SMJE 3153
CONTROL SYSTEM

The Root Locus

ROOT LOCUS (RL): CONTENT

Intro → Concept → Procedure → Root Locus & MATLAB

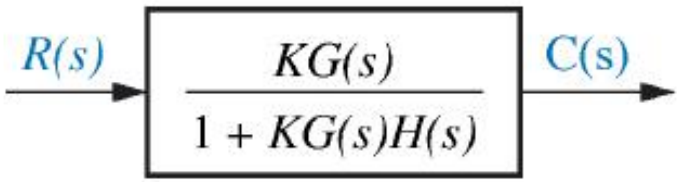
Introduction: Root Locus

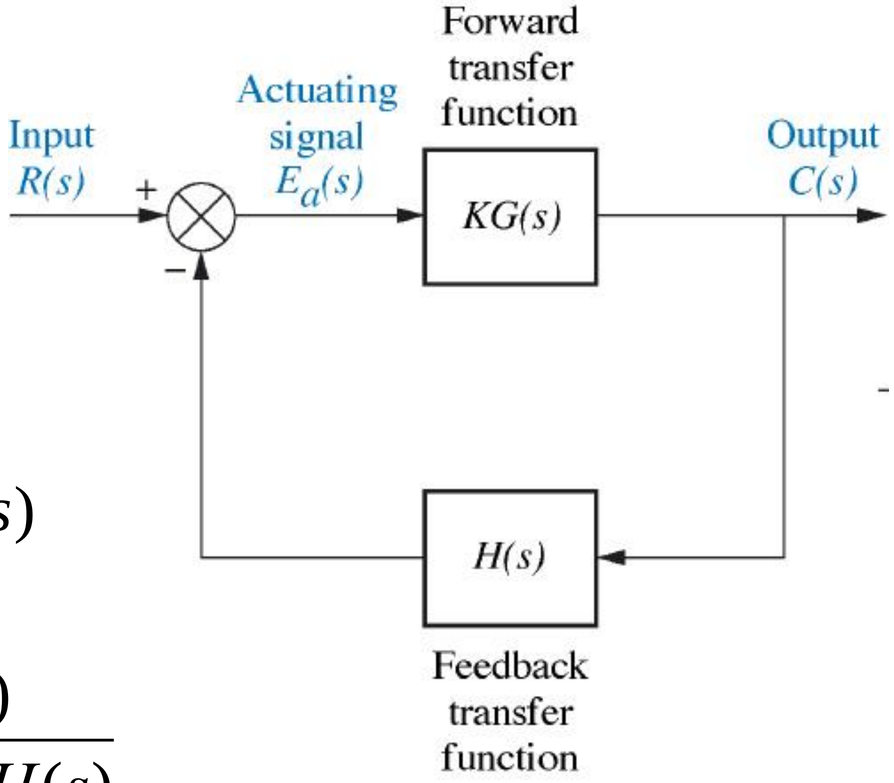
- Root locus (RL) is, a powerful method of analysis and design for stability and transient response.
- RL is a graphical representation of the closed-loop poles as a system parameter is varied.
- Thus, RL gives the transient response specifications (OS, T_p, T_s, e_{ss}, stability) as a system parameter is varied.

Concept: The Control System Problem

- Poles are important in control systems as they determine time response specifications and stability.
- OL poles are easily found and they do not change with changes in the system gain.
- CL poles are difficult to find especially for higher order systems.
- Moreover, CL poles change with changes in the system gain.

- CL control system:





$$OLTF = KG(s)H(s)$$

$$CLTF = \frac{KG(s)}{1+KG(s)H(s)}$$

- Consider a system with

$$G(s) = \frac{N_G(s)}{D_G(s)}; H(s) = \frac{N_H(s)}{D_H(s)}$$

- Then, the CLTF

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

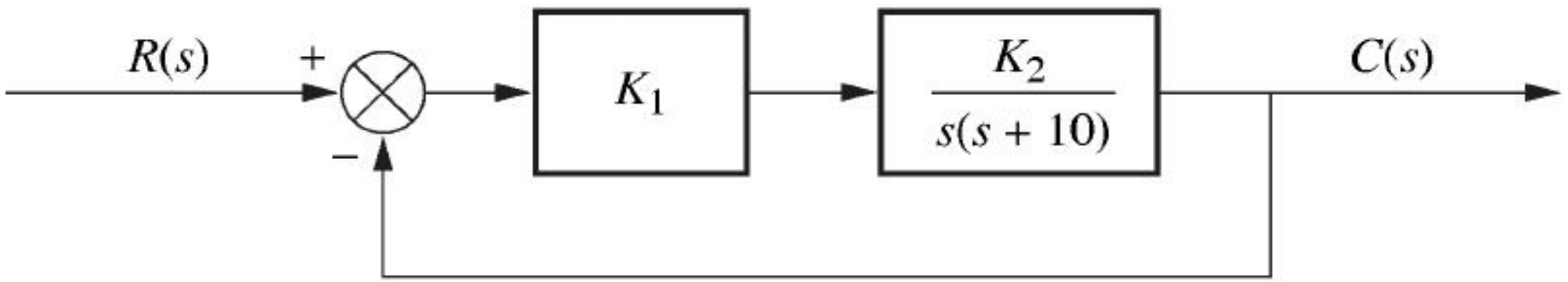
- Zeros of *T(s)*: zeros of *G(s)* and poles of *H(s)*
- Poles of *T(s)*: not immediately known and affected by *K.*

- Example:

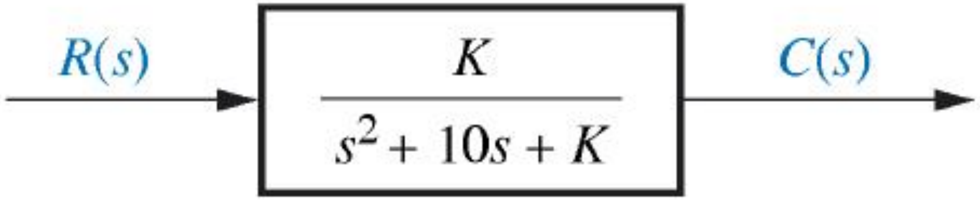
  For the closed-loop system with $G(s) = (s+1)/(s(s+2)$ and $H(s)=(s+3)/(s+4)$ and a gain, K find OL and CL poles and zeros.
- OLTF: $KG(s)H(s)$.
  - OL poles: 0, -2, -4 ; OL zeros: -1, -3.
- CLTF: $K(s+1)(s+4)/(s^3+(6+K)s^2+(8+4K)s+3K)$
  - CL zeros: -1, -4; CL poles: affected by K
- Root locus gives a graphical presentation of the CL poles as K varies.

Concept: Defining the Root Locus

- Consider the closed-loop system below. How the gain, K affected the closed-loop poles and system specifications?



- CLTF:

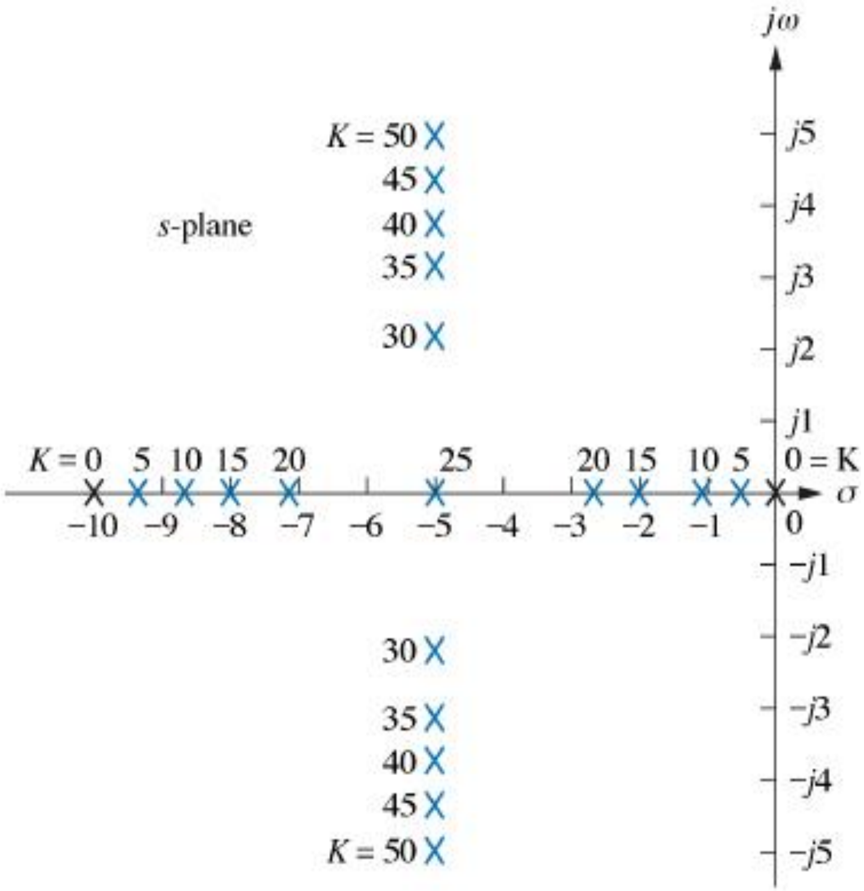


where $K = K_1 K_2$

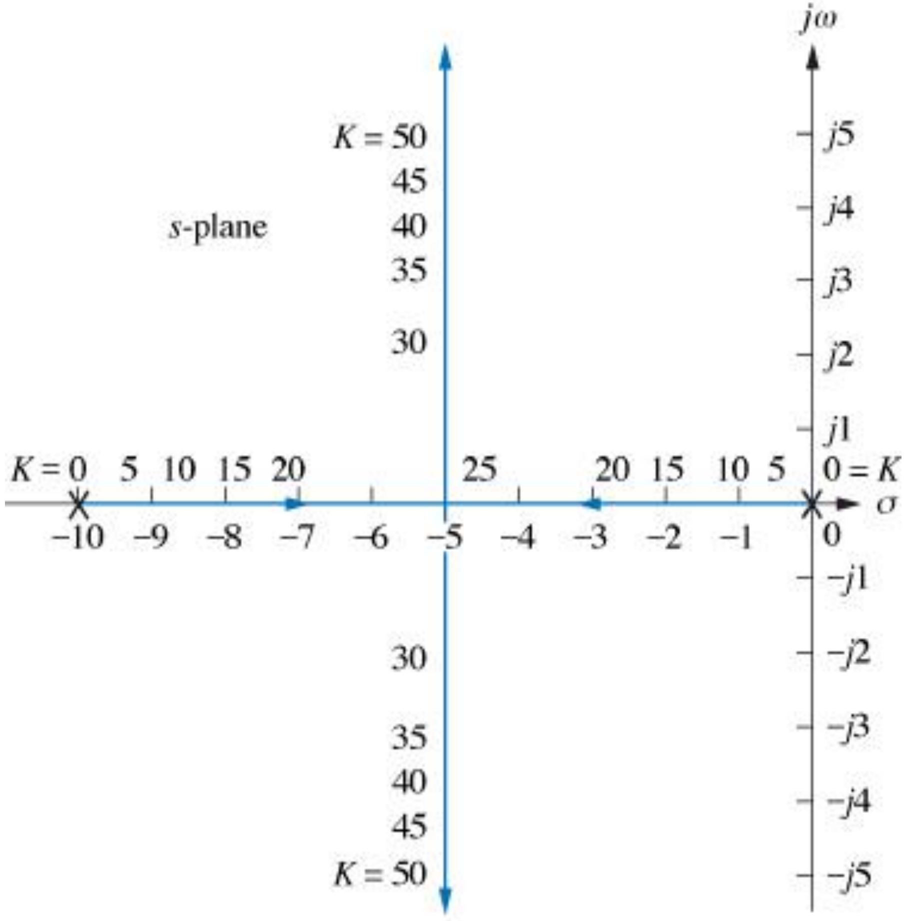
- Pole location as a function of the gain, K

K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

- CL poles at -10 moves toward the right and CL poles at 0 moves toward the left as K increases.
- They meet at -5, breakaway from the real axis, move into complex plane. One moves upward and the other moves downwards.



- Individual CL pole locations are removed and their paths are represented with lines.
- This is a representation of a path of the CL poles as the gain is varied.
- It is known as root locus.



(b)

- The root locus show:
- K < 25, the poles are real, **overdamped**.

- K = 25, the poles are real and multiple (overlapped), **critically damped**.

- K > 25, complex poles, **underdamped**.
- For the underdamped system, regardless the value of the gain, the real parts are always the same. Thus the settling time remains the same.

- Since the root locus never crosses into the RHP, the system is always stable for all value of K.
- Root locus is useful for higher order systems.

Concept: Properties of Root Locus

- Can we draw a root locus for higher order systems?
- Properties of the root locus will be helpful to sketch a root locus for higher order systems.
- Consider a closed-loop system with a feedback, *H(s)*.
- CLTF:

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

- A pole, s exists the characteristic polynomial (denominator) becomes zero, or

$$KG(s)H(s) = -1 = 1\angle(2k+1)180^0;\ k = 0, \pm1, \pm2, \ldots$$

- Alternatively,
  - Magnitude: $|KG(s)H(s)| = 1$
  - Angle: $\angle KG(s)H(s) = (2k+1)180^0$

- Thus, a pole of the closed-loop system causes the angle of *KG(s)H(s)* to be an odd multiple of 180^0.
- The magnitude of *KG(s)H(s)* must be unity. Thus the gain, K can be calculated as

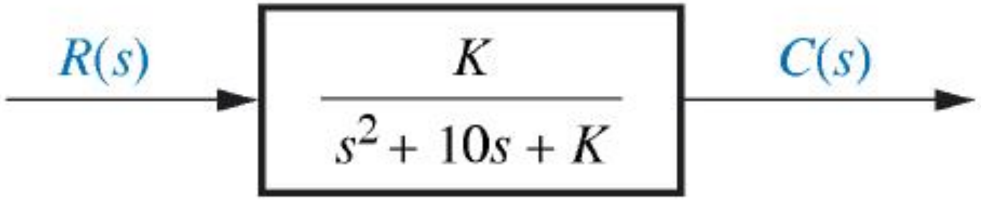
$$K = \frac{1}{|G(s)||H(s)|}$$

- For the previous example, CL poles exists at
  s = -9.47, -0.53 when the gain, K = 5.

$$KG(s)H(s) = \frac{K}{s(s+10)}$$

- Substituting the pole, s = -9.47 and K =5 yields $KG(s)H(s)$ = -1.
- All the CL poles must satisfy the requirement.

- CLTF:



where $K = K_1K_2$

- Pole location as a function of the gain, K

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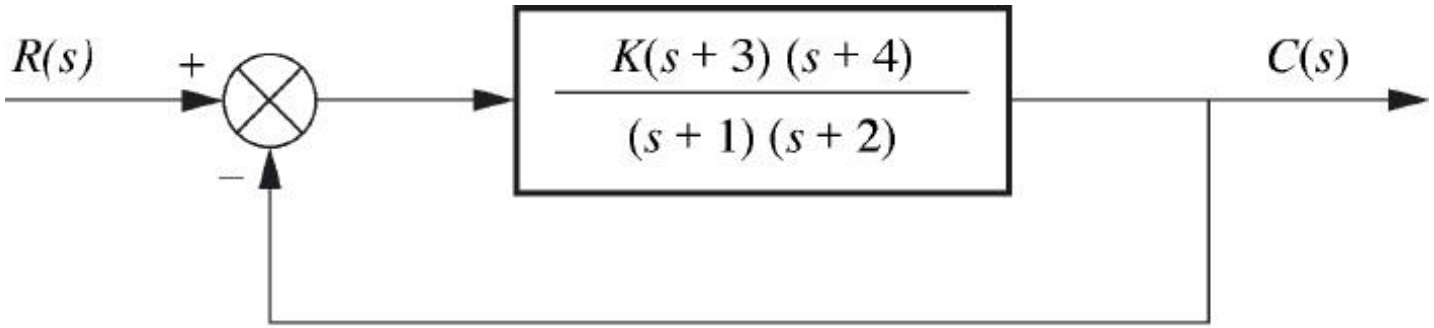
- Consider the CL system



- Examine whether the point, **s = -2 + j3** is a closed loop pole of the system.
- To be a CL pole, angle and gain criteria must be fulfilled.

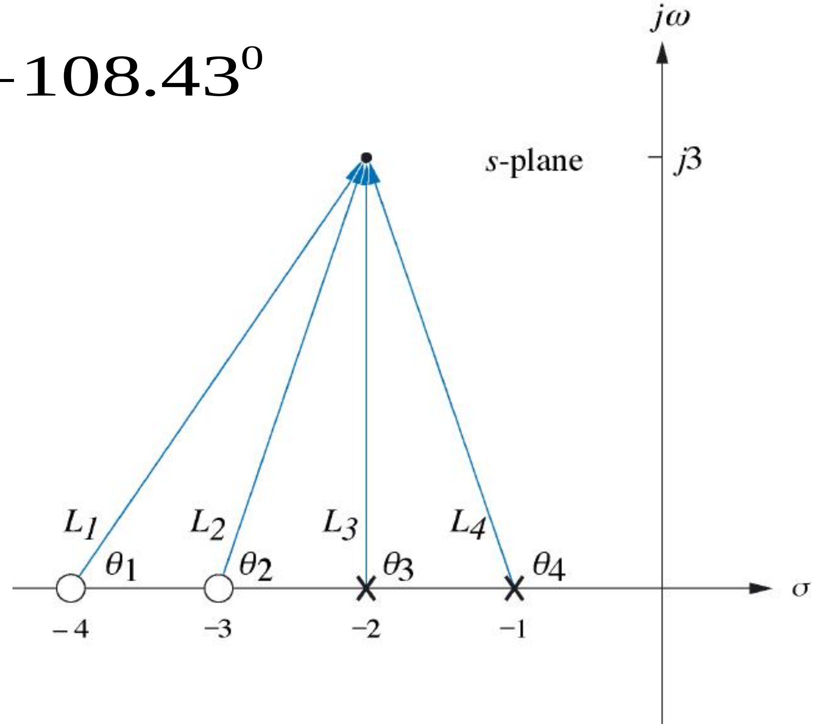
- Angle criteria: Angle of zeros – angle of poles = odd multiple of 180^0.

$$Angle = \theta_1 + \theta_2 - \theta_3 - \theta_4$$
$$= 56.31^0 + 71.57^0 - 90^0 - 108.43^0$$
$$= -70.55^0 \neq 180^0$$

- Thus -2 + j3 is not a CL pole for any gain.
- Examine a new point

$$s = -2 + j(\sqrt{2}/2)$$

- The angle = 180^0.

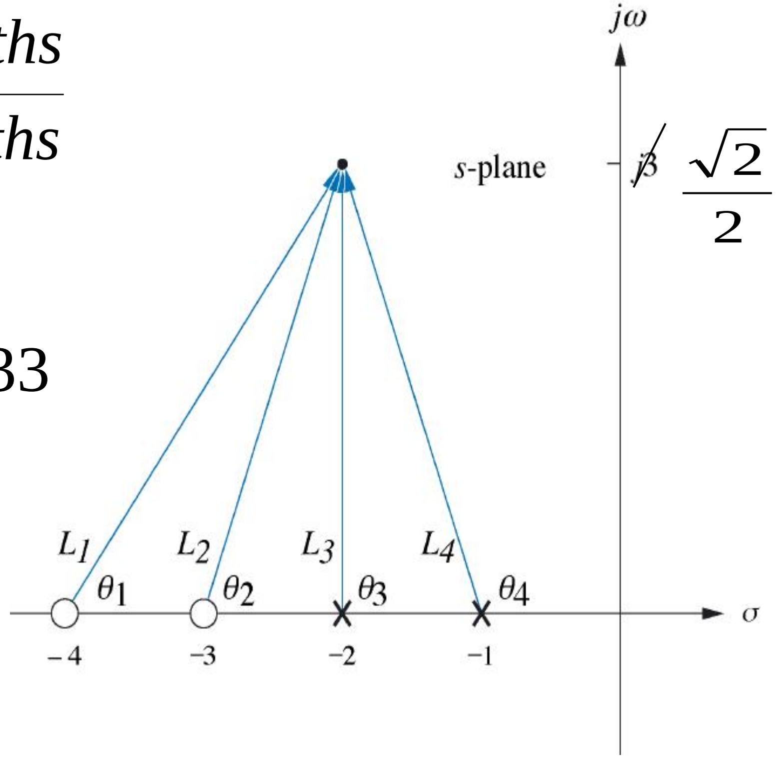


- We have to evaluate the gain.

$$K = \frac{1}{|G(s)||H(s)|} = \frac{\prod pole\,lengths}{\prod zero\,lengths}$$

$$K = \frac{L_3 L_4}{L_1 L_2} = \frac{\frac{\sqrt{2}}{2}(1.22)}{(2.12)(1.22)} = 0.33$$

- Thus it is a point on the root locus for a gain of 0.33.



Example 1

- Given a unity feedback that has a forward transfer function

$$G(s) = \frac{K(s+2)}{(s^2+4s+13)}$$

Determine if a point, -3 + j0 is on the root locus.
If the point is on the root locus, find the gain, K that satisfy the requirement.

Answer: Yes, K = 10

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Example 2

- Given a unity feedback that has a forward transfer function

$$G(s)=\frac{K(s+2)(s^2+2s+17)}{(s+10)(s^2+9)}$$

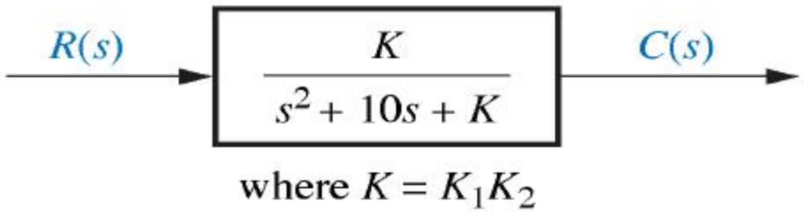
Determine the value of gain, K to be used if one of the CL poles must be at s = -9.

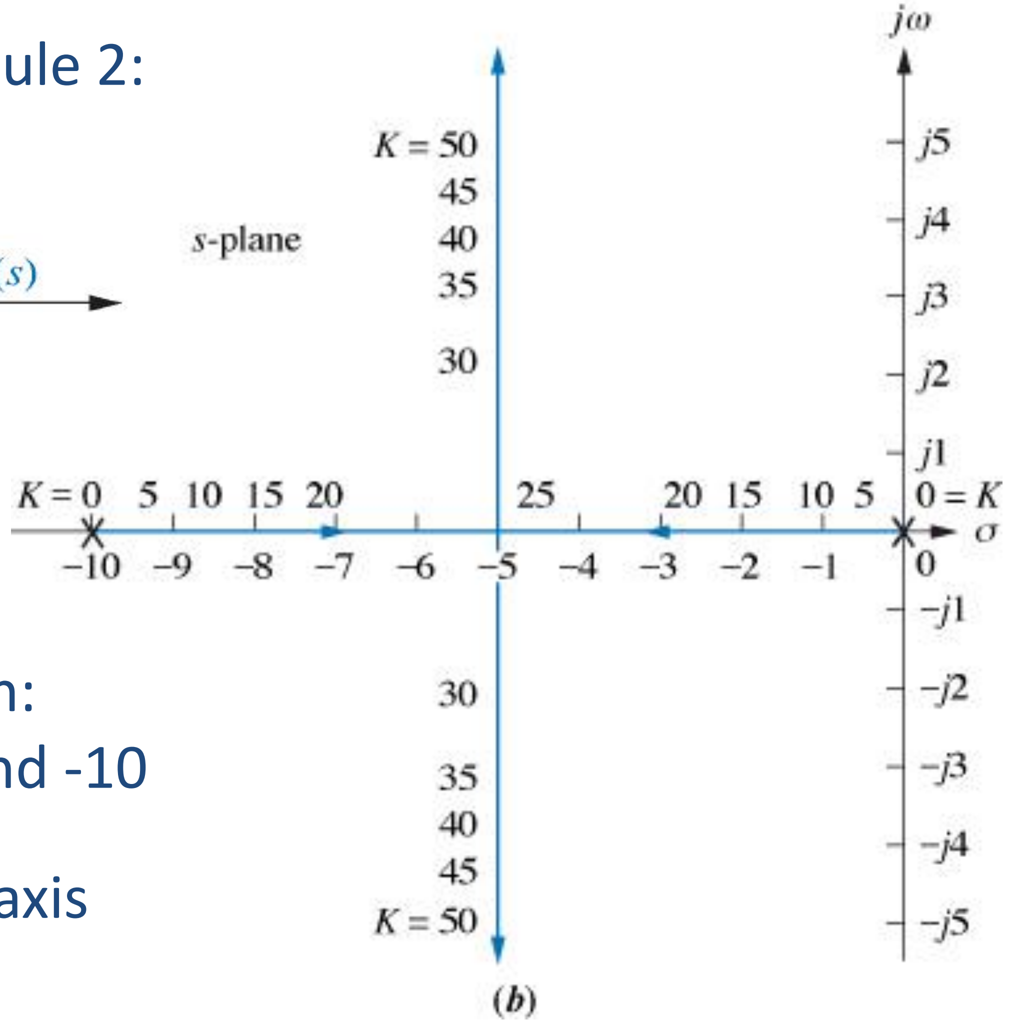
- Answer: K = 0.161

Procedure: Sketching the Root Locus

- The following rules allow us to sketch the root locus. **After Sketching, - the root locus can be refined.**
- **Rule 1, Number of Branches**: *The number of branches are equal to the number of closed-loop poles*.
- **Rule 2, Symmetry** : *The root locus is symmetrical about the real axis*.
- **Rule 3, Real-axis Segment**: *On the real axis, for K > 0, the root locus exists to the left of an odd number of real-axis, finite OL poles and/or zeros*.

- Example of Rule 1 and Rule 2:

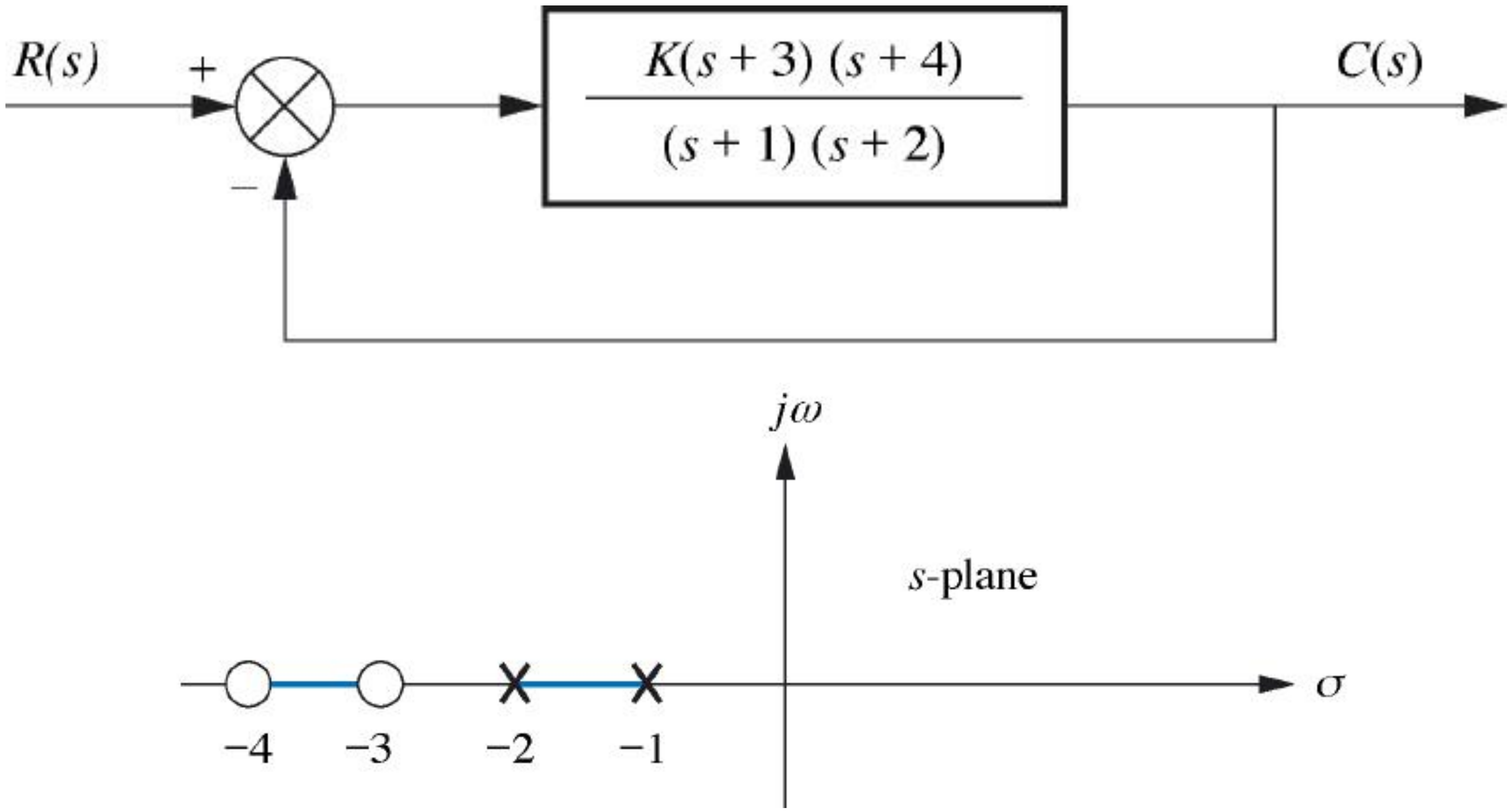




- Two branches are shown: originates from origin and -10

- Symmetrical about real axis

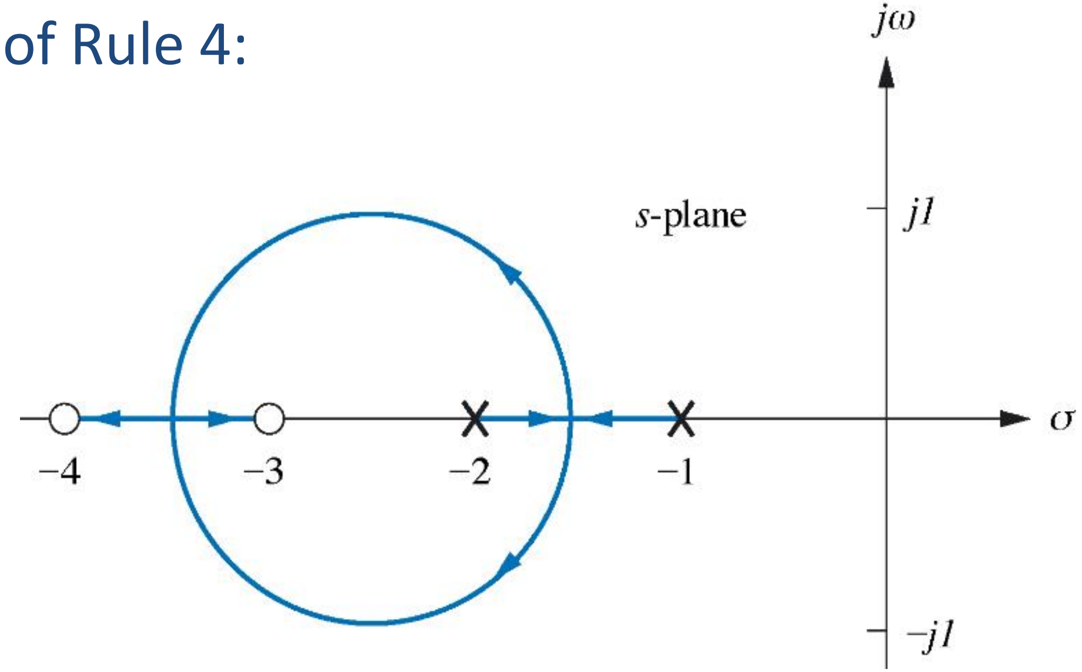
- Example of Rule 3:



- On the real axis, for K>0, the RL exists to the left of -1 and -3

- **Rule 4, Starting and Ending Point**: *The root locus begins at the finite and infinite poles of G(s)H(s) at K=0 and ends at the finite and infinite zeros of G(s)H(s) at K → ∞.*
- Example: KG(s)H(s) = K/[s(s+1)(s+2)] have three finite poles and three infinite zeros.
- Where are the infinite zeros? Rule 5 helps us to locate these zeros at infinity.

- Example of Rule 4:



- These poles and zeros are the open-loop poles and zeros.
- The root locus begins at the finite poles at -1 and -2

and ends at the finite zeros at -3 and -4.

- **Rule 5, Behavior at Infinity**: *The root locus approaches straight lines as asymptotes as the locus approaches infinity.*
  - Number of asymptotes = # finite poles - # finite zeros.
  - Real-axis intercept,

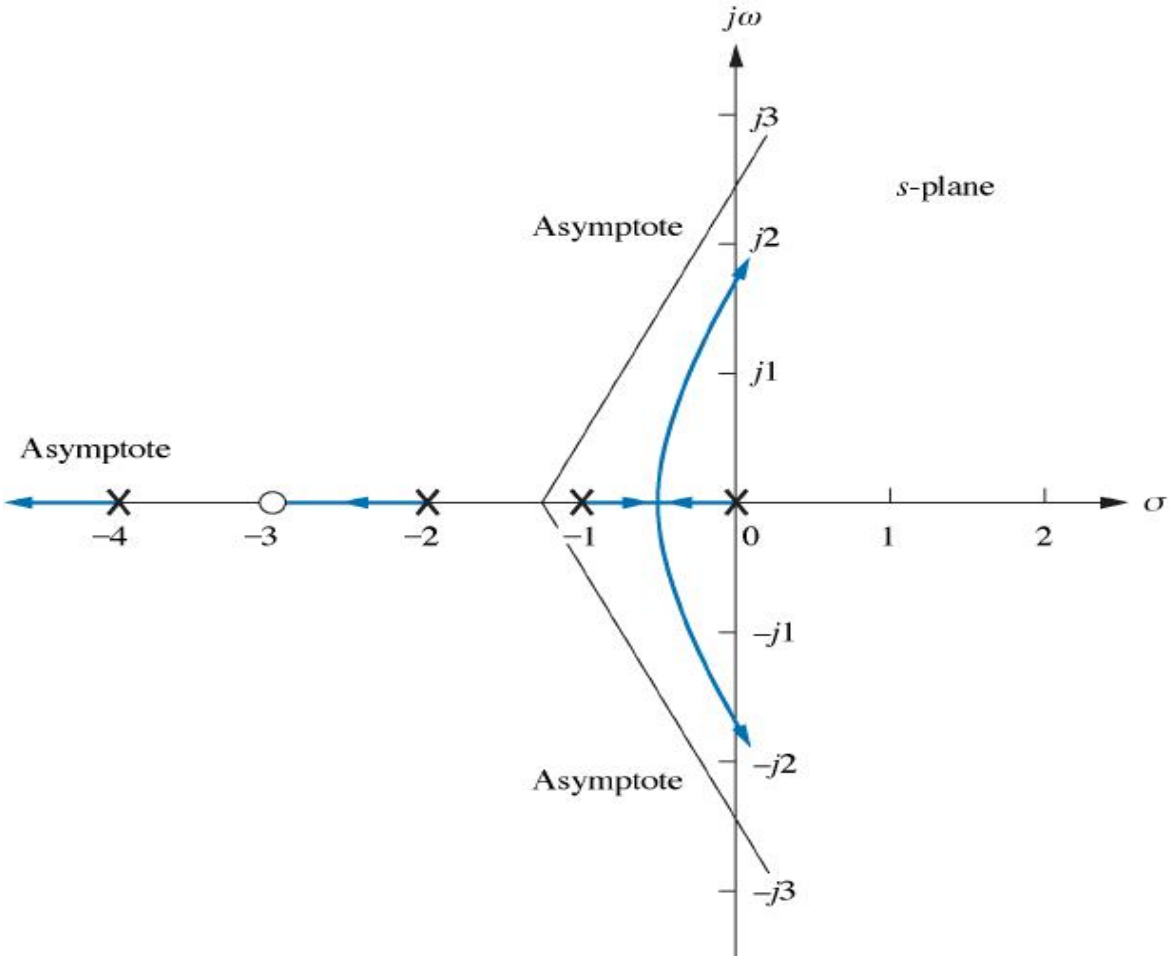
$$\sigma_a = \frac{\sum finite\ poles - \sum finite\ zeros}{\#\ finite\ poles - \#\ finite\ zeros}$$

  - Angle

$$\theta_a = \frac{(2k+1)\pi}{\#\ finite\ poles - \#\ finite\ zeros}$$

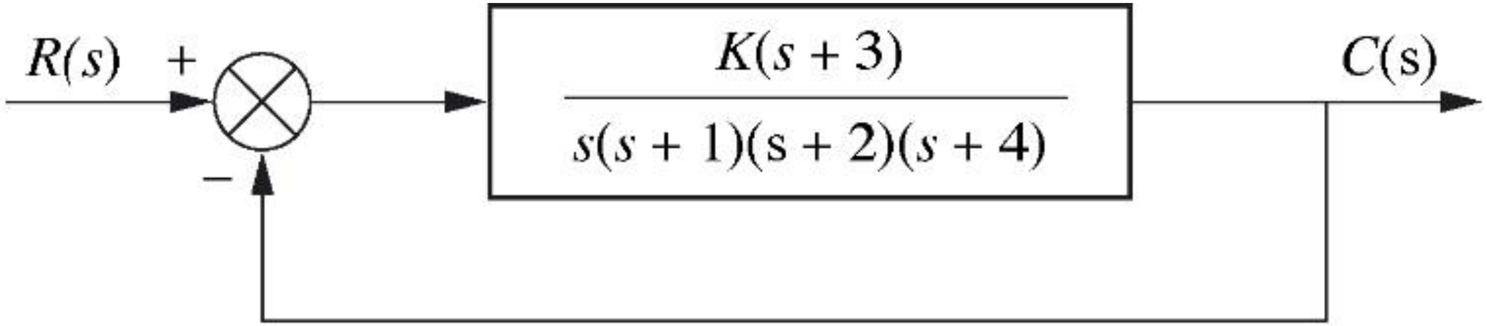
where k = 0, $\pm$1, $\pm$2 .....

- Example of Rule 5:



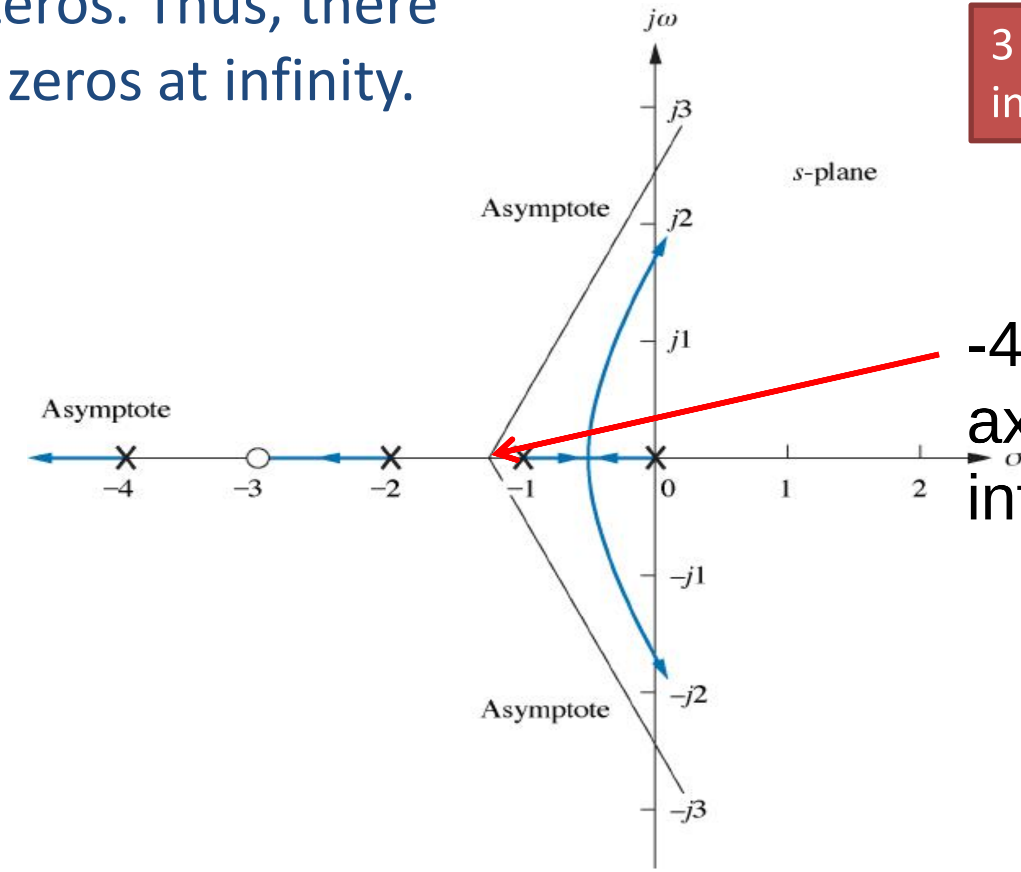
Example 3

Sketch the root locus for the system using Rules 1-5.



Solution

- There are more OL poles and OL zeros. Thus, there must be zeros at infinity.



3 zeros at infinity

-4/3, real-axis intercept

Example 4

Sketch the root locus for a unity feedback system that has the forward transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

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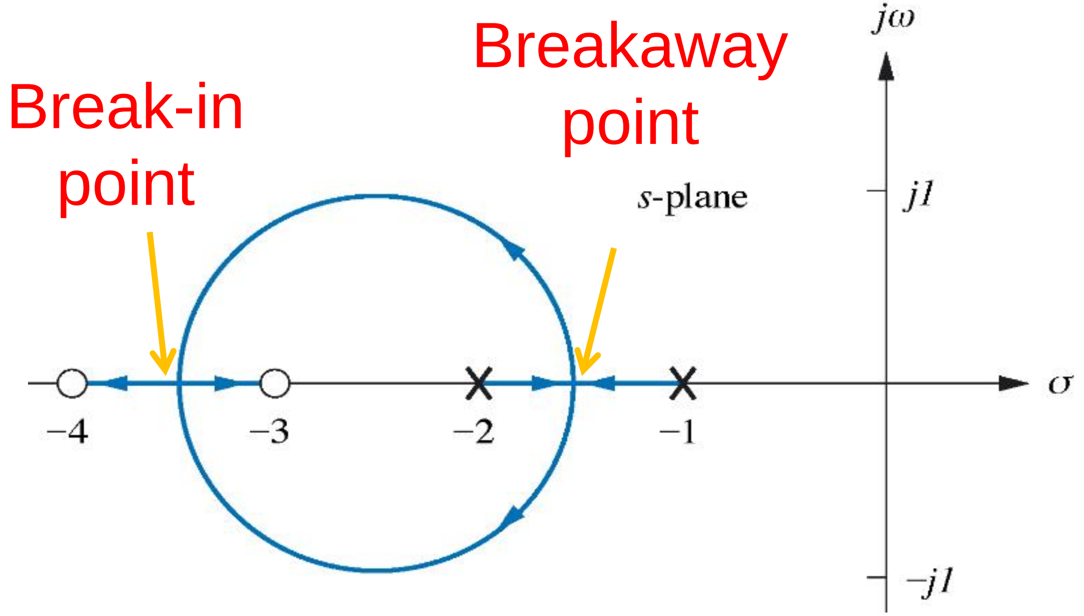
Procedure: Refining the Sketch

- **Rule 6, Real-axis Breakaway and Break-in Points**: The root locus breaks away from the real axis at a point of maximum gain and breaks into the real axis at a point where the gain is minimum.
- The breakaway point occurs at a point of maximum gain on the real axis between the OL poles.
- The gain at the break-in point is the minimum gain on the real axis between the two zeros.



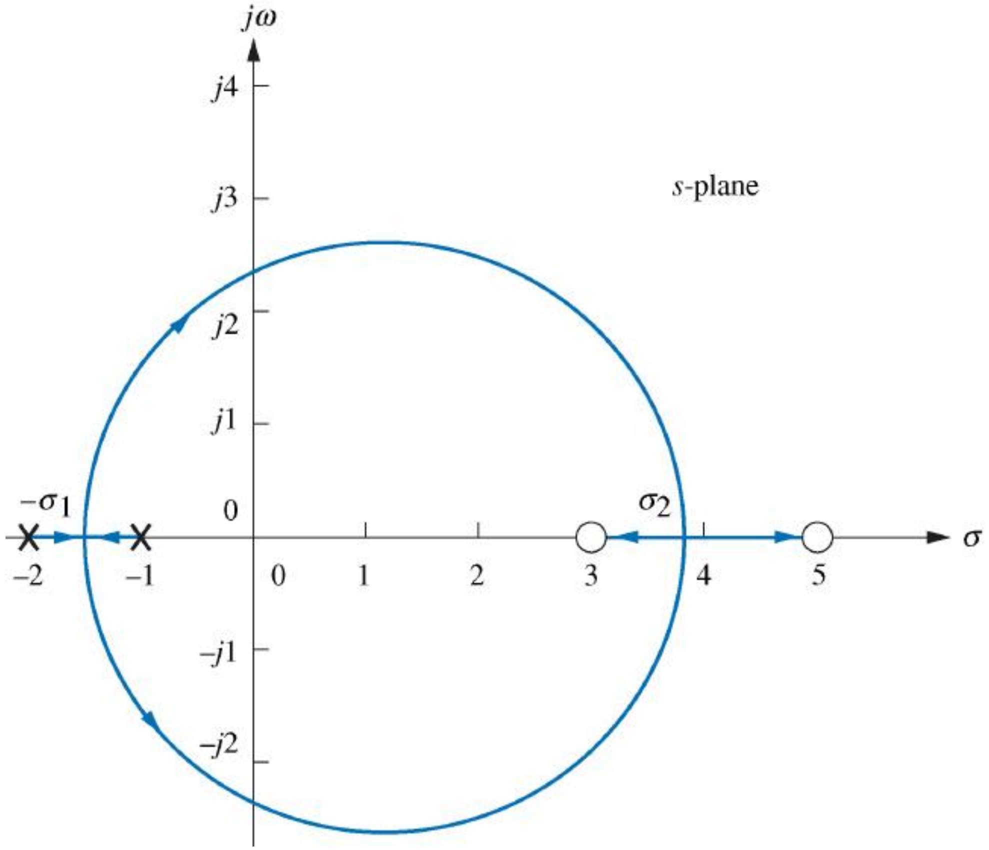
- To find the point s = σ;
  - Maximize and minimize the gain, K through differentiation.

$$\frac{d}{ds}(G(s)H(s)) = 0$$

- Solve for s.

- The s obtained from above is the breakpoint, σ

- Example: Find the breakaway and break-in points of the root locus.



- To find the points, take the derivative of the equation, equate with zero and solve.

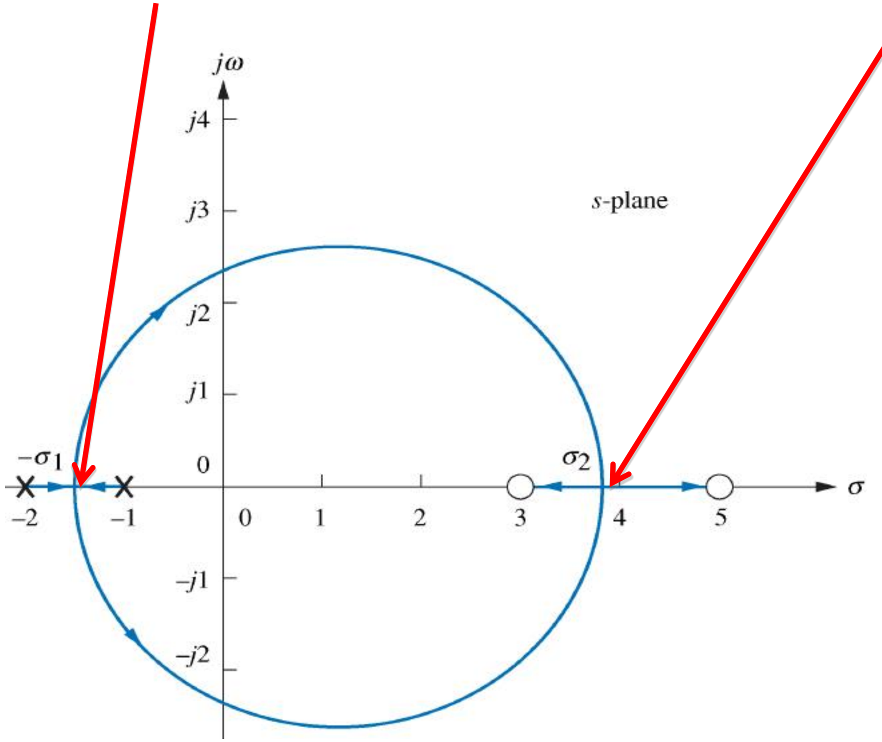
$$KG(s)H(s) = -\frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)}$$

- Letting K=1, gives

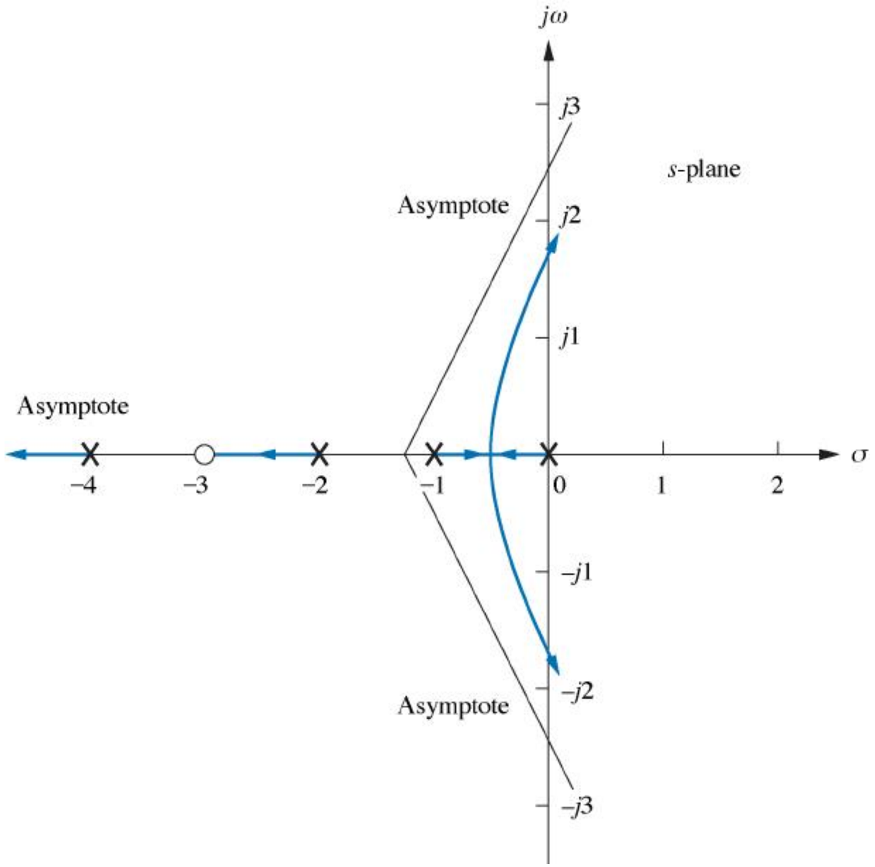
$$\frac{d}{ds}G(s)H(s) = \frac{d(s^2-8s+15)}{ds(s^2+3s+2)}$$

$$11\sigma^2 - 26\sigma - 61 = 0$$

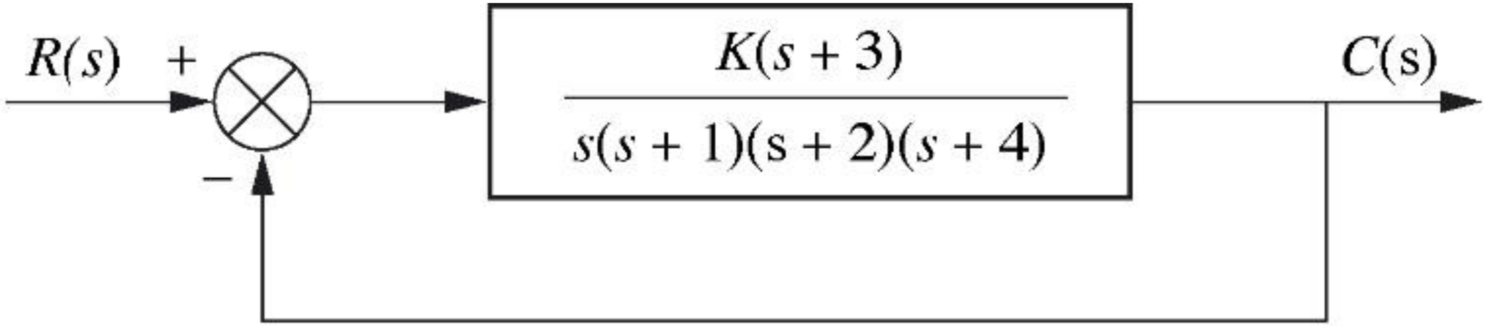
- Solving yields σ = -1.45 (breakaway point) and 3.82 (break-in point).



- **Rule 7:**
- **The jω - crossing**: *Routh-Hurwitz can be used to find the jω-crossing*.
- The $j\omega$-axis crossing is a point on the root locus that separate the stable and unstable operations.
- Use RH criterion to find the point.



- *Example*: For the system, find the frequency and gain, K, for which the root locus crosses the imaginary axis. For what range of K is the system stable.



- CLTF: $$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$$

- RH table

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	$90-K$	$21K$	
s^1	$\frac{-K^2-65K+720}{90-K}$		
s^0	$21K$		

- ROZ yield the possibility for imaginary axis poles.

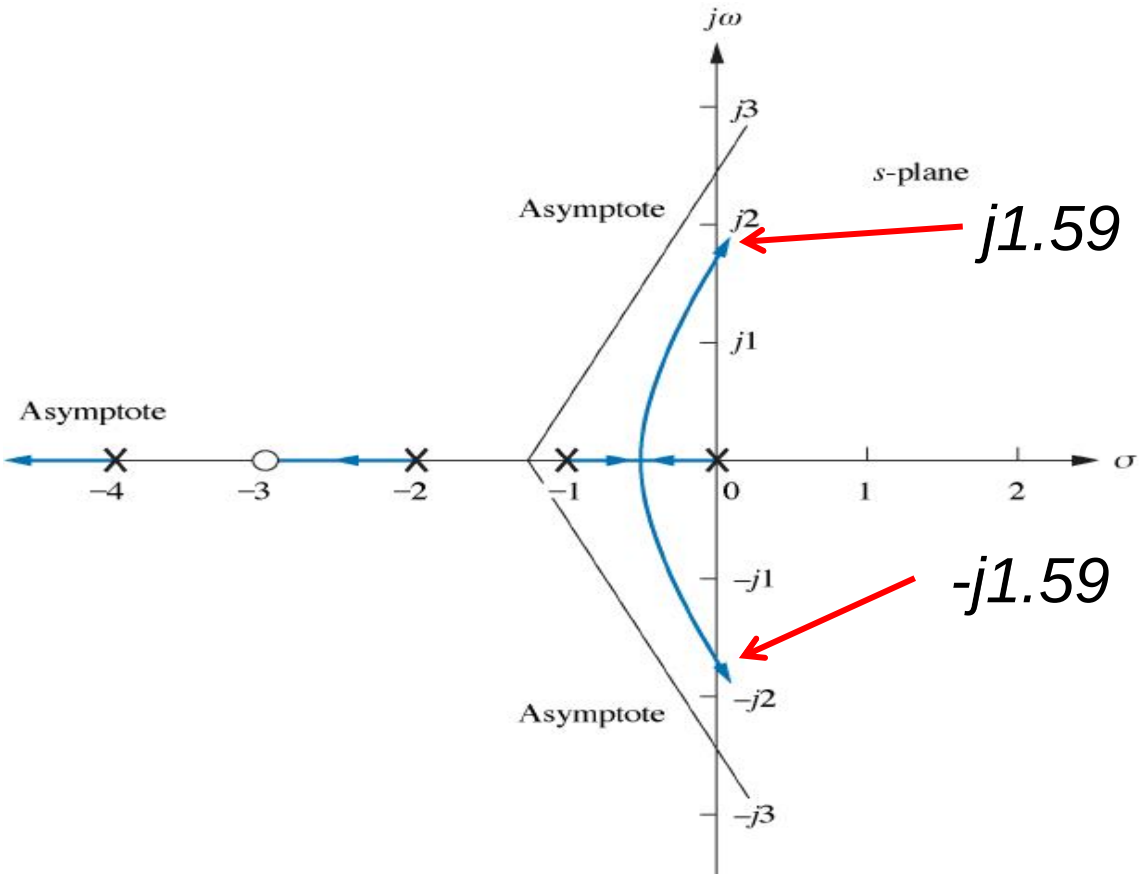
$$-K^2 - 65K + 720 = 0; \quad K = 9.65$$

- Forming the even polynomial with K,

$$(90-K)s^2+21K=80.35s^2+202.7=0$$
$$s=\pm j1.59$$

- Hence, the root locus crosses the $j\omega$-axis at

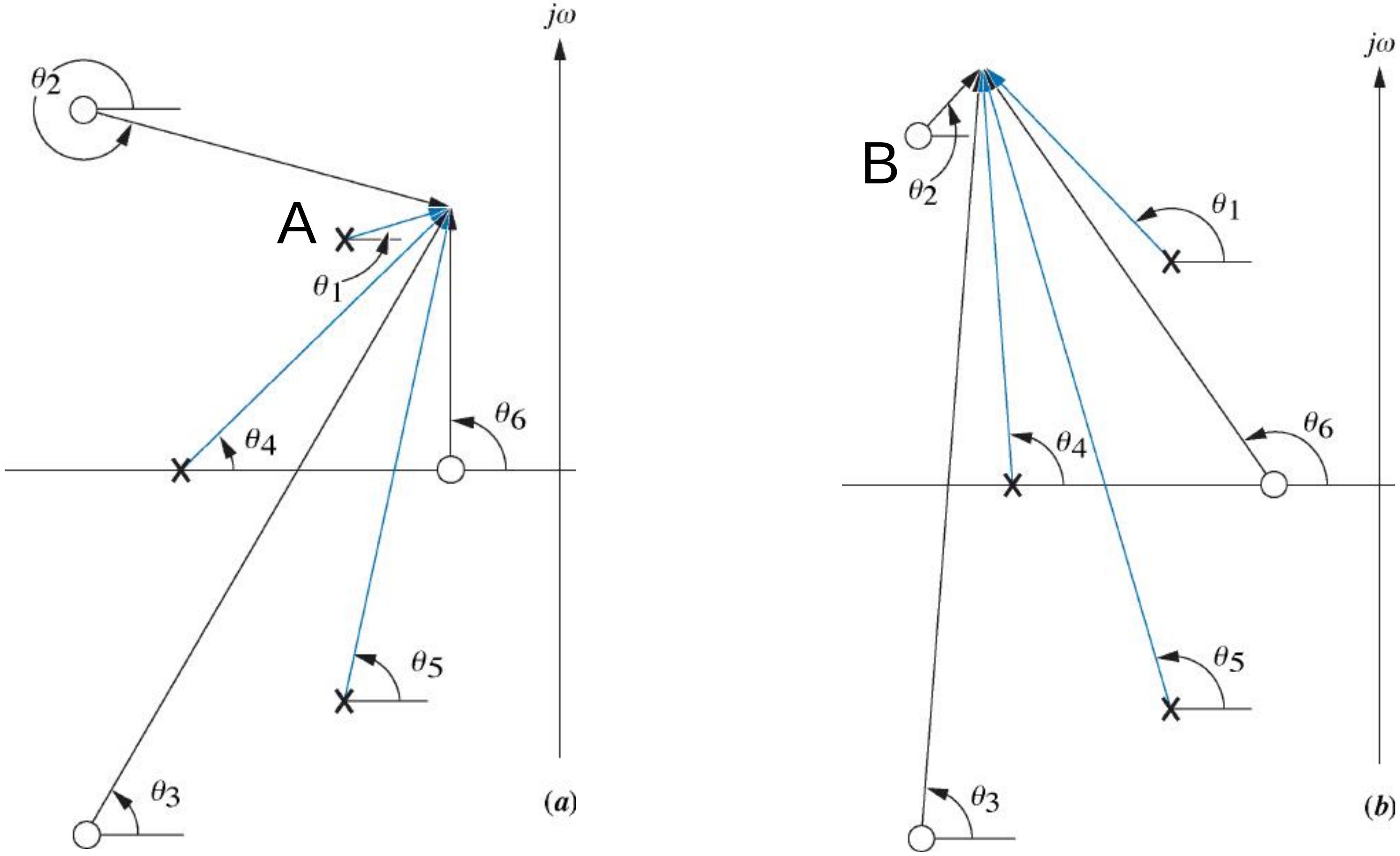
  s= $\pm$*j1.59*.
- The system is stable for 0 ≤ K < 9.65.


jω
j3
j2
j1
−j1
−j2
−j3
s-plane
Asymptote
Asymptote
Asymptote
σ
−4
−3
−2
−1
0
1
2
j1.59
-j1.59

- **Rule 8, Angles of Departure and Arrival:** Angles of departure (pole) and angles of arrival (zero) can be calculated to further refine the sketch.
- Used for complex OL poles and zeros.
- The root locus departs from complex OL poles and arrives at complex OL zeros.
- The angles can be found using the angle property:

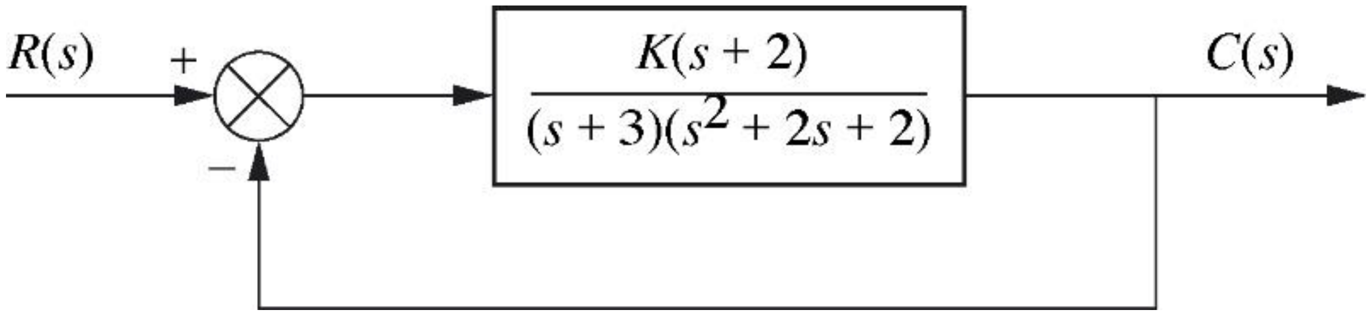
$$\theta_a = \sum(\text{angles of all poles to B}) - \sum(\text{angles of all zeros to B}) + 180^0$$

$$\theta_d = \sum(\text{angles of all zeros to A}) - \sum(\text{angles of all poles to A}) + 180^0$$


jω
θ2
A
θ1
θ6
θ4
θ5
θ3
(a)
jω
B
θ2
θ1
θ4
θ6
θ5
θ3
(b)

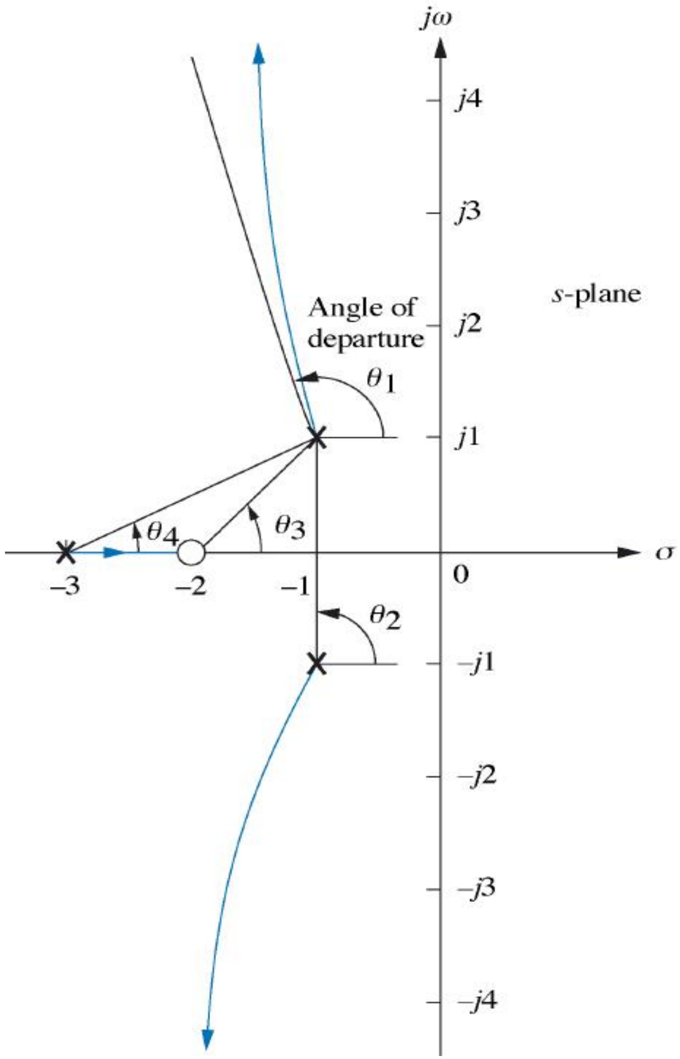
Example 5

- **Example:** For the unity feedback system, find the angle of departure from the complex poles and sketch the root locus.



Solution 5

- Departure angle, θ_1=
  $\sum$*(angles of all zeros to the pole)* - $\sum$ *(angles of all poles to the pole)+ 180*^{0}
  $\theta_1 = \theta_3 - \theta_2 - \theta_4 + 180^0$
  $\theta_1 = -251.6^0 = 108.4^0$



- **Rule 9, Plotting and calibrating the root locus:** All points on the root locus satisfy the magnitude and angle properties.
- We may want to locate points on the root locus and their gains. For example, when the root locus crosses the line representing 20 % OS.
- The gain, K at any point on the root locus is given by

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod finite\ poles\ lengths}{\prod finite\ zero\ lengths}$$

Example 6

- For a unity feedback system that has the forward transfer function,

$$G(s) = \frac{K(s+2)}{s^2 - 4s + 13}$$

- do the following:
  - Sketch the root locus
  - Find the angle of departure from the complex poles.

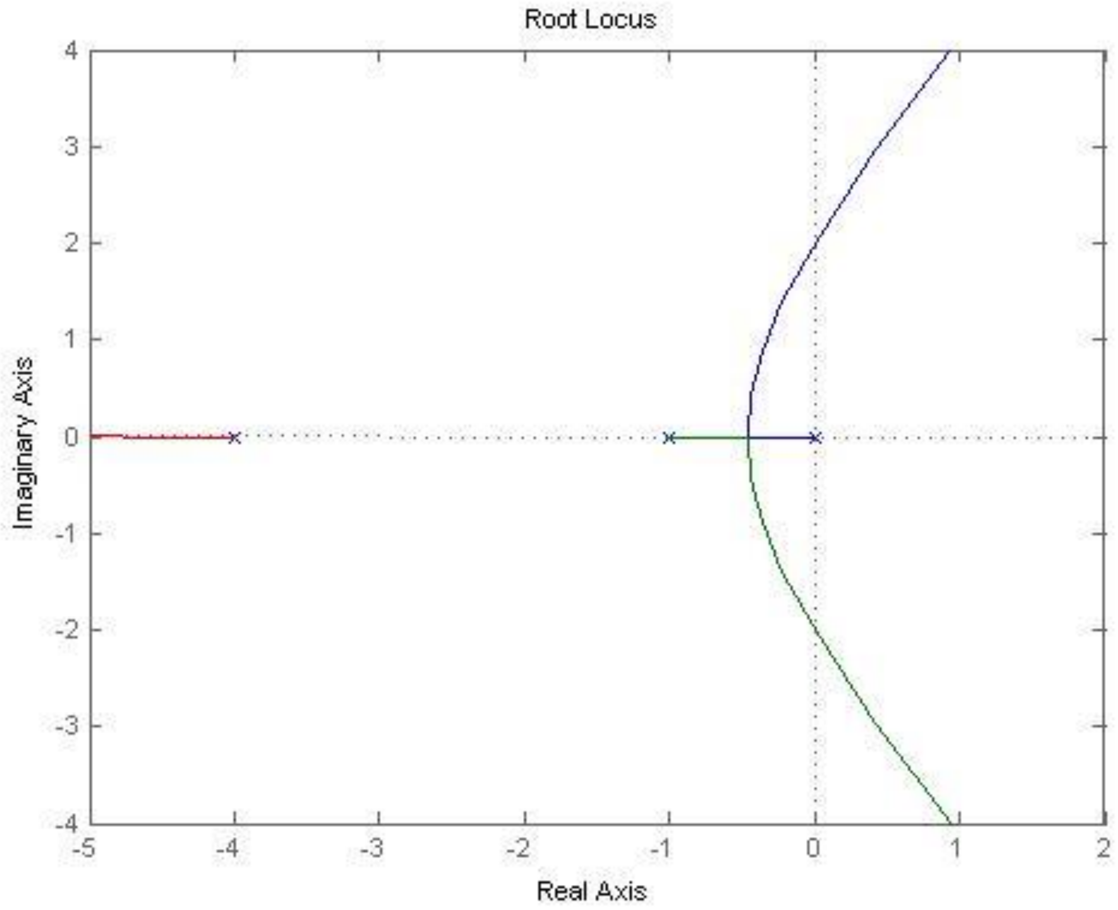
Example 7

- For a unity feedback system that has the forward transfer function,

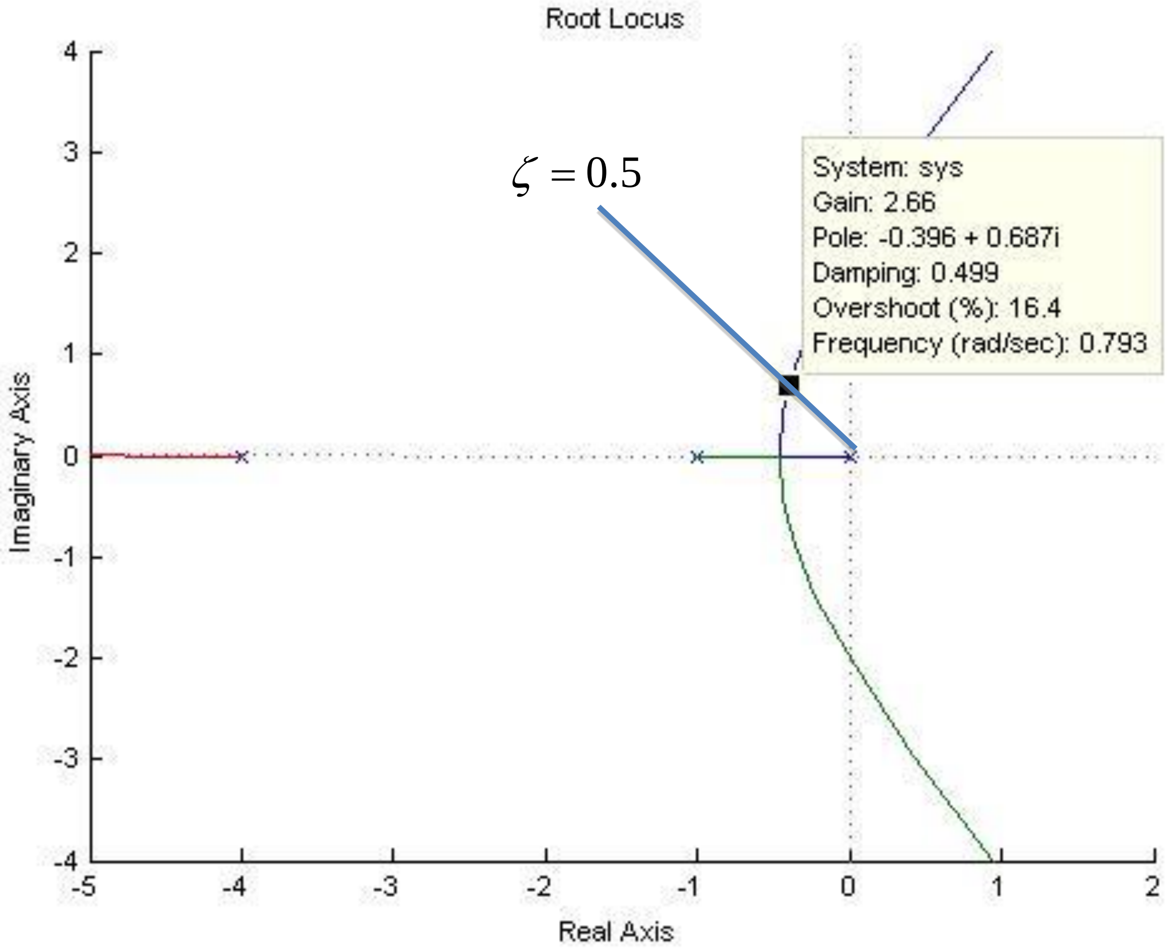
$$G(s) = \frac{K}{s(s+1)(s+4)}$$

- Do the following:
  - Sketch the root locus
  - Find K so that the system operates at damping ratio of 0.5. (use: 180 – $\cos^{-1}$(theta))

Example 7 (solution)

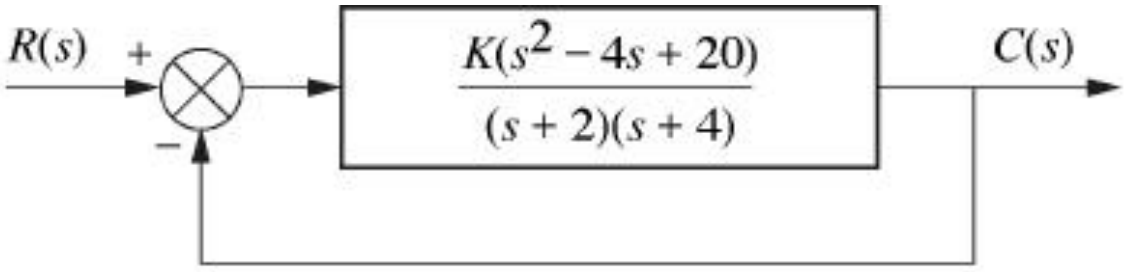


Example 7 (solution)



Example 8

Sketch the root locus for the system and find the following:



- The point and gain where the locus crosses the 0.45 damping ratio line.
- The range of *K* within which the system is stable.

Root Locus Using Matlab