Phys 210A,B Giovanni Zocchi
Notes: David Rodriguez

9/28/2006

E & M

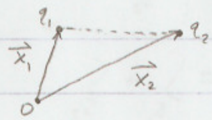
we use Gaussian units (cgs) (Old book edition)

4-5 problems / week from book (Jackson)

Recommended: Look at Feynman Lectures

Electrostatics

Coulomb's law



$$\vec{F} = k \frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|^2} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}$$ this is force on q_1 by q_2

Gaussian: $k = 1$ MKS: $\frac{1}{4\pi\epsilon_0}$

$$\vec{E}(\vec{x}) = \frac{\vec{F}}{q_1}$$

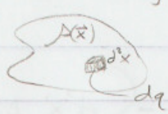
Field from charge at $\vec{x}_1$: $$\vec{E}(\vec{x}) = \frac{q_1}{|\vec{x} - \vec{x}_1|^3}(\vec{x} - \vec{x}_1)$$

Superposition: many charges $q_i, \vec{x}_i$

$$\vec{E}(\vec{x}) = \sum_i \frac{q_i}{|\vec{x} - \vec{x}_i|^3}(\vec{x} - \vec{x}_i)$$ $\therefore$ Linear theory

Continuous distribution of charges

$$dq = \rho(\vec{x})\, d^3x$$



$$\vec{E}(\vec{x}) = \int d^3\vec{x}'\, \rho(\vec{x}') \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}$$

$$\rho(\vec{x}) = \sum_i q_i\, \delta(\vec{x} - \vec{x}_i)$$

$\delta(x-a) = 0$ if $x \neq a$ $\quad \int_{-\infty}^{+\infty} \delta(x-a)\, dx = 1$

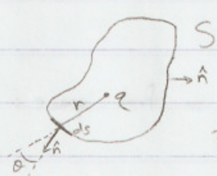
for any $f(x)$ $$\int_{-\infty}^{+\infty} f(x)\, \delta(x-a)\, dx = f(a)$$

Ex: $\int dx\, f(x)\, \delta'(x) = -f'(0)$ Integrate by parts

$$\vec{E}(\vec{x}) = \int d^3x' \sum_i q_i\, \delta(\vec{x}' - \vec{x}_i) \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} = \sum_i q_i \frac{(\vec{x} - \vec{x}_i)}{|\vec{x} - \vec{x}_i|^3}$$

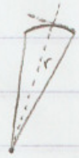
$\delta(\vec{x}) = \delta(x)\,\delta(y)\,\delta(z)$

Gauss law



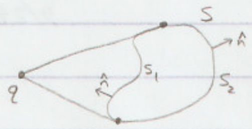
$$\int_S \vec{E} \cdot \hat{n}\, ds = ?$$

$$\vec{E} \cdot \hat{n}\, ds = \frac{q}{r^2} \cos\theta\, ds \qquad \cos\theta\, ds = r^2\, d\Omega$$ solid angle



$$\text{solid angle} = \frac{\text{projection of surface} \perp \text{to direction}}{r^2} \qquad \Omega = \frac{A_\perp}{r^2} \qquad d\Omega = \frac{ds\cos\theta}{r^2}$$

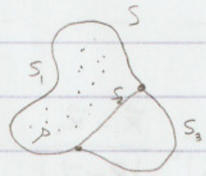
$$\therefore \int_S \vec{E} \cdot \hat{n}\, ds = q \int d\Omega = 4\pi q$$



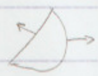
$$\int_{S_1+S_2} \vec{E}\cdot\hat{n}\,ds = \int_{S_1} d\Omega - \int_{S_2} d\Omega = 0$$

since $\hat{n}$ is in different direction

Charges outside Gaussian surface dont contribute.



$$\int_{S_2+S_3} = 0 = \int_{S_3} - \int_{S_2} \Longrightarrow \int_{S_3} = \int_{S_2}$$



$$\int_{S_1+S_3} = \int_{S_1} + \int_{S_3} = \int_{S_1} + \int_{S_2}$$

$$\int_S \vec{E}\cdot\hat{n}\,ds = 4\pi \int_V \rho(\vec{x})\,d^3x$$

In general: $\displaystyle\int_S \vec{A}(\vec{x})\cdot\hat{n}\,ds = \int_V (\vec{\nabla}\cdot\vec{A})\,d^3x$ $\quad$ S encloses V

$$\int_V d^3x\left(\vec{\nabla}\cdot\vec{E} - 4\pi\rho(\vec{x})\right) = 0 \quad \forall\, V \Longrightarrow \boxed{\vec{\nabla}\cdot\vec{E} = 4\pi\rho(\vec{x})}$$

for all

Electrostatic boundary

$$\vec{E}(\vec{x}) = \int d^3\vec{x}\,\rho(\vec{x}')\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = -\int d^3x'\,\rho(\vec{x}')\,\vec{\nabla}_x\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

grad with respect to x

$$= \vec{\nabla}_x(\text{something}) \Longrightarrow \boxed{\vec{\nabla}\times\vec{E} = 0} \qquad \forall\, f(x) \quad \nabla\times(\nabla f) = 0$$

for electrostatics

Given $\vec{E}$ (static) $\exists$ $\phi(\vec{x})$ such that $\boxed{\vec{E} = -\vec{\nabla}\phi}$

there exists

$$\boxed{\phi(\vec{x}) = \int d^3x'\,\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}} \qquad \text{single charge: } \phi(\vec{x}) = \frac{q}{|\vec{x}-\vec{x}_1|}$$

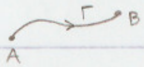
this choice makes $\phi \to 0$ as $r \to \infty$

$\nabla^2\phi = -4\pi\rho(\vec{x})$ Poisson eq. (solution in free space: no boundary conditions)

If $\rho = 0$, $\nabla^2\phi = 0$ Laplace eq.

Potential Energy of charge in a field

$$W = -\int_A^B \vec{F}\cdot d\vec{l}$$



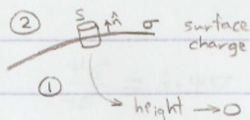
Work you do to move charge from A to B

$$W = -q\int_\Gamma \vec{E}\cdot d\vec{l} = q\int_\Gamma \vec{\nabla}\phi\cdot d\vec{l} = q\left[\phi(B) - \phi(A)\right]$$

∴ potential energy is $q\phi$

$$E_{pot} = q\phi$$

Boundary conditions:



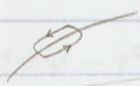
2 different media

$$\int \vec{E}\cdot\hat{n}\,ds = (\vec{E}_2\cdot\hat{n})S - (\vec{E}_1\cdot\hat{n})S = 4\pi\sigma S$$

$$\boxed{(\vec{E}_2 - \vec{E}_1)\cdot\hat{n} = 4\pi\sigma}$$

Discontinuity in normal component of $\vec{E}$ if $\sigma \neq 0$

tangential component ?



HW 1 (New Edition) 1.1, 1.3, 1.5, 1.10, 1.11

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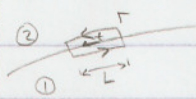


$$\oint_\Gamma \vec{E}\cdot d\vec{l} = \vec{E}_2\cdot\hat{t}L - \vec{E}_1\cdot\hat{t}L = 0 \qquad \oint \vec{E}\cdot d\vec{l} = \int (\nabla\times\vec{E})\cdot\hat{n}\,dS = 0$$

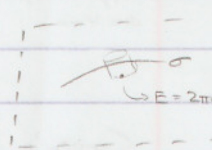
$$\boxed{(\vec{E}_2 - \vec{E}_1)\cdot\hat{t} = 0}$$

$$\boxed{\nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} = -4\pi\,\delta(\vec{x} - \vec{x}')}$$

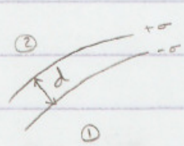
$\frac{1}{|\vec{x}-\vec{x}'|}$ is $\phi(\vec{x})$ due to unit charge in $\vec{x}'$

$$\boxed{\nabla^2\phi = -4\pi\rho}$$

$$\phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$



$$\nabla_x^2\,\phi(\vec{x}) = \int d^3x'\,\rho(\vec{x}')\,\nabla^2\left(\frac{1}{|\vec{x} - \vec{x}'|}\right) = -4\pi\rho(\vec{x})$$



$\sigma d = D$ dipole per unit surface

$$E_{in} = 4\pi\sigma \qquad \Delta\phi = 4\pi\sigma d = 4\pi D$$

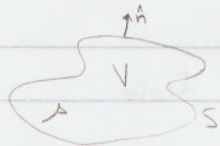
Summary:

$$\nabla\cdot\vec{E} = 4\pi\rho \qquad \nabla\times\vec{E} = 0 \qquad \vec{E} = -\nabla\phi \qquad \nabla^2\phi = -4\pi\rho$$

free space solution (no B.C.s)

$$\phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

General situation



charge distribution and boundaries

Solve $\nabla^2\phi = -4\pi\rho$ in V with B.C. on S

Green's theorem

$\vec{A}(\vec{x})$ vector field $\qquad \int_V d^3x\, \nabla\cdot\vec{A} = \int_S (\vec{A}\cdot\hat{n})\, ds$

$\vec{A} = \phi\nabla\psi \qquad \phi, \psi$ scalar functions

$$\int_V d^3x \left(\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi\right) = \int_S (\phi\nabla\psi)\cdot\hat{n}\, ds$$

notation $\nabla\psi\cdot\hat{n} = \frac{\partial\psi}{\partial n}$

$$= \int_S \phi\frac{\partial\psi}{\partial n} ds$$

$\phi \leftrightarrow \psi$ then subtract $\longrightarrow \int_V d^3x\,(\phi\nabla^2\psi - \psi\nabla^2\phi) = \int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) ds$

Boundary conditions which specify ϕ inside

1) ϕ given on S (Dirichlet b.c.)
2) $\frac{\partial\phi}{\partial n}$ given on S (Neumann b.c.)

Suppose you have ϕ_1, ϕ_2 that satisfy $\nabla^2\phi = -4\pi\rho$ and $\phi_1 = \phi_2$ on S

$u = \phi_1 - \phi_2 \qquad \nabla^2 u = 0 \qquad u = 0$ on S

Apply $\mathcal{X}$ ($\phi = \psi = u$) $\qquad \int_V d^3x\, |\nabla u|^2 = 0 \qquad \therefore\ \nabla u = 0$ inside V $\quad \therefore\ u =$ constant

$\Rightarrow \phi_1 = \phi_2 +$ constant

Green's function for $\nabla^2\phi = -4\pi\rho$

$\nabla_x^2 G(\vec{x}, \vec{x}') = -4\pi\,\delta(\vec{x} - \vec{x}')$ (unit point charge @ $\vec{x}'$)

Particular solution $\frac{1}{|\vec{x} - \vec{x}'|}$

General solution $\qquad G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}') \qquad$ where $\nabla^2 F = 0$

Green theorem with $\phi = \Phi$ $\psi = G$

$$\int_V d^3x' \left(\phi \nabla^2 G - G \nabla^2 \phi\right) = \int_V d^3x' \left((-4\pi)\phi(x')\,\delta(\vec{x}-\vec{x}') + G(\vec{x},\vec{x}')\, 4\pi \rho(x')\right)$$

$$= \int_S \left(\phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'}\right) ds$$

$$\boxed{\phi(\vec{x}) = \int_V d^3x'\, G(\vec{x},\vec{x}')\,\rho(\vec{x}') - \frac{1}{4\pi}\int_S \left(\phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'}\right) ds}$$

If ϕ given on S : choose $G=0$ on S

$$\phi(\vec{x}) = \underbrace{\int_V d^3x'\, G(\vec{x},\vec{x}')\,\rho(\vec{x}')}_{\text{volume term}} - \underbrace{\frac{1}{4\pi}\int_S \phi \frac{\partial G}{\partial n'} ds}_{\text{surface term}}$$

$\phi\big|_S = f(\vec{x})$

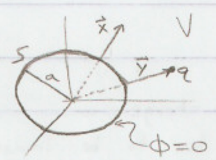
This is what we put

G solution to point charge + simple (uniform) b.c. on S

namely $G = 0$ on S (Dirichlet)

Ex: Find ϕ for $q=1$ @ x', b.c. $\phi = 0$ on $x=0$ ... method of images gives you answer $\longrightarrow$ Green's function

Green funct. for sphere (outside)



Charge q @ $\vec{y}$

Image charge q' @ $\vec{y}'$

$$\phi(\vec{x}) = \frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|}$$

$\longleftarrow$ Green function with (and $q=1$)

$\vec{x} = x\,\hat{m}_1$ $\quad \vec{y} = y\,\hat{m}_2$ $\quad \vec{y}' = y'\,\hat{m}_2$

$$\phi(\vec{x}) = \frac{q}{x\left|\hat{m}_1 - \frac{y}{x}\hat{m}_2\right|} + \frac{q'}{y'\left|\hat{m}_2 - \frac{x}{y'}\hat{m}_1\right|}$$

$$\phi\big|_{x=a} = \frac{q}{a\left|\hat{m}_1 - \frac{y}{a}\hat{m}_2\right|} + \frac{q'}{y'\left|\hat{m}_2 - \frac{a}{y'}\hat{m}_1\right|} = 0$$

If $\frac{y}{a} = \frac{a}{y'}$ then $|\sim|$ are same

If $\frac{q}{a} = -\frac{q'}{y'}$ and $\frac{y}{a} = \frac{a}{y'}$, $\phi\big|_{x=a} = 0$

$\therefore$ $\underline{y' = \frac{a^2}{y}}$ $\quad q' = -\frac{y'}{a}q = -\frac{a^2}{ya}q$ $\quad \therefore$ $\underline{q' = -\frac{a}{y}q}$

$$G = \frac{1}{|\vec{x}-\vec{y}|} - \frac{a}{y}\frac{1}{\left|\vec{x} - \frac{a^2}{y^2}\vec{y}\right|}$$

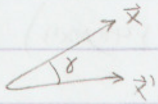
$\nabla^2 G = -4\pi\,\delta(\vec{x}-\vec{y})$

$$G = \frac{1}{x\left|\hat{m} - \frac{y}{x}\hat{m}_2\right|} - \frac{a}{y}\frac{y}{a^2}\frac{1}{\left|\hat{m}_2 - \frac{x}{a^2}y\,\hat{m}_1\right|}$$

$\frac{1}{a}$

10/5/2006

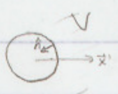
$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a/x'}{\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}$$



$\vec{x} \to x, \theta, \phi \qquad \vec{x}' \to x', \theta', \phi' \qquad \gamma = f(\theta, \phi, \theta', \phi')$

$$|\vec{x}-\vec{x}'|^2 = x^2 + x'^2 - 2xx'\cos\gamma \qquad \frac{-1}{\left|\frac{x'}{a}\vec{x} - \frac{a}{x'^2}\vec{x}'\right|}$$

$$G(\vec{x},\vec{x}') = \left(x^2+x'^2-2xx'\cos\gamma\right)^{-1/2} - \left(\left(\frac{x'x}{a}\right)^2 + a^2 - 2xx'\cos\gamma\right)^{-1/2}$$



$$\frac{\partial G}{\partial n'} = -\left.\frac{\partial G}{\partial x'}\right|_{x'=a} = -\frac{1}{a}\frac{x^2-a^2}{(x^2+a^2-2ax\cos\gamma)^{3/2}}$$

Solution:

$$\Phi(x,\theta,\phi) = -\int_S a^2\, d\Omega'\, \Phi(a,\theta',\phi')\left(-\frac{1}{a}\right)\frac{x^2-a^2}{[\ ]^{3/2}} + \int_V d^3x'\, \rho\, G$$

$$d\Omega' = d\phi'\, d(\cos\theta')$$

Problem: conducting, grounded sphere + point charge

what is surface charge on sphere?

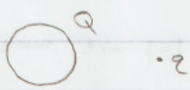
$$\boxed{-\left.\frac{\partial\Phi}{\partial n}\right|_S = 4\pi\sigma}$$

$$-\left.\frac{\partial G}{\partial x}\right|_{x=a} = -\frac{1}{a}\frac{x'^2-a^2}{(x'^2+a^2-2ax\cos\gamma)^{3/2}}$$

total charge on sphere has to be image charge

$$\sigma(\theta) = -\frac{q}{4\pi a}\frac{z^2-a^2}{(z^2+a^2-2az\cos\theta)^{3/2}}$$

Insulated conducting sphere total charge Q + point charge q $\quad \Phi = ?$



use grounded sphere q' and add extra charge $Q-q'$ @ center of sphere

grounded sphere q' + $\cdot\, Q-q'$ = sphere Q

Comp. Exam Problem

Spherically symmetric $\phi(r) = \frac{f(r)}{r}$ $\quad$ $f(r) \to A \quad r \to 0$ $\quad$ f is non singular (no singularities)

$f(r) \to B \quad r \to \infty$

a) total charge ?

$r \to \infty \quad \phi \sim \frac{B}{r} \implies Q_{tot} = B$

b) identify point charges: f is non singular ϕ has no singularities except at $r=0$

$r \to 0 \quad \phi \sim \frac{A}{r} \quad \therefore$ point charge $q = A$ @ $r = 0$

c) find $\rho(\vec{x})$

$$\nabla^2 \phi = -4\pi\rho$$

$$\nabla\phi = \frac{1}{r}\nabla f + f\nabla\left(\frac{1}{r}\right) \qquad \nabla^2\phi = \nabla\left(\frac{1}{r}\right)\cdot\nabla f + \frac{1}{r}\nabla^2 f + \nabla f\cdot\nabla\left(\frac{1}{r}\right) + f\,\nabla^2\left(\frac{1}{r}\right)$$

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\vec{r}) \qquad \nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}(r^{-1})\hat{r} = -\frac{\hat{r}}{r^2} \qquad \nabla f(r) = f'(r)\,\hat{r}$$

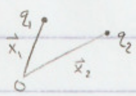
$$\nabla^2 f(r) = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial f}{\partial r} = \frac{1}{r^2}\left(2r\frac{\partial f}{\partial r} + r^2\frac{\partial^2 f}{\partial r}\right) = \frac{2}{r}f' + f''$$

$$2\nabla\left(\frac{1}{r}\right)\cdot\nabla f = 2\left(-\frac{\hat{r}}{r^2}\right)\cdot\left(f'\hat{r}\right) = -\frac{2}{r^2}f'$$

$$\nabla^2\phi = \cancel{-\frac{2}{r^2}f'} + \cancel{\frac{2}{r^2}f'} + \frac{1}{r}f'' - 4\pi f\,\delta(\vec{r}) = \frac{1}{r}f''(r) - 4\pi f(r)\,\delta(\vec{r})$$

$$= \frac{1}{r}f''(r) - 4\pi A\,\delta(\vec{r}) = -4\pi\rho$$

Electrostatic energy



$$W = \frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|} = \phi(\vec{x}_2)\,q_2 = \phi(\vec{x}_1)\,q_1$$

charges q_i, x_i $\qquad W = \frac{1}{2}\sum_{i\neq j}^{N}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$

continuous ρ

$$W = \frac{1}{2}\int d^3x \int d^3x' \frac{\rho(\vec{x})\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{1}{2}\int d^3x\,\rho(\vec{x})\,\phi(\vec{x})$$

in terms of E:

$$W = \frac{1}{2}\int d^3x\left(-\frac{1}{4\pi}\right)\phi\,\nabla^2\phi = \frac{1}{8\pi}\int d^3x\,\nabla\phi\cdot\nabla\phi = \int d^3x\,\frac{|\vec{E}|^2}{8\pi}$$

energy density $\boxed{u = \frac{E^2}{8\pi}}$ $\quad$ $\boxed{u = \frac{1}{8\pi}\left(|\vec{E}|^2 + |\vec{B}|^2\right)}$

force on sphere?

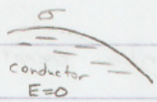
image charge q' @ y' $\leftarrow$ force between charges

$q' = -\frac{a}{y}q \qquad y' = \frac{a^2}{y}$

$$F = \frac{qq'}{|y-y'|^2} = \frac{q^2 a}{y^3}\frac{1}{\left(1-\frac{a^2}{y^2}\right)^2} \propto \frac{1}{y^3}$$

HW 2 : 2.1, 2.7, 2.23, 2.26   10/10/2006

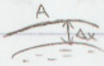
Read Feynman Lectures on Gauss theory – Electrostatic fields



Pressure on conductor?

$E = 4\pi\sigma$

Displace surface by Δx



$\Delta W = \frac{1}{8\pi} E^2 A \Delta x$ change in energy

small wrt radius of curvature

$$\Delta W = \frac{(4\pi)^2}{8\pi}\sigma^2 A \Delta x \qquad F = \frac{\Delta W}{\Delta x} \qquad P = \frac{F}{A} = \frac{\Delta W}{\Delta x}\frac{1}{A} = 2\pi\sigma^2$$

Orthogonal functions

$\overset{}{a} \quad \overset{}{b} \rightarrow x \qquad \{u_n(x)\}_n$ Can we represent any $f(x)$ in interval $[a,b]$ in terms of $\{u_n\}$?

Define a scalar product

$$\langle f, g \rangle = \int_a^b dx\, f(x)\, g^*(x)$$

set of orthonormal functions

$$\int_a^b dx\, u_n(x)\, u_m^*(x) = \delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$\{u_n\}$ is complete if $\forall f(x)$ in $[a,b]$: $f(x) = \sum_n a_n u_n(x)$

$$a_n = \int_a^b dx\, f(x)\, u_n^*(x) \qquad \lim_{N\to\infty} \int_a^b \left| f(x) - \sum_{n=1}^N a_n u_n(x) \right|^2 dx = 0$$

$$f(x) = \sum_n \left(\int_a^b dx'\, f(x')\, u_n^*(x') \right) u_n(x) \implies \sum_n u_n(x)\, u_n^*(x') = \delta(x - x') \quad \text{completeness relation}$$

Example:

$$\left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{2n\pi x}{a}\right), \sqrt{\frac{2}{a}} \cos\left(\frac{2n\pi x}{a}\right) \right\}_{n=1,2,\ldots}$$

complete in $\left[-\frac{a}{2}, \frac{a}{2}\right]$

$$\frac{2}{a}\int_{-a/2}^{a/2} dx\, \sin\left(\frac{2n\pi x}{a}\right) \sin\left(\frac{2m\pi x}{a}\right) = \delta_{nm}$$

$$f(x) = \sum_n A_n \sin\left(\frac{2n\pi x}{a}\right) + B_n \cos\left(\frac{2n\pi x}{a}\right) \quad \text{Fourier series}$$

$$A_n = \frac{2}{a}\int_{-a/2}^{a/2} f(x) \sin\left(\frac{2n\pi x}{a}\right) dx$$

$a \to \infty$ $\quad \frac{2\pi n}{a} \to k$ (continuous variable) $\quad \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \right\}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}$$ Fourier integral $$A(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx} \frac{1}{\sqrt{2\pi}}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-ik'x}\, dx = \delta(k-k')$$ orthonormal

$$\int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = 2\pi\, \delta(x-x')$$ completeness relation

Integral representations of δ: $\int dx\, e^{i(k-k')x} = 2\pi\, \delta(k-k')$

Separation of Variables

$\nabla^2 \phi = 0$ in rectangular coord.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \phi(x,y,z) = X(x)\, Y(y)\, Z(z)$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$ each $\frac{1}{n}\frac{d^2 n}{dz^2}$ = constant

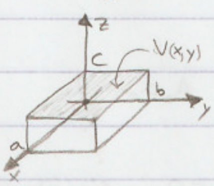
$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2 \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2$$

$$-\alpha^2 - \beta^2 + \gamma^2 = 0$$

Suppose α, β are real

$$X(x) \sim e^{\pm i\alpha x} \qquad Y(y) \sim e^{\pm i\beta y} \qquad \gamma^2 = \alpha^2 + \beta^2 \qquad Z(z) \sim e^{\pm \gamma z}$$

$$\phi \sim \sum_{\alpha,\beta} A_{\alpha\beta}\, e^{\pm i\alpha x}\, e^{\pm \beta y}\, e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$$



Solve $\nabla^2\phi = 0$ inside box

b.c. $\phi = 0$ on all faces except $z = c$ where $\phi = V(x,y)$

$$X \sim \sin\left(\frac{\pi n x}{a}\right) \qquad Y \sim \sin\left(\frac{\pi m y}{b}\right) \qquad \gamma_{n,m} = \sqrt{\left(\frac{2\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2} = \pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

$$Z \sim e^{\gamma z} - e^{-\gamma z} \Rightarrow Z(0) = 0 \qquad Z = \tfrac{1}{2}\left(e^{\gamma z} - e^{-\gamma z}\right) = \sinh(\gamma z)$$

$$\phi = \sum_{n,m} A_{n,m} \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n y}{b}\right) \sinh(\gamma_{n,m} z) \sqrt{\frac{2}{a}}\sqrt{\frac{2}{b}}$$ Err: $2\pi n \Rightarrow \pi n$

$n, m = 1, 2, 3, \ldots$

$n, m = 0 \Rightarrow \phi = 0$ true if $V(x,y) = 0$

Last b.c.

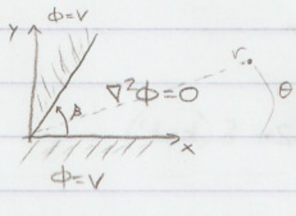
$$\Phi(x,y,c) = V(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n y}{b}\right) \sinh(\gamma_{nm} c) \sqrt{\frac{2}{a}}\sqrt{\frac{2}{b}}$$

$$A_{nm} \sinh(\gamma_{nm} c) = \int_{y=0}^{b}\int_{x=0}^{a} dx\,dy\; V(x,y) \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi m y}{b}\right) \sqrt{\frac{2}{a}}\sqrt{\frac{2}{b}}$$

normalization

10/12/2006

Separation of Variables



Cylindrical coords.
interested in Φ for $r \to 0$
Φ finite for $r \to 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 \qquad \text{b.c.}\quad \Phi(\theta=0) = \Phi(\theta=\beta) = V \quad \forall\, r$$

$$\Phi = R(r)\Psi(\theta)$$

$$r\frac{1}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Psi}\frac{d^2\Psi}{d\theta^2} = 0 \qquad \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \nu^2$$

$$\frac{1}{\Psi}\frac{d^2\Psi}{d\theta^2} = -\nu^2$$

$$\Psi(\theta) = A_\nu \sin(\nu\theta) + B_\nu \cos(\nu\theta) \qquad \text{try } R \sim r^\alpha$$

$$\alpha^2 r^\alpha = \nu^2 r^\alpha \longleftrightarrow \alpha^2 = \nu^2 \qquad \alpha = \pm\nu \qquad \therefore\ R(r) = a_\nu r^\nu + b_\nu r^{-\nu}$$

If $\nu = 0$

$$\Psi'' = 0 \implies \Psi(\theta) = A_0 + B_0\theta$$

$$r\frac{dR}{dr} = \text{const} \qquad R(r) = a_0 + b_0 \ln r$$

Φ finite as $r \to 0 \implies b_0 = 0$

$$\theta = 0 \qquad \Phi = A_0 a_0 + \sum_\nu \left(a_\nu r^\nu + b_\nu r^{-\nu}\right) B_\nu = V \qquad B_\nu = 0 \qquad A_0 a_0 = V$$

$$\theta = \beta \qquad \Phi = (A_0 + B_0\beta)a_0 + \sum_\nu \left(a_\nu r^\nu + b_\nu r^{-\nu}\right) A_\nu \sin(\nu\beta) = V \qquad B_0 = 0 \qquad \sin(\nu\beta) = 0$$

$$\nu\beta = n\pi \quad n = 1,2,3,\ldots \qquad \nu = \frac{n\pi}{\beta} \qquad b_\nu = 0 \quad (\Phi \text{ finite as } r \to 0)$$

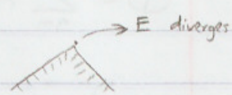
$$\Phi(r,\theta) = V + \sum_n A_n r^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi}{\beta}\theta\right)$$

for $r \to 0$

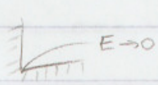
$$\Phi \sim V + A_1 r^{\frac{\pi}{\beta}} \sin\left(\frac{\pi}{\beta}\theta\right)$$

$$E_r = -\frac{\partial \Phi}{\partial r} \qquad E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} \implies E \sim r^{\frac{\pi}{\beta} - 1}$$

if $\beta > \pi$



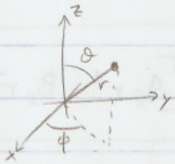
$\beta < \pi$



$\beta = 2\pi$

$\to E \sim r^{-1/2}$

$\nabla^2 \Phi = 0$ in spherical coords r, ϕ, θ



$$\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2}$$

$$\Phi = \frac{U(r)}{r} P(\theta)\, Q(\phi) \qquad \frac{r^2}{\Phi} \times \Phi$$

$$r^2 \frac{1}{U}\frac{d^2 U}{dr^2} + \frac{1}{\sin\theta}\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{1}{Q}\frac{d^2 Q}{d\phi^2} = 0 \qquad \times \sin^2\theta$$

$$\frac{1}{Q}\frac{d^2 Q}{d\phi^2} = -m^2 \implies Q(\phi) = e^{\pm i m \phi} \qquad m \text{ integer if } 0 \le \phi \le 2\pi$$

$$\frac{r^2}{U}\frac{d^2 U}{dr^2} + \frac{1}{\sin\theta}\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0$$

$$\frac{r^2}{U}\frac{d^2 U}{dr^2} = \ell(\ell+1) \qquad \frac{d^2 U}{dr^2} - \frac{\ell(\ell+1)}{r^2} U = 0$$

$$\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right)\frac{1}{\sin\theta} + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right) P = 0$$

$$U \sim r^\alpha \implies \alpha(\alpha-1) r^{\alpha-2} - \ell(\ell+1) r^{\alpha-2} = 0 \qquad \alpha(\alpha-1) = \ell(\ell+1)$$

$$\implies \alpha = \ell+1 \quad \alpha = -\ell$$

$$\therefore U = A r^{\ell+1} + B r^{-\ell} \qquad \frac{U}{r} = A r^{\ell} + B r^{-(\ell+1)}$$

change variable $x = \cos\theta$ $\quad \frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta \frac{d}{dx} \qquad \frac{1}{\sin\theta}\frac{d}{d\theta} = -\frac{d}{dx}$

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right) P = 0$$

case $m=0$ azimuthal symmetry

$$P(x) = \sum_{n=1}^{\infty} a_n x^n \qquad \frac{d}{dx}\left(n a_n x^{n-1} - n a_n x^{n+1}\right) + \ell(\ell+1) a_n x^n = 0$$

$$\underbrace{n(n-1) a_n x^{n-2}}_{n \to n+2} - n(n+1) a_n x^n + \ell(\ell+1) a_n x^n = 0$$

$$(n+2)(n+1) a_{n+2} - \left(n(n+1) - \ell(\ell+1)\right) a_n = 0$$

Recursion relation $\quad \boxed{a_{n+2} = \frac{n(n+1) - \ell(\ell+1)}{(n+2)(n+1)} a_n}$

want to converge @ $x=1$ $\implies$ ℓ must be integer $\implies$ finite sum

$P_\ell(x) \to$ polynomials of order ℓ

$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2 - 1)$ $\qquad$ Legendre Polynomials

Choose arbitrary const. so that $P_\ell(1) = 1$ $\qquad x \in [-1, 1]$

Other solution is for $\theta = 0, \pi$ $\quad \underline{\sin\theta = 0}$ $\quad$ not well behaved @ $\theta = 0, \pi$

$$\boxed{m=0: \quad \Phi(r,\theta)=\sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)}$$

$\{P_l\}$ is a complete set in $[-1,1]$ $\therefore$ $f(x)=\sum_{l=0}^{\infty} a_l P_l(x)$ $\quad x\in[-1,1]$

orthogonal

$\int_{-1}^{1} P_l P_{l'}\, dx = 0$ if $l \neq l'$ $\qquad a_l = ? \int_{-1}^{1} f(x) P_l(x)\, dx$

normalization Normalization is (2l+1)/2

Orthogonality $\quad \frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + l(l+1)P_l = 0$

$\times P_{l'} \Longrightarrow \quad \int dx\, P_{l'} \frac{d}{dx}[\;] + \int dx\, l(l+1) P_l P_{l'} = 0$

Int by parts

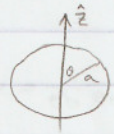
$0 - \int dx \frac{dP_{l'}}{dx}[\;] + \int \odot = 0$

Copy and $l \leftrightarrow l'$ then subtract $\rightarrow \left(l(l+1) - l'(l'+1)\right)\int_{-1}^{1} dx\, P_l P_{l'} = 0$

if $l \neq l'$, then $\int = 0$

Orthonormal, complete set $\left\{\sqrt{\frac{2l+1}{2}}\, P_l(x)\right\}$ $\qquad$ Find expression of δ-function in terms of P_l's

Example



$\Phi(r=a) = V(\theta) \qquad \nabla^2\Phi = 0$ solve for $r \leq a$

Φ finite for $r\to 0$ so $B_l = 0$

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = V(\theta) \qquad A_l a^l = \frac{2l+1}{2}\int_{-1}^{1} d(\cos\theta)\, V(\theta)\, P_l(\cos\theta)$$

HW 3: 2.9, 3.1, 3.3, 3.6

10/17/2006

Spherical coord., azimuthal symmetry

$$\phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

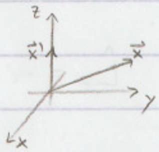
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Suppose you know solution on z-axis $(\theta=0)$ $\phi(z)$

If I can write it as: $\phi(z) \sim \sum_l a_l z^l + b_l z^{-(l+1)}$

Then solution is that times $P_l(\cos\theta)$
(in whole space)

Example: $\dfrac{1}{|\vec{x}-\vec{x}'|}$



$\Longrightarrow$ symmetric around z (azimuthal sym.)
is sol. of $\nabla^2\phi = 0$ for $\vec{x} \neq \vec{x}'$

$$|\vec{x}| > |\vec{x}'|: \quad \frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{z-z'} = \frac{1}{z}\,\frac{1}{1-\frac{z'}{z}} = \frac{1}{z}\sum_{l=0}^{\infty}\left(\frac{z'}{z}\right)^l$$

$$a_l = 0 \qquad b_l = z'^l \qquad \sum_{l=0}^{\infty}\frac{x'^l}{x^{l+1}} P_l(\cos\theta) = \frac{1}{|\vec{x}-\vec{x}'|}$$

$m \neq 0$

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \left(l(l+1) - \frac{m^2}{1-x^2}\right)P = 0 \qquad x\in[-1,1]$$

Try $P \sim x^\alpha \sum_{n=0}^{\infty} a_n x^n$ Look @ $x \simeq 1$ $\quad x = 1-y$ $\quad y>0$, $y \ll 1$

$$\frac{d}{dx} \to \frac{d}{dy} \qquad 1-x^2 = (1+x)(1-x) \simeq 2y$$

$$2y\frac{d}{dy}\left(2y\frac{dP}{dy}\right) + \left(\overset{\to 0}{l(l+1)2y} - m^2\right)P = 0 \qquad 4y\frac{d}{dy}\left(y\frac{dP}{dy}\right) - m^2 P = 0$$

$$P \sim y^\alpha \qquad 4\alpha^2 - m^2 = 0 \Longrightarrow \alpha = \pm\frac{m}{2}$$

Ansatz: $P(x) = (1-x)^{\frac{m}{2}}\sum_{n=0}^{\infty} a_n x^n \longrightarrow$ find recursion relation for a_n

$l \to$ positive integer $\qquad m \to 0, \pm1, \pm2, \dots, \pm l$

$P_{lm}(x)$ complete set for any fixed m in $[-1,1]$; Orthogonal

$\underbrace{P_{lm} Q_m}_{\text{normalization}} = Y_{lm}(\theta,\phi)$ complete set on sphere surface, orthonormal $\qquad 0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$

Spherical Harmonics

Orthonormal: $$\int_0^{2\pi}\int_{-1}^{1} Y_{lm}(\theta,\phi)\, Y^*_{l'm'}(\theta,\phi)\, d\phi\, d(\cos\theta) = \delta_{ll'}\,\delta_{mm'}$$

completeness $$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi') = \delta(\theta-\theta')\,\delta(\phi-\phi')$$

$$\boxed{\text{Spherical coordinates} \quad \nabla^2\phi = 0 \qquad \phi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right)Y_{\ell m}(\theta,\phi)}$$

Any function $g(\theta,\phi) = \sum_{\ell,m} A_{\ell m} Y_{\ell m}(\theta,\phi)$ $\qquad A_{\ell m} = \int_0^{2\pi}\int_{-1}^{1} g(\theta,\phi)\, Y^*_{\ell m}(\theta,\phi)\, d\phi\, d(\cos\theta)$

Connection to QM

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \left(V(\vec{x}) - E\right)\psi = 0$$

stationary state

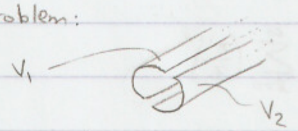
H atom $\quad V(x) = -\frac{e^2}{|\vec{x}|}$

$$\hat{L}^2 Y_{\ell m} = \ell(\ell+1) Y_{\ell m} \qquad \hat{L}_z Y_{\ell m} = m Y_{\ell m}$$

$$\vec{L} = \vec{r}\times\vec{p} = -i\,\vec{r}\times\nabla$$

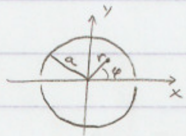
$$\hat{L}^2 = -\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \qquad \hat{L}_z = -i\frac{\partial}{\partial\phi}$$

Problem:



2 cylindrical shells, find ϕ inside $\nabla^2\phi = 0$

2D problem polar coord in plane



$0 \le r \le a$

$0 \le \varphi \le 2\pi$

b.c. $\quad r = a$

$0 < \varphi < \pi \qquad \phi = V_1$

$\pi < \varphi < 2\pi \qquad \phi = V_2$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} = 0 \qquad \phi(r,\varphi) = R(r)P(\varphi)$$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{P}\frac{d^2P}{d\varphi^2} = 0 \qquad \frac{1}{P}P'' = -k^2 \qquad P \sim A_k\sin(k\varphi) + B_k\cos(k\varphi)$$

$k \to$ integer so $P(\varphi) = P(\varphi + 2\pi)$ single valued sol^n

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - k^2 R = 0 \qquad R \sim r^\alpha \implies \alpha^2 - k^2 = 0 \qquad \alpha = \pm k \qquad R \sim r^k + r^{-k}$$

$k = 0, 1, 2, \ldots \quad r \to 0 \quad \phi$ is finite so $\cancel{r^{-k}} \implies R \sim r^k$

$$\phi(r,\varphi) = \sum_{k=0}^{\infty} r^k\left(A_k\sin(k\varphi) + B_k\cos(k\varphi)\right)$$

b.c.

$$r = a \qquad \sum_{k=0}^{\infty} a^k\left(A_k\sin(k\varphi) + B_k\cos(k\varphi)\right) = \begin{cases} V_1 & 0 < \varphi < \pi \\ V_2 & \pi < \varphi < 2\pi \end{cases}$$

$$a^k A_k = \frac{1}{\pi}\int_0^{\pi} V_1\sin(k\varphi)\,d\varphi + \frac{1}{\pi}\int_{\pi}^{2\pi} V_2\sin(k\varphi)\,d\varphi$$

normalization: $\frac{2}{[\text{interval}]}$ for sin, cos

except Fourier series first term (extra $\frac{1}{2}$ factor)

$$= -\frac{V_1}{\pi k}\left(\cos(k\pi) - 1\right) + \frac{V_2}{\pi k}\left(\cos(k\pi) - 1\right) = \begin{cases} 0 & k\ \text{even} \\ & \end{cases}$$

$k=0 \quad \frac{V_1+V_2}{2}$

k=odd

$$= \frac{2}{\pi k}\left(V_1 - V_2\right) \qquad \cos(k\pi) = (-1)^k$$

@ $r = 0$ ϕ is average of surfaces $\frac{V_1 + V_2}{2}$

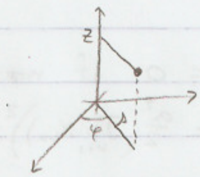
Repeat for B_k using $\cos(k\varphi)$ ... find $B_k = 0$

$k = 2n+1 \qquad n = 0, 1, 2, 3, \dots$

$$\Phi(r,\varphi) = \frac{V_1+V_2}{2} + \sum_{n=0}^{\infty} \frac{2}{\pi}\frac{V_1-V_2}{(2n+1)\,a^{2n+1}} \sin((2n+1)\varphi)\, r^{2n+1}$$

10/19/2006

Solutions of $\nabla^2\Phi = 0$ in Cylindrical coordinates



$$(\rho, \varphi, z): \quad \nabla^2\Phi = \frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\Phi = R(\rho)\, Q(\varphi)\, Z(z)$$

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{1}{R}\frac{dR}{d\rho} + \frac{1}{\rho^2}\frac{1}{Q}\frac{d^2Q}{d\varphi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = +k^2 \qquad Z \sim e^{\pm kz}$$

$$\frac{1}{Q}\frac{d^2Q}{d\varphi^2} = -\nu^2 \qquad Q \sim A_\nu \sin(\nu\varphi) + B_\nu \cos(\nu\varphi)$$ ν - integer so that Φ is single valued

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0$$

$x = \rho k \qquad \frac{d}{d\rho} = k\frac{d}{dx} \qquad \frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$

try $R \sim x^\alpha \sum_n a_n x^n \longrightarrow$ Solution $J_\nu(x)$, $J_{-\nu}(x)$ Bessel functions

If ν is integer, J_ν, $J_{-\nu}$ are linearly dependent so need second linear indep. solution:

$$\begin{cases} N_\nu(x) = \dfrac{1}{\sin(\nu\pi)}\left[J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)\right] \\ J_\nu(x) \end{cases}$$

Neumann function or Bessel function of 2^{nd} kind

basis vectors in space of solutions to Bessel eq.

Limiting form: $x \ll 1$ $(x \geq 0)$

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \frac{\nu^2}{x^2}R = 0 \qquad R \sim x^\alpha$$

$$\alpha(\alpha-1) + \alpha - \nu^2 = 0 \qquad \alpha = \pm\nu \implies R \sim x^{\pm\nu}$$

$x \ll 1$: $\quad J_\nu \sim x^\nu \qquad N_\nu \sim x^{-\nu} \quad \nu \neq 0 \qquad N \to \infty$ as $x \to 0$

$\qquad N_\nu \sim \ln x \quad \nu = 0$

$x \gg 1, \nu \qquad J_\nu \sim \frac{1}{\sqrt{x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \qquad N_\nu \sim \frac{1}{\sqrt{x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$

$J_\nu(x)$ has infinite # of zeros $\quad J_\nu(x_{\nu n}) = 0 \quad n = 1, 2, \ldots$

$x_{\nu n} \simeq n\pi + (\nu - \frac{1}{2})\frac{\pi}{2}$

$\rho \in [0, a]: \quad \left\{ J_\nu\left(\frac{x_{\nu n}}{a}\rho\right),\ n = 1, 2, 3, \ldots \right\}$ is a complete set

$\left\{ \sqrt{\rho}\, J_\nu\left(\frac{x_{\nu m}}{a}\rho\right) \right\}$ are orthogonal $\quad \int \rho\, J_{\nu n} J_{\nu m}\, d\rho = 0$ if $n \neq m$

$= \frac{a^2}{2}\left(J_{\nu+1}(x_{\nu n})\right)^2$ if $n = m$

can write any $f(\rho)$ for $\rho \in [0, a]$

$$f(\rho) = \sum_n A_n J_{\nu n}\left(\frac{x_{\nu n}}{a}\rho\right) \qquad A_n = \frac{2}{a^2 (J_{\nu+1}(x_{\nu n}))^2} \int_0^a d\rho\, \rho\, f(\rho)\, J_{\nu n}\left(\frac{x_{\nu n}}{a}\rho\right)$$

Example: 

$\nabla^2 \phi = 0$ inside cylinder

b.c.s: $\phi = 0$ on surface

$\phi = V(\rho, \varphi)$ @ $z = L \quad 0 < \rho < a$

φ-dep $\sim A_m \sin(m\varphi) + B_m \cos(m\varphi) \quad$ m integer

z-dep $\sim \sinh(kz) \qquad \rho$-dep $\sim J_m(k\rho) + \cancel{N_m(k\rho)}$

ϕ finite for $\rho \to 0$

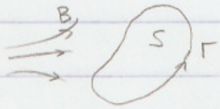
$\phi = 0$ for $\rho = a \quad \therefore \quad J_m(ka) = 0 \qquad ka = x_{mn} \implies k = \frac{x_{mn}}{a} \qquad J_m\left(\frac{x_{mn}}{a}\rho\right)$

$$\Phi = \sum_{m,n} \left(A_{mn} \sin(m\varphi) + B_{mn} \cos(m\varphi)\right) J_m\left(\frac{x_{mn}}{a}\rho\right) \sinh\left(\frac{x_{mn}}{a} z\right)$$

$$\sinh\left(\frac{x_{mn}}{a}L\right) A_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(x_{mn})} \int_0^{2\pi}\int_0^a \rho \sin(m\varphi)\, J_m\left(\frac{x_{mn}}{a}\rho\right) V(\rho, \varphi)\, d\rho\, d\varphi$$

Time-dependent fields 10/24/2006

Faraday Induction



MKS : $k=1$
cgs : $k=\frac{1}{c}$

flux of B

$$\mathcal{E} = \oint_\Gamma \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\int_S \vec{B}\cdot\hat{n}\,ds = -\frac{1}{c}\frac{d}{dt}\Phi_S^{(B)}$$

(E, B have same dimensions in Gaussian)

$$I = \frac{\mathcal{E}}{R}$$

electromotive force

current, resistance

(uninterrupted circuit)

Lorentz force $\vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)$

$\uparrow\vec{v}$ q $\quad \vec{F} = q\,\frac{\vec{v}}{c}\times\vec{B}$

Can get Faraday Induction from Lorentz force + Galilean transformation

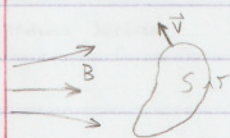
Frame of charge: $\vec{v}'=0 \qquad \vec{F}' = \vec{F} = q\,\frac{\vec{v}}{c}\times\vec{B}$

Must be $\vec{E}'$ appearing because $v'\times B = 0 \implies \vec{E}' = \frac{\vec{v}}{c}\times\vec{B}$ so $q\vec{E}' = \vec{F}' = \vec{F} = q\,\frac{\vec{v}}{c}\times\vec{B}$

In general: $\boxed{\vec{E}' = \vec{E} + \frac{\vec{v}}{c}\times\vec{B}}$ true for $\frac{v}{c} \ll 1$

relativistic: $\vec{E}' = \gamma\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) + \underset{\text{order}}{\mathcal{O}\left(\frac{v}{c}\right)^2}$

$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \beta = \frac{v}{c}$



B static, circuit moving @ $\vec{v}$

rest frame of circuit: $\vec{E} = \frac{\vec{v}}{c}\times\vec{B}$

$$\oint_\Gamma \vec{E}\cdot d\vec{l} = \oint_\Gamma \left(\frac{\vec{v}}{c}\times\vec{B}\right)\cdot d\vec{l} = \int_S \nabla\times\left(\frac{\vec{v}}{c}\times\vec{B}\right)\cdot\hat{n}\,ds$$

$$\nabla\times(\vec{a}\times\vec{b}) = \vec{a}(\nabla\cdot\vec{b}) - \vec{b}(\nabla\cdot\vec{a}) + (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b}$$

$\nabla\cdot\vec{B} = 0 \qquad \vec{v}$ is constant $\implies \nabla\cdot\frac{\vec{v}}{c} = 0 \qquad (\vec{B}\cdot\nabla)\frac{\vec{v}}{c} = 0 \qquad \implies \nabla\times\left(\frac{\vec{v}}{c}\times\vec{B}\right) = -\left(\frac{\vec{v}}{c}\cdot\nabla\right)\vec{B}$

$$= -\int_S \left(\frac{\vec{v}}{c}\cdot\nabla\right)\vec{B}\cdot\hat{n}\,ds \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla \qquad \vec{B} \text{ static } \frac{\partial\vec{B}}{\partial t} = 0$$

$$= -\frac{1}{c}\int_S \left(\frac{d\vec{B}}{dt}\right)\cdot\hat{n}\,dS = -\frac{1}{c}\frac{d}{dt}\int_S \vec{B}\cdot\hat{n}\,ds = \oint_\Gamma \vec{E}\cdot d\vec{l}$$

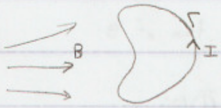
$$\oint_\Gamma \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\int_S \vec{B}\cdot\hat{n}\,ds \qquad S \text{ fixed}, \underset{\vec{v}=0}{} \quad B = B(t)$$

$$\int_S (\nabla\times E)\cdot\hat{n}\,ds = -\frac{1}{c}\int_S \frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\,ds \implies \int_S \left(\nabla\times E + \frac{1}{c}\frac{\partial B}{\partial t}\right)\cdot\hat{n}\,dS = 0$$

$$\boxed{\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}}$$

Energy Associated with Magnetic Field

Magnetostatics



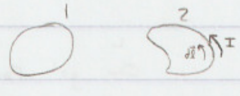
turn on $\vec{B}$: induce $\boxed{\mathcal{E} = -\frac{1}{c}\frac{d}{dt}\Phi(B)}$

$$\frac{dW}{dt} = \mathcal{E} I = \frac{I}{c}\frac{d}{dt}\Phi(B)$$

$$\delta W = \frac{I}{c}\,\delta\,\Phi(B) = \frac{I}{c}\,\delta\int_S (\nabla\times\vec{A})\cdot\hat{n}\,ds \qquad \vec{B} = \nabla\times\vec{A}$$

$$= \frac{I}{c}\,\delta\left[\oint_\Gamma \vec{A}\cdot d\vec{l}\right] \qquad W = \frac{I}{c}\oint_\Gamma \vec{A}\cdot d\vec{l}$$

Consider 2 loops



$\vec{J}$ = current density $\qquad I = \int J\,d\sigma$ crosssection

$$W = \frac{1}{c}\int\oint_{\Gamma_2} \vec{A}\cdot\vec{J}\,d\vec{l}\,d\sigma$$

In general : $\boxed{W = \frac{1}{2c}\int \vec{A}\cdot\vec{J}\,d^3x}$ Analogous to $W = \frac{1}{2}\int \rho\,\phi\,d^3x$

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}\left(+\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right) \quad \to 0 \text{ statics}$$

$$\frac{1}{2c}\frac{c}{4\pi}\int \vec{A}\cdot(\nabla\times\vec{B})\,d^3x$$

$$\int \vec{A}\cdot(\nabla\times\vec{B})\,d^3x = \int \vec{B}\cdot\underbrace{(\nabla\times\vec{A})}_{\vec{B}}\,d^3x \qquad \nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b})$$

$$\int \nabla\cdot\vec{A}\times\vec{B}\,d^3x = \int_S (\vec{A}\times\vec{B})\cdot\hat{n}\,ds = 0$$

take surface to infinity
localized currents & fields

$$W = \frac{1}{8\pi}\int |\vec{B}|^2\,d^3x$$

energy density $\quad u = \frac{1}{8\pi}\left(|\vec{E}|^2 + |\vec{B}|^2\right)$

Before Maxwell

$$\nabla\cdot\vec{E} = 4\pi\rho \qquad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} \qquad \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \nabla\cdot\vec{B} = 0$$

Necessity of new term: $\nabla\cdot\nabla\times\vec{B} = 0 = \frac{4\pi}{c}\nabla\cdot\vec{J}$ $\quad \nabla\cdot\vec{J} = 0$ false with time varying quantities

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0 \qquad \frac{4\pi}{c}\left(-\frac{\partial\rho}{\partial t}\right) = -\frac{4\pi}{c}\frac{1}{4\pi}\frac{\partial}{\partial t}(\nabla\cdot\vec{E}) \implies \boxed{\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}}$$

Static solutions in free space

$$\nabla\times\vec{E} = 0 \implies \vec{E} = -\nabla\phi \qquad \nabla^2\phi = -4\pi\rho \qquad \phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$$

$$\boxed{\vec{B} = \nabla\times\vec{A}} \qquad \nabla\times(\nabla\times\vec{A}) = -\nabla^2\vec{A} + \nabla(\nabla\cdot\vec{A})$$

static situation: Coulomb gauge so that $\nabla\cdot\vec{A} = 0$

$$\boxed{\begin{gathered}\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J} \\ \vec{A}(\vec{x}) = \frac{1}{c}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'\end{gathered}}$$

Time dependant 10/26/2006

$\boxed{\nabla\cdot\vec{E} = 4\pi\rho \qquad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{\partial\vec{E}}{\partial t}\frac{1}{c} \qquad \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \nabla\cdot\vec{B} = 0}$

$\vec{B} = \nabla\times\vec{A}$

$\nabla\times\vec{E} + \frac{1}{c}\nabla\times\frac{\partial\vec{A}}{\partial t} = 0 \qquad \nabla\times\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0 \qquad \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\nabla\phi$

$\boxed{\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}}$

Gauge transformation

$\vec{A} \to \vec{A}' = \vec{A} + \nabla\Lambda \qquad \nabla\times\vec{A}' = \vec{B}$ since $\nabla\times\nabla\Lambda = 0$

$\phi \to \phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} \qquad \vec{E}' = -\nabla\phi + \cancel{\frac{1}{c}\frac{\partial}{\partial t}\nabla\Lambda} - \frac{1}{c}\frac{\partial A}{\partial t} - \cancel{\frac{1}{c}\frac{\partial\nabla\Lambda}{\partial t}} = \vec{E}$

$\nabla\cdot\vec{E} = -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = 4\pi\rho$

$\nabla\times\vec{B} = -\nabla^2\vec{A} - \nabla(\nabla\cdot\vec{A}) + \nabla\frac{\partial\phi}{c\,\partial t} + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \frac{4\pi}{c}\vec{J}$

$\boxed{\boxed{\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0} \text{ Lorentz gauge}} \qquad \partial_\mu A^\mu = 0 \qquad A^\mu = (\phi, \vec{A})$

$-\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 4\pi\rho \qquad -\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \frac{4\pi}{c}\vec{J}$ 2 non homogeneous wave equations

with transformation

$\nabla\cdot\vec{A} + \nabla^2\Lambda + \frac{1}{c}\frac{\partial\phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t} = 0 \qquad \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\left(\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)$

Still have freedom to make additional gauge transformation such that $\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0$

$\nabla^2\Psi(\vec{x},t) - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi\, f(\vec{x},t)$ source function

Fourier Transform $\Psi(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\Psi}(\vec{x},\omega)e^{-i\omega t}\,d\omega \qquad (\nabla^2 + k^2)\Psi(\vec{x},\omega) = -4\pi\, f(\vec{x},\omega)$ $\quad k^2 = \omega^2/c^2$

Find Green's function: $(\nabla^2 + k^2)G(\vec{x},\vec{x}';\omega) = -4\pi\,\delta(\vec{x}-\vec{x}')$ free space: $G(R,\omega)$ $R = |\vec{x}-\vec{x}'|$

in spherical coords: $\frac{1}{R}\frac{d^2}{dR^2}(RG) + k^2 G = -4\pi\,\delta(R)$

for $R \neq 0$ $(\vec{x}\neq\vec{x}')$ $\frac{d^2}{dR^2}(RG) + k^2 RG = 0 \implies RG = e^{\pm ikR} \qquad G(R,\omega) = A_+\frac{e^{ikR}}{R} + A_-\frac{e^{-ikR}}{R}$

for $R \to 0$ $G(R,\omega) = \frac{1}{R}$ $\left(kR\to 0 \quad \nabla^2 G = -4\pi\,\delta(R)\right)$ spherical wave

Green's function for wave eqn:

$\nabla^2 G(\vec{x},\vec{x}';t,t') - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\,\delta(\vec{x}-\vec{x}')\,\delta(t-t')$

$G = \int\frac{d\omega}{2\pi}G(\vec{x},\vec{x}';\omega,t')e^{-i\omega t} \qquad \delta(t-t') = \int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}$

$(\nabla^2 + k^2)\,G(\vec{x},\vec{x}';\omega,t') = -4\pi\,\delta(\vec{x}-\vec{x}')e^{i\omega t'}$

Solution: $G = \dfrac{e^{\pm ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} e^{i\omega t'}$

$$G(\vec{x},\vec{x}';t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\frac{\omega}{c}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} e^{i\omega t'} e^{-i\omega t}\, d\omega = \frac{1}{2\pi}\frac{1}{R}\int e^{-i\omega\left((t-t')-\frac{1}{c}R\right)} d\omega$$

$$G(\vec{x},\vec{x}';t,t') = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\Big[t' - \Big(t - \underbrace{\frac{|\vec{x}-\vec{x}'|}{c}}_{\text{time of propagation of disturbance}}\Big)\Big]$$

$\tau = t - \dfrac{|\vec{x}-\vec{x}'|}{c}$ retarded time $\qquad G = \dfrac{1}{|\vec{x}-\vec{x}'|}\delta(t'-\tau)$

fields at $\vec{x},t$ depend on sources at $\vec{x}'$ and time $= t - \dfrac{|\vec{x}-\vec{x}'|}{c} = \tau$

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f$$

$$\psi(\vec{x},t) = \iint \frac{\delta\left(t' - \left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right)}{|\vec{x}-\vec{x}'|} f(\vec{x}',t')\, d^3x'\, dt'$$

$$\psi(\vec{x},t) = \int f\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\frac{1}{|\vec{x}-\vec{x}'|}\, d^3x' = \int \frac{\left[f(\vec{x}',t')\right]_{ret}}{|\vec{x}-\vec{x}'|}\, d^3x'$$

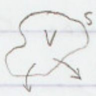
$$\boxed{\Phi(\vec{x},t) = \int \frac{\rho\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}\, d^3x'} \qquad \boxed{\vec{A}(\vec{x},t) = \int \frac{\vec{J}\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}\, d^3x'}$$

free space solution (no boundaries)

Conservation of Energy $\qquad$ fields, currents, charges

Power transferred to charges $\quad P = \vec{F}\cdot\vec{v} = q\vec{v}\cdot\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) = q\vec{v}\cdot\vec{E}$

$\vec{J} = nq\vec{v}$ $\quad$ n: # charges / unit volume $\quad P = \vec{J}\cdot\vec{E} \implies P = \int \vec{J}\cdot\vec{E}\, d^3x$ $\quad$ energy per unit time transferred to charges



① fields with energy density $u = \frac{1}{8\pi}(E^2 + B^2)$

② energy current (radiation) $\vec{P}$ $\qquad$ ③ kinetic energy of charges $\left(\frac{d}{dt} \to \vec{J}\cdot\vec{E}\right)$

$$\frac{\partial}{\partial t}\int_V u\, d^3x + \int_S \vec{P}\cdot\hat{n}\, ds + \int_V \vec{J}\cdot\vec{E}\, d^3x = 0 \implies \frac{\partial u}{\partial t} + \nabla\cdot\vec{P} + \vec{J}\cdot\vec{E} = 0$$

$\to$ find: $\vec{P} = \frac{c}{4\pi}(\vec{E}\times\vec{B})$ $\quad$ Poynting vector

11/2/2006

$$\nabla^2 G = -4\pi\delta(\vec{x}-\vec{x}') \implies G \begin{cases} \sim 1/r & \text{3-D} \\ \sim \ln r & \text{2-D} \end{cases}$$

$\leftarrow$ ~~test~~ problem or use 3-D and include $\int_{-\infty}^{\infty} dz'$

Conservation of Energy

$$\frac{\partial u}{\partial t} + \nabla\cdot\vec{P} + \vec{J}\cdot\vec{E} = 0 \qquad u = \frac{1}{8\pi}(E^2+B^2)$$

$\vec{P}$: $\frac{\text{energy}}{\text{time}\cdot\text{area}}$

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\frac{4\pi}{c}\vec{J}\cdot\vec{E} = \vec{E}\cdot(\nabla\times\vec{B}) - \frac{1}{c}\underbrace{\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}}_{\frac{1}{2}\frac{\partial}{\partial t}(E^2)}$$

$$\nabla\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{B})$$

$$-\nabla\cdot(\vec{E}\times\vec{B}) + \vec{B}\cdot\underbrace{(\nabla\times\vec{E})}_{-\frac{1}{c}\frac{\partial B}{\partial t}}$$

$$-\frac{1}{c}\frac{1}{2}\frac{\partial}{\partial t}(B^2)$$

$$\therefore\ \frac{\partial}{\partial t}\left(\frac{1}{8\pi}(E^2+B^2)\right) + \nabla\cdot\vec{P} - \nabla\cdot(\vec{E}\times\vec{B})\frac{c}{4\pi} - \cancel{\frac{1}{8\pi}\frac{\partial B^2}{\partial t}} - \cancel{\frac{1}{8\pi}\frac{\partial E^2}{\partial t}} = 0$$

$$\nabla\cdot\vec{P} = \frac{c}{4\pi}\nabla\cdot(\vec{E}\times\vec{B}) \implies \boxed{\vec{P} = \frac{c}{4\pi}(\vec{E}\times\vec{B})}$$ Poynting vector

energy current

Conservation of Momentum

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

Kinetic $\vec{p}$ per unit volume

$$\frac{d\vec{p}_{ki}}{dt} = \rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B} \qquad \frac{\text{momentum}}{\text{time}\cdot\text{volume}}$$

$$\frac{d\vec{p}_{ki}}{dt} + \frac{\partial}{\partial t}\vec{p}_f + \text{div}(\text{tensor}) = 0$$

$\frac{d\vec{p}_{ki}}{dt}$: carried by charges; $\vec{p}_f$: field $\vec{p}$, carried by fields

$$\nabla\cdot\vec{E} = -4\pi\rho \qquad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\frac{d\vec{p}_{ki}}{dt} = -\frac{1}{4\pi}\vec{E}(\nabla\cdot\vec{E}) + \frac{1}{4\pi}(\nabla\times\vec{B})\times\vec{B} - \frac{1}{4\pi c}\frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

$$\vec{E}(\nabla\cdot\vec{E}) + \vec{B}\underbrace{(\nabla\cdot\vec{B})}_{0}$$

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\underbrace{\frac{\partial\vec{B}}{\partial t}}_{cE\times(\nabla\times E)}$$

$$\frac{d\vec{p}_{ki}}{dt} = -\frac{1}{4\pi c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \frac{1}{4\pi}\left[\vec{E}(\nabla\cdot\vec{E}) + \vec{B}(\nabla\cdot\vec{B}) - \vec{E}\times(\nabla\times\vec{E}) - \vec{B}\times(\nabla\times\vec{B})\right]$$

to show: $[\partial] = \text{div}(\text{tensor})$

totally anti-symm. tensor

$$\varepsilon_{ijk}:\quad \varepsilon_{123} = 1 \qquad \varepsilon_{ijk} = -\varepsilon_{jik} \qquad (\vec{a}\times\vec{b})_i = \varepsilon_{ijk}a_j b_k \qquad (\nabla\times\vec{a})_i = \varepsilon_{ijk}\partial_j a_k$$

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \qquad \vec{a}\cdot\vec{b} = a_i b_k \delta_{ik} = a_i b_i$$

$$(\text{div}(\text{tensor}))_i = \partial_k T_{ik}$$

$$[\partial]_i = E_i\partial_k E_k + B_i\partial_k B_k - \varepsilon_{ijk}E_j\varepsilon_{klm}\partial_l E_m - \varepsilon_{ijk}B_j\varepsilon_{klm}\partial_l B_m$$

$$= \partial_k(E_iE_k) - (\partial_kE_i)E_k + \partial_k(B_iB_k) - (\partial_kB_i)B_k + \underbrace{(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})E_j\partial_l E_m}_{E_j\partial_i E_j - E_j\partial_j E_i}$$

$\vdots$

$$4\pi T_{ik} = E_iE_k + B_iB_k - \frac{1}{2}(E^2+B^2)\delta_{ik}$$ Maxwell stress tensor

HW 4 : 6.1, 6.11, 6.14a, 6.15, 6.18 — 11/7/2006

Cons. momentum

$P_{mech} = \frac{\text{momentum of charges}}{\text{volume}}$

$$-\frac{d\vec{P}_{mech}}{dt} = \frac{\partial P_{fields}}{\partial t} + \text{div}\left(\text{Maxwell Stress Tensor}\right)$$

$\vec{P}_{mech} = \vec{P}_{ki}$ $\quad \vec{P}_{fields} = \vec{P}_f$ momentum carried by fields

$$\frac{d\vec{P}_{ki}}{dt} = \rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B} \qquad \vec{P}_f = \frac{1}{4\pi c}(\vec{E}\times\vec{B}) = \frac{1}{c^2}\underbrace{\vec{S}}_{\text{Poynting, energy current}}$$

$u = \frac{1}{8\pi}(E^2+B^2)$ $\quad S$ $\quad \frac{u}{c} \sim \vec{P}_f$ $\quad uc \sim \vec{S}$

$$[\partial_x]_i = \partial_k\left(E_iE_k + B_iB_k - \frac{1}{2}(E^2+B^2)\delta_{ik}\right) = \partial_k T_{ik}\, 4\pi$$

$$\boxed{T_{ik} = \frac{1}{4\pi}\left(E_iE_k + B_iB_k - \frac{1}{2}(E^2+B^2)\delta_{ik}\right)} \quad \text{Maxwell Stress Tensor}$$

$$\frac{\partial}{\partial t}\left[\frac{1}{4\pi c}(\vec{E}\times\vec{B})\right]_i + \partial_k T_{ik} = -\left[\frac{d\vec{P}_{ki}}{dt}\right]_i$$

$$\int_V d^3x\left[-\frac{d}{dt}P\right]_i = \int_V d^3x\left[\frac{1}{4\pi}(\vec{E}\times\vec{B})\right]_i + \int_S T_{ik}\, n_k\, dS$$

S $\hat{n}$ V

$T_{ik}n_k$: ith component of $\frac{\text{momentum}}{\text{surface} \cdot \text{time}}$ flowing in direction n_k

---

Transformation: rotation R

scalar: $S \xrightarrow{R} S' = S$ $\qquad$ vector: $A_i \xrightarrow{R} A_i' = R_{ik}A_k$ $\qquad x_i' = R_{ik}x_k$

tensor of rank 2: $T_{ik} \xrightarrow{R} T_{ik}' = R_{il}R_{km}T_{lm}$

$$\Downarrow \quad R_{ik} = \frac{\partial x_i'(x_1,x_2,x_3)}{\partial x_k}$$

$\vec{A}$ is vector if it transforms like a vector

parity P: $\vec{x} \to -\vec{x}$ $\quad \therefore$ $\vec{E} \to -\vec{E}$ $\quad \phi \to \phi$ $\quad \rho \to \rho$

$\nabla\cdot E = 4\pi\rho$ $\quad \vec{A} \to -\vec{A}$ $\quad \vec{B} \to \vec{B}$ (pseudovector) $\nabla\times A$ $\quad \vec{J} \to -\vec{J}$

$- \; - \; + \; +$

Maxwell's Eqn invariant under parity

T: $t \to -t$ $\quad \therefore$ $\vec{E} \to \vec{E}$ $\quad \vec{A} \to -\vec{A}$ $\quad \vec{J} \to -\vec{J}$ $\quad \vec{B} \to -\vec{B}$ $\quad \rho \to \rho$ $\quad \phi \to \phi$

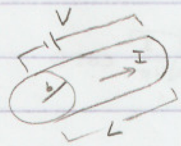
created by currents

$\nabla\times B = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial E}{\partial t}$ $\quad + \; - \; - \; + \; -$

M.E. invariant under CPT

C: charge inversion $\quad \vec{E} \to -\vec{E}$ $\quad \rho \to -\rho$ $\quad \phi \to -\phi$

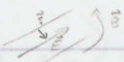
$q \to -q$ $\quad \vec{B} \to -\vec{B}$ $\quad \vec{J} \to -\vec{J}$

Problem:  steady state current I, pot. diff V

a) Calculate $\vec{S}$ @ surface of wire $\qquad E = \frac{V}{L}$

$$B\, 2\pi r = \frac{4\pi}{c} I \longrightarrow B = \frac{2I}{cr}$$

$$\int_S (\nabla \times B)\cdot ds = \int_\Gamma B\, dl \qquad \int_S J = I$$



$$\vec{S} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right) \qquad S = -\frac{V}{L}\frac{2I}{cb}\hat{r}\frac{c}{4\pi} \text{ into wire}$$

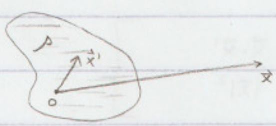
b) $\frac{\text{Power}}{\text{Length}}$ dissipated: $\frac{VI}{L}$

$$\underset{\text{energy current}}{S}\, 2\pi b = \frac{V}{L}\frac{2I}{\cancel{c}\cancel{b}}\frac{\cancel{c}}{\cancel{4\pi}}\cancel{2\pi b} = \frac{VI}{L}$$

since steady state Energy from V converted to heat (dissipated)

otherwise charges would accelerate

Multipole Expansion (Chpt 4)



$\phi = ?$ $\quad |\vec{x}| > |\vec{x}'|$

first term $\sim \frac{1}{r}$ (total charge) $\qquad$ second term $\sim \frac{1}{r^2}$ (dipole)

Spherical

$$\phi(r,\theta,\varphi) = \sum_{l,m}\left(\cancel{A_{lm}} r^l + B_{lm} r^{-(l+1)}\right) Y_{lm}(\theta,\varphi)$$

$0 \quad \phi \to 0 \;\; r\to\infty$

$$\boxed{\phi(r,\theta,\varphi) = \sum_{lm}\frac{4\pi}{2l+1} q_{lm}\frac{1}{r^{l+1}} Y_{lm}(\theta,\varphi)} \qquad \sum_{lm} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}$$

$l=0$ monopole $\qquad l=1$ dipole

Find q_{lm} in terms of ρ $\qquad \phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$

$r = |\vec{x}|$
$r > r'$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{lm}\frac{4\pi}{2l+1}\frac{r'^l}{r^{l+1}} Y_{lm}(\theta,\varphi) Y^*_{lm}(\theta',\varphi')$$

$$\phi(\vec{x}) = \sum_{lm}\frac{4\pi}{2l+1}\left[\int Y^*_{lm}(\theta',\varphi')\, r'^l \rho(\vec{x}')\, d^3x'\right]\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

$$\therefore \boxed{q_{lm} = \int \rho(\vec{x}')\, r'^l\, Y^*_{lm}(\theta',\varphi')\, d^3x'} \qquad q_{l,-m} = (-1)^m q^*_{lm} \qquad Y_{l-m} = (-1)^m Y^*_{lm}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}}\ \sin\theta\ e^{i\varphi} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$q_{00} = \frac{1}{\sqrt{4\pi}}\int \rho(\vec{x}')\, d^3x' = \frac{Q}{\sqrt{4\pi}}$$

$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$

$$\sin\theta\, e^{i\varphi} = \sin\theta\,(\cos\theta - i\sin\theta) = \frac{1}{r}(x - iy) \qquad \vec{p} = \int d^3x'\,\rho(\vec{x}')\,\vec{x}'$$

$$q_{11} = -\int d^3x'\,\rho(\vec{x}')\, r'\,\frac{1}{r'}(x' - iy')\sqrt{\frac{3}{8\pi}} = -\sqrt{\frac{3}{8\pi}}(p_x - ip_y) \qquad q_{1-1} = \sqrt{\frac{3}{8\pi}}(p_x + ip_y)$$

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{i2\varphi} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}(x+iy)^2\frac{1}{r^2}$$

Quadrupole moment $\quad \boxed{Q_{ij} = \int \rho(\vec{x}')\left(3x_i'x_j' - r'^2\delta_{ij}\right) d^3x'}$

Cartesian Coordinates

$$\Phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|}\frac{1}{\left[1 - 2\frac{xx'+yy'+zz'}{x^2+y^2+z^2}\ \cdots\right]^{1/2}} \overset{|\vec{x}|\gg|\vec{x}'|}{=} \frac{1}{|\vec{x}|}\left[1 + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2}\ \cdots\right]$$

$$\Phi(\vec{x}) \simeq \underbrace{\frac{1}{|\vec{x}|}\int \rho(\vec{x}')\, d^3x'}_{\frac{q}{r}} + \underbrace{\frac{\vec{x}}{|\vec{x}|^3}\cdot\int \rho(\vec{x}')\,\vec{x}'\, d^3x'}_{\frac{\vec{p}\cdot\vec{x}}{r^3}}$$

11/9/2006

$$\boxed{\Phi(\vec{x}) = \frac{q}{|\vec{x}|} + \frac{\vec{p}\cdot\vec{x}}{|\vec{x}|^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{|\vec{x}|^5} + \ldots}$$

$$E_r = -\frac{\partial\Phi}{\partial r} = \sum_{lm} 4\pi\frac{(l+1)}{(2l+1)} q_{lm}\frac{Y_{lm}(\theta,\varphi)}{r^{l+2}}$$

$E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \ldots \qquad$ take case of dipole $\vec{p}$ along $\hat{z}$ $\quad l=1 \quad m=0$

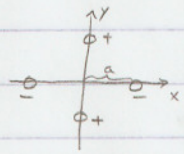
$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$E_r = 4\pi\frac{2}{3}q_{10}\sqrt{\frac{3}{4\pi}}\frac{\cos\theta}{r^3} \qquad q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z \quad \therefore \quad E_r = \frac{2p\cos\theta}{r^3}$$

$$E_\theta = \frac{p\sin\theta}{r^3} \qquad E_\varphi = 0$$

$$\boxed{\vec{E}(\vec{x}) = \frac{3\hat{x}(\vec{p}\cdot\vec{x}) - \vec{p}}{|\vec{x}|^3} \quad \text{field of dipole in cartesian}}$$

Quadrupole



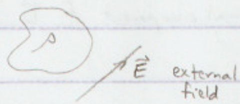
monopole: $Q = 0$

dipole: $p_x = a(-q) + (-a)(-q) = 0 = p_y = p_z$

quadrupole:

$$Q_{11} = -q(3a^2 - a^2) + -2a^2 q = -4a^2 q$$

Electrostatic Energy



slowly varying $\vec{E}$ (accross region with ρ)

No $\frac{1}{2}$ since this is not energy of charges
It is energy by field

$$\boxed{W = \int \phi(\vec{x})\rho(\vec{x})\, d^3x}$$

Summation Notation

$$\phi(\vec{x}) = \phi(\vec{0}) + \vec{x}\cdot\nabla\phi(\vec{0}) + \frac{1}{2}\frac{\partial^2\phi}{\partial x_i \partial x_j}(\vec{0})\, x_i x_j$$

$$\frac{\partial E_j}{\partial x_i}\delta_{ij} = \nabla\cdot\vec{E} = 0$$

external field not produced by ρ

$$W = q\,\phi(\vec{0}) - \vec{E}(\vec{0})\cdot\vec{p} - \frac{1}{2}\frac{1}{3}\frac{\partial E_j}{\partial x_i}(\vec{0})\int d^3x\,\rho(x)\left[3x_i x_j - x^2\delta_{ij}\right]$$

$$W = q\,\phi(\vec{0}) - \underbrace{\vec{p}\cdot\vec{E}(\vec{0})}_{\text{energy of dipole in external field}} - \underbrace{\frac{1}{6}\frac{\partial E_j}{\partial x_i}(\vec{0})\,Q_{ij}}_{\text{energy of quadrupole in external field}}$$

dielectric

molecules in $\vec{E} \to$ develop $\vec{p}$

material in $E \to$ develops $\frac{\text{dipole moment}}{\text{volume}} = \vec{P}$ (polarization)

write ϕ in material : contribution from "extra charges" (free charges) $\sim \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$

+ contribution from polarization $\sim \int \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} d^3x'$

$$\phi(\vec{x}) = \int\left\{\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right\}d^3x'$$

Integration by parts...

$$= \int\left\{\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \vec{P}(\vec{x}')\cdot\nabla'\frac{1}{|\vec{x}-\vec{x}'|}\right\}d^3x' = \int\frac{1}{|\vec{x}-\vec{x}'|}\underbrace{\left[\rho(\vec{x}') - \nabla'\cdot\vec{P}(\vec{x}')\right]}_{\text{effective } \rho_{\text{eff}}}d^3x'$$

$\nabla\cdot\vec{E} = 4\pi\rho \xrightarrow{\text{inside}} \nabla\cdot\vec{E} = 4\pi\left(\rho - \nabla\cdot\vec{P}\right)$

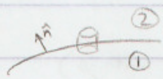
$\nabla\cdot\left(\vec{E} + 4\pi\vec{P}\right) = 4\pi\rho_{extra}$ Electrostatics in the material $\nabla\times\vec{E} = 0$

usual situation $\vec{P}\propto\vec{E} \implies \boxed{\vec{P} = \chi\vec{E}}$ susceptibility

$\nabla\cdot\left[(1+4\pi\chi)\vec{E}\right] = 4\pi\rho_{extra}$ $\boxed{1+4\pi\chi = \epsilon \text{ dielectric const.}}$

$\epsilon\vec{E} = \vec{D} \implies \boxed{\nabla\cdot\vec{D} = 4\pi\rho_{extra}}$

b.c.



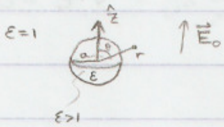
$(\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = 0$ if $\rho_{extra} = 0$ Normal component of $\vec{D}$ continuous

$\boxed{(\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = 4\pi\sigma_{extra}}$

$\boxed{(\vec{E}_2 - \vec{E}_1)\times\hat{n} = 0}$ Tangential component (from $\nabla\times\vec{E} = 0$)

Dielectric sphere in Uniform external field $(\varepsilon = \epsilon)$

no free charges $\rho_{extra} = 0$



b.c. $r\to\infty$ $\vec{E} = E_0\hat{z}$ $\phi \to -E_0 r\cos\theta$

$r\to 0$ ϕ finite $\underbrace{1\frac{\partial\phi_{out}}{\partial r}\Big|_{r=a}}_{-D_{out}} = \underbrace{\varepsilon\frac{\partial\phi_{in}}{\partial r}\Big|_{r=a}}_{-D_{in}}$ normal component

$\frac{1}{r}\frac{\partial\phi_{out}}{\partial\theta}\Big|_{r=a} = \frac{1}{r}\frac{\partial\phi_{in}}{\partial\theta}\Big|_{r=a}$ tangential component

$$\phi = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta)$$

$$\phi_{out} = A_1 r\cos\theta + A_0 + \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) \qquad A_1 = -E_0$$

$$\phi_{in} = \sum_{l=0}^{\infty} C_l r^l P_l(\cos\theta)$$

$$\frac{\partial\phi_{out}}{\partial r}\Big|_a = -E_0\cos\theta - \sum_l (l+1)B_l a^{-(l+2)}P_l(\cos\theta) = \varepsilon\sum_l l C_l a^{l-1}P_l(\cos\theta) = \varepsilon\frac{\partial\phi_{in}}{\partial r}\Big|_a$$

$l=1$ $-E_0 - 2B_1 a^{-3} = \varepsilon C_1$ $B_l = C_l = 0$ for $l\neq 1$

$\frac{1}{a}\frac{\partial\phi}{\partial\theta}\Big|_a = E_0\sin\theta - B_1 a^{-3}\sin\theta = -C_1\sin\theta$

$-2B_1 a^{-3} - \varepsilon C_1 = E_0$

$-B_1 a^{-3} + C_1 = -E_0$

$-2(E_0 + C_1) - \varepsilon C_1 = E_0$ $-C_1(2+\varepsilon) = 3E_0$

$C_1 = \frac{-3E_0}{2+\varepsilon}$

$B_1 = a^3(C_1 + E_0) = a^3\left(\frac{-3E_0}{2+\varepsilon} + E_0\right)$ $\frac{-3E_0 + 2E_0 + \varepsilon E_0}{-E_0}$

$B_1 = \frac{\varepsilon - 1}{\varepsilon + 2}a^3 E_0$ $\vec{P} = \frac{\vec{p}}{\frac{4}{3}\pi a^3} = \frac{3}{4\pi}E_0\frac{\varepsilon-1}{\varepsilon+2}\hat{z}$

$\phi_{in} = \frac{-3E_0}{2+\varepsilon}r\cos\theta$ $\phi_{out} = -E_0 r\cos\theta + \underbrace{\frac{\varepsilon-1}{\varepsilon+2}\frac{a^3}{r^2}E_0\cos\theta}_{\text{dipole}}$ $\vec{p} = \frac{\varepsilon-1}{\varepsilon+2}a^3E_0\hat{z}$

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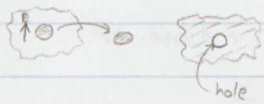
$\vec{E}_{in} = \frac{3}{\varepsilon+2} E_0 \hat{z}$ $\quad (< E_0 \text{ for } \varepsilon > 1)$ $\quad \vec{P} = \frac{3}{4\pi} E_0 \frac{\varepsilon - 1}{\varepsilon+2} \hat{z}$

macroscopically $\vec{P} = \chi \vec{E}$ $\quad \varepsilon = 1 + 4\pi\chi$

microscopically $\vec{p} = \gamma \vec{E}$ dipole moment aquired by molecule in $\vec{E}$

γ ← molecular polarizability

Relation $\chi \leftrightarrow \gamma$?



Field in a hole in a dielectric.

Superposition

$\vec{E} = \vec{E}_{hole} + \vec{E}_{plug}$ (uniformly polarized sphere)

$\vec{E}$ inside uniformly polarized sphere

$$\vec{E}_{plug} = \vec{E}_{in} - \vec{E}_0 = E_0\left(\frac{3}{\varepsilon+2} - 1\right)\hat{z} = \frac{1-\varepsilon}{2+\varepsilon} E_0 \hat{z} = -\frac{4\pi}{3}\vec{P}$$

$\vec{E}_{hole} = \vec{E} + \frac{4\pi}{3}\vec{P}$ $\quad \vec{p} = \gamma \vec{E}_{hole}$ $\quad n \; \frac{\#\text{ molecules}}{\text{volume}}$ $\quad \vec{P} = n\gamma \vec{E}_{hole}$

$\vec{P} = n\gamma\left(\vec{E} + \frac{4\pi}{3}\vec{P}\right)$ $\quad \chi = n\gamma\left(1 + \frac{4\pi}{3}\chi\right)$

$\chi\left(1 - \frac{4\pi}{3}n\gamma\right) = n\gamma$ $\quad \boxed{\chi = \frac{n\gamma}{1 - \frac{4\pi}{3}n\gamma}}$ Clausius–Mossotti

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Magnetostatics (Chapter 5)

$\nabla \cdot \vec{B} = 0$ $\quad \nabla \times \vec{B} = \frac{4\pi}{c}\vec{J}$ $\quad \nabla \cdot \vec{J} = 0$ $\quad \vec{B} = \nabla \times \vec{A}$

$\nabla \times (\nabla \times \vec{A}) = \frac{4\pi}{c}\vec{J}$ $\quad$ gauge: $\nabla \cdot \vec{A} = 0$ $\quad \left(\text{previously } \frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A} = 0\right)$

$\nabla^2 \vec{A} = -\frac{4\pi}{c}\vec{J}$

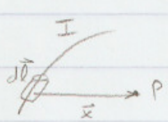
Solution: $\boxed{\vec{A}(\vec{x}) = \frac{1}{c}\int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'}$ in Cartesian coords.

$B_i = (\nabla \times A)_i = \varepsilon_{ikm}\partial_k A_m = \varepsilon_{ikm}\frac{1}{c}\int J_m(\vec{x}')\, \partial_k \frac{1}{|\vec{x} - \vec{x}'|} d^3x'$

$\partial_k \frac{1}{|\vec{x} - \vec{x}'|} = -\frac{(\vec{x} - \vec{x}')_k}{|\vec{x} - \vec{x}'|^3}$

acts on x (not x') $\frac{\partial}{\partial x_k}$

$\boxed{\vec{B} = \frac{1}{c}\int \frac{\vec{J}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} d^3x'}$ Biot–Savart Law



$d\vec{B} = \frac{I}{c}\frac{d\vec{l} \times (\vec{x})}{|\vec{x}|^3}$ integrate over wire

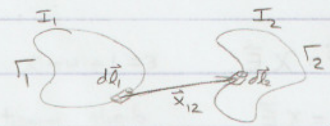
Special case: no currents in region of interest

$\nabla \cdot \vec{B} = 0$ $\quad \nabla \times \vec{B} = 0$ and boundary conditions

$\vec{B} = -\nabla \Phi_M \implies \nabla^2 \Phi_M = 0$

Force between loops of current $\quad \vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$

$$\vec{F} = \frac{1}{c}\int d^3x \left(\vec{J}(\vec{x}) \times B(\vec{x})\right)$$

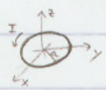


$$\vec{B} = \frac{I_1}{c}\int_{\Gamma_1} \frac{d\vec{l}_1 \times (\vec{x}_{12})}{|\vec{x}_{12}|^3}$$

$$\vec{F}_{12} = \frac{I_1 I_2}{c^2}\int_{\Gamma_2}\int_{\Gamma_1} \frac{d\vec{l}_2 \times \left(d\vec{l}_1 \times (\vec{x}_{12})\right)}{|\vec{x}_{12}|^3}$$

$\vec{B}$ from a circular current loop

Spherical $\hat{\Phi} = \hat{\varphi}$



$$\vec{J} = J\,\delta(r-a)\,\underbrace{\delta(\cos\theta)}_{\delta(\theta-\pi/2)}\,\hat{\varphi} \qquad I = \int J\,\delta(r-a)\,\delta(\cos\theta)\, r\, d(\cos\theta)\, dr$$

$$I = Ja \qquad J = \frac{I}{a}$$

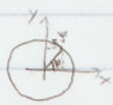
$$A_\varphi = \frac{1}{c}\int r'^2\, dr'\, \overbrace{d(\cos\theta')\, d\varphi'}^{d\Omega'}\, \frac{I}{a}\, \delta(r'-a)\, \delta(\cos\theta')\, [\;]^{-1/2}$$

Spherical: $\boxed{|\vec{x}-\vec{x}'| = \left(r^2 + r'^2 - 2rr'\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')\right)\right)^{1/2}}$

$$= \frac{Ia}{c}\int_0^{2\pi} d\varphi' \left(a^2 + r^2 - 2ar\left(\sin\theta\cos(\varphi-\varphi')\right)\right)^{-1/2}$$

WRONG since $\vec{A} = \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$ is in cartesian

$$\nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} \;\not\Leftrightarrow\; \overset{\text{Ex:}}{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_r}{\partial r}\right)} = \frac{4\pi}{c}J_r$$



$$\vec{J} = -J\sin\varphi\,\hat{x} + J\cos\varphi\,\hat{y}$$

Azimuthal symmetry so choose $\varphi = 0$
(calculate $\vec{A}$ on x-z plane)

$$A_z = 0 \qquad A_x \sim \int_0^{2\pi} d\varphi' \frac{\overset{\text{odd}}{\sin\varphi'}}{[\cdots \underset{\text{even}}{\cos\varphi'}]^{1/2}} = 0$$

$$A_y = \frac{Ia}{c}\int_0^{2\pi} d\varphi' \frac{\cos\varphi'}{\left(a^2 + r^2 - 2ar\sin\theta\cos\varphi'\right)^{1/2}} \qquad A_x = A_z = 0 \qquad A_y \to A_\varphi$$

$$\sim \left(1 - 2\frac{a}{r}\sin\right)^{-1/2}$$

far from loop $r \gg a$
$$A_\varphi = \frac{Ia}{c}\int_0^{2\pi} d\varphi' \frac{1}{r}\cos\varphi'\left(1 + \frac{a}{r}\sin\theta\cos\varphi'\right)$$

$$\int_0^{2\pi} d\varphi' \cos^2\varphi' = \pi \qquad A_\varphi = \frac{I\pi}{c}\frac{a^2}{r^2}\sin\theta \qquad \vec{B} = \nabla\times\vec{A}$$

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\varphi) = \frac{I\pi}{c}\frac{a^2}{r^3}2\cos\theta \qquad B_\theta = \frac{-1}{r}\frac{\partial}{\partial r}(rA_\varphi) = \frac{I\pi}{c}\frac{a^2}{r^3}\sin\theta \qquad B_\varphi = 0$$

Field of dipole $\boxed{\vec{m} = \pi a^2 \frac{I}{c}\hat{z}}$

$$B_r = 2m\frac{\cos\theta}{r^3} \qquad B_\theta = m\frac{\sin\theta}{r^3} \qquad B_\varphi = 0$$

$$m = \frac{1}{c}I\,(\text{area})$$

11/16/2006

Multipole Expansion (localized current distrib.)

$$\vec{A}(\vec{x}) = \frac{1}{c}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} \qquad \frac{1}{|\vec{x}-\vec{x}'|} \overset{|\vec{x}|\gg|\vec{x}'|}{=} \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + \dots$$

Monopole term

$$\vec{A}(\vec{x}) = \frac{1}{c}\frac{1}{|\vec{x}|}\int \vec{J}(\vec{x}')d^3x' = 0 \qquad \nabla\cdot\vec{J} = 0 \leftarrow \text{static } \frac{\partial J}{\partial t} = 0$$

Integration by parts

$$\int \vec{J}(\vec{x}')d^3x' = 0 \qquad \int x_j \frac{\partial J_i}{\partial x_i} d^3x = -\int J_i \frac{\partial x_j}{\partial x_i} d^3x + 0 = -\int J_j\, d^3x = 0$$

$$\vec{A}(\vec{x}) = \frac{1}{c}\frac{1}{|\vec{x}|^3}\int d^3x' (\vec{x}\cdot\vec{x}')\vec{J}(\vec{x}')$$

$$A_i = \frac{1}{c}\frac{\vec{x}}{|\vec{x}|^3}\cdot\int d^3x'\, \vec{x}' J_i = -\frac{1}{2}\left[\vec{x}\times\int(\vec{x}'\times\vec{J})d^3x'\right]_i \frac{1}{c}\frac{1}{|\vec{x}|^3}$$

$$\vec{m} = \frac{1}{2c}\int \vec{x}'\times\vec{J}(\vec{x}')\, d^3x' \qquad \therefore\ \vec{A}(\vec{x}) = \frac{\vec{m}\times\vec{x}}{|\vec{x}|^3} \quad \text{dipole term}$$

magnetic dipole moment

Define magnetization $\vec{M}$ (mag dipole/unit volume) $\boxed{\vec{M}(\vec{x}) \equiv \frac{1}{2c}(\vec{x}\times\vec{J})}$ $\quad \vec{m} = \int \vec{M}\, d^3x$

$$\vec{B} = \nabla\times\vec{A} = \frac{3\vec{x}(\vec{x}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}$$

Integrate by parts (first term) $\quad -\left(x_j J_k \frac{\partial x_i}{\partial x_k} + x_j x_i \overset{0}{\frac{\partial J_k}{\partial k}}\right)$

$$\int d^3x\left[x_i J_k \frac{\partial x_j}{\partial x_k} + x_j J_k \frac{\partial x_i}{\partial x_k}\right] = \int d^3x\left[-x_j\frac{\partial}{\partial x_k}(x_i J_k) + x_j J_k\frac{\partial x_i}{\partial x_k}\right] = 0$$

$$= \int d^3x\,(x_i J_j + x_j J_i) \leftarrow \text{stuff in } \frac{\partial}{\partial x_k} \qquad J_k\frac{\partial x_i}{\partial x_k} = \frac{\partial}{\partial x_k}(x_i J_k) - x_i \underbrace{\frac{\partial J_k}{\partial x_k}}_{\to 0}$$

$$\int(\vec{x}\cdot\vec{x}')\vec{J}\, d^3x \longrightarrow x_k\int d^3x'\, J_i x_k' = \frac{1}{2}x_k\int\underbrace{(J_i x_k' - J_k x_i')}_{J_m x_n'(\delta_{im}\delta_{kn} - \delta_{km}\delta_{in})} d^3x'$$

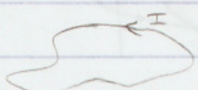
$$\therefore = \frac{1}{2}\varepsilon_{lik}\varepsilon_{lmn} x_k\int J_m x_n'\, d^3x' \qquad = \varepsilon_{lik}\varepsilon_{lmn}J_m x_n'$$

$$= -\frac{1}{2}\varepsilon_{ilk}\, x_k\int(\vec{J}\times\vec{x}')_l\, d^3x' = \frac{1}{2}\left(\vec{x}\times\int(\vec{J}\times\vec{x}')d^3x'\right)_i = \vec{x}\cdot\int\vec{x}' J_i\, d^3x$$

Gaussian units: $\quad [E] = [B] = [M] = [P]$

$$\vec{M} = \frac{1}{2c}(\vec{x}\times\vec{J}) \qquad [M] = \frac{t}{L} L \frac{e}{tL^2} = \frac{e}{L^2} = [E] \qquad \vec{E} = \frac{Qq}{r^2}\hat{r}$$

Exercise: for any plane current loop



$m = \frac{I}{c}$(area of loop) in direction $\perp$ loop

Energy of dipole $U = -\vec{m}\cdot\vec{B}$

Moving (discrete) charges

$$\vec{J} = \sum_i q_i \vec{v}_i\, \delta(\vec{x}-\vec{x}_i)$$

$$\vec{m} = \frac{1}{2c}\int d^3x \sum_i q_i (\vec{x}\times\vec{v}_i)\,\delta(\vec{x}-\vec{x}_i) = \frac{1}{2c}\sum_i q_i(\vec{x}_i\times\vec{v}_i)$$

$q_i = q$ $\quad m_i = m$ (mass) $\quad \therefore\ \vec{m} = \frac{q}{2cm}\sum_i(\vec{x}_i\times\vec{p}_i) = \frac{q}{2mc}\vec{L}$

total angular momentum

Relation between orbital angular momentum and $\vec{m}$: $\boxed{\vec{m} = \frac{q}{2mc}\vec{L}}$

Material acquires $\vec{M}$ in presence of $\vec{B}$

Contributions to $\vec{A}$: $\sim \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$ $\quad \sim \int d^3x' \frac{\vec{M}\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$

excess (free) currents $\quad$ internal dipoles

$$\vec{A}(\vec{x}) = \frac{1}{c}\int d^3x' \left\{ \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + c\,\frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} \right\}$$

$$= \frac{1}{c}\int d^3x' \left\{ \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + c\vec{M}(\vec{x}')\times\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) \right\}$$

$$\varepsilon_{ijk} M_j \partial_k \frac{1}{|\vec{x}-\vec{x}'|} \xrightarrow[\text{Parts}]{\text{Integrate}} -\varepsilon_{ijk}(\partial_k M_j)\frac{1}{|\vec{x}-\vec{x}'|} = \frac{\nabla'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$\vec{A}(\vec{x}) = \frac{1}{c}\int d^3x' \frac{\vec{J}(\vec{x}) + c\nabla'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$ solution to $\nabla^2\vec{A} = -\frac{4\pi}{c}(\vec{J} + c\nabla\times\vec{M})$

$\nabla\times\vec{B} = \frac{4\pi}{c}(\vec{J} + c\nabla\times\vec{M})$ $\qquad \vec{J}$: free current

$\vec{H} = \vec{B} - 4\pi\vec{M} \implies \nabla\times\vec{H} = \frac{4\pi}{c}\vec{J}$ $\qquad$ need to calculate $\vec{M}$ (relation between $\vec{B}$ & $\vec{H}$)

$\vec{B} = \mu\vec{H}$ for 1) paramagnetic materials $\mu > 1$ (permanent dipoles align to field)

$\xrightarrow{\quad} \vec{B}$
$\rightarrow$

2) diamagnetic materials $\mu < 1$ (no permanent dipoles)

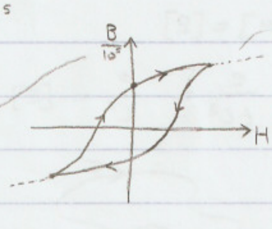
$\xrightarrow{\quad} \vec{B}$
$\leftarrow$

Small effect: $\mu = 1 \pm \sim 10^{-5}$

afterwards, keeps increasing with more H $\qquad \vec{H} = \vec{B} - \cancel{4\pi\vec{M}}$ constant

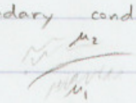
3) Ferromagnetic materials

"μ" $\sim 10^5$



no $B = F(H)$

Boundary conditions: B_n continuous H_t continuous for $(\vec{J}_f = 0)$



normal tangent

Solving Magnetostatic problems

1) $\vec{B} = \mu \vec{H}$ $\vec{B} = \nabla \times \vec{A}$ $\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J}$ $\nabla \times \vec{B} = \frac{4\pi}{c}\mu \vec{J}$

$\nabla^2 A = -\frac{4\pi}{c}\mu \vec{J}$

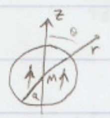
Special case $\vec{J} = 0$ $\nabla \times \vec{H} = 0$ $\vec{H} = -\nabla \Phi_M$ $\nabla^2 \Phi_M = 0$

2) Ferromagnets with fixed magnetization given $\vec{M}(\vec{x})$

$\vec{J} = 0$ $\nabla \cdot \vec{B} = 0$ $\nabla \cdot \vec{H} = -4\pi \nabla \cdot \vec{M}$ $\nabla \times \vec{H} = 0$ $\vec{H} = -\nabla \Phi_M$ $\rho_M = -\nabla \cdot \vec{M}$

$\nabla \cdot \vec{H} = 4\pi \rho_M$ $\nabla^2 \Phi_M = -4\pi \rho_M$ $\hat{n}$ σ_M $\sigma_M = \vec{M} \cdot \hat{n}\big|_S$

Problem: uniformly magnetized sphere 11/21/2006



uniform M inside $\vec{M} = M_0 \hat{z}$ inside, $\vec{M} = 0$ outside

$\vec{J} = 0$ no external currents

$\rho_M = -\nabla \cdot \vec{M} = 0$ in volume $\sigma_M = \vec{M} \cdot \hat{r} = M_0 \cos\theta$

$\nabla^2 \Phi_M = 0$ $H = -\nabla \Phi_M$ $\Phi_M = \int_{\text{sphere}} ds' \frac{\sigma_M(\vec{x}')}{|\vec{x} - \vec{x}'|}$

$$\Phi_M = a^2 \int d\Omega' \frac{\sigma_M(\theta')}{|\vec{x} - \vec{x}'|} \qquad \frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) Y^*_{lm}(\theta',\varphi')$$

Azimuthal symmetry $m=0$ $Y_{10} \sim \cos\theta$ $\Rightarrow$ only have $l=1$ $m=0$

$$\Phi_M(r,\theta,\varphi) = a^2 \frac{4\pi}{3} \frac{r_<}{r_>^2} \underbrace{\sqrt{\frac{3}{4\pi}} \cos\theta}_{Y_{10}} \int M_0 \cos\theta' \underbrace{\sqrt{\frac{3}{4\pi}} \cos\theta'}_{Y^*_{10}} d\Omega'$$

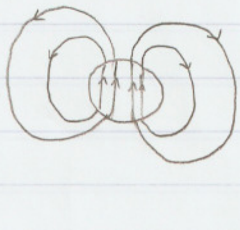
$\Phi_M = \frac{4\pi}{3} M_0 a^2 \frac{r_<}{r_>^2} \cos\theta$ inside $r_< = r$ $r_> = a$ $\Phi_M = \frac{4\pi}{3} M_0 r \cos\theta$ $\vec{H}_{in} = -\frac{4\pi}{3} M_0 \hat{z}$

$$\vec{B}_{in} = \vec{H}_{in} + 4\pi \vec{M}_{in} = -\frac{4\pi}{3} M_0 \hat{z} + 4\pi M_0 \hat{z} = \frac{8\pi}{3} M_0 \hat{z}$$

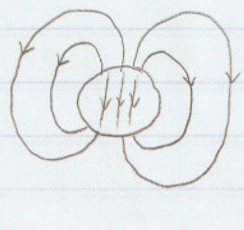
outside $r_< = a$ $r_> = r$

$\Phi_M = \frac{4\pi}{3} M_0 \frac{a^3}{r^2} \cos\theta$ dipole $\vec{m} = \frac{4\pi}{3} M_0 a^3 \hat{z}$ $\vec{B}_{out} = \vec{H}_{out}$

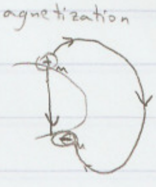
B lines



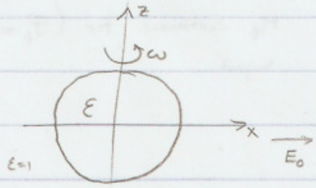
H lines



seem to originate from magnetization charges



Problem 6.8



$\vec{\omega} = \omega \hat{z}$

$\vec{E} = E_0 \hat{x}$

$J \propto 2q\omega\delta\cos\varphi = \omega p \cos\varphi$ (microscopic)

$\nabla\cdot\vec{D} = 4\pi\rho_{free}$ $\quad \boxed{-\nabla\cdot\vec{P} = \rho_{pol}}$ $\quad \vec{D} = \vec{E} + 4\pi\vec{P}$ $\quad \nabla\cdot\vec{E} = 4\pi(\rho_{free} + \rho_{pol})$

$P=0$, P: $\int_V \nabla\cdot\vec{P}\, dV = -q = \int_S \vec{P}\cdot\hat{n}\, dS = -\vec{P}\cdot\hat{n}\, A \Rightarrow \boxed{\vec{P}\cdot\hat{n} = \sigma_{pol}}$

$J = \sigma_{pol} V\, \delta(r-a)$ $\quad \vec{M} = \frac{1}{2c}\vec{x}\times\vec{J}$ $\quad \boxed{-\nabla\cdot\vec{M} = \rho_m}$ $\quad \boxed{\vec{M}\cdot\hat{n} = \sigma_m}$

$\Phi_M = a^2 \int \frac{\sigma_m}{|\vec{x}-\vec{x}'|}\, d\Omega'$

$\nabla\cdot\vec{E} = 4\pi\rho$ $\quad \nabla\cdot\vec{B} = 0$ $\quad \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ $\quad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

$\vec{B} = \nabla\times\vec{A}$ $\quad \vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ $\quad$ gauge $\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$

$\rightarrow$ gave us wave eq $\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\rho$

free space solution $\quad \Phi(\vec{x},t) = \int \frac{\rho\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}\, d^3x'$

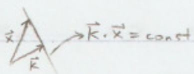
$\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B})$ energy current $\quad u = \frac{1}{8\pi}\left(|\vec{E}|^2 + |\vec{B}|^2\right)$ energy density

Plane Wave Solutions (in vacuum)

$\rho = 0 \quad \vec{J} = 0$ $\quad \nabla\times(\nabla\times\vec{E}) = -\nabla^2\vec{E} + \nabla(\cancel{\nabla\cdot\vec{E}})^0 = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\vec{B}) = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$

$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$ $\quad$ same for $\vec{B}$

$\nabla^2 u(\vec{x},t) - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$ $\quad$ Plane wave soln: $u(\vec{x},t) = A e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ $\quad |\vec{k}| = \frac{\omega}{c}$



$A \in \mathbb{C}$

11/28/2006

Maxwell's Eq. in vacuum

$\nabla\cdot\vec{E}=0 \qquad \nabla\cdot\vec{B}=0 \qquad \nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \nabla\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0$ same for $\vec{B}$

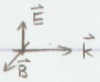
Plane wave solution: $\vec{E}=\vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad |\vec{k}|=\frac{\omega}{c}$

$\vec{B}=\vec{B}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \vec{E}_0,\vec{B}_0\in\mathbb{C}^3$

physical fields $Re(\vec{E})$

$\nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \implies i\vec{k}\times\vec{E}_0=-\frac{1}{c}(-i\omega\vec{B}_0) \qquad \vec{k}\times\vec{E}_0=\frac{\omega}{c}\vec{B}_0 \qquad \vec{k}=\frac{\omega}{c}\hat{k}$

$\vec{B}_0=\hat{k}\times\vec{E}_0 \qquad B\perp k,E$ also $\vec{k}\times\vec{B}_0=-\frac{\omega}{c}\vec{E}_0 \implies \vec{E}_0=-\hat{k}\times\vec{B}_0 \qquad E\perp k,B$



plane wave, in vacuum $|\vec{E}_0|=|\vec{B}_0|$

energy flow $\vec{S}=\frac{c}{4\pi}(\vec{E}\times\vec{B})$ energy current

time average $\langle\vec{S}\rangle=\frac{1}{2}\frac{c}{4\pi}Re\{\vec{E}\times\vec{B}^*\}$

$\langle\vec{S}\rangle=\frac{c}{8\pi}|\vec{E}_0|^2\hat{k} \qquad u=\frac{1}{8\pi}(|\vec{E}|^2+|\vec{B}|^2)$

$\langle\vec{S}\rangle=c\langle u\rangle \qquad \langle u\rangle=\frac{1}{2}\frac{1}{8\pi}2|\vec{E}_0|^2=\frac{1}{8\pi}|\vec{E}_0|$

energy transported at velocity c

$\vec{E}_0=(E_1,E_2,E_3) \qquad E_i\in\mathbb{C} \qquad |\vec{E}_0|=\vec{E}_0\cdot\vec{E}_0^*=\sum_k\left((E_k^r)^2+(E_k^i)^2\right)$

$E_1=E_1^r+iE_1^i$

physical $|\vec{E}|^2=|Re\{\vec{E}\}|^2=\left(\vec{E}=\vec{E}_0e^{i\omega t}\right)=|Re\{\vec{E}_0\}\cos\omega t-Im\{\vec{E}_0\}\sin\omega t|^2$

$(i\sin\omega t)(i\vec{E}_0^i)$

$\langle|\vec{E}|^2\rangle=\frac{1}{2}|Re\{\vec{E}_0\}|^2+\frac{1}{2}|Im\{\vec{E}_0\}|^2=\frac{1}{2}|\vec{E}_0|^2=\frac{1}{2}\vec{E}_0\cdot\vec{E}_0^*$

Rule: $\vec{Q}=\vec{A}e^{i\omega t} \qquad \langle\vec{Q}\rangle=\frac{1}{2}\vec{A}\cdot\vec{A}^*$

Plane waves in medium (ε,μ)

$\nabla\cdot\vec{D}=0 \qquad \nabla\times\vec{H}=\frac{1}{c}\frac{\partial\vec{D}}{\partial t} \qquad \vec{D}=\varepsilon\vec{E} \qquad \vec{B}=\mu\vec{H} \qquad \vec{P}=\chi\vec{E}$ more generally $P_i=\chi_{ik}E_k$

$\nabla^2\vec{H}-\frac{\varepsilon\mu}{c^2}\frac{\partial^2\vec{H}}{\partial t^2}=0 \qquad c\to\frac{c}{n} \qquad n=\sqrt{\varepsilon\mu}$ index of refraction

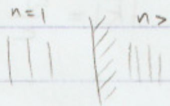
same for H, B, E, D

in terms of E, B: $\quad \nabla\cdot\vec{E}=0 \quad \nabla\cdot\vec{B}=0 \quad \nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \quad \nabla\times\vec{B}=\frac{\varepsilon\mu}{c}\frac{\partial\vec{E}}{\partial t}$

$\vec{E}=\vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad |\vec{k}|=n\frac{\omega}{c} \qquad \vec{E}=\vec{E}_0\, e^{i(n\vec{k}\cdot\vec{x}-\omega t)}$ $\quad$ same for $\vec{B}$

physical path $\longrightarrow$ optical path = physical path $\times\, n$

$\lambda=\frac{2\pi}{|\vec{k}|}$



shorter wavelength
same frequency

$i n\vec{k}\times\vec{B}_0=\frac{n^2}{c}\left(-i\omega\,\vec{E}_0\right) \qquad \vec{E}_0=-\frac{1}{n}\,\hat{k}\times\vec{B}_0 \qquad |\vec{B}_0|=n\,|\vec{E}_0|$

energy transport

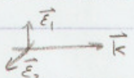
$\vec{S}=\frac{c}{4\pi}\left(\vec{E}\times\vec{H}\right) \qquad u=\frac{1}{8\pi}\left(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right)$

$\langle\vec{S}\rangle=\frac{c}{8\pi}\left(\vec{E}\times\vec{H}^*\right)=\frac{c}{8\pi}\frac{n}{\mu}|\vec{E}_0|^2\vec{k}$

$\langle u\rangle=\frac{1}{8\pi}\,\frac{1}{2}\left(\vec{E}\cdot\vec{D}^*+\vec{B}\cdot\vec{H}^*\right)=\frac{1}{8\pi}\,\frac{1}{2}\left(\varepsilon|\vec{E}_0|^2+\frac{\overset{\varepsilon\mu}{n^2}}{\mu}|\vec{E}_0|^2\right)=\frac{1}{8\pi}\,\varepsilon\,|\vec{E}_0|^2$

$\langle\vec{S}\rangle=\langle u\rangle$ (velocity) $\longrightarrow \frac{cn}{\mu\varepsilon}=\frac{c\,n}{n^2}=\frac{c}{n}$

Polarization



$|\vec{\varepsilon}_1|=|\vec{\varepsilon}_2|=1$

$\vec{E}_1=\vec{\varepsilon}_1\,E_1\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ (unit vector $\vec{\varepsilon}\in\mathbb{R}^3$) $\qquad \vec{E}_2=\vec{\varepsilon}_2\,E_2\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$\vec{E}=\left(\vec{\varepsilon}_1E_1+\vec{\varepsilon}_2E_2\right)e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad E_1, E_2\in\mathbb{C}$

Case a) E_1, E_2 have same phase $\qquad E_1=R_1e^{i\varphi} \quad R_1,\varphi\in\mathbb{R}$

$E_2=R_2e^{i\varphi}$

$\vec{E}=\left(R_1\vec{\varepsilon}_1+R_2\vec{\varepsilon}_2\right)e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ $\qquad$ Linear Polarization in direction $R_1\vec{\varepsilon}_1+R_2\vec{\varepsilon}_2$

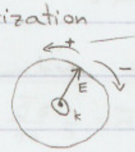
amplitude $=\left(R_1^2+R_2^2\right)^{1/2}$

$e^{i\pi/2}=i$

Case b) E_1, E_2 are $90°$ out of phase, $R_1=R_2=R \qquad E_1=Re^{i\varphi} \qquad E_2=Re^{i(\varphi\pm\pi/2)}$

$\vec{E}=\left(\vec{\varepsilon}_1\pm i\vec{\varepsilon}_2\right)R\,e^{i(\vec{k}\cdot\vec{x}-\omega t+\varphi)}$ $\qquad$ Circular Polarization

$\vec{x}=0 \quad \varphi=0 \qquad \mathrm{Re}\{\vec{E}\}=\vec{\varepsilon}_1R\cos(\omega t)\overset{i(-i)}{\pm}\vec{\varepsilon}_2R\sin(\omega t)$



\+ positive helicity $(S_k=1)$

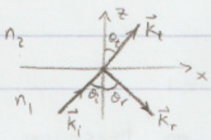
− negative helicity $(S_k=-1)$

angular momentum in k direction

Bases: $\quad \{\vec{\varepsilon}_1,\vec{\varepsilon}_2\} \quad$ or $\quad \{\vec{\varepsilon}_1\pm i\vec{\varepsilon}_2\}$

Case c) $\quad$ Elliptical Polarization

Reflection and Refraction



$n_i = \sqrt{\epsilon_i \mu_i}$

$\vec{E}_i = \vec{E}_0^i e^{i(n_1 \vec{k}_i \cdot \vec{x} - \omega t)}$ $\quad \vec{B}_i = n_1 \hat{k}_i \times \vec{E}_0^i$ $\quad \vec{E}_r = \vec{E}_0^r e^{i(n_1 \vec{k}_r \cdot \vec{x} - \omega t)}$ $\quad \vec{E}_t = \vec{E}_0^t e^{i(n_2 \vec{k}_t \cdot \vec{x} - \omega t)}$

B.C.s: D_n, B_n cont. $\quad E_t, H_t$ cont.

frequency is same $\omega_i = \omega_r = \omega_t = \omega$ to satisfy BC $\forall\, t$ @ $z=0$

$n_1 k_{ix} = n_1 k_{rx} = n_2 k_{tx}$ to satisfy BC $\forall\, x, y$ @ $z=0$

$n_1 |\vec{k}_i| \sin\theta_i = n_1 |\vec{k}_r| \sin\theta_r = n_2 |\vec{k}_t| \sin\theta_t \qquad |\vec{k}_i| = |\vec{k}_r| = |\vec{k}_t| = \frac{\omega}{c}$

$\boxed{\Rightarrow \theta_i = \theta_r \qquad n_1 \sin\theta_i = n_2 \sin\theta_t}$

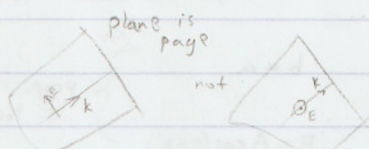
Reflection $\qquad$ Snell's law

11/30/2006

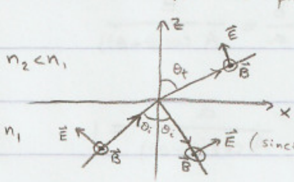
Friday Dec 8 ~3pm class $\qquad$ Final Dec 12 3-6pm $\qquad$ No Tues. class

HW 6: 5.6, 5.13, 5.17, 7.2, 7.28

Polarization $\parallel$ to plane of incidence



← Polarization $\perp$ to plane



$\vec{E}$ (since $n_2 < n_1$) $\rightarrow$ look at E_x component $\qquad \overleftarrow{t} = \overleftarrow{i} + \overrightarrow{r}$ $\qquad$ no phase change

phase difference if $n_2 > n_1 \rightarrow \pi$ phase change in reflected wave

E_t cont: $\quad E_0^i \cos\theta_i - E_0^r \cos\theta_i = E_0^t \cos\theta_t$

H_t cont: $\quad \frac{n_1}{\mu_1} E_0^i + \frac{n_1}{\mu_1} E_0^r = \frac{n_2}{\mu_2} E_0^t$ $\qquad R = \frac{E_0^r}{E_0^i} \qquad T = \frac{E_0^t}{E_0^i}$

reflection and transmission coefficients

$+R\cos\theta_i + T\cos\theta_t = +\cos\theta_i$

$\mu_1 = \mu_2 = 1 \qquad R n_1 - T n_2 = -n_1$

$$\boxed{R_{\parallel} = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} \qquad T_{\parallel} = \frac{2 n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t}}$$

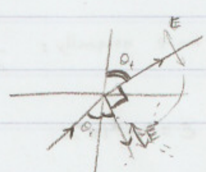
Brewster Angle

$R_{\parallel} = 0$ (not $R_\perp$) $\quad \therefore \vec{E}$ polarized $\perp$ to plane

$n_2 \cos\theta_i = n_1 \cos\theta_t \Rightarrow \boxed{\tan\theta_B = \frac{n_2}{n_1}}$

$\theta_i + \theta_t = \pi/2$

$\left.\begin{array}{l}\sin\theta_t = \cos\theta_i \\ \sin\theta_i = \cos\theta_t\end{array}\right\}$ Snell's law $\rightarrow n_2 \cos\theta_i = n_1 \cos\theta_t$



light produced from oscillations in materials

Total Internal Reflection

$\theta \equiv \theta_i \qquad \theta' \equiv \theta_t \qquad n_1 > n_2$

$$\sin\theta' = \frac{n_1}{n_2}\sin\theta$$

Critical angle $\theta_c \rightarrow \sin\theta_c = \frac{n_2}{n_1} < 1$

such that $\theta' = \pi/2$ no transmission

total internal reflection

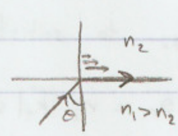
$\theta > \theta_c \quad \sin\theta' > 1 \quad \theta'$ is complex

$\sin\theta' \in \mathbb{R} \qquad \cos\theta' = \sqrt{1 - \underbrace{\sin^2\theta'}_{>1}} = i\sqrt{\sin^2\theta' - 1} \qquad \cos\theta'$ is pure imaginary

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\vec{E}_t = \vec{E}_0^t \exp\left\{ i\left(n_2 k x \sin\theta_t + n_2 k z \underbrace{\cos\theta_t}_{\text{imaginary}} - \omega t\right)\right\} \rightarrow e^{-n_2 k z\sqrt{\sin^2\theta_t - 1}}$$

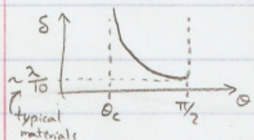
$$\vec{E}_t = \vec{E}_0^t e^{i(n_1 k \sin\theta x - \omega t)} \underbrace{e^{-zk\sqrt{n_1^2\sin^2\theta - n_2^2}}}_{\text{damped}}$$



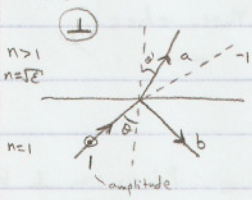
$E_t \sim e^{-\frac{z}{2\delta}}$ penetration length δ

2 since energy $\propto |\vec{E}|^2$

$$2\delta = \frac{1}{k\sqrt{n_1^2\sin^2\theta - n_2^2}} \qquad k = \frac{2\pi}{\lambda} \qquad \delta = \frac{\lambda}{4\pi\sqrt{n_1^2\sin^2\theta - n_2^2}}$$

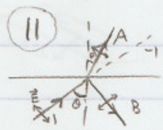


1 way to focus wave in a plane



wave radiated by material to cancel incident wave

3 ways materials radiate: a, b, -1



$b \propto a$

$B \propto A \underbrace{\cos(\theta+\theta')}_{\text{need component in dir. of B}}$

prop. constant is the same $\Rightarrow \frac{b}{a} = \frac{B}{A\cos(\theta+\theta')}$

$\left.\begin{array}{l} -1 \propto a \\ -1 \propto A\cos(\theta-\theta') \end{array}\right\} \frac{a}{A\cos(\theta-\theta')} = 1$

Energy conservation $\quad 1 = \varepsilon a^2 + b^2$

$1 = \varepsilon A^2 + B^2$

$$b^2 = \frac{\sin^2(\theta-\theta')}{\sin^2(\theta+\theta')} = R_\perp^2$$

Dispersion

model for $\varepsilon(\omega)$: harmonically bound charge

$$\ddot{x} + \underbrace{\gamma\dot{x}}_{\substack{\text{damping} \\ \text{to keep resonance finite}}} + \omega_0^2 x = -\frac{e}{m}E(x,t) \qquad E(x,t) = E_0 e^{-i\omega t} \quad \left(\text{range of variation of } x \ll \lambda\right)$$

$E(x,t)$ ← field of wave

$x \sim e^{-i\omega t} \qquad x_0(-\omega^2 - i\gamma\omega + \omega_0^2) = -\frac{eE_0}{m} \qquad x_0 = \frac{-eE_0}{m}\frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)}$

dipole moment

$$p = -ex_0 = \frac{e^2}{m}\frac{E_0}{(\omega_0^2 - \omega^2 - i\gamma\omega)} \qquad \vec{P} = \chi\vec{E} \qquad \chi(\omega) = N\frac{e^2}{m}\frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

N: # charges per unit volume

more generally: $\chi(\omega) = \sum_j \frac{N\, e^2/m\, f_j}{\omega_j^2 - \omega^2 - i\gamma\omega}$ $\qquad \sum_j f_j = Z$ ← # e^- in atom

f_j: oscillator strength

$\varepsilon = 1 + 4\pi\chi$

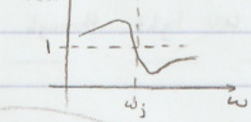
$$\varepsilon = 1 + 4\pi\chi = 1 + \sum_j \frac{4\pi Ne^2/m}{\omega_j^2 - \omega^2 - i\omega\gamma} f_j$$

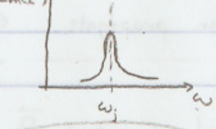
$\gamma \ll \omega_j$ typically

$\therefore$ ε is almost real except at resonances where $\omega = \omega_j$

$\omega \ll \omega_j$ $\quad Re(\varepsilon) > 1$

$\omega \gg \omega_j$ $\quad Re(\varepsilon) < 1$





12/7/2006

Final Exam comprehensive up to reflection/refraction

no dispersion

Dispersion

$$\varepsilon(\omega) = 1 + \sum_j \frac{4\pi N \frac{e^2}{m} f_j}{\omega_j^2 - i\gamma\omega - \omega^2}$$

charges/unit volume

$\sum_j f_j = Z$ — # e^- in atom; f_j: oscillator strength

large $Im\{\varepsilon\} \rightarrow$ absorption

$n = \sqrt{\varepsilon}$ $\qquad E \sim e^{i(nkx - \omega t)}$

$n = a + ib \qquad n^2 = a^2 - b^2 + 2iab$

$Re\{\varepsilon\} = a^2 - b^2 \qquad Im\{\varepsilon\} = 2ab$

$E \sim e^{i(akx - \omega t)} e^{-bkx}$ $\qquad$ attenuation length $\ell = \frac{1}{bk} = \frac{\lambda \, 2\sqrt{Re\{\varepsilon\}}}{2\pi \, Im\{\varepsilon\}}$

$b \ll a \qquad Re\{\varepsilon\} \simeq a^2 \qquad Im\{\varepsilon\} \simeq 2\sqrt{Re\{\varepsilon\}}\, b$

Relation to DC conductivity:

$\omega \rightarrow 0$ $\quad$ a) $\omega_j \neq 0 \;\; \forall j$ (no free e^-) $\qquad \varepsilon \rightarrow 1 + \sum_j \frac{4\pi N e^2/m}{\omega_j^2} f_j$ real, >1

b) free electrons $\omega_0 = 0$ $\qquad \varepsilon = 1 + \underbrace{\sum_{\omega_j \neq 0} \frac{4\pi N e^2/m \, f_j}{\omega_j^2}} - \frac{4\pi N e^2/m \, f_0}{\omega(\omega + i\gamma)}$

$$\varepsilon = \varepsilon_0 + i\frac{4\pi N e^2/m \, f_0}{\omega(\gamma - i\omega)}$$

singular for $\omega \rightarrow 0$

1) $\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$

$\vec{J} = \sigma\vec{E} \qquad \vec{D} = \varepsilon_0 \vec{E}$

or : 2) $\nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \qquad \vec{D} = \varepsilon\vec{E}$ — includes singular part

$$\frac{4\pi}{c}\sigma\vec{E} + \frac{1}{c}\varepsilon_0(-i\omega)\vec{E} = \frac{-i\omega}{c}\left(\varepsilon_0 + i\frac{4\pi\sigma}{\omega}\right)\vec{E} = \frac{-i\omega}{c}\varepsilon\vec{E}$$

$\varepsilon = \varepsilon_0 + i\frac{4\pi}{\omega}\sigma \qquad \therefore \boxed{\sigma = \frac{Ne^2 f_0}{m(\gamma - i\omega)}}$

singularity for $Im\{\varepsilon\}$ for $\omega \rightarrow 0$ has strength $4\pi\sigma$

No damping $(\gamma=0)$ ex: Plasma

$$\varepsilon = \varepsilon_0 - \frac{4\pi N e^2}{m\omega^2} = \varepsilon_0 - \frac{\omega_p^2}{\omega^2}$$ $\boxed{\text{Plasma frequency} \quad \omega_p^2 = \frac{4\pi N e^2}{m}}$

$\omega \ll \omega_p$ $\quad \varepsilon < 0 \quad n = \sqrt{\varepsilon} \rightarrow$ imaginary no wave propagation : reflected

$\omega > \omega_p$ wave can propagate | ex: UV light through metals

---

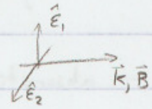
Plasma + External Constant $\vec{B}$

Special case: $\hat{k} \parallel \vec{B}$

$$m\ddot{\vec{x}} = q\left(\underset{\uparrow \text{ wave}}{\vec{E}} + \frac{\dot{\vec{x}}}{c} \times \underset{\uparrow \text{ external}}{\vec{B}}\right)$$ $\vec{B}$ of wave is small $\sim \frac{v}{c}$.

Ansatz $\quad \vec{E} = E_0(\hat{\varepsilon}_1 \pm i\hat{\varepsilon}_2)\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ circularly polarized wave

$$\simeq E_0(\hat{\varepsilon}_1 \pm i\hat{\varepsilon}_2)\, e^{-i\omega t} \qquad (\lambda \gg x)$$



$$\vec{x} = x_0(\hat{\varepsilon}_1 \pm i\hat{\varepsilon}_2)\, e^{-i\omega t}$$

$$m\ddot{\vec{x}} - \frac{q}{c}(\dot{\vec{x}} \times \vec{B}) = qE_0(\hat{\varepsilon}_1 \pm i\hat{\varepsilon}_2)\, e^{-i\omega t}$$

$$(\hat{\varepsilon}_1 \pm i\hat{\varepsilon}_2) \times \vec{B} = \pm i(\hat{\varepsilon}_1 \pm i\hat{\varepsilon}_2) B$$

$$x_0\left(-m\omega^2 \mp q\frac{\omega}{c}B\right) = qE_0 \qquad x_0 = \frac{qE_0}{-m\omega^2 \mp q\frac{\omega}{c}B}$$

$$\vec{x} = \frac{-q}{m\omega\left(\omega \pm \frac{qB}{mc}\right)}\vec{E} \qquad q = -e \qquad \boxed{\frac{eB}{mc} = \omega_L \quad \text{Larmor frequency}}$$

$$\vec{x} = \frac{e}{m\omega(\omega \mp \omega_L)}\vec{E} \qquad \varepsilon = 1 - 4\pi\frac{e^2 N}{m\omega(\omega \mp \omega_L)}$$

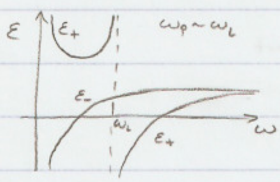
N: # charges/unit volume

$$\varepsilon_\pm = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_L)}$$

Birefringent
2 indeces of refraction
(for positive and negative helicity)

$\omega \to 0 \quad \varepsilon_\pm \to 1 \pm \frac{\omega_p^2}{\omega\omega_L}$

$\omega \to \omega_L \quad \varepsilon_- = 1 + \frac{\omega_p^2}{2\omega\omega_L} \qquad \varepsilon_+ = 1 + \frac{\omega_p^2}{\omega(\omega - \omega_L)} \rightarrow +\infty,\ -\infty$



upper sign: ε_+ positive helicity
lower sign: ε_- negative helicity

$\omega \to 0 \quad \varepsilon_+ \simeq \frac{\omega_p^2}{\omega\omega_L} \qquad n = \sqrt{\varepsilon} \simeq \frac{\omega_p}{\sqrt{\omega\omega_L}} \qquad V_{phase} = \frac{1}{n}\frac{\omega}{c} = \frac{\omega}{c}\frac{\sqrt{\omega\omega_L}}{\omega_p}$

$V_g = \frac{d\omega}{dk} \qquad n\frac{\omega}{c} = k \qquad k = \frac{\omega}{c}\frac{\omega_p}{\sqrt{\omega\omega_L}} \qquad \frac{1}{V_g} = \frac{dk}{d\omega} = \frac{1}{2}\frac{\omega_p}{\sqrt{\omega\omega_L}}\frac{1}{c} \qquad V_g \to 0$ as $\omega \to 0$
$\sim \sqrt{\omega}$

Page 39 and 40 are the original division in my notes for Phys 210A and 210B.
They contain no information and are not included here.

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1/9/2007

HW 1 Chpt 7: 12, 13, 22, 23

Dispersion

$\varepsilon = \varepsilon(\omega)$

Model: harmonically bound charges

$$\varepsilon = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \qquad \omega_p^2 = \frac{4\pi N e^2}{m}$$

a) no free charges $\omega_0 \neq 0$ resonances $\omega \to \omega_0$ absorption (large $\mathrm{Im}(\varepsilon)$)

b) free charges $\omega_0 = 0$ no damping (plasma) $\gamma = 0$ $\varepsilon \simeq 1 - \frac{\omega_p^2}{\omega^2}$

for $\omega < \omega_p$ wave is excluded ($\varepsilon < 0$ $n = \sqrt{\varepsilon} \to$ pure imaginary)

with damping $\gamma \neq 0$ $\varepsilon = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega}$ has $\frac{1}{\omega}$ singularity

for $\omega \to 0$, related to conductivity σ $\vec{J} = \sigma \vec{E}$

$$\varepsilon = 1 + i\frac{4\pi\sigma}{\omega} \qquad \sigma = \sigma(\omega)$$

$\sigma \to$ constant as $\omega \to 0$ (DC conductivity)

EM wave propagation in a plasma with $\vec{B}$

$$\varepsilon_{\pm} = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_L)}, \qquad \omega_L = \frac{eB}{mc} \quad \text{Larmor freq.}$$

ionosphere is birefringent $\varepsilon_+ \neq \varepsilon_-$ $\to$ positive helicity $\to$ negative helicity

EM wave in conductive medium

$$\varepsilon = \varepsilon_0 + i\frac{4\pi\sigma}{\omega} \qquad n = \sqrt{\varepsilon} \qquad E \sim e^{i(nkx - \omega t)} \qquad k = \frac{\omega}{c}$$

$\to$ attenuation length good conductor $\frac{\sigma}{\omega} \gg 1$ poor conductor $\frac{\sigma}{\omega} \ll 1$

$\to$ skin depth $\delta = \frac{c}{\sqrt{2\pi\sigma\omega}}$

mag. field $\gg$ electric field $\frac{|\vec{H}_0|}{|\vec{E}_0|} = |n| \sim \sqrt{\frac{\sigma}{\omega}} \gg 1$ (for good conductors)

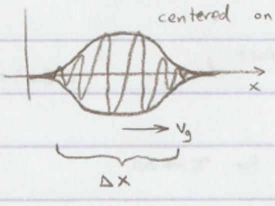
$n = n(\omega)$ $\quad E \sim e^{i(n(\omega)k_0 x - \omega t)}$ $\quad k_0 = \frac{\omega}{c}$ $\quad n(\omega)k_0 = k(\omega)$

$E \sim e^{i(kx - \omega(k)t)}$ $\quad k = \frac{n\omega}{c}$ $\quad \omega(k)$ dispersion relation

phase velocity $v_p = \frac{\omega}{c}$

group velocity $v_g = \left.\frac{d\omega}{dk}\right|_{k_0}$

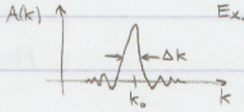
centered on wavevector k_0



E or B

$$u(x,t) = \int \frac{1}{\sqrt{2\pi}} A(k)\, e^{i(kx - \omega(k)t)}\, dk$$

Ex. for A(k)



$\boxed{\Delta x \Delta k \sim 1}$ fundamental relation

width of pulse

weights $|u|^2$, $|A|^2$

$$\langle x \rangle = \int x\, |u|^2 dx \qquad \langle k \rangle = \int k\, |A|^2 dk$$

$\Delta x = (\langle (x - \langle x \rangle)^2 \rangle)^{1/2}$ similar for Δk $\quad \therefore \boxed{\Delta x \Delta k \geq \tfrac{1}{2}}$ like in QM

$\Delta x \Delta k = \tfrac{1}{2}$ for Gaussian packets $\quad u \sim e^{-\frac{x^2}{2\sigma^2}} e^{i k_0 x}$

taking $\Delta x = \sigma$

$$A(k) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \underbrace{e^{-ikx} e^{ik_0 x}}_{e^{-i(k-k_0)x}} e^{-\frac{x^2}{2\sigma^2}} dx \sim e^{-2\sigma^2 (k-k_0)^2}$$

(Fourier Transform)

$$\mathcal{F}\left\{ e^{-\frac{x^2}{2\sigma^2}} \right\} \sim e^{-2\sigma^2 k^2}$$

$e^{-\frac{k^2}{2(\Delta k)^2}}$ $\quad 2\sigma^2 = \frac{4\sigma^2}{2} \quad \therefore \Delta k = \frac{1}{2\sigma}$

$$\Delta x \Delta k = \sigma \frac{1}{2\sigma} = \frac{1}{2}$$

Kramers – Kronig Relations

$$c(\omega) = a(\omega)\, b(\omega) \qquad C(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} c(\omega) e^{-i\omega t} d\omega \qquad A(t),\ B(t)$$

convolution $\quad C(t) = \int_{-\infty}^{\infty} A(\tau)\, B(t-\tau)\, d\tau$

$$c(\omega) = \int C(t) e^{i\omega t} dt \qquad a(\omega) b(\omega) = \int A(t) e^{i\omega t} dt \int B(t') e^{i\omega t'} dt'$$

$$C(t) = \int dt \int dt'\, A(t) B(t') \underbrace{\int \frac{d\omega}{2\pi} e^{i\omega(t+t'-t'')}}_{\delta(t+t'-t'')} = \int_{-\infty}^{\infty} dt\, A(t)\, B(t''-t)$$

$$\vec{D}(\vec{x},\omega) = \varepsilon(\omega) \vec{E}(\vec{x},\omega) \qquad \varepsilon = 1 + 4\pi \chi(\omega)$$

$$\vec{D}(\vec{x},\omega) = \vec{E}(\vec{x},\omega) + 4\pi \chi(\omega) \vec{E}(\vec{x},\omega) \qquad \vec{D}(\vec{x},\omega) = \int \vec{D}(\vec{x},t) e^{+i\omega t} dt$$

$$\vec{D}(\vec{x},t) = \vec{E}(\vec{x},t) + \int_{-\infty}^{\infty} G(\tau)\, \vec{E}(\vec{x}, t-\tau)\, d\tau \qquad G = \mathcal{F}^{-1}\{4\pi \chi(\omega)\}$$

casuality $\quad G(\tau) = 0$ for $\tau < 0$ so that D(now) doesn't depent on E(future)

$$G(\tau) = \int \frac{1}{2\pi} (\varepsilon(\omega) - 1) e^{-i\omega\tau} d\omega \qquad \varepsilon(\omega) = 1 + \int G(\tau) e^{i\omega\tau} d\tau$$

G is finite $\quad G \to 0$ for $\tau \to \infty$

$\to \varepsilon$ analytic function in $\mathrm{Im}(\omega) \geq 0$

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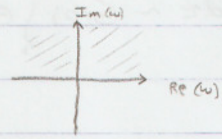
Kramers – Kronig

$$\varepsilon(\omega) = 1 + \int_{-\infty \to 0}^{\infty} G(\tau)\, e^{i\omega\tau}\, d\tau \qquad G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\varepsilon(\omega)-1)\, e^{-i\omega\tau}\, d\tau$$

$G(\tau) = 0$ for $\tau < 0$ $\qquad \vec{D}(\omega) = \varepsilon(\omega)\vec{E}(\omega)$

$$\vec{D}(t) = \vec{E}(t) + \int_{-\infty}^{\infty} G(\tau)\,\vec{E}(t-\tau)\, d\tau$$



with $Im(\omega) \neq 0$ $\quad e^{i\omega t} \sim e^{-Im(\omega)t}\, e^{iRe(\omega)t}$ $\quad \therefore \int_0^{\infty} G e^{i\omega\tau} \to$ converges

exponential damping

on real axis ($Im(\omega)=0$) need $G \to 0$ for $\tau \to \infty$

$\varepsilon(\omega)$ is analytic for $Im(\omega) \geq 0$

Analytic function $\quad W(z) = U(x,y) + iV(x,y) \qquad z = x + iy$

↳ derivative exists in any direction

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{dw}{dz} = \lim_{\Delta x \to 0} \frac{W(z+\Delta x) - W(z)}{\Delta x} = \lim_{\Delta y \to 0} \frac{W(z+i\Delta y) - W(z)}{i\Delta y}$$

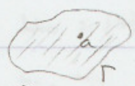
$$\frac{1}{\Delta x}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x = \frac{1}{i\Delta y}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y$$

Real: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

Im: $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

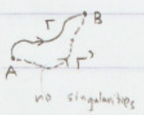
Cauchy: $\quad f(z)$ analytic (no singularities) inside and on Γ

$$\oint_\Gamma \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$$



$$\oint_\Gamma \frac{f(z)}{z-a}\, dz = \begin{cases} 2\pi i\, f(a) & a \text{ inside } \Gamma \\ \pi i\, f(a) & a \text{ on } \Gamma \quad \text{(Principal Part)} \\ 0 & a \text{ outside } \Gamma \end{cases}$$

f analytic $\quad \oint_\Gamma f(z)\, dz = 0$



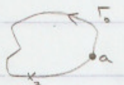
$$\int_{A\,\Gamma}^{B} f(z)\, dz = \int_{A\,\Gamma'}^{B} f(z)\, dz$$

$$\oint_\Gamma \frac{f(z)}{z-a}\, dz = \oint_\gamma \frac{f(z)}{z-a}\, dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{\cancel{re^{i\theta}}}\, \cancel{r}\, i\, \cancel{e^{i\theta}}\, d\theta$$

$z - a = re^{i\theta} \qquad dz = ri\, e^{i\theta} d\theta$

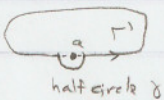
$$= i\int_0^{2\pi} f(a+re^{i\theta})\, d\theta \quad \text{take } \lim_{r\to 0} \Rightarrow 2\pi i\, f(a)$$

Taking the Principal Part



$$\oint_{\Gamma_0} \frac{f(z)}{z-a}\, dz$$

$$= \lim_{\delta\to 0}\left(\int_{z_0}^{a-\delta} (\;) + \int_{a+\delta}^{z_0} (\;)\right)$$



$\Gamma' + \gamma = \Gamma$

$$\oint_\Gamma (\;) = 2\pi i\, f(a) = \int_{\Gamma'} (\;) + \int_\gamma (\;)$$

$\lim_{\delta\to 0}$: principal part $\qquad \int_\gamma \to \pi i f(a)$

$\therefore\ \pi i f(a)$

$$\oint_{\Gamma_0} \frac{f(z)}{z-a}\, dz = \pi i f(a)$$

$$\oint_\Gamma \frac{\varepsilon(\omega)-1}{\omega-a}\,d\omega = 2\pi i\,(\varepsilon(a)-1)$$

(ω) big circle Γ

$$= \int_{\text{Circle}} (\ ) + \int_{-\infty}^{\infty} \frac{\varepsilon(\omega)-1}{\omega-a}\,d\omega$$

(first term → 0; second term: real axis)

$\varepsilon(\omega)-1$ $\quad \omega = i\alpha \quad \alpha\in\mathbb{R}$

$\hookrightarrow = \frac{1}{\alpha}$ for large α

$$\varepsilon(\omega)-1 = \int_0^\infty G(\tau)e^{i\omega\tau}d\tau \sim \int_0^\infty G(\tau)e^{-\alpha\tau}d\tau \sim G(0)\int_0^\infty e^{-\alpha t}d\tau \sim G(0)\frac{1}{\alpha}$$

($\alpha\to\infty$)

R: radius of circle

$\frac{\varepsilon(\omega)-1}{\omega} \sim \frac{1}{R^2} \qquad R = |\omega| \qquad \therefore \int_{\text{Circle}} \frac{1}{R^2} \to 0$ as $R\to\infty$

on real axis $\quad \varepsilon - 1 \to 0$ for $\omega\to\infty$ at least as $1/\omega^2$

$\omega\in\mathbb{R} \qquad \varepsilon(\omega) = 1 + \frac{1}{i\pi}P\left(\int_{-\infty}^{\infty}\frac{\varepsilon(\omega')-1}{\omega'-\omega}d\omega'\right)$

(P: principal part)

$$\boxed{\mathrm{Re}\{\varepsilon(\omega)\} = 1 + \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\{\varepsilon(\omega')\}}{\omega'-\omega}d\omega'} \qquad \boxed{\mathrm{Im}\{\varepsilon(\omega)\} = -\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Re}\{\varepsilon(\omega')\}-1}{\omega'-\omega}d\omega'}$$

Kramers-Kronig Relations

Complex conjugates

$\varepsilon^*(\omega^*) = \varepsilon(-\omega)$ from $\varepsilon(\omega) = 1 + \int_0^\infty G(\tau)e^{i\omega\tau}d\tau$

$\mathrm{Re}\{\varepsilon\}$ is even in ω $\quad \mathrm{Im}\{\varepsilon\}$ is odd

$$\frac{\mathrm{Im}(\varepsilon)\,(\omega'+\omega)}{(\omega'-\omega)(\omega'+\omega)} = \frac{\overbrace{\mathrm{Im}(\varepsilon)}^{\text{even}}\overbrace{(\omega'+\omega)}^{\text{odd}}}{\underbrace{\omega'^2-\omega^2}_{\text{even}}} \Rightarrow \boxed{\mathrm{Re}\{\varepsilon(\omega)\} = 1 + \frac{2}{\pi}P\int_0^\infty\frac{\omega'\,\mathrm{Im}\{\varepsilon(\omega')\}}{\omega'^2-\omega^2}d\omega'}$$

$$\frac{\overbrace{(\mathrm{Re}(\varepsilon)-1)}^{\text{even}}\overbrace{(\omega'+\omega)}^{\text{odd even}}}{\underbrace{\omega'^2-\omega^2}_{\text{even}}} \Rightarrow \boxed{\mathrm{Im}\{\varepsilon(\omega)\} = -\frac{2\omega}{\pi}P\int_0^\infty\frac{\mathrm{Re}\{\varepsilon(\omega')\}-1}{\omega'^2-\omega^2}d\omega'}$$

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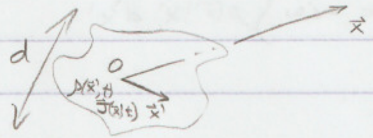
Generation of E.M. Waves by Oscillating Sources

Localized sources



$$\rho(\vec{x},t)=\rho(\vec{x})e^{-i\omega t} \qquad J(\vec{x},t)=J(\vec{x})e^{-i\omega t}$$

$$\vec{A}(\vec{x},t)=\frac{1}{c}\int d^3x'\int dt'\ \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\,\delta\left[t'-\left(t-\frac{|\vec{x}-\vec{x}'|}{c}\right)\right]$$

$$=\frac{1}{c}\int d^3x'\ \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}e^{-i\omega t}e^{ik|\vec{x}-\vec{x}'|} \qquad \frac{\omega}{c}=k \qquad \vec{A}(\vec{x},t)=\vec{A}(\vec{x})e^{-i\omega t}$$

$$\hookrightarrow =\frac{1}{c}\int d^3x'\ \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}e^{ik|\vec{x}-\vec{x}'|}$$

$\underline{d \ll \lambda}$ $\quad \lambda=\frac{2\pi}{k}=\frac{2\pi c}{\omega}$ long wavelength limit

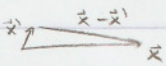
1) $d \ll |\vec{x}| \ll \lambda$ Near field zone static fields
2) $d \ll |\vec{x}| \sim \lambda$ Intermediate zone
3) $d \ll \lambda \ll |\vec{x}|$ Radiation zone

Near zone $\quad |\vec{x}-\vec{x}'| \ll \lambda \quad k|\vec{x}-\vec{x}'| \ll 1 \quad \therefore\ e^{ik|\vec{x}-\vec{x}'|}\simeq 1$

$$\vec{A}(\vec{x},t)=\underbrace{\frac{1}{c}\int d^3x'\ \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}}_{\text{static solution}}\ e^{-i\omega t}$$

oscillating at ω

Radiation zone $\quad |\vec{x}| \gg |\vec{x}'| \quad r=|\vec{x}| \quad \frac{1}{|\vec{x}-\vec{x}'|}\simeq\frac{1}{r}$



$|\vec{x}-\vec{x}'|\simeq r-\vec{x}'\cdot\hat{n}$

unit vector in $\vec{x}$ dir : $\hat{n}=\frac{\vec{x}}{|\vec{x}|}$

$k|\vec{x}-\vec{x}'|=kr-k\hat{n}\cdot\vec{x}'$

$$\vec{A}(\vec{x})=\frac{1}{c}\frac{e^{ikr}}{r}\int\vec{J}(\vec{x}')e^{-ik\hat{n}\cdot\vec{x}'}d^3x'$$

"spherical wave" with direction-dependant amplitude

$\vec{B}=\nabla\times\vec{A}$ $\qquad \nabla\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$ $\quad (\vec{J}=0$ here$)$ $\qquad \vec{E}=\frac{i}{k}\nabla\times\vec{B}$

$$=-i\frac{\omega}{c}\vec{E}$$

$$e^{-ik\vec{x}'\cdot\hat{n}}\simeq 1-ik\hat{n}\cdot\vec{x}'$$

electric dipole radiation $\qquad$ magnetic dipole + electric quadrupole radiation

No monopole radiation : monopole can't 'pulsate' and conserve charge

Electric Dipole $\quad \vec{A}(\vec{x}) = \frac{1}{c}\frac{e^{ikr}}{r}\underbrace{\int \vec{J}(\vec{x}')\, d^3x'}_{\text{not zero} \;\rightarrow\; \nabla\cdot\vec{J} - i\omega\rho = 0}$ $\quad \nabla\cdot\vec{J} = i\omega\rho$

$$\left[\int \vec{J}(\vec{x}')\, d^3x'\right]_i = \int J_\ell \overbrace{\frac{\partial x_i'}{\partial x_\ell'}}^{\delta_{i\ell}} d^3x' \overset{\text{integration by parts}}{=} -\int x_i (\nabla\cdot\vec{J})\, d^3x' = -i\omega \int \rho(\vec{x}')\, x_i'\, d^3x'$$

$$\therefore \int \vec{J}(\vec{x}')\, d^3x' = -i\omega \underbrace{\int \vec{x}'\rho(\vec{x}')\, d^3x'}_{\vec{p} \text{ electric dipole moment}} = -i\omega\vec{p}$$

$$\vec{A}(\vec{x},t) = -ik\frac{e^{ikr}}{r}\vec{p}\, e^{-i\omega t}$$

Fields (radiation zone) $\quad (\nabla\times A)_i \sim \varepsilon_{i\ell m}\, \partial_\ell \frac{e^{ikr}}{r} p_m$

$$\partial_\ell\left(\frac{e^{ikr}}{r}\right) = \frac{r\, ik\, e^{ikr}\, \partial_\ell(r) - e^{ikr}\, \partial_\ell(r)}{r^2} \qquad \partial_\ell(r) = \frac{x_\ell}{r} \quad \text{unit vector in } \ell \text{ direction}$$

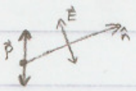
$$= \frac{e^{ikr}}{r^3} x_\ell (ikr - 1) \qquad \begin{matrix}\text{in radiation} \\ kr >> 1\end{matrix} \qquad \therefore \; ik\frac{e^{ikr}}{r} n_\ell = \partial_\ell \frac{e^{ikr}}{r}$$

$$(\nabla\times\vec{A})_i = -ik\, \varepsilon_{i\ell m}\frac{e^{ikr}}{r}\, ik\, n_\ell\, p_m$$

$$\nabla\times\vec{A} = \boxed{k^2\frac{e^{ikr}}{r}(\hat{n}\times\vec{p})\, e^{-i\omega t} = \vec{B}}$$

$$\vec{E} = \frac{i}{k}\nabla\times\vec{B} \qquad \nabla\times\left(\frac{e^{ikr}}{r}(\hat{n}\times\vec{p})\right) = ik\frac{e^{ikr}}{r}\hat{n}\times(\hat{n}\times\vec{p})$$

$$\vec{E} = -k^2\frac{e^{ikr}}{r}\hat{n}\times(\hat{n}\times\vec{p}) = -\hat{n}\times\vec{B} = \boxed{\vec{B}\times\hat{n} = \vec{E}} \qquad \vec{B} = k^2\frac{e^{ikr}}{r}(\hat{n}\times\vec{p})\, e^{-i\omega t}$$

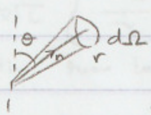


polarization in $\vec{p}, \hat{n}$ plane

Power radiated $\quad \vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B})$

$$\vec{S} = \frac{1}{2}\frac{c}{4\pi}\,\mathrm{Re}\{\vec{E}\times\vec{B}^*\} \text{ (time averaged)} \qquad \vec{E}\perp\vec{B} \quad |\vec{E}| = |\vec{B}| \quad \mathrm{Re}\{E\times B^*\} = |\vec{E}\times\vec{B}| = |\vec{E}|^2 = |\vec{B}|^2$$

$$\boxed{\vec{S} = \frac{c}{8\pi}\frac{k^4}{r^2}|\hat{n}\times\vec{p}|^2\hat{n}}$$

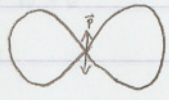


power radiated into $d\Omega$

$$dP = \vec{S}\cdot\hat{n}\, r^2\, d\Omega$$

$$\frac{dP}{d\Omega} = \frac{c}{8\pi}k^4|\hat{n}\times\vec{p}|^2$$

$$\boxed{\frac{dP}{d\Omega} = \frac{c}{8\pi}k^4|\vec{p}|^2\sin^2\theta}$$



Dipole Radiation Pattern

Total power $\left(\int d\Omega\right)$

$$\int \sin^2\theta\, d\Omega = \frac{8\pi}{3} \qquad \boxed{P_{tot} = \frac{c}{3}k^4|\vec{p}|^2} \qquad P \propto k^4$$

2nd term (magnetic dipole + electric quadrupole)

$$\vec{A}(\vec{x}) = -\frac{1}{c} ik \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}')(\hat{n}\cdot\vec{x}')\, d^3x'$$

$$(\vec{x}'\times\vec{J})\times\hat{n} = (\hat{n}\cdot\vec{x}')\vec{J} - (\hat{n}\cdot\vec{J})\vec{x}'$$

symmetric $\vec{J} \leftrightarrow \vec{x}'$

$$\vec{J}(\hat{n}\cdot\vec{x}') = \frac{1}{2}\Big[\underbrace{(\vec{x}'\times\vec{J})\times\hat{n}}_{\text{magnetic dipole}} + \underbrace{(\hat{n}\cdot\vec{J})\vec{x}' + \vec{J}(\hat{n}\cdot\vec{x}')}_{\text{electric quadrupole}}\Big]$$

$$\vec{M} = \frac{1}{2c}(\vec{x}\times\vec{J}) \qquad \vec{m} = \int \vec{M}\, d^3x \qquad \vec{A}(\vec{x}) = -\frac{ik}{2c}\frac{e^{ikr}}{r}\int(\vec{x}'\times\vec{J})\times\hat{n}\, d^3x'$$

$$\vec{A}(\vec{x}) = -ik\frac{e^{ikr}}{r}(\vec{m}\times\hat{n})$$

$$\vec{B} = \nabla\times\vec{A} = k^2\frac{e^{ikr}}{r}(\hat{n}\times\vec{m})\times\hat{n} \qquad \vec{E} = \vec{B}\times\hat{n}$$

polarization perpendicular to $\vec{m}, \hat{n}$ plane

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$$\int\left[(\hat{n}\cdot\vec{x})\vec{J} + (\hat{n}\cdot\vec{J})\vec{x}\right]_i d^3x = \int\left(n_\ell x_\ell J_i + n_\ell J_\ell x_i\right) d^3x$$

$$\int\Big(n_\ell x_\ell J_m \underbrace{\frac{\partial x_i}{\partial x_m}}_{\to\,\delta_{im}} + n_\ell J_\ell x_i\Big) d^3x \overset{\text{integrate by parts}}{=} \int d^3x\Big(\underbrace{-n_\ell\frac{\partial}{\partial x_m}(x_\ell J_m)x_i}_{-\left(n_\ell x_\ell \frac{\partial J_m}{\partial x_\ell} + n_\ell J_\ell\right)x_i} + n_\ell J_\ell x_i\Big)$$

$$\therefore = \int d^3x\,\big(-\underbrace{n_\ell x_\ell}_{(\hat{n}\cdot\vec{x})} x_i\,\partial_m J_m\big)$$

$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \qquad \nabla\cdot\vec{J} = i\omega\rho$$

$$= -i\omega\int\rho(\hat{n}\cdot\vec{x})x_i\, d^3x$$

$$\vec{A}(\vec{x}) = -\frac{1}{2}k^2\frac{e^{ikr}}{r}\int\rho(\vec{x}')(\hat{n}\cdot\vec{x}')\vec{x}'\, d^3x'$$

$$\vec{B} = \nabla\times\vec{A} \qquad \nabla\times\left(\frac{e^{ikr}}{r}(\hat{n}\cdot\vec{x}')\vec{x}'\right) = ik\,\hat{n}\times(\quad)$$

$$\vec{B} = ik(\hat{n}\times\vec{A}) \qquad \vec{E} = \vec{B}\times\hat{n}$$

$$\vec{B} = -\frac{i}{2}k^3\frac{e^{ikr}}{r}\int\rho(\vec{x}')(\hat{n}\cdot\vec{x}')(\hat{n}\times\vec{x}')\, d^3x' \qquad B_\ell \sim \int \varepsilon_{\ell\alpha\beta}\, n_\alpha x'_\beta\, n_k x'_k\, \rho(\vec{x}')\, d^3x'$$

$$\sim \frac{1}{3}\int d^3x'\,\varepsilon_{\ell\alpha\beta}\, n_\alpha n_k\,\underbrace{\rho(\vec{x}')\big(3x'_\beta x'_k - \overbrace{r^2\delta_{\beta k}}^{\text{zero}:\ \varepsilon_{\ell\alpha\beta}n_\alpha n_\beta = 0}\big)}_{Q_{\beta k}}$$

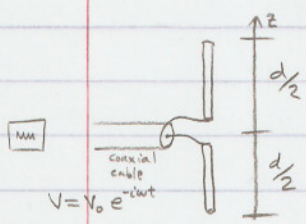
$$B_\ell = -\frac{i}{6}k^3\frac{e^{ikr}}{r}\varepsilon_{\ell\alpha\beta}\, n_\alpha \underbrace{n_k Q_{\beta k}}$$

$$(\vec{Q})_\beta = \sum_k Q_{\beta k} n_k$$

$$\vec{B} = -\frac{i}{6}k^3\frac{e^{ikr}}{r}(\hat{n}\times\vec{Q})$$

power radiated will be k^6 for quadrupole (dipole power $\propto k^4$)

Example: Antenna



$\lambda \gg d$

$I(z,t) = I(z)\,e^{-i\omega t}$ Approx: $I(z) = I_0\left(1 - \frac{2|z|}{d}\right)$

(to satisfy continuity

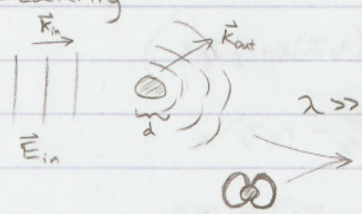
Continuity: $\nabla\cdot\vec J = i\omega\rho$ (× cross-sectional area) $\longrightarrow$ $\frac{dI}{dz} = i\omega\rho'$ ← charge per unit length

$\frac{dI}{dz} = \mp I_0 \frac{2}{d}$ ($z>0$ / $z<0$) $\quad \rho'(z) = \frac{-i}{\omega}\mp I_0\frac{2}{d} = \pm\frac{2iI_0}{\omega d}$

$$p = \int_{-d/2}^{d/2} z\rho'(z)\,dz = \frac{2iI_0}{\omega d}\,\frac{1}{2}\,\frac{d^2}{4}\,2 = \frac{iI_0 d}{2\omega}$$

Electric dipole $\quad \frac{dP}{d\Omega} = \frac{c}{8\pi}k^4|p|^2\sin^2\theta = \frac{c}{8\pi}k^4\sin^2\theta\,\frac{I_0^2 d^2}{4\omega^2} = \frac{I_0^2 d^2}{32\pi c}k^2\sin^2\theta$

Scattering



$\vec E_{in} = \vec\varepsilon_{in}E_0\,e^{i\vec k_{in}\cdot\vec x}e^{-i\omega t}$ (polarization) $\quad \vec B_{in} = \hat n_{in}\times\vec E_{in} \quad \hat n = \frac{\vec k}{|\vec k|}$

$\lambda \gg d$

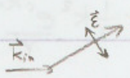
$\vec E_{out} = k^2\frac{e^{ikr}}{r}(\hat n\times\vec p)\times\hat n \quad \vec B_{out} = \hat n_{out}\times\vec E_{out}$

$\vec S = \frac{c}{8\pi}\mathrm{Re}\{\vec E\times\vec B^*\}$
$\vec S\cdot\hat n$?
$(\vec E\times\vec B)\cdot\hat n = |\vec E\times\vec B|$

$$\frac{(\text{power radiated into } d\Omega,\ \hat n \text{ with polarization } \vec\varepsilon)}{(\text{incident current [power/area]})} = d\sigma\ [\text{area}]$$

$$= \frac{\frac{c}{8\pi}\overbrace{|\vec E_{out}\cdot\hat\varepsilon|^2}^{\text{polarization } \vec\varepsilon}\ \overbrace{r^2 d\Omega}^{\text{area}}}{\frac{c}{8\pi}|E_0|^2}$$

$[(\hat n\times\vec p)\times\hat n]\cdot\vec\varepsilon$



$\vec p \parallel \vec E_{in}$

$(\vec p - (\vec p\cdot\hat n)\hat n)\cdot\vec\varepsilon = \vec\varepsilon\cdot\vec p$

$$= \frac{k^4|\vec\varepsilon\cdot\vec p|^2\,d\Omega}{|E_0|^2}$$

$|\vec p| \propto E_0 \implies \frac{d\sigma}{d\Omega}$ independent on E_0, $\propto k^4$

$$\boxed{\frac{d\sigma}{d\Omega} = k^4\frac{|\vec\varepsilon\cdot\vec p|^2}{|E_0|^2}}$$

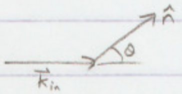
Dielectric Sphere $\quad \vec p = \frac{\varepsilon-1}{\varepsilon+2}a^3\vec E_{in} \quad \vec E_{in} = \vec\varepsilon_{in}E_0\,e^{i\vec k_{in}\cdot\vec x}e^{-i\omega t}$

$$\frac{d\sigma}{d\Omega} = k^4\left(\frac{\varepsilon-1}{\varepsilon+2}\right)^2 a^6|\vec\varepsilon\cdot\vec\varepsilon_{in}|^2$$

- need to average over incident polarization
- then sum over scattered polarization

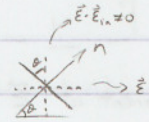
Dielectric sphere continued

– average over incident polarization



a) $\vec{\varepsilon} \perp$ scattering plane $(\vec{k}_{in}, \hat{n})$ $\nearrow \hat{n}$ $\otimes \vec{\varepsilon}$

b) $\vec{\varepsilon} \parallel$ scattering plane $\longrightarrow$



$\vec{\varepsilon}\cdot\vec{\varepsilon}_{in} \neq 0$ $\quad$ $\vec{\varepsilon}\cdot\vec{\varepsilon}_{in} = 0$

a) $\langle |\vec{\varepsilon}\cdot\vec{\varepsilon}_{in}|^2 \rangle = \langle \cos^2\varphi \rangle = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\varphi \, d\varphi = \frac{1}{2}$

φ parametrizes direction $\vec{E}_{in}$ in plane $\perp \vec{k}_{in}$

b) $\langle |\vec{\varepsilon}\cdot\vec{\varepsilon}_{in}|^2 \rangle = \langle |\cos\theta \cos\varphi|^2 \rangle = \frac{1}{2}\cos^2\theta$

$$\frac{d\sigma_\perp}{d\Omega} = \frac{1}{2} k^4 a^6 \left(\frac{\varepsilon-1}{\varepsilon+2}\right)^2 \qquad \frac{d\sigma_\parallel}{d\Omega} = \frac{1}{2} k^4 a^6 \left(\frac{\varepsilon-1}{\varepsilon+2}\right)^2 \cos^2\theta$$

– sum over polarization:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega} = \frac{1}{2} k^4 a^6 \left(\frac{\varepsilon-1}{\varepsilon+2}\right)^2 (1+\cos^2\theta)$$

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} k^4 a^6 \left(\frac{\varepsilon-1}{\varepsilon+2}\right)^2 = \frac{\text{power out}}{\text{power/area in}} = \frac{\dot{N}_{out}}{\dot{N}/A^{\,in}}$$

$\#$ photons/time

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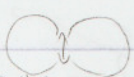
$\theta = \pi/2 \quad \frac{d\sigma_\parallel}{d\Omega} = 0$

polarized $\perp$ to scattering plane



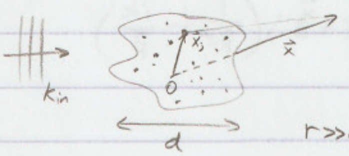
light polarized parallel to ground when scattered

$\sigma \propto k^4 \sim \frac{1}{\lambda^4}$ shorter wavelengths are scattered more $\to$ blue sky, red sunset

$$\dot{N}_{out} = \sigma_{tot} \frac{\dot{N}_{in}}{A}$$

$\dot{N} \to \#$ photons/time

Collection of Scatterers (long wavelength)



1 scatterer: $\vec{E}_{out} = k^2 \frac{e^{ikr}}{r} (\hat{n}\times\vec{p})\times\hat{n}$

$r \gg d$ $\qquad$ $\hat{n} = \frac{\vec{r}}{r}$

$|\vec{x}-\vec{x}_j| \simeq r - \vec{x}_j\cdot\hat{n}$

$$\frac{1}{|\vec{x}-\vec{x}_j|} \simeq \frac{1}{r} \qquad e^{ik|\vec{x}-\vec{x}_j|} \simeq e^{ikr} e^{-ik\vec{x}_j\cdot\hat{n}}$$

keep track of phases

$\to k_{out}$

$\vec{p} \sim e^{i\vec{k}_{in}\cdot\vec{x}_j}$ $\qquad$ $\vec{k}_{out} = k\hat{n}$

$$\vec{E}_{out} = k^2 \frac{e^{ikr}}{r} \Big[\underbrace{(\hat{n}\times\vec{p})\times\hat{n}}_{\text{without phase}} \Big] \sum_j e^{-ik\vec{x}_j\cdot\hat{n}} \underbrace{e^{i\vec{k}_{in}\cdot\vec{x}_j}}_{\text{phase of } \vec{p}}$$

$\vec{q} = \vec{k}_{out} - \vec{k}_{in}$ $\qquad$ $\sum_j \to \sum_j e^{-i\vec{q}\cdot\vec{x}_j}$ $\qquad$ $\boxed{\frac{d\sigma}{d\Omega} = k^4 \frac{|\vec{\varepsilon}\cdot\vec{p}|^2}{E_0^2} \mathcal{F}(\vec{q})}$

$\boxed{\mathcal{F}(\vec{q}) = \left|\sum_j e^{-i\vec{q}\cdot\vec{x}_j}\right|^2}$ form factor

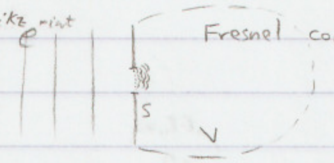
$\rho = \sum_j \delta(\vec{x}-\vec{x}_j)$ $\qquad$ $\left|\int d^3x\, \rho\, e^{-i\vec{q}\cdot\vec{x}}\right|^2 = \mathcal{F}(\vec{q})$

Fourier transform of distribution (squared)

Scattering $\lambda \gg d$ Diffraction $\lambda \ll d$

Scalar Diffraction Theory

$\vec{E} = \vec{E}_0 e^{ikz} e^{-i\omega t}$



Fresnel construction – each point is source of spherical waves

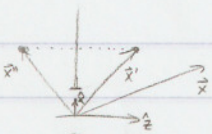
ψ = component of $\vec{E}$ $\qquad \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\psi = 0 \qquad \psi \sim e^{-i\omega t}$

$\left[\nabla^2 + k^2\right]\psi = 0$

Helmholtz Eq

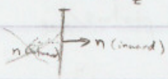
free space Green function $\quad (\nabla^2 + k^2)\, G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$

$$G(\vec{x},\vec{x}') = \frac{e^{ikR}}{R} \qquad R = |\vec{x}-\vec{x}'| \qquad \tilde{R} = |\vec{x}-\vec{x}''|$$



$$G_D(\vec{x},\vec{x}') = \frac{e^{ikR}}{R} - \frac{e^{ik\tilde{R}}}{\tilde{R}}$$

Solution of G and is zero on screen



$$\int_V \left(G\nabla^2\psi - \psi\nabla^2 G\right) d^3x' = \int_S \left(G\frac{\partial\psi}{\partial n} - \psi\frac{\partial G}{\partial n}\right) ds'$$

LHS: $\displaystyle\int d^3x' \left(-G\psi k^2 - \psi\left(-k^2 G - 4\pi\delta(\vec{x}-\vec{x}')\right)\right) = 4\pi\,\psi(\vec{x})$

$G \to \frac{e^{ikr}}{r} \qquad \psi \to \frac{e^{ikr}}{r}$ for large r $\qquad \int_S = \int_{\text{screen}} + \underbrace{\int_{\text{half sphere}}}_{\to 0}$ for large r (falls faster than r^2)

RHS:

$$+\int_{\text{Screen}} \psi(\vec{x}')\,\frac{\partial G_D}{\partial n'}\,ds'$$

inward normal

$$R = |\vec{x}-\vec{x}'| = \left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{1/2} \qquad \tilde{R} = |\vec{x}-\vec{x}''| = \left((x-x')^2 + (y-y')^2 + (z+z')^2\right)^{1/2}$$

$$\psi(\vec{x}) = \frac{E_0}{4\pi}\int_{\text{Hole}} dx'dy'\,\frac{\partial G_D}{\partial z'}$$

$$\frac{\partial}{\partial z'}\left(\frac{e^{ikR}}{R}\right) = \frac{e^{ikR}}{R}(z-z')\left(\frac{1}{R^2} - \frac{ik}{R}\right)$$

$$\frac{\partial}{\partial z'}\left(\frac{e^{ik\tilde{R}}}{\tilde{R}}\right) = \frac{e^{ik\tilde{R}}}{\tilde{R}}(z+z')\left(\frac{1}{\tilde{R}^2} - \frac{ik}{\tilde{R}}\right)$$

$$\left.\frac{\partial G_D}{\partial z'}\right|_{z'=0} = 2\,\frac{e^{ikR}}{R}(-ik)\,\frac{z}{R}$$

$$\boxed{\psi(\vec{x}) = \frac{-ik}{2\pi}\int_{\text{Hole}} \frac{e^{ikR}}{R}\,\frac{z}{R}\,\psi(x',y')\,dx'dy'} \qquad \text{or} \qquad \psi = \frac{-ik}{2\pi}\underbrace{E_0}_{\psi(x',y')}\int_{\text{Hole}} \frac{e^{ikR}}{R}\,\frac{z}{R}\,dx'dy'$$

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$$\psi(\vec{x}) = \frac{-ik}{2\pi} \iint_{\text{Hole}} \psi(x', y') \frac{e^{ikR}}{R} \frac{z}{R} \, dx' dy'$$

Approx: $\frac{1}{R} \simeq \frac{1}{r}$ $\frac{z}{R} \simeq \cos\theta$ (looking far from hole)

phase $kR \simeq k(r - \vec{x}' \cdot \hat{n})$ $\psi(\vec{x}) \simeq \frac{-ik}{2\pi} E_0 \frac{e^{ikr}}{r} \cos\theta \int e^{-ik\vec{x}' \cdot \hat{n}} dx' dy'$

Circular Hole

$\vec{x}' = (r'\cos\varphi', r'\sin\varphi', 0)$ $\hat{n} = (\sin\theta, 0, \cos\theta)$

$\vec{x}' \cdot \hat{n} = r' \sin\theta \cos\varphi'$

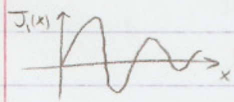
$$\int_0^a r' dr' \int_0^{2\pi} d\varphi' \, e^{-ikr'\sin\theta\cos\varphi'}$$ $$J_0(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\lambda\cos\varphi} d\varphi$$

Bessel functions

$$\int x J_0(x) dx = x J_1(x)$$

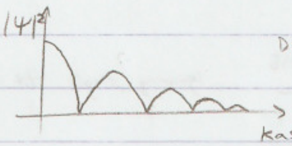
$$\therefore \int_0^a r' 2\pi J_0(kr'\sin\theta) dr' = \frac{2\pi}{(k\sin\theta)^2} ka\sin\theta \, J_1(ka\sin\theta) = \frac{2\pi a^2}{ka\sin\theta} J_1(ka\sin\theta)$$

$u = kr'\sin\theta$ $du = k\sin\theta \, dr'$ $r' = \frac{u}{k\sin\theta}$



$\lim_{x \to 0} \frac{J_1(x)}{x} = 1$ $ka \gg 1$ intensity confined in foward direction $\psi \simeq 0$ if $\theta \neq 0$

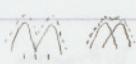
intensity $\propto |\psi|^2$



Diffraction pattern

first minimum $J_1(x) = 0$ for $x = 3.8$ $\frac{2\pi}{\lambda} a \sin\theta = ka\sin\theta \simeq 3.8 \simeq 1.2\pi$

$\Rightarrow \boxed{\sin\theta = 0.6 \frac{\lambda}{a}} \longrightarrow 1.2 \frac{\lambda}{D} = \sin\theta$ criterion for angular resolution



Problem 9.1 $\rho(x,t)$ is periodic $T = \frac{2\pi}{\omega}$

$$\rho(\vec{x}, t) = \frac{1}{2}\rho_0(\vec{x}) + \sum_{n=1}^{\infty} \text{Re}\{\rho_n(\vec{x}) e^{-in\omega t}\}$$
Fourier series

$$\rho_n(\vec{x}, \omega) = \frac{2}{T} \int_0^T \rho(\vec{x}, t) e^{in\omega t} dt$$

$$\text{Re}\left\{\frac{2}{T} \int \rho(\vec{x}, t') e^{in\omega t'} e^{-in\omega t} dt'\right\} = \frac{2}{T} \int dt' \rho(\vec{x}, t') \cos(n\omega t)\cos(n\omega t') + \frac{2}{T} \int dt' \rho(\vec{x}, t') \sin(n\omega t)\sin(n\omega t')$$

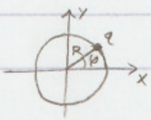
$$A_n \cos(n\omega t) = A_n \cos\left(\frac{2\pi n t}{T}\right)$$

$$f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n t}{T}\right) + B_n \sin\left(\frac{2\pi n t}{T}\right)$$

Fourier decomposition of $f(t)$

calculate multipole moments using ρ_n

one charge



$\vec{x}_q = (R\cos\omega t, R\sin\omega t, 0)$

$\varphi = \omega t$

$\rho(\vec{x},t) = q\,\delta(\vec{x}-\vec{x}_q)$

$\rho(r,\varphi,t) = q\,\delta(r-R)\,\delta(\varphi-\omega t)\,\frac{1}{r}\,\delta(z)$ cylindrical coords

$$\rho_n(r,\varphi,z) = \frac{q}{r}\,\delta(r-R)\,\delta(z)\,\frac{2\omega}{2\pi}\int_0^{2\pi/\omega}\delta(\varphi-\omega t)\,e^{in\omega t}\,dt = \frac{1}{\pi}\frac{q}{r}\,\delta(r-R)\,\delta(z)\,e^{in\varphi}$$

$$q_{lm} = \int r^l \rho\, Y^*_{lm}\, d^3x \qquad q_{00} = \frac{1}{\sqrt{4\pi}}\,\frac{1}{2}\,\frac{q}{\pi}\int r\,dr\int dz\int_0^{2\pi} d\varphi\,\frac{\delta(r-R)}{r}\,\delta(z)\,e^{in\varphi}$$

$n=0$ otherwise $\int e^{in\varphi} = 0$

$q_{00} = \frac{1}{\sqrt{4\pi}}\,q$ for $n=0$ $\quad q_{00} = 0$ for $n \neq 0$

$n=0 \rightarrow$ no radiation

$l=1$ $\quad Y^*_{1m}$: $e^{-i\varphi}$ $\;m=1$ $\rightarrow$ for $n=1$ this is only nonzero term

$e^{i\varphi}$ $\;m=-1$

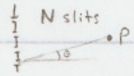
1 $\;m=0$

$$q_{11} = \int dr\,r\int dz\int d\varphi\,\sqrt{\frac{3}{8\pi}}\,\sin\theta\,e^{-i\varphi}\,r\,q\,\frac{\delta(r-R)}{r}\,\underbrace{\delta(z)}_{\theta=\pi/2}\,e^{i\varphi}\,\frac{1}{\pi}$$

$q_{11} = -\sqrt{\frac{3}{8\pi}}\,2qR$ $\qquad$ dipole radiates at freq. ω $(n=1)$

$q_{11} = -\sqrt{\frac{3}{8\pi}}\,(p_x - ip_y) \qquad q_{1-1} = \sqrt{\frac{3}{8\pi}}\,(p_x + ip_y) \qquad p_x = qR \quad p_y = iqR$

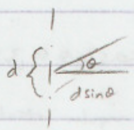
Problems concerning Diffraction Gratings



N slits

distance large compared to size

$$\psi \sim \sum_{m=0}^{N-1} e^{i\delta m} = \frac{1-e^{i(N-1)\delta}}{1-e^{i\delta}}$$ ignoring single slit contribution



$e^{ikr} \rightarrow e^{ikd\sin\theta}$

$kd\sin\theta = \delta$

Diffraction from 1 slit

$$\int e^{-ik\vec{x}'\cdot\hat{n}}\,dx'dy'\cos\theta \sim \cos\theta\int_{-a/2}^{a/2} dy'\,e^{-ik\sin\theta\,y'} = \cos\theta\,\frac{i\frac{a}{2}}{\frac{a}{2}k\sin\theta}\,2\sin\left(k\tfrac{a}{2}\sin\theta\right)$$

$\vec{x}'\;\hat{n}$

single dimension

translationally invariant along x (?)

$\sim \cos\theta\,\frac{\sin\alpha}{\alpha} \qquad \alpha = k\frac{a}{2}\sin\theta$

Grating with thermal noise $\quad$ spacing average d but Gaussian dist'n $\quad \int_{-a/2+\varepsilon}^{a/2+\varepsilon} dy'\,e^{iky'\sin\theta} \qquad P(\varepsilon) \propto e^{-\frac{\varepsilon^2}{2\sigma^2}}$

$\langle e^{ik\sin\theta\,(\varepsilon_m-\varepsilon_n)} \rangle$

Special Relativity

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Invariance

Newton's laws are invariant under Galilean tranformation — can't tell if moving

$$\frac{dp}{dt} = -\frac{dU}{dx} \qquad p=p(t) \qquad U=U(x) \qquad p = m\frac{d}{dt}x_p(t)$$

position of particle

$$x' = x - vt \qquad t'=t \implies x = x'+vt' \qquad t=t' \qquad x=x(x',t') \qquad t=t(x',t')$$

$$x_p' = x_p - vt \qquad \frac{d}{dt} = \frac{d}{dt'}\frac{dt'}{dt} + \frac{d}{dx'}\frac{dx'}{dt} = \frac{d}{dt'}$$

in this case (particle position not fn of position) not a field

$$\frac{d}{dx} = \frac{d}{dx'}\frac{dx'}{dx} + \frac{d}{dt'}\underbrace{\frac{dt'}{dx}}_{0} = \frac{d}{dx'} \qquad U(x(x')) = U'(x') \qquad \text{Ex: } U=\tfrac{1}{2}kx^2 \quad U'(x') = \tfrac{1}{2}k(x'+vt)^2$$

U invariant

$$p' = p - mv \qquad \frac{d}{dt'}(p'+mv) = -\frac{d}{dx'}U'(x') \qquad \frac{dp'}{dt'} = -\frac{dU'(x')}{dx'}$$

speed of frame = constant

$$m\ddot{x}_p = -kx_p$$

$$\frac{d^2}{dt^2}(x_p'+vt) = -\frac{k}{m}(x_p'+vt) \qquad \frac{d^2x_p'}{dt^2} = -\frac{k}{m}x_p' - \frac{k}{m}vt$$

the differential equation changes
→ choosing a frame with spring stationary makes eqn's easier

Wave Eq: → not invariant, not a problem – not fundamental eq.

$$\frac{1}{c^2}\frac{\partial}{\partial t^2}\Psi(x,t) - \frac{\partial^2}{\partial x^2}\Psi(x,t) = 0 \qquad \Psi \text{ pressure field (sound waves)}$$

Ψ invarient $\quad \Psi(x(x'),t(t')) = \Psi'(x',t')$ same function regardless of frame

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}$$

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)\Psi' - \frac{\partial^2}{\partial x'^2}\Psi' = 0$$

there is a preffered frame ($v=0$)

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \left(1-\frac{v^2}{c^2}\right)\frac{\partial^2}{\partial x'^2} - \frac{2v}{c^2}\frac{\partial^2}{\partial t'\partial x'}\right)\Psi' = 0$$

Form of wave eq not invariant

Schrödinger Eq:

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}\Psi + V(x)\Psi \qquad \Psi' = \Psi e^{i\theta(t)\varphi(x)}$$

→ invariant

$$\frac{d\vec{p}}{dt} = -\nabla U \qquad \text{invariant under rotations}$$

$U \to$ scalar $\quad U(x(x')) = U'(x') \qquad \vec{p} \to$ vector $\quad$ (ex.) $p_1' = p_1\cos\theta + p_2\sin\theta \quad$ not true that $p_1' = p_1$

Lagrangian $\mathcal{L} = \sum_i \left\{ \frac{1}{2} m \dot{x}_i^2 \right\} - \frac{1}{2} \sum_{i \neq j} \left\{ V(|\vec{x}_i - \vec{x}_j|) \right\}$

invariant under translation $x' = x - \lambda \quad t' = t$

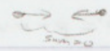
$V|(x_i' + \lambda) - (x_j' + \lambda)| = V(|\vec{x}_i' - \vec{x}_j'|) \qquad \mathcal{L}(\dot{x}, x) = \mathcal{L}(\dot{x}, x - \lambda)$ invariant under translation

corresponds to total momentum conservation $\frac{dp_{tot}}{dt} = 0 \quad \vec{P}_{tot} = \sum_i \vec{p}_i \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i}$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0 \qquad \frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial x_i} \qquad \frac{dP_{tot}}{dt} = \frac{d}{dt}\sum p_i = \sum_i \frac{\partial \mathcal{L}}{\partial x_i} = \sum_i \left(-\tfrac{1}{2}\right)\frac{\partial}{\partial x_i}\sum_{l \neq m} V(|\vec{x}_l - \vec{x}_m|)$$

$= 0$ Σ(force on particle i due to all others)

by Newton's 3rd law all cancel



$V(|\vec{x}_1 - \vec{x}_2|) = V(z) \qquad z = |\vec{x}_1 - \vec{x}_2|$

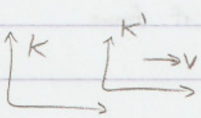
$$\frac{\partial V}{\partial x_2} = \frac{\partial V}{\partial z}\frac{\partial z}{\partial x_2} = -\frac{\partial V}{\partial z} \qquad \frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial z} \qquad \sum_i \frac{\partial V}{\partial x_i} = 0$$

If $\mathcal{L}(x) = \mathcal{L}(x - \lambda) \ \forall \lambda$

$$\mathcal{L}(x-\lambda) \simeq \mathcal{L}(x) - \underbrace{\sum_j \lambda \frac{\partial \mathcal{L}}{\partial x_j}}_{\text{is zero}}$$

$$\sum_j \frac{\partial \mathcal{L}}{\partial x_j} = 0 \Rightarrow \frac{dp_{tot}}{dt} = 0$$

Postulates
1) Physics is same in different inertial systems
2) Speed of light independent of relative motion of source and observer



At $t=0$ origins coincide

Flash of light at $t=0$ $\vec{x}=0$

in K: see shell of light $x^2 + y^2 + z^2 = (ct)^2$

in K': see shell of light $x'^2 + y'^2 + z'^2 = (ct')^2$

Seek transformation so that $(ct)^2 - |\vec{x}|^2 = s^2$ is invariant

$x' = ax + bt \qquad t' = \alpha x + \beta t'$

$x' = a(x - vt) \qquad t' = b(t - g\frac{x}{v}) \qquad (ct')^2 - x'^2 = (cb(t - g\frac{x}{v}))^2 - (a(x-vt))^2 = ct^2 - x^2$

$a = \gamma \quad b = \gamma \quad g = \beta^2$

$$\beta = \frac{v}{c} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad x' = \gamma(x - \beta ct) \quad ct' = \gamma(ct - \beta x)$$

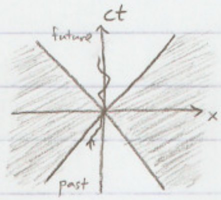
$x_0 = ct \qquad x_1 = x$

$x_0' = \gamma(x_0 - \beta x_1) \qquad x_1' = \gamma(x_1 - \beta x_0)$

$A^\alpha = (A^0, \vec{A}) \qquad A^{0^2} - |\vec{A}|^2 = A \cdot A = A_\alpha A^\alpha$ modulus of this vector is invariant

$\rightarrow$ length of 4-vector

$A \cdot B = A_\alpha B^\alpha = A^0 B^0 - \vec{A} \cdot \vec{B}$



Event = point in space time

$$s^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2$$

can be causally connected $\rightarrow$ $s^2 > 0$ "time-like" separation $\rightarrow$ $\exists$ Lorentz transformation such that events occur at same place (future/past cone)

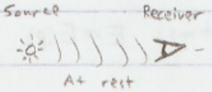
not causally connected $\rightarrow$ $s^2 < 0$ "space-like" separation $\rightarrow$ $\exists$ Lorentz transformation such that events occur at same time (elsewhere)

$s^2 = 0$ "light-like" separation (on light cone)

2/1/2007

Maxwell Eqs. — not invariant under Galilean $\longrightarrow$ these are fundamental eqs.

$\rightarrow$ can find out if you are moving ... but you can't ?



M.E.: speed of light = c

Mechanics: speed of sound = c_s

Source moves at vel. v — M.E.: speed of light = c — Mechanics: speed of sound = c_s — same as before

Receiver moves at v — M.E.: measure: c — Mechanics: measure $c_s - V$

$\rightarrow$ can't tell if you are moving $\rightarrow$ consistent with principle of relativity but not with Galilean addition of velocities

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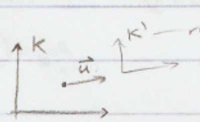
Proper time



K: $d\vec{x} = \vec{u}\,dt$ $\qquad ds^2 = c^2dt^2 - dx^2 = c^2dt^2 - u^2dt^2 = (c^2 - u^2)\,dt^2 = c^2dt^2(1-\beta^2)$

K': $d\vec{x}' = 0$ $\qquad ds'^2 = c^2 d\tau^2 \qquad ds^2 = ds'^2 \qquad \cancel{c^2}d\tau^2 = \cancel{c^2}dt^2(1-\beta^2) \qquad d\tau = dt\sqrt{1-\beta^2}$

$$\boxed{d\tau = \frac{dt}{\gamma}}$$ proper time — time in rest frame of particle $\rightarrow$ is the shortest

$$\tau_2 - \tau_1 = \int_{t_1}^{t_2} \frac{dt}{\gamma(t)} < t_2 - t_1 \quad \text{since } \gamma \geq 1$$

Doppler Shift

$E \sim e^{i(\vec{k}\cdot\vec{x} - \omega t)} = e^{i\varphi}$ — phase is invariant — measure of how many waves $\rightarrow$ # does not change

$\vec{k}\cdot\vec{x} - \omega t = \vec{k}'\cdot\vec{x}' - \omega' t'$ $\qquad$ 4-vector: $\left(\frac{\omega}{c}, \vec{k}\right)$ $\quad \therefore$ product of 2 4-vectors

$\frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \vec{\beta}\cdot\vec{k}\right)$ (if β in $\hat{x}$: $\beta k_x = \beta k_1$) $\qquad k_{\parallel}' = \gamma\left(k_{\parallel} - \beta\frac{\omega}{c}\right)$ ($\parallel$ to β) $\qquad \vec{k}_{\perp}' = \vec{k}_{\perp}$ ($\perp$ to β)

$|\vec{k}| = \frac{\omega}{c} \qquad |\vec{k}'| = \frac{\omega'}{c}$ for light

$\omega' = \gamma\left(\omega - v\frac{\omega}{c}\right)$ ··· for $\vec{\beta} \parallel \vec{k}$ ··· $\omega' = \gamma\left(1 - \frac{v}{c}\right)\omega = \frac{1 - v/c}{\sqrt{1 - v^2/c^2}}\,\omega = \boxed{\sqrt{\frac{1 - v/c}{1 + v/c}}\,\omega = \omega'}$

$\vec{\beta} \perp \vec{k} \qquad \omega' = \gamma\omega$

Energy & Momentum

$\left(\frac{E}{c}, \vec{p}\right)$ is a 4-vector

$E = \gamma m c^2$ m – rest mass $\vec{p} = \gamma m \vec{v}$ $\vec{v} = \frac{d\vec{x}}{dt}$

$\frac{E^2}{c^2} - |\vec{p}|^2$ is invariant

$\longrightarrow$ in rest frame $= \frac{(mc^2)^2}{c^2} = m^2c^2$ $E^2 = p^2c^2 + m^2c^4$

ultrarelativistic γ is large $\gamma \gg 1$ $v \simeq c$ $p \simeq \gamma m c$

$\frac{E^2}{c^2} - \gamma^2 m^2 c^2 = m^2 c^2$ $\frac{E^2}{c^2} = p^2 + m^2c^2 \simeq p^2 \Rightarrow E \simeq pc$ at high energies

At low energies, $\gamma \simeq 1$, $\beta \ll 1$ $E = c\sqrt{p^2 + m^2c^2} = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} \simeq mc^2\left(1 + \frac{1}{2}\frac{p^2}{m^2c^2}\right)$

$E \simeq mc^2 + \frac{p^2}{2m}$ at low energies

Different dispersion relations: $E = E(p)$

$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{\beta} \times \vec{B}\right)$

Space-Time Geometry

x'^{α} $\alpha = 0,1,2,3$ $x'^{\alpha} = x'^{\alpha}(x^0, x^1, x^2, x^3)$ $s' = s$ scalar

$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta}$ contravariant $A'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} A_{\beta}$ covariant

$A^{\alpha} = (A^0, \vec{A})$ $B^{\alpha} = (B^0, \vec{B})$ $A \cdot B = A^0 B^0 - \vec{A} \cdot \vec{B}$ $A_{\alpha} = (A^0, -\vec{A})$

metric tensor: $g_{\alpha\beta} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ $A \cdot B = g_{\alpha\beta} A^{\alpha} B^{\beta}$ $A_{\alpha} = g_{\alpha\beta} A^{\beta}$ $\therefore\ A \cdot B = A_{\alpha} B^{\alpha}$

$ds^2 = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta}$ distance $T_{\alpha\beta} = g_{\beta\delta}\, g_{\alpha\gamma}\, T^{\gamma\delta}$

2/6/2007

$|A|^2 = A \cdot A = A_{\alpha} A^{\alpha} = g_{\alpha\beta} A^{\alpha} A^{\beta} = g^{\alpha\beta} A_{\alpha} A_{\beta} = A^{0\,2} - |\vec{A}|^2$ $g_{\alpha\beta} = g^{\alpha\beta}$

$g_{\alpha\beta}\, g^{\beta\gamma} = \delta_{\alpha}^{\ \gamma}$

Operator corresponding to ∇

$\frac{\partial}{\partial x^{\alpha}} \equiv \partial_{\alpha}$ $\frac{\partial}{\partial x'^{\alpha}} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\beta}}$ transforms like A_{α} $\partial_{\alpha} = \left(\frac{\partial}{\partial x^0}, \nabla\right)$ $\partial^{\alpha} = \left(\frac{\partial}{\partial x^0}, -\nabla\right)$

$\partial^{\alpha}\partial_{\alpha} = \frac{\partial^2}{\partial x^{0\,2}} - \nabla^2 \equiv \Box$ operator for wave equation. $\Box \Psi = 0$

"Divergence": $\partial^{\alpha} A_{\alpha} = \frac{\partial A^0}{\partial x^0} + \nabla \cdot \vec{A}$ Lorentz invariant $\partial^{\alpha} A_{\alpha} = \partial_{\alpha} A^{\alpha}$

Current 4-vector

$J^{\alpha} = (c\rho, \vec{J})$ $\partial_{\alpha} J^{\alpha} = \underbrace{\frac{\partial \rho}{\partial t}}_{\frac{c\,\partial\rho}{c\,\partial t}} + \nabla \cdot \vec{J} = 0$ Continuity Eq. $\partial_{\alpha} J^{\alpha} = 0$

In Lorentz gauge: $\frac{1}{c}\frac{\partial\phi}{\partial t}+\nabla\cdot\vec{A}=0 \rightarrow \partial_\alpha A^\alpha=0$ $\quad J^\alpha=(c\rho,\vec{J})$

$\left(\frac{1}{c^2}\frac{\partial}{\partial t^2}-\nabla^2\right)\begin{matrix}\phi\\ \vec{A}\end{matrix}=4\pi\begin{matrix}\rho\\ \frac{\vec{J}}{c}\end{matrix}$ $\quad\therefore\quad A^\alpha=(\phi,\vec{A})\qquad \Box A^\alpha=\frac{4\pi}{c}J^\alpha$

A Lorentz transformation $\quad T^{\alpha\beta}$ tensor $\quad\therefore\quad \boxed{T'=AT\,{}^TA}$

${}^TA_{ij}=A_{ji}$ transpose $\qquad A^\dagger_{ij}=A^*_{ji}$ hermitian conjugate $\qquad$ symmetric ${}^TA=A$ $\quad$ hermitian $A^\dagger=A$

real numbers: orthogonal matrix ${}^TA=A^{-1}$ preserve scalar products $\qquad$ antisymmetric ${}^TA=-A$

$a=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\qquad a\cdot b=(a)\begin{pmatrix}b\end{pmatrix}={}^Tab\qquad a'=Aa\quad b'=Ab$

$a'\cdot b'={}^Ta'b'={}^Ta{}^TAAb={}^Tab$ if ${}^TAA=I\quad {}^TA=A^{-1}$

${}^T(Aa)={}^Ta{}^TA$

unitary matrix $\quad A^\dagger=A^{-1}$

$a\cdot b=a^\dagger b$

Tensor T_{ik} under orthogonal transf. $\qquad b=Ta\qquad b'=T'a\qquad T'=?$

$a'=Aa\qquad b'=Ab$

$b'=T'a'\Rightarrow\quad Ab=T'Aa$

$b=A^{-1}T'Aa\qquad T=A^{-1}T'A\qquad T'=ATA^{-1}$

if A is orthogonal: $\quad T'=AT\,{}^TA$

A is Lorentz transformation $\quad a'=Aa\quad b'=Ab\quad a=(\ )$ 4-vector

$a\cdot b={}^T(ga)b=a'\cdot b'={}^T(gAa)Ab={}^Ta{}^T(gA)Ab={}^Ta{}^TAgAb\qquad {}^Tg=g$

metric

${}^Tagb={}^Ta{}^TAgAb\qquad\therefore\quad {}^TAgA=g$ for a Lorentz transformation

$gg=I$

$g{}^TAgA=I\quad\therefore\quad\boxed{g{}^TAg=A^{-1}}$

$b^\alpha=T^{\alpha\beta}a_\beta\qquad b'^\alpha=T'^{\alpha\beta}a'_\beta\qquad a_\beta=g_{\beta\gamma}a^\gamma$

$b=Tga\qquad a=(\ )\quad b=(\ )\qquad a'=Aa\quad b'=Ab$

$b'=T'ga'\qquad Ab=T'gAa\qquad b=A^{-1}T'gAa\qquad A^{-1}T'gA=Tg$ $\quad$ x left A, right A^{-1} both sides

$T'g=ATgA^{-1}\qquad T'=ATgA^{-1}g=AT{}^TA\quad\therefore\quad T'=AT{}^TA$ proved

---

$\vec{E}=-\nabla\phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\qquad\vec{B}=\nabla\times\vec{A}$

$\partial^\alpha=\left(\frac{1}{c}\frac{\partial}{\partial t},-\nabla\right)$

$E_x=-\frac{\partial\phi}{\partial x}-\frac{1}{c}\frac{\partial A_x}{\partial t}=+\partial^1A^0-\partial^0A^1=-(\partial^0A^1-\partial^1A^0)\qquad E_y=-(\partial^0A^2-\partial^2A_0)$

$B_x=\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}=-\partial^2A^3+\partial^3A^2=-(\partial^2A^3-\partial^3A^2)\qquad B_y=\partial^1A^3-\partial^3A^1$

$$\boxed{F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha=\begin{pmatrix}0&-E_x&-E_y&-E_z\\E_x&0&-B_z&B_y\\E_y&B_z&0&-B_x\\E_z&-B_y&B_x&0\end{pmatrix}}$$

antisymmetric

$F' = A F\, {}^{T}\!A$

Lorentz transf. with v along x

$\tilde{x}^0 = \gamma(x^0 - \beta x_1)$ $\quad \tilde{x}^1 = \gamma(x^1 - \beta x^0)$ $\quad \tilde{x}^2 = x^2$ $\quad \tilde{x}^3 = x^3$ $\quad (\tilde{x}) = A(x)$

$$A = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2/8/2007

$$F' = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & 0 \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & \\ & & \cdot & \\ & & & \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & & 0 \\ & & & \\ 0 & & & \end{pmatrix}$$

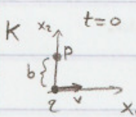
$x \to 1$
$y \to 2$
$z \to 3$

$$F' = \begin{pmatrix} 0 & -E_1 & -\gamma E_2 + \gamma\beta B_3 & -\gamma E_3 - \gamma\beta B_2 \\ & 0 & \gamma\beta E_2 - \gamma B_3 & \gamma\beta E_3 + \gamma B_2 \\ \text{antisymmetric} & & 0 & -B_1 \\ & & & 0 \end{pmatrix}$$

$E_1' = E_1$ $\quad E_2' = \gamma(E_2 - \beta B_3)$ $\quad E_3' = \gamma(E_3 + \beta B_2)$

$B_1' = B_1$ $\quad B_2' = \gamma(B_2 + \beta E_3)$ $\quad B_3' = \gamma(B_3 - \beta E_2)$

Fields of a charge in uniform motion:



$\vec{v} = v\hat{x}_1$ $\quad$ P: observation point in K $\quad P = (0, b, 0)$

K': rest frame of charge $\quad x^{0'} = \gamma(x^0 - \beta x^1)$ $\quad x^{1'} = \gamma(x^1 - \beta x^0)$

$x^0 = \gamma(x^{0'} + \beta x^{1'})$ $\quad x^1 = \gamma(x^{1'} + \beta x^{0'})$

$x_1' = \gamma(0 - \beta ct)$ $\quad ct' = \gamma(ct - (0))$

$P' = (-\gamma v t, b, 0) = (-vt', b, 0)$

fields in K': $\quad \vec{E} = q\dfrac{\hat{r}_1}{r^2}$ $\quad r'^2 = (vt')^2 + b^2$

$E_1' = q\dfrac{-vt'}{((vt')^2 + b^2)^{3/2}}$ $\quad E_y' = q\dfrac{b}{((vt')^2 + b^2)^{3/2}}$ $\quad E_z' = 0$ $\quad \vec{B}' = 0$

in K:

$E_1 = E_1'$ $\quad E_2 = \gamma E_2'$ $\quad E_3 = \gamma E_3' = 0$ $\quad B_1 = B_2 = 0$ $\quad B_3 = \gamma\beta E_2' = \beta E_2$

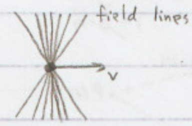
$E_1 = \dfrac{-q\gamma v t}{((\gamma v t)^2 + b^2)^{3/2}}$ $\quad E_2 = \dfrac{q\gamma b}{((\gamma v t)^2 + b^2)^{3/2}}$ $\quad B_3 = \dfrac{q\gamma\beta b}{((\gamma v t)^2 + b^2)^{3/2}}$

$t=0$ $\quad E_x = E_z = 0$ $\quad E_y = \gamma\dfrac{q}{b^2}$ $\quad$ for $\gamma \gg 1$ $\quad \beta \simeq 1$ $\quad B_z \simeq E_y$ $\quad$ looks like flash of a plane wave short times $\quad t \ll \frac{b}{\gamma v}$ $\quad E_x \simeq \frac{q}{b^2}\left(\frac{-\gamma v t}{b}\right) = \gamma \, \frac{q}{b^2}\left(\frac{-vt}{b}\right)$ $\quad E_y \simeq \gamma \frac{q}{b^2}$

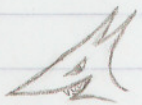
$\to$ see Coulomb field enhanced by γ $\qquad$ enhanced $\quad$ coulomb $\quad \sim\cos\theta$

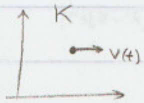
long times $\quad t \gg \frac{b}{\gamma v}$ $\quad E_x \simeq \frac{1}{\gamma^2}\frac{q}{(vt)^2}$

$\to$ see Coulomb field suppressed by $\frac{1}{\gamma^2}$ $\quad \therefore$



E_x averages out since it changes sign





$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad x' = \gamma(x - vt)$$

$$dx = \gamma(dx' + v\,dt') \qquad dt = \gamma\left(dt' + \frac{v}{c^2}dx'\right)$$

$$\frac{dx}{dt} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} \qquad \boxed{u = \frac{u' + v}{1 + \frac{vu}{c^2}}}$$

$$\frac{d}{dt}\frac{dx}{dt} = \frac{dt'}{dt}\frac{d}{dt'}\left(\frac{dx}{dt}\right) + \underbrace{\frac{dx'}{dt}\frac{d}{dx'}\cancel{\left(\frac{dx}{dt}\right)}}_{0} \qquad \frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)$$

if $\frac{dx'}{dt'} = 0$ (frame of moving particle) $\quad \frac{dx}{dt} = v \quad \frac{dt'}{dt} = \frac{1}{\gamma} \quad \left[dt' = \frac{dt}{\gamma}\right]$

$$\frac{d}{dt}\frac{dx}{dt} = \gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)\frac{\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)\frac{d^2x'}{dt'^2} - \left(\frac{dx'}{dt'} + v\right)\frac{v}{c^2}\frac{d^2x'}{dt'^2}}{\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)^2} = \frac{1}{\gamma^3}\frac{\frac{d^2x'}{dt'^2}}{\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)^3}$$

Moving Charges (Chpt 14) $\qquad$ 2/15/2007

Potential of a moving charge $\quad \phi(\vec{x}, t) = \int d^3x' \, \frac{\rho(\vec{x}', t')}{|\vec{x} - \vec{x}'|} \qquad t' = t - \frac{|\vec{x} - \vec{x}'|}{c}$

charge q with vel. $\vec{v}$

trajectory $\vec{x}_0(t)$ of charge $\qquad \rho(\vec{x}', t') = q\,\delta(\vec{x}' - \vec{x}_0(t'))$

$$\Phi(\vec{x}, t) = \int q\,\frac{\delta(\vec{x}' - \vec{x}_0(t'))}{|\vec{x} - \vec{x}'|}\,d^3x' \qquad \text{change of variables} \quad \vec{y} = \vec{x}' - \vec{x}_0(t') \qquad \Phi = \int d^3y \, \overset{\text{det(Jacobian)}}{\det(J)} \, \frac{q\,\delta(\vec{y})}{|\vec{x} - \vec{y} - \vec{x}_0(t')|}$$

$$\Phi = \det(J)(\vec{x}' = \vec{x}_0(t'))\,\frac{q}{|\vec{x} - \vec{x}_0(t')|} = \int \det(J)(\vec{x}')\,\frac{q}{|\vec{x} - \vec{x}'|}\,\delta(\vec{x}' - \vec{x}_0(t'))\,d^3x'$$

$$\boxed{J_{ij} = \left(\frac{\partial y_i}{\partial x'_j}\right)^{-1}} \text{ inverse of matrix} \qquad \boxed{\det\left(\left(\frac{\partial y_i}{\partial x'_j}\right)^{-1}\right) = \frac{1}{\det\left(\frac{\partial y_i}{\partial x'_j}\right)}}$$

$$\frac{\partial y_i}{\partial x'_j} = \underset{\uparrow \, \frac{\partial x'_i}{\partial x'_j}}{\delta_{ij}} - \frac{\partial x_{0i}(t')}{\partial x'_j} \qquad \frac{\partial x_{0i}(t')}{\partial x'_j} = \frac{d\,x_{0i}(t')}{dt'}\frac{\partial t'}{\partial x'_j} = c\,\overset{\text{vel of particle at retarded time}}{\beta_i(t')}\,\frac{1}{c}\,n_j = \beta_i n_j$$

$$\frac{\partial t'}{\partial x'_j} = -\frac{1}{c}\frac{\partial}{\partial x'_j}\left((x_1 - x'_1)^2 + (x_2 - x'_2)^2 + \dots\right)^{1/2} = +\frac{1}{c}\,\frac{1}{2}\,\frac{2(x_j - x'_j)}{|\vec{x} - \vec{x}'|}$$

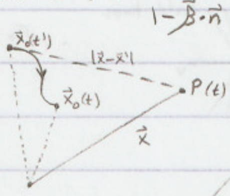
$$\boxed{\frac{\partial y_i}{\partial x'_j} = \delta_{ij} - \beta_i n_j} \qquad n_j = \frac{x_j - x'_j}{|\vec{x} - \vec{x}'|} \quad \vec{n} = \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}$$

$$\frac{\partial y_i}{\partial x'_j} = \begin{pmatrix} 1-\beta_1 n_1 & -\beta_1 n_2 & -\beta_1 n_3 \\ -\beta_2 n_1 & 1-\beta_2 n_2 & -\beta_2 n_3 \\ -\beta_3 n_1 & -\beta_3 n_2 & 1-\beta_3 n_3 \end{pmatrix} \qquad \det(\,) = 1-\beta_1 n_1 - \beta_2 n_2 - \beta_3 n_3 = 1-\vec\beta\cdot\vec n$$

$$\det(J) = \frac{1}{1-\vec\beta\cdot\vec n} \qquad \therefore \quad \boxed{\phi(\vec x, t) = \left[\frac{q}{|\vec x - \vec x_0|}\,\frac{1}{1-\vec\beta\cdot\vec n}\right]_{ret}} \qquad \vec n = \frac{\vec x - \vec x_0}{|\vec x - \vec x_0|} \qquad t' = t - \frac{|\vec x - \vec x_0(t')|}{c}$$

retarded time

defined implicitly



Liénard–Wiechert potentials

$$\vec A = \frac{1}{c}\int \frac{\vec J(\vec x', t')}{|\vec x - \vec x'|}\, d^3x' \qquad t' = t - \frac{|\vec x - \vec x'|}{c} \qquad \vec J = q\vec v\, \delta(\vec x' - \vec x_0(t'))$$

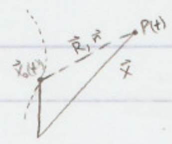
$$\boxed{\vec A = \left[\frac{q\vec\beta}{|\vec x - \vec x_0|(1-\vec\beta\cdot\vec n)}\right]_{ret}}$$

$$\vec E = -\nabla\phi - \frac{1}{c}\frac{\partial \vec A}{\partial t} \qquad \vec B = \nabla\times\vec A$$

complicated   ex. $\frac{\partial\phi}{\partial x_1} \sim \frac{\partial}{\partial x_1}\frac{1}{|\vec x - \vec x_0|} = -\frac{1}{2}\frac{\frac{\partial}{\partial x_1}\{(x_1-x_{01})^2+\dots\}}{|\vec x - \vec x_0|^3}$

both depend on x_1

$$\frac{\partial x_0(t')}{\partial x_1} = \underbrace{\frac{dx_{01}}{dt'}}_{\text{vel.}}\frac{\partial t'}{\partial x_1}$$



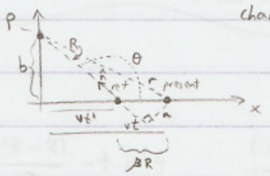
actually, 3

$$\boxed{\vec E = q\left[\frac{\vec n - \vec\beta}{\gamma^2(1-\vec\beta\cdot\vec n)^2 R^2}\right]_{ret} + \frac{q}{c}\left[\frac{\vec n \times \{(\vec n - \vec\beta)\times\dot{\vec\beta}\}}{(1-\vec\beta\cdot\vec n)^3 R}\right]_{ret}}$$

static field   radiation field ($\dot\beta$)

$$R = |\vec x - \vec x_0| \qquad t' = t - \frac{R(t')}{c} \qquad \vec n = \frac{\vec R}{R}$$

$$\boxed{\vec B = [\vec n \times \vec E]_{ret}}$$



charge q $\vec\beta \parallel \hat x$

$$\vec\beta\cdot\vec n = \beta\cos\theta \qquad t' = t - \frac{R}{c} \qquad vt' = vt - \beta R$$

$$(R(1-\vec\beta\cdot\vec n))^2 = (R - R\beta\cos\theta)^2 = r^2 - a^2 \qquad r^2 = b^2 + (vt)^2$$

$a = \beta R\sin\theta = \beta b$ $r^2 - a^2 = b^2 + (vt)^2 - \beta^2 b^2 = \frac{b^2}{\gamma^2} + (vt)^2$

$\beta_2 = 0 \quad n_2 = \sin\theta$

$$E_2 = \frac{q\,b}{\gamma^2\left(\frac{b^2}{\gamma^2} + (vt)^2\right)^{3/2}} = \frac{q\,b\,\gamma}{(b^2 + (\gamma v t)^2)^{3/2}} \quad \text{as before}$$

Radiation by Charges

2/20/2007

radiation part:

$$\vec{E} = \frac{q}{c}\left[\frac{\vec{n}\times\{(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{(1-\vec{\beta}\cdot\vec{n})^3 R}\right]_{ret} \qquad \vec{B} = [\vec{n}\times\vec{E}]_{ret} \qquad t' = t - \frac{R(t')}{c} \text{ retarded time}$$

Power Radiated non relativistic $\beta \ll 1$

$$\vec{E} \simeq \frac{e}{c}\left(\frac{\vec{n}\times(\vec{n}\times\dot{\vec{\beta}})}{R}\right)_{ret} \qquad \vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B}) \qquad \vec{S}\cdot\vec{n} = \frac{c}{4\pi}|\vec{E}|^2 = \frac{c}{4\pi}\frac{e^2}{c^2}\frac{1}{R^2}|\vec{n}\times(\vec{n}\times\dot{\vec{\beta}})|^2$$

$$dP = \vec{S}\cdot\vec{n}\, R^2 d\Omega \qquad \frac{dP}{d\Omega} = \frac{e^2}{4\pi c}|\vec{n}\times(\vec{n}\times\dot{\vec{\beta}})|^2$$

$$\therefore |\vec{n}\times(\vec{n}\times\dot{\vec{\beta}})| = \sin^2\theta\cdot\dot{\beta}^2 = \frac{\dot{v}^2}{c^2}\sin^2\theta$$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}|\dot{v}|^2\sin^2\theta \qquad \int\sin^2\theta\, d\Omega = \frac{8\pi}{3} \qquad \therefore \boxed{P_{tot} = \frac{2}{3}\frac{e^2}{c^3}|\dot{v}|^2} \quad \text{Larmor result}$$

β not small:

$$\boxed{P_{tot} = -\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right)} \qquad \tau \text{ — proper time} \qquad d\tau = \frac{dt}{\gamma}$$

$$p^\mu = \left(\frac{E}{c},\vec{p}\right) \qquad \frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2 - \left|\frac{d\vec{p}}{d\tau}\right|^2 \qquad \frac{E^2}{c^2} - |\vec{p}|^2 = m^2c^2$$

$$\frac{2}{c^2}E\frac{dE}{d\tau} - 2\vec{p}\cdot\frac{d\vec{p}}{d\tau} = 0 \qquad \frac{dE}{d\tau} = c^2\frac{\vec{p}}{E}\cdot\frac{d\vec{p}}{d\tau}$$

$$= \left(\vec{\beta}\cdot\frac{d\vec{p}}{d\tau}\right)^2 - \left|\frac{d\vec{p}}{d\tau}\right|^2 = \frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} \qquad \vec{p} = \gamma m\vec{v} \quad E = \gamma mc^2 \qquad \frac{dE}{d\tau} = \vec{v}\cdot\frac{d\vec{p}}{d\tau}$$

for $\beta \ll 1 \rightarrow -\left|\frac{dp}{d\tau}\right|^2 = -\left|\frac{dp}{dt}\right|^2 = -m^2|\dot{v}|^2 \rightarrow$ get Larmor result

Total power radiated by relatinistic particles:

a) $\dot{\vec{\beta}} \parallel \vec{\beta}$ (linear accelerator): $\quad \frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = (\beta^2-1)\left|\frac{d\vec{p}}{d\tau}\right|^2$

$$P_{tot} = \frac{2}{3}\frac{e^2}{m^2c^3}\frac{1}{\gamma^2}\left|\frac{d\vec{p}}{d\tau}\right|^2 = \frac{2}{3}\frac{e^2}{m^2c^3}\left|\frac{d\vec{p}}{dt}\right|^2$$

in terms of energy gain per length: $\frac{dE}{dx}$ $\qquad \frac{E^2}{c^2} - p^2 = mc^2 \qquad 2\frac{1}{c^2}E\frac{dE}{dt} - p2\frac{dp}{dt} = 0$

$$\frac{dE}{dt} = \frac{dx}{dt}\frac{dE}{dx} = v\frac{dE}{dx} \qquad v\frac{dE}{dx} = c^2\frac{v}{c^2}\frac{dp}{dt} \qquad \therefore \frac{dE}{dx} = \frac{dp}{dt}$$

$$\therefore P_{tot} = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2$$

power supplied: $\frac{dE}{dt} = v\frac{dE}{dx}$ $\qquad$ ratio of radiated power (power lost) to supplied: $\quad \frac{P}{dE/dt} = \frac{2}{3}\frac{e^2}{m^2c^3}\frac{1}{v}\frac{dE}{dx}$ $\quad$ ($v \sim c$)

$$\frac{P}{dE/dt} = \frac{2}{3}\frac{e^2/mc^2}{mc^2}\frac{dE}{dx} \ll 1 \rightarrow \text{in practice } \frac{dE}{dx} \ll \frac{mc^2}{e^2/mc^2} \sim \frac{0.5\text{ MeV}}{10^{-13}\text{ cm}} \sim 0.5\times10^{13}\text{ MeV/cm}$$

$\therefore$ radiation losses are negligible in linear accelerators

$\frac{e^2}{mc^2}$ – "classical radius of e^-"

b) $\dot{\vec{\beta}} \perp \vec{\beta}$ (circular accelerator): $\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = -\left|\frac{d\vec{p}}{d\tau}\right|^2$

$P_{tot} = \frac{2}{3}\frac{e^2}{m^2c^3}\left|\frac{d\vec{p}}{d\tau}\right|^2$ in lab $\frac{d\vec{p}}{dt} = \vec{\omega}\times\vec{p} = \frac{1}{\gamma}\frac{d\vec{p}}{d\tau}$ $\therefore \left|\frac{d\vec{p}}{d\tau}\right| = \gamma\omega p$

$\therefore P_{tot} = \frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2\omega^2 p^2$

energy loss per turn: $P_{tot}\frac{2\pi}{\omega} = \delta E = \frac{4\pi}{3}\frac{e^2}{m^2c^3}\omega p^2\gamma^2$

$\omega R = v \simeq c$ $\omega = \frac{c}{R}$ $p = \gamma m v \simeq \gamma m c$ $E = \gamma m c^2$ $\gamma = \frac{E}{mc^2}$

$p^2 = \gamma^2 m^2 c^2 = \frac{E^2}{c^2}$ $\therefore \delta E = \frac{4\pi}{3}\frac{e^2}{m^2c^3}\frac{c}{R}\frac{E^2}{c^2}\gamma^2 = \frac{4\pi}{3}\frac{e^2}{m^2c^3}\frac{c}{R}\frac{E^4}{c^2\, m^2c^4}$

$\delta E = \frac{4\pi}{3}\frac{e^2}{R}\frac{E^4}{(mc^2)^4}$ 1 GeV e^- in $R = 100\text{m}$: $\delta E \sim 10\text{ eV}$ can get larger with $E\uparrow$ or $R\downarrow$

Angular Distribution of Radiation by Fast Particle

$$\vec{S}\cdot\vec{n}(t) = \frac{c}{4\pi}\frac{e^2}{c^2}\frac{1}{R^2}\left[\frac{\vec{n}\times((\vec{n}-\vec{\beta})\times\dot{\vec{\beta}})}{(1-\vec{\beta}\cdot\vec{n})^3}\right]^2_{ret}$$

"instantaneous" power $\frac{d\tilde{P}}{d\Omega} = \frac{e^2}{4\pi c}\left[\ \right]^2_{ret}$ actual instantaneous power: $\frac{dP}{d\Omega} = \frac{d\tilde{P}}{d\Omega}\frac{dt}{dt'}$ $t' = t - \frac{R(t')}{c}$

(not quite: $\int \frac{d\tilde{P}}{d\Omega}d\Omega \neq P_{tot}$

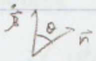
$t = t' + \frac{R(t')}{c}$ $\frac{dt}{dt'} = 1 + \frac{1}{c}\frac{dR(t')}{dt'}$ $R(t') = |\vec{x}-\vec{x}_0(t')| = \sqrt{(x_1-x_{01})^2+\ldots}$

$\frac{dR(t')}{dt'} = \frac{1}{|\vec{x}-\vec{x}_0|}(-1)\frac{d\vec{x}_0}{dt'}\cdot(\vec{x}-\vec{x}_0) = -\vec{v}\cdot\vec{n}$

$\therefore \boxed{\frac{dt}{dt'} = 1 - \vec{\beta}\cdot\vec{n}(t')}$

$$\therefore \boxed{\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\frac{|\vec{n}\times((\vec{n}-\vec{\beta})\times\dot{\vec{\beta}})|^2}{(1-\vec{\beta}\cdot\vec{n})^5}}$$

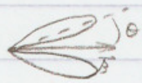
Check: $\int\frac{dP}{d\Omega}d\Omega = P_{tot}$ special case: $\dot{\vec{\beta}} \parallel \vec{\beta}$



$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}|\dot{\vec{\beta}}|^2\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$ $2\pi\int_{-1}^{1}d(\cos\theta)\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} = 2\pi\frac{4}{3}\frac{1}{(1-\beta^2)^3}$ $\therefore P = \frac{2}{3}\frac{e^2}{c^3}\frac{|\dot{\vec{v}}|^2}{(1-\beta^2)^3} = \frac{2}{3}\frac{e^2}{c^3}\gamma^6\dot{v}^2$

$\frac{dp}{dt} = \gamma m\dot{v} + mv\frac{d\gamma}{dt} = \gamma mc\dot{\beta} + \gamma^3 mc\beta^2\dot{\beta} = \gamma mc\dot{\beta}(1+\beta^2\gamma^2) = \gamma^3 mc\dot{\beta}$ $\therefore P = \frac{2}{3}\frac{e^2}{c^3}\gamma^6\dot{v}^2$ ✓

$1 + \frac{\beta^2}{1-\beta^2} = \frac{1-\beta^2+\beta^2}{1-\beta^2} = \frac{1}{1-\beta^2} = \gamma^2$



$\theta \sim \frac{1}{\gamma}$

2/22/2007

Aiya
Eärendil
Elenion
Ancalima

$$\vec{E} = \frac{e}{c}\left(\frac{\vec{n}\times\left((\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{R\,(1-\vec{\beta}\cdot\vec{n})^3}\right)_{ret} \qquad \vec{B} = \vec{n}\times\vec{E}$$

Energy radiated between $t' = T_1, T_2$ $\qquad E = \int_{T_1+\frac{R(T_1)}{c}}^{T_2+\frac{R(T_2)}{c}} \frac{d\tilde{P}}{d\Omega}(t)\,dt \qquad \frac{d\tilde{P}}{d\Omega}(t) = \frac{e^2}{4\pi c}\left[\ \right]^2_{ret}$

$$dt = \frac{dt}{dt'}dt' = (1-\vec{\beta}\cdot\vec{n})\,dt'$$

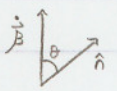
$$t = t' + \frac{R(t')}{c} \qquad E = \int_{T_1}^{T_2} \frac{e^2}{4\pi c}\left[\ \right]^2 dt'(1-\vec{\beta}\cdot\vec{n}) \qquad \therefore \frac{dP}{d\Omega} = (1-\vec{\beta}\cdot\hat{n})\frac{d\tilde{P}}{d\Omega}$$

$$\boxed{\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left|\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)\right|^2}{(1-\vec{\beta}\cdot\hat{n})^5}}$$

Pattern of Radiation

Simple cases

$\dot{\vec{\beta}} \parallel \vec{\beta}$:



$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}|\dot{v}|^2\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \qquad \text{max of } \frac{dP}{d\Omega}:\ \frac{d}{d\theta}\frac{\sin^2\theta}{(\ )^5} = 0$$

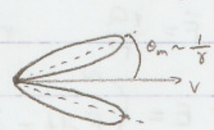
$$\rightarrow \cos\theta_m = \frac{1}{3\beta}\left(-1+\sqrt{1+15\beta^2}\right) \qquad \beta\simeq 1 \quad \beta = 1-\varepsilon \quad \beta^2 \simeq 1-2\varepsilon = 1-2(1-\beta)$$

$$\boxed{1-\beta \simeq \frac{1}{2}(1-\beta^2) = \frac{1}{2\gamma^2}} \qquad \cos\theta_m = 1-\frac{\theta^2}{2} = \frac{1}{3(1-\varepsilon)}\left[-1+\sqrt{1+15-30\varepsilon}\right]$$

$$\sqrt{16-30\varepsilon} \qquad 4\sqrt{1-\tfrac{15}{8}\varepsilon} \simeq 4\left(1-\tfrac{15}{16}\varepsilon\right)$$

$$1-\frac{\theta^2}{2} \simeq \frac{1}{3}(1+\varepsilon)\left(3-\frac{15}{4}\varepsilon\right) = (1+\varepsilon)\left(1-\frac{5}{4}\varepsilon\right) \simeq 1-\frac{1}{4}\varepsilon$$

$$\frac{\theta^2}{2} \simeq \frac{1}{4}\varepsilon \qquad \theta^2 \simeq \frac{\varepsilon}{2} = \frac{1-\beta}{2} = \frac{1}{4\gamma^2} \qquad \therefore \boxed{\theta_m \simeq \frac{1}{2\gamma}}$$



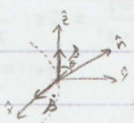
10 GeV e^- $\gamma = 10^4$ almost parallel beam with high intensity

$$1-\beta\left(1-\frac{\theta^2}{2}\right) = 1-\beta+\beta\left(\frac{1}{2\gamma}\right)^2 \simeq \frac{1}{2\gamma^2}+\frac{1}{8\gamma^2} = \frac{5}{8}\frac{1}{\gamma^2}$$

peak power $\displaystyle \left.\frac{dP}{d\Omega}\right|_{\theta_m} \sim \left(\frac{1}{2\gamma}\right)^2\left(\frac{8}{5}\right)^5\gamma^{10}\frac{e^2}{4\pi c^3}|\dot{v}|^2 \sim (*)\frac{e^2}{4\pi c^3}|\dot{v}|^2\gamma^8 \propto \underline{\gamma^8}$

$$P \sim \left(\frac{dP}{d\Omega}\right)_m \Delta\Omega \qquad \Delta\Omega \propto \theta_m^2 \qquad \therefore P \sim \gamma^8\frac{1}{\gamma^2} \sim \gamma^6$$

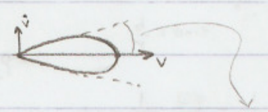
$\dot{\vec{\beta}} \perp \vec{\beta}$:



$$\vec{\beta} = (0,0,\beta) \qquad \dot{\vec{\beta}} = (\dot{\beta},0,0) \qquad \hat{n} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

$$|\cdot|^2 = \dot{\beta}^2\left\{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\varphi\right\} \qquad \frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}|\dot{v}|^2\frac{1}{(1-\beta\cos\theta)^3}\left(1-\frac{\sin^2\theta\cos^2\varphi}{\gamma^2(1-\beta\cos\theta)^2}\right)$$

$\theta_m = 0$ $\qquad$ peak power $\displaystyle \left.\frac{dP}{d\Omega}\right|_{\theta_m} = \frac{e^2\dot{v}^2}{4\pi c^3}8\gamma^6 \propto \underline{\gamma^6}$



$\theta \ll 1$ $\displaystyle \frac{1}{(1-\beta\cos\theta)^3} \simeq \frac{1}{\left(1-\beta+\beta\frac{\theta^2}{2}\right)} = \frac{1}{\frac{1}{2\gamma^2}+\frac{\theta^2}{2}} = \frac{2\gamma^2}{1+\theta^2\gamma^2}$ $\qquad$ opening angle $\displaystyle \frac{\theta^2}{2} \sim \frac{1}{2\gamma^2}$ $\quad \theta \sim \frac{1}{\gamma}$

total power at peak $\displaystyle P \sim \left(\frac{dP}{d\Omega}\right)_m \Delta\Omega \sim \gamma^6\frac{1}{\gamma^2} \sim \gamma^4$

$$\dot{\vec{\beta}} \perp \vec{\beta}:\ P = \frac{2}{3}\frac{e^2}{c^3}|\dot{v}|^2\gamma^4 \qquad \dot{\vec{\beta}} \parallel \vec{\beta}:\ P = \frac{2}{3}\frac{e^2}{c^3}|\dot{v}|^2\gamma^6$$

in terms of applied force

circular motion $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{v}) = \gamma m \frac{d\vec{v}}{dt}$ $\quad |\dot{\vec{v}}|^2 = \frac{F^2}{\gamma^2 m^2}$

linear motion $\vec{F} = \frac{d}{dt}(\gamma m\vec{v}) = \gamma m \frac{dv}{dt} + mv\frac{d\gamma}{dt}$ $\quad \frac{d\gamma}{dt} = \gamma^3 \frac{\beta}{c}\dot{v}$ $\quad \therefore |\dot{\vec{v}}|^2 = \frac{F^2}{\gamma^6 m^2}$

$$= \gamma m\dot{v} + m\beta^2\dot{v}\gamma^3 = \gamma m\dot{v}(1+\gamma^2\beta^2) = \gamma^3 m\dot{v}$$

$$1+\frac{\beta^2}{1-\beta^2} = \frac{1-\beta^2+\beta^2}{1-\beta^2} = \frac{1}{1-\beta^2} = \gamma^2$$

$\therefore P_\perp = \frac{2}{3}\frac{e^2}{c^3}\frac{F^2}{m^2}\gamma^2$ $\qquad P_\parallel = \frac{2}{3}\frac{e^2}{c^3}\frac{F^2}{m^2}$

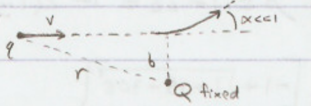
$\rightarrow$ power radiated comes from $\dot{\beta}_\perp$ $(\gamma \gg 1)$

$\dot{v}_\perp = \frac{v^2}{R} \simeq \frac{c^2}{R}$

2/27/2007

HW 6: 14. 4, 5, 7a,b, 12

Power radiated in large impact parameter collision



small deflection

a) Non-relativistic $\beta \ll 1$ $\qquad$ Larmor formula: $P = \frac{2}{3}\frac{q^2}{c^3}|\dot{\vec{v}}|^2$

$\vec{F} = \frac{qQ}{r^2}$ $\qquad r^2 = (vt)^2 + b^2$ $\qquad \vec{F} = m\dot{\vec{v}}$ $\qquad \dot{v} = \frac{F}{m} = \frac{qQ}{mr^2} = \frac{qQ}{m((vt)^2+b^2)}$

$$E = \int_{-\infty}^{\infty} P\,dt = \frac{2}{3}\frac{Q^2 q^4}{m^2 c^3}\int_{-\infty}^{\infty}\frac{dt}{((vt)^2+b^2)^2}$$

$$\text{Integral} = \frac{1}{b^4}\frac{b}{v}\int_{-\infty}^{\infty}\frac{d(\frac{vt}{b})}{(1+(\frac{vt}{b})^2)^2} = \underbrace{\frac{1}{vb^3}}_{\text{important part} \therefore E\propto\frac{1}{vb^3}}\underbrace{\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2}}_{\text{a number}: \alpha\sim 1} = \frac{\alpha}{vb^3}$$

$$\boxed{E = \alpha\frac{2}{3}\frac{Q^2 q^4}{m^2 c^3 b^3 v}}$$

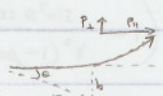
$\int_\Gamma \frac{dz}{1+z^2}$

$\cdot (z+i)(z-i)$ $\qquad \frac{f(z)}{z-i}$ $\qquad \int_\Gamma \frac{dz}{1+z^2} = 2\pi i\,\underset{f(i)}{\frac{1}{2i}} = \pi$

$2\pi i\,\underbrace{(\text{Residue at pole})}$ $f(\text{pole})$ $\quad f = \frac{1}{z+i}$

$$\int_\Gamma \frac{f(z)\,dz}{(z-z_0)^2} = 2\pi i f'(z_0) \text{ ?}$$

$\therefore \int\frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$ $\qquad f(z) = \frac{1}{(z+i)^2}$ $\quad f'(z) = \frac{-2}{(z+i)^3}$



small deflection $\qquad p_\parallel \gg p_\perp$ $\qquad \frac{p_\perp}{p_\parallel} \ll 1$ $\qquad p_\parallel = mv$ $\qquad p_\perp = \int_{-\infty}^{\infty}\frac{dp_\perp}{dt}dt$ $\qquad \frac{dp_\perp}{dt} = \frac{qQ}{r^2}\frac{b}{r}$

$\sin\theta = \frac{b}{r}$

$$p_\perp = \int_{-\infty}^{\infty}\frac{qQb}{((vt)^2+b^2)^{3/2}}dt = \frac{qQb}{b^3}\frac{b}{v}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{3/2}} = \frac{qQ}{vb}\ (\cancel{=})$$

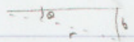
$p_\perp \sim \frac{qQ}{vb}$

$\frac{qQ}{vb}\frac{1}{mv} = \underbrace{\frac{qQ}{b}}_{PE}\underbrace{\frac{1}{mv^2}}_{KE} \ll 1$

$$\rightarrow \text{yes: } \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!}\underset{n^{th}\text{ derivative}}{f^{(n)}(z_0)}$$

b) Ultrarelativistic $\gamma \gg 1$ use $\dot{v}_\perp$ $P = \frac{2}{3}\frac{e^2}{c^3}\dot{v}^2\gamma^4$ for $\dot{\vec{\beta}} \perp \vec{\beta}$

$\frac{d\vec{p}}{dt} = \frac{qQ}{r^2}$ $\vec{p} = \gamma m \vec{v}$ $\frac{d}{dt}(\gamma m \vec{v}) = \gamma m \dot{\vec{v}}_\perp$ considering only perpendicular motions

$\dot{v}_\perp^2 = \left(\frac{qQ}{r^2}\frac{b}{r}\right)^2 \frac{1}{\gamma^2 m^2}$  $\sin\theta = \frac{b}{r}$ $E = \frac{2}{3}\frac{q^2}{c^3}\gamma^4 \frac{q^2Q^2}{\gamma^2 m^2} b^2 \underbrace{\int_{-\infty}^{\infty} \frac{dt}{((vt)^2+b^2)^3}}_{\frac{1}{b^6}\frac{b}{v}\underbrace{\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^3}}_{A = \frac{3\pi}{8}}}$ $v \sim c$

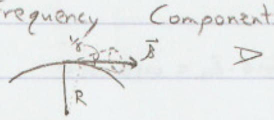
$\dot{v}_\perp^2 \propto \frac{1}{r^6}$ $r^2 = (vt)^2 + b^2$

$$\boxed{E = \frac{2}{3}\frac{q^4 Q^2}{m^2 c^4}\frac{\gamma^2}{b^3} A}$$

Contribution from $\dot{\vec{\beta}} \parallel \vec{\beta}$ $P = \frac{2}{3}\frac{q^2}{c^3}\dot{v}_\parallel^2 \gamma^6$ $\frac{dp_\parallel}{dt} = \gamma m \dot{v} + m v \dot{\gamma}$

$P = \frac{2}{3}\frac{q^2}{c^3}\frac{F_\parallel^2}{m^2}$ small in ultrarelativistic limit $F_\parallel = \frac{qQ}{r^2}\frac{vt}{r}$

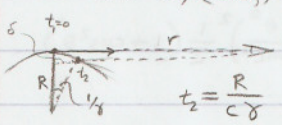
Frequency Components



pulse width Δt $\Delta E \Delta t \sim h$ $\Delta E = h\Delta\nu$ $\Delta\nu \sim \frac{1}{\Delta t}$

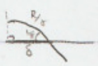
light contains frequencies at least up to $\frac{1}{\Delta t}$

one would say (wrong): $R\frac{1}{\gamma} = c\Delta t$ $\Delta t = \frac{R}{\gamma c}$ wrong



$\tilde{t}_1 = 0 + \frac{r}{c} = \frac{r}{c}$ light gets to observer

$t_2 = \frac{R}{c\gamma}$ $\tilde{t}_2 = t_2 + \frac{r-\delta}{c}$ if $\delta = \frac{R}{\gamma}$ $\Delta\tilde{t} = \tilde{t}_2 - \tilde{t}_1 = \frac{r}{c} + \frac{R}{\gamma c} - \frac{R}{c\gamma} - \frac{r}{c} = 0$ wrong



$\delta = \frac{R}{\gamma}\cos(1/\gamma) \simeq \frac{R}{\gamma}\left(1 - \frac{1}{2\gamma^2}\right)$ $\Delta\tilde{t} = \tilde{t}_2 - \tilde{t}_1 = \frac{R}{c\gamma} + \cancel{\frac{r}{c}} - \frac{R}{\gamma c} + \frac{R}{2\gamma^3 c} - \cancel{\frac{r}{c}} = \frac{R}{2\gamma^3 c}$

$\Delta t \sim \frac{R}{\gamma^3 c}$ $\Delta\nu \sim \frac{c}{R}\gamma^3$ high freq. can get x-rays from synchrotron radiation

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3/1/2007

Thomson Scattering

EM wave scattering off a free charge

nonrelativistic: ok if $\frac{h\nu}{c} \ll mc$ $(h\nu \ll mc^2)$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\left|\underbrace{\hat{n}\times(\hat{n}\times\dot{\vec{v}})}_{\text{polarization}}\right|^2$$ incoming wave: $\vec{E} = \vec{\varepsilon}_{in} E_0 e^{i(\vec{k}_0\cdot\vec{x} - \omega t)}$

$\dot{\vec{v}} = \frac{e\vec{E}}{m} = \frac{e}{m}E_0\vec{\varepsilon}_{in} e^{i(\vec{k}_0\cdot\vec{x}-\omega t)}$ if amplitude of motion of charge is $\ll \lambda$: $\dot{\vec{v}} \simeq \frac{eE_0}{m}\vec{\varepsilon}_{in} e^{-i\omega t} e^{i\,\text{const}}$

Power radiated (in $\hat{n}$) with polarization $\vec{\varepsilon}_{out}$

$$\frac{dP}{d\Omega}(\vec{\varepsilon}_{out}) = \frac{e^2}{4\pi c^3}\left|\vec{\varepsilon}_{out}\cdot\left[\hat{n}\times(\hat{n}\times\dot{\vec{v}})\right]\right|^2$$ $\vec{\varepsilon}_{out}\cdot\left\{(\hat{n}\cdot\dot{\vec{v}})\hat{n} - \dot{\vec{v}}\right\} = \vec{\varepsilon}_{out}\cdot\dot{\vec{v}}$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\left|\vec{\varepsilon}_{out}\cdot\dot{\vec{v}}\right|^2$$

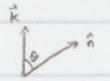
$$\frac{dP}{d\Omega}(\vec{\varepsilon}_{in},\vec{\varepsilon}_{out}) = \frac{e^4E^2}{4\pi m^2c^3}\,\underbrace{\frac{1}{2}}_{\text{time averaged} \to \frac{1}{2}|\vec{A}\cdot\vec{B}^*|^2}\,\left|\vec{\varepsilon}_{out}\cdot\vec{\varepsilon}_{in}^{\,*}\right|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{dP}{d\Omega}\underbrace{\frac{8\pi}{cE^2}}_{(\text{flux in})^{-1}} \qquad \therefore \qquad \frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2\left|\vec{\varepsilon}_{out}\cdot\vec{\varepsilon}_{in}^{\,*}\right|^2$$

flux in $= S = \frac{c}{8\pi}E^2$

$\frac{e^2}{mc^2}$ = "classical e⁻ radius"

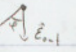
average over initial polarizations ($\vec{\varepsilon}_{in}$) then sum over $\vec{\varepsilon}_{out}$



a) $\vec{\varepsilon}_{out} \perp$ scattering plane

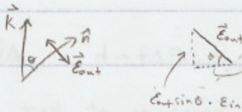
$\otimes\,\vec{\varepsilon}_{out}$ $\qquad \vec{\varepsilon}_{out}\cdot\vec{\varepsilon}_{in} = \cos\varphi$

"top down" ($\vec{k}$ out of page)



$$\langle|\vec{\varepsilon}_{out}\cdot\vec{\varepsilon}_{in}|\rangle = \frac{1}{2} \qquad \langle\cos^2\varphi\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\varphi\, d\varphi$$

b) $\vec{\varepsilon}_{out} \parallel$ scattering plane



$\varepsilon_{out}\sin\theta\cdot\varepsilon_{in} = 0$ $\qquad \varepsilon_{out}\cos\theta$

$$\vec{\varepsilon}_{out}\cos\theta\cdot\vec{\varepsilon}_{in} = \cos\theta\cos\varphi$$

$$\langle\cos^2\theta\cos^2\varphi\rangle = \cos^2\theta\,\langle\cos^2\varphi\rangle = \frac{1}{2}\cos^2\theta$$

$$\frac{d\sigma_\perp}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2\frac{1}{2} \qquad \frac{d\sigma_\parallel}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2\frac{1}{2}\cos^2\theta \qquad \frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2\frac{1}{2}\left(1+\cos^2\theta\right)$$

If relativistic $\longrightarrow$ get Compton scattering

$\left(\frac{h\nu}{c}, \hbar\vec{k}_{in}\right) \rightsquigarrow$ $\qquad (mc, 0)$ e⁻ $\qquad \rightsquigarrow \left(\frac{h\nu'}{c}, \hbar\vec{k}_{out}\right)$

e⁻ $(\gamma mc, \gamma m\vec{v})$

Motion of Particles in EM Fields (Chpt 12)

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) \qquad \vec{p} = \gamma m\vec{v} \qquad \frac{d\overset{\text{energy}}{\mathcal{E}}}{dt} = e\vec{E}\cdot\vec{v} \qquad \mathcal{E} = \gamma mc^2 \qquad P = \vec{F}\cdot\vec{v}$$

$\frac{d\gamma}{dt} = \gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}$

$$\frac{d\mathcal{E}}{dt} = mc^2\frac{d\gamma}{dt} = \gamma^3 mc^2\,\vec{\beta}\cdot\dot{\vec{\beta}} \qquad \vec{v}\cdot\frac{d\vec{p}}{dt} = e\vec{E}\cdot\vec{v} \qquad \frac{d\vec{p}}{dt} = \gamma m\frac{d\vec{v}}{dt} + m\vec{v}\frac{d\gamma}{dt} = \gamma mc\dot{\vec{\beta}} + mc\vec{\beta}(\gamma^3\vec{\beta}\cdot\dot{\vec{\beta}})$$

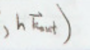
$$\vec{v}\cdot\frac{d\vec{p}}{dt} = mc^2\beta^2\gamma^3\vec{\beta}\cdot\dot{\vec{\beta}} + \gamma mc^2\vec{\beta}\cdot\dot{\vec{\beta}} = \gamma mc^2\vec{\beta}\cdot\dot{\vec{\beta}}\underbrace{(\beta^2\gamma^2+1)}_{\frac{\beta^2}{1-\beta^2}+1 = \frac{1}{1-\beta^2} = \gamma^2} = \gamma^3 mc^2\vec{\beta}\cdot\dot{\vec{\beta}} = \frac{d\mathcal{E}}{dt}$$

$$\therefore\ \frac{d\mathcal{E}}{dt} = \vec{v}\cdot\frac{d\vec{p}}{dt}$$

1. Uniform, constant $\vec{B}$

$$\frac{d\mathcal{E}}{dt} = 0 \qquad \gamma = \text{const.} \qquad \gamma m\frac{d\vec{v}}{dt} = \frac{e}{c}\vec{v}\times\vec{B} \qquad \frac{d\vec{v}}{dt} = \vec{v}\times\frac{e\vec{B}}{\gamma mc} = \vec{v}\times\vec{\omega} \qquad \vec{\omega} = \frac{e\vec{B}}{\gamma mc}$$

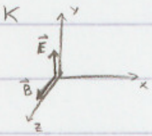
circular motion

trajectory is helix along $\vec{B}$ $\quad$ 

radius of circle $\qquad \underbrace{p_\perp}_{\perp \text{ to } \vec{B}} \qquad v_\perp = \omega R \qquad \frac{p_\perp}{\gamma m} = \frac{eB}{\gamma mc}R \qquad \boxed{R = \frac{cp_\perp}{eB}}$

2. Crossed $\vec{E}$ & $\vec{B}$ (uniform, constant) $\vec{E} \perp \vec{B}$

K



$\vec{E} = E\hat{y}$ $\quad \vec{B} = B\hat{z}$

Lorentz transf. $\vec{\beta} = \beta\hat{x}$

1) $|\vec{B}| > |\vec{E}|$ $\quad \beta = \frac{E}{B}$

K' $\quad E_1' = E_1 \quad E_2' = \gamma(E_2 - \beta B_3) \quad E_3' = \gamma(E_3 + \beta B_2)$

$E_1' = 0 \quad E_2' = \gamma(E - \beta B) = \gamma\left(E - \frac{E}{B}B\right) = 0 \quad E_3' = 0$

$B_1' = 0 \quad B_2' = 0 \quad B_3' = \gamma(B_3 - \beta E_2) = \gamma\left(B - \frac{E}{B}E\right) = \gamma B\left(1 - \frac{E^2}{B^2}\right) = \gamma B(1-\beta^2) = \frac{1}{\gamma}B$

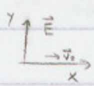
$\vec{\beta} = \frac{\vec{E} \times \vec{B}}{B^2}$ for $|\vec{B}| > |\vec{E}|$ $\rightarrow$ eliminates E field

in K' trajectory is helix along $\vec{B}$

in K trajectory: circle around B lines + drift $\vec{\beta}$

2) $|\vec{E}| > |\vec{B}|$ $\quad \beta = \frac{B}{E}$ $\quad \ldots \quad \vec{B} = 0 \quad \vec{E} = \frac{1}{\gamma}E\hat{y}$

Static, uniform $\vec{E}$ $\quad \vec{v}_0 \perp \vec{E}$



$\vec{E} = E\hat{y} \quad \vec{v}_0 = v_0\hat{x}$

$\frac{d\vec{p}}{dt} = q\vec{E} \qquad \frac{d}{dt}(\gamma m v_x) = 0 \qquad \frac{d}{dt}(\gamma m v_y) = qE$

$\gamma m v_x = \gamma_0 m v_0 \qquad \gamma m v_y = qEt \qquad$ calculate $\gamma(t)$

$\frac{d\varepsilon}{dt} = mc^2\frac{d\gamma}{dt} = qEv_y \qquad mc^2\frac{d\gamma}{dt} = qE\,\frac{qEt}{\gamma m} \qquad \gamma\frac{d\gamma}{dt} = \frac{q^2E^2}{m^2c^2}t$

$\frac{1}{2}(\gamma^2 - \gamma_0^2) = \frac{q^2E^2}{m^2c^2}\frac{1}{2}t^2 \qquad \gamma(t) = \sqrt{\gamma_0^2 + \frac{q^2E^2}{m^2c^2}t^2}$

$v_x = \frac{\gamma_0 v_0}{\gamma(t)} = v_0\left(1 + \left(\frac{qE}{\gamma_0 mc}\right)^2 t^2\right)^{-1/2}$ $\qquad$ characteristic time $\tau = \frac{\gamma_0 mc}{qE}$

$v_x(t) = \frac{v_0}{\sqrt{1 + (t/\tau)^2}} \qquad v_y(t) = \frac{qE}{\gamma_0 m}\frac{t}{\sqrt{1+(t/\tau)^2}}$

3/6/2007

As $t \to \infty \quad v_x \to 0 \quad v_y \to c$

short times $(t \ll \tau)$

$v_x \simeq v_0\left(1 - \frac{1}{2}\left(\frac{t}{\tau}\right)^2\right) \qquad v_y \simeq \frac{qE}{\gamma_0 m}t\left(1 - \frac{1}{2}\left(\frac{t}{\tau}\right)^2\right)$

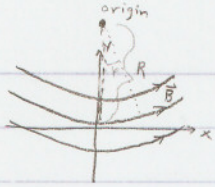
$v_x \simeq v_0 \qquad v_y \simeq \frac{qE}{\gamma_0 m}t \qquad \therefore\ x(t) \simeq v_0 t \quad y(t) \simeq \frac{qE}{2\gamma_0 m}t^2$

$\rightarrow$ parabola (as non-rel)

long times $(t \gg \tau)$

$v_x \simeq \frac{v_0\tau}{t} \qquad v_y \simeq \frac{qE}{\gamma_0 m}t\,\frac{\tau}{t} = c \qquad x(t) \simeq v_0\tau\ln\left(\frac{t}{\tau}\right) \quad y(t) \simeq ct$

trajectory: $y = c\tau e^{\frac{x}{v_0\tau}}$



locally: B lines are parallel with curvature $\sim \frac{1}{R}$

cylindrical coord. at origin (ρ, φ, z) $\qquad \vec{B} = B_\varphi \hat{\varphi}$

$\nabla \times \vec{B} = 0$ (no currents) $\Longrightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\varphi) = 0 \quad \therefore B_\varphi = B_0 \frac{R}{\rho} \qquad \nabla \cdot \vec{B} = 0$

$\gamma = \text{const.} \quad \frac{d}{dt}(\gamma m c^2) = 0$ no electric fields

$$\frac{d\vec{p}}{dt} = \gamma m \frac{d\vec{v}}{dt} = \frac{e}{c}\vec{v} \times \vec{B} \qquad \vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{z} \qquad \dot{\vec{v}} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{\varphi} + \ddot{z}\hat{z}$$

$\vec{v} \times \vec{B} = B_\varphi \dot{\rho}\,\hat{z} - B_\varphi \dot{z}\,\hat{\rho}$

$$\hat{\rho}: \quad \ddot{\rho} - \rho\dot{\varphi}^2 = \frac{-eB_0}{\gamma m c}\frac{R}{\rho}\dot{z} \qquad \omega_B = \frac{eB_0}{\gamma m c}$$

$$\hat{\varphi}: \quad \rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi} = 0 \qquad \hat{z}: \quad \ddot{z} = \frac{eB_0}{\gamma m c}\frac{R}{\rho}\dot{\rho}$$

$$\frac{d}{dt}(\rho^2\dot{\varphi}) = 2\rho\dot{\rho}\dot{\varphi} + \rho^2\ddot{\varphi} = \rho(\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi}) = 0 \quad \therefore \rho^2\dot{\varphi} = \text{constant} = R v_\parallel$$

$$\frac{d}{dt}(\ln\rho) = \frac{\dot{\rho}}{\rho} \propto \ddot{z} \qquad \therefore \dot{z} = \omega_B R \ln\left(\frac{\rho}{R}\right) + v_0$$

integration constant

$$\frac{d}{dt}\left(\ln\left(\frac{\rho}{R}\right)\right) = \frac{R}{\rho}\frac{d}{dt}\left(\frac{\rho}{R}\right) = \frac{\dot{\rho}}{\rho}$$

ansatz: $\rho(t) = R + y(t) \qquad y \ll R$

$\longrightarrow$ small curvature $\qquad$ probable motion:

this should be y

$$\ln\left(\frac{\rho}{R}\right) = \ln\left(1 + \frac{\rho}{R}\right) \simeq \frac{y}{R} \qquad \therefore \dot{z} \simeq \omega_B y + v_0$$

$$\hat{\rho}: \quad \ddot{y} - \frac{R^2 v_\parallel^2}{\rho^3} = -\omega_B \frac{R}{\rho}(\omega_B y + v_0) \qquad \frac{1}{\rho^3} = \frac{1}{(R+y)^3} = \frac{1}{R^3(1+y/R)^3} \simeq \frac{1}{R^3}\left(1 - 3\frac{y}{R}\right) \qquad \frac{1}{\rho} \simeq \frac{1}{R}\left(1 - \frac{y}{R}\right)$$

$$\ddot{y} - \frac{v_\parallel^2}{R}\left(1 - 3\frac{y}{R}\right) = -\omega_B\left(1 - \frac{y}{R}\right)(\omega_B y + v_0) \qquad \ddot{y} + y\left(3\frac{v_\parallel^2}{R^2} - \omega_B\frac{v_0}{R} + \omega_B^2\right) = \frac{v_\parallel^2}{R} - \omega_B v_0$$

Drop y^2 terms $\qquad$ oscillator

$\ddot{x} + \omega^2 x = \alpha \qquad x = A\sin(\omega t) + \frac{\alpha}{\omega^2}$ simple harmonic oscillator around displaced equilibrium

$$\frac{\alpha}{\omega^2} = \langle y \rangle = \frac{\frac{v_\parallel^2}{R} - \omega_B v_0}{3\frac{v_\parallel^2}{R^2} + \omega_B^2 - \omega_B\frac{v_0}{R}} \qquad \therefore \langle \dot{z} \rangle = \omega_B \langle y \rangle + v_0 \quad \text{drift motion}$$

say: $\omega_B R \gg v_\parallel, v_0 \Longrightarrow \langle y \rangle \simeq \frac{v_\parallel^2}{\omega_B^2 R} - \frac{v_0}{\omega_B} \qquad \langle \dot{z} \rangle \simeq \frac{v_\parallel^2}{R\omega_B}$ curvature drift

If $v_\parallel = 0$, $\langle \dot{z} \rangle = \frac{v_0^2}{R\omega_B}$ still a drift $\qquad$ no drift if $v_\parallel = v_0 = 0$

Another case: $\vec{B}$ has a gradient $\nabla_\perp B \perp \vec{B}$

$\vec{B} \parallel \hat{z}$ $\nabla B \parallel \hat{x}$ $\gamma = \text{const.}$

$$\gamma m \frac{d\vec{v}}{dt} = \frac{e}{c}\vec{v}\times\vec{B} \qquad \dot{\vec{v}} = \frac{e}{\gamma mc}\vec{v}\times\vec{B} \qquad \vec{B} = \underbrace{\vec{B}_0}_{\text{uniform}} + \underbrace{\vec{B}_1(\vec{x})}_{\text{first order perturbation}} = \Big(B_0 + \frac{\partial B}{\partial x}\underbrace{dx}_{\text{location of particle}}\Big)\hat{z}$$

$$\dot{\vec{v}} = \vec{v}\times\vec{\omega}(\vec{x}) \qquad \vec{\omega} = \frac{e}{\gamma mc}\left(\vec{B}_0 + \vec{B}_1\right) = \omega_0 + \omega_1$$

$$\vec{v} = \underbrace{\vec{v}_0(t)}_{\text{unperturbed (we know sol'n)}} + \vec{v}_1(t) \qquad \dot{\vec{v}}_1 = \vec{v}_0\times\vec{\omega}_1 + \vec{v}_1\times\vec{\omega}_0 \qquad \omega_0 = \frac{eB_0}{\gamma mc} \quad \omega_1 = \frac{e}{\gamma mc}\frac{\partial B}{\partial x}dx$$

look for a drift: $\langle\dot{\vec{v}}_1\rangle = 0$ $\quad\langle\vec{v}_1\rangle \neq 0$ $\quad\langle\vec{v}_0\times\vec{\omega}_1\rangle = -\langle\vec{v}_1\times\vec{\omega}_0\rangle = -\langle\vec{v}_1\rangle\times\vec{\omega}_0$

Solution to unperturbed problem: $x_0(t) = \left(a\cos(\omega_0 t), -a\sin(\omega_0 t), 0\right)$

$v_0 = \left(-a\omega_0\sin(\omega_0 t), -a\omega_0\cos(\omega_0 t), 0\right)$ $\qquad dx \to a\cos(\omega_0 t)$

$$\vec{v}_0\times\vec{\omega}_1 = \frac{-e}{\gamma mc}\frac{\partial B}{\partial x}a\cos(\omega_0 t)\left(a\omega_0\cos(\omega_0 t)\,\hat{x} - a\omega_0\sin(\omega_0 t)\,\hat{y}\right)$$

$$\langle\vec{v}_0\times\vec{\omega}_1\rangle = -\hat{x}\,\frac{1}{2}\frac{e}{\gamma mc}\frac{\partial B}{\partial x}a^2\omega_0 = -\hat{x}\,\frac{1}{2}a^2\omega_0^2\frac{1}{B_0}\frac{\partial B}{\partial x}$$

$$-\langle\vec{v}_1\rangle\times\vec{\omega}_0 \implies \boxed{\langle\vec{v}_1\rangle = \hat{y}\,\frac{1}{2}a^2\omega_0\frac{1}{B_0}\frac{\partial B}{\partial x}}$$

3/8/2007

Lagrangian of Particles

Action: non rel.: $\quad S = \int_{t_1}^{t_2}\mathcal{L}(\dot{x},x)\,dt$ $\qquad \vec{x}(t)$ trajectory

$$\delta S = 0 \implies \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} = 0 \qquad \text{equation of motion}$$

free particle: $\mathcal{L} = \frac{1}{2}mv^2$

free particle, relativistic $\qquad$ ask that S is a Lorentz scalar

$dt = \gamma\,d\tau$ in frame of particle (rest frame) $\qquad S = \int_{\tau_1}^{\tau_2}\gamma\mathcal{L}\,d\tau$ $\qquad \gamma\mathcal{L}$ is L-scalar

$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$ $\qquad p_\mu p^\mu = m^2c^2$ $\qquad \gamma\mathcal{L} \propto m^2c^2$ constant $\qquad \mathcal{L} \propto \frac{1}{\gamma} = \sqrt{1-\beta^2}$

want $\beta\to 0$ $\quad\mathcal{L}\to\frac{1}{2}mv^2$ $\qquad \sqrt{1-\beta^2} \simeq 1 - \frac{1}{2}\frac{v^2}{c^2}$ $\quad\therefore\quad \boxed{\mathcal{L} = -mc^2\sqrt{1-\beta^2}}$ relativistic $\mathcal{L}$ for free particle

$$\frac{\partial\mathcal{L}}{\partial v} = \frac{\partial}{\partial v}\left(-mc^2\sqrt{1-\frac{v^2}{c^2}}\right) = -mc^2\left(\frac{1}{2}\right)\left(-\frac{2v}{c^2}\right)\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma mv$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial v}\right) = \frac{d}{dt}(\gamma mv) = 0 \qquad \text{true}$$

interaction part of $\mathcal{L}$: $\mathcal{L}_{int}$ in EM field

non-rel: $\mathcal{L}_{int} = -q\phi$ rel: $\gamma\mathcal{L}_{int}$ is L-scalar

$\mathcal{L}_{int}$ will depend on fields (linear), $\vec{v}$, and $\vec{x}$ (through fields $\vec{A}(\vec{x})$, $\phi(\vec{x})$)

$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$ $A^\mu = (\phi, \vec{A})$ $p_\mu A^\mu \Rightarrow \gamma\mathcal{L}_{int} \propto p_\mu A^\mu = \gamma mc\phi - \vec{p}\cdot\vec{A}$

$\mathcal{L}_{int} \propto mc\phi - \frac{1}{\gamma}\vec{p}\cdot\vec{A} \to -q\phi$ non-rel

$\mathcal{L}_{int} = -q\phi + \frac{q}{\gamma mc}\vec{p}\cdot\vec{A} = -q\phi + \frac{q}{c}\vec{v}\cdot\vec{A}$ $\mathcal{L}_{int} = -q\phi + q\frac{\vec{v}}{c}\cdot\vec{A}$ relativistic $\mathcal{L}_{int}$

$$\boxed{\mathcal{L} = -mc^2\sqrt{1-\beta^2} - q\phi + q\frac{\vec{v}}{c}\cdot\vec{A}}$$ relativistic $\mathcal{L}$ with EM fields

$\frac{\partial\mathcal{L}}{\partial v_i} = \gamma m v_i + \frac{q}{c}A_i = p_i + \frac{q}{c}A_i$ $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial v_i}\right) = \frac{dp_i}{dt} + \frac{q}{c}\frac{dA_i}{dt}$

$\frac{\partial\mathcal{L}}{\partial x_i} = -q\frac{\partial\phi}{\partial x_i} + \frac{q}{c}v_j\frac{\partial A_j}{\partial x_i}$ $\frac{dp_i}{dt} = -\frac{q}{c}\frac{dA_i}{dt} + \frac{q}{c}v_j\frac{\partial A_j}{\partial x_i} - q\frac{\partial\phi}{\partial x_i}$

$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ $\vec{B} = \nabla\times\vec{A}$ $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla$

$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times(\nabla\times\vec{A})\right)$ $\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + v_j\frac{\partial A_i}{\partial x_j}$

$(\vec{v}\times(\nabla\times\vec{A}))_i = \varepsilon_{ijk}v_j\varepsilon_{klm}\frac{\partial A_m}{\partial x_l}$
$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})v_j\frac{\partial A_m}{\partial x_l}$
$= v_j\frac{\partial A_j}{\partial x_i} - v_j\frac{\partial A_i}{\partial x_j}$

$$\frac{dp_i}{dt} = q\Bigg(\underbrace{-\frac{\partial\phi_i}{\partial x_i} - \frac{1}{c}\frac{\partial A_i}{\partial t}}_{E_i} + \underbrace{\frac{v_j}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)}_{\frac{1}{c}(\vec{v}\times\vec{B})_i}\Bigg)$$

$\therefore \frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$ true

Hamiltonian of Particles

$\mathcal{L} = \mathcal{L}(x,v) \to H = H(x,P)$ $P_i = \frac{\partial\mathcal{L}}{\partial v_i}$ $H = \sum_i P_i v_i - \mathcal{L}$

conjugate momentum

$P_i = p_i + \frac{q}{c}A_i$ $\vec{P} = \gamma m\vec{v} + \frac{q}{c}\vec{A}$

free particle: $H = \sqrt{m^2c^4 + p^2c^2}$ with EM fields $\boxed{H = \sqrt{m^2c^4 + c^2\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2} + q\phi}$

non-rel: $\mathcal{L} = \frac{1}{2}mv^2 - q\phi + q\frac{\vec{v}}{c}\cdot\vec{A}$ $H = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2 + q\phi$ $\left(= \frac{p^2}{2m} + q\phi\right)$

connection with Quantum Mechanics: $\hat{\vec{P}} = -i\nabla$

non-rel: with EM $\hat{H} = \frac{1}{2m}\left(-i\nabla - \frac{q}{c}\vec{A}\right)^2 + q\phi$

Example: charge in uniform magnetic field

$\vec{B} = B_0 \hat{z}$ $\quad$ $\vec{A} = A_\varphi \hat{\varphi}$ in cyl. coords $\quad$ $\nabla \times \vec{A} = B_0 \hat{z} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\varphi)\hat{z}$

$\frac{\partial}{\partial \rho}(\rho A_\varphi) = \rho B_0$ $\quad$ $\rho A_\varphi = \frac{1}{2} B_0 \rho^2 + \underbrace{\text{const}}_{\text{set}=0}$ $\quad$ $A_\varphi = \frac{1}{2} B_0 \rho$

stationary states $\hat{H}\Psi = E\Psi$ $\quad$ $q = -e$, $e > 0$

$$\left(-\nabla^2 + \frac{e^2}{c^2}A^2 - i\frac{2e}{c}\vec{A}\cdot\nabla\right)\Psi = 2mE\Psi \qquad \Psi(\rho,\varphi,z) = \Psi(\rho,\varphi)e^{ikz}$$

$$\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{e^2}{c^2}\underbrace{A_\varphi^2}_{\frac{1}{4}B_0^2\rho^2} - i\frac{2e}{c}\frac{1}{2}B_0\rho\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)\Psi = 2mE\Psi$$

circular motion so use: $\Psi(\rho,\varphi) = R(\rho)e^{il\varphi}$ $\quad$ l: integer

$$\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{l^2}{\rho^2} + \frac{e^2B_0^2}{4c^2}\rho^2 + \frac{eB_0}{c}l\right)R = 2mER$$

Ansatz $R(\rho) \sim \rho^n e^{-\frac{\rho^2}{2\alpha^2}}$ $\quad$ similar to S.H.O. $\quad$ (wont find all levels)

$$-n^2\rho^{n-2} + \frac{n+2}{\alpha^2}\rho^n + \frac{n}{\alpha^2}\rho^n - \frac{1}{\alpha^4}\rho^{n+2} + \left(\frac{l^2}{\rho^2} + \frac{eB_0}{c}l + \frac{e^2B_0^2}{4c^2}\rho^2\right)\rho^n = 2mE\rho^n$$

$\rho^{n-2}(-n^2 + l^2) = 0 \Rightarrow n = l$

$\rho^n\left(\frac{n+2}{\alpha^2} + \frac{n}{\alpha^2} + \frac{eB_0}{c}\overset{l}{n}\right) = 2mE\rho^n$ $\quad$ $\left(\frac{2}{\alpha^2} + \frac{eB_0}{c}\right)n + \frac{2}{\alpha^2} = 2mE$

$\rho^{n+2}\left(-\frac{1}{\alpha^4} + \frac{e^2B_0^2}{4c^2}\right) = 0 \Rightarrow \alpha^2 = \frac{2c}{eB_0}$ $\quad$ $\frac{2}{\alpha^2} = \frac{eB_0}{c}$

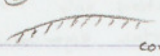
$2mE = \frac{2eB_0}{c}n + \frac{eB_0}{c}$ $\quad$ $E = \underbrace{\frac{eB_0}{mc}}_{\omega_B}\left(n + \frac{1}{2}\right)$ $\quad$ $E = \omega_B\left(n + \frac{1}{2}\right)$ $\quad$ have missed some solutions

should have used $R \sim \left(\sum a_n\rho^n\right)e^{-\frac{\rho^2}{2\alpha^2}}$

Waveguides

3/13/2007

(2) vacuum

a)  (1) conductor $\sigma = \infty$ $\quad$ $E_1 = 0$ $\quad$ $H_1 = 0$ $\quad$ $E_2^{(N)} = 4\pi\Sigma$ (surface charge) $\quad$ $\oint \vec{H}\cdot d\vec{l} = \frac{4\pi}{c}j\Delta l\Delta s \Rightarrow H_2 = \frac{4\pi}{c}K$

$H_2\Delta l$ $\quad$ $K = \lim_{\Delta s\to 0} j\Delta s$

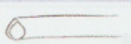
E_2 is normal, H_2 is tangential

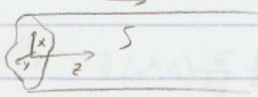
b) if σ = finite $\quad$ $\varepsilon = \varepsilon_0 + i\frac{4\pi\sigma}{\omega}$ $\quad$ there is a penetration depth

$n = \sqrt{\varepsilon}$ $\quad$ good conductor $\frac{4\pi\sigma}{\varepsilon_0\omega} \gg 1$ $\quad$ $n = \sqrt{\frac{4\pi\sigma}{2\omega}} + i\sqrt{\frac{4\pi\sigma}{2\omega}}$

$E \sim e^{i(nkx - \omega t)}$ $\quad$ $\mathrm{Im}\{n\}k = \frac{1}{\delta} = \frac{\omega}{c}\sqrt{\frac{2\pi\sigma}{\omega}} = \sqrt{\frac{2\pi\sigma\omega}{c^2}}$ $\quad$ $\delta = \frac{c}{\sqrt{2\pi\sigma\omega}}$ skin depth

fields drop to zero in region of size δ

waveguide  metal pipe (hollow)



in cavity: $\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ $\quad \nabla\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

B.C. $E_{\parallel}\big|_S = 0 \quad B_{\perp}\big|_S = 0$ $\qquad \nabla\cdot\vec{E}=0 \quad \nabla\cdot\vec{B}=0$

tangential, at surface

Ansatz: $\vec{E}(\vec{x},t) = \vec{E}(x,y)\,e^{i(kz-\omega t)}$ $\qquad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$ same for $\vec{B}$

$\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2} \qquad \therefore \left(\nabla_t^2 - k^2 + \frac{\omega^2}{c^2}\right)\vec{E}(x,y) = 0$

$\vec{E}(x,y) = \vec{E}_t(x,y) + E_z(x,y)\,\hat{z}$ (transverse) $\qquad$ B.C. $E_z\big|_S = 0$ ($E \perp$ surface) $\quad \vec{B}_t\cdot\hat{n}\big|_S = 0$ ($B \parallel$ surface)

reduce problem to eqs + b.c. for E_z, B_z only

$\vec{B} = \vec{B}_t + B_z\hat{z} \qquad \vec{B}_t = B_x\hat{x} + B_y\hat{y}$

$$\nabla\times\vec{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial_x & \partial_y & \partial_z \\ B_x e^{\sim} & B_y e^{\sim} & B_z e^{\sim} \end{vmatrix} = \begin{pmatrix} \left(\frac{\partial B_z}{\partial y} - ikB_y\right)\hat{e}_1 \\ +\left(+ikB_x - \frac{\partial B_z}{\partial x}\right)\hat{e}_2 \\ \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{e}_3 \end{pmatrix} e^{\sim}$$

$$\nabla\times\vec{B} = \left(\nabla_t\times(B_z\hat{e}_3) + ik\,\hat{e}_3\times\vec{B}_t + \nabla_t\times\vec{B}_t\right)e^{\sim}$$

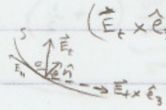
$$\nabla_t\times(B_z\hat{e}_3) + ik\hat{e}_3\times\vec{B}_t + \nabla_t\times\vec{B}_t = -i\frac{\omega}{c}\left(\vec{E}_t + E_z\hat{e}_3\right)$$

$\Longrightarrow \nabla_t\times(B_z\hat{e}_3) + ik\hat{e}_3\times\vec{B}_t = -i\frac{\omega}{c}\vec{E}_t \qquad \nabla_t\times\vec{B}_t = -i\frac{\omega}{c}E_z\hat{e}_3$

$(\times\,\hat{e}_3) \rightarrow \nabla_t B_z = i\frac{\omega}{c}\vec{E}_t\times\hat{e}_3 + ik\vec{B}_t \qquad (*)$

Doing the same with $\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$: $\quad \nabla_t E_z = -i\frac{\omega}{c}\vec{B}_t\times\hat{e}_3 + ik\vec{E}_t$

$(*)\cdot\hat{n}\big|_S \qquad \nabla_t B_z\cdot\hat{n}\big|_S = \frac{\partial B_z}{\partial n}\Big|_S \qquad (\vec{E}_t\times\hat{e}_3)\cdot\hat{n}\big|_S = E_{\parallel}\big|_S = 0$ (b.c.)



$\vec{B}_t\cdot\hat{n}\big|_S = 0$ (b.c.)

$\therefore \frac{\partial B_z}{\partial n}\Big|_S = 0 \qquad E_z\big|_S = 0$

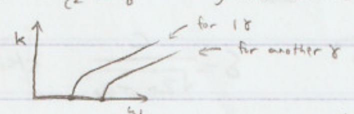
$\left(\nabla_t^2 - k^2 + \frac{\omega^2}{c^2}\right)E_z(x,y) = 0$ (same for B_z) $\qquad -k^2 + \frac{\omega^2}{c^2} = \gamma^2 \rightarrow$ solutions only for certain eigenvalues γ_n

To simultaneously satisfy eigenvalue problem:

TM mode: $B_z = 0 \qquad$ TE mode: $E_z = 0 \qquad$ expect $\gamma^2 > 0$

TM waves: $\vec{B}_t = \frac{\omega}{ck}(\vec{E}_t\times\hat{e}_3) \qquad \nabla_t E_z = -i\frac{\omega^2}{c^2 k}\vec{E}_t + ik\vec{E}_t = i\left(\frac{-\omega^2}{c^2 k} + k\right)\vec{E}_t \Longrightarrow \vec{E}_t = i\frac{ik}{\gamma^2}\nabla_t E_z$

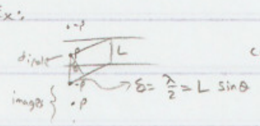
$k^2 = \frac{\omega^2}{c^2} - \gamma^2$ $\quad$ given mode, γ: there is a cutoff freq $\omega_0 = c\gamma$ $\quad$ if $\omega < \omega_0 \rightarrow$ no propagation



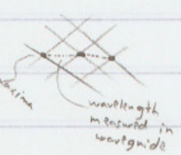
phase vel $\quad v_p = \frac{\omega}{k} = \frac{1}{k}c\sqrt{\gamma^2+k^2} = c\sqrt{1+\frac{\gamma^2}{k^2}} > c \rightarrow \infty$ at cutoff

group vel $\quad v_g = \frac{d\omega}{dk} \qquad \frac{2\omega\,d\omega}{c^2} = 2k\,dk \qquad \frac{d\omega}{dk} = c^2\frac{k}{\omega} = \frac{c}{\sqrt{1+\frac{\gamma^2}{k^2}}} < c \rightarrow 0$ at cutoff

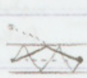
Ex:



constructive interference $\frac{\lambda}{2L}$

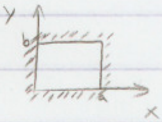


or :



only special angles give constructive interference

Example: rectangular waveguide



TE waves: $E_z = 0$ $\quad \vec{B}_t = \frac{ik}{\gamma^2}\nabla_t B_z$

$B_z = \psi(x,y)\, e^{i(kz-\omega t)}$ $\quad \left.\frac{\partial\psi}{\partial n}\right|_s = 0$ $\quad \nabla^2\psi + \gamma^2\psi = 0$

$\frac{\partial\psi}{\partial x} = 0$ for $\begin{matrix} x=0 \\ x=a \end{matrix}$ $\quad \frac{\partial\psi}{\partial y} = 0$ for $\begin{matrix} y=0 \\ y=b \end{matrix}$ $\quad B_x = i\frac{k}{\gamma^2}\frac{\partial\psi}{\partial x}e^{i()}$ $\quad B_y = i\frac{k}{\gamma^2}\frac{\partial\psi}{\partial y}e^{i()}$

$\vec{E}_t = \frac{\omega}{ck}\vec{B}_t \times \hat{e}_3$ $\quad E_x = \frac{\omega}{ck}B_y$ $\quad E_y = -\frac{\omega}{ck}B_x$ $\quad E_z = 0$

$\psi(x,y) = A\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right)$ $\quad \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 = \gamma_{nm}^2$ $\quad n,m = 0,1,2,\ldots$

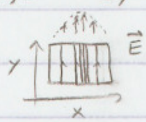
Dispersion relation: $-k^2 + \frac{\omega^2}{c^2} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$ $\quad$ Cutoff Freq $(k=0)$ $\quad \omega_{nm} = c\,\gamma_{nm}$ $\quad$ lowest freq. possible

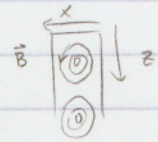
$n=1 \quad m=0$: $\gamma^2 = \left(\frac{\pi}{a}\right)^2$ $\quad k^2 = \frac{\omega^2}{c^2} - \left(\frac{\pi}{a}\right)^2$ $\quad$ Cutoff: $\omega_0 = c\frac{\pi}{a}$ $\quad k^2 = \frac{1}{c^2}(\omega^2 - \omega_0^2)$ $\quad \frac{\pi}{a}i\frac{k}{\gamma^2} = \sqrt{\frac{\omega^2}{\omega_0^2} - 1}$

$1 = \frac{\omega^2}{c^2k^2}\left(1 - \frac{\omega_0^2}{\omega^2}\right)$

$B_z = A\cos\left(\frac{\pi}{a}x\right)e^{i(kz-\omega t)}$ $\quad B_x = -iA\sqrt{\frac{\omega^2}{\omega_0^2} - 1}\,\sin\left(\frac{\pi}{a}x\right)e^{\sim}$ $\quad B_y = 0$

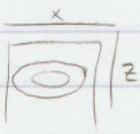
$E_z = 0$ $\quad E_x = 0$ $\quad E_y = \frac{-1}{\sqrt{1 - \frac{\omega_0^2}{\omega^2}}}B_x$





for $\omega \sim \omega_0$

for $\omega > \omega_0$



for $\omega \gg \omega_0$ $\quad B_x \gg B_z$ (factor $\frac{\omega}{\omega_0}$)

$B_x \simeq -iA\frac{\omega}{\omega_0}\sin\left(\frac{\pi}{a}x\right)e^{\sim}$ $\quad E_y \simeq iA\frac{\omega}{\omega_0}\sin\left(\frac{\pi}{a}x\right)e^{\sim}$
$(E_y \to -B_x)$

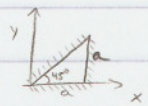
$B_x \perp E_y$
$\longrightarrow$ plane wave traveling along z

Coaxial cable : TEM mode: $E_z = B_z = 0$ (only in coaxial cable)

$\nabla_t \times \vec{B}_t = 0$ $\quad \nabla_t \times \vec{E}_t = 0$ $\quad \nabla_t \cdot \vec{E} = 0$ $\quad \nabla_t^2\vec{E}_t = 0$ $\quad \left(\nabla_t^2 - k^2 + \frac{\omega^2}{c^2}\right)\vec{E}_t = 0 \implies k = \frac{\omega}{c}$

no cutoff freq.

Problem:



TE waves $\quad E_z = 0$ $\quad B_z = \psi(x,y)\,e^{i(kz-\omega t)}$

$(\nabla^2 + \gamma^2)\psi = 0$

B.C. $\left.\frac{\partial\psi}{\partial n}\right|_s = 0$ $\quad \frac{\partial\psi}{\partial x} = 0$ @ $x=a$ $\quad \frac{\partial\psi}{\partial y} = 0$ @ $y=0$ $\quad \nabla\psi\cdot(-1,1)\Big|_s = \left(-\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\right)\Big|_s = 0$ @ $x=y$

$\psi = XY$ $\quad X = \cos\left(\frac{n\pi}{a}x\right) + \sin\left(\frac{n\pi}{a}x\right)$ $\quad Y = \cos\left(\frac{m\pi}{a}y\right) + \sin\left(\frac{m\pi}{a}y\right)$

$\gamma^2 = \left(\frac{\pi}{a}\right)^2(n^2+m^2)$ $\quad \omega = \frac{c\pi}{a}\sqrt{n^2+m^2}$ Cutoff