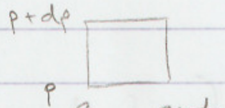
Astro 270 Ferdinand Coroniti
Notes: David Rodriguez

Note $\underline{u} = \vec{u}$ $\quad \underline{\underline{\pi}} = \overline{\overline{\pi}}$ 10/2/2006

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$ q_α : generalized coord. of particle α

$$f^{(N)}(q_\alpha, p_\alpha, t)\, d^3q_1 \cdots d^3q_N\, d^3p_1 \cdots d^3p_N$$

Liouville theorem



$$\frac{\partial f^{(N)}}{\partial t} + \sum_\alpha \left[\frac{\partial}{\partial q_\alpha} \cdot \left(\dot{q}_\alpha f^{(N)}\right) + \frac{\partial}{\partial p_\alpha} \cdot \left(\dot{p}_\alpha f^{(N)}\right) \right] = 0$$

$$\frac{\partial f^{(N)}}{\partial t} + \sum_\alpha \left(\dot{q}_\alpha \cdot \frac{\partial f^{(N)}}{\partial q_\alpha} + \dot{p}_\alpha \cdot \frac{\partial f^{(N)}}{\partial p_\alpha} \right) = 0 = \frac{df^{(N)}}{dt}$$ Convective Derivative

Single particle distribution

$$f^{(1)}(q_1, p_1, t) = \int f^{(N)}(\,)\, d^3q_2 \cdots d^3p_2 \cdots d^3p_N$$

$$= \frac{\partial f^{(1)}}{\partial t} + \dot{q}_1 \cdot \frac{\partial f^{(1)}}{\partial q_1} + \underset{\substack{\text{integrating}\\ \text{over all}\\ \text{space}}}{0} + \sum \int \dot{p}_1 \cdot \frac{\partial f^{(1)}}{\partial p_1}\, d^3q_{\ldots}\, d^3p_{\ldots} = 0$$

force depends on all q, p

$$+ \langle F_1 \rangle \cdot \frac{\partial f'}{\partial p_1} + \sum \int$$

take $\langle F_1 \rangle + \delta F_1$

average independent of q, p — small fluctuating part

$$\langle F_1 \rangle \cdot \frac{\partial f'}{\partial p_1} = -\sum \delta F_1 \cdot \frac{\partial f^N}{\partial p} = \left.\frac{\delta f}{\delta t}\right|_{coll}$$

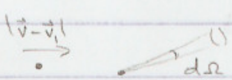
$$\boxed{\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{F}{m} \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\delta f}{\delta t}\right)_{coll}}$$ Boltzmann Eq

if $\left.\left(\frac{\delta f}{\delta t}\right)\right|_{coll} = 0 \Rightarrow$ Vlasov Eq

Collision: Losses

$$\left.\frac{\delta f}{\delta t}\right|_{loss} d^3v\, d^3r = -|\vec{v} - \vec{v}_1|\, f(\vec{v})\, d^3v\, d^3r \left(\frac{d\sigma}{d\Omega} d\Omega\right) f(\vec{v}_1)\, d^3v_1$$

loss (+ for gain) — prob of scattering — # particles with v_1



For total: integrate

$$\left.\frac{\delta f}{\delta t}\right|_{gain} d^3v\, d^3r = |\vec{v}' - \vec{v}_1'|\, f(\vec{v}')\, d^3v'\, d^3r \left(\frac{d\sigma}{d\Omega} d\Omega\right) f(\vec{v}_1)\, d^3v_1'$$

$$\left.\frac{\delta f}{\delta t}\right|_{coll} = \int |v-v_1| \left(\frac{d\sigma}{d\Omega} d\Omega\right)\left(f(\vec{v}')f(\vec{v}_1') - f(\vec{v})f(\vec{v}_1)\right) d^3v_1$$

Mean free path $\lambda_{mfp} = \dfrac{1}{n_{coll}\,\sigma_{tot}}$

$\tau_{coll} = \dfrac{\lambda_{mfp}}{v}$ $\quad v \sim \sqrt{\dfrac{kT}{m}}$

Moments of Boltzmann Eq $\int (\cdot) f\, d^3v$

3 moments
- m mass
- mv momentum
- $\frac{mv^2}{2}$ KE

$$\int \left(\frac{\delta f}{\delta t}\right)_{coll} \begin{pmatrix} m \\ mv \\ \frac{mv^2}{2} \end{pmatrix} d^3v = 0$$

Conserves all 3 moments

First :

$$\frac{\partial}{\partial t}\underbrace{\int m f\, d^3v}_{\rho = mn} + \frac{\partial}{\partial x_i}\underbrace{\int m v_i f\, d^3v}_{m u_i \text{ avg vel}} + \frac{F_i}{m}\underbrace{\int m \frac{\partial f}{\partial v_i} d^3v}_{\to 0} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \qquad u_i = \frac{1}{\rho}\int m v_i f\, d^3v$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0}$$ Continuity

$$\frac{\partial \rho}{\partial t} + u\cdot\nabla\rho + \rho\,\nabla\cdot\vec{u} = 0 \qquad \underbrace{\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)}_{\text{Convective Derivative}}\rho = \frac{d\rho}{dt}$$

$$\frac{d\rho}{dt} + \rho\,\nabla\cdot\vec{u} = 0$$

Momentum:

$$\frac{\partial}{\partial t}\int m v_j f\, d^3v + \frac{\partial}{\partial x_i}\int m v_i v_j f\, d^3v + F_i \int v_j \frac{\partial f}{\partial v_i} d^3v = 0$$

Integrate by parts

$$\frac{\partial}{\partial t}(\rho u_j) \qquad -F_i \int \underbrace{\frac{\partial v_j}{\partial v_i}}_{\delta_{ij}} f\, d^3v = -\rho\frac{F_i}{m}$$

$v_j = v_j' + u_j$

$\int v_i' f\, d^3v = 0$ Average to zero

$$m\int (v_i'+u_i)(v_j'+u_j) f\, d^3v = \rho u_i u_j + \underbrace{m\int v_i' v_j' f\, d^3v}_{\sigma_{ij}^2}$$

σ_{ij}^2 velocity dispersion tensor ordinary/ideal gas $\quad P = nkT$

$$\sigma_{ij}^2 = P\delta_{ij} - \Pi_{ij}$$

$\Pi_{ij} \rightarrow$ viscosity tensor

force density $\frac{F}{m}$

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}\left(\rho u_i u_j + \sigma_{ij}^2\right) = \rho F_i$$

$$\boxed{\frac{\partial}{\partial t}(\rho \vec{u}) + \nabla\cdot(\rho\vec{u}\vec{u}) = -\nabla P + \nabla\cdot\overline{\overline{\Pi}} + \rho\vec{F}}$$ Conservation of Momentum

$$\vec{u}\underbrace{\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u})\right)}_{=0 \text{ Continuity}} \qquad \boxed{\rho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = \rho\frac{d\vec{u}}{dt} = -\nabla P + \nabla\cdot\overline{\overline{\Pi}} + \rho\vec{F}}$$

Energy:

$$\frac{\partial}{\partial t}\int\frac{mv^2}{2}f\,d^3V + \frac{\partial}{\partial x_i}\int v_i\frac{mv^2}{2}f\,d^3V + F_i\int\frac{mv^2}{2}\frac{\partial f}{\partial v_i}d^3V = 0$$

Integrate by parts

$$-F_i\int m v_i f\,d^3V = -\rho\underbrace{\vec{u}\cdot\vec{F}}_{\text{power}(F\cdot v)}$$

$$v^2 = (v'+u)^2 = v'^2 + 2v'u + u^2$$

$$\frac{\partial}{\partial t}\int\frac{m}{2}\left(u^2 + 2v'u + v'^2\right)f\,d^3V$$

$$= \frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + e\right)$$

e: internal energy $du = \int\frac{mv'^2}{2}f\,d^3V \qquad e = \frac{P}{\gamma-1}$

$$\int(v_i'+u_i)\frac{m}{2}\left(v'^2 + 2v_j'u_j + u^2\right)f\,d^3V = \underbrace{\rho u_i\frac{u^2}{2}}_{\text{Flux of KE}} + u_i e + \sigma_{ij}^2 u_i + \underbrace{q_i}_{\text{Heat flow}}$$

q_i: $\int v_i'\frac{m}{2}v'^2 f\,d^3V$

$$= u_i\underbrace{(e+P)}_{h} - u_i\Pi_{ij} + q_i + \rho u_i\frac{u^2}{2}$$

$$h = e + P = \frac{\gamma P}{\gamma-1} \qquad \text{enthalpy } H$$

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \frac{P}{\gamma-1}\right) + \nabla\cdot\left(\rho\vec{u}\frac{u^2}{2} + \vec{u}\frac{\gamma P}{\gamma-1} - \vec{u}\cdot\overline{\overline{\Pi}} + \vec{q}\right) = \rho\vec{u}\cdot\vec{F}}$$

Conservation of energy

Other forms:

$$\frac{u^2}{2}\underbrace{\left(\frac{\partial}{\partial t}\rho + \nabla\cdot\rho v\right)}_{=0 \text{ Continuity}} + \rho\left(\frac{\partial\frac{u^2}{2}}{\partial t} + \vec{u}\cdot\nabla\frac{u^2}{2}\right) = \rho\vec{u}\cdot\vec{F} + \nabla\cdot(\vec{u}\cdot\overline{\overline{\Pi}}) - \nabla\cdot\vec{q}$$

$$= -\vec{u}\cdot\nabla P + \vec{u}\cdot\nabla\overline{\overline{\Pi}} + \rho\vec{u}\cdot\vec{F}$$ from momentum

$$\frac{1}{\gamma-1}\left(\frac{\partial P}{\partial t}+\gamma\,\nabla\cdot(P\vec{u})\right)-\vec{u}\cdot\nabla P$$

$$\gamma P\nabla\cdot\vec{u}+\gamma\vec{u}\cdot\nabla P$$

$$\underbrace{\left(\frac{\gamma}{\gamma-1}-1\right)}_{=\frac{1}{\gamma-1}}\vec{u}\cdot\nabla P$$

$$\underbrace{\frac{1}{\gamma-1}\left(\frac{\partial P}{\partial t}+\vec{u}\cdot\nabla P\right)+\frac{\gamma P}{\gamma-1}\nabla\cdot\vec{u}}_{LHS}=-\vec{u}\cdot\nabla\cdot\overline{\overline{\pi}}+\nabla\cdot(\vec{u}\cdot\overline{\overline{\pi}})-\nabla\cdot\vec{q}$$

$$-u_j\frac{\partial \pi_{ij}}{\partial x_i}+\frac{\partial}{\partial x_i}(u_j\pi_{ij})$$

$$=\pi_{ij}\frac{\partial u_j}{\partial x_i}=\overline{\overline{\pi}}:\nabla\vec{u}$$

$$\boxed{\overline{\overline{\pi}}:\nabla\vec{u}-\nabla\cdot\vec{q}=\frac{1}{\gamma-1}\left(\frac{\partial P}{\partial t}+\vec{u}\cdot\nabla P\right)+\frac{\gamma P}{\gamma-1}\nabla\cdot\vec{u}}\quad 2^{nd}\text{ form}$$

$$\frac{P}{\gamma-1}\left(\frac{1}{P}\frac{dP}{dt}-\frac{\gamma}{\rho}\frac{d\rho}{dt}\right)=\frac{P}{\gamma-1}\frac{d}{dt}\left(\ln\left(\frac{P}{\rho^\gamma}\right)\right)$$

$$S=\frac{k_B}{\gamma-1}\ln\left(\frac{P}{\rho^\gamma}\right)\qquad\therefore\quad\boxed{\frac{P}{k_B}\frac{dS}{dt}=\overline{\overline{\pi}}:\nabla\vec{u}-\nabla\cdot\vec{q}}\quad 3^{rd}\text{ form}$$

entropy

convective derivative

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\vec{v}=0$$

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}\right)=-\nabla P+\nabla\cdot\overline{\overline{\pi}}$$

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2}+\frac{P}{\gamma-1}\right)+\nabla\cdot\left(\rho\vec{v}\frac{v^2}{2}+\frac{\gamma\vec{v}}{\gamma-1}P-\vec{v}\cdot\overline{\overline{\pi}}+\vec{q}\right)=0$$

$$\frac{1}{\gamma-1}\left(\frac{dP}{dt}+\gamma P\,\nabla\cdot\vec{v}\right)=\overline{\overline{\pi}}:\nabla\vec{v}-\nabla\cdot\vec{q}$$

$$\frac{P}{k_B}\frac{dS}{dt}=\overline{\overline{\pi}}:\nabla\vec{v}-\nabla\cdot\vec{q}$$

Taking moments destroys some detail in f in order to get average values which are easier to use

10/4/2006

Viscous stress tensor

$$\Pi_{ij} = \underset{\uparrow \text{shear}}{\mu}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \underbrace{\eta \frac{\partial v_i}{\partial x_j}\delta_{ij}}_{-\frac{2}{3}\mu \frac{\partial v_i}{\partial x_i}} \qquad \eta = \beta - \frac{2}{3}\mu$$

β ← Bulk viscosity

for gases $\beta = 0$

$$\overleftrightarrow{\Pi} = \mu\left(\nabla\vec{v} + (\nabla\vec{v})^T\right) - \frac{2}{3}\mu\, \nabla\cdot\vec{v}\, \overleftrightarrow{I}$$

$\hat{x}\hat{x}\frac{\partial v_x}{\partial x} \quad \hat{x}\hat{y}\frac{\partial v_x}{\partial y}$

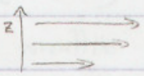
$$\overleftrightarrow{\Pi}_{xx} = \hat{x}\cdot\overleftrightarrow{\Pi}\cdot\hat{x} = \mu\frac{\partial v_x}{\partial x} + \cdots$$

Krook collision operator

$$\left.\frac{\delta f}{\delta t}\right|_{coll} = -\nu\left(f - f_0\right) \qquad \nu = \frac{1}{\tau_{coll}} \qquad f_0 = \frac{n}{\pi^{3/2}a^3}\exp\left(-\frac{1}{a^2}(\vec{v}-\vec{u})^2\right) \qquad a^2 = \frac{2k_BT}{m}$$

Maxwellian Distribution

$\vec{u} = u(z)\,\hat{e}_x$



Steady flow $\frac{\partial f}{\partial t} = 0$ No external forces $\frac{\vec{F}}{m}\cdot\frac{\partial f}{\partial \vec{v}} = 0$

$$v_z\frac{\partial f}{\partial z} = -\nu\left(f - f_0\right) \qquad \text{(From Boltzmann)}$$

$$\frac{\bar{v}f}{L} : \nu f \sim \frac{f}{\tau} \sim \frac{\bar{v}}{\lambda_{mfp}} f \qquad L \gg \lambda_{mfp}$$

(avg v)

f driven by collisions to be f_0

but since $\frac{\bar{v}f}{L} \neq 0$ f will be driven to $f = f_0 + f_1$

$$v_z\frac{\partial f_0}{\partial z} = -\nu f_1$$

↳ (f_1 small: ignore)

$$f_1 = -\frac{v_z}{\nu}\left(\frac{-2(v_x - u_x)}{a^2}\frac{\partial u_x}{\partial z} f_0\right)$$

$$f_1 = -\frac{2v_z}{\nu a^2}\overbrace{(v_x - u_x)}^{v_x'}\frac{\partial u_x}{\partial z} f_0$$

$$\Pi_{xz} = -m\int v_z' v_x' f\, d^3v = \frac{2m}{\nu a^2}\frac{\partial u_x}{\partial z}\int v_z'^2 v_x'^2 f_0\, d^3v$$

f ↳ $f_0 + f_1$; f_0 cancels out (anti sym with $v_z'v_x'$)

$\int v_z'^2 v_x'^2 f_0\, d^3v = n\left(\frac{kT}{m}\right)\left(\frac{kT}{m}\right)$

$$\Pi_{xz} = \frac{2mn}{\nu\left(\frac{2k_BT}{m}\right)}\left(\frac{kT}{m}\right)^2\frac{\partial u_x}{\partial z} = \underbrace{p\frac{kT}{m\nu}}_{\mu}\frac{\partial u_x}{\partial z} = \mu\frac{\partial u_x}{\partial z}$$

$$\mu = \rho \frac{kT}{m\nu} \qquad \left[\frac{kT}{m\nu}\right] = \frac{L^2}{T} \text{ units } = [D] \text{ diffusion coefficient units}$$

$$\mu = \rho D$$

Heat flow

internal Energy density

$x+\lambda$ ——— $e(x+\lambda)$

λ_{mfp} $\downarrow q_-$

x ——— $e(x)$ collide, move and transfer heat

$\uparrow q_+$

$x-\lambda$ ——— $e(x-\lambda)$ $[q]$ $\frac{ergs}{cm^2\ s}$

Procedure useful to calculate flux of anything at x

$$q_+ = \frac{\bar{v}\, e(x-\lambda)}{3\,(2)} \qquad q_- = -\frac{\bar{v}\, e(x+\lambda)}{6}$$

3 dir. only upward

total heat flux: $q = q_+ + q_- = -\frac{\bar{v}}{6}\left(e(x+\lambda) - e(x-\lambda)\right)$ Taylor expand

$$\left(e(x) + \frac{\partial e}{\partial x}\lambda\right) - \left(e(x) - \frac{\partial e}{\partial x}\lambda\right)$$

$$\boxed{q = -\frac{\bar{v}\lambda}{3}\frac{\partial e}{\partial x}}$$

$[\bar{v}\lambda] = \frac{L^2}{T}$ diffusion coefficient $\therefore$ a diffusive term

ideal gas $e = \frac{P}{\gamma-1} = \frac{n k_B T}{\gamma - 1}$ $\qquad q = -\frac{n k_B \bar{v}\lambda}{3(\gamma-1)}\frac{\partial T}{\partial x} = -K_T \frac{\partial T}{\partial x}$

thermal conductivity

photon gas $e = aT^4$

$\bar{v} = c$ $\qquad q = -\frac{4}{3}\frac{acT^3}{K_{op}\rho}\frac{\partial T}{\partial x}$

$\lambda_{mfp} = \frac{1}{K_{op}\rho}$

opacity

Linearization

$\rho = \rho_0 + \delta\rho(\vec{x},t)$ $\qquad \frac{\delta\rho}{\rho_0} \ll 1$ $\qquad P = P_0 + \delta P(\vec{x},t)$

average fluctuations

$\vec{v} = 0 + \delta\vec{v}(\vec{x},t)$ frame moving @ average vel.

$\vec{\vec{\Pi}} = \vec{q} = 0$ (for now)

$$\frac{\partial \delta\rho}{\partial t} + \nabla\cdot(\rho_0 + \delta\rho)\delta\vec{v} = 0 \qquad \frac{\partial\delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\vec{v} + \cancel{\nabla\cdot\delta\rho\,\delta\vec{v}}^{\,0} = 0$$

small

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\vec{v} = 0 \quad \text{Continuity}$$

$$(\rho_0 + \delta\rho)\left(\frac{\partial \delta\vec{v}}{\partial t} + \cancelto{0}{\delta\vec{v}\cdot\nabla\delta\vec{v}}\right) = -\nabla \delta P$$

$$\rho_0 \frac{\partial \delta\vec{v}}{\partial t} = -\nabla \delta P \qquad \text{Momentum}$$

$$\frac{\partial \delta P}{\partial t} + \cancelto{0}{\delta\vec{v}\cdot\nabla P_0} + \gamma(P_0 + \cancelto{0}{\delta P})\nabla\cdot\delta\vec{v} = 0$$

$$\frac{\partial \delta P}{\partial t} + \gamma P_0 \nabla\cdot\delta\vec{v} = 0 \qquad \frac{\partial \delta P}{\partial t} - \frac{\gamma P_0}{\rho_0}\frac{\partial \delta\rho}{\partial t} = 0 \quad \text{using continuity}$$

$$\frac{\partial}{\partial t}\Big(\underbrace{\delta P - \frac{\gamma P_0}{\rho_0}\delta\rho}_{=0 \text{ if isentropic}}\Big) = 0 \qquad \text{energy}$$

$$S = \frac{k_B}{\gamma - 1}\ln\left(\frac{P}{\rho^\gamma}\right) \qquad dS = \frac{k_B}{\gamma-1}\left(\frac{dP}{P} - \frac{\gamma\, d\rho}{\rho}\right) = \frac{k_B}{\gamma-1}\left(\frac{\delta P}{P_0} - \frac{\gamma\, d\rho}{\rho}\right) = 0$$

if $dS = 0$

$$\rho_0\frac{\partial \delta\vec{v}}{\partial t} = -\frac{\gamma P_0}{\rho_0}\nabla\delta\rho \qquad \frac{\partial \delta\rho}{\partial t} = -\rho_0 \nabla\cdot\delta\vec{v} \quad \text{Continuity}$$

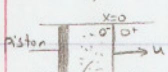
$$\nabla\cdot \;\Rightarrow\; \rho_0 \frac{\partial}{\partial t}(\nabla\cdot\delta\vec{v}) = -\frac{\gamma P_0}{\rho_0}\nabla^2\rho \longrightarrow \frac{\partial}{\partial t}\left(\frac{\partial \rho}{\partial t}\right) = -\frac{\gamma P_0}{\rho_0}\nabla^2\rho$$

$$\Longrightarrow \frac{\partial^2\delta\rho}{\partial t^2} = \frac{\gamma P_0}{\rho_0}\nabla^2\delta\rho \qquad c_s^2 = \frac{\gamma P_0}{\rho_0} = \left.\frac{dP}{d\rho}\right|_s$$

Plane $\quad \delta\rho(\vec{x},t) = \tilde{\delta\rho}\, e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad \nabla = i\vec{k} \quad \nabla^2 = -k^2 \quad \frac{\partial}{\partial t} = -i\omega \quad \frac{\partial^2}{\partial t} = -\omega^2$

$$\Longrightarrow -\omega^2 = -k^2 c_s^2 \qquad \omega = \pm k c_s \qquad \frac{\omega}{k} = c_s \quad \text{non dispersive}$$

Step wave $\quad$ 1-D $\quad \Pi = q = 0 \qquad$ short discontinuity



$\delta\rho = \delta P = \delta v = 0$

$\delta\rho \neq 0 \quad \delta P \neq 0 \quad \delta v \neq 0$

$$\xi = x - ut \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \qquad \frac{\partial}{\partial t} = -u\frac{\partial}{\partial \xi}$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\vec{v} = 0 \Longrightarrow -u\frac{\partial\delta\rho}{\partial\xi} + \rho_0\frac{\partial\delta\vec{v}}{\partial\xi} = 0$$

$$\int_{0^-}^{0^+} d\xi \frac{\partial}{\partial\xi}\left[-u\,\delta\rho + \rho_0\,\delta v\right] = 0 \Longrightarrow u\,\delta\rho - \rho_0\,\delta v = 0$$

$$\frac{\delta\rho}{\rho_0} = \frac{\delta v}{u}$$

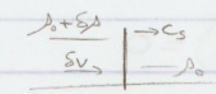
$$\rho_0 \frac{\partial \delta \vec{v}}{\partial t} = -\nabla \delta P = -c_s^2 \nabla \delta \rho$$

$$-\rho_0 u \frac{\partial \delta \vec{v}}{\partial \xi} = -c_s^2 \frac{\partial \delta \rho}{\partial \xi} \qquad \int_{0^-}^{0^+} d\xi \frac{\partial}{\partial \xi}\left[-\rho_0 u \delta \vec{v} + c_s^2 \delta \rho\right] = 0$$

$$\rho_0 u \delta v = c_s^2 \delta \rho \quad \Longrightarrow \quad \frac{\delta \rho}{\rho_0} = \frac{u \delta v}{c_s^2} = \frac{\delta v}{u}$$

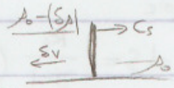
$$\left(\frac{u}{c_s^2} - \frac{1}{u}\right)\delta v = 0 \quad \Longrightarrow \quad c_s^2 = u^2$$ this discontinuity travels @ sound speed

$\frac{\delta \rho}{\rho_0} > 0$ compression $\delta v > 0$



(push piston)

$\frac{\delta \rho}{\rho_0} < 0$ rarefaction $\delta v < 0$



(pull piston)

10/9/2006

1-D viscosity $\vec{q} = 0$ $\xrightarrow{x}$ Fourier representation $\nabla = ik$ $\frac{\partial}{\partial t} = -i\omega$

Continuity

$$-i\omega \delta\rho + \rho_0 ik \delta v = 0 \quad \Longrightarrow \quad \frac{\delta \rho}{\rho_0} = \frac{k}{\omega} \delta v$$

Momentum

$$-i\omega \rho_0 \delta v_x = -ik\delta P + ik \delta \Pi_{xx}$$

$$\delta \Pi_{xx} = \rho_0 D ik \delta v$$

$$\omega \rho_0 \delta v_x = k \delta P - ik^2 D \rho_0 \delta v$$

Entropy

$$\frac{1}{\gamma - 1}\left[-i\omega \delta P + i\gamma P_0 k \delta v\right] = \delta \Pi_{xx} ik \delta v_x \sim \delta v^2 - 0$$ linearization (entropy conserved)

$$\delta P = \frac{\gamma P_0}{\omega} k \delta v$$

$$\omega \delta v = \frac{k^2}{\omega} \underbrace{\frac{\gamma P_0}{\rho_0}}_{c_s^2} \delta v - ik^2 D \delta v \qquad \underbrace{(\omega^2 + i\omega k^2 D - k^2 c_s^2)}_{=0} \delta v = 0$$

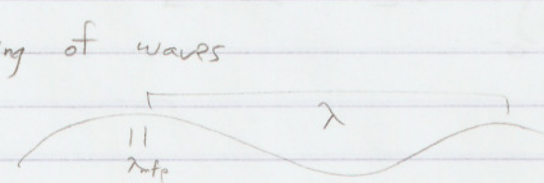
$$\boxed{\omega = -\frac{ik^2 D}{2} \pm \sqrt{-\left(\frac{k^2 D}{2}\right)^2 + k^2 c_s^2}}$$

If Dissipative small $k^2 D << k c_s \longrightarrow \omega = \pm k c_s - i \frac{k^2 D}{2}$

$$e^{-i\omega t} = e^{\mp ikc_s} e^{-\frac{k^2 D}{2} t}$$

$\hookrightarrow$ damping of waves

$k \lambda_{mfp} << 1$ $\lambda > \lambda_{mfp}$

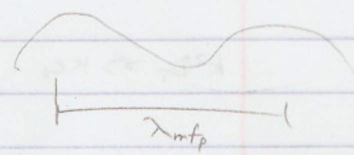


$\therefore$ small damping compared to $k \lambda_{mfp} >> 1$

Wavelength is short so $k^2 D >> k c_s$

$$\omega = -i\frac{k^2 D}{2} \pm i\frac{k^2 D}{2}\underbrace{\left(1 - \frac{4k^2 c_s^2}{(k^2 D)^2}\right)^{1/2}}$$

$k\lambda_{mfp} >> 1$ $\qquad \lambda_{mfp} \gtrsim \lambda$



$$\approx \left(1 - \frac{2k^2 c_s^2}{(k^2 D)^2}\right)$$

$$\omega_- = -ik^2 D \qquad \omega_+ = -i\frac{c_s^2}{D}$$

strong damping $\qquad$ weak damping

compare $kD : c_s$

$$k\bar{v}\lambda_{mfp} : \sim\sqrt{\frac{kT}{m}} \longrightarrow k\lambda_{mfp} : 1$$

$\bar{v} \sim \sqrt{\frac{kT}{m}}$

Since $\lambda_{mfp} \sim \lambda$ collisions can transfer momentum faster than wave $\Rightarrow$ damping

---

Heat conduction $\overleftrightarrow{\Pi} = 0 \qquad \vec{q} = -\kappa_T \nabla T$

$$\frac{\delta\rho}{\rho_0} = \frac{k}{\omega}\delta v \qquad \omega\rho_0 \delta v = k\,\delta P$$

$$\frac{1}{\gamma - 1}\left[-i\omega\,\delta P + \gamma P_0\, ik\,\delta v\right] = -\nabla\cdot\delta\vec{q} = -ik(-ik)\kappa_T\,\delta T$$

entropy not conserved

$$\omega\,\delta P - \gamma P_0 k\,\delta v = -ik^2(\gamma-1)\kappa_T\,\delta T$$

ideal gas $\qquad T = \frac{m}{k_B}\frac{P}{\rho} \qquad \delta T = \frac{m}{k_B}\left(\frac{\delta P}{\rho_0} - \frac{P_0\,\delta\rho}{\rho_0^2}\right)$

$$= -ik^2\underbrace{(\gamma-1)\frac{\kappa_T m}{k_B \rho_0}}_{D_T}\left(\delta P - P_0\frac{\delta\rho}{\rho_0}\right) = -ik^2 D_T\left(\delta P - P_0\frac{k}{\omega}\delta v\right)$$

D_T thermal diffusion coefficient

$$\omega\left(1 + \frac{ik^2 D_T}{\omega}\right)\delta P = k\gamma P_0\,\delta v\left(1 + \frac{ik^2 D_T}{\omega\gamma}\right)$$

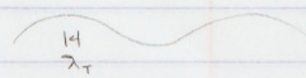
$$\omega\,\delta v = \frac{k^2\gamma P_0}{\rho_0\,\omega}\left(\frac{1 + \frac{ik^2 D_T}{\omega\gamma}}{1 + \frac{ik^2 D_T}{\omega}}\right)\delta v$$

$$\omega^2 = k^2 c_s^2\,\frac{\left(1 + \frac{ik^2 D_T}{\omega\gamma}\right)}{\left(1 + \frac{ik^2 D_T}{\omega}\right)}$$

Limits

$k^2 D_T \ll k c_s \qquad \omega^2 = k^2 c_s^2 \left(1 + i \frac{k^2 D}{\omega}\left(\frac{1}{\gamma} - 1\right)\right)$

$k \lambda_T \ll 1$



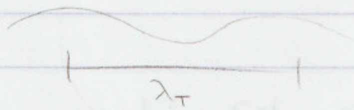
$\omega = \pm k c_s \left(1 - i \frac{1}{2} \frac{k^2 D}{|k c_s|} \frac{(\gamma - 1)}{\gamma}\right)$ weak damping

$k^2 D_T \gg k c_s \qquad 1 + \frac{i k^2 D_T}{\omega} \approx \frac{i k^2 D_T}{\omega}$

$\therefore \quad \omega^2 = \frac{k^2 c_s^2}{\gamma} = k^2 \frac{P_0}{\rho_0}$ no damping

as if $\gamma = 1 \rightarrow$ isothermal

$k \lambda_T \gg 1$

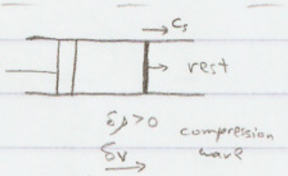


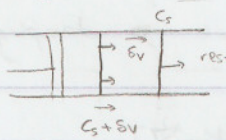
$\delta T = 0$

heat conducted fast so no T fluctuations

$\rightarrow$ isothermal

Thermal conduction takes speed to isothermal value



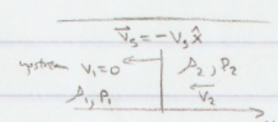


2^{nd} wave will catch up with 1^{st} and give its energy

Compression waves steepen
not rarefaction waves

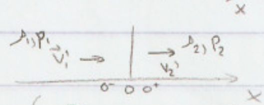
Shock $\rightarrow$ nonlinear sound wave

LAB frame



$\vec{V} = \vec{V}' + \vec{V}_s$

Shock Frame



$\vec{V}_1' = \vec{V}_s$

$V_2' = V_s - V_2$

$\frac{\partial}{\partial t} = 0$ everything steady in shock frame

1-D : $\frac{\partial}{\partial x} \hat{x} = \nabla$

ignore $\overleftrightarrow{\Pi}, \vec{q}$

Continuity $\frac{d}{dx}(\rho v)=0$ $\qquad \int_{0^-}^{0^+} dx \frac{d}{dx}(\rho v) = \rho_2 v_2 - \rho_1 v_1 = 0$

$$\Rightarrow \rho_1 v_1 = \rho_2 v_2 \quad ①$$

$$\int_{0^-}^{0^+} dx\, \rho_1 v_1 \frac{dv}{dx} = -\frac{dP}{dx}$$

$$\rho_1 v_1 (v_2 - v_1) + (P_2 - P_1) = 0 \quad ②$$

$$\frac{d}{dx}\left(\rho_1 v_1 \frac{v^2}{2} + \frac{\gamma}{\gamma-1} vP\right) = 0 \qquad \rho_1 v_1 \frac{(v_2^2 - v_1^2)}{2} + \frac{\gamma}{\gamma-1}(v_2 P_2 - v_1 P_1) = 0 \quad ③$$

$$\frac{(v_2 - v_1)(v_2 + v_1)}{2} + \frac{\gamma}{\gamma-1} \frac{v_2(P_1 - \rho_1 v_1 (v_2 - v_1)) - v_1 P_1}{\rho_1 v_1} = 0$$

$$(v_2 - v_1)\left[\frac{(v_2+v_1)}{2} + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1 v_1} - \frac{\gamma}{\gamma-1} v_2\right] = 0$$

$$v_2 \underbrace{\left(\frac{1}{2} - \frac{\gamma}{\gamma-1}\right)}_{\frac{\gamma+1}{2(\gamma-1)}} + \frac{v_1}{2} + \frac{\gamma}{\gamma-1} v_1 \frac{P_1}{\rho_1 v_1^2} = 0$$

$$M^2 = \frac{v_1^2}{c_{s_1}^2}$$

$$(v_2 - v_1)\left(v_2 - \frac{\gamma-1}{\gamma+1} v_1 - \frac{2}{\gamma+1} v_1 \frac{1}{M^2}\right) = 0 \qquad \boxed{v_2 = v_1\left(\frac{\gamma-1}{\gamma+1} + \frac{2}{\gamma+1}\frac{1}{M^2}\right)} \quad ①$$

$$② \quad \boxed{\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{1}{\frac{\gamma-1}{\gamma+1} + \frac{2}{\gamma+1}\frac{1}{M^2}}} \qquad P_2 = P_1 - \rho_1 v_1 (v_2 - v_1)$$

$$v_2 - v_1 = -\frac{2}{\gamma+1} v_1 \left(1 - \frac{1}{M^2}\right) \quad \text{vel in LAB} \qquad \boxed{P_2 = P_1 + \frac{2}{\gamma+1}\rho_1 v_1^2 \left(1 - \frac{1}{M^2}\right)} \quad ③$$

①, ②, ③ Rankine–Hugoniot relations (jump conditions) $\qquad \frac{k_B T_2}{m} = \frac{P_2}{\rho_2}$

$\rho_2 > \rho_1 \qquad P_2 > P_1 \qquad c_{s_2} > v_2$ ← properties of shocks

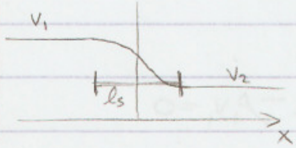
Hypersonic limit

$M^2 \gg 1$ $\qquad v_2 = v_1\left(\frac{\gamma-1}{\gamma+1}\right) = \frac{v_1}{4} \qquad \frac{\rho_2}{\rho_1} \simeq \frac{\gamma+1}{\gamma-1} = 4$ — ideal monatomic gas $\quad \frac{5/3+1}{5/3-1} = \frac{8/3}{2/3} = 4$

$c_{s_1} \to 0$

$$P_2 = \frac{2}{\gamma+1}\rho_1 v_1^2 \qquad \frac{k_B T_2}{m} = \frac{P_2}{\rho_2} = \frac{2}{\gamma+1}\frac{\gamma-1}{\gamma+1} v_1^2 = \frac{3}{16} v_1^2$$

$\vec{\vec{\Pi}} \neq 0$ shock not infinitely thin



$$\frac{d}{dx}(\rho v) = 0$$

$-\infty$ x $\rho(x)\, v(x)$

$$\rho(x)\, v(x) = \rho_1 v_1 \quad ①$$

② $\rho_1 v_1 \frac{dv}{dx} + \frac{dP}{dx} = \frac{d}{dx}\left(\Pi_{xx} = \rho D \frac{dv}{dx}\right)$ $\qquad \left.\frac{dv}{dx}\right|_{x\to-\infty} \to 0$

$$\rho_1 v_1 \left(v(x) - v_1\right) + P(x) - P_1 = \rho(x) D \frac{dv}{dx}$$

③ $\frac{d}{dx}\left(\rho_1 v_1 \frac{v^2}{2} + \frac{\gamma}{\gamma-1} vP - v_1 \rho_1 D \frac{dv}{dx}\right) = 0$

$$\rho_1 v_1 \left(\frac{v^2 - v_1^2}{2}\right) + \frac{\gamma}{\gamma-1}\left(vP - v_1 P_1\right) = \rho_1 v_1 D \frac{dv}{dx}$$

$$\frac{(v-v_1)(v+v_1)}{2} + \frac{\gamma}{\gamma-1}\,\frac{(v-v_1)P_1 - \rho_1 v_1 (v-v_1) v + \rho_1 v_1 D \frac{dv}{dx}}{\rho_1 v_1} = D\frac{dv}{dx}$$

$$(v-v_1)\left(\frac{v+v_1}{2} + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1 v_1} - \frac{v\gamma}{\gamma-1}\right) = D\frac{dv}{dx}\left(1 - \frac{\gamma}{\gamma-1}\right)$$

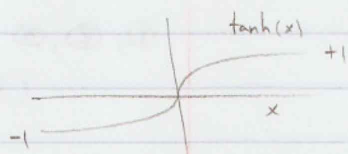
$$(v-v_1)\left(-\frac{(\gamma+1)}{2(\gamma-1)}v + \frac{v_1}{2} + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1 v_1}\right) = -\frac{D}{\gamma-1}\frac{dv}{dx}$$

$$(v-v_1)\left(v - \underbrace{\left[\frac{\gamma-1}{\gamma+1}v_1 + \frac{2v_1}{\gamma+1}\frac{\gamma P_1}{\rho_1 v_1^2}\right]}_{v_2}\right) = \frac{2}{\gamma+1} D \frac{dv}{dx}$$

$$(v-v_1)(v-v_2) = \frac{2}{\gamma+1} D \frac{dv}{dx}$$

put origin where $v = \bar{v} = \frac{v_1+v_2}{2}$ $\qquad v = v' + \bar{v}$

$v - v_1 = v' - \frac{v_1-v_2}{2}$ $\qquad v - v_2 = v' + \frac{v_1-v_2}{2}$



$v'^2 - \left(\frac{v_1-v_2}{2}\right)^2 = \frac{2}{\gamma+1} D \frac{dv'}{dx}$ $\qquad \tilde{v} = \frac{v'}{\left(\frac{v_1-v_2}{2}\right)}$

$\tilde{v}^2 - 1 = \frac{2}{\gamma+1}\frac{D\,(2)}{v_1-v_2}\frac{d\tilde{v}}{dx}$ $\qquad \int_0^{\tilde{v}} \frac{d\tilde{v}}{\tilde{v}^2-1} = \frac{\gamma+1}{4}(v_1-v_2)\int_0^x \frac{dx}{D(x)}$

$$-\tanh^{-1}(\tilde{v})\Big|_0^{\tilde{v}}$$

$\tilde{v} = -\tanh\left(\frac{\gamma+1}{4}(v_1-v_2)\int_0^x \frac{dx}{D(x)}\right)$ $\qquad v(x) = \frac{v_1+v_2}{2} - \frac{v_1-v_2}{2}\tanh\left(\frac{\gamma+1}{4}(v_1-v_2)\int_0^x \frac{dx}{D}\right)$

$x \to -\infty \quad v(x) \to v_1$

$x \to \infty \quad v(x) \to v_2$

$$\frac{l_s}{D}\frac{\gamma+1}{4}(v_1-v_2) \sim 1$$ scale of shock (l_s)

$$\frac{v_1 l_s}{D} \sim \frac{v_1}{\bar{v}}\frac{l_s}{\lambda_{mfp}}$$ $\underline{l_s \simeq \lambda_{mfp}}$ collisions change entropy and velocity accross shock

10/11/2006

Isothermal shock

$\vec{v_1}$	Shock	$\vec{v_2}$	cool	$\vec{v_3}$
ρ_1		ρ_2		ρ_3
T_1		$T_2 > T_1$		$T_3 = T_1$

ISM heating / cooling → equilibrium

cooling function $\rho\mathcal{L}(\rho,T)$ $\frac{\text{ergs}}{\text{gm s}}$ that are radiated away (to go back to equilibrium)

$$\rho v\frac{dv}{dx} = -\frac{dP}{dx} \qquad \rho v\frac{d v^2/2}{dt} + \frac{\gamma}{\gamma-1}\frac{d}{dx}(Pv) = -\rho\mathcal{L}$$

$$\rho v^2\frac{dv}{dx} + \frac{\gamma}{\gamma-1}P\frac{dv}{dx} + \frac{\gamma}{\gamma-1}v\left(\frac{dP}{dx} = -\rho v\frac{dv}{dx}\right) = -\rho\mathcal{L}$$

$$\rho v^2\frac{dv}{dx}\underbrace{\left(1-\frac{\gamma}{\gamma-1}\right)}_{\frac{-1}{\gamma-1}} + \frac{\gamma P}{\gamma-1}\frac{dv}{dx} = -\rho\mathcal{L}$$

$$\underbrace{\left(\frac{\gamma P}{\rho}-v^2\right)}_{>0}\underbrace{\frac{dv}{dx}}_{<0} = \underbrace{-(\gamma-1)\mathcal{L}}_{<0}$$ $v_2 < c_{s_2}$ (shown in HW)

cooling flow slows down $\therefore$ $\frac{d\rho}{dx} > 0$ compresses $\rho_3 > \rho_2$

Appears as (if you can't resolve)

$\vec{v_1}$	‖	$\vec{v_2}$
ρ_1		ρ_2
T_1		T_1
c_{s_1}		c_{s_1}

'isothermal'

energy goes to radiation rather than heat

Rankine-Hugoniot

$$\rho_1 v_1 = \rho_2 v_2 \qquad \rho v\frac{dv}{dx} + \frac{dP}{dx} = 0 \qquad \rho v(v_2-v_1) + P_2 - P_1 = 0$$

$$\rho_2 v_2^2 - \rho_1 v_1^2 + c_s^2(\rho_2-\rho_1) = 0 \qquad \cancel{v_2}\cancel{\rho_2}\frac{(v_2^2+c_s^2)}{v_2} = \cancel{\rho_1}\frac{(v_1^2+c_s^2)}{v_1}\cancel{v_1}$$

$$(v_2^2 + c_s^2)\, v_1 = v_2\,(v_1^2 + c_s^2)$$

$$(v_2 - v_1)\, v_1 v_2 = (v_2 - v_1)\, c_s^2 \qquad v_2 = \frac{c_s^2}{v_1}$$

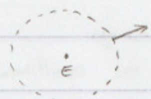
$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{v_1^2}{c_s^2} = M^2$$

Previously: $\dfrac{\rho_2}{\rho_1} = \dfrac{1}{\frac{\gamma-1}{\gamma+1} + \frac{2}{\gamma+1}\frac{1}{M^2}}$ if $\gamma = 1 \Rightarrow \dfrac{\rho_2}{\rho_1} = M^2$

Ex. molecular cloud $c_s \sim 1\ \text{km/s}$ Supernova $v \sim 10{,}000\ \text{km/s}$

$\therefore\ M \sim 10^4 \Rightarrow \dfrac{\rho_2}{\rho_1} \sim 10^8$ impressive compression

Supernova blast wave



$\rho_1\ P_1\ v_1 = 0$ (LAB)

Assume shock wave is spherical
material is homogeneous outside

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho v r^2) = 0$$

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) = -\frac{\partial P}{\partial r}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \frac{P}{\gamma-1}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho v r^2 \frac{v^2}{2} + \frac{\gamma r^2}{\gamma-1} v P\right) = 0$$

$$[\rho_1] = \frac{M}{L^3} \qquad [P_1] = \frac{M}{T^2 L} \qquad [E] = \frac{ML^2}{T^2}$$

$\left[\dfrac{E}{P_1}\right] = L^3 \qquad r_0 = \left(\dfrac{E}{P_1}\right)^{1/3} \qquad \left[\dfrac{P_1}{\rho_1}\right] = \dfrac{L^2}{T^2} \quad c_s^2 = \dfrac{P_1}{\rho_1}$ basic speed

basic time basic length

$c_s t_0 = r_0$

$$\rho(r,t) = \rho\left(\frac{r}{r_0}, \frac{t}{t_0}\right)$$

$M^2 \gg 1 \qquad \dfrac{\rho_2}{\rho_1} = \dfrac{\gamma+1}{\gamma-1} \qquad P_2 = \dfrac{2}{\gamma+1}\rho_1 v_1^2$

As if $c_{s_1} \to 0 \quad P_1 \to 0$ but we lose r_0, c_s, t_0

So need other quantity to have dimensionless quantities

$[\xi] = 1 \qquad [\xi] = r\, t^{l} E^{m} \rho^{n}$

$1 : L\, T^{l} \left(\frac{ML^2}{T^2}\right)^m \left(\frac{M}{L^3}\right)^n$

M: $m + n = 0 \qquad m = -n \qquad M^0 = 1$ $\qquad m = -\frac{1}{5}$

T: $l - 2m = 0 \qquad l = 2m = -2n$ $\qquad l = -\frac{2}{5}$

L: $1 + 2m - 3n = 0 \qquad 1 - 5n = 0 \implies n = \frac{1}{5}$

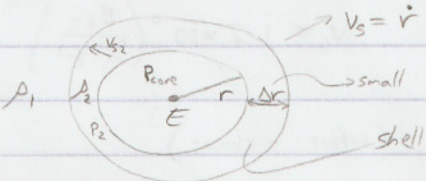
$$r = \xi \left(\frac{E t^2}{\rho_1}\right)^{1/5}$$

$$\rho(r,t) = \rho(\xi)$$

similarity solutions

r,t used to normalize

Sedov–Taylor solution



$$M_{shell} = \frac{4\pi}{3} r^3 \rho_1 = 4\pi r^2 \Delta r\, \rho_2$$

$$\therefore \frac{\Delta r}{r} = \frac{1}{3}\left(\frac{\rho_1}{\rho_2}\right) \sim \frac{1}{12}$$

LAB   Shock

$$V_{2_L} = -V_{2_S} + V_s = -\frac{\gamma-1}{\gamma+1} V_s + V_s \qquad V_2 = \frac{2}{\gamma+1} V_s$$

$$\frac{d}{dt}\Big[\underbrace{M_{shell} V_{shell}}_{\text{momentum}}\Big] = \underbrace{4\pi r^2 P_{core}}_{\text{area}\cdot\text{pressure} = \text{force}} \qquad \frac{d}{dt}\left(\frac{4\pi r^3}{3}\rho_1 \frac{2}{\gamma+1}\dot r\right) = 4\pi r^2 \alpha P_2$$

~ constant

Since we dont know P_{core} : $\quad P_{core} \simeq \alpha P_2$

(swindle)

$$\frac{d}{dt}\left(\frac{4\pi r^3 \rho_1}{3}\frac{2}{\gamma+1}\dot r\right) = 4\pi r^2 \alpha \frac{2}{\gamma+1}\rho_1 \dot r^2$$

$$\frac{d}{dt}(r^3 \dot r) = 3\alpha r^2 \dot r^2 = 3\alpha (r^3 \dot r)\frac{\dot r}{r}$$

$$\frac{d(r^3\dot r)}{r^3 \dot r} = 3\alpha \frac{dr}{r} \qquad r^3 \dot r = a r^{3\alpha} \qquad \boxed{\dot r = a r^{3\alpha - 3}}$$

$$E = \frac{M_{shell} V_{shell}^2}{2} + \frac{4\pi}{3} r^3 \frac{\alpha P_2}{\gamma - 1} = \frac{4\pi}{3} r^3 \rho_1 \left(\frac{2}{\gamma+1}\right)^2 \frac{\dot r^2}{2} + \frac{4\pi}{3} r^3 \frac{\alpha \rho_1 \dot r^2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)$$

$$= \underbrace{\frac{4\pi}{3}\left(\frac{2}{\gamma+1}\right)\left(\frac{1}{\gamma+1} + \frac{\alpha}{\gamma-1}\right)\rho_1}_{\text{const}} \underbrace{r^3 \dot r^2}_{a^2 r^{6\alpha - 3}}$$

const $\qquad \therefore \;$ constant $\implies \alpha = \frac{1}{2}$

$$a = (\sim)\left(\frac{E}{\rho_1}\right)^{1/2} \qquad \text{since} \quad E = (\sim)\rho_1 a^2$$

$\dot{r} = a r^{-3/2}$ $\quad r^{3/2}\dot{r} = a \quad \frac{2}{5} r^{5/2} = at$

$$r = \left(\frac{5}{2}at\right)^{2/5} = \boxed{1.12\left(\frac{Et^2}{\rho_1}\right)^{1/5} = r}$$

ξ from $\gamma = 5/3$ and stuff like that

$$V_s = \dot{r} = \frac{2}{5}(\ )\left(\frac{E}{\rho_1 t^3}\right)^{1/5}$$

SN $\quad E = 10^{51}\text{ ergs} = E_{51} \qquad \rho_1 = 1.67\times10^{-24}\, n_1$

$t = 3\times10^7 \tau$ (years)

$$r = 0.33\,\text{pc}\left(\frac{E_{51}\tau^2}{n_1}\right)^{1/5} \qquad V_s = 1.2\times10^{10}\left(\frac{E_{51}}{n_1\tau^3}\right)^{1/5}\ \text{cm/s}$$

Sedov

good for: $M_{shell} > M_{ejecta}$ $\quad$ Sedov phase $\quad$ (before: ~chaos)

$\tau = 100\text{ yrs} \quad r \sim 2\text{ pc} \quad V_s \sim 10^4\text{ km/s}$

$\tau \sim 10^4\text{ yrs} \quad r \sim 10\text{-}12\text{ pc} \quad V_s \sim 500\text{ km/s} \qquad \frac{k_B T_2}{m} = \frac{3}{16}V_s^2$

$T_2 \sim 3\times10^6\text{ K} \longrightarrow$ X-rays

$\tau \sim 10^5\text{ yrs}$ (late phase) $\quad$ core cools, $P_{core}\downarrow \quad \frac{d}{dt}(MV)_{shell} \approx 0 \quad$ radiation phase, snowplow

Can also be used for AGN (accretion on) $\quad E = Lt$ (luminosity)

Non-relativistic MHD
($E \ll B$)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad \rho \overbrace{\frac{d\vec{v}}{dt}}^{\text{convective}} = -\nabla P + \frac{\vec{J} \times \vec{B}}{c}$$

$$d\vec{F} = I\, d\vec{l} \times \frac{\vec{B}}{c} = JA\, d\vec{l} \times \vec{B}/c$$

ideal $\Pi = q = 0$

$$\frac{\partial}{\partial t}\left(\underbrace{\rho v^2}_{\frac{1}{2}(?)} + \frac{p}{\gamma - 1} + \underbrace{\frac{B^2}{8\pi}}_{\varepsilon_{mo}}\right) + \nabla \cdot \left(\underbrace{\rho \vec{v} \frac{v^2}{2}}_{S_{mo}} + \frac{\gamma}{\gamma - 1} \vec{v} P + \frac{c\, \vec{E} \times \vec{B}}{4\pi}\right) = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} \qquad \nabla \cdot \vec{B} = 0$$

Ohm's law

$$\underbrace{\vec{E}'}_{\text{Moving}} = \underbrace{\vec{E}}_{\text{LAB}} + \frac{\vec{v} \times \vec{B}}{c} = \underbrace{\eta}_{\text{resistivity}} \vec{J} = \frac{1}{\underbrace{\sigma}_{\text{conductivity}}} \vec{J}$$

$$-\frac{1}{c}\frac{\partial B}{\partial t} = -\nabla \times \left(\frac{v \times B}{c}\right) + \nabla \times \left(\eta \frac{c}{4\pi} \nabla \times B\right)$$

$$\frac{\partial B}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{\eta c^2}{4\pi} \nabla^2 \vec{B} \qquad \left[\frac{\eta c^2}{4\pi}\right] = \frac{L^2}{T} = D_B \quad \text{magnetic diffusion coefficient}$$

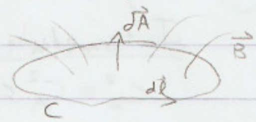
$$\frac{v_{ch} B}{L} : \frac{\eta c^2}{4\pi}\frac{B}{L^2}$$

Magnetic Reynolds $\quad r_m = \dfrac{\eta c^2}{4\pi v_{ch}}$

usually η is small in astro conditions

$$\Rightarrow \vec{E} + \frac{\vec{v} \times \vec{B}}{c} = 0 \qquad E_{\parallel} = 0$$

$$\oint_C \left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right) \cdot d\vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A}$$



$$0 = \frac{d}{dt}\int \vec{B} \cdot d\vec{A}$$

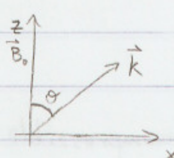
Frozen-in (Alfven)
field lines move with plasma

Momentum Eq:

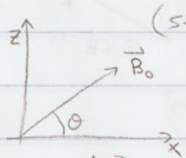
$$\rho \frac{d\vec{v}}{dt} = -\nabla P - \frac{\vec{B} \times (\nabla \times \vec{B})}{4\pi} = -\nabla \Big(P + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic pressure}}\Big) + \underbrace{\frac{B}{4\pi} \cdot \nabla B}_{\text{magnetic tension}}$$

10/16/2006

Linear waves



-OR-



$\nabla = \hat{e}_x \dfrac{\partial}{\partial x}$

(Step Approach)

$\xi = x - ut$ in wave frame

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial t} = -u\frac{\partial}{\partial \xi}$$

$$-u\frac{\partial \delta\rho}{\partial \xi} + \rho_0 \frac{\partial \delta v_x}{\partial \xi} = 0 \qquad u\,\delta\rho - \rho_0\, \delta v_x = 0 \qquad \boxed{\frac{\delta\rho}{\rho_0} = \frac{\delta v_x}{u}}$$

$$\frac{B^2}{8\pi} = \frac{(\vec{B}_0 + \delta\vec{B})^2}{8\pi}$$

$$\rho_0 u \frac{\partial \delta\vec{v}}{\partial \xi} = -\hat{e}_x \frac{\partial}{\partial \xi}\left[\delta P + \frac{\vec{B}_0 \cdot \delta\vec{B}}{4\pi}\right] + \frac{B_{0x}}{4\pi}\frac{\partial}{\partial \xi}\delta\vec{B} \qquad \text{Momentum Eq.}$$

$$\nabla \cdot \delta\vec{B} = \frac{\partial}{\partial \xi}\delta B_x = 0$$

$$\rho_0 u\, \delta\vec{v} = \hat{e}_x\left(c_s^2\, \delta\rho + \frac{B_{0z}\,\delta B_z}{4\pi}\right) - \frac{B_{0x}}{4\pi}\delta\vec{B}$$

① $$\rho_0 u\, \delta v_x = \rho_0 c_s^2 \frac{\delta v_x}{u} + \frac{B_{0z}\,\delta B_z}{4\pi}$$

② $$\rho_0 u\, \delta v_y = -\frac{B_{0x}}{4\pi}\delta B_y$$ ③ $$\rho_0 u\, \delta v_z = -\frac{B_{0z}}{4\pi}\delta B_z$$

Ohm's law $$-u\frac{\partial \delta\vec{B}}{\partial \xi} = +B_{0x}\frac{\partial \delta\vec{v}}{\partial \xi} - \vec{B}_0 \frac{\partial \delta v_x}{\partial \xi} \qquad \frac{\partial \delta B_x}{\partial \xi} = 0$$

④ $$u\,\delta B_y = -B_{0x}\,\delta v_y$$ ⑤ $$u\,\delta B_z = -B_{0x}\,\delta v_z + B_{0z}\,\delta v_x$$

One normal mode is in $\hat{y}$ (oscillations in $\hat{y}$ indep. of x, z)

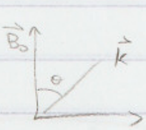
$$\delta v_x = \delta v_z = \delta B_z = 0$$

$$\delta v_y = -\frac{B_{0x}}{4\pi\rho_0 u}\delta B_y \qquad u\,\delta B_y = \frac{B_{0x}^2}{4\pi\rho_0 u}\delta B_y$$

$$\left(u^2 - \frac{B_{0x}^2}{4\pi\rho_0 u}\right)\delta B_y = 0$$

$$\frac{\omega^2}{k^2} = u^2 = \frac{B_{0x}^2}{4\pi\rho_0} = \frac{B_0^2\cos^2\theta}{4\pi\rho_0}$$ Alfven (Shear, Torsional) Intermediate wave

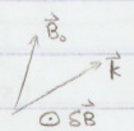
$$\boxed{C_A^2 = \frac{B_0^2}{4\pi\rho_0}} \qquad \omega^2 = k^2 C_A^2 \cos^2\theta \qquad C_I^2 = C_A^2\cos^2\theta$$



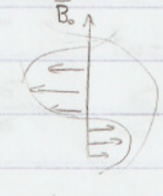
$k\cos\theta = k_\parallel$ $\quad \omega = k_\parallel C_A \quad \frac{\partial\omega}{\partial k_\parallel} = C_A \quad \frac{\partial\omega}{\partial k_\perp} = 0$ wave doesnt propogate accross mag. field

$k\sin\theta = k_\perp$

$\delta\vec{B} \perp$ plane $\vec{B}_0, \vec{k}$

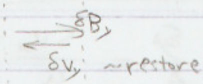


Transverse mode



"pluck the string"
jiggle plasma

$\delta\rho = \delta P = 0$ Incompressible
→ cannot form shock wave



pure EnM wave travelling along B lines

Now use ①, ③, ⑤

$$u\,\delta B_z = \frac{B_{ox}^2}{4\pi\rho_0 u}\,\delta B_z + B_{oz}\,\delta v_x$$

$$\left(u^2 - \frac{B_{ox}^2}{4\pi\rho_0}\right)\delta B_z = u B_{oz}\,\delta v_x \qquad \delta B_z = \frac{u B_{oz}\,\delta v_x}{u^2 - \frac{B_{ox}^2}{4\pi\rho_0}}$$

$$u\,\delta v_x = \frac{C_s^2}{u}\,\delta v_x + \frac{u B_{oz}^2/4\pi\rho_0}{u^2 - \frac{B_{ox}^2}{4\pi\rho_0}}\,\delta v_x$$

$$u^2\left(u^2 - \frac{B_{ox}^2}{4\pi\rho_0}\right) = C_s^2\left(u^2 - \frac{B_{ox}^2}{4\pi\rho_0}\right) + u^2\,\frac{B_z^2}{4\pi\rho_0}$$

$$u^4 - u^2\left(C_A^2 + C_s^2\right) + C_s^2 C_A^2 \cos^2\theta = 0 \qquad C_A^2 = \frac{B_0^2}{4\pi\rho_0}$$

$$\frac{\omega^2}{k^2} = \frac{C_A^2 + C_s^2}{2} \pm \sqrt{\left(\frac{C_A^2+C_s^2}{2}\right)^2 - C_s^2 C_A^2\cos^2\theta}$$

+ Fast (magnetosonic wave)

− Slow wave

$$\sqrt{\left(\frac{C_A^2 - C_s^2}{2}\right)^2 + C_A^2 C_s^2 \sin^2\theta}$$

$$\beta = \frac{2}{\gamma}\frac{C_s^2}{C_A^2} = \frac{8\pi P_0}{B_0^2}$$ ratio of pressure to mag. pressure

$\beta \gg 1$ dominated by P, $C_s > C_A$ $\qquad$ $\beta \ll 1$ dominated by B, $C_A > C_s$

$\theta = 0$ $\quad$ $\beta < 1$ $\quad$ Alfven $C_I = C_A$

Fast wave $\left.\frac{\omega}{k}\right)_+ = C_F = C_A$ $\qquad$ Slow wave $\left.\frac{\omega}{k}\right)_- = C_{SL} = C_s$

$\beta > 1$ $\quad$ $\left.\frac{\omega}{k}\right)_+ = C_F = C_s$ $\qquad$ $\left.\frac{\omega}{k}\right)_- = C_{SL} = C_A$

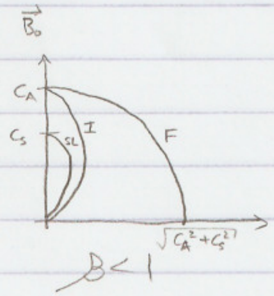
$\theta = \frac{\pi}{2}$ $\quad$ $\left.\frac{\omega}{k}\right)_+ = \sqrt{C_A^2 + C_s^2} = C_F$ $\qquad$ $\left.\frac{\omega}{k}\right)_- = 0$

$$\theta \le \frac{\pi}{2} \qquad \sqrt{\sim} \approx \frac{C_A^2 + C_s^2}{2}\left(1 - \frac{2 C_s^2 C_A^2\cos^2\theta}{C_A^2 + C_s^2}\right)$$

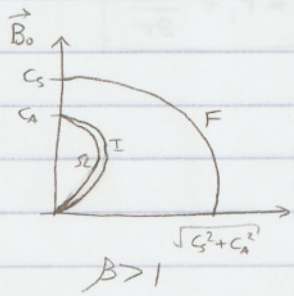
$$C_{SL} = \sqrt{\frac{C_A^2 C_s^2\cos^2\theta}{C_A^2 + C_s^2}}$$

$\xrightarrow{\beta \ll 1}$ $C_s\cos\theta$ $\quad$ $\omega = k_\parallel C_s$

$\xrightarrow{\beta \gg 1}$ $C_A\cos\theta$



$\beta < 1$



$\beta > 1$

$$C_A^2 + C_s^2 = \frac{\gamma P_0 + \frac{B_0^2}{4\pi}}{\rho_0}$$ in Fast wave: both ordinary + mag. P act together

$$\delta B_z = \frac{u B_{0z} \delta v_x}{\frac{\omega^2}{k^2} - C_I^2}$$ $\delta\rho > 0$, $\delta v_x > 0$

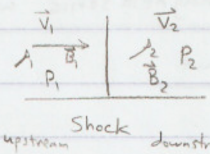
F: $\delta B_z > 0$, $\delta P > 0$, $\frac{\omega}{k} > C_I$

SL: $\delta P > 0$, $\delta B_z < 0$, $\frac{\omega}{k} << C_I$ (B decreases)

F, SL polarized in plane of k, B_0

Neither I, SL propagate accross $\vec{B}$

Evolutionary Conditions



F: $V_1 > C_{F_1}$ $\quad C_{I_2} \le V_2 < C_{F_2}$

SL: up $C_{I_1} \ge V_1 > C_{SL1}$ $\quad$ down $V_2 < C_{SL2}$

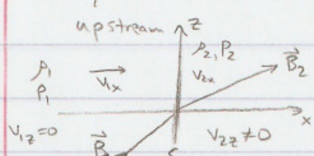
if $\beta > 1$ not much space for V_1

SL shocks not as strong

Co-planarity

$\vec{B}_2$ must be in plane $\vec{B}_1, \vec{V}_1(x)$ (cannot change plane)

Oblique slow shock



work out Rankine-Hugoniot relations

$$\frac{d}{dx}(\rho v_x) = 0 \qquad \rho_2 V_{x2} = \rho_1 V_{x1}$$

$$\rho_1 V_{x1} \frac{d\vec{v}}{dx} = -\hat{e}_x \frac{d}{dx}\left(P + \frac{B^2}{8\pi}\right) + \frac{B_x}{4\pi}\frac{d}{dx}\vec{B}$$

$$\underbrace{\rho_1 V_{x1}(V_{x2} - V_{x1})}_{\sim 0 \text{ (small vel for SL)}} = -\left(P_2 - P_1 + \frac{1}{8\pi}\left(B_{z2}^2 - B_{z1}^2\right)\right)$$ $B_{z2} \to 0$ Switch-off slow shock

$$\boxed{P_2 = P_1 + \frac{B_{z1}^2}{8\pi}}$$

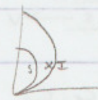
$$\rho_1 v_{x1} v_{z2} = \frac{B_x}{4\pi}\left(B_{z2} - B_{z1}\right) \Longrightarrow \rho_1 v_{x1} v_{z2} = -\frac{B_x B_{z1}}{4\pi}$$

$$\vec{E} = -\frac{\vec{v}\times\vec{B}}{c} \qquad \nabla\times\vec{E} = 0 \qquad \frac{dE_y}{dx} = 0 \qquad E_y = -\frac{1}{c}\left(v_z B_x - v_x B_z\right)$$

$v_{z1} = 0 \quad B_{z2} = 0$

$$E_y = \frac{v_{x1} B_{z1}}{c} \qquad v_{z2} = -\frac{v_{x1} B_{z1}}{B_x} \qquad \frac{B_{z1}}{B_x} \gg 1 \quad \text{(oblique)}$$

$$v_{z2} = -\frac{B_x B_{z1}}{4\pi\rho_1 v_{x1}} \qquad v_{x1}^2 = \frac{B_x^2}{4\pi\rho_1} \qquad v_{x1} = \frac{B_x}{\sqrt{4\pi\rho_1}} = C_{I_1}$$



max strength slow shock is switch-off shock

$$v_{z2} = -\frac{B_{z1}}{\sqrt{4\pi\rho_1}} \approx -C_{A_1}$$

Take B_1 and convert to kinetic energy (v_z)

$$v_{1x} \ll C_{A_1} \Longrightarrow v_{2z} = C_{A_1}$$

Conservation of energy

$$\frac{d}{dx}\left[\rho_1 v_{x1}\frac{v^2}{2} + \frac{\gamma}{\gamma-1}v_x P + \frac{cE_y B_z}{4\pi}\right] = 0$$

$$\rho_1 v_{x1}\frac{v_{2z}^2}{2} + \frac{\gamma}{\gamma-1}\left(v_{x2}P_2 - v_{x1}P_1\right) + \frac{cE_y}{4\pi}\left(\cancelto{0}{B_{z2}} - B_{z1}\right) = 0$$

$$\frac{v_{2z}^2}{2} = \frac{1}{2}\frac{B_{1z}^2}{4\pi\rho_1} \qquad (?) \quad -\frac{v_{x1}B_{z1}^2}{4\pi}$$

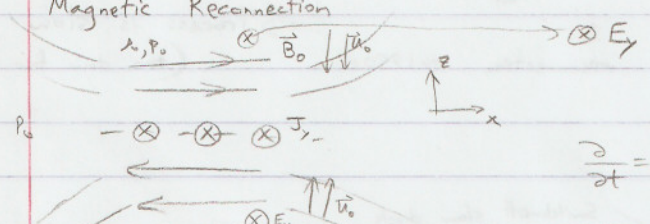
$$\frac{\gamma}{\gamma-1}\left(v_{x2}P_2 - v_{x1}P_1\right) = \frac{v_{x1}^2 B_{z1}^2}{8\pi} \qquad v_{x2} = v_{x1}\frac{P_1 + \frac{\gamma-1}{\gamma}\frac{B_{1z}^2}{8\pi}}{P_2 = P_1 + \frac{B_{1z}^2}{8\pi}}$$

$$\gamma = 5/3 \qquad \text{low } \beta \qquad v_{x2} \to v_{x1}\frac{\gamma-1}{\gamma}$$

$$\frac{\gamma}{\gamma-1} = 5/2 \qquad \beta \ll 1 \qquad v_{x2} = \frac{2}{5}v_{x1}$$

10/18/2006

Magnetic Reconnection



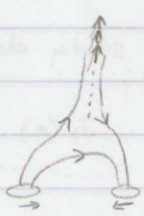
$$\vec{v} = c\frac{\vec{E}\times\vec{B}}{B^2}$$

$$\frac{\partial}{\partial t} = 0 \qquad \vec{B}\times\vec{E} = -\vec{E}\times\vec{v}\times\vec{B}$$

plasma + field lines move to center

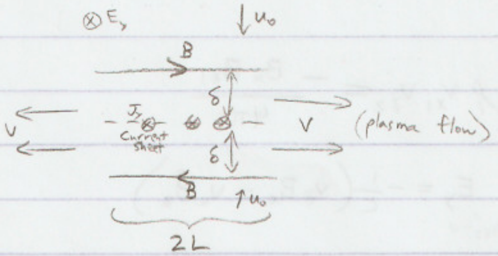
B_0 fields annihilate (removal of Poynting energy) → Energy goes to kinetic / heat

First studied in solar flares



Wrong

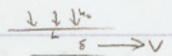
early model



How fast is energy converted?

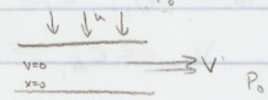
Consider ρ_0 constant

$$\rho_0 u_0 L = \rho_0 v \delta \implies u_0 L = v\delta$$



$u_0 \ll c_s, c_A$ $\qquad -\frac{d}{dz}\left(P + \frac{B^2}{8\pi}\right) = 0$ Hydrostatic Equilibrium

$$P_0 + \frac{B_0^2}{8\pi} = P(0)$$



$$\rho v \frac{dv}{dx} = -\frac{dP}{dx}$$

$$P_0 + \frac{\rho_0 v^2}{2} = P(0) \qquad \frac{d}{dx}\left(\rho_0 \frac{v^2}{2} + P\right) = 0$$

$$\frac{\rho_0 v^2}{2} = P(0) - P_0 = \frac{B_0^2}{8\pi} \implies v = \frac{B_0}{\sqrt{4\pi\rho_0}} = c_A \qquad \delta = L\frac{u_0}{c_A} = M_A L$$

Alfven Mach #

Ampere: $\quad \frac{dB_x}{dz} = \frac{4\pi}{c} J_y \qquad \frac{B_0}{\delta} = \frac{4\pi}{c}\sigma E_y \qquad E_y = \frac{u_0 B_0}{c}$

$$\frac{B_0}{\delta} = \frac{4\pi}{c^2} u_0 B_0 \sigma \implies \delta = \frac{c^2}{4\pi\sigma u_0} = \frac{c^2}{4\pi\sigma c_A M_A} = \frac{r_M}{M_A}$$

Magnetic Reynolds length

$$M_A L = \frac{r_M}{M_A} \implies M_A = \sqrt{\frac{r_M}{L}} = \sqrt{\frac{1}{R_M}}$$

Magnetic Reynolds number

Answer to P-S

In astro: $L \sim$ large $\quad r_M \sim$ small $\quad$ ($\sigma \sim$ high) $\implies R_M \sim 10^{12} - 10^{20}$

ionized hot

$\implies$ Process is slow.

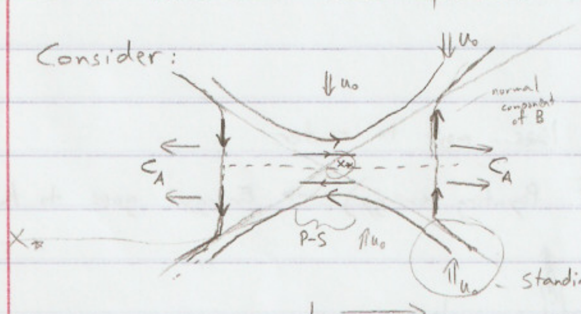
Parker - Sweet model of magnetic reconnection ~1950s

P-S

(too slow for solar flares)

1964 Petshek solved problem

Consider:



Channel opens out $\to$ more stuff in + out

standing shocks annihilate field

But far: $\xrightarrow{B}$ $\xleftarrow{\quad}$ so take $J_y(x)$ to set up $\succ\!\prec$

$\delta B_x < 0 \to$ field bends in

$$u_0 X_* = \delta V = C_A = C_A \frac{r_m}{M_A} \qquad X_* = \frac{r_m}{M_A^2}$$

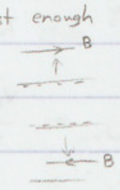
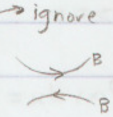
$$\frac{\delta B_x}{B_0} = -\frac{4}{\pi} M_A \ln\left(\frac{L}{X_*}\right) \qquad B_x \downarrow : V = \frac{B_x}{\sqrt{4\pi\rho_0}} \downarrow \qquad u_0 = \frac{cE_y}{B_x} \uparrow \Longrightarrow \text{choke}$$

$$\therefore \delta B_x \geq -\frac{1}{2}$$

$$\therefore \max M_A \approx \frac{\pi}{8} \frac{1}{\ln\left(\frac{L}{r_m} M_A^2\right)} \approx \frac{\pi}{8} \frac{1}{\ln(R_m)} \qquad M_A \leq \frac{1}{10}$$

ignore it

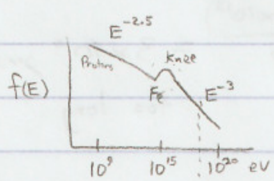
Now it works → fast enough

Rarefaction wave  pulls fields in 

Cosmic Rays

$10^9\,eV \leq E \leq 10^{20}\,eV$

Distribution function $f(p) \propto \frac{1}{p^q}$ $\qquad f(E) \propto \frac{1}{E^{q-2}}$



10^{18} → extragalactic (gyro radius larger than disk of galaxy)

Isotropic one part in 1000 $\left(\frac{1}{10^3}\right)$ $\qquad u/c \lesssim 10^{-3}$

How long in galaxy? use ${}^{10}_{4}Be$ know production rate (collisions w/Fe etc) ${}^{10}_{4}Be$ halflife $\sim 1.6 \times 10^6\,yr$

$\tau \sim 2 \times 10^7$ yrs

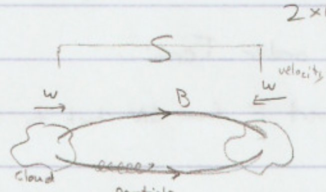
energy density $\epsilon_{CR} \sim \epsilon_{gas} \sim \epsilon_{mag} \sim \epsilon_{\gamma} \sim 10^{-12} \frac{ergs}{cm^3}$ (γ: photons)

$$V_{gal} = \pi R^2 h \simeq 3\,(2\times10^4)^2(500)\ pc^3 = 6\times10^{11}\ pc^3 \simeq 2\times10^{67}\ cm^3$$

20 Mpc

$$V\epsilon = E = 2\times10^{55}\ ergs \qquad \frac{2\times10^{55}}{2\times10^7\,(3\times10^7)} = \frac{E}{\tau} = 3\times10^{40}\ ergs/s$$

1949 – Fermi



bounces back + forth (Magnetic Mirror)

proposed particles get energy by bouncing back + forth

$J = \oint P_{\parallel}\, dS$ $\qquad$ 2nd adiabatic invariant

$$\frac{\Delta P_{\parallel}}{P_{\parallel}} = -\frac{\Delta S}{S} = -\frac{w\,t_b}{\frac{1}{2}V_{\parallel}\,t_b} = -2\frac{w}{V_{\parallel}}$$

bounce time

$\gamma = \sqrt{1 + p^2/m^2c^2}$ $\qquad \gamma \Delta\gamma = p_\parallel \Delta p_\parallel / m^2c^2$

$$\frac{\Delta\gamma}{\gamma} = \frac{p_\parallel}{\gamma^2}\frac{\Delta p_\parallel}{m^2c^2} = -\frac{p_\parallel^2 w}{\gamma^2 m^2 c^2 v_\parallel} = -\frac{v_\parallel w}{c^2}$$

First order Fermi Acceleration $\boxed{\dfrac{\Delta\gamma}{\gamma} = -\dfrac{v_\parallel w}{c^2}}$

Higher $E \to$ Higher kick

But in astro $\langle w \rangle = 0$ $\qquad \xrightarrow{v_\parallel} \overset{w}{\to} \Rightarrow$ lose E

Prob of collision $\quad P(x)\,dx = \dfrac{dx}{v} \qquad P_{head\,on} = \dfrac{v_\parallel + w}{2v_\parallel} \qquad P_{overtake} = \dfrac{v_\parallel - w}{2v_\parallel}$

More likely that hit head on + gain energy

$$\frac{\Delta\gamma}{\gamma} = \frac{v_\parallel w}{c^2}\left(\frac{v_\parallel + w}{2v_\parallel} - \frac{v_\parallel - w}{2v_\parallel}\right) = \frac{w^2}{c^2}$$

$\dfrac{\Delta\gamma}{\gamma} = \dfrac{w^2}{c^2}$ 2^{nd} order Fermi acceleration

$$\frac{1}{\tau_{acc}} = \frac{1}{\gamma}\frac{\langle\Delta\gamma\rangle}{\Delta t} = \frac{w^2}{c^2}\frac{c}{l} \qquad \tau_{acc} = \frac{cl}{w^2} \sim \frac{(3\times10^{10})(3\times10^{19})}{10^{12}\ 3\times10^7} = 3\times10^{10}\ \text{yrs}$$

too long

So this doesnt work

Supernova Shocks $\quad E = 10^{51}$ ergs $\qquad$ rate $1/30\ 1/yr$

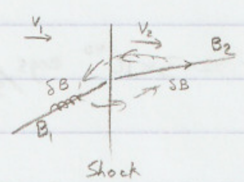
$$\dot{E}_{SN} = \frac{10^{51}}{30(3\times10^7)} \simeq 3\times10^{42}\ \text{erg/s} \qquad r_{SN} \sim 200\,pc$$

$$V_{SN} = \tfrac{4\pi}{3} r_{SN}^3 \simeq 8\times10^{62}\ cm^3 \qquad N_{SN} = \frac{\overset{\text{CR time}}{2\times10^7}}{\underset{\text{SN rate}}{30}} = 7\times10^5$$

$$V_{total} = 7\times10^5\ 8\times10^{62} = 6\times10^{68}\ cm^3 > V_{gal} \sim 2\times10^{67}$$

CR blasted by SN as they travel the galaxy

1977 Axford Skadram $\to$ shocks accelerate



CR sent back and forth by $\delta\vec{B}$

$\Rightarrow$ 1^{st} order Fermi

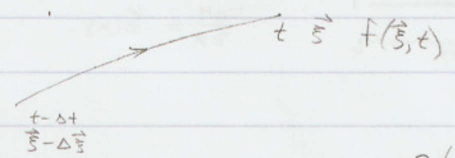
$\dfrac{v_1 - v_2}{v_1}$ $\qquad$ Convert E from SN shock to high E particles

10/23/2006

Derivation of Fokker-Planck

$\vec{\xi} = (\vec{x}, \vec{p})_i \quad i = 1 \ldots 6$

momentum



$f(\vec{\xi}-\Delta\vec{\xi}, t-\Delta t)$

$P(\vec{\xi}-\Delta\vec{\xi}; \Delta\vec{\xi})$ = probability that particle @ $\vec{\xi}-\Delta\vec{\xi}$, $t-\Delta t$ makes it to $\vec{\xi}$, t

$$f(\vec{\xi}, t) = \int P(\vec{\xi}-\Delta\vec{\xi}; \Delta\vec{\xi}) f(\vec{\xi}-\Delta\vec{\xi}, t-\Delta t)\, d\Delta\vec{\xi} \quad \text{Pauli Master Eqn}$$

exact

Δt - small $\quad \Delta\vec{\xi}$ - small

$$f(\vec{\xi}, t) = \int d(\Delta\vec{\xi}) \left[f(\vec{\xi}, t) - \frac{\partial f}{\partial t}\Delta t - \frac{\partial f}{\partial \xi_i}\Delta\xi_i + \frac{1}{2}\frac{\partial^2 f}{\partial \xi_i \partial \xi_k}\Delta\xi_i \Delta\xi_k \right] \left[P(\vec{\xi}, \Delta\vec{\xi}) - \frac{\partial P}{\partial \xi_i}\Delta\xi_i + \frac{1}{2}\frac{\partial^2 P}{\partial \xi_i \partial \xi_k}\Delta\xi_i\Delta\xi_k \right]$$

$$\int P(\xi - \Delta\xi; \Delta\xi)\, d\Delta\xi = 1$$

$$= f(\vec{\xi}, t) - \frac{\partial f}{\partial t}\Delta t + \int d\Delta\vec{\xi} \left[\left(-P\frac{\partial f}{\partial \xi_i} - f\frac{\partial P}{\partial \xi_i} \right)\Delta\xi_i + \frac{1}{2}\left(P\frac{\partial^2 f}{\partial \xi_i \partial \xi_k} + f\frac{\partial^2 P}{\partial \xi_i \partial \xi_k} + \frac{\partial f}{\partial \xi_i}\frac{\partial P}{\partial \xi_k} + \frac{\partial P}{\partial \xi_i}\frac{\partial f}{\partial \xi_k} \right)\Delta\xi_i\Delta\xi_k \right]$$

$$\frac{\partial f}{\partial t} = \int d\Delta\vec{\xi} \left[-\frac{\partial}{\partial \xi_i}\left(\frac{\Delta\xi_i P f}{\Delta t} \right) + \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_k}\left(\frac{\Delta\xi_i \Delta\xi_k}{2\Delta t} P f \right) \right]$$

$$\frac{\Delta\xi_i}{\Delta t} = \bar{\dot{\xi}}_i + \left.\frac{\Delta\xi_i}{\Delta t}\right|_{random}$$

Avg stream vel. in phase space

$$\left\langle \frac{\Delta\xi_i \Delta\xi_k}{2\Delta t} \right\rangle \quad \text{statistical average}$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} + \frac{\partial}{\partial \xi_i}\left(\bar{\dot{\xi}}_i f \right) = \frac{\partial}{\partial \xi_i}\left(-\frac{\langle \Delta\xi_i \rangle}{\Delta t} f \right) + \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_k}\left(D_{ik} f \right) \quad \text{Fokker-Planck Eqn}$$

diffusion term

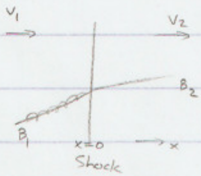
$f = 1$

$$0 = -\frac{\partial}{\partial \xi_i}\left(\frac{\langle \Delta\xi_i \rangle}{\Delta t} + \frac{\partial D_{ik}}{\partial \xi_k} \right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} + \frac{\partial}{\partial \xi_i}\left(\bar{\dot{\xi}}_i f \right) = \frac{\partial}{\partial \xi_i}\left(D_{ik}\frac{\partial f}{\partial \xi_k} \right)$$

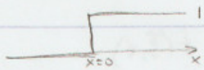
$$\frac{df}{dt} = \left.\frac{\delta f}{\delta t}\right|_{coll}$$

Cosmic Rays



CR bounce back and forth

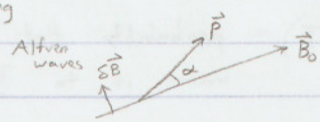
$D_{xx} \equiv D$



$$\frac{\partial H}{\partial x} = \delta(x)$$

$$\underbrace{\vec{u}}_{vel} = V_1\left(1 - \frac{V_1 - V_2}{V_1} H(x)\right)$$

Scattering



α - pitch angle $\qquad P_{\parallel} = p\cos\alpha \qquad P_{\perp} = p\sin\alpha$

$$\frac{d\vec{p}}{dt} = \frac{q}{\gamma mc}\vec{p}\times\vec{B} \qquad \text{equation of motion} \quad F = q(\vec{v}\times\vec{B}) \qquad \vec{v} = \frac{\vec{p}}{\gamma m}$$

$$\frac{dP_{\parallel}}{dt} = \frac{q}{\gamma mc}\left(\vec{p}\times(\vec{B}_0 + \delta\vec{B})\right)\cdot\hat{B}_0 = \frac{q}{\gamma mc} P_{\perp}\,\delta B = -\overbrace{\sin\alpha\, p}^{P_{\perp}}\frac{d\alpha}{dt}$$

$$\frac{d\alpha}{dt} = \frac{q\,\delta B}{\gamma mc} \qquad \Delta\alpha = \frac{q\,\delta B}{\gamma mc}\Delta t \qquad \langle\Delta\alpha\rangle = 0$$

$$\underbrace{\frac{\langle(\Delta\alpha)^2\rangle}{2\Delta t}}_{\nu} = \frac{q^2\langle\delta B^2\rangle}{\gamma^2 m^2 c^2}\frac{\Delta t^2}{2\Delta t} \qquad \boxed{\nu = \frac{q^2\langle\delta B^2\rangle}{2\gamma^2 m^2 c^2}\frac{\pi}{(\Delta k_{\parallel})V_{\parallel}}} \quad \text{exact}$$

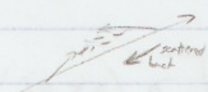
pitch angle diff. coeff.

$$\Delta t \sim \pi / k_{\parallel}V_{\parallel}$$

time to cross half a wavelength

we need spatial diffusion

$$\tan\alpha = P_{\perp}/P_{\parallel}$$



$$\Delta\alpha \sim P_{\perp}/P_{\parallel}$$

Scattering time $\quad t_s = \dfrac{(P_{\perp}/P_{\parallel})^2}{2\nu}$ $\qquad \Delta S = V_{\parallel} t_s \qquad D = \dfrac{\langle(\Delta s)^2\rangle}{2t_s}$

$$D = \frac{V_{\parallel}^2 t_s}{2} = \frac{V_{\parallel}^2}{4\nu}\frac{P_{\perp}^2}{P_{\parallel}^2} = \frac{\cancel{V_{\parallel}^2}\, p^2\sin^2\alpha}{4\nu\,\gamma^2 m^2 V_{\parallel}^2} = \boxed{\frac{p^2}{8\gamma^2 m^2}\underbrace{\int_0^{\pi}\frac{\sin^2\alpha}{\nu(\alpha)}\sin\alpha\, d\alpha}_{\text{averaging over pitch angle}} = D} \quad \text{exact}$$

δV

n density of C.R.

$$N = n\,\delta V \qquad \frac{dN}{dt} = 0 = \delta V\frac{dn}{dt} + n\frac{d\,\delta V}{dt} \qquad \frac{1}{n}\frac{dn}{dt} = -\frac{1}{\delta V}\frac{d\,\delta V}{dt} = -\nabla\cdot\underbrace{\vec{u}}_{vel.}$$

from continuity

element in phase space $\quad d^3p\, d^3x \underset{\text{prop.}}{\propto} p^3\,\delta V$ if isotropic $\qquad \underbrace{\dfrac{d}{dt}(p^3\delta V) = 0}_{\text{conserved}}$

$\nabla\cdot\vec{u} = 0$ if incompress.

$$3p^2\frac{dp}{dt}\delta V + p^3\frac{d\,\delta V}{dt} = 0 \qquad \frac{dp}{dt} = -\frac{p}{3}\frac{d(\ln\delta V)}{dt} = -\frac{p}{3}\nabla\cdot\vec{u}$$

$P V^{\gamma}$ = constant (pressure, volume)

$$\frac{dp}{dt} = -\frac{p}{3}\nabla\cdot\vec{u}$$

compression force

spherical coord

$$\frac{\partial f}{\partial t} + \nabla\cdot(uf) - \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 \frac{p}{3}\nabla\cdot u f\right) = \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial f}{\partial t} + u\cdot\nabla f + \cancel{f(\nabla\cdot u)} - \cancel{(\nabla\cdot u)f} - \frac{p}{3}\nabla\cdot u\frac{\partial f}{\partial p}$$

$$\frac{\partial f}{\partial t} + \vec{u}\cdot\nabla f - \underbrace{\frac{p}{3}\nabla\cdot\vec{u}}_{\text{only imp @ } x=0}\frac{\partial f}{\partial p} = \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right)$$

Parker – Jokipii 1968–70
Blandford, Ostriker 1978

b.c. $x\to\infty$ $f^{>}\to f_2(p)$ accelerated CR

$x\to-\infty$ $f^{<}\to f_1(p)$ injection spectrum

Steady state $\frac{\partial f}{\partial t}=0$

$x\neq 0$ $\frac{\partial}{\partial x}\underbrace{\left(uf - D\frac{\partial f}{\partial x}\right)}_{\text{net flux}} = 0$ $\qquad uf - D\frac{\partial f}{\partial x} = C(p)$ constant wrt x, but not p

$$\frac{\partial f}{\partial x} - \frac{u}{D}f = -\frac{C(p)}{D}$$

$$\frac{\partial}{\partial x}\left(f e^{-\int_0^x \frac{u}{D}dx}\right) = \frac{-C(p)}{D}e^{-\int\sim} = \frac{C(p)}{u}\frac{\partial}{\partial x}e^{-\int\sim} \qquad f = A e^{\int_0^x \frac{u}{D}dx} + \frac{C(p)}{u}$$

$\underset{x\to\infty}{f^{>}} = f_2(p)$ $\quad e^{S}\to\infty$ so $A=0$ $\qquad f^{<}\underset{x\to-\infty}{\longrightarrow} f_1(p) = \frac{C^{<}(p)}{u}$ $\quad e^{S}\to 0$

At $x=0$

$$f^{<} = A + f_1 = f^{>} = f_2 \implies A = f_2 - f_1$$

$$f^{>} = f_2 \qquad f^{<} = (f_2-f_1)e^{\int_0^x \frac{u}{D}dx} + f_1$$

$$\int_{0^-}^{0^+}dx\left(\frac{\partial}{\partial x}(uf) - f\frac{\partial u}{\partial x} - \frac{p}{3}\underbrace{\frac{\partial u}{\partial x}}_{\text{has } \delta(x)\ \left(=\frac{\partial H}{\partial x}\right)}\frac{\partial f}{\partial p} = \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right)\right)$$

f uniform in 0^+ so $\left.\frac{\partial f}{\partial x}\right|_{0^+}=0$

$$\cancel{(v_2-v_1)f_2} - \cancel{f_2(v_2-v_1)} + \frac{p}{3}(v_1-v_2)\frac{\partial f_2}{\partial p} = -D\left.\frac{\partial f}{\partial x}\right|_{0^-}$$

$$= -D\frac{\partial}{\partial x}\left((f_2-f_1)e^{\int\sim}+f_1\right)\Big|_{0^-}$$

$$= -D(f_2-f_1)\frac{v_1}{D} = -(f_2-f_1)v_1$$

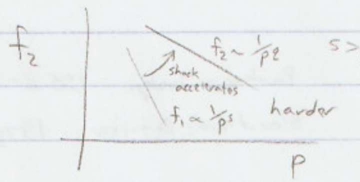
$$\frac{\partial f_2}{\partial p} + \underbrace{\frac{3v_1}{v_1-v_2}}_{q}\frac{1}{p}f_2 = \underbrace{\frac{3v_1}{v_1-v_2}}_{q}\frac{1}{p}f_1(p) \qquad \frac{\partial}{\partial p}\left(f_2 p^q\right) = q p^{q-1} f_1$$

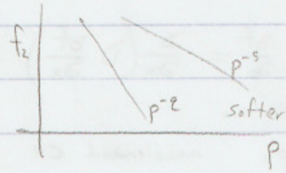
$$f_2(p) = \frac{q}{p^q}\int_0^p (p')^{q-1} f_1(p')\,dp'$$

Answer is indep of D (diffusion)

Ex: $f_1(p) = \delta(p - p_0) \Longrightarrow f_2 = \dfrac{q}{p^q}$

$f_1(p) = \dfrac{1}{p^s} \quad p \geq p_0 \qquad f_2(p) = \dfrac{q}{q-s}\dfrac{1}{p^q}\left[p^{q-s} - p_0^{\,q-s}\right]$





$s < q$

$f_2 = 1/p^s$ shock doesnt change it

If the shock can harden spectrum → it will; otherwise no change

so wont soften it

$$q = \frac{3}{1 - v_2/v_1} = \frac{3}{1 - 1/4} = 4$$

Hypersonic, ideal

limit given by Rankine–Hugoniot relations

(theory OK for $10^9 - 10^{14}$ CR excite Alfven waves → self consistent model)

10/25/2006

Solar / Stellar Wind

Parker Solution

steady state symmetric $\frac{\partial}{\partial\varphi} = \frac{\partial}{\partial\theta} = 0 \qquad \frac{\partial}{\partial t} = 0$ Ideal no viscosity, heat condition

Continuity

$\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(\rho v r^2) = 0 \qquad \rho v r^2 = \text{const}$

Momentum $\quad v\dfrac{dv}{dr} = -\dfrac{1}{\rho}\dfrac{dP}{dr} - \dfrac{GM_*}{r^2}$

Energy $\quad \dfrac{1}{r^2}\dfrac{d}{dr}\Bigg(\underbrace{\rho v r^2}_{\text{const}}\underbrace{\left(\frac{v^2}{2} + \frac{\gamma}{\gamma-1}\frac{P}{\rho} - \frac{GM_*}{r}\right)}_{E = \text{const specific energy}}\Bigg) = 0$

Entropy $\quad \dfrac{S - S_0}{k_B} = \dfrac{1}{\gamma-1}\ln\left(\dfrac{P}{\rho^\gamma}\dfrac{\rho_0^{\,\gamma}}{P_0}\right)$

$$c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_s = \frac{\gamma P}{\rho}$$

$P = \rho\dfrac{c_s^2}{\gamma} \propto \rho^\gamma \qquad \rho^{\gamma-1} \propto c_s^2 \;\rightarrow\; \rho = \beta (c_s^2)^{\frac{1}{\gamma-1}} \qquad P = \dfrac{\beta}{\gamma}(c_s^2)^{\frac{\gamma}{\gamma-1}}$

$$\frac{P}{\rho^\gamma} = \frac{\beta}{\gamma}(c_s^2)^{\frac{\gamma}{\gamma-1}}\frac{1}{\beta^\gamma}\frac{1}{(c_s^2)^{\gamma/\gamma-1}} = \frac{1}{\gamma}\frac{1}{\beta^{\gamma-1}}$$

$\dfrac{S - S_0}{k_B} = \ln\left(\dfrac{\beta_0}{\beta}\right) \qquad \dfrac{\beta_0}{\beta} = \exp\left(\dfrac{S - S_0}{k_B}\right)$

$v = Mc_s \qquad \rho v r^2 = \beta (c_s)^{\frac{2}{\gamma-1}} M c_s r^2 = \text{const} = \beta_0 (c_{s0})^{\frac{2}{\gamma-1}} M_0 c_{s0} r_0^2$

arbitrarily set to $1 = M_0$

$$\frac{2}{\gamma-1} + 1 = \frac{\gamma+1}{\gamma-1}$$

$$c_s = c_{s0}\left(\frac{\rho_0}{\rho}\right)^{\frac{\gamma-1}{\gamma+1}}\left(\frac{r_0^2}{M r^2}\right)^{\frac{\gamma-1}{\gamma+1}}$$

$$\frac{\gamma}{\gamma-1} - 1 = \frac{\gamma - \gamma + 1}{\gamma-1} = \frac{1}{\gamma-1}$$

$$v\frac{dv}{dr} = \frac{1}{2}\frac{d}{dr}\left(M^2 c_s^2\right) = \frac{c_s^2}{2}\frac{dM^2}{dr} + \frac{M^2}{2}\frac{dc_s^2}{dr} = RHS$$

$$\frac{dP}{\rho} = \left(\frac{\cancel{\rho}}{\cancel{\gamma}}\right)\frac{\gamma}{\gamma-1}\left(c_s^2\right)^{\frac{\gamma}{\gamma-1}-1} dc_s^2 \, \frac{1}{\cancel{\rho}\left(c_s^2\right)^{\frac{1}{\gamma-1}}} = \frac{1}{\gamma-1}\, dc_s^2$$

$$RHS = -\frac{1}{\gamma-1}\frac{dc_s^2}{dr} - \frac{GM_*}{r^2} = \frac{c_s^2}{2}\frac{dM^2}{dr} + \frac{M^2}{2}\frac{dc_s^2}{dr}$$

$$\frac{1}{2}\frac{dM^2}{dr} + \left(\frac{M^2}{2} + \frac{1}{\gamma-1}\right)\frac{1}{c_s^2}\frac{dc_s^2}{dr} = -\frac{GM_*}{c_s^2 r^2} \qquad \frac{1}{c_s^2}\frac{dc_s^2}{dr} = \frac{d\ln c_s^2}{dr}$$

$$\frac{d\ln c_s^2}{dr} = -\frac{\gamma-1}{\gamma+1}\left(\frac{1}{M^2}\frac{dM^2}{dr} + \frac{4}{r}\right)$$

$$\frac{1}{2}\frac{dM^2}{dr} - \frac{\gamma-1}{\gamma+1}\left(\frac{M^2}{2} + \frac{1}{\gamma-1}\right)\left(\frac{1}{M^2}\frac{dM^2}{dr} + \frac{4}{r}\right) = -\frac{GM_*}{c_s^2 r^2}$$

$$\left(\frac{\gamma-1}{\gamma+1}\right)\frac{4}{r}\left(\frac{M^2}{2} + \frac{1}{\gamma-1}\right) - \frac{GM_*}{c_s^2 r^2} = \left(\frac{1}{2} - \frac{\gamma-1}{2(\gamma+1)} - \frac{1}{\gamma+1}\frac{1}{M^2}\right)\frac{dM^2}{dr}$$

$$\left(1 - \frac{1}{M^2}\right)\frac{dM^2}{dr} = \frac{4}{r}(\gamma-1)\left(\frac{M^2}{2} + \frac{1}{\gamma-1}\right) - \frac{GM_*(\gamma+1)}{c_s^2 r^2}$$

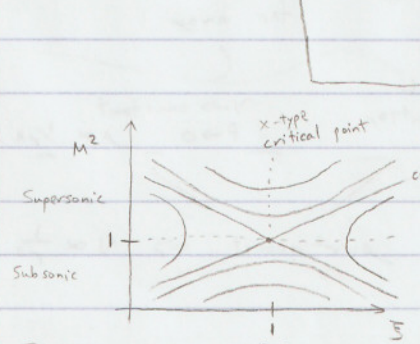
$$E + \frac{GM_*}{r} = \frac{M^2 c_s^2}{2} + \frac{c_s^2}{\gamma-1} \cdots \frac{\gamma P}{\rho} \qquad \frac{1}{c_s^2} = \frac{\frac{M^2}{2} + \frac{1}{\gamma-1}}{E + \frac{GM_*}{r}}$$

$$\left(\frac{M^2-1}{M^2}\right)\frac{dM^2}{dr} = \left(\frac{M^2}{2} + \frac{1}{\gamma-1}\right)\left[\frac{4(\gamma-1)}{r} - \frac{GM_*(\gamma+1)}{r^2\left(E + \frac{GM_*}{r}\right)}\right] = \left(\frac{M^2}{2} + \frac{1}{\gamma-1}\right)\left[\frac{\frac{4(\gamma-1)E}{r} + \frac{4(\gamma-1)GM_*}{r^2} - \frac{GM_*(\gamma+1)}{r^2}}{E + \frac{GM_*}{r}}\right]$$

$$= 4\left(\frac{M^2(\gamma-1)}{2} + 1\right)\left[\frac{E}{r} - \frac{(5-3\gamma)}{4(\gamma-1)}\frac{GM_*}{r^2}\right]\left(\frac{1}{E + \frac{GM_*}{r}}\right)$$

$$\xi = r/r_0$$

$$\boxed{\frac{M^2-1}{M^2}\frac{dM^2}{d\xi} = 4\left(\frac{M^2(\gamma-1)}{2} + 1\right)\left[\frac{\frac{E}{\xi} - \frac{(5-3\gamma)}{4(\gamma-1)}\frac{GM_*}{r_0\xi^2}}{E + \frac{GM_*}{r_0\xi}}\right]}$$



$E>0$

$\xi \to \infty \quad (M^2-1)\frac{dM^2}{dr} > 0$

$\xi \to 0 \quad (\qquad) < 0$

$M^2 = 1 \implies 0 \cdot \frac{dM^2}{d\xi} = \text{const} \implies \frac{dM^2}{d\xi} = \infty$ except @

infinite slope for all but

If $E = \frac{(5-3\gamma)}{4(\gamma-1)}\frac{GM_*}{r_0}$ @ $\xi = 1$

still unknown constant, so OK

$$\frac{M^2-1}{M^2}\,\frac{dM^2}{d\xi} = 4\left(\frac{M^2(\gamma-1)}{2}+1\right)\left(\frac{5-3\gamma}{4(\gamma-1)}\right)\left(\frac{1}{\xi}-\frac{1}{\xi^2}\right)\frac{1}{\frac{5-3\gamma}{4(\gamma-1)}+1/\xi}$$

only solution where M goes from subsonic to supersonic or vice versa

Otherwise solutions remain sub/super sonic

$$\frac{5-3\gamma}{4(\gamma-1)}\frac{GM_*}{r_0}+\frac{GM_*}{r_0} = c_{s0}^2\left(\overset{M^2=1}{\frac{1}{2}}+\frac{1}{\gamma-1}\right) \implies c_{s0}^2 = \frac{1}{2}\frac{GM_*}{r_0}$$

de Laval Nozzle
→ sub/supersonic transition
} similar to above

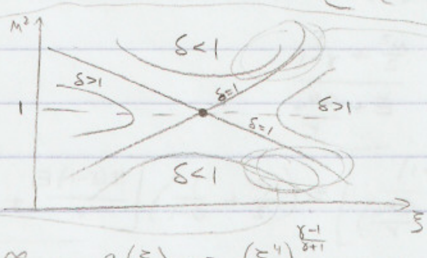
Ideal monatomic gas $\gamma = 5/3$, Nozzle disappears $M=1$ everywhere

Algebraic Equation

$$c_s^2\left(\frac{M^2}{2}+\frac{1}{\gamma-1}\right) = E + \frac{GM_*}{\xi r_0} = \frac{5-3\gamma}{4(\gamma-1)}\frac{GM_*}{r_0}+\frac{GM_*}{\xi r_0}$$

$$\left(\frac{M^2}{2}+\frac{1}{\gamma-1}\right)c_{s0}^2\underbrace{\left(\frac{\rho_0^2}{\beta^2}\right)^{\frac{\gamma-1}{\gamma+1}}}_{\delta}\left(\frac{1}{M^2\xi^4}\right)^{\frac{\gamma-1}{\gamma+1}} = \frac{GM_*}{r_0}\left(\frac{5-3\gamma}{4(\gamma-1)}+\frac{1}{\xi}\right) \qquad \frac{GM_*}{2r_0} = c_{s0}^2$$

$$\boxed{\delta\left(\frac{1}{M^2}\right)^{\frac{\gamma-1}{\gamma+1}}\left(\frac{M^2}{2}+\frac{1}{\gamma-1}\right) = \left(\xi^4\right)^{\frac{\gamma-1}{\gamma+1}}\left[\frac{5-3\gamma}{2(\gamma-1)}+\frac{2}{\xi}\right]} \qquad \delta\, f(M^2) = g(\xi)$$



All but @ critical point

$\xi \to \infty \qquad g(\xi) \to (\xi^4)^{\frac{\gamma-1}{\gamma+1}} \qquad M^2 \to \infty \qquad f(M^2) \propto (M^2)^{1-\frac{\gamma-1}{\gamma+1}} = (M^2)^{\frac{2}{\gamma+1}} = (M^4)^{\frac{1}{\gamma+1}}$

$\therefore\ M \propto \xi^{\gamma-1}$

$$c_s \propto \left(\frac{1}{M\xi^2}\right)^{\frac{\gamma-1}{\gamma+1}} \propto \left(\frac{1}{\xi^{\gamma+1}}\right)^{\frac{\gamma-1}{\gamma+1}} \to \frac{1}{\xi^{\gamma-1}} \qquad M = \frac{v}{c_s} \sim \xi^{\gamma-1} \implies v \to \text{constant}$$

$v\dfrac{dv}{dr} = -\dfrac{1}{\rho}\dfrac{dP}{dr} - \dfrac{GM_*}{r^2}$ $\qquad$ $c_s \to 0$, $P \to 0$, $r \to \infty$ $\Big\}$ $v \to$ const $\qquad$ supersonic winds go to constant speed far away

$\rho \propto 1/r^2$

$M^2 \to 0 \qquad \left(\frac{1}{M^2}\right)^{\frac{\gamma-1}{\gamma+1}} \propto \left(\xi^4\right)^{\frac{\gamma-1}{\gamma+1}} \implies M \propto \frac{1}{\xi^2}$ $\qquad$ wind solution $\qquad$ $v \to$ constant, $P \to 0$, $\rho \propto 1/r^2$ ... since $\rho v r^2 = $ const

$c_s \propto \left(\frac{1}{M\xi^2}\right)^{\frac{\gamma-1}{\gamma+1}} \to$ const.

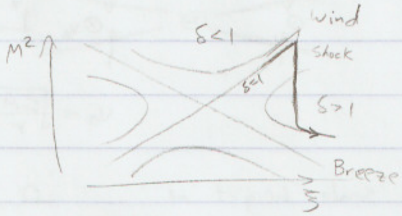
$c_s \to$ const
$P \to$ const $\quad \rho \to$ const $\qquad \rho v r^2 =$ const $\implies v \propto \dfrac{1}{r^2}$

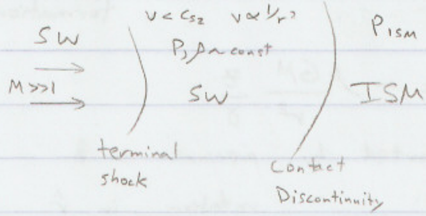
→ Breeze solution $\quad v \propto 1/r^2$

$\rho, c_s, P \to$ const

$\rho v^2 \propto 1/r^2$ $\quad P_{ISM} \neq 0 \Rightarrow$ breeze? no $\rightarrow$ wind, supersonic

1966 Axford





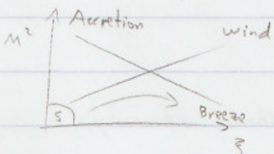
$\rho v^2 = P_{ISM}$ $\qquad$ 1 AU $\quad \rho_1 \left(\frac{1}{r^2}\right) v^2 = P_{SM}$ $\qquad n \sim 5\ cm^{-3}$ $\quad v = 500$ km/s $\quad P_{ISM} \sim 10^{-12}$ (cgs units)

$$r_{shock} = \left(\frac{\rho_1 v^2}{P_{ISM}}\right)^{1/2} = \left(\frac{5(1.6\times10^{-24})(25\times10^{14})}{10^{-12}}\right)^{1/2} = 140\ AU$$

Voyager I passed it

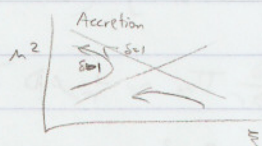
Next Monday 2-3 pm $\qquad$ 10/30/2006

Accretion



$$c_{s_0}^2 = \frac{GM_*}{2r_0} \qquad r_0 = \frac{2GM_* c^2}{4c^2 c_{s_0}^2} = \frac{R_s}{4}\frac{c^2}{c_{s_0}^2}$$

Schwartzchild radius

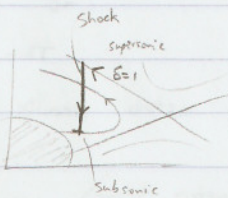


$$v\frac{dv}{dr} = -\cancel{\frac{dP}{\rho\, dr}}^{\,0} - \frac{GM}{r} \qquad v^2 = \frac{2GM}{r}$$

$$\frac{v^2}{c^2} = \frac{R_s}{r} = \frac{1}{r_*} \qquad \frac{v}{c} = \frac{1}{\sqrt{r_*}}$$

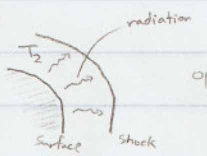
$4\pi r^2 \rho v = \dot{M}$ $\quad$ mass accretion rate

$$\rho = \frac{\dot{M}}{4\pi c R_s^2 r_*^{3/2}}$$



$$\frac{kT_2}{m_{p_{nuc}}} = \frac{3}{16} v_1^2 = \frac{3}{16}\frac{c^2}{r_*}$$

$kT_2 \sim \frac{1}{4} m_p c^2$ $\quad$ very hot



opacity: mostly e^- scattering

$\vec{k}$, $\vec{k}_s$ $\quad \Delta\vec{p}_e = \hbar(\vec{k} - \vec{k}_s) \quad \cdot\hat{r} \qquad \Delta p_{er} = \frac{h\nu}{c}(1 - \cos\theta_s)$ $\quad \theta_s$

$t_s = \frac{1}{n_e \sigma_T c}$ $\qquad dn_\nu = \frac{I_\nu\, d\Omega\, d\nu}{h\nu c}$ $\qquad$ Thomson scattering $\quad \sigma_T \sim \frac{8\pi r_e^2}{3}$

force (radiation)

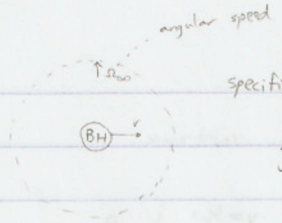
$$\frac{dp}{dt} = (n_e \sigma_T c)\int \frac{\frac{h\nu}{c}(1-\cos\theta) I_\nu\, d\nu\, d\Omega}{h\nu c} = \frac{n_e\sigma_T}{c} S_{rad} \rightarrow \frac{L}{4\pi r^2}$$

radiation flux

$$\rho v\frac{dv}{dr} = -\frac{dP}{dr} - \frac{GM_*\rho}{r^2} + \frac{n_e\sigma_T}{c}\frac{L}{4\pi r^2}$$

Eddington limit: $\boxed{L_{ED} = \frac{4\pi G M_* m_p c}{\sigma_T}}$ $\quad$ when $-\frac{GM_*\rho}{r^2} + \frac{n_e\sigma_T}{c}\frac{L}{4\pi r^2} = 0$ $\quad$ ($\rho = n_e m_p$)

this is max L but it doesn't always limit mass flow



specific angular momentum $j_\infty = R^2 \Omega_\infty$

$j_H = r^2 \Omega = r v_\phi \qquad j_H << j_\infty$

spins faster as gets closer

formation of disk

$$\frac{dP}{dz} = -\rho \frac{GM}{r^2} \frac{z}{r}$$

$v_\phi = \sqrt{\frac{GM}{r}}$

Supported by pressure in $\hat{z}$

" " rotation (centripetal) in $\hat{r}$

scale height $H \qquad \Omega = \sqrt{\frac{GM}{r^3}}$

$$\frac{dP}{dz} \sim \frac{P}{H} = -\rho \frac{\Omega^2 H}{\gamma} \qquad H^2 = \frac{\gamma P}{\rho \Omega^2} = \frac{c_s^2}{\Omega^2}$$

$v_\phi = r\Omega$

$\frac{GM}{r} = r^2 \frac{GM}{r^3}$ ✓

Thin: $\frac{H}{r} = \frac{c_s}{r\Omega} = \frac{c_s}{v_\phi} << 1$ Thin

Keplerian disk $\qquad j = \sqrt{GMr} \propto \sqrt{r} \qquad$ gas must loose j to fall in

$\uparrow \; \dot{r} \; \downarrow$ $\Omega(r)$ (frame) $\qquad$ shear flow $\qquad v_2 \quad v_1 > v_2$

allows transfer of j (with shear viscosity)

$$\rho \frac{\partial v_\phi}{\partial t} \sim \rho \frac{v_\phi}{\tau_D} \sim (\nabla \cdot \overset{\approx}{\Pi}) \cdot \hat{\phi} \sim \frac{\partial}{\partial r} \Pi_{r\phi} \sim \frac{1}{r} \rho D \frac{\partial v_\phi}{\partial r} \sim \frac{\rho D v_\phi}{r^2}$$

$\tau_D \sim \frac{r^2}{D}$ diffusion timescale $\qquad \tau_D \sim \frac{\Omega r^2}{\underbrace{c_s \lambda_{mfp}}_{\sim D}} = \left(\frac{r}{H}\right)^2 \underbrace{\frac{H}{\lambda_{mfp}}}_{>>1 \text{ (astro)}} \longrightarrow \infty$

Thin $\left(\frac{r}{H}\right)^2 >> 1$

~~Collisional~~ Ordinary viscosity negligible in astro scales since τ_D is too long

$\hat{r} \cdot \overset{\approx}{\Pi} \cdot \hat{\phi} = \Pi_{r\phi} = \Pi_{\phi r}$ (symmetric tensor)

$$\overset{\approx}{\Pi} = \mu\left(\nabla\vec{v} + (\nabla\vec{v})^T\right) - \tfrac{2}{3}\mu \, \nabla\cdot\vec{v} \, \overset{\approx}{I}$$

Thin Disk

$\frac{\partial}{\partial t} = 0$ Steady state $\qquad \frac{\partial}{\partial \phi} = 0$ axisymm $\qquad \frac{\partial}{\partial z} >> \frac{\partial}{\partial r}$ larger in r than z (thin)

$v_z = 0$

$\vec{v} = v_r \hat{r} + v_\phi \hat{\phi} \qquad v_\phi >> c_s$ (thin) $\qquad c_s >> v_r$

$v_\phi >> c_s >> v_r$

Now to find $\Pi_{r\phi}$ $\qquad \hat{r} \cdot \overset{\approx}{I} \cdot \hat{\phi} = 0$

$\nabla\vec{v}$ Dyad $\qquad$ unit tensor

$$\hat{r}\cdot\overset{\approx}{\Pi}\cdot\hat{\phi} = \mu\left(\hat{r}\cdot\nabla(v_\phi\hat{\phi})\cdot\hat{\phi} + \underbrace{\hat{\phi}\cdot\nabla(v_\phi\hat{\phi})\cdot\hat{r}}_{(\nabla v)^T}\right) \quad v_\phi \gg v_r$$

$$= \mu\left(\frac{\partial}{\partial r}(v_\phi\hat{\phi})\cdot\hat{\phi} + \frac{1}{r}\frac{\partial}{\partial\phi}(v_\phi\hat{\phi})\cdot\hat{r}\right) = \mu\left(\frac{\partial v_\phi}{\partial r} - \underbrace{\frac{v_\phi}{r}}_{\Omega}\right) = \Pi_{r\phi}$$

$$\boxed{\frac{\partial\hat{\phi}}{\partial\phi} = -\hat{r} \qquad \frac{\partial\hat{r}}{\partial\phi} = \hat{\phi}}$$

but $\frac{\partial v_\phi}{\partial\phi} = 0$

$$\Pi_{r\phi} = -\tfrac{3}{2}\mu\Omega$$

$v_\phi = \sqrt{\frac{GM}{r}}$ $\quad \frac{\partial v_\phi}{\partial r} = \underbrace{\frac{(GM)^{1/2}}{r^{3/2}}}_{\Omega}\cdot -\frac{1}{2}$

$\frac{\partial v_\phi}{\partial r} = -\frac{1}{2}\Omega$

Shakura – Sunyaev Equations 1973 (Thin Disk Accretion)

continuity: $\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{\partial}{\partial z}\cancel{(\rho v_z)} = 0$ $\quad$ $v_z = 0$

v_r ~ indep of z

$\rho v_r r = $ constant (mass flux)

$$\underbrace{\int_{-H}^{H} dz\,\rho}_{\Sigma}\, v_r r \qquad \boxed{\underbrace{-2\pi r}_{\text{all disk}}\,\Sigma\, v_r = \dot{M}}$$

mass accretion in disk

<0 accretion

surface density

Momentum eq:
$$\rho\vec{v}\cdot\nabla\vec{v} = -\nabla P - \frac{GM\rho}{R^2}\hat{R} + \nabla\cdot\overset{\approx}{\Pi}$$
spherical R

$$\vec{v}\cdot\nabla\vec{v} = \left(v_r\frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial\phi}\right)\left(v_r\hat{r} + v_\phi\hat{\phi}\right) = \left(v_r\frac{\partial v_r}{\partial r} - \underbrace{\frac{v_\phi^2}{r}}_{\text{centripetal}}\right)\hat{r} + \left(v_r\frac{\partial v_\phi}{\partial r} + \underbrace{v_r\frac{v_\phi}{r}}_{\text{coriolis}}\right)\hat{\phi}$$

$\hat{r}$: $\quad \rho v_r\underbrace{\cancel{\frac{\partial v_r}{\partial r}}}_{\text{small}} - \rho\frac{v_\phi^2}{r} = -\underbrace{\cancel{\frac{\partial P}{\partial r}}}_{c_s \ll v_\phi} - \frac{GM}{r^2}\rho \Rightarrow v_\phi^2 = \frac{GM}{r}$ keplerian flow

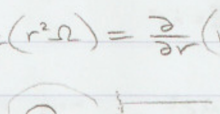
$\hat{\phi}$: $\quad \rho v_r\left(\frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\right) = \frac{\rho v_r}{r}\frac{\partial}{\partial r}\underbrace{(rv_\phi)}_{\substack{j\\\text{specific}\\\text{angular momentum}}} = (\nabla\cdot\overset{\approx}{\Pi})\cdot\hat{\phi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\Pi_{\phi r})$

$$(\nabla\cdot\overset{\approx}{\Pi})\cdot\hat{\phi} = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r\Pi_{r\phi}\hat{\phi})\cdot\hat{\phi}}_{\frac{1}{r}\frac{\partial}{\partial r}(r\Pi_{r\phi})} + \underbrace{\frac{1}{r}\frac{\partial}{\partial\phi}(\Pi_{\phi r}\hat{r})\cdot\hat{\phi}}_{\frac{\Pi_{\phi r}}{r}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\Pi_{\phi r})$$

$\Pi_{\phi r} = \Pi_{r\phi}$

$$\overbrace{(\rho v_r r)}^{\text{constant}}\frac{\partial}{\partial r}(r^2\Omega) = \frac{\partial}{\partial r}(r^2\Pi_{\phi r}) \qquad \text{now } \int_{r_0}^{r}$$

boundary conditions



r_0 last stable orbit

$\Pi_{r\phi}\big|_{r_0} = 0$ $\quad$ nothing inside r_0 that exerts viscous force

$$\boxed{(\rho v_r r)(\underbrace{r^2\Omega}_{\sqrt{GMr}} - \underbrace{r_0^2\Omega_0}_{\sqrt{GMr_0}}) = r^2\Pi_{\phi r}}$$

$$(\rho v_r r)\,r^2\Omega\underbrace{\left(1 - \sqrt{\frac{r_0}{r}}\right)}_{\equiv S} = r^2\Pi_{\phi r} \qquad * -2\pi \text{ and } \int_{-H}^{H}dz \Rightarrow \boxed{\dot{M}\Omega S = -2\pi\int_{-H}^{H}\Pi_{r\phi}\,dz}$$

$\hat{z}$: $\quad 0 = -\frac{\partial P}{\partial z} - \frac{\rho G M z}{r^3}$ $\quad \boxed{\frac{\partial P}{\partial z} = -\rho \Omega^2 z}$ Hydrostatic Equilibrium

$$\boxed{P = \underbrace{\frac{\rho k T}{\mu m_p}}_{\text{gas}} + \underbrace{\frac{1}{3} a T^4}_{\text{radiation}}}$$

$r^2\Omega^2 = \frac{GM}{r}$ $\quad c_s \ll v_\phi$ $\quad$ radiation (heat flow)

Energy : $\quad \frac{1}{r}\frac{\partial}{\partial r}\left[\rho v_r r\left(\frac{v_\phi^2}{2} + \cancel{\frac{\gamma}{\gamma-1}\frac{P}{\rho}} - \frac{GM}{r}\right) - \underbrace{r v_\phi \Pi_{r\phi}}_{-r^2\Omega\Pi_{r\phi}}\right] + \frac{\partial q_z}{\partial z} = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\rho v_r r\left(\frac{-GM}{2r}\right) - \Omega(\rho v_r r)(r^2\Omega - r_0^2\Omega_0)\right] = -\frac{\partial q_z}{\partial z}$$

$$\frac{(\rho v_r r)}{r}\frac{\partial}{\partial r}\left[-\frac{3}{2}\frac{GM}{r} + \sqrt{\frac{GM}{r^3}}\, r_0^2\Omega_0\right] = \frac{3}{2}(\rho v_r r)\left(\underbrace{\frac{GM}{r^3}}_{\Omega^2} - \frac{\sqrt{GM}}{r^3 r^{1/2}}\sqrt{GM r_0}\right)$$

$$= \frac{3}{2}(\rho v_r r)\Omega^2\underbrace{\left(1 - \sqrt{\frac{r_0}{r}}\right)}_{\equiv S} = \frac{3}{2}(\rho v_r r)\Omega^2 S$$

$$\frac{3}{2}(\rho v_r r)\Omega^2 S = -\frac{\partial q_z}{\partial z} \qquad \int_0^H dz$$

$$-\frac{3}{2}\left(\int_0^H \rho\, dz\right) v_r r\,\Omega^2 S = q_z(z=H) \qquad q_z(z=0) = 0$$

$= \frac{1}{2}\int_{-H}^{H}\rho\, dz = \frac{1}{2}\Sigma$

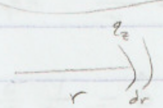
$$\boxed{q_z = \frac{3}{8\pi}\dot{M}\Omega^2 S}$$

radiative flux @ surface

$-\frac{3}{4}\left(v_r r \Sigma = -\frac{\dot{M}}{2\pi}\right)\Omega^2 S$

11/1/2006

$H = \frac{c_s}{\Omega}$



Annulus $\quad dL = 2\pi r\, dr\; q_z\,(2)$ $\quad$ 2 sides: up/down

$x = r/r_0$

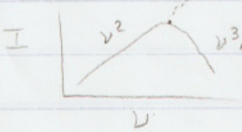
$$dL = 4\pi r\, dr\; \frac{3}{8\pi}\dot{M}\frac{GM}{r^3}\left(1 - \sqrt{\frac{r_0}{r}}\right) = \frac{3}{2}\dot{M}\frac{GM}{r_0}\frac{dx}{x^2}\left(1 - \frac{1}{\sqrt{x}}\right)$$

$$L = \int_1^\infty dL \qquad L = \frac{3}{2}\dot{M}\frac{GM}{r_0}\left(-\frac{1}{x} + \frac{2}{3x^{3/2}}\right)\Big|_1^\infty = \frac{1}{2}\dot{M}\frac{GM}{r_0}$$

$$\boxed{L = \frac{1}{4}\frac{2GM}{r_0 c^2}c^2\dot{M} = \frac{1}{4}\frac{R_s}{r_0}\dot{M}c^2}$$

Note: $E = mc^2 \quad \frac{dE}{dt} = \dot{M}c^2$

$r_0 = 3R_s \quad L \sim \frac{1}{12}\dot{M}c^2 \qquad r_0 = \frac{1}{2}R_s \quad L \sim \frac{1}{2}\dot{M}c^2$ $\qquad$ more efficient than nuclear burning

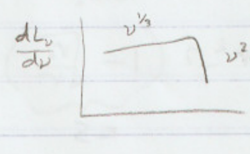


$h\nu_{max} = 2.7kT$

$$q \propto \Omega^2 \propto \frac{1}{r^3} = \sigma_{SB} T_s^4 \Rightarrow T_s \propto 1/r^{3/4}$$

Annulus $\quad \nu \sim T \sim 1/r^{3/4} \qquad r \propto \frac{1}{\nu^{4/3}} \qquad dr \propto \frac{d\nu}{\nu^{7/3}}$

$$dL_\nu \propto r\frac{dr}{r^3} = \frac{dr}{r^2} \propto \frac{d\nu}{\nu^{7/3}}\nu^{8/3} = \nu^{1/3}d\nu$$



no more material inside r_0 so cutoff

AGN – Quasars $\quad T_0 \sim 2\times10^5 K$

radiation pressure

Radiation flux $\quad \boxed{S_z = -\frac{4}{3}\frac{caT^3}{\kappa\rho}\frac{\partial T}{\partial z} = -\frac{c}{\kappa\rho}\frac{\partial P_r}{\partial z}}$

Thomson opacity

$$\kappa_T = \frac{\sigma_T}{m_p} = 0.4 \qquad \text{Free-free opacity} \quad \kappa_{ff} = 0.1\, n\, T^{-7/2}$$

number density

Structure of Disk : Inner region $\kappa = \kappa_T \quad P = P_r$

Middle region $\kappa = \kappa_T \quad P = P_g$ (gas pressure)

Outer region $\kappa = \kappa_{ff} \quad P = P_g$

Inner region

$$S_z = -\frac{c}{\kappa\rho}\frac{\partial P_r}{\partial z} = -\frac{c}{\kappa}\Omega^2 z \qquad \text{From } \frac{\partial P}{\partial z} = -\rho\,\Omega^2 z \quad \text{Hydrostatic Equil.}$$

$$= -2\pi\int_0^z \rho\, dz\; V_r\, \Omega^2 S\, \frac{3}{8\pi} \qquad \rho = \text{constant wrt } z \text{ in inner region so that } \int_0^z \rho\, dz = \rho z$$

S_z has to be q_z $\quad \frac{\partial q_z}{\partial z} = -\frac{3}{2}\rho V_r r \Omega^2 S$

What about $\dot{M}\Omega S = -2\pi\int_{-H}^{H} \Pi_{r\phi}\, dz$?

$$\Pi_{r\phi} = -\frac{3}{2}\rho \overset{\mu}{D}\Omega = -\frac{3}{2}\rho\, c_s \lambda_{mfp}\Omega \qquad \Pi_{r\phi} \text{ for molecular collisions too small}$$

Convection $\quad \bar{H} \sim \lambda_{mfp} \qquad \Pi_{r\phi} = -\rho c_s H\Omega = -\rho c_s^2 = -P$ pressure

$\therefore \Pi_{r\phi} = -\alpha P = -\alpha\rho c_s^2 \qquad$ α–disks

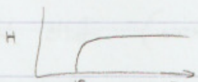
constant

using $\lambda_{mfp} \sim H$ allows angular momentum, material, etc

$\therefore \boxed{\dot{M}\Omega S = 2\pi\alpha\Sigma c_s^2} = 2\pi\alpha P H$ to move by viscosity and create disk

$$\frac{3}{8\pi}\dot{M}\Omega^2 S = \frac{c}{\kappa_T}\Omega^2 H \qquad q_z = S_z|_{z=H} \qquad \text{Inner Region}$$

$$H = \left(\frac{3\kappa_T}{8\pi c}\right)\dot{M} S$$



$$c_s^2 = H^2\Omega^2 = \left(\frac{3\kappa_T}{8\pi c}\right)^2 \dot{M}^2\Omega^2 S^2$$

$$P_r = \frac{1}{3}aT_c^4 = \frac{\dot{M}\Omega S}{2\pi\alpha H} = \frac{1}{2\pi\alpha}\frac{8\pi c}{3\kappa_T}\Omega \propto \frac{1}{r^{3/2}} \qquad T_c \propto \frac{1}{r^{3/8}} \qquad T_s \propto \frac{1}{r^{3/4}}$$

$$\Sigma = \frac{\dot{M}\Omega S}{2\pi\alpha c_s^2} = \frac{1}{2\pi\alpha}\left(\frac{8\pi c}{3\kappa_T}\right)^2\frac{1}{\dot{M}\Omega S} \propto r^{3/2} \qquad \Sigma = 2\rho H \quad \therefore \rho \propto r^{3/2}$$

$$v_r = \frac{-\dot{M}}{2\pi \Sigma r} \propto \frac{1}{r^{5/2}}$$ density goes down as you go in since material speeds up

Optical depth $\tau_T = \int_0^H \frac{dz}{\lambda_T} = \int_0^H \rho \kappa_T \, dz = \frac{1}{2}\kappa_T \Sigma \geq 1$ optically Thick

Thomson scattering conserves photon number, $E \sim$ constant

$\kappa_T \gg \kappa_{ff}$

$\ell = N\lambda_T$

$\Delta z = \sqrt{N}\,\lambda_T$

$\rho \kappa_{ff} = 1/\lambda_a$ absorption

$N = \frac{\lambda_a}{\lambda_T}$ # of steps before absorbed

True Absorption $\lambda_* = \sqrt{\frac{\lambda_a}{\lambda_T}}\,\lambda_T = \sqrt{\lambda_a \lambda_T}$ for Blackbody $\tau > 1$

$$\tau_* = \int_0^H \frac{dz}{\sqrt{\lambda_a\lambda_T}} = \int_0^H \rho\sqrt{\kappa_T \kappa_{ff}}\, dz$$ True optical depth

lots of absorption and reemission optically thick

Quasars $\tau_* < 1$

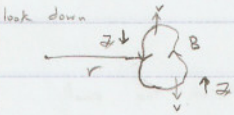
Inner region not optically thick (τ_*)

Middle region is optically thick

If $\tau \sim 1 \rightarrow$ Brehmstralung spectrum (?)

Big hole in assumption: $\Pi_{r\phi} = -\alpha P$

look down



Maxwell stress tensor $\frac{B_r B_\phi}{4\pi}$ analogous to $\Pi_{r\phi}$

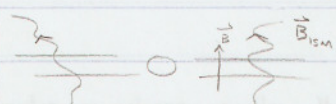
$2x$: stress causes change in vel

slow stuff

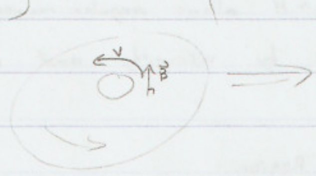
$z \downarrow \uparrow \longrightarrow j$ moves out

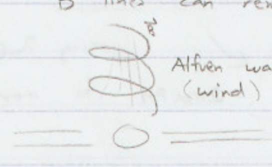
$\uparrow$ angular momentum

Fast stuff $\downarrow \uparrow z \longleftarrow j$ moves in



$\vec{B}$ lines can remain connected to ISM





Alfven wave (wind)

circularly polarized (carries angular momentum)

Wind takes j and mass with it

Wind can turn into jets

Problem:

$\vec{B}$ lines (stuck to plasma) falls in $\frac{B^2}{8\pi}$ increases

QSO $B \sim 10^4 G$ very high pressures

or $\uparrow\downarrow$ magnetic reconnection

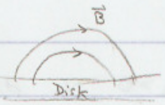
BH:

$E_{\parallel} R_s = 10^{20}$ volts

Black hole threaded with $\vec{B}$ lines (not rotating)

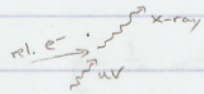
relativistic wind of e^-, e^+

(rotating BH, $\vec{B}$ not inside – stuck in ergosphere)



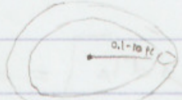
Corona $\quad T \sim 10^9 - 10^{12}\ K$

$m_e c^2 \quad m_p c^2$ so might have e^-, e^+ in corona



Outer
$T \sim 10^4 K$ $\quad M_{disk} > M_{BH}$ disk becomes self gravitating $\Rightarrow$ unstable, forms clumps

might form stars



dust torus (?)

11/6/2006

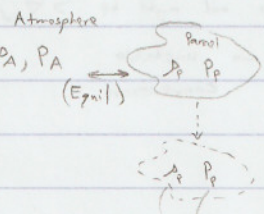
Instabilities

Stars:

$$\nabla P = \frac{dP}{dr}\hat{r} = -\frac{GM(r)\rho}{r^2}\hat{r} \quad \cdot \text{ Hydrostatic Equil.}$$

Radiative transfer $\quad q_r = -\frac{4}{3}\frac{caT^3}{\kappa\rho}\frac{\partial T}{\partial r} \qquad 4\pi r^2 q_r = L$

Is star stable?



$\downarrow \vec{g}$

Adiabatic (fast change)
no heat flow

P_p equilibrium after soundwave, not necessarily true for ρ_p

Will parcel sink or rise back (stable)?

Thermal Buoyancy Instability

$\rho(P, S)$

$$d\rho_p = \frac{\partial\rho}{\partial P}\Big|_S dP_p + \frac{\partial\rho}{\partial S}\Big|_P \underbrace{dS_p}_{0 \text{ Adiabatic}}$$

$$d\rho_A = \frac{\partial\rho}{\partial P}\Big|_S dP_A + \frac{\partial\rho}{\partial S}\Big|_P dS_A \qquad dP_p = dP_A \quad \text{Sound wave} \to \text{equilibrate}$$

$$d\rho_A - d\rho_p = \frac{\partial\rho}{\partial S}\Big|_P dS_A$$

$$dE = TdS - PdV$$

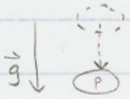
$$H = E + PV \qquad dH = TdS + VdP \qquad V = \frac{m}{\rho} \quad \frac{1}{\rho} = V \text{ specific volume}$$

$$dH = \frac{\partial H}{\partial S}\Big|_P dS + \frac{\partial H}{\partial P}\Big|_S dP = TdS + \frac{1}{\rho}dP$$

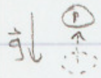
$$\frac{\partial^2 H}{\partial P \partial S} = \frac{\partial^2 H}{\partial S \partial P}$$

$$\frac{\partial T}{\partial P}\Big|_S = \frac{\partial}{\partial S}\left(\frac{1}{\rho}\right)\Big|_P = -\frac{1}{\rho^2}\frac{\partial\rho}{\partial S}\Big|_P \qquad \therefore \boxed{d\rho_A - d\rho_p = -\rho^2 \underbrace{\frac{\partial T}{\partial P}\Big|_S}_{>0} dS_A}$$

$$d\rho_A - d\rho_p = -\rho^2 \frac{\partial T}{\partial P}\bigg|_s dS_A$$



$d\rho_A - d\rho_p < 0$ parcel is denser $\Rightarrow$ sink $\Rightarrow$ $dS_A > 0$ in direction of $\vec{g}$



If $d\rho_A - d\rho_p > 0 \Rightarrow$ rise $dS_A < 0$ in direction of $-\vec{g}$

Both of these are unstable

$$P = \frac{\rho}{\mu m_p} kT \qquad dP_p = \frac{k}{\mu m_p}\left[T\,d\rho_p + \rho\, dT_p\right] = \frac{k}{\mu m_p}\left[T\,d\rho_A + \rho\, dT_A\right] = dP_A$$

$$\underline{}\quad T$$

$$d\rho_A - d\rho_p = \rho\left(-\frac{dT_A}{T} + \frac{dT_p}{T}\right) = -\rho^2 \frac{\partial T}{\partial P}\bigg|_s dS_A$$

$$dr$$

$\frac{dT}{dr} < 0$ temp decreases outward

$$-\frac{dT_A}{T\,dr} = \nabla_{rad} \qquad \frac{1}{T}\frac{dT_p}{dr} = -\nabla_{AD}$$

$$\boxed{\nabla_{rad} - \nabla_{Ad} = -\rho \underbrace{\frac{\partial T}{\partial P}\bigg|_s}_{>0} \frac{dS_A}{dr}}$$

$$\frac{dS_A}{dr} < 0 \quad \text{unstable} \quad \nabla_{rad} - \nabla_{ad} > 0$$

If κ goes large ∇_{rad} large and might be $> \nabla_{ad}$

Temp gradiant too steep $\rightarrow$ unstable convection

Accretion Disk

$$S_z = -\frac{c}{\kappa\rho}\frac{\partial P_r}{\partial z} = \frac{c}{\kappa}\Omega^2 z \qquad \rho = \text{const. w/ } z$$

$$dE = T\,dS - P\,dV$$

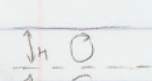
$$d(aT^4V) = T\,dS - \tfrac{1}{3}aT^4\,dV = V\,4aT^3\,dT + aT^4\,dV$$

$$\underline{4aT^3V\,dT + \tfrac{4}{3}aT^4\,dV = T\,dS} \qquad dS = d\left(\tfrac{4}{3}aT^3V\right)$$

$$T$$

radiation: $\boxed{S_{rad} = \frac{4}{3}\frac{aT^3}{\rho}}$

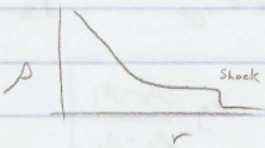
$$S_z = -\frac{c}{\kappa\rho}\underbrace{\left(\frac{4}{3}aT^3\frac{\partial T}{\partial z}\right)}_{\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{3}aT^4\right)} \qquad \frac{dS_{rad}}{dz} \propto \frac{\partial T}{\partial z} < 0 \quad \underline{\text{unstable}} \quad \underline{\text{bad}}$$



Rayleigh – Taylor Instability

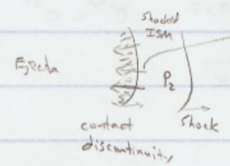
$\vec{g} \downarrow$ $\uparrow^{z} \rightarrow_{x}$ $\quad \rho_2 > \rho_1$ / ρ_1 $\qquad$ unstable (top would sink)

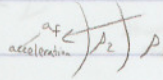
Supernova shockwave $\qquad t < 10^2$ yrs $\qquad v \sim$ const



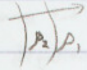
After $\sim 10^2$ yrs $\qquad P_{ejecta} < P_2$

$\quad \hookrightarrow$ compression (reverse shock)





comoving frame $\qquad m\vec{a}' = \vec{F} - m\vec{a}_f$

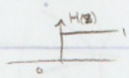


$\rightarrow \vec{g}_{et}$ effective gravity

$\therefore$ Shock breaks up

$$e^{ikx - i\omega t} \begin{cases} \delta\rho \\ \delta\vec{v} \end{cases}$$

$$\rho_0 = \rho_2 H(z) + \rho_1 H(-z)$$



$$\frac{d\rho_0}{dz} = (\rho_2 - \rho_1)\,\delta(z)$$

Slow (low-freq) $\qquad$ time long compared to sound wave time scale

$$\nabla \cdot \delta\vec{v} = 0 \qquad \text{incompressible}$$

$$\rho = \rho_0 + \delta\rho \qquad \vec{v} = \delta\vec{v} \qquad P = P_0 + \delta P$$

Continuity:

$$-i\omega\,\delta\rho + \rho_0 \overset{0}{\cancel{\nabla\cdot\delta\vec{v}}} + \delta v \cdot \nabla\rho_0 = 0 \qquad \delta\rho = -i\,\frac{\delta v_z}{\omega}\,\frac{\partial\rho_0}{\partial z}$$

$\nabla\cdot(\rho v)$

Momentum:

not zero, can have gradients in pressure

$$-i\omega\rho_0\,\delta\vec{v} = -\nabla\delta P + \vec{g}\,\delta\rho \qquad \nabla\cdot\delta\vec{v} = ik\,\delta v_x + \frac{d\,\delta v_z}{dz} = 0$$

$\quad \hookleftarrow$ effective $\vec{g}$

$$\delta v_x = -\frac{1}{ik}\frac{d\,\delta v_z}{dz}$$

$$\hat{x}: \quad -i\omega\rho_0\,\delta v_x = -ik\,\delta P \qquad \delta P = \frac{\omega}{k}\rho_0\,\delta v_x = -\frac{\omega}{ik^2}\rho_0 \frac{d\,\delta v_z}{dz}$$

$$\hat{z}: \quad -i\omega\rho_0\,\delta v_z = -\frac{d\,\delta P}{dz} - g\,\delta\rho$$

$$-i\omega\rho_0\,\delta v_z = -\frac{d}{dz}\left(-\frac{\omega\rho_0}{ik^2}\frac{d}{dz}(\delta v_z)\right) - g\left(-i\,\frac{\delta v_z}{\omega}\frac{\partial\rho_0}{\partial z}\right) \qquad \left(*\,\frac{ik^2}{\omega}\right)$$

$$\frac{d}{dz}\left(\rho_0 \frac{d}{dz}(\delta v_z)\right) - k^2\rho_0\,\delta v_z = \frac{k^2 g}{\omega^2}\frac{\partial\rho_0}{\partial z}\delta v_z = \frac{k^2}{\omega^2} g\,(\rho_2 - \rho_1)\,\delta(z)\,\delta v_z$$

$$\frac{d^2\delta v_z}{dz^2} - k^2\delta v_z = 0 \qquad (z \neq 0) \qquad \delta v_z^{>} = \tilde{\delta v}_z\, e^{-kz} \qquad z > 0$$

$\rho_0 =$ const above or below

$$\delta v_z^{<} = \tilde{\delta v}_z\, e^{kz} \qquad z < 0$$

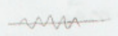
$$\int_{0^-}^{0^+} \frac{d}{dz}\left(\rho_0 \frac{d\,\delta v_z}{dz}\right) dz - \cancel{\nu}^{\,0} \quad \text{After Int.} \quad = \frac{k^2 g}{\omega} \int_{0^-}^{0^+} dz\,(\rho_2-\rho_1)\,\delta v_z\,\delta(z) = \frac{k^2 g}{\omega^2}(\rho_2-\rho_1)\,\tilde{\delta v_z}$$

$$\rho_0 \frac{d\,\delta v_z}{dz}\bigg|_{0^+} - \rho_0 \frac{d\,\delta v_z}{dz}\bigg|_{0^-} = -k\rho_2\,\delta\tilde{v}_z - k\rho_1\,\delta\tilde{v}_z = -k\,\delta\tilde{v}_z\,(\rho_2+\rho_1)$$

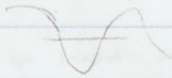
$$\boxed{\omega^2 = -kg\,\frac{\rho_2-\rho_1}{\rho_2+\rho_1}} \qquad \rho_2-\rho_1>0 \qquad \omega^2<0 \implies \text{unstable}$$

$\omega = i\gamma \qquad e^{i\omega t} = e^{\gamma t}$

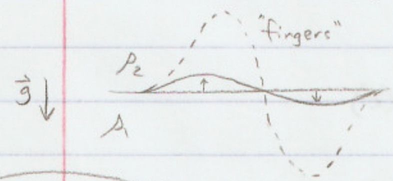
$$\underbrace{\omega = i\gamma}_{\text{growth rate}} \qquad \gamma = \sqrt{kg\,\frac{\rho_2-\rho_1}{\rho_2+\rho_1}} \propto \sqrt{k}$$

 short wavelength

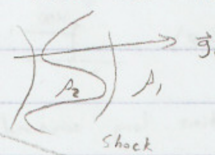
not dangerous, but grow fast

 long wavelength → dangerous → major changes in equil

grow slowest

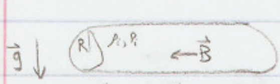


modes that matter are these that can reach out (longer wavelength)



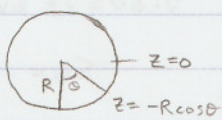
11/8/2006

Isolated $\vec{B}$ field in plasma



$P_0, \rho_0 \quad$ no $\vec{B}_0$

$$P_0 = P_i + \frac{B^2}{8\pi} \qquad \text{Pressure balance after magnetosonic wave}$$



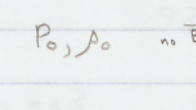
$$P_0(z) = P(0) - \rho_0 g z = P(0) + \rho_0 g R\cos\theta$$

$$\int dF_z^{ext} = \int_0^{2\pi} (P(0) + \rho_0 g R\cos\theta)\cos\theta\,R\,d\theta\,l \qquad F_z^{ext} = \pi R^2 l \rho_0 g \quad \text{net upward force from outside pressure}$$

$$F_z^g = -\pi R^2 l \rho_i g \quad (\text{gravity}) \qquad F_z^{net} = \pi R^2 l g(\rho_0-\rho_i) \quad \text{bouyancy force}$$

$$\rho_0 > \rho_i \qquad \underset{(\text{thin tube})}{T_0 \sim T_i} \qquad P_0 > P_i \qquad \frac{\rho_0}{\mu m_p}kT > \frac{\rho_i}{\mu m_p}kT$$

↳ Tube rises



experience drag force

$$\underset{\text{coefficient}}{C_D}\,\underbrace{\rho_0 V^2}_{\text{ram pressure}}\,\underbrace{(\pi R l)}_{\text{area}} = F_{drag}$$

$F_{drag} = F^{net}$ $\quad C_D \rho_o V^2 \pi R l = \pi R^2 l g (\rho_o - \rho_i)$

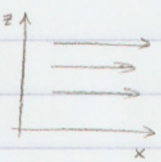
$$V^2 = \frac{gR}{C_D}\frac{\rho_o-\rho_i}{\rho_o} = \frac{gR}{C_D}\frac{\mu m_p}{KT}\frac{\overbrace{P_o - P_i}^{B^2/8\pi}}{\rho_o} = \frac{\mu m_p g R}{KT 2 C_D} C_A^2 = \frac{1}{2C_D}\left(\frac{R}{H}\right) C_A^2 = V^2$$

$$\frac{B^2}{4\pi\rho_o} = C_A^2$$

Put $\vec{B}$ fields in gravitational bound atmosphere $\rightarrow$ they rise

Parker Instability

Equil. $\vec{g}\downarrow$



$B_x(z) = B$

Isothermal

$$\frac{d}{dz}\Big(P_o + \underbrace{\frac{B_o^2}{8\pi}}_{\substack{\alpha P_o \\ \text{constant}}}\Big) = -\rho_o g$$

$(1+\alpha)\frac{KT}{\mu m_p}\frac{d\rho_o}{dz} = -\rho_o g \qquad \frac{1}{\rho_o}\frac{d\rho_o}{dz} = -\frac{1}{H} = \frac{-\mu m_p g}{KT(1+\alpha)} \qquad \rho_o = \tilde{\rho}_o e^{-z/H}$

$B_o = \hat{B}_o e^{-\frac{z}{2H}} \qquad g = \frac{KT}{\mu m_p}\frac{(1+\alpha)}{H} = \frac{1}{H}\left(C_S^2 + \frac{C_A^2}{2}\right) \qquad \alpha = \frac{B_o^2}{8\pi}\frac{1}{P_o} = \frac{\mu m_p}{KT}\frac{1}{2}\frac{B_o^2}{4\pi\rho_o} = \underbrace{\frac{\mu m_p}{KT}}_{C_s^2}\frac{1}{2}C_A^2$

$K_y = 0 \qquad \delta V_y = \delta B_y = 0 \qquad$ No Alfven wave

$f(z) e^{ikx - i\omega t}$

Continuity $\quad -i\omega\,\delta\rho + \rho_o \nabla\cdot\delta\vec{V} + \underbrace{\delta\vec{V}\cdot\nabla\rho_o}_{\delta V_z \frac{\partial\rho_o}{\partial z}} = 0$

Choose $\delta\rho \propto e^{-z/H} \qquad \delta\vec{V} \neq f(z) \qquad$ so $e^{-z/H}$ is common factor

$\tilde{\rho}_o \rightarrow \rho_o \qquad -i\omega\,\delta\rho + \rho_o ik\,\delta V_x - \delta V_z \frac{1}{H}\rho_o = 0 \qquad$ local equation in z

$$\boxed{\frac{\delta\rho}{\rho_o} = \frac{k}{\omega}\delta V_x + i\frac{\delta V_z}{H\omega}}$$

Ohm's law $\quad \frac{\partial B}{\partial t} = -\vec{V}\cdot\nabla\vec{B} + \vec{B}\cdot\nabla\vec{V} - \vec{B}\,\nabla\cdot\vec{V} \qquad -i\omega\,\underbrace{\delta\vec{B}}_{\text{choose } \propto e^{-\frac{z}{2H}}} = -\delta\vec{V}\cdot\nabla\vec{B}_o + \underbrace{\vec{B}_o\cdot\nabla\delta\vec{V}}_{\propto e^{-z/2H}} - \vec{B}_o\nabla\cdot\delta\vec{V}$

$$-i\omega\,\delta B_x = -\delta V_z\frac{\partial B_o}{\partial z} + \cancel{ik B_o \delta V_x} - \cancel{B_o ik\,\delta V_x}$$
$$= \frac{\delta V_z B_o}{2H}$$

$$\boxed{\delta B_x = i\frac{\delta V_z B_o}{2H\omega}}$$

$-i\omega\,\delta B_z = 0 + ik B_o \delta V_z - 0$

$$\boxed{\delta B_z = -\frac{k}{\omega}B_o\,\delta V_z}$$

Momentum $\quad -i\omega\rho_0\,\delta\vec{v} = -\nabla\left(\delta P + \frac{B_0\,\delta B_x}{4\pi}\right) + \frac{\overbrace{\vec{B}_0\cdot\nabla\vec{\delta B}}^{f(x)}}{4\pi} + \frac{\overbrace{\vec{\delta B}\cdot\nabla\vec{B}_0}^{f(z)}}{4\pi} + \delta\rho\,\vec{g}$

isothermal P $\quad c_s^2 = \frac{\delta P}{\delta\rho}$

$\hat{x}$: $\quad -i\omega\rho_0\,\delta v_x = -ik\left(c_s^2\,\delta\rho\right) - ik\cancel{\frac{B_0}{4\pi}\delta B_x} + ik\cancel{\frac{B_0}{4\pi}\delta B_x} + \frac{\delta B_z}{4\pi}\frac{\partial B_0}{\partial z}$

$$\delta v_x = \frac{kc_s^2}{\omega}\frac{\delta\rho}{\rho_0} + \frac{i}{\omega\rho_0}\underbrace{\left(\frac{-B_0}{2H}\right)}_{\frac{\partial B_0}{\partial z}}\underbrace{\left(\frac{-kB_0}{\omega 4\pi}\delta v_z\right)}_{\delta B_z}$$

$$\boxed{\delta v_x = \frac{kc_s^2}{\omega}\frac{\delta\rho}{\rho} + i\frac{k\,c_A^2}{2\omega^2 H}\delta v_z}$$

$$\frac{\delta\rho}{\rho_0} = \frac{k}{\omega}\left(\frac{kc_s^2}{\omega}\frac{\delta\rho}{\rho_0} + i\frac{k\,c_A^2}{2H\omega^2}\delta v_z\right) + i\frac{\delta v_z}{H\omega}$$

$$\frac{\delta\rho}{\rho_0} = \frac{i\,\delta v_z}{H\omega}\left(\frac{k^2c_A^2}{2\omega^2}+1\right)\frac{1}{1-\frac{k^2c_s^2}{\omega^2}} = \frac{\frac{i\,\delta v_z}{H\omega}\left(k^2\frac{c_A^2}{2}+\omega^2\right)}{\omega^2 - k^2c_s^2}$$

$\hat{z}$: $\quad -i\omega\rho_0\,\delta v_z = \frac{1}{H}\left(c_s^2\,\delta\rho + \frac{B_0\,\delta B_x}{4\pi}\right) + \frac{B_0\,ik\,\delta B_z}{4\pi} - \overbrace{\frac{\delta\rho}{H}\left(c_s^2 + \frac{c_A^2}{2}\right)}^{\delta\rho g}$ $\quad$ ① cancels

$$= -\frac{\delta\rho}{2H}c_A^2 + \frac{B_0}{4\pi H}\left(\frac{i\,\delta v_z\,B_0}{2H\omega}\right) + \underbrace{\frac{ikB_0}{4\pi}\left(-\frac{kB_0}{\omega}\delta v_z\right)}_{k^2\frac{B_0^2}{4\pi\rho_0}\delta v_z,\ \frac{B_0^2}{4\pi\rho_0} = c_A^2} \quad * \left(i\omega\frac{1}{\rho_0}\right)$$

Plug it in

$$\left(\omega^2 - k^2c_A^2 + \frac{c_A^2}{2H^2}\right)\delta v_z = \frac{-i\omega\,c_A^2}{2H}\frac{\delta\rho}{\rho_0}$$

$$\left(\omega^2 - k^2c_s^2\right)\left(\omega^2 - k^2c_A^2 + \frac{c_A^2}{2H^2}\right) = \frac{-i\omega\,c_A^2}{2H}\left(\frac{i\,\delta v_z}{H\omega}\right)\left(\frac{k^2c_A^2}{2}+\omega^2\right) = \frac{c_A^2}{2H^2}\left(\frac{k^2c_A^2}{2}+\omega^2\right)$$

ω^2 ② $\quad$ ②

$$\omega^4 - \omega^2\left(k^2c_s^2 + k^2c_A^2\right) + k^4c_A^2c_s^2 \underbrace{- k^2\frac{c_s^2c_A^2}{2H^2} - \frac{k^2c_A^4}{4H^2}}_{} = 0$$

new pieces: atmosphere inhomogeneous

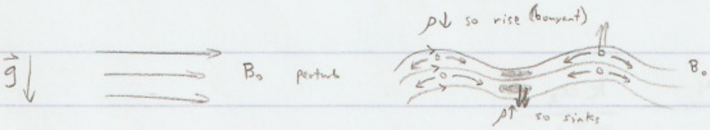
if $H \to \infty$, recover fast/slow waves

$$\omega^2 = \frac{k^2}{2}\left(c_s^2+c_A^2\right) \pm \left[\underbrace{\left(\frac{k^2}{2}\left(c_A^2+c_s^2\right)\right)^2 - k^4c_A^2c_s^2 + \frac{k^2c_A^2}{2H^2}\left(c_s^2+\frac{c_A^2}{2}\right)}_{}\right]^{1/2}$$

if $>0 \implies \omega^2 < 0$ unstable

Instability $\quad -k^2c_s^2 + \frac{1}{2H^2}\left(c_s^2 + \frac{c_A^2}{2}\right) > 0$

$$\boxed{k^2H^2 < \frac{1}{2}\left(1 + \frac{c_A^2}{2c_s^2}\right)} \quad \underline{\text{Unstable}}$$



Parker / Magnetic Rayleigh-Taylor Instability

Part of sunspot formation and present in galaxies 

possible formation of clouds

11/13/2006

Kelvin-Helmholtz Instability (shear flows)

z, x: $\rho_2 \longrightarrow \vec{u}_2$, $\rho_1 \longrightarrow \vec{u}_1$ $\quad \vec{g}=0 \quad e^{ikx-i\omega t} \quad \delta\vec{v},\ \delta P,\ \delta\rho$ (shear flow)

Motions slow ($\ll c_s$) $\Rightarrow$ Incompressible (ignore propagation of sound wave) $\quad \nabla\cdot\delta\vec{v}=0$

$k_y=0$ for simplicity $\qquad \vec{u}_0=\hat{x}\left(u_2 H(z)+u_1 H(-z)\right) \qquad \rho_0=\rho_2 H(z)+\rho_1 H(-z)$

Continuity $\quad -i\omega\,\delta\rho+\underbrace{\vec{u}_0\cdot\nabla\delta\rho}_{ik u_0\,\delta\rho}+\rho_0\cancel{\nabla\cdot\delta\vec{v}}^{\,0}+\underbrace{\delta\vec{v}\cdot\nabla\rho_0}_{\delta v_z \frac{\partial\rho_0}{\partial z}}=0$

$\bar{\omega}=\omega-k u_0(z)$
doppler shifted ω

$$-i\bar{\omega}\,\delta\rho+\delta v_z\frac{\partial\rho_0}{\partial z}=0 \qquad \delta\rho=\frac{-i\,\delta v_z}{\bar{\omega}}\frac{\partial\rho_0}{\partial z}\ \ (\sim\delta(z))$$

$$\nabla\cdot\delta\vec{v}=ik\,\delta v_x+\frac{\partial v_z}{\partial z}=0$$

$$-\omega i\rho_0\,\delta\vec{v}+\rho_0\vec{u}_0\cdot\nabla\delta\vec{v}+\rho_0\,\delta\vec{v}\cdot\nabla\vec{u}_0=-\nabla P$$

$\hat{x}$: $\quad -i\bar{\omega}\rho_0\,\delta v_x+\rho_0\,\delta v_z\frac{\partial u_0}{\partial z}=-ik\,\delta P \qquad \hat{z}$: $\quad -i\omega\,\delta v_z=-\frac{\partial\,\delta P}{\partial z}$

$$\delta v_z=\frac{-i}{\bar{\omega}\rho_0}\frac{\partial\,\delta P}{\partial z}$$

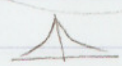
$$ik\underbrace{\left(\frac{k}{\bar{\omega}\rho_0}\delta P-\frac{i}{\bar{\omega}}\delta v_z\frac{\partial u_0}{\partial z}\right)}_{\delta v_x}+\frac{\partial\,\delta v_z}{\partial z}=0 \qquad i\frac{k^2\,\delta P}{\bar{\omega}\rho_0}+\frac{k}{\bar{\omega}}\frac{\partial u_0}{\partial z}\left(\frac{-i}{\bar{\omega}\rho_0}\frac{\partial\,\delta P}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{-i}{\bar{\omega}\rho_0}\frac{\partial\,\delta P}{\partial z}\right)=0$$

$$\frac{d}{dz}\left(\frac{1}{\bar{\omega}\rho_0}\frac{\partial\,\delta P}{\partial z}\right)-\frac{k^2\,\delta P}{\bar{\omega}\rho_0}=-\frac{k}{\bar{\omega}^2\rho_0}\frac{\partial u_0}{\partial z}\frac{\partial\,\delta P}{\partial z}$$

$$z\neq0 \quad \frac{d^2\,\delta P}{dz^2}-k^2\,\delta P=0 \Rightarrow \underset{z>0}{\delta P^>=\tilde{\delta P}\,e^{-kz}} \qquad \underset{z<0}{\delta P^<=\tilde{\delta P}\,e^{+kz}}$$

$$\int_{0^-}^{0^+}dz \Rightarrow \frac{1}{\bar{\omega}_2\rho_2}\left(-k\tilde{\delta P}\right)-\frac{1}{\bar{\omega}_1\rho_1}\left(k\tilde{\delta P}\right)=-k\int_{-\epsilon}^{0^-}\frac{dz}{\bar{\omega}_1^2\rho_1}\left(k\tilde{\delta P}\right)\left(-u_1\,\delta(z)\right)-k\int_{0^+}^{\epsilon}\frac{dz}{\bar{\omega}_2^2\rho_2}\left(-k\tilde{\delta P}\right)\left(u_2\,\delta(z)\right)$$

$$\frac{1}{\bar{\omega}_2\rho_2}+\frac{1}{\bar{\omega}_1\rho_1}=\frac{-ku_1}{\bar{\omega}_1^2\rho_1}-\frac{ku_2}{\bar{\omega}_2^2\rho_2}$$

since  δP changes slope @ 0 so $\frac{\partial\,\delta P}{\partial z}$ discontinuous

$$\frac{\omega - ku_2}{\bar{\omega}_2^2\rho_2} + \frac{\omega - ku_1}{\bar{\omega}_1^2\rho_2} = \frac{-ku_1}{\bar{\omega}_1^2\rho_1} - \frac{ku_2}{\bar{\omega}_2^2\rho_2}$$

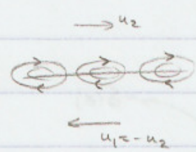
$$\omega\left[\frac{1}{\bar{\omega}_2^2\rho_2} + \frac{1}{\bar{\omega}_1^2\rho_2}\right] = 0$$

$$\bar{\omega}_2^2\rho_2 = -\bar{\omega}_1^2\rho_1 \qquad \bar{\omega}_2 = \pm i\sqrt{\frac{\rho_1}{\rho_2}}\,\bar{\omega}_1 \qquad \omega\left(1 \mp i\sqrt{\frac{\rho_1}{\rho_2}}\right) = k\left(u_2 \mp i\sqrt{\frac{\rho_1}{\rho_2}}\,u_1\right)$$

$$\omega = k\,\frac{\left(u_2 \mp i\sqrt{\frac{\rho_1}{\rho_2}}\,u_1\right)\left(1 \pm i\sqrt{\frac{\rho_1}{\rho_2}}\right)}{1 + \frac{\rho_1}{\rho_2}} = k\,\frac{\left(\left(u_2 + \frac{\rho_1}{\rho_2}u_1\right) \pm i\sqrt{\frac{\rho_1}{\rho_2}}\,(u_2 - u_1)\right)}{1 + \frac{\rho_1}{\rho_2}}$$

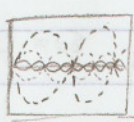
$$\boxed{\omega = k\,\frac{(\rho_2 u_2 + \rho_1 u_1) \pm i\sqrt{\rho_1\rho_2}\,(u_2 - u_1)}{\rho_1 + \rho_2}}$$

$\mathrm{Im}\,\omega > 0$ unstable for $u_2 - u_1 \gtrless 0$



Long wavelengths are again the important ones

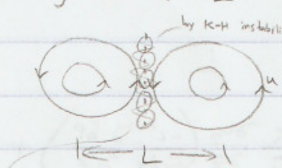
$\lambda\uparrow \quad k\downarrow$



Kolmogorov $\longrightarrow$ Subsonic $\nabla\cdot\vec{v} = 0$ $(\nabla\cdot\vec{B} = 0)$ eddies (convection cells)

Turbulence   Homogeneous, Isotropic 3-D   Steady state

Large Scale $- L$ inject energy (perturb it)   viscosity $= 0$ (large scale)



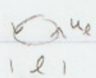
energy

$$\epsilon = \rho\frac{u^2}{2} \qquad \dot{\epsilon} = \frac{\epsilon}{L/u} = \frac{\rho u^3}{2L}$$

$$\text{time} \sim \frac{L}{u}$$

Small eddies form by Kelvin Helmholtz instability   Energy transferred to smaller eddies

Small Scale



$$\epsilon_\ell = \rho\frac{u_\ell^2}{2} \qquad \text{lifetime} \quad \tau_\ell \sim \frac{\ell}{u_\ell} \quad \text{turn over time}$$

$$\dot{\epsilon}_\ell = \frac{\epsilon_\ell}{\tau_\ell} = \frac{\rho u_\ell^3}{2\ell}$$

Kolmogorov:

$$\dot{\epsilon}_L = \dot{\epsilon}_\ell \qquad \rho\frac{u^3}{2L} = \rho\frac{u_\ell^3}{2\ell} \qquad u_\ell = u\left(\frac{\ell}{L}\right)^{1/3}$$

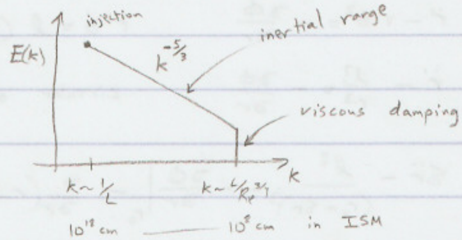
In smaller scales, viscosity can dampen it

$$\vec{u} = \int e^{i\vec{k}\cdot\vec{x}}\,\vec{u}_k\, d^3k \qquad \vec{k}\cdot\vec{u}_k = 0$$

$$\left\langle \frac{\rho u^2}{2} \right\rangle = \int dk\; E(k) \qquad \epsilon_\ell = \int_k^\infty dk'\, E(k') \qquad k \sim 1/\ell$$

$$\tau_\ell = \frac{\ell}{u_\ell} \sim \ell^{2/3} \sim \frac{1}{k^{2/3}} = \tau_k \qquad \frac{1}{\tau_k}\int_k^\infty dk'\, E(k') = \dot{\epsilon}_\ell = \text{const.}$$

$$\int_k^\infty dk\; E(k) \propto \frac{1}{k^{2/3}} \qquad \boxed{E(k) \propto \frac{1}{k^{5/3}}}$$



$$\frac{1}{\gamma-1}\left(\frac{dP}{dt} + \gamma P \cancel{\nabla\cdot\vec{v}}^{\,0}\right) = -\overleftrightarrow{\Pi} : \nabla\vec{v} - \nabla\vec{q}$$

$$\frac{d\epsilon}{dt} = \dot{\epsilon} \sim -\rho D\left(\frac{\partial v}{\partial x}\right)^2 \qquad \frac{\rho u_{\ell_0}^3}{\ell_0} \simeq \rho D\,\frac{u_{\ell_0}^2}{\ell_0^2}$$

$$\left(\frac{\rho u_\ell^3}{2\ell}\right)$$

At what scale, viscosity is as important as $\dot{\epsilon}$

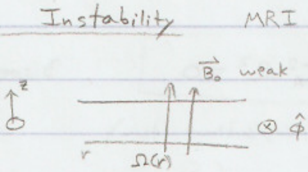
$$u_{\ell_0} \sim \frac{D}{\ell_0} = u\left(\frac{\ell_0}{L}\right)^{1/3} \qquad \left(\frac{\ell_0}{L}\right)^{4/3} = \frac{D}{Lu} = Re^{-1} \quad \text{Reynold's Number}$$

$$\frac{\ell_0}{L} \sim \frac{1}{R_e^{3/4}} \qquad Re = \frac{Lu}{D} \sim \frac{Lu}{\lambda_{mfp} c_s}$$

11/15/2006

Magneto Rotational Instability MRI

Accretion Disk



$\beta = \frac{8\pi P}{B^2}$ $\quad \beta \gg 1$

$\otimes\ \hat{\phi} \qquad \vec{u}_0 = u_0(r)\,\hat{\phi} = r\Omega\,\hat{\phi} \qquad \nabla\cdot\delta\vec{v} = 0$ incomp.

$\frac{\partial}{\partial\phi} = 0$ axisymmetric $\qquad \frac{\partial}{\partial r} = 0$ both equil (P_0, T_0, B_0) but not $\underline{\Omega(r)}$ $\qquad \frac{\partial}{\partial z} = ik \qquad \frac{\partial}{\partial t} = -i\omega$

$\delta\rho = \delta P = 0$ follow from $\nabla\cdot\delta\vec{v} = 0$ etc.

$\delta B_z = 0 \qquad \delta B_r,\ \delta B_\phi,\ \delta v_r,\ \delta v_\phi \qquad \delta v_z = 0$

Ohm's Law

$$-i\omega\,\delta\vec{B} = -\vec{u}_0\cdot\nabla\delta\vec{B} + \vec{B}_0\cdot\nabla\delta\vec{v} + \delta\vec{B}\cdot\nabla\vec{u}_0 = -\frac{u_0}{r}\frac{\partial}{\partial\phi}\left(\delta B_r\hat{r} + \delta B_\phi\hat{\phi}\right) + ikB_0\delta\vec{v}$$
$$+ \left(\delta B_r\frac{\partial}{\partial r} + \frac{\delta B_\phi}{r}\frac{\partial}{\partial\phi}\right)u_0\hat{\phi}$$

$\hat{r}:\quad -i\omega\,\delta B_r = \cancel{\Omega\,\delta B_\phi} + ikB_0\,\delta v_r - \cancel{\frac{u_0}{r}}^{\,\Omega}\cancel{\delta B_\phi} = ikB_0\,\delta v_r$

$\hat{\phi}:\quad -i\omega\,\delta B_\phi = -\Omega\,\delta B_r + ikB_0\,\delta v_\phi + \delta B_r\frac{\partial}{\partial r}(r\Omega) = ikB_0\,\delta v_\phi + r\,\delta B_r\frac{\partial\Omega}{\partial r}$

Momentum

$$-i\omega\,\delta\vec{v} + \vec{u}_0\cdot\nabla\delta\vec{v} + \delta\vec{v}\cdot\nabla\vec{u}_0 = \frac{\vec{B}_0\cdot\nabla\delta\vec{B}}{4\pi\rho_0} = ik\frac{B_0}{4\pi\rho_0}\delta\vec{B}$$

$$-i\omega\,\delta\vec{v} + \frac{u_0}{r}\frac{\partial}{\partial\phi}\left(\delta v_r\hat{r} + \delta v_\phi\hat{\phi}\right) + \left(\delta v_r\frac{\partial}{\partial r} + \frac{\delta v_\phi}{r}\frac{\partial}{\partial\phi}\right)(u_0\hat{\phi}) =$$

$$\hat{r}: \quad -i\omega\,\delta v_r \overbrace{- \Omega\,\delta v_\phi - \Omega\,\delta v_\phi}^{-2\Omega\,\delta v_\phi} = ik\,\frac{B_0\,\delta B_r}{4\pi\rho_0}$$

$$\hat{\phi}: \quad -i\omega\,\delta v_\phi + \Omega\,\delta v_r + \delta v_r \frac{\partial}{\partial r}(r\Omega) = ik\,\frac{B_0\,\delta B_\phi}{4\pi\rho_0}$$

define $\dfrac{\kappa^2}{2\Omega} = \Omega + \dfrac{\partial}{\partial r}(r\Omega)$ $\quad \kappa^2 = 4\Omega^2 + 2\Omega r\dfrac{\partial\Omega}{\partial r}$ epicyclic frequency ($= \Omega^2$ for Keplerian)

$\ddot{r} - r\dot{\theta}^2 = -\dfrac{\partial\Phi}{\partial r}$ $\quad r^2\dot{\theta} = l$ (constant)

$\ddot{r} - \dfrac{l^2}{r^3} = -\dfrac{\partial\Phi}{\partial r}$ $\quad$ circular orbit $\ddot{r} = 0$ $\quad$ perturb it $\quad r = r_0 + \delta r(t)$

$\delta\ddot{r} - \dfrac{l^2}{(r_0+\delta r)^3} = -\dfrac{\partial\Phi}{\partial r}\Big|_{r_0} - \dfrac{\partial}{\partial r}(r\Omega^2)\Big|_{r_0}\delta r$ $\quad$ (Taylor expand) $\quad -\dfrac{l^2}{r_0^3} = -\dfrac{\partial\Phi}{\partial r}\Big|_{r_0}$

$\delta\ddot{r} - \dfrac{l^2}{r_0^3}\left(1 - 3\dfrac{\delta r}{r_0}\right) = $ $\quad \delta\ddot{r} + \dfrac{3l^2}{r_0^4}\delta r + \dfrac{\partial}{\partial r}(r\Omega^2)\,\delta r = 0$

$\delta\ddot{r} + (\underbrace{3\Omega^2 + \Omega^2 + 2r\Omega\frac{\partial\Omega}{\partial r}}_{4\Omega^2 + 2\Omega r\frac{\partial\Omega}{\partial r} = \kappa^2})\delta r = 0$ $\quad \delta\ddot{r} + \kappa^2\delta r = 0$

$\therefore \quad \hat{\phi}: \quad -i\omega\,\delta v_\phi + \dfrac{\kappa^2}{2\Omega}\delta v_r = ik\,\dfrac{B_0\,\delta B_\phi}{4\pi\rho_0}$

$\vdots$

$\gg$ $\boxed{(\omega^2 - k^2C_A^2)^2 - \underbrace{(\omega^2 - k^2C_A^2)\kappa^2}_{\text{needs to be arranged (add/subtract term)}} - 4k^2C_A^2\Omega^2 = 0}$ $\quad$ Dispersion relation $\quad$ Slow/Intermediate waves

$\gg$ $\boxed{\omega^2 = k^2C_A^2 + \dfrac{\kappa^2}{2} \pm \sqrt{\dfrac{\kappa^4}{4} + 4k^2C_A^2\Omega^2}}$ $\quad$ (Always ω^2; time invariant)

$\omega^2 < 0$ unstable: $\quad \left(k^2C_A^2 + \dfrac{\kappa^2}{2}\right)^2 < \dfrac{\kappa^4}{4} + 4k^2C_A^2\Omega^2$

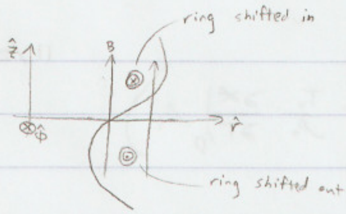
$k^4C_A^4 + k^2C_A^2\kappa^2 + \cancel{\dfrac{\kappa^4}{4}} < \cancel{\dfrac{\kappa^4}{4}} + 4k^2C_A^2\Omega^2$ $\quad k^2C_A^2 < 4\Omega^2 - \kappa^2 = 3\Omega^2$ Keplerian

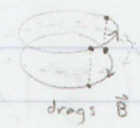
$kC_A < \sqrt{3}\,\Omega$

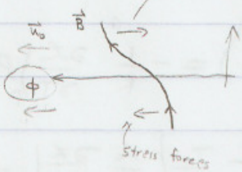
Max $\omega^2 < 0$ $\quad \dfrac{\partial\omega^2}{\partial k^2C_A^2} = 1 - \dfrac{2\Omega^2}{\sqrt{\frac{\kappa^4}{4} + k^2C_A^2\Omega^2}} = 0$ $\quad \overset{\Omega^4/4}{\cancel{\dfrac{\kappa^4}{4}}} + k^2C_A^2\Omega^2 = 4\Omega^4$

$4\Omega^2 - \tfrac{1}{4}\Omega^2 = \dfrac{15}{16}\Omega^2 = k^2C_A^2$ $\quad \omega^2 = \dfrac{15}{16}\Omega^2 + \dfrac{\Omega^2}{2} - 2\Omega^2 = -\dfrac{9}{16}\Omega^2$

$\omega_u = \pm i\,\dfrac{3}{4}\Omega$







stress forces (slows down plasma) $j\downarrow$

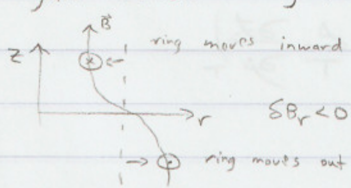
K: wants to move inwards

$\delta B_\phi > 0$

(speeds up plasma) $j\uparrow$

Keplerian: wants to move out due to j

angular momentum $j = r^2 \Omega$ $\qquad j\uparrow \propto \sqrt{r}$ ~ $r^2 \frac{\sqrt{}}{\sqrt{r^3}}$



$$\frac{\partial B_\phi}{\partial t} = ik B_0 \delta v_\phi + r \frac{\partial \Omega}{\partial r} \delta B_r$$

tries to stabilize $\qquad > 0 \qquad$ feedback process → reinforces

but: $k c_A < \sqrt{3}\,\Omega$

$$\Pi_{r\phi} = \left\langle \frac{\delta B_\phi \, \delta B_r}{4\pi} \right\rangle < 0 \qquad \Pi_{r\phi} = -\frac{3}{2} \rho D \Omega \quad \text{(Keplerian disk w/ ordinary viscosity)}$$

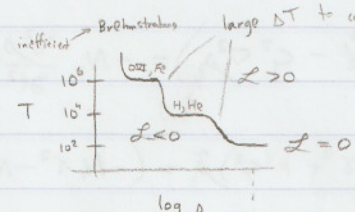
Transports angular momentum around the disk.

Thermal Instability in ISM

ISM:

$$\frac{\rho}{m} \mathcal{L}(\rho, T) = \Lambda - \Gamma \qquad \frac{\text{ergs}}{\text{cm}^3 \text{ s}}$$

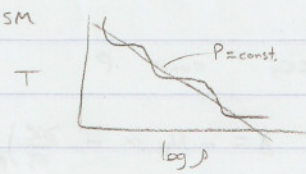
Λ: cooling function $\qquad \Gamma$: heating function

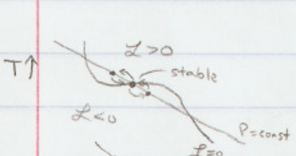


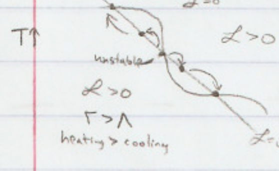
↑ molecular lines $\quad n \propto e^{-\varepsilon/kT}$ radiators

← reduce density, T ~ constant plateau

P = constant in ISM







$\mathcal{L} > 0 \Rightarrow \Lambda > \Gamma$

$$\left.\frac{\partial \mathcal{L}}{\partial T}\right|_p < 0 \quad \text{unstable}$$

3 phases: dark molecular clouds 10^2 — stable
hot phase 10^4
coronal phase 10^6

$$\rho = \rho_0 + \rho_1(r,t) \qquad P = P_0 + P_1 \qquad \vec{v} = 0 + \vec{v}_1$$

(perturbation $\delta\rho$)

Continuity: $\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot v_1 = 0 \qquad$ Momentum: $\rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\nabla P_1 \qquad \nabla \cdot ()$

$$-\frac{\partial^2 \rho_1}{\partial t^2} = -\nabla^2 P_1$$

$$\frac{1}{\gamma - 1} \left(\frac{\partial P_1}{\partial t} + \gamma P_0 \nabla \cdot \vec{v}_1 \right) = -\delta \left(\frac{\rho}{m} \mathcal{L}(\rho, t) \right)$$

$$\frac{1}{\gamma-1} \frac{\partial}{\partial t} \left(P_1 - \underbrace{\frac{\gamma P_0}{\rho_0}}_{c_s^2} \rho_1 \right) = -\frac{\rho_0}{m} \delta\mathcal{L}(\rho, t) = -\frac{\rho_0}{m} \left(\left.\frac{\partial \mathcal{L}}{\partial T}\right|_\rho T_1 + \left.\frac{\partial \mathcal{L}}{\partial \rho}\right|_T \rho_1 \right)$$

$$T = \frac{m}{k} \frac{P}{\rho} \quad \text{Ideal gas} \qquad T_1 = \frac{m}{k} \left(\frac{P_1}{\rho_0} - \frac{P_0}{\rho_0} \frac{\rho_1}{\rho_0} \right)$$

11/20/2006

$$\frac{1}{\gamma-1}\frac{\partial}{\partial t}\left(P_1 - c_s^2\rho_1\right) = -\frac{1}{k}\left.\frac{\partial \mathcal{L}}{\partial T}\right)_p P_1 - \frac{\rho_0}{m}\left(\left.\frac{\partial \mathcal{L}}{\partial \rho}\right|_T \rho_1 - \frac{T_0}{\rho_0}\left.\frac{\partial \mathcal{L}}{\partial T}\right)_p \rho_1\right)$$

$$= -\frac{1}{k}\left.\frac{\partial \mathcal{L}}{\partial T}\right|_p P_1 - \frac{T_0}{m}\left[\frac{\rho_0}{T_0}\left.\frac{\partial \mathcal{L}}{\partial \rho}\right|_T - \left.\frac{\partial \mathcal{L}}{\partial T}\right)_p\right]\rho_1$$

$$\left.\frac{d\mathcal{L}}{dT}\right|_p = \left.\frac{\partial \mathcal{L}}{\partial T}\right|_p \frac{dT}{dT} + \left.\frac{\partial \mathcal{L}}{\partial \rho}\right|_T \left.\frac{d\rho}{dT}\right)_p = \left.\frac{\partial \mathcal{L}}{\partial T}\right)_p - \frac{\rho}{T}\left.\frac{\partial \mathcal{L}}{\partial \rho}\right)_T$$

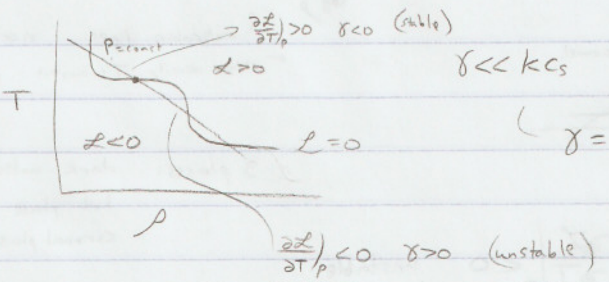
$$\therefore = -\frac{1}{k}\left.\frac{\partial \mathcal{L}}{\partial T}\right)_p P_1 + \frac{T_0}{m}\left.\frac{d\mathcal{L}}{dT}\right|_p \rho_1$$

$$N_v = \frac{\gamma-1}{k}\left.\frac{\partial \mathcal{L}}{\partial T}\right)_p \qquad N_p = \frac{\gamma-1}{\gamma k}\left.\frac{\partial \mathcal{L}}{\partial T}\right|_p$$

$$\frac{\partial}{\partial t}\left(P_1 - \rho_1 c_s^2\right) = -N_v P_1 + N_p c_s^2 \rho_1$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \rho_1}{\partial t^2} - c_s^2 \nabla^2 \rho_1\right) = -N_v \frac{\partial^2 \rho_1}{\partial t^2} + N_p c_s^2 \nabla^2 \rho_1 \qquad \rho_1 = \tilde{\rho}_1 e^{\gamma t + i\vec{k}\cdot\vec{r}}$$

$$\therefore \quad \gamma\left(\gamma^2 + k^2 c_s^2\right)\hat{\rho}_1 = \left(-N_v \gamma^2 - N_p k^2 c_s^2\right)\hat{\rho}_1$$



$\gamma \ll k c_s$ const P

$$\gamma = -N_p \propto -\left.\frac{\partial \mathcal{L}}{\partial T}\right)_p$$

Jean's Instability

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \qquad \rho\frac{d\vec{v}}{dt} = -\nabla P - \rho\nabla\phi \qquad \nabla^2\phi = 4\pi G\rho$$

Equil. $\rho_0 = \text{const}$ $\quad P_0 = \text{const}$ $\quad$ but $\nabla^2\phi \neq 0 \implies$ no equil.

Jean's Swindle : ignore fact that there is no equil

$$\rho = \rho_0 + \rho_1 \qquad P = P_0 + P_1 \qquad \vec{v} = \vec{v}_1 \qquad \phi = \phi_0 + \phi_1$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla\cdot\vec{v}_1 = 0 \qquad \rho_0\frac{\partial \vec{v}_1}{\partial t} = -c_s^2\nabla\rho_1 - \rho_0\nabla\phi_1 \qquad \nabla^2\phi_1 = 4\pi G\rho_0$$

$\nabla\cdot$

$$\frac{\partial}{\partial t}\left(\rho_0(\nabla\cdot\vec{v})\right)$$

$$-\frac{\partial^2\rho_1}{\partial t^2} = -c_s^2\nabla^2\rho_1 - 4\pi G\rho_0\rho_1 \qquad \omega^2 = k^2c_s^2 - 4\pi G\rho_0$$

unstable $\omega^2<0$ $\quad k^2c_s^2 < 4\pi G\rho_0 \quad k < \sqrt{\frac{4\pi G\rho_0}{c_s^2}} \equiv k_J = \frac{2\pi}{\lambda_J}$ — Jean's wavelength

$\omega_J = 4\pi G\rho_0$

---

in Cosmology : background is evolving

Transform coordinates $\vec r(t) = a(t)\vec x$ $\qquad a(t_0)=1$

a: scale factor, $\vec x$: comoving coordinates

$$\left.\frac{\partial}{\partial t}\right)_{\vec r} = \left.\frac{\partial}{\partial t}\right)_{\vec x} + \frac{\partial}{\partial\vec x}\cdot\left.\frac{\partial\vec x}{\partial t}\right)_{\vec r} = \frac{\partial}{\partial t} - \frac{\dot a}{a}\vec x\cdot\frac{\partial}{\partial\vec x} \qquad \frac{\dot a}{a}=H$$

$$\left.\frac{\partial\vec x}{\partial t}\right)_{\vec r} = -\frac{\dot a}{a^2}\vec r = -\frac{\dot a}{a}\vec x$$

@ constant x

$$\left.\frac{\partial}{\partial\vec r}\right)_t = \frac{1}{a}\frac{\partial}{\partial\vec x} \qquad \vec v = \dot a\vec x + \vec u \quad \text{relative comoving}$$

Convective derivative $\left.\frac{\partial}{\partial t}\right)_{\vec r} + \vec v\cdot\left.\frac{\partial}{\partial\vec r}\right)_t = \frac{\partial}{\partial t} - \frac{\dot a}{a}\vec x\cdot\frac{\partial}{\partial\vec x} + \frac{\dot a\vec x+\vec u}{a}\cdot\frac{\partial}{\partial\vec x} = \frac{\partial}{\partial t} + \frac{\vec u}{a}\cdot\frac{\partial}{\partial\vec x}$

$$\frac{\partial\rho}{\partial t} + \frac{\vec u}{a}\cdot\frac{\partial\rho}{\partial\vec x} + \frac{\rho}{a}\frac{\partial}{\partial\vec x}\cdot(\dot a\vec x+\vec u) = 0 \qquad \frac{\partial\rho}{\partial t} + \rho\nabla\cdot\vec v = 0$$

$$\frac{\partial\rho}{\partial t} + \frac{3\dot a}{a}\rho + \frac{\vec u}{a}\cdot\frac{\partial\rho}{\partial\vec x} + \frac{\rho}{a}\frac{\partial}{\partial\vec x}\cdot\vec u = 0$$

$$\rho\frac{d\vec v}{dt} = -\nabla P - \rho\nabla\phi \qquad \rho\left(\left.\frac{\partial}{\partial t}\right)_x + \frac{\vec u}{a}\cdot\frac{\partial}{\partial\vec x}\right)(\dot a\vec x+\vec u) = \rho\left(\ddot a\vec x + \frac{\partial\vec u}{\partial t} + \frac{\vec u}{a}\cdot\frac{\partial\vec u}{\partial\vec x} + \frac{\dot a}{a}\vec u\right)$$

$$= -\frac{1}{a}\frac{\partial P}{\partial\vec x} - \frac{\rho}{a}\frac{\partial\phi}{\partial\vec x}$$

$\nabla^2\phi = 4\pi G\rho$

$$\frac{\nabla_x^2\phi}{a^2} = 4\pi G\rho$$

$$\rho = \rho_0(t)\left(1+\delta(x,t)\right) \qquad \delta = \frac{\rho-\rho_0}{\rho_0} \qquad (\text{Previously for us: } \delta\rho = \rho_0\delta)$$

$P = P_0(t) + \delta P \qquad \phi = \phi_0 + \delta\phi \qquad \vec u = \mathcal{O}(\delta) \qquad \rho_0, P_0$ uniform

Cont: $$\underbrace{(1+\delta)\left[\frac{\partial\rho_0}{\partial t} + \rho_0\frac{3\dot a}{a}\right]}_{0^{th}\text{ order}} + \underbrace{\rho_0\frac{\partial\delta}{\partial t} + \cancel{\rho_0\frac{\vec u}{a}\cdot\frac{\partial\delta}{\partial\vec x}} + \frac{\rho_0}{a}\nabla\cdot\vec u}_{1^{st}\text{ order}} = 0$$

2^{nd} order small

Each order must be zero
$$\frac{\partial\rho_0}{\partial t} + \rho_0\frac{3\dot a}{a} = 0 \qquad \rho_0 a^3 = \text{const.} = \rho_0(t_0)$$

conservation of mass

$$\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\cdot\vec u = 0 \qquad 1^{st}\text{ order}$$

Momentum

$$\rho_0(1+\delta)(\ddot a\vec x) + \rho_0\left(\frac{\partial\vec u}{\partial t} + \frac{\dot a}{a}\vec u\right) = -\frac1a\nabla\delta P - \rho_0(1+\delta)\frac1a\nabla(\phi_0+\overbrace{\cancel{\phi_1}}^{2^{nd}\text{ order}}) - \frac{\rho_0}{a}\nabla\phi_1$$

0th: $\ddot a\vec x = -\frac1a\nabla\phi_0$ $\quad \nabla\cdot \quad = \quad \ddot a = -\frac13\frac1a\nabla^2\phi_0 = \boxed{-\frac{4\pi G}{3}\rho_0 a = \ddot a}$ Einstein Eq.

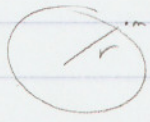
$\delta P = \rho_0 c_s^2\delta$ $\quad$ 1st: $\frac{\partial\vec u}{\partial t} + \frac{\dot a}{a}\vec u = -\frac1a c_s^2\nabla\delta - \frac1a\nabla\phi_1$ $\quad \nabla\cdot$

and using $\dot\delta + \frac1a\nabla\cdot\vec u = 0$ (Continuity), $\nabla\cdot\vec u = -a\dot\delta$

$$\frac{\partial}{\partial t}(-a\dot\delta) - \frac{\dot a}{a}(a\dot\delta) = -\frac{c_s^2}{a}\nabla^2\delta - \frac1a\nabla^2\phi_1$$

$$-a\ddot\delta - \dot a\dot\delta - \dot a\dot\delta \qquad \ddot\delta + \frac{2\dot a}{a}\dot\delta = \frac{c_s^2}{a^2}\nabla^2\delta + \overbrace{\frac{1}{a^2}\nabla^2\phi_1}^{4\pi G\rho_0\delta}$$

$$\ddot\delta + \frac{2\dot a}{a}\dot\delta = \frac{c_s^2}{a^2}\nabla^2\delta + 4\pi G\rho_0\delta$$



$E = \frac{m\dot r^2}{2} - \frac{GM(r)m}{r}$ $\qquad M(r) = \frac{4\pi}{3}r^3\rho_0(t)$ $\qquad r = ar_0$ $\qquad \dot r = \dot a r_0$

$$E = \frac m2 r_0^2\left(\dot a^2 - \frac{8\pi G\rho_0}{3}a^2\right) = -\frac m2 r_0^2 \underbrace{k}_{\text{curvature}}c^2$$

$\boxed{\dot a^2 + kc^2 = \frac{8\pi G\rho_0}{3}a^2}$ Friedman Eq.

$k<0$ open
$k=0$ flat
$k>0$ close

$\dot a^2 + kc^2 = \frac{8\pi G\rho_0}{3}a^2 + \frac{\Lambda}{3}a^2$ ← Cosmological constant

$k=0$ $\quad \dot a^2 = \frac{8\pi G\rho_0}{3}a^2$ $\quad \dot a_0^2 = \frac{8\pi G\rho_0(t_0)}{3}a_0^2$ $\quad \rho_0(t_0) = \frac{3H_0^2}{8\pi G} = \rho_{c0}$ critical density @ $t=t_0$

$\rho_0(t) = \frac{\rho_0(t_0)}{a^3}$ $\quad \therefore\ \dot a^2 = \frac{8\pi G\rho_0(t_0)}{3a}$ $\quad \sqrt a\,\dot a = \sqrt{\frac{8\pi G\rho_0(t_0)}{3}}$ $\quad \frac23 a^{3/2}\propto t$ $\quad a\propto t^{2/3}$

$\boxed{\frac{\dot a}{a} = \frac23\frac1t}$ $\qquad 4\pi G\rho_0(t) = \frac32\frac{\dot a^2}{a^2} = \frac23\frac{1}{t^2}$

Ignore $\frac{c_s^2}{a^2}\nabla^2\delta$ $(\propto\nabla^2 P)$

$\ddot\delta + \frac{4}{3t}\dot\delta - \frac{2}{3t^2}\delta = 0$ $\qquad k=0$ $\qquad \delta\sim t^n$

$t^{n-2}\left(n(n-1) + \frac43 n - \frac23\right) = 0$ $\qquad n^2 + \frac13 n - \frac23 = 0$ $\qquad n = -\frac16 \pm\sqrt{\frac1{36}+\frac23} = -\frac16 \pm \frac56$

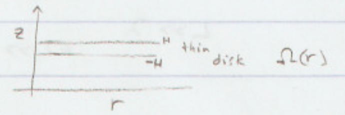
$n_- = -1$ (damped soln) $\qquad n_+ = \frac23$ $\qquad \delta\propto t^{2/3}\propto a(t)$

Non Cosmology: $\delta\rho\propto e^{\omega t}$ faster than $t^{2/3}$ $\qquad$ Jean's Instability in Cosmology

No T fluctuations in CMB → cold dark matter
⇒ no large ρ fluctuations, how can there be?

Spiral Density Waves

$\rho = \sigma \, \delta(z)$ — $\uparrow$ effective surface density $\sigma(r,\varphi)$



$$\frac{\partial \sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\sigma u_r) + \frac{1}{r}\frac{\partial}{\partial \varphi}(\sigma u_\varphi) = 0 \qquad \text{Continuity}$$

$$\int_{-H}^{H} P\, dz = \Pi \qquad \text{Mom.:} \quad \sigma\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla\Pi - \sigma\nabla\Phi$$

$$\nabla^2 \Phi = 4\pi G \sigma \, \delta(z)$$

Equil. $\frac{\partial}{\partial t} = 0 = \frac{\partial}{\partial \varphi} \qquad u_r = 0 \qquad u_{\varphi_0} = \Omega r$

Mom: $$-\sigma_0 \frac{u_{\varphi_0}^2}{r} = -\frac{\partial \Pi_0}{\partial r} - \sigma_0 \frac{\partial}{\partial r}\Phi_0$$

$$\sigma = \sigma_0 + \sigma_1 \qquad \Pi = \Pi_0 + \Pi_1 \qquad \vec{u} = u_0\hat{\varphi} + \vec{u}_1 \qquad \Phi = \Phi_0 + \Phi_1$$

Cont. $$\frac{\partial \sigma_1}{\partial t} + \Omega \frac{\partial \sigma_1}{\partial \varphi} + \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_0 u_{r_1}) + \frac{\sigma_0}{r}\frac{\partial}{\partial \varphi}(u_{\varphi_1}) = 0$$

Mom: $$\sigma_0\left(\frac{\partial \vec{u}_1}{\partial t} + \vec{u}_0\cdot\nabla\vec{u}_1 + \vec{u}_1\cdot\nabla\vec{u}_0\right) - \sigma_1\underbrace{\frac{u_{\varphi_0}^2}{r}}_{\vec{u}_0\cdot\nabla\vec{u}_0} = -\nabla\Pi_1 - \sigma_0\nabla\Phi_1 - \sigma_1\nabla\Phi_0$$

from MRI

$\hat{r}$: $$\sigma_0\left(\frac{\partial u_{r_1}}{\partial t} + \Omega\frac{\partial u_{r_1}}{\partial \varphi} - 2\Omega u_{\varphi_1}\right) - \sigma_1\frac{u_{\varphi_0}^2}{r} = -\frac{\partial \Pi_1}{\partial r} - \sigma_0\frac{\partial \Phi_1}{\partial r} - \sigma_1\frac{\partial \Phi_0}{\partial r}$$

Trick $\quad d\Pi = \sigma\, dh \qquad \frac{\partial \Pi_0}{\partial r} = \sigma_0\frac{\partial h_0}{\partial r} \qquad \frac{\partial \Pi_1}{\partial r} = \sigma_0\frac{\partial h_1}{\partial r} + \sigma_1\frac{\partial h_0}{\partial r}$

From equil: $$\underbrace{\frac{u_{\varphi_0}^2}{r} = \frac{\partial h_0}{\partial r} + \frac{\partial \Phi_0}{\partial r}}_{\mathscr{Y}} \qquad -\sigma_1\left(\mathscr{Y}\right) = 0$$

$\therefore$ left with $-\sigma_0\frac{\partial h_1}{\partial r} - \sigma_0\frac{\partial \Phi_1}{\partial r}$

$$\therefore \quad \frac{\partial u_{r_1}}{\partial t} + \Omega\frac{\partial u_{r_1}}{\partial \varphi} - 2\Omega u_{\varphi_1} = -\frac{\partial h_1}{\partial r} - \frac{\partial \Phi_1}{\partial r}$$

take $h = h(\sigma)$

$h_1 = \frac{\partial h_0}{\partial \sigma}\sigma_1$

Define $\quad c_s^2 = \sigma_0\frac{\partial h_0}{\partial \sigma} = \text{const} \implies h_1 = c_s^2\frac{\sigma_1}{\sigma_0}$

$$\therefore \quad \frac{\partial u_{r_1}}{\partial t} + \Omega\frac{\partial u_{r_1}}{\partial \varphi} - 2\Omega u_{\varphi_1} = -c_s^2\frac{\partial}{\partial r}\left(\frac{\sigma_1}{\sigma_0}\right) - \frac{\partial \Phi_1}{\partial r}$$

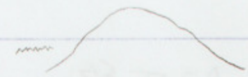
$\hat{\varphi}$: $$\left(\frac{\partial u_{\varphi_1}}{\partial t} + \Omega\frac{\partial u_{\varphi_1}}{\partial \varphi} + \frac{\kappa^2}{2\Omega}u_{r_1}\right) = -\frac{c_s^2}{r}\frac{\partial}{\partial \varphi}\left(\frac{\sigma_1}{\sigma_0}\right) - \frac{1}{r}\frac{\partial \Phi_1}{\partial \varphi}$$

Poisson $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi_1}{\partial \varphi^2} + \frac{\partial^2 \Phi_1}{\partial z^2} = 4\pi G\sigma_1\,\delta(z)$$

An aside: WKB Approximation

$$\frac{d^2\psi}{dx^2} + (E - V(x))\psi = 0 \qquad \frac{1}{V}\frac{\partial V}{\partial x} = \frac{1}{L} \text{ scale length}$$

$L >> \lambda$ $\qquad kL >> 1$



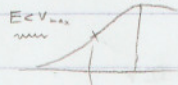
$$\psi(x) = \tilde{\psi}(x)\, e^{i\int^x k(x)dx}$$

slow varying function: $\frac{1}{\tilde\psi}\frac{d\tilde\psi}{dx} = \frac{1}{L}$ $\qquad \frac{d\psi}{dx} = \left(ik\tilde\psi + \frac{d\tilde\psi}{dx}\right) e^{i\int^x k dx}$

$$\frac{d^2\psi}{dx^2} \simeq \left(-k^2\tilde\psi + 2ik\frac{d\tilde\psi}{dx} + i\frac{dk}{dx}\psi + \underbrace{\frac{d^2\tilde\psi}{dx^2}}_{\text{small}}\right) e^{i\int k dx}$$

lowest order $\quad k^2(x) = E - V(x) \qquad$ first order $\quad 2ik\frac{d\tilde\psi}{dx} + i\frac{dk}{dx} = 0$

$$\tilde\psi(x) \propto \frac{1}{\sqrt{k(x)}}$$



V_{max}, $E < V_{max}$

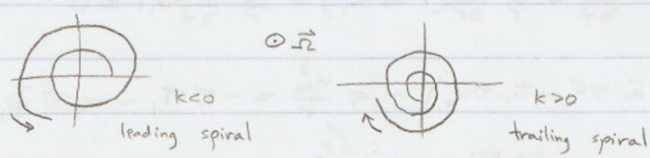
$E = V_{max}$ $\quad k \to 0 \quad \lambda \to \infty$ bad for WKB

WKB Turning Point $\qquad$ waves reflect @ $k=0$

Now using WKB $\qquad \frac{1}{\sigma_0}\frac{\partial\sigma_0}{\partial r} << \frac{1}{\sigma_1}\frac{\partial\sigma_1}{\partial r} \qquad \sigma_1 = \tilde\sigma_1 e^{-i\omega t + im\varphi + ikr}$ (mode # $m=1,2,\ldots$)

Phase $\quad \psi = -\omega t + m\varphi + kr \qquad$ fixed time $\quad d\psi = m\,d\varphi + k\,dr = 0$ for constant phase

$$\frac{d\varphi}{dr} = -\frac{k}{m}$$

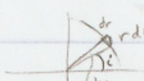


$\odot\vec\Omega$

$k<0$ leading spiral $\qquad k>0$ trailing spiral $\qquad$ galaxy $\odot\vec\Omega$

fixed r $\quad d\psi = -\omega dt + m\,d\varphi \qquad \frac{d\varphi}{dt} = \frac{\omega}{m}$ pattern speed

if $\frac{d\varphi}{dt} = \Omega \to$ corotation



$$\tan i = \frac{dr}{r\,d\varphi} = \frac{m}{kr} \qquad \text{WKB } kr >> 1 \; (r >> \lambda) \qquad \frac{m}{kr} << 1 \implies \text{tightly wound spirals}$$

Cont: $\quad -i(\omega - m\Omega)\sigma_1 + \sigma_0 i k u_{r_1} = 0 \quad$ (dropping higher order terms) $\left(\sim \frac{1}{r} \to 0 \quad \frac{\partial}{\partial r} \to 0\right)$

$$u_{r_1} = \frac{\omega - m\Omega}{k}\frac{\sigma_1}{\sigma_0}$$

Poisson: $$\frac{\partial^2\Phi_1}{\partial r^2} + \frac{\partial^2\Phi_1}{\partial z^2} = 4\pi G\sigma_1\delta(z) = \frac{\partial^2\Phi_1}{\partial z^2} - k^2\Phi_1 \implies \Phi_1^z = \tilde\Phi_1 e^{\mp|k|z}$$

$$-2|k|\tilde\Phi_1 = 4\pi G\sigma_1 \implies \tilde\Phi_1 = -\frac{2\pi G}{|k|}\sigma_1$$

Mom φ: $$-i(\omega - m\Omega)u_{\varphi_1} + \frac{\kappa^2}{2\Omega}u_{r_1} = 0 \qquad u_{\varphi_1} = -i\frac{\kappa^2}{2\Omega}\frac{u_{r_1}}{\omega - m\varphi} \implies u_{\varphi_1} = -i\frac{\kappa^2}{2\Omega}\frac{1}{k}\frac{\sigma_1}{\sigma_0}$$

Mom r: $$-i(\omega - m\Omega)u_{r_1} - 2\Omega u_{\varphi_1} = -ikc_s^2\frac{\sigma_1}{\sigma_0} - ik\Phi_1 \qquad -i\frac{(\omega-m\Omega)^2}{k}\frac{\sigma_1}{\sigma_0} + i\frac{\kappa^2}{k}\frac{\sigma_1}{\sigma_0} = -ikc_s^2\frac{\sigma_1}{\sigma_0} - ik\left(\frac{-2\pi G}{|k|}\frac{\sigma_1}{\sigma_0}\right)\sigma_0$$

$\ominus * ik$

~ doppler shifted

epicyclic freq.

$$(\omega - m\Omega)^2 - \kappa^2 - k^2 c_s^2 + 2\pi G \sigma_0 |k| = 0$$ Dispersion relation

(Jean's: $\omega^2 = k^2 c_s^2 - 4\pi G \rho_0$)

Case $m=0$

$$\omega^2 = \kappa^2 + k^2 c_s^2 - 2\pi G \sigma_0 |k|$$

$k \to 0 \implies \omega^2 = \kappa^2 = 4\Omega^2 + r\frac{\partial \Omega^2}{\partial r}$ $\quad j = r^2\Omega$ angular momentum

$$= 4\frac{j^2}{r^4} + r\frac{\partial}{\partial r}\left(\frac{j^2}{r^4}\right) = \frac{1}{r^3}\frac{\partial j^2}{\partial r}$$

Stable $\omega^2 > 0$ if $\frac{\partial j}{\partial r} > 0$ Rayleigh criterion

angular momentum <u>increases outward</u>

$\kappa^2 > 0$

$$\frac{\omega^2}{\kappa^2} = 1 + \frac{k^2 c_s^2}{\kappa^2} - 2\pi\frac{G\sigma_0}{\kappa^2}|k| \qquad \frac{\kappa^2}{2\pi G\sigma_0} = k_T$$ Toomre wave number

rotation, pressure

$$= 1 + \frac{k^2}{k_T^2}\left(\frac{\kappa^4}{4\pi^2 G^2 \sigma_0^2}\right)\frac{c_s^2}{\kappa^2} - \frac{|k|}{k_T} \qquad \frac{\kappa^2 c_s^2}{(\pi G \sigma_0)^2} = Q^2$$ Toomre Q-factor

self gravity

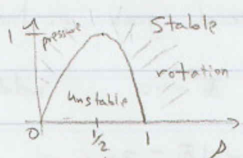
$$\frac{\omega^2}{\kappa^2} = 1 + \frac{Q^2}{4}\frac{k^2}{k_T^2} - \frac{|k|}{k_T}$$ $Q \geq 1$ stable (to be shown)

11/27/2006

$$\frac{(\omega - m\Omega)^2}{\kappa^2} = 1 + \frac{Q^2}{4}\frac{k^2}{k_T^2} - \frac{|k|}{k_T}$$

$m=0$ stable $\to \frac{\omega^2}{\kappa^2} > 0 \qquad \rho = \frac{k_T}{|k|} \qquad 1 + \frac{Q^2}{4}\frac{1}{\rho^2} - \frac{1}{\rho} > 0$

$\rho^2 - \rho + \frac{Q^2}{4} \geq 0 \qquad Q \geq 2\sqrt{\rho(1-\rho)}$



$\infty > \frac{|k|}{k_T} > 1$

$Q \geq 1$ stable

$m \neq 0$ spiral density waves

$$\nu^2 = \frac{(\omega - m\Omega)^2}{\kappa^2} = 1 + \frac{Q^2}{4}\frac{k^2}{k_T^2} - \frac{|k|}{k_T} \qquad \boxed{\frac{|k|}{k_T} = \frac{2}{Q^2}\left(1 \pm \sqrt{1 - Q^2(1-\nu^2)}\right)}$$

$+$: $k<0$ short leading; $k>0$ short trailing

$-$: $k<0$ long leading; $k>0$ long trailing

$+$: short wavelength mode

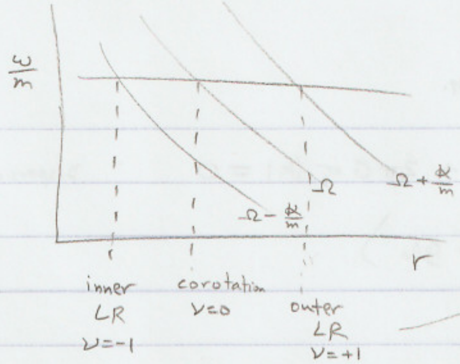
$-$: long " "

If $\nu^2 = 1 \quad \nu = \pm 1 \quad |k| \to 0$ WKB turning point $\qquad$ reflection point

$$\frac{\omega - m\Omega}{\kappa} = \pm 1 \qquad \frac{\omega}{m} = \Omega \pm \frac{\kappa}{m}$$

galaxy rotation

$\frac{\omega}{m} = \Omega \pm \frac{\kappa}{m}$



Lindblad resonance

At Lindblad resonance, leading $\leftrightarrow$ trailing
change

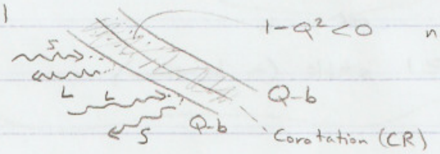
Long waves don't propagate outside resonance $\nu^2 > 1$ so $|k| < 0 \leftarrow$ impossible

At LR, short $\rightarrow$ damped (Landau damping) based on kinetic theories
Stars get/give energy from wave

Q-barrier $1 - Q^2(1-\nu^2) = 0$ $\quad \nu^2 = \frac{Q^2-1}{Q^2} \Rightarrow \underset{S}{|k|_+} = \underset{L}{|k|_-}$
have same phase speed

if $Q \gg 1$ $\quad \nu^2 = 1 - 1/Q^2 \simeq 1$ close to LR

$Q \gtrsim 1$



$1 - Q^2 < 0$ no wave propagation

At Q-b, $L \leftrightarrow S$

Angular Momentum

Toomre Trick

place external grav perturbation $\Phi^{ext} = \tilde{\Phi}_e \exp\left[-i(\omega + i\gamma)t + im\varphi + ikr\right]$ $\quad \gamma \geq 0$

$t \to -\infty$ $\quad \Phi^{ext} \to 0$ adds Energy to system

Poisson $-|k|\Phi_1 = 2\pi G \sigma_1$ $\qquad \sigma_1 = -\frac{|k|\Phi_1}{2\pi G}$

Momentum r: $\quad -i\frac{(\omega + i\gamma - m\Omega)^2}{k}\frac{\sigma_1}{\sigma_0} + \left(i\frac{\kappa^2}{k} + \frac{ik c_s^2}{k}\right)\frac{\sigma_1}{\sigma_0} = -ik\left(\Phi_1 + \Phi_1^{ext}\right)$

$$\left((\omega + i\gamma - m\Omega)^2 - \kappa^2 - k^2 c_s^2\right)\left(\frac{-|k|\Phi_1}{2\pi G}\right)_{\sigma_0} = k^2\left(\Phi_1 + \Phi_1^{ext}\right)$$

$$\underbrace{-\left[\quad\right]\frac{1}{2\pi G \sigma_0 |k|}}_{\frac{1}{D}}\Phi_1 = \Phi_1 + \Phi_1^{ext}$$

$$D(\omega + i\gamma) \equiv \frac{-2\pi G \sigma_0 |k|}{(\omega + i\gamma - m\Omega)^2 - \kappa^2 - k^2 c_s^2}$$

$\gamma = 0 \quad D = 1$

$$\frac{\Phi_1}{D} = \Phi_1 + \Phi^{ext} \qquad \Phi^{ext} = -\frac{D-1}{D}\Phi_1$$

if $\gamma/\omega \ll 1$ $\quad D(\omega + i\gamma) \simeq \underbrace{D(\omega)}_{1} + \frac{\partial D}{\partial \omega}(i\gamma) \qquad D - 1 = i\gamma \frac{\partial D}{\partial \omega}$

$$\Phi^{ext} = -i\gamma \frac{\partial D}{\partial \omega}\Phi_1$$

torque $\tau_z^{ext} = (\vec{r} \times \vec{F}^{ext}) \cdot \hat{z}$ $\quad$ In cylindrical $F \to F_\varphi$

$$rF_\varphi^{ext} = r\left(\frac{-1}{r}\frac{\partial \Phi_1^{ext}}{\partial \varphi}\right) = -im(-i\gamma)\frac{\partial D}{\partial \omega}\Phi_1 = -m\gamma \frac{\partial D}{\partial \omega} Re(\Phi_1)$$

Mass of disk: $(\sigma_0 + \sigma_1)$

$$\int_{-\infty}^{\infty}\int_0^r\int_0^{2\pi} Re(\sigma_0 + \sigma_1)\left(-m\gamma\frac{\partial D}{\partial \omega}Re(\Phi_1)\right) d\varphi\, r dr\, dt = J_{wave}$$

total angular momentum put into wave by external force

$$\iiint \left(\frac{-|k|\, Re(\Phi_1)}{2\pi G}\right)\left(-m\gamma \frac{\partial D}{\partial \omega} Re(\Phi_1)\right) r dr\, d\varphi\, dt \qquad \Phi_1 \sim e^{\gamma t - i\omega t + im\varphi + ikr}$$

$$e^{2\gamma t}\cos^2(\omega t - m\varphi - kr) \qquad \int_{-\infty}^{t=0} e^{2\gamma t}\, dt = \frac{1}{2\gamma}$$

cancels with γ

$$2\pi \int_0^\infty \underbrace{\frac{|k| m}{8\pi G}\frac{\partial D}{\partial \omega}|\Phi_1|^2}_{\text{angular momentum per unit area}} r dr = J_{wave}$$

(?) from $\int_0^{2\pi} \cos^2(x) dx$

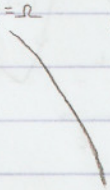
Set $\gamma = 0$ (Doesn't appear)

$$\frac{\partial D}{\partial \omega} = \frac{4\pi G \sigma_0 |k| (\omega - m\Omega)}{[\ \ ]^2} \propto J_{wave}$$

$\frac{\omega}{m} = \Omega$

$\frac{\omega}{m} < \Omega$ $\quad J < 0$

$\frac{\omega}{m} > \Omega$ $\quad J > 0$



CR

angular momentum changes at Corotation

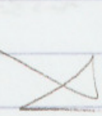
Group Velocity

$D(\omega, k, r) = 1$ $\quad$ Dispersion relation

$$dD = \frac{\partial D}{\partial \omega}d\omega + \frac{\partial D}{\partial k}dk + \frac{\partial D}{\partial r}dr = 0$$

$dr = 0$ $\quad \frac{d\omega}{dk} = -\frac{\partial D/\partial k}{\partial D/\partial \omega}$ $\leftarrow$ group velocity $\qquad \frac{\partial |k|}{\partial k} = sgn(k)$ sign of k

$$\frac{\partial D}{\partial k} = -\frac{2\pi G \sigma_0\, sgn(k)}{[\ \ ]} - \frac{2\pi G \sigma_0 |k|\, 2k c_s^2}{[\ \ ]^2}$$



$k = |k|\, sgn(k)$

$$\frac{\partial D}{\partial k} = -\frac{(2\pi G \sigma_0\, \mathrm{sgn}(k))}{[\;]^2}\Big(\underbrace{[\;]}_{-2\pi G\sigma_0|k|} + 2k^2 c_s^2\Big)$$

$-2\pi G \sigma_0 |k|$ since $D = 1$

$$= -\frac{(4\pi G\sigma_0|k|)\,\mathrm{sgn}|k|}{[\;]^2}\left(|k| c_s^2 - \pi G\sigma_0\right) = -\frac{4\pi G\sigma_0|k|\,\pi G\sigma_0\,\mathrm{sgn}\,k}{[\;]^2}\left(\frac{|k| c_s^2}{\pi G \sigma_0} - 1\right)$$

$$k_T = \frac{\kappa^2}{2\pi G\sigma_0} \qquad \frac{\partial D}{\partial k} = \sim\left(\frac{|k|}{k_T}\frac{\kappa^2 c_s^2}{2(\pi G\sigma_0)^2} - 1\right) = \sim\left(\frac{Q^2}{2}\frac{|k|}{k_T} - 1\right)$$

$$Q^2 = \frac{\kappa^2 c_s^2}{(\pi G\sigma_0)^2}$$

$$\frac{|k|}{k_T} = \frac{2}{Q^2}\left(1 \pm \sqrt{1 - Q^2(1-\nu^2)}\right) \quad \therefore \quad \frac{\partial D}{\partial k} = \sim\left(\pm\sqrt{1-Q^2(1-\nu^2)}\right)$$

$$\frac{d\omega}{dk} = -\frac{\partial D/\partial k}{\partial D/\partial \omega} = \frac{\pi G\sigma_0\,\mathrm{sgn}(k)\left(\pm\sqrt{1-Q^2(1-\nu^2)}\right)}{\omega - m\Omega} = v_g$$

$\nu < 0$ $\quad \nu = 0$ $\quad \nu > 0$

$\frac{\omega}{m} = \Omega$

S–L ⇝ | ⇜ S–L

S–T ⇜ | ⇝ S–T

L–L ⇜ | ⇝ L–L

L–T ⇝ | ⇜ L–T

$J < 0$ $\quad$ CR $\quad$ $J > 0$

→ r

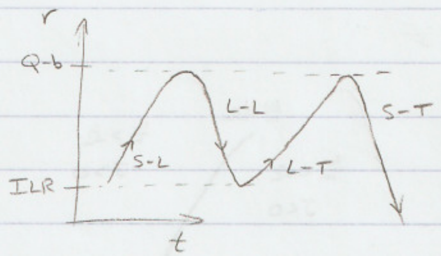
S: + $\quad$ L: $k < 0$ $\quad$ $v_g < 0$ outside CR

$v_g > 0$ inside CR

T: $k > 0$ $\quad$ $v_g > 0$ outside CR

$v_g < 0$ inside CR

$$\omega - m\Omega = \left(\frac{\omega}{m} - \Omega\right)m = \nu m \kappa$$



Describes unwinding of spiral wave then rewinding in opposite sense (L→T)

WASER $\quad$ Toomre

wave amplification ...

$Q=1$

L–T ⇝ $j = -1$

...... S–T ⇜ $j = -2$ (conserve j) | ⇝ S–T $j = +1$

⇜ | ⇝ L–T .... $j = -2$

S–T ⇜ $j = -4$ | ⇝ S–T $j = +2$

$Q = 1$ $\quad$ CR

another Q-barrier

WASER Mechanism

$$j \propto |\Phi|^2$$

amplitude of wave increases

$Q > 1$ $\quad$ Q-b $\quad$ Q-b

L–T ⇝ $j = -1$

S–T ⇜ $j = -(1+\epsilon)$ | ⇝ S–T $j = \epsilon$

$-(1+\epsilon)^2$ $\quad$ CR $\quad$ ⇝ S–T $j = \epsilon(1+\epsilon)$

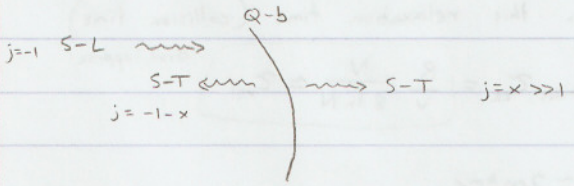
ineffective for large Q

Massive disks σ_0 large so that $Q \sim 1$

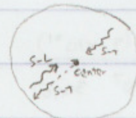
Swing (Goldreich–Lyndan–Bell)

non-WKB effect

Swing Mechanism



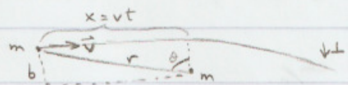
Many galaxies have no ILR $\quad \Omega - \frac{\kappa}{m} < 0$



swings through center

11/29/2006

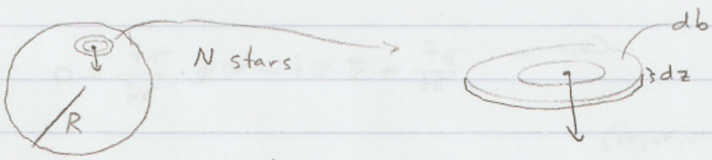
Collisions Between Stars



$$m\frac{dv_\perp}{dt} = \frac{Gm\cos\theta}{r^2}$$

$$\frac{dv_\perp}{dt} = \frac{Gmb}{(x^2+b^2)^{3/2}} \qquad \delta v_\perp = Gmb\int_{-\infty}^{\infty}\frac{dt}{(x^2+b^2)^{3/2}} = \frac{2Gmb}{v}\underbrace{\int_0^\infty \frac{dx}{(x^2+b^2)^{3/2}}}_{\left.\frac{x}{b^2\sqrt{x^2+b^2}}\right|_0^\infty} = \frac{2Gm}{vb}$$

$$\frac{\delta v_\perp}{v} = \frac{2Gm}{v^2 b} = \frac{Gm^2/b}{mv^2/2} = \frac{\Phi^{grav}}{K_{kinetic}} \ll 1$$



$2\pi b\,db$ = area

$$dN_{coll} = n_* \, dz\, 2\pi b\, db$$

density

$$\int n_* dz \simeq \frac{N}{\pi R^2} = \sigma_* \quad \text{surface density}$$

$\langle \delta v_\perp \rangle = 0$ (different directions equally contribute)

$$\frac{\langle \delta v_\perp^2\rangle}{v^2} = \left(\frac{2Gm}{v^2 b}\right)^2\left(\frac{N}{\pi R^2}\right)2\pi b\, db$$

$$= \frac{8G^2m^2}{v^4R^2}N\int_{b_{min}=\frac{2Gm}{v^2}}^{R}\frac{db}{b}$$

--- not ∞ so not diverge

closest approach $\quad \frac{mv^2}{2} \sim \frac{Gm^2}{b_{min}}$

$$b_{min} = \frac{2Gm}{v^2}$$

$$\frac{\langle \delta v_\perp^2\rangle}{v^2} = \frac{8G^2m^2}{v^4R^2}N\ln\left(\frac{Rv^2}{2Gm}\right)$$

Virial Theorem $\quad 2\left(\frac{M\langle v^2\rangle}{2}\right) - \frac{GM^2}{R} = 0 \qquad M = Nm$

Good approx.

$$\langle v^2\rangle \sim \frac{GM}{R} \qquad \frac{\langle \delta v_\perp^2\rangle}{v^2} = \frac{8\cancel{G^2}m^2N}{\cancel{G^2}(mN)^2}\ln\left(\frac{N}{2}\right) = \frac{8}{N}\ln\left(\frac{N}{2}\right) \simeq \boxed{\frac{8}{N}\ln(N) \simeq \frac{\langle \delta v_\perp^2\rangle}{v^2}}$$

crossing time $\tau_{CR} \sim \frac{R}{V}$ collision – relaxation

$N_{passes} \frac{\langle \delta v_\perp^2 \rangle}{V^2} \sim 1 \leftarrow$ 1 collision within this relaxation time (collision time)

$$\tau_{rel} = N_{passes}\, \tau_{CR} = \boxed{\frac{R}{V} \frac{N}{8 \ln N} \simeq \tau_{rel}}$$ Good approx.

Galaxy $N \sim 10^{11}$ $R \sim 20\,kpc$ $V \sim 200\,km/s$

$\tau_{CR} \sim \frac{20\,(3 \times 10^{21})}{2\times10^7\ \ 3\times10^7} \sim 10^8\,yrs$ $\tau_{rel} \sim 10^8 \frac{10^{11}}{10\ 10^2} \sim \frac{10^{19}}{10^3} \sim 10^{16}\,yrs$ for 1 collision

Globular Cluster $N \sim 10^5$ $R \sim 10\,pc$ $V \sim 1\,km/s$

$\tau_{CR} \sim \frac{3\times10^{19}}{10^5\ 3\times10^7} \sim 10^7\,yrs$ $\tau_{rel} \sim 10^7 \frac{10^5}{10(20)} \sim 5\times10^9\,yrs$ → borderline collisions

Galaxy Clusters $N \sim 10^3$ $R \sim Mpc$ $V \sim 10^8\,cm/s$

$\tau_{CR} \sim \frac{3\times10^{24}}{3\times10^7\ 10^8} \sim 10^9\,yrs$ $\tau_{rel} \sim 10^9 \frac{10^3}{10\ 10} \sim 10^{10}\,yrs$ → collisions common

→ virialized → spherical distribution

Collisionless Boltzmann Equation (Vlasov Eq.)

$\vec{w} = (\vec{x}, \vec{v})$ $\dot{\vec{x}} = \vec{v}$ $\dot{\vec{v}} = -\nabla\Phi$

$\frac{\partial f}{\partial t} + \sum_\alpha \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = 0$ Cartesian: $\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f - \nabla\Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0$

Cylindrical: $\vec{w} = (r, \varphi, z, v_r, v_\varphi, v_z)$

$$\frac{\partial f}{\partial t} + \dot{r}\frac{\partial f}{\partial r} + \dot{\varphi}\frac{\partial f}{\partial \varphi} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_r \frac{\partial f}{\partial v_r} + \dot{v}_\varphi \frac{\partial f}{\partial v_\varphi} + \dot{v}_z \frac{\partial f}{\partial v_z} = 0$$

$\dot{v}_\varphi = r\ddot{\varphi} + \dot{r}\dot{\varphi} = -\frac{v_r v_\varphi}{r} - \frac{1}{r}\frac{\partial \Phi}{\partial \varphi}$

$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = -\frac{1}{r}\frac{\partial\Phi}{\partial\varphi}$

$v_r = \dot{r}$ $v_\varphi = r\dot{\varphi}$ $v_z = \dot{z}$ $\ddot{r} - r\dot{\varphi}^2 = -\frac{\partial\Phi}{\partial r}$ $\ddot{r} = \dot{v}_r = r\dot{\varphi}^2 - \frac{\partial\Phi}{\partial r} = \frac{v_\varphi^2}{r} - \frac{\partial\Phi}{\partial r}$

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \frac{v_\varphi}{r}\frac{\partial f}{\partial \varphi} + v_z \frac{\partial f}{\partial z} + \left(\frac{v_\varphi^2}{r} - \frac{\partial\Phi}{\partial r}\right)\frac{\partial f}{\partial v_r} - \frac{1}{r}\left(v_\varphi v_r + \frac{\partial\Phi}{\partial r}\right)\frac{\partial f}{\partial v_\varphi} - \frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0$$

Spherical: $(r, \theta, \varphi, v_r, v_\theta, v_\varphi)$

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \frac{v_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial f}{\partial\varphi} + \left(\frac{v_\theta^2 + v_\varphi^2}{r} - \frac{\partial\Phi}{\partial r}\right)\frac{\partial f}{\partial v_r} + \left(\frac{v_\varphi^2 \cot\theta - v_r}{r} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)\frac{\partial f}{\partial v_\theta}$$
$$- \left(\frac{v_\varphi(v_\theta\cot\theta + v_r)}{r} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\right)\frac{\partial f}{\partial v_\varphi} = 0$$

$$\ddot{\vec{x}}_i = -\nabla\Phi = -G\sum_{\substack{j\\ i\neq j}} \frac{m_j(\vec{x}_i-\vec{x}_j)}{|\vec{x}_i-\vec{x}_j|^3}$$

Energy conserved (conservative force)

E integrals of motion $I(\vec{x}(t),\vec{v}(t))$ constants of motion

Jean's Theorem

$\frac{\partial}{\partial t}=0$ $\quad f \to f(I_\alpha(\vec{x},\vec{v}))$ — function of constants of motion

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla - \nabla\Phi\cdot\frac{\partial}{\partial\vec{v}}$$

$$\frac{df}{dt} = 0 = \underbrace{\frac{\partial f}{\partial t}}_{0} + \sum_\alpha \frac{\partial f}{\partial I_\alpha}\underbrace{\frac{dI_\alpha}{dt}}_{0}$$

0 constant of motions

$\therefore$ $f(I_\alpha)$ satisfies Boltzmann eq. $\leftarrow$ Jean's Theorem

$\rho = \int f(\vec{x},\vec{v})\,d^3v \qquad \nabla^2\Phi = 4\pi G\rho$ self-consistent problem

1-D system $\qquad \uparrow z \qquad \frac{\partial}{\partial z}\neq 0$ others $=0$

$E = \frac{v_z^2}{2} + \Phi(z) \qquad f(z,v_z) = f(E)$

$$v\frac{\partial f}{\partial z} - \frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v} = 0$$

$$v\frac{\partial f}{\partial E}\frac{\partial E}{\partial t} - \frac{\partial\phi}{\partial z}\frac{\partial f}{\partial E}\frac{\partial E}{\partial v}$$

$$v\frac{\partial f}{\partial E}\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial z}\frac{\partial f}{\partial E}v = 0$$

$$f(E) = \frac{\rho_0}{\sqrt{2\pi\sigma^2}}e^{-E/\sigma^2} = \frac{\rho_0}{\sqrt{2\pi\sigma^2}}e^{-\frac{v^2}{2\sigma^2}-\frac{\Phi}{\sigma^2}}$$

σ = velocity dispersion (non-ordered "thermal" motion)
↳ term not used in astro ($\sigma^2 \sim \frac{kT}{m}$)

$$\rho = \frac{\rho_0}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{v^2}{2\sigma^2}}e^{-\frac{\Phi}{\sigma^2}}dv = \rho_0 e^{-\frac{\Phi}{\sigma^2}}$$

$$\frac{d^2\Phi}{dz^2} = 4\pi G\rho_0 e^{-\frac{\Phi}{\sigma^2}} \qquad 2\psi = \Phi/\sigma^2$$

$$\underbrace{\left(\frac{2\sigma^2}{4\pi G\rho_0}\right)}_{\lambda^2}\frac{\partial^2\psi}{\partial z^2} = e^{-2\psi} \qquad \xi = z/\lambda$$

$$\frac{d^2\psi}{d\xi^2} = e^{-2\psi}$$

B.C.s $\quad \psi(0)=0 \quad \psi'(0)=0$

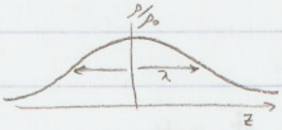
$$\frac{d\psi}{d\xi}\frac{d^2\psi}{d\xi^2} = e^{-2\psi}\frac{d\psi}{d\xi} = \frac{d}{d\xi}\left(\frac{1}{2}\left(\frac{d\psi}{d\xi}\right)^2\right) \qquad \int_{\xi=0}^{\xi} d\left(\frac{1}{2}\left(\frac{d\psi}{d\xi}\right)^2\right) = \int_0^\psi e^{-2\psi}d\psi$$

$$\frac{1}{2}\left(\frac{d\psi}{d\xi}\right)^2 = \frac{1}{2}\left(1-e^{-2\psi}\right) \qquad \frac{d\psi}{d\xi} = \sqrt{1-e^{-2\psi}} \qquad \int_0^\psi \frac{d\psi}{\sqrt{1-e^{-2\psi}}} = \int_0^\xi d\xi$$

$u = e^{-\psi} \qquad \psi = \ln(1/u) = -\ln u \qquad d\psi = -\frac{du}{u}$

$$\int_1^u \frac{-du}{u\sqrt{1-u^2}} = \pm\xi \qquad -\operatorname{sech}^{-1}u\Big|_1^u \qquad u = \operatorname{sech}(\xi) \qquad \rho = \rho_0 e^{-2\psi} = \rho_0 u^2$$

$\rho = \rho_0 \,\mathrm{sech}^2\left(\frac{z}{\lambda}\right)$ $\quad \psi = \ln\left(\cosh\left(\frac{z}{\lambda}\right)\right)$ $\quad \Phi = 2\sigma^2 \ln\left(\cosh\left(\frac{z}{\lambda}\right)\right)$



$\lambda^2 = \frac{2\sigma^2}{4\pi G \rho_0}$ OK for disks of spiral galaxies

12/4/2006

Final
1 Blastwaves / Supernova
2 Winds
3 Accretion Disk
4 Kelvin-Helmoltz Instability
5 Stuff now

Example in spherical

steady state $\frac{\partial}{\partial t} \to 0$

$E = \frac{v^2}{2} + \Phi$ constant of motion

spherical symmetry $\Phi(r)$ no θ, φ dependance in φ

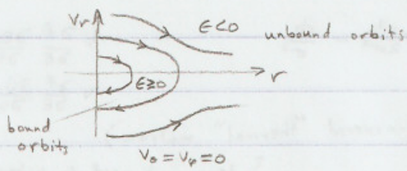
$f = f(E)$ $\quad \frac{\partial f}{\partial (\,)} = \frac{\partial f}{\partial E}\frac{\partial E}{\partial (\,)}$

$v_r \frac{\partial f}{\partial r} = v_r \frac{\partial \Phi}{\partial r}$ $\quad -\frac{\partial \Phi}{\partial r}\frac{\partial f}{\partial v_r} = -\frac{\partial \Phi}{\partial r} v_r$ etc. all terms cancel

$\nabla^2 \Phi = 4\pi G \rho = 4\pi G \int f(E)\, d^3v$ Integrodifferential Eq.

$E = -\epsilon$ $\quad \Phi = -\psi$ (bound systems have negative energy)

$\epsilon = \psi - \frac{v^2}{2}$



with no collisions we expect no stars in unbound system (after some time)

Plummer

$$f(\epsilon) = \begin{cases} F\,\epsilon^{n-3/2} & \epsilon > 0 \\ 0 & \epsilon < 0 \end{cases} \qquad n \geq \tfrac{1}{2}$$

$\rho = \int f(\epsilon)\, d^3v = 4\pi \int_0^{\sqrt{2\psi}} f(\epsilon)\, v^2\, dv$ $\quad \frac{v^2}{2} = \psi - \epsilon$ $\quad v_{max} = \sqrt{2\psi}$

$v\,dv = -d\epsilon$ $\quad v = \sqrt{2(\psi - \epsilon)}$ $\quad \therefore \rho = 4\pi F \int_0^{\psi} \sqrt{2(\psi-\epsilon)}\, \epsilon^{n-3/2}\, d\epsilon$

$\epsilon = \psi x$ $\quad d\epsilon = \psi\, dx$

$$4\pi F \sqrt{2} \int_0^1 \sqrt{\psi}\sqrt{1-x}\, \psi^{n-3/2} x^{n-3/2}\, \psi\, dx = 4\pi F\sqrt{2}\, \psi^n \underbrace{\int_0^1 x^{n-3/2}\sqrt{1-x}\, dx}_{C_n \text{ a number}}$$

$\therefore \boxed{\rho = 4\pi F \sqrt{2}\, C_n\, \psi^n}$

Hydrostatic Equil $\quad \frac{\partial P}{\partial r} = -\rho \frac{\partial \Phi}{\partial r} = \rho \frac{\partial \psi}{\partial r}$

Polytrope $P = k\rho^\gamma$ $\quad k\gamma\rho^{\gamma-1}\frac{d\rho}{dr} = \rho\frac{\partial\psi}{\partial r}$ $\quad \rho^{\gamma-2}d\rho \propto d\psi$

$$\rho^{\gamma-1} \propto \psi \quad \rho \propto \psi^{\frac{1}{\gamma-1}}$$

$\therefore \frac{1}{\gamma-1} = n \quad \gamma = 1 + \frac{1}{n}$

$n > \frac{1}{2} \quad \gamma < 3$ (if $\gamma = 3$ gas is incompressible)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \underset{\downarrow \; b^2\partial\tilde{\psi}/\partial r}{\frac{\partial\psi}{\partial r}}\right) = -(4\pi)^2 G F\sqrt{2}\, C_n \underset{\downarrow \; \tilde{\psi}^n\psi_0^{n-1}b^2}{\psi^n} \qquad \tilde{\psi} = \psi/\psi_0 \qquad s = r/b$$

$$\frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2\frac{\partial\tilde{\psi}}{\partial s}\right) = -\tilde{\psi}^n \quad \tilde{\psi} > 0$$
$$= 0 \quad \tilde{\psi} < 0$$

Lane-Emden Eq.

Analytical solns for $n = 1, 5, \infty$

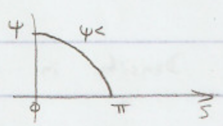
$n=1 \quad \tilde{\psi}(0) = 0 \quad \tilde{\psi}'(0) = 0$

$\psi = \frac{u}{s} \quad \psi' = \frac{u'}{s} - \frac{u}{s^2} \quad s^2\psi' = su' - u \quad (s^2\psi')' = su''$

$\frac{(s^2\psi')'}{s^2} = -\frac{u}{s} \quad u'' + u = 0 \quad \psi^< = \frac{\sin s}{s} \quad \psi^{<\prime} = \frac{\cos s}{s} - \frac{\sin s}{s^2}$

this satisfies bc.

$\psi^<(s=\pi) = 0 \quad \rho = 0$ stellar distribution ends



$s^2\frac{\partial\psi^>}{\partial s} = a \quad \psi^> = -\frac{a}{s} + b$

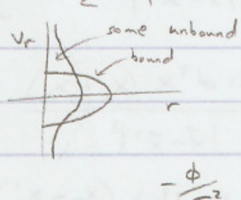
$\frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2\frac{\partial\psi}{\partial s}\right) = 0 \quad \psi^{>\prime} = \frac{a}{s^2} \quad \frac{a}{\pi^2} = \psi^{<\prime}(\pi) \Rightarrow a = -\pi$

$\psi^>(\pi) = \psi^<(\pi)$: $0 = -\frac{a}{\pi} + b \Rightarrow b = -1 \quad \therefore \psi^> = \pi/s - 1 \quad \sim \frac{1}{r}$ potential

---

Spherical Symm $\phi(r)$ $\quad E = \frac{v^2}{2} + \phi$

$f(E) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} e^{-E/\sigma^2}$



$$\rho = \frac{4\pi\rho_0}{(2\pi\sigma^2)^{3/2}}\int_0^\infty v^2 e^{\frac{-v^2}{2\sigma^2} - \frac{\phi}{\sigma^2}}dv = \rho_0 e^{-\phi/\sigma^2}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 4\pi G\rho_0 e^{-\phi/\sigma^2}$$

$\frac{\partial P}{\partial r} = -\rho\frac{\partial\phi}{\partial r}$ $\quad$ isothermal, T = constant $\quad \frac{kT}{m}\frac{\partial\rho}{\partial r} = -\rho\frac{\partial\phi}{\partial r} \quad \frac{d\rho}{\rho} = -\frac{m}{kT}d\phi$

$$\rho = \rho_0 e^{-\frac{m\phi}{kT}} \qquad \sigma^2 = \frac{kT}{m}$$

$$\left(\frac{\sigma^2}{\rho}\frac{d\rho}{dr} = -\frac{d\phi}{dr}\right) r^2 \qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{\sigma^2}{\rho}\frac{d\rho}{dr}\right) = -\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -4\pi G\rho$$

$$\rho = \rho_0\left(\frac{r}{r_0}\right)^\alpha \qquad \frac{\sigma^2}{r^2}\frac{d}{dr}\left(r^2\frac{\alpha}{r}\right) = -4\pi G\rho_0\left(\frac{r}{r_0}\right)^\alpha$$

$$\frac{\alpha\sigma^2}{r^2} = -4\pi G\rho_0\left(\frac{r}{r_0}\right)^\alpha \qquad \alpha = -2 \qquad 4\pi G\rho_0 r_0^2 = 2\sigma^2 \qquad \rho_0 r_0^2 = \frac{\sigma^2}{2\pi G}$$

$$\therefore \boxed{\rho = \rho_0\left(\frac{r_0}{r}\right)^2} \qquad \text{Isothermal Sphere}$$

$$M(r) = 4\pi\int_0^r \rho r^2 dr = 4\pi\rho_0 r_0^2 r$$

Concludes Final Material

Jean's Equations

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f - \nabla\phi\cdot\frac{\partial f}{\partial\vec{v}} = 0$$ Stars satisfy this eq

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho\bar{v}_i) = 0 \qquad \frac{\partial}{\partial t}(\rho\bar{v}_j) + \frac{\partial}{\partial x_i}(\rho\overline{v_iv_j}) = -\rho\frac{\partial\phi}{\partial x_j}$$

$$\rho\bar{v}_i\bar{v}_j + \rho\sigma_{ij}^2$$

Oort 1930s

$$\frac{\partial P}{\partial z} = -\rho\frac{\partial\phi}{\partial z}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial P}{\partial z}\right) = -\frac{\partial^2\phi}{\partial z^2} = -4\pi G\rho$$

$$P = \rho\sigma^2$$

Density in plane of galaxy $\rho \sim 0.15\ M_\odot/pc^3$

Stellar density $\rho_* \sim \frac{1}{2}\rho$

$$\int(\text{Momentum } x_k)\, d^3x$$

$$W_{jk} = -\int d^3x\,\rho\, x_k\frac{\partial}{\partial x_j}\int\frac{-G\rho(x')\,d^3x'}{|\vec{x}-\vec{x}'|} = -\int d^3x\,\rho(x)\,x_k\int\frac{\rho(x')\,d^3x'(x_j-x_j')}{|\vec{x}-\vec{x}'|^3}G$$

$$= -G\iint d^3x'\,\rho(x')\,x_k'\,\frac{\rho(x)\,d^3x\,(x_j'-x_j)}{|\vec{x}-\vec{x}'|^3}$$ Add

$$2W_{jk} = -G\iint d^3x\,\rho(x)\,\rho(x')\,d^3x'\,\frac{(x_k-x_k')(x_j-x_j')}{|\vec{x}-\vec{x}'|^3} \implies W_{jk} = W_{kj}$$

grav. PE is symmetric

$$\mathrm{Tr}(W) = \frac{1}{2}\int\rho(x)\,d^3x\,\phi(x) \qquad \text{total grav. PE}$$

$$\int d^3x \, x_k \frac{\partial}{\partial x_i}\left(\rho \overline{v_i v_j}\right) \overset{\text{integrate by parts}}{=} -\int d^3x \, \rho \, \overline{v_k v_j} = -2K_{jk} \quad \text{Also symmetric}$$

$$\int d^3x \, x_k \frac{\partial}{\partial t}\left(\rho \bar{v}_j\right)$$

Moment of inertia tensor $I_{jk} = \int \rho \, x_j x_k \, d^3x$

$$\frac{\partial I_{jk}}{\partial t} = \int \frac{\partial \rho}{\partial t} x_j x_k d^3x \overset{\text{(continuity)}}{=} -\int \frac{\partial}{\partial x_i}\left(\rho \bar{v}_i\right) x_j x_k d^3x \quad \begin{array}{c}\text{Integrate}\\ \text{by parts}\end{array}$$

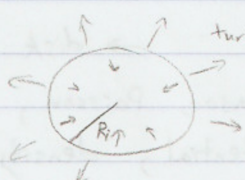
$$= \int \rho \bar{v}_i \left(\delta_{ij} x_k + x_j \delta_{ik}\right) = \int \rho \bar{v}_j x_k + \rho \bar{v}_k x_j$$

$$\frac{\partial^2 I_{jk}}{\partial t^2} = \int \left(\underbrace{x_k \frac{\partial}{\partial t}\left(\rho \bar{v}_j\right)}_{= W_{jk} + 2K_{jk}} + x_j \frac{\partial}{\partial t}\left(\rho \bar{v}_k\right) \right)$$

$$\boxed{\frac{1}{2}\frac{\partial^2 I_{jk}}{\partial t^2} = W_{jk} + 2K_{jk}} \quad \text{Tensor Virial Theorem}$$

Scalar $\partial/\partial t = 0$ $\quad Tr(W_{jk}) + 2\,Tr(K_{jk}) = 0 \qquad 2\bar{K} + \bar{W} = 0$

Perturbation in expanding universe



turn-around point → gravity stops local expansion

$$E = -\tfrac{3}{5}\frac{GM^2}{R_i} = \bar{K} + \bar{W}$$

Virialized $\quad 2\bar{K} + \bar{W} = 0$

$$E = \tfrac{1}{2}\bar{W}_{final} = -\tfrac{1}{2}\,\tfrac{3}{5}\frac{GM}{R_f} \qquad R_f = \tfrac{1}{2}R_0$$

<u>Violent relaxation</u> $\quad f(E) = e^{-E/\sigma^2} \Rightarrow \rho = \rho_0\left(\frac{r_0}{r}\right)^2$ cold dark matter

Final Material 12/6/2006

1) Supernova blast waves
   spherical <u>inflowing</u> wind, asymptotic solutions
   Rankine-Hugoniot relations, solve them for limits
   Solve for time evolution of supernova

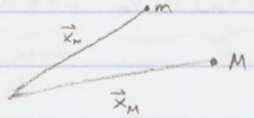
2) Stellar winds → Jets (not 4π steradian) derive then Solve wind eqns
   wind equation, critical points, sketch of solutions

3) Accretion Disk
   → class notes given: local fluid dynamic eq, and answers of
   Construct macroscopic equations ($\int_{-H}^{H} dz$ etc)
   Solve for a simple model of viscosity (inner region)

4) Kelvin-Helmholtz → class notes
   incompressible derive eq for perturbed pressure and solve
   Find, solve dispersion relation

5) Stellar dynamics → disk
   given: 1-D Vlasov, Poisson, distribution function
   Solve for potential, density

Collisions



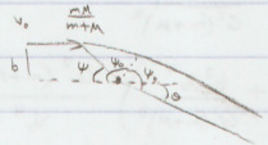
$\vec{r} = \vec{x}_m - \vec{x}_M$ $\qquad$ $\vec{v} = \vec{v}_m - \vec{v}_M$ $\qquad$ $\Delta\vec{V}_{CM} = 0$

$\Delta\vec{v} = \Delta\vec{v}_m - \Delta\vec{v}_M$

$m\Delta\vec{v}_m + M\Delta\vec{v}_M = 0$

$$\boxed{\Delta\vec{v}_M = -\frac{m}{m+M}\Delta\vec{v}}$$

1 body



$\boxed{2\psi_0 = \pi + \theta}$ $\qquad$ $\frac{mM}{m+M}\ddot{\vec{r}} = -\frac{GmM}{r^2}\hat{r}$

$\ddot{r} - r\dot{\psi}^2 = -\frac{G(m+M)}{r^2}$

$r\ddot{\psi} + 2\dot{r}\dot{\psi} = 0$ $\qquad$ $L = r^2\dot{\psi}$

$\dot{r} = \dot{\psi}\frac{dr}{d\psi} = \frac{L}{r^2}\frac{dr}{d\psi}$ $\qquad$ $r = 1/u$ $\qquad$ $= \frac{-L^2}{r^2u^2}\frac{du}{d\psi}$

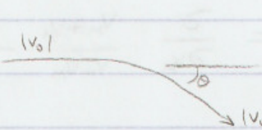
$\therefore \quad \frac{-L^2}{r^2}\frac{d^2u}{d\psi^2} - \frac{L^2}{r^3} = -\frac{G(m+M)}{r^2}$ $\qquad$ $\frac{d^2u}{d\psi^2} + u = \frac{G(M+m)}{L^2 = b^2v_0^2}$

$\frac{1}{r} = a\cos(\psi - \psi_0) + \frac{G(m+M)}{b^2v_0^2}$ $\qquad$ $r \to \infty \quad \psi \to 0$

$a\cos\psi_0 = -\frac{G(m+M)}{b^2v_0^2}$ $\qquad$ $-\frac{\dot{r}}{r^2} = a\frac{L}{r^2}\sin(\psi - \psi_0)$ $\qquad$ $\dot{r} \to -v_0$

$$\boxed{\tan\psi_0 = -\frac{bv_0^2}{G(m+M)}}$$ $\qquad$ $a\sin\psi_0 = 1/b$

$\perp v_0$



$(\Delta v)_\perp = v_0\sin\theta$

$= v_0\sin(2\psi_0 - \pi)$

$= -v_0\sin(2\psi_0) = -2v_0\sin\psi_0\cos\psi_0 = -v_0\, 2\frac{\tan\psi_0}{1+\tan^2\psi_0}$

$\langle \Delta v_\perp \rangle = 0$

$\Delta v_\parallel = v_0\cos\theta - v_0 = v_0(\cos(2\psi_0 - \pi) - 1) = -v_0(\cos 2\psi_0 + 1) = -2v_0\cos^2\psi_0$

$\Delta v_\parallel = -\frac{2v_0}{1+\tan^2\psi_0}$

$\Delta\vec{v}_{M_\parallel} = \frac{2m}{m+M}\frac{\vec{v}_0}{(1+\tan^2\psi_0)}$

Flux of incoming stars

$$= \vec{v}_0 f(\vec{v}_m)\, d^3 v_m \qquad \frac{\#\text{ collisions}}{\text{sec}} = \underbrace{\vec{v}_0 f(\vec{v}_m)\, d^3 v_m}_{\text{flux}}\ \underbrace{2\pi b\, db}_{\text{area}}$$

$$\frac{d\vec{v}_M}{dt} = \frac{2m}{m+M} \iint \frac{v_0}{1+\tan^2\psi_0}\, \vec{v}_0 f(\vec{v}_m)\, d^3 v_m\, 2\pi b\, db$$

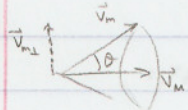
$$\left\langle \frac{d\vec{v}_M}{dt} \right\rangle = \frac{4\pi m}{m+M} \int d^3 v_m\, v_0 \vec{v}_0 f(\vec{v}_m) \int_0^{b_{max}\sim R} \frac{b\, db}{1+\frac{b^2 v_0^4}{G^2(m+M)^2}}$$

$$\frac{1}{2}\ln\left(1+\frac{b_{max}^2 v_0^4}{G^2(m+M)^2}\right)\frac{G^2(m+M)^2}{v_0^4}$$

$$\Lambda = \frac{b_m v_0^2}{G(m+M)} \qquad \frac{1}{2}\ln(1+\Lambda^2) \simeq \frac{1}{2}\ln\Lambda^2 = \ln\Lambda$$

$$= 4\pi G^2 m(m+M) \int \frac{v_m^2\, dv_m \sin\theta\, d\theta\, d\varphi\, \ln\Lambda}{v_0^3}\ \vec{v}_0\, f(v_m)$$

$\vec{v}_0 \to \vec{v}_m - \vec{v}_M$; $f(v_m)$ assumed to be isotropic



perpendicular pieces go away

$$\left\langle \frac{d\vec{v}_M|_{\parallel \vec{v}_M}}{dt} \right\rangle = 4\pi(2\pi)G^2 m(m+M) \int v_m^2\, dv_m\, f(v_m) \int_0^{\pi} \frac{\sin\theta\, d\theta\, (v_m\cos\theta - v_M)}{(v_m^2 + v_M^2 - 2 v_m v_M \cos\theta)^{3/2}}\ \overline{\ln\Lambda}$$

$\ln\Lambda$ depends on v_m, θ so just use average

$$\frac{\partial}{\partial v_M} \int_0^{\pi} \frac{\sin\theta\, d\theta}{(\ \ )^{1/2}}$$

$$\frac{\partial}{\partial v_M}\left(- \left.\frac{\sqrt{v_m^2 + v_M^2 - 2 v_m v_M x}}{v_m v_M}\right|_{x=-1}^{+1} \right) = \frac{\partial}{\partial v_M}\ \frac{v_m + v_M}{v_m v_M} - \begin{cases} \dfrac{v_m - v_M}{v_m v_M} & v_m > v_M \\[2ex] \dfrac{v_M - v_m}{v_m v_M} & v_m < v_M \end{cases}$$

$$= \begin{cases} 0 & v_m > v_M \\ -\dfrac{2}{v_M^2} & v_m < v_M \end{cases}$$

constant, not part of integral

$$= -\frac{(4\pi)^2 G^2 m(m+M)}{v_M^2} \int_0^{v_M} v_m^2\, dv_m\, f(v_m)\ \overline{\ln\Lambda}$$

$$f(v_m) = \frac{n}{(2\pi\sigma^2)^{3/2}}\, e^{-\frac{v_m^2}{2\sigma^2}} \quad \text{Maxwellian}$$

$$\left\langle \frac{dv_M}{dt} \right\rangle = -\frac{4\pi G^2 m(m+M)}{v_M^2}\ \overline{\ln\Lambda}\ n\left(\text{erf}(x) - \frac{2}{\sqrt{\pi}}\, x\, e^{-x^2}\right) \qquad x = \frac{v_M}{\sqrt{2\sigma^2}}$$

drag coefficient, dynamic friction, slowing down rate

Appears in Fokker-Planck Equation $\quad \left\langle \frac{\Delta\vec{v}}{\Delta t} \right\rangle = \frac{\partial}{\partial \vec{v}} \cdot \vec{D}$

$$\left\langle \frac{dv_M}{dt} \right\rangle = -\frac{v_M}{\tau_{rel}} \sim \frac{4\pi G^2 2 m^2 n \ln\Lambda}{v_M^2} \quad (m = M)$$

relaxation time

$$\tau_{rel} \sim \frac{v_m^3}{2(4\pi) G^2 m^2 n \ln\Lambda} \frac{v_m}{R} \underbrace{\left(\frac{R}{v_m}\right)}_{\tau_{CR}}$$

τ_{CR} crossing time

virialized $\quad v^2 = \frac{GM}{R} = \frac{GNm}{R}$

$$\Lambda = \frac{Rv^2}{G 2m} = \frac{N}{2}$$

$$\tau_{rel} \sim \frac{N^2 \tau_{CR}}{2 \underbrace{\frac{(4\pi)}{3} n R^3}_{N} \ln\Lambda (3)} = \frac{N}{6 \ln\Lambda} \tau_{CR}$$

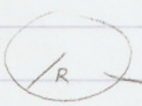
$$\therefore \quad \tau_{rel} \sim \frac{N}{6 \ln N} \tau_{CR}$$

Virialized star collection

$E = \bar{K} + \bar{W} \qquad 2\bar{K} + \bar{W} = 0$

$E = -\bar{K} = -N \left\langle \frac{mv^2}{2} \right\rangle = -\frac{3}{2} NkT$ bound

specific heat $\quad c = \frac{\partial E}{\partial T} = -\frac{3}{2} Nk$



heat bath $\quad dQ = -|dQ|$

$dQ = c\,dT \qquad -|dQ| = -|c|\,dT \quad \therefore \; dT > 0$

extract heat $\rightarrow$ T goes up

extract heat by taking out star $\rightarrow$ stars move faster (collapse, $R\downarrow$) $\rightarrow$ mean kinetic energy goes up

$\rightarrow$ Gravothermal Instability

$v_{esc}^2 = \frac{2GM}{R} \qquad \langle v^2 \rangle = \frac{3}{5} \frac{GM}{R} \qquad v_{esc}^2 \sim \frac{10}{3} \langle v^2 \rangle$

cluster should evaporate $\sim 10^2 \tau_{rel}$

as cluster collapses, $\langle v \rangle \uparrow$ hard (close) collisions 'core collapse'