HW 1 Solutions

Stat 61

Problem 1

(b) The marginal density of X is the joint pdf integrated over all possible y-values.

$$f_X(x) = \int_0^1 \frac{6}{7}(x+y)^2 dy$$
$$= \left(\frac{2}{7}(x+y)^3\right)\Big|_0^1$$
$$= \frac{2}{7}(x+1)^3 - \frac{2}{7}x^3$$
$$= \frac{6}{7}x^2 + \frac{6}{7}x + \frac{2}{7}$$

The marginal density of Y is the joint pdf integrated over all possible x-values.

$$f_Y(y) = \int_0^1 \frac{6}{7}(x+y)^2 dx$$
$$= \left(\frac{2}{7}(x+y)^3\right)\Big|_0^1$$
$$= \frac{2}{7}(y+1)^3 - \frac{2}{7}y^3$$
$$= \frac{6}{7}y^2 + \frac{6}{7}y + \frac{2}{7}$$

(c) The conditional distribution of Y given X is the joint pdf divided by the marginal distribution of X:

$$f(y|X=x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{6}{7}(x+y)^2}{\frac{6}{7}x^2 + \frac{6}{7}x + \frac{2}{7}} = \frac{3(x+y)^2}{3x^2+3x+1}$$

The conditional distribution of X given Y is the joint pdf divided by the marginal distribution of Y:

$$f(x|Y=y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{6}{7}(x+y)^2}{\frac{6}{7}y^2 + \frac{6}{7}y + \frac{2}{7}} = \frac{3(x+y)^2}{3y^2+3y+1}$$

Problem 2

Given:

$$F(x) = 1 - x^{-\alpha}$$

$$x \geq 1$$

The probability density function is the derivative of the cdf:

$$f(x) = F'(x) = \frac{d}{dx}(1 - x^{-\alpha}) = -(-\alpha)x^{-\alpha-1} = \alpha x^{-\alpha-1}$$

(a) The expected value is the sum/integral of the product of each possibility x with its probability $f(x)$:

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_{1}^{+\infty} x \alpha x^{-\alpha-1} dx = \int_{1}^{+\infty} \alpha x^{-\alpha} dx$$

$$= \left(\frac{\alpha x^{-\alpha+1}}{-\alpha+1} \right) \Bigg|_{1}^{+\infty} = -\frac{\alpha(1^{-\alpha+1})}{-\alpha+1} = \frac{\alpha}{\alpha-1}$$

Problem 3

Given:

$$E(X) = \mu$$

$$Var(X) = \sigma^2$$

$$Z = \frac{X - \mu}{\sigma}$$

To proof: $E(Z) = 0$ and $Var(Z) = 1$

Proof: For the linear combination $W = aX + b$, the mean and variance have the following properties:

$$E(W) = aE(X) + b$$

$$Var(W) = a^2 Var(X)$$

Use these properties with $a = \frac{1}{\sigma}$ and $b = -\frac{\mu}{\sigma}$ to determine the mean and variance:

$$E(Z) = E\left(\frac{X - \mu}{\sigma}\right) = E\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right) = \frac{1}{\sigma}E(X) - \frac{\mu}{\sigma} = \frac{1}{\sigma}(\mu) - \frac{\mu}{\sigma} = 0$$

$$Var(Z) = Var\left(\frac{X - \mu}{\sigma}\right) = Var\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right) = \left(\frac{1}{\sigma}\right)^2 Var(X) = \frac{1}{\sigma^2}(\sigma^2) = 1$$

$\square$