# **HW 1 Solutions**

## Stat 61

#### Problem 1

(b) The marginal density of X is the joint pdf integrated over all possible y-values.

$$egin{align} f_X(x) &= \int_0^1 rac{6}{7} (x+y)^2 dy \ &= \left. \left(rac{2}{7} (x+y)^3
ight)
ight|_0^1 \ &= rac{2}{7} (x+1)^3 - rac{2}{7} x^3 \ &= rac{6}{7} x^2 + rac{6}{7} x + rac{2}{7} \end{aligned}$$

The marginal density of Y is the joint pdf integrated over all possible x-values.

$$egin{align} f_Y(y) &= \int_0^1 rac{6}{7} (x+y)^2 dx \ &= \left(rac{2}{7} (x+y)^3
ight)igg|_0^1 \ &= rac{2}{7} (y+1)^3 - rac{2}{7} y^3 \ &= rac{6}{7} y^2 + rac{6}{7} y + rac{2}{7} \end{aligned}$$

(c) The conditional distribution of Y given X is the joint pdf divided by the marginal distribution of X:

$$f(y|X=x) = rac{f(x,y)}{f_X(x)} = rac{rac{6}{7}(x+y)^2}{rac{6}{7}x^2 + rac{6}{7}x + rac{2}{7}} = rac{3(x+y)^2}{3x^2 + 3x + 1}$$

The conditional distribution of X given Y is the joint pdf divided by the marginal distribution of Y:

$$f(x|Y=y) = rac{f(x,y)}{f_Y(y)} = rac{rac{6}{7}(x+y)^2}{rac{6}{7}y^2 + rac{6}{7}y + rac{2}{7}} = rac{3(x+y)^2}{3y^2 + 3y + 1}$$

### Problem 2

Given:

$$F(x) = 1 - x^{-lpha}$$
  $x \geq 1$ 

The probability density function is the derivative of the cdf:

$$f(x)=F'(x)=rac{d}{dx}(1-x^{-lpha})=-(-lpha)x^{-lpha-1}=lpha x^{-lpha-1}$$

(a) The expected value is the sum/integral of the product of each possibility x with its probability f(x):

$$egin{aligned} E(X) &= \int_{-\infty}^{+\infty} x f(x) dx = \int_{1}^{+\infty} x lpha x^{-lpha-1} dx = \int_{1}^{+\infty} lpha x^{-lpha} dx \ &= \left. \left(rac{lpha x^{-lpha+1}}{-lpha+1}
ight)
ight|_{1}^{+\infty} = -rac{lpha (1^{-lpha+1})}{-lpha+1} = rac{lpha}{lpha-1} \end{aligned}$$

#### Problem 3

Given:

$$E(X) = \mu$$
  $Var(X) = \sigma^2$   $Z = rac{X - \mu}{\sigma}$ 

To proof: E(Z)=0 and Var(Z)=1

Proof: For the linear combination W=aX+b, the mean and variance have the following properties:

$$E(W)=aE(X)+b$$

$$Var(W) = a^2 Var(X)$$

Use these properties with  $a=rac{1}{\sigma}$  and  $b=-rac{\mu}{\sigma}$  to determine the mean and variance:

$$E(Z) = E\left(rac{X-\mu}{\sigma}
ight) = E\left(rac{1}{\sigma}X - rac{\mu}{\sigma}
ight) = rac{1}{\sigma}E(X) - rac{\mu}{\sigma} = rac{1}{\sigma}(\mu) - rac{\mu}{\sigma} = 0$$

$$Var(Z) = Var\left(rac{X-\mu}{\sigma}
ight) = Var\left(rac{1}{\sigma}X - rac{\mu}{\sigma}
ight) = \left(rac{1}{\sigma}
ight)^2 Var(X) = rac{1}{\sigma^2}(\sigma^2) = 1$$