) Gravitational Collapse

$\rightarrow$ requires big gas disks + fast cooling

$\downarrow$

$M_{tot} = M_{core} + M_{env}$

$$L = \frac{G M_{core} \dot{M}_{core}}{R_c}$$

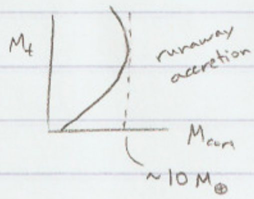
$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho \qquad \frac{L}{4\pi r^2} = \frac{-16}{3} \frac{\sigma T^3}{\kappa \rho} \frac{dT}{dr}$$

$$P \sim \frac{\rho}{\mu m_H} kT$$

$$T \sim \frac{\mu}{k} \frac{G M_t \, m_p}{r} \qquad \rho \sim \frac{64\pi G}{3\kappa L} \left(\frac{\mu m_p}{k} G M_t\right)^4 \frac{1}{r^3}$$

$$M_{env} = \int_{R_{core}}^{R_{out}} 4\pi r^2 \rho \, dr \sim \frac{M_t^4}{L} \qquad L \sim M_{core}^{2/3} \dot{M}_{core}$$

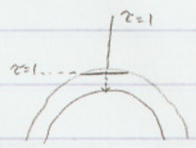
$$M_t = M_{core} + (\ldots) \frac{M_t^4}{M_{core}^{2/3} \dot{M}_{core}}$$



larger $\rightarrow$ cant maintain hydrostatic envelope
$\rightarrow$ runaway accretion
$\rightarrow$ sucks up all surrounding gas and stops growing

Exoplanets

Transit

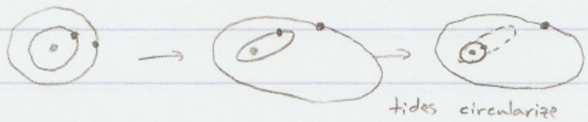


radius is slightly larger than normally defined radius
- not enough to account for large radii in some planets

etc. weather on exoplanets ?!

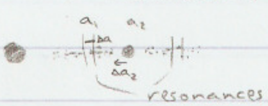
Planetary Migration

1) Planet - Planet Scattering



doesnt account for all properties

2) Planet - Planetesimal Scattering



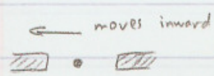
kick out planetesimals, planet drifts inwards

$$(2\pi a_1 \Delta a \Sigma) \frac{GM_*}{2a_1} \sim \frac{GM_* M_p}{2a_2^2} \Delta a_2 \implies \Sigma \sim \frac{M_p}{2\pi a_2^2} \text{ needed}$$

requires too much mass in disk

3) Gas Disk Migration

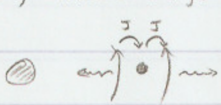
Type I — large rocks, small planets migrate (too fast to form planets)



(prefered) Type II — Jupiter-sized, make large gap not waves, migration is by viscosity

$$(\omega - m\Omega_p)^2 = \kappa^2$$

Ω_p: planet orbital freq



planet launches waves which transfer angular momentum