HW 5 Solutions

Stat 61

Problem 1

(a)

Let n be the sample size, and let $X_1, \ldots, X_n$ be independent identically distributed random variables with the same probability distribution function (the one described in the exercise).

To find the MLE of θ, we first define the likelihood function:

$$\operatorname{lik}(\theta) = f(x_1, \ldots, x_n \mid \theta) = f(x_1 \mid \theta) \cdots f(x_n \mid \theta)$$

$$= P(X = x_1 \mid \theta) \cdots P(X = x_n \mid \theta).$$

Substituting $n = 10$, given values of X, and the definition of the probability distribution function of X yields

$$\operatorname{lik}(\theta) = P(X = 0 \mid \theta)^2 \cdot P(X = 1 \mid \theta)^3 \cdot P(X = 2 \mid \theta)^3 \cdot P(X = 3 \mid \theta)^2$$

$$= \left(\frac{2}{3} \cdot \theta\right)^2 \cdot \left(\frac{1}{3} \cdot \theta\right)^3 \cdot \left(\frac{2}{3} \cdot (1 - \theta)\right)^3 \cdot \left(\frac{1}{3} \cdot (1 - \theta)\right)^2,$$

since the values 0 and 3 appeared two times in the sample while values 1 and 2 appeared three times.

It's easier to work with the natural logarithm of the given expression, so we define

$$l(\theta) = \ln(\operatorname{lik}(\theta))$$

$$= 2 \cdot \ln\left(\frac{2}{3} \cdot \theta\right) + 3 \cdot \ln\left(\frac{1}{3} \cdot \theta\right) + 3 \cdot \ln\left(\frac{2}{3} \cdot (1 - \theta)\right) + 2 \cdot \ln\left(\frac{1}{3} \cdot (1 - \theta)\right),$$

and we need to find its global maximum on the interval $[0, 1]$ (where θ can take on values).

The derivative of l is

$$l'(\theta) = \frac{2}{\theta} + \frac{3}{\theta} - \frac{3}{1 - \theta} - \frac{2}{1 - \theta} = \frac{5 - 10 \cdot \theta}{\theta \cdot (1 - \theta)}.$$

Stationary points are the null points of the above derivative, so

$$l'(\theta) = 0 \iff \frac{5 - 10 \cdot \theta}{\theta \cdot (1 - \theta)} = 0 \iff 5 - 10 \cdot \theta = 0 \iff \theta = \frac{1}{2}.$$

Problem 1

(b)

Remember that the posterior density can be found as

$$f_{\Theta|X}(\theta \mid x) = \frac{f_{X,\Theta}(x,\theta)}{f_X(x)} = \frac{f_{X|\Theta}(x \mid \theta) \cdot f_\Theta(\theta)}{f_X(x)}\,. \tag{1}$$

Here, X denotes the observed sample $(X_1, \ldots, X_{10})$. The density $f_{X|\Theta}(x \mid \theta)$ was found in the *c*) part (it's actually the likelihood function), and we'll repeat it here:

$$f_{X|\Theta}(x \mid \theta) = \left(\frac{2}{3} \cdot \theta\right)^2 \cdot \left(\frac{1}{3} \cdot \theta\right)^3 \cdot \left(\frac{2}{3} \cdot (1-\theta)\right)^3 \cdot \left(\frac{1}{3} \cdot (1-\theta)\right)^2$$

$$= \frac{2^5}{3^{10}} \cdot \theta^5 \cdot (1-\theta)^5\,.$$

So, we only need to find the unconditional density of X, but that shouldn't be a problem, since we know that

$$f_X(x) = \int_{\mathbb{R}} f_{X|\Theta}(x \mid \theta) \cdot f_\Theta(\theta)\, d\theta = \int_0^1 f_{X|\Theta}(x \mid \theta)\,,$$

since the density function of Θ is

$$f_\Theta(\theta) = 1, \quad \theta \in [0,1].$$

Instead of direct evaluating of this dreadful looking integral, we can remember the **beta** density, and the fact that

$$\int_0^1 x^{a-1} \cdot (1-x)^{b-1}\, dx = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)},$$

where a and b are some positive real numbers.

Also, if a is a non-negative integer, then remember that

$$\Gamma(a) = (a-1)!\,.$$

Solution source: QuizletPlus

Applied to our case, this yields that

$$f_X(x) = \frac{2^5}{3^{10}} \cdot \int_0^1 \theta^5 \cdot (1-\theta)^5 \, d\theta = \frac{2^5}{3^{10}} \cdot \frac{\Gamma(6)\cdot\Gamma(6)}{\Gamma(12)} = \frac{2^5}{3^{10}} \cdot \frac{5!\cdot 5!}{11!}$$

$$= \frac{8}{40,920,957} \approx 1.9 \cdot 10^{-7} \,.$$

Finally, substituting into (1) and using the fact that $f_\Theta(\theta) = 1$, for any $\theta \in [0,1]$, gives us the desired posterior density:

$$f_{\Theta|X}(\theta \mid x) = \frac{\frac{2^5}{3^{10}} \cdot \theta^5 \cdot (1-\theta)^5 \cdot 1}{\frac{8}{40{,}920{,}957}} = \boxed{2772 \cdot \theta^5 \cdot (1-\theta)^5}\,.$$

We may recognize this as a beta density with the parameters $a = 6$ and $b = 6$.

Solution source: QuizletPlus

Problem 2

To find the method of moments estimate of θ, we should start by finding the expected value of X, which is a continuous random variable, with the density function as given in the exercise.

Using the definition of the expected value of a continuous random variable, we have that

$$E(X) = \int_{\theta}^{+\infty} x \cdot f(x)\,dx = \int_{\theta}^{+\infty} x \cdot e^{-(x-\theta)}\,dx = \begin{vmatrix} u = x & dv = e^{-(x-\theta)}\,dx \\ du = dx & v = -e^{-(x-\theta)} \end{vmatrix}$$

$$= \underbrace{-x \cdot e^{-(x-\theta)}\Big|_{\theta}^{+\infty}}_{=\theta} + \int_{\theta}^{+\infty} e^{-(x-\theta)}\,dx = \theta - e^{-(x-\theta)}\Big|_{\theta}^{+\infty} = \theta + 1\,.$$

So, from the above expression we can express θ as

$$\theta = E(X) - 1\,.$$

Now, the method of moments simply suggests writing the sample mean in place of $E(X)$, and that would be the method of moments estimate of θ.

Therefore, our desired estimate is

$$\boxed{\hat{\theta} = \overline{X} - 1}\,.$$

Solution source: QuizletPlus

Problem 3

Let n be the sample size, and let $X_1, \ldots, X_n$ be independent identically distributed random variables with the same density function (the one described in the exercise).

To find the MLE of θ, we first define the likelihood function:

$$\operatorname{lik}(\theta) = f(x_1, \ldots, x_n \mid \theta) = f(x_1 \mid \theta) \cdots f(x_n \mid \theta).$$

Substituting the definition of the density function of X yields

$$\operatorname{lik}(\theta) = e^{-(x_1-\theta)} \cdot e^{-(x_2-\theta)} \cdots e^{-(x_n-\theta)} = e^{n\theta-(x_1+\cdots x_n)}.$$

It's easier to work with the natural logarithm of the given expression, so we define

$$l(\theta) = \ln(\operatorname{lik}(\theta)) = n \cdot \theta - \sum_{i=1}^{n} x_i .$$

Notice that this log-likelihood is defined this way only when $\theta \leq x_1, \ldots, x_n$, because the density f is non-zero only when $\theta \leq x$ (and is 0 otherwise). If θ is greater than some x_i, then the log-likelihood is not defined, because the likelihood is 0.

So, we can notice that the log-likelihood is an affine function (i.e. its graph is a straight line), which is strictly increasing, since the coefficient next to θ is n, which is always positive.

Therefore, the log-likelihood is the greatest when θ is the greatest possible, but with the constraint that $\theta \leq x_1, \ldots, x_n$.

So, if we set that θ is the minimum of those n x_i's, then it should be clear that at that point the log-likelihood is the greatest.

In other words, we have that

$$\boxed{\tilde{\theta} = X_{(1)}}$$

is the MLE for θ, where $X_{(1)}$ denotes the minimum of the sample, i.e. $X_{(1)} = \min_{1 \leq i \leq n} X_i$.

Solution source: QuizletPlus