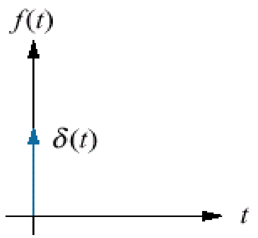
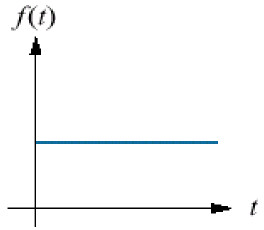
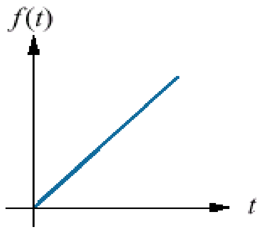
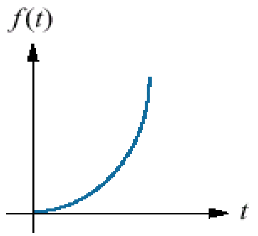
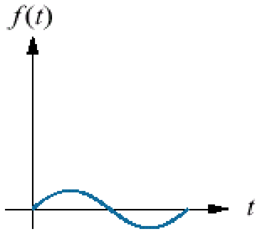
SMJE 3153
PID Controllers Design

Chapter Outline

Introduction

- Root locus (RL) is a powerful tool for design of control systems.
- We will study four design techniques:
  - Gain adjustment
  - Proportional-Integral (PI) controller
  - Proportional-Derivative (PD) controller
  - Proportional-Integral-Derivative (PID) controller

Re-visit: Input Signals

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)\,dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

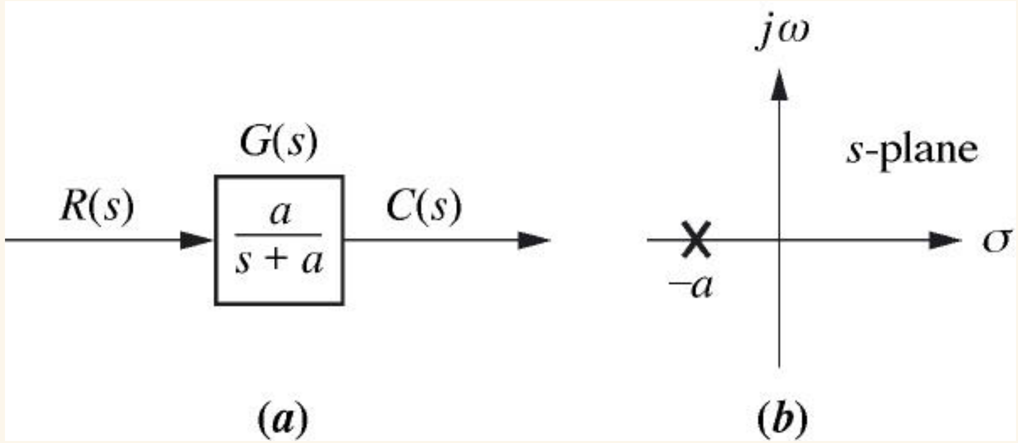
Re-visit: Poles and Zeros

- Poles: roots of the denominator of a transfer function.
- Zeros: roots of the numerator of a transfer function.
- Poles and zeros can be mapped on an s-plane (pole: x, zero: o).
- Important in control systems.

Re-visit: First Order System

- Transfer function:

$$\frac{C(s)}{R(s)} = \frac{a}{(s+a)}$$



- With a unit step input,

$$r(t) = \frac{1}{s}, c(t) = 1 - e^{-at}$$

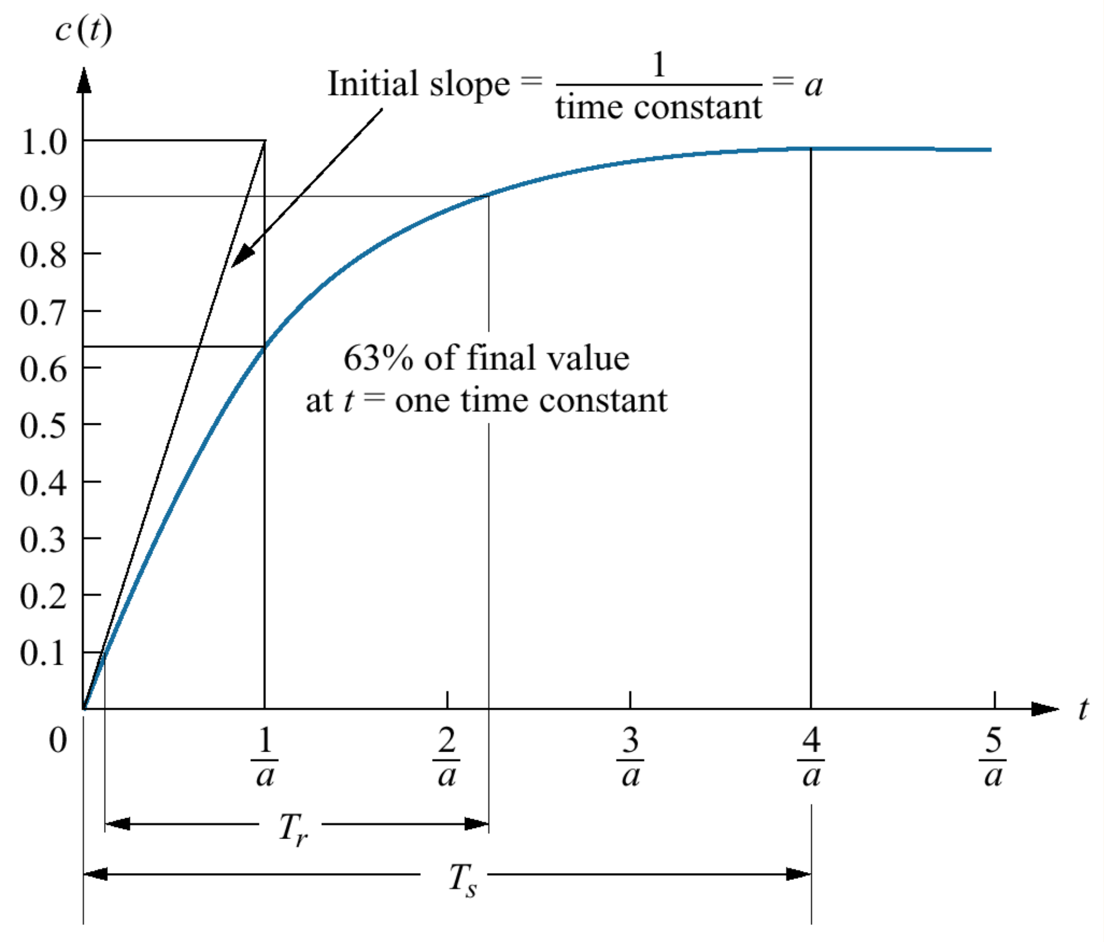
Re-visit: First Order System

- Time response
- Specifications:

  time constant,

  rise time, settling

  time.

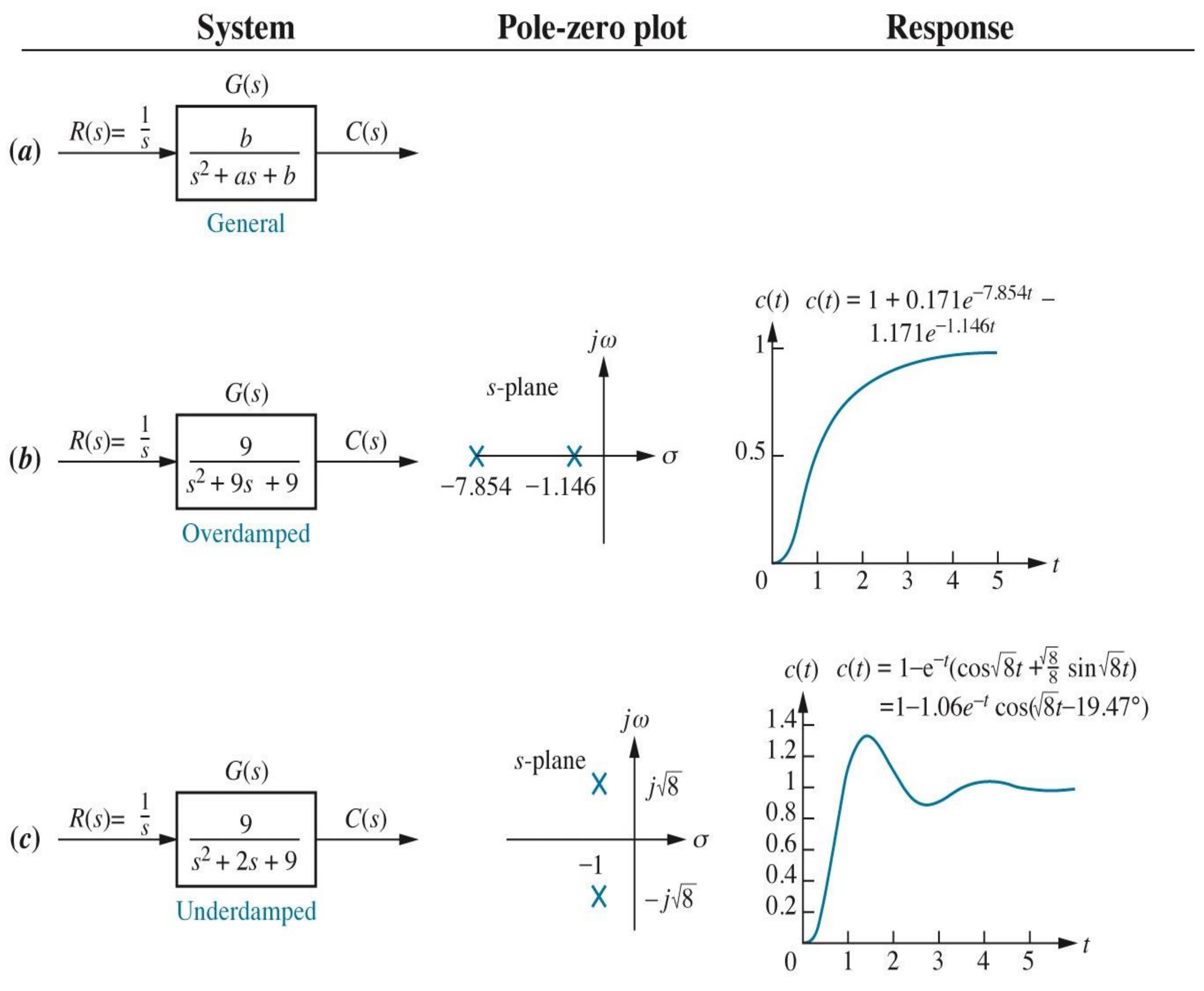


Revisit: Second Order System

- Transfer function:

$$G(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

- The second order system responses are determined by damping ratio, ζ and natural frequency, ω_n
- Time responses can be categorised into overdamped, underdamped, undamped and critically damped based on the damping ratio and pole locations.


System
Pole-zero plot
Response
G(s)
R(s)= 1/s
b / s^2 + as + b
C(s)
(a)
General
(b)
9 / s^2 + 9s + 9
Overdamped
s-plane
jω
σ
−7.854
−1.146
c(t) c(t) = 1 + 0.171e^−7.854t − 1.171e^−1.146t
(c)
9 / s^2 + 2s + 9
Underdamped
j√8
−j√8
−1
c(t) c(t) = 1−e^−t(cos√8t + √8/8 sin√8t)
=1−1.06e^−t cos(√8t−19.47°)
t

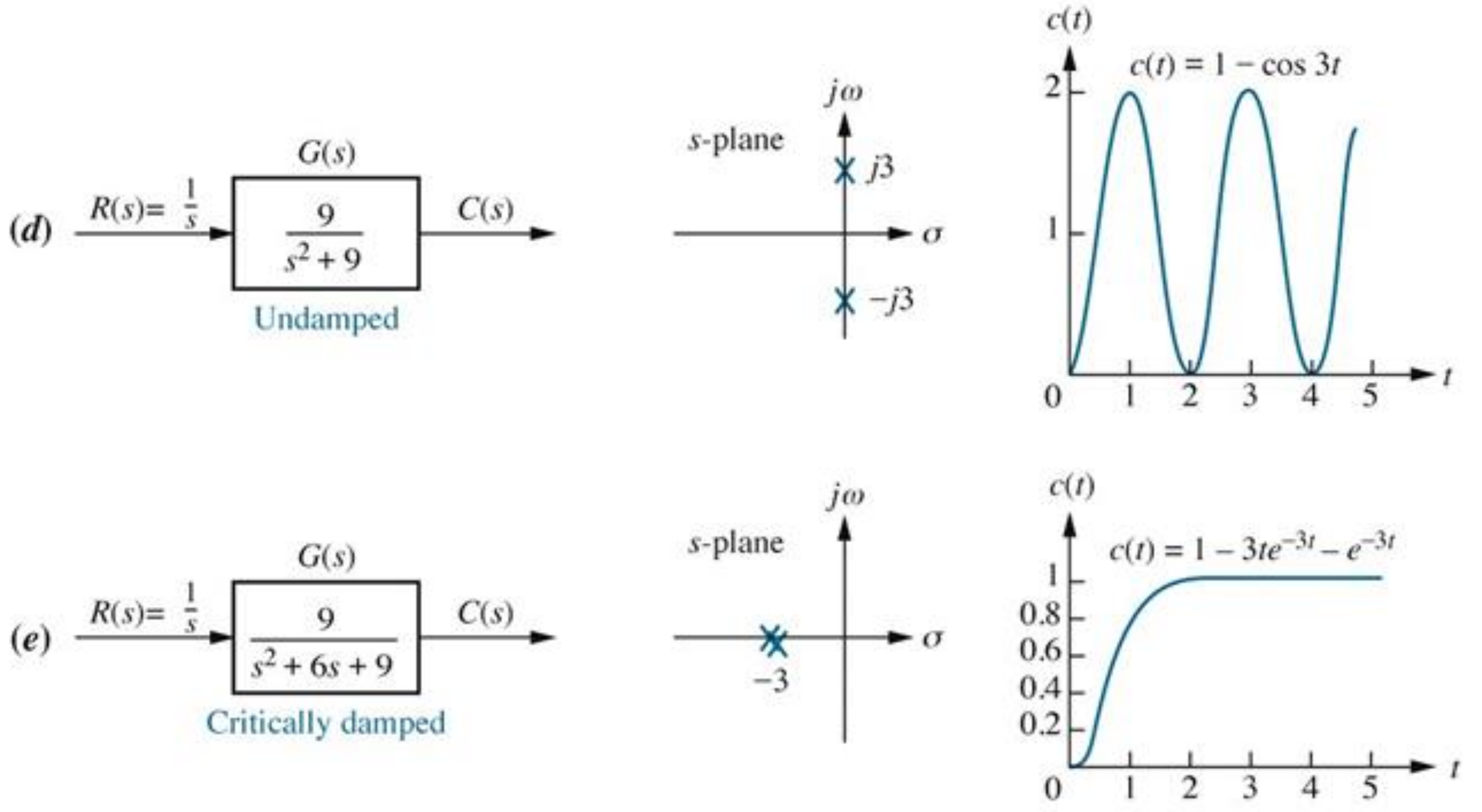
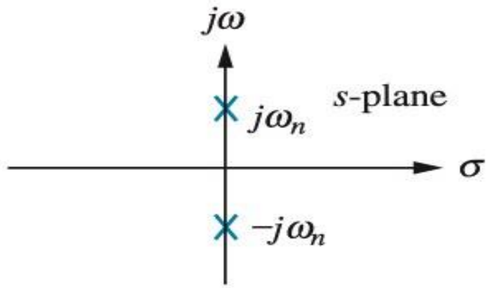
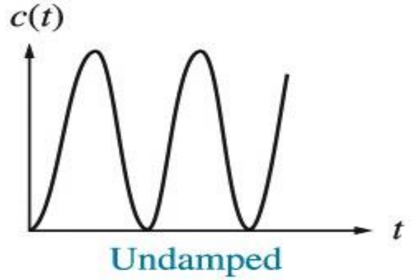
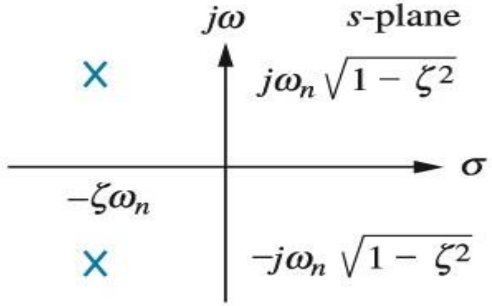
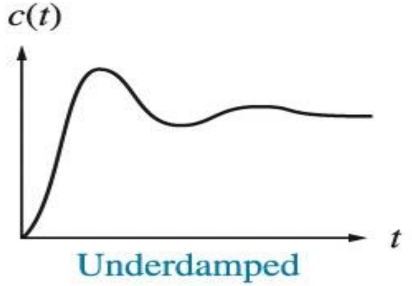
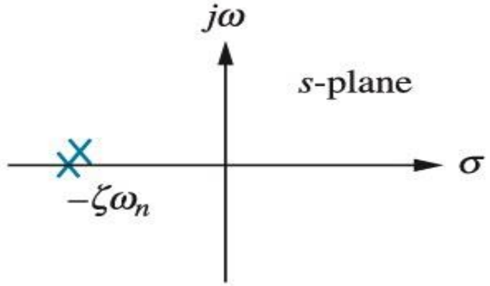
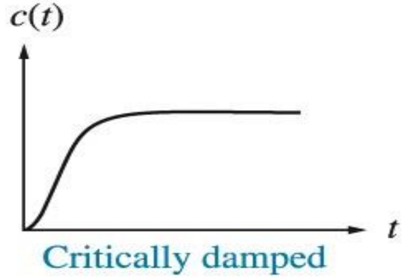
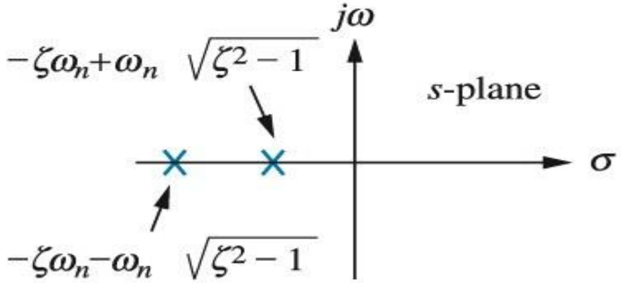
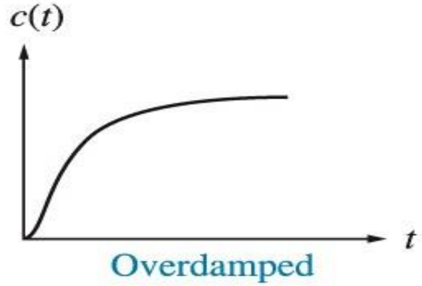
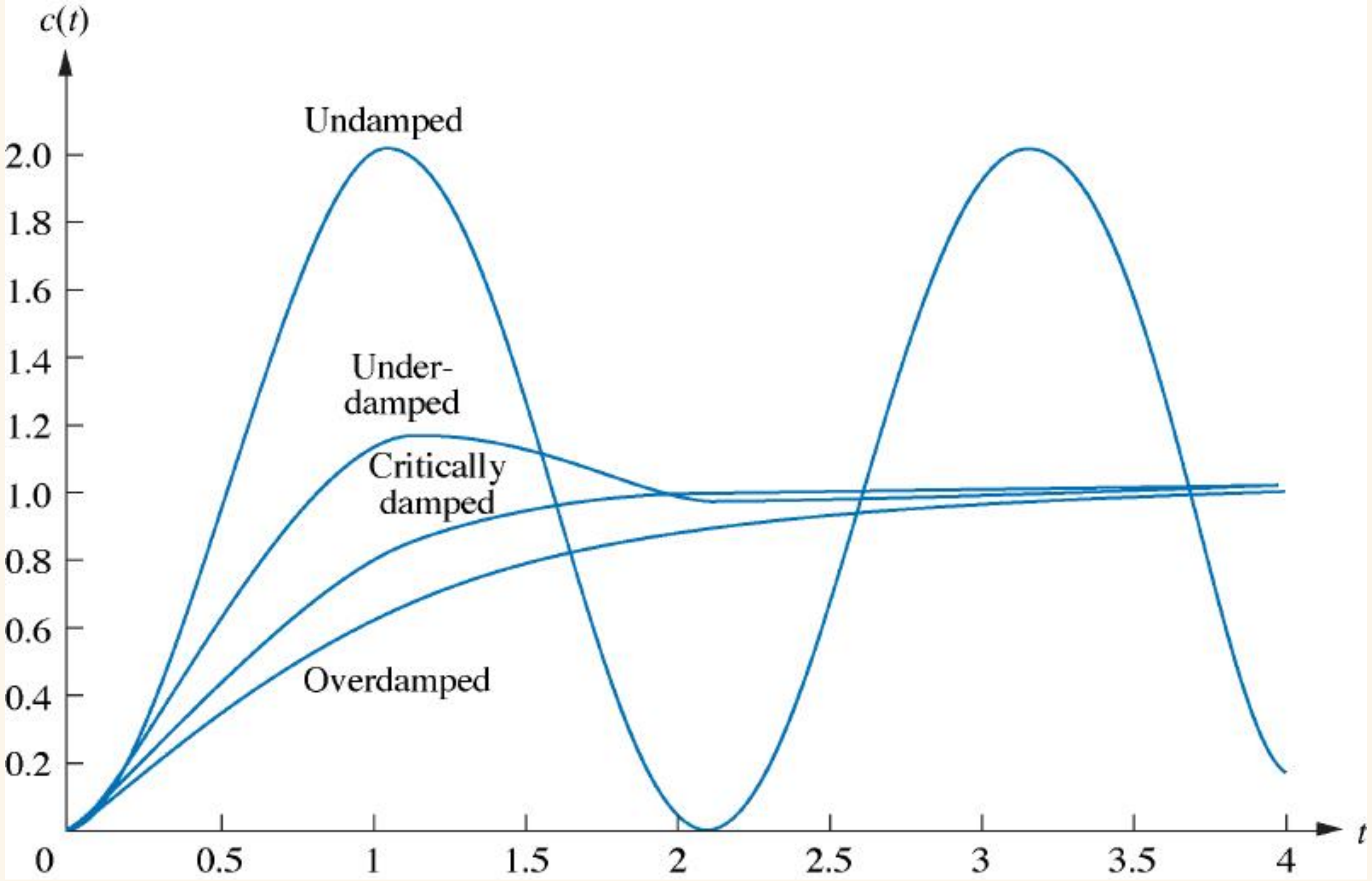


Figure 4.7cde

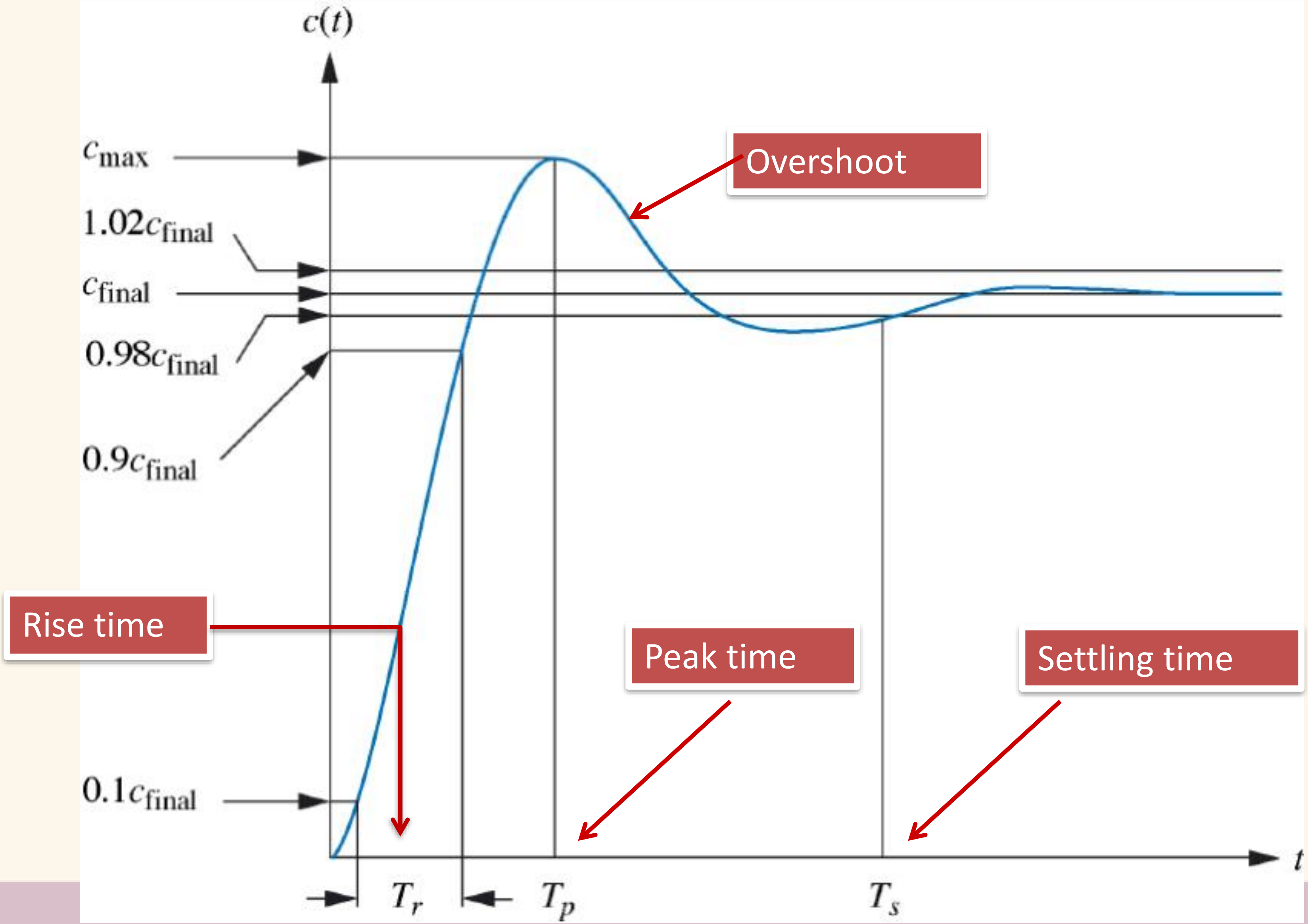
ζ	Poles	Step response
0		
$0 < \zeta < 1$		
$\zeta = 1$		
$\zeta > 1$		

Re-visit: 2nd Order System Response

- Responses:
- Example



Re-visit: Underdamped Response



Re-visit: Performance Specifications

- Peak time,

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

- Percent overshoot,

$$OS = e^{-\frac{\zeta\omega_n}{\sqrt{1-\zeta^2}}} \times 100$$

- Settling time,

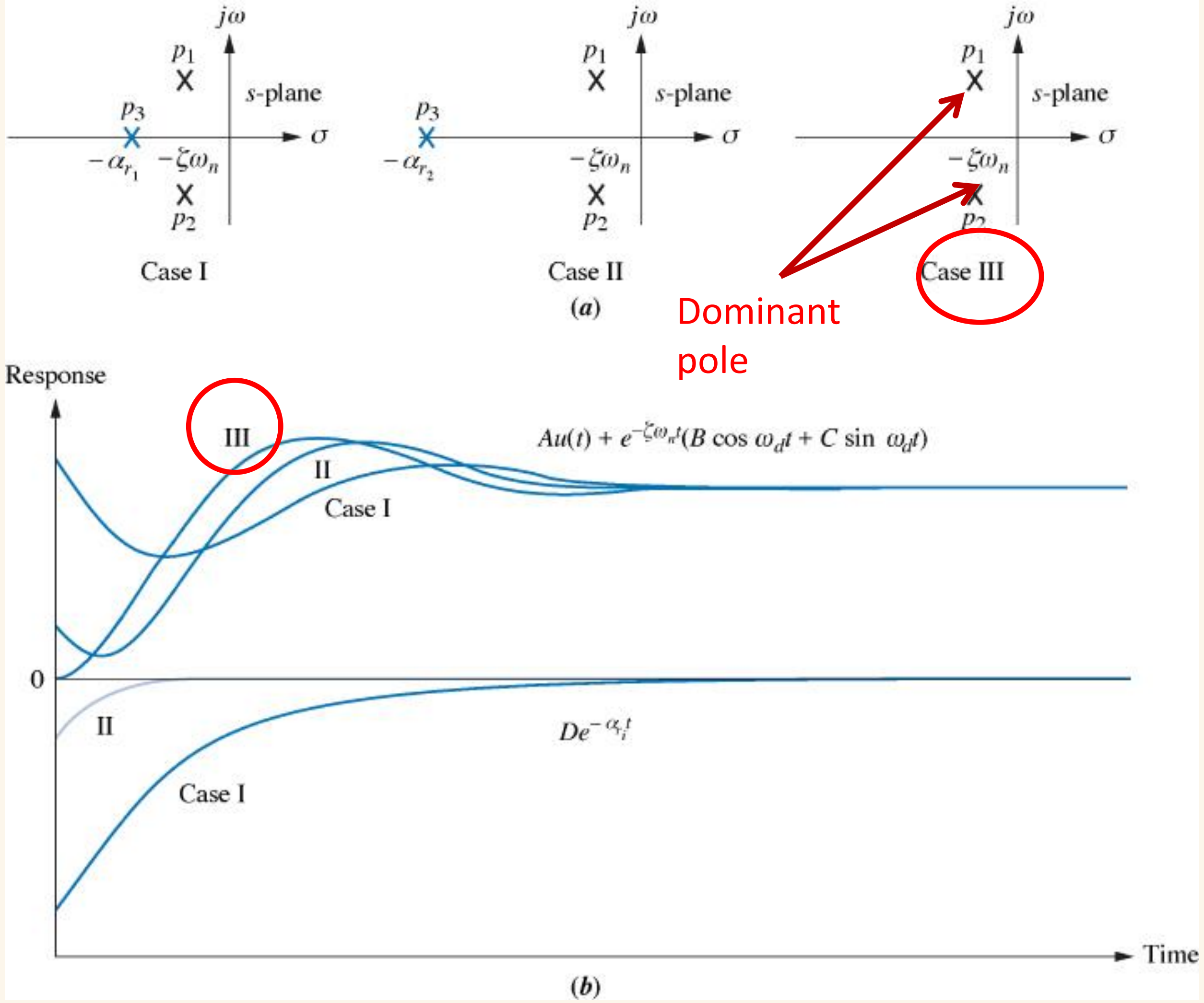
$$T_s = \frac{4}{\zeta\omega_n}$$

Re-visit: Additional Poles

- The formulae were derived based on a purely second order system. Therefore, the formulae valid only for second order system without zero.
- However, under certain conditions, a system with more that 2 poles can be approximated as a 2^{nd} order system.
- In this case, the complex poles are known as dominant poles.

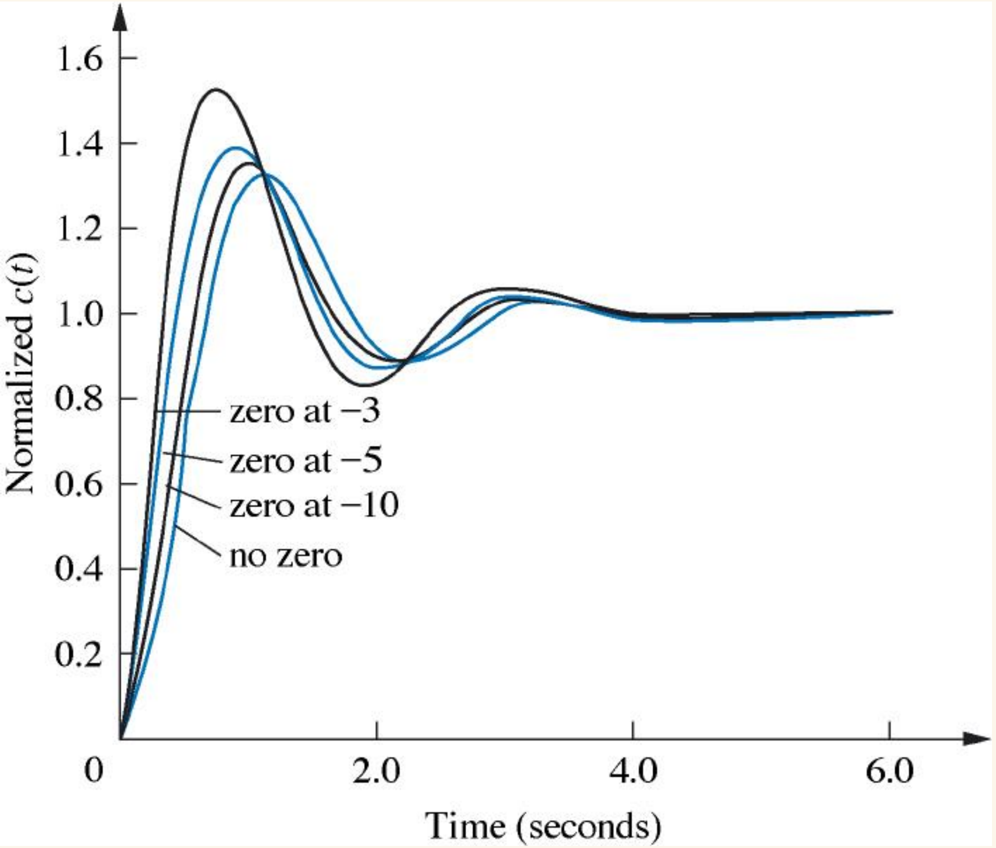
Re-visit: Additional Poles

- Consider 3 cases:
- Case III behaves as a 2nd order response
- Approximation is valid if the third pole more than 5 times farther to the left
- p1 and p2 are dominant poles.



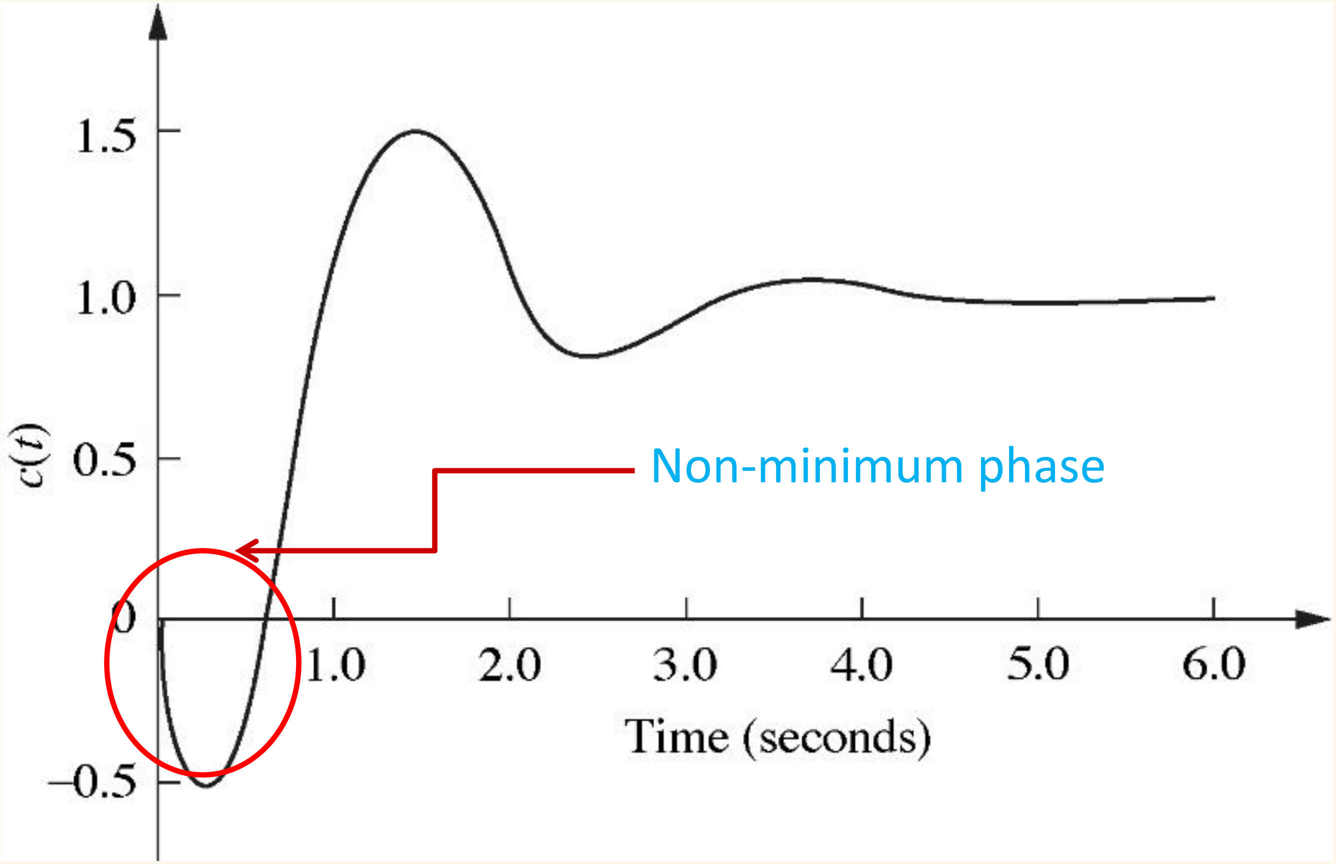
Additional Zeros

- The zeros affect the amplitude of response but do not affect the nature of the response.
- Consider a 2nd order system with poles, $s_{1,2} = -1 \pm j2.828$. Adding zero at -3, -5 and -10.
- The closer the zero to the dominant poles, the greater the effect on the transient response



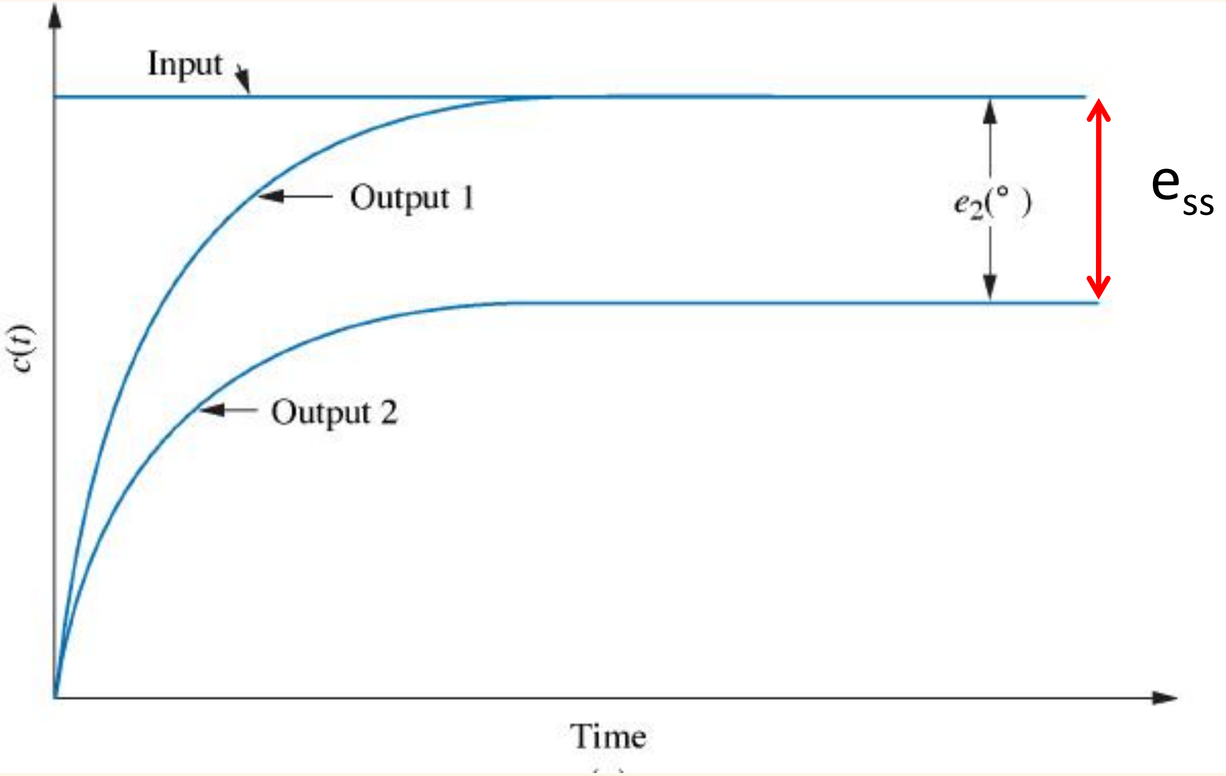
Additional Zeros

- Adding a positive zero,
- Example



Re-visit: Steady-State Errors

- Steady-state error (e_{ss}) is the difference between the input and output for a prescribed test input as t → ∞.



Steady-State Errors

- e_{ss} can be calculated using the final value theorem.

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

- where $E(s) = R(s) - C(s)$
- e_{ss} depends on the input and system type.
- Static error constants: position constant (K_p), velocity constant (K_v) and acceleration constant (K_a).

$$\boxed{e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}}$$

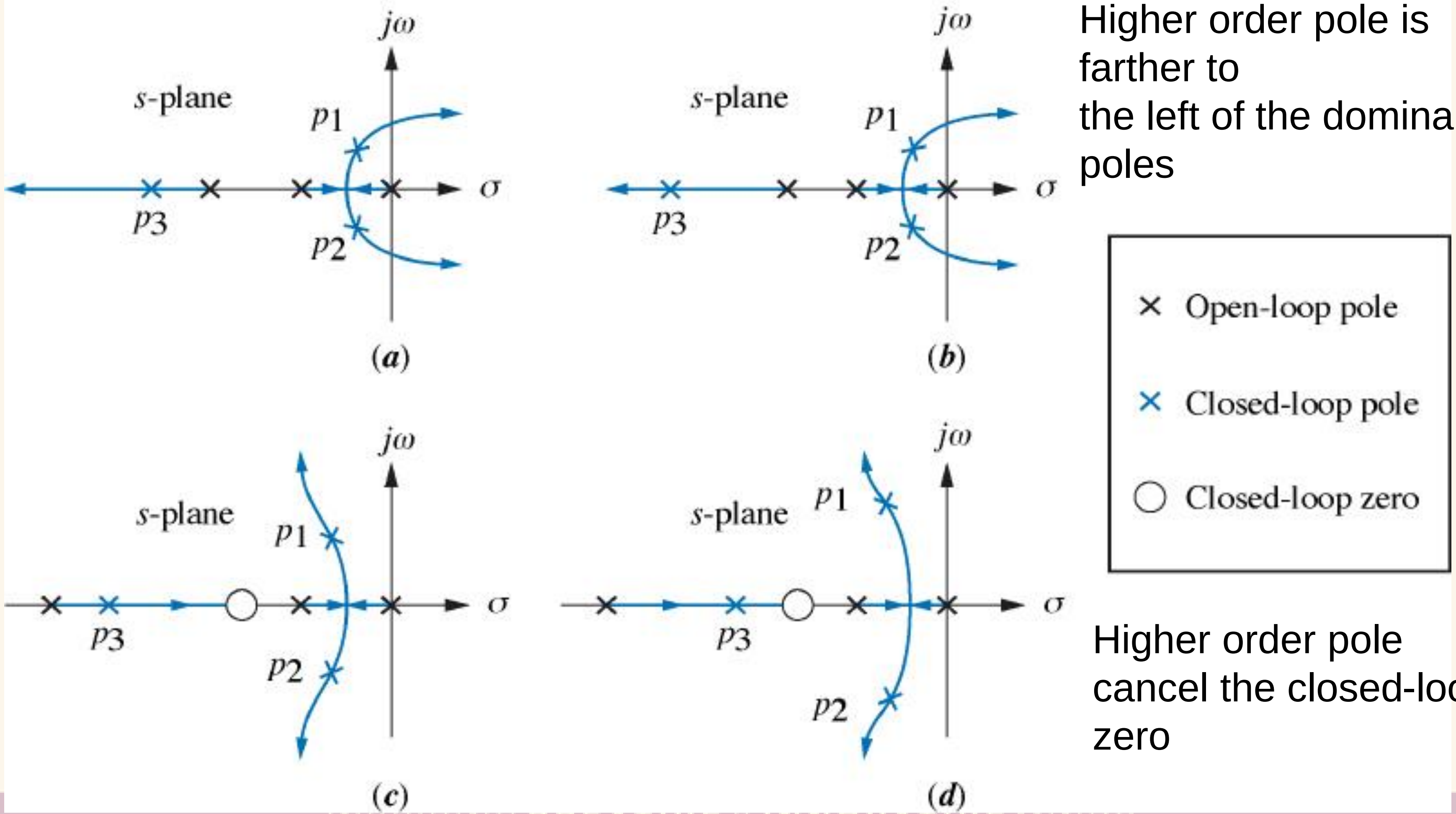
2.2
Design with Gain Adjustment

Second-order Approximation

- The formula for the time response specifications only valid for a pure second order system without zero.
- Conditions for second order approximation:
  - Higher order poles are *five* time farther into the left that the dominant second order poles.
  - Closed-loop zeros are nearly cancelled by the close proximity of higher order poles.

Second-order Approximation

- Figures (b) and (d) yield better approximation.



Higher order pole is farther to
the left of the dominant poles

Higher order pole cancel the closed-loop zero

Design with gain adjustment

- Design procedure for higher order systems :
  - Sketch the root locus.
  - Assume the system is a second order system without zero and find the gain to meet the specifications.
  - Justify the second-order approximation by finding higher order poles.

Example 1

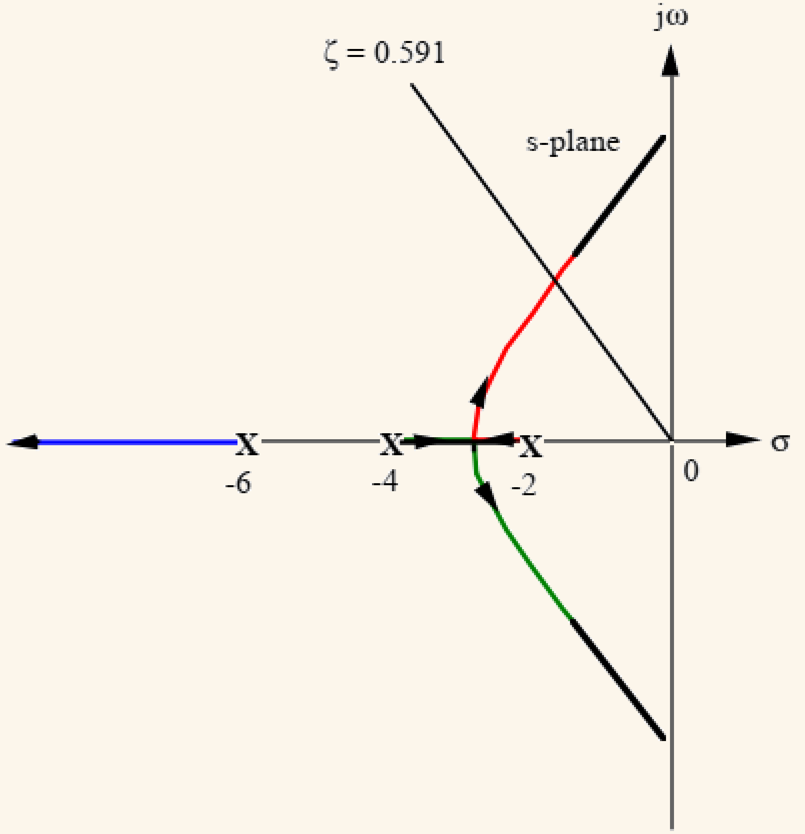
- For a unity feedback system that has the forward-path transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

a) Sketch the root locus
b) Design the value of K to yield 10% OS
c) Estimate the settling time, peak time and steady-state error for the value of K in (b).
d) Determine the validity of your second-order approximation.

Solution 1

- $K = 45.55$
- $T_s = 1.97$ s, $T_p = 1.13$ s, $e_{ss}=0.51$
- Second order approximation is *not valid*.



b. Searching along the $\zeta = 0.591$ (10% overshoot) line for the 180^0 point yields $-2.028+j2.768$ with $K = 45.55$.

c. $T_s = \dfrac{4}{|\text{Re}|} = \dfrac{4}{2.028} = 1.97$ s; $T_p = \dfrac{\pi}{|\text{Im}|} = \dfrac{\pi}{2.768} = 1.13$ s;

$\omega_n T_r = 1.8346$ from the rise-time chart and graph in Chapter 4. Since ω_n is the radial distance to the pole, $\omega_n = \sqrt{2.028^2 + 2.768^2} = 3.431$. Thus, $T_r = 0.53$ s;

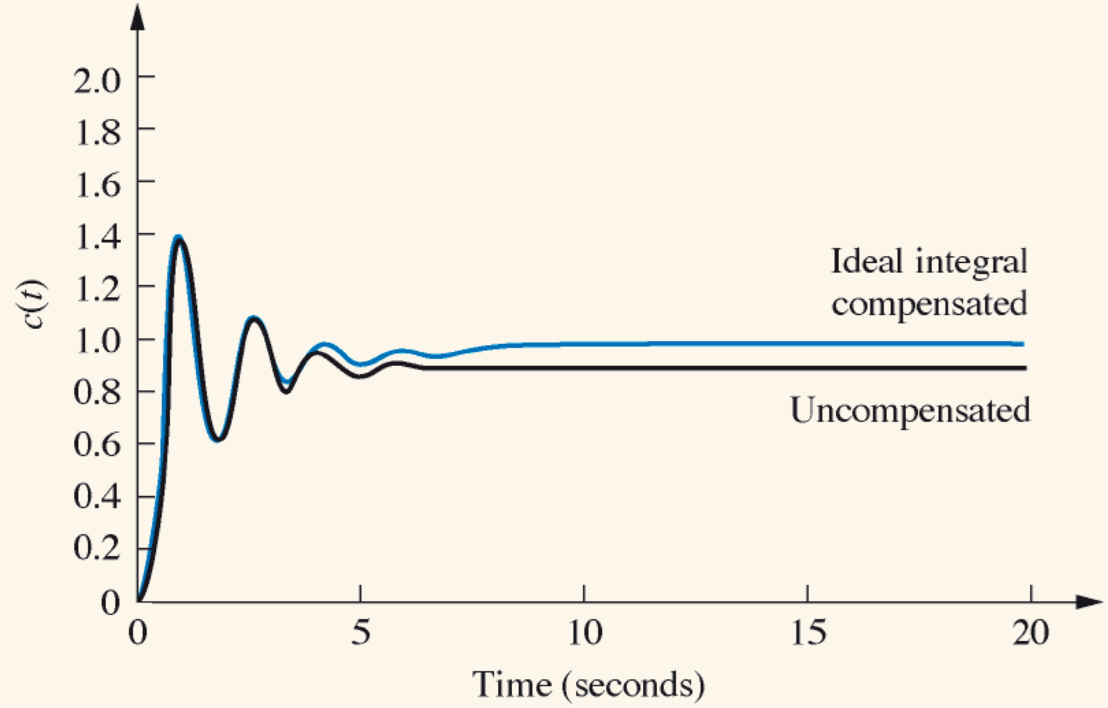
since the system is Type 0, $K_p = \dfrac{K}{2*4*6} = \dfrac{45.55}{48} = 0.949$. Thus,

$$e_{step}(\infty) = \frac{1}{1+K_p} = 0.51.$$

d. Searching the real axis to the left of –6 for the point whose gain is 45.55, we find –7.94. Comparing this value to the real part of the dominant pole, -2.028, we find that it is not five times further. The second-order approximation is not valid.

Design with gain adjustment

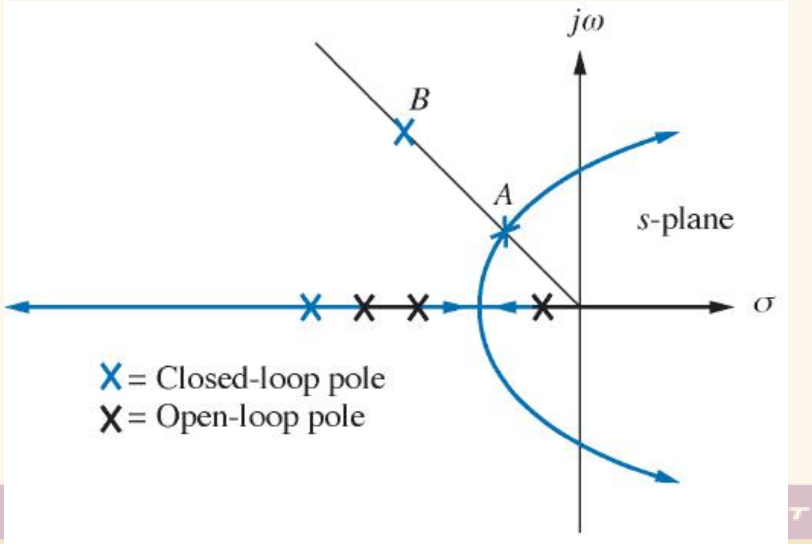
- The root locus allows us to choose a proper gain to meet a transient response specification.
- As the gain is varied, we move through different regions of response.
- However, we are limited to those responses that exists along the root locus only.

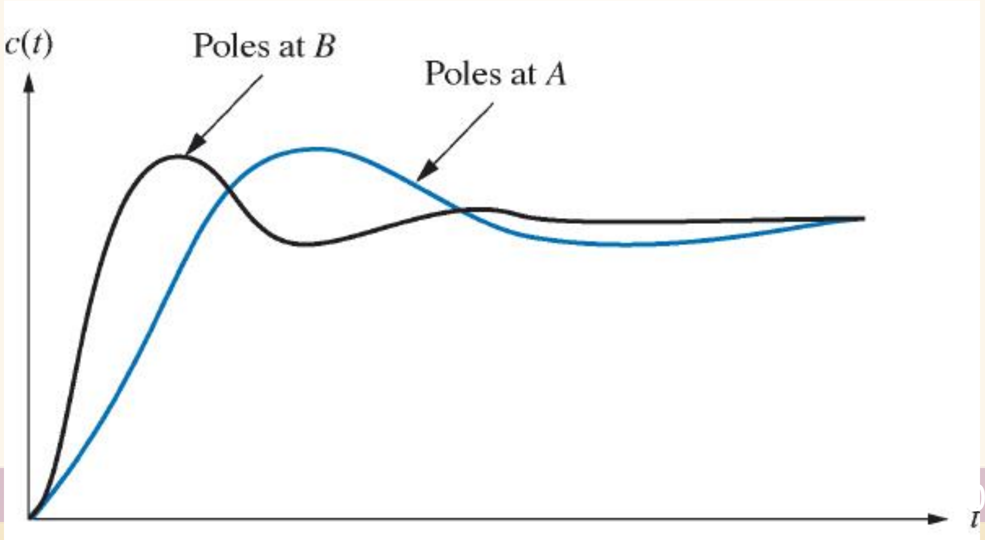


2.3 Improving the Steady State Error (USE PI CONTROLLER)

Improving the System Response

- Flexibility in the design can be increased if we can design for response that are not on the root locus.
- Consider the desired transient response defined by OS and settling time.
- With gain adjustment, we can only obtain settling time at A to satisfy the desired OS.
- Point B cannot be on the root locus with the gain adjustment.
- A controller/compensator has to be designed.





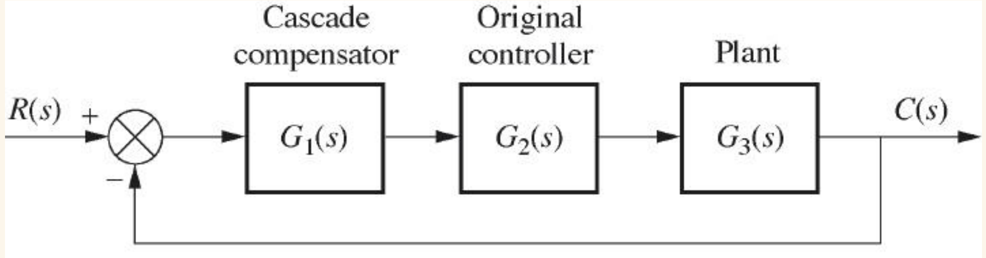
Improving Steady-state Error

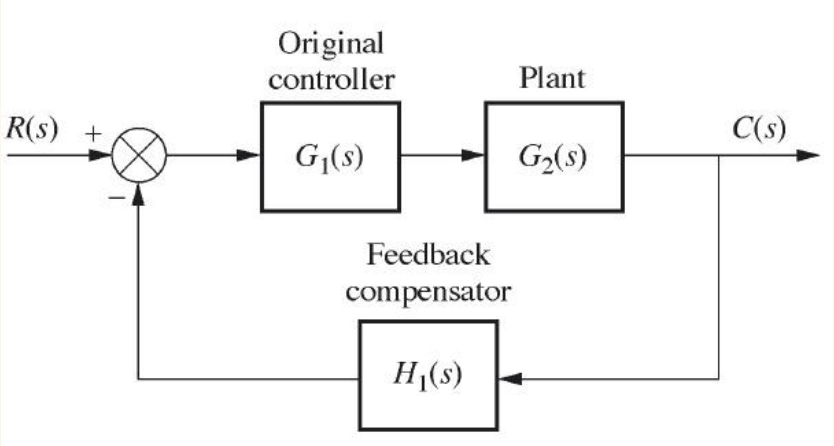
- Compensators can also be used to improve steady-state error.
- By increasing the gain, the ess reduces but OS increases.
- By reducing the gain, OS reduces but ess increases.
- Compensators can be designed to meet the transient response and steady-state error simultaneously.

Configurations

- Two configurations of compensation:
  - Cascade compensation: The compensation network is placed in cascade with the plant.
  - Feedback compensation: The compensator is placed in the feedback path

- Both methods change the OL poles and zeros, thus creating a new root locus that goes through the desired CL pole location.





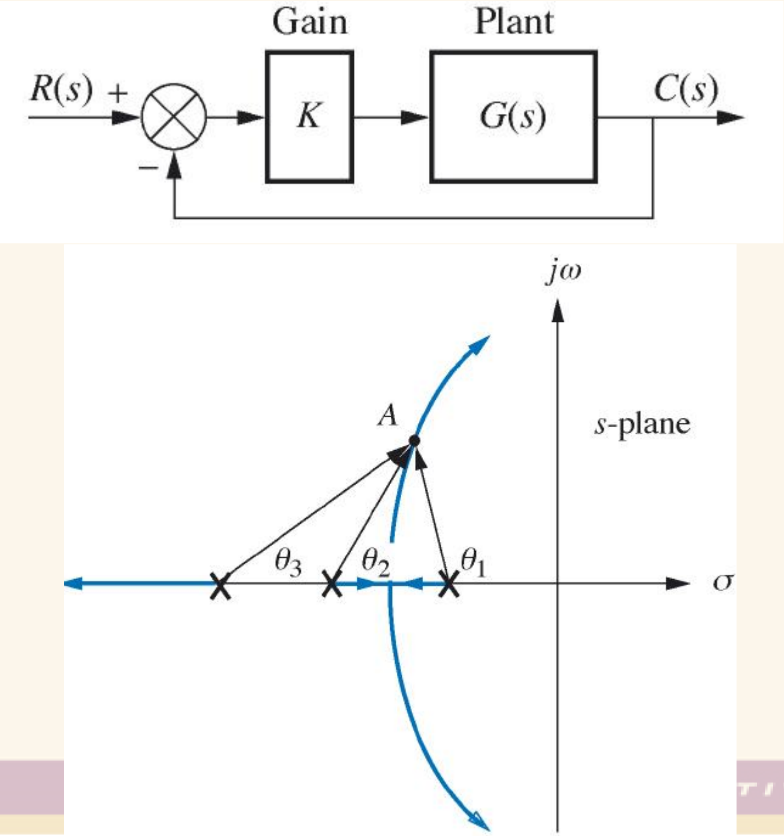
Proportional-Integral (PI) Controller/Compensator

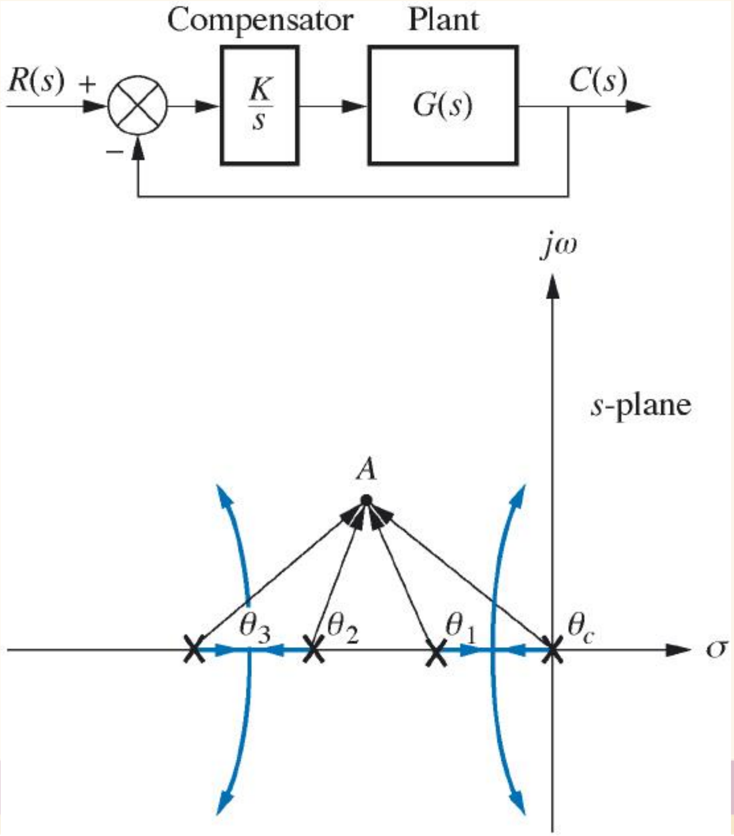
- PI controller/compensator is used to improve steady-state error.
- e_{ss} can be improved by placing an open-loop pole at the origin as this increases the system type by one.

		Type 0		Type 1		Type 2	
Input	Steady-state error formula	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Proportional-Integral (PI) Controller/Compensator

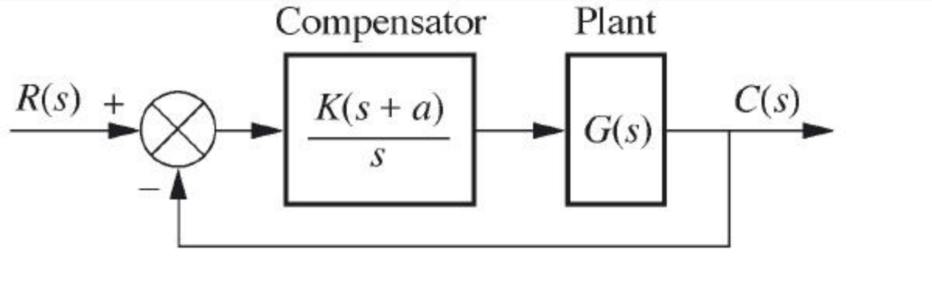
- Consider a system operating at a desirable response with CL pole at A.
- By adding a pole to increase the system type, A is no longer a CL pole.

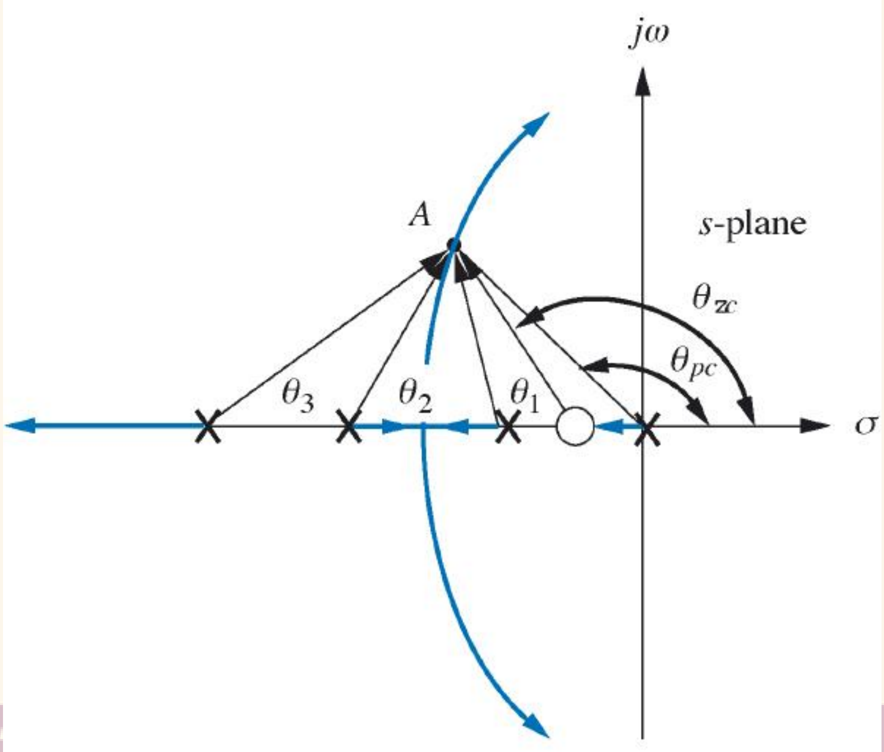




Proportional-Integral (PI) Controller/Compensator

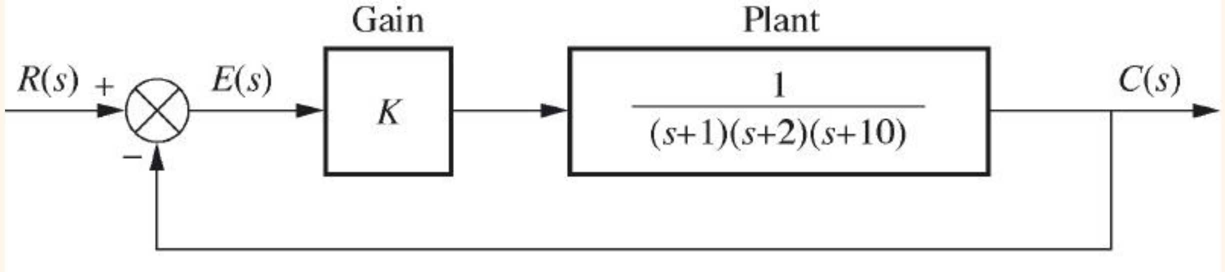
- To solve, a zero close to the pole at the origin has to be added. Thus A is now a CL pole.
- Thus, we have improved the e_{ss} without affecting the transient response. This is known as *PI controller/compensator*.



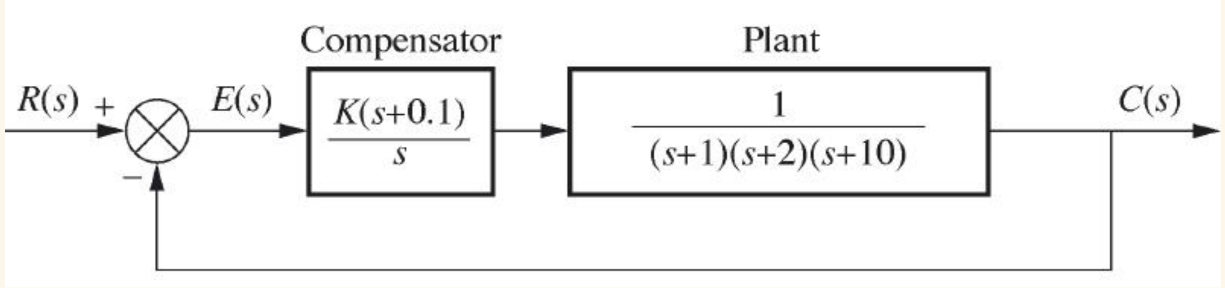


Example 2

- Given a system operating with a damping ratio of 0.174, show that the addition of the PI compensator reduces the e_{ss} to zero for a unit step input without affecting transient response.

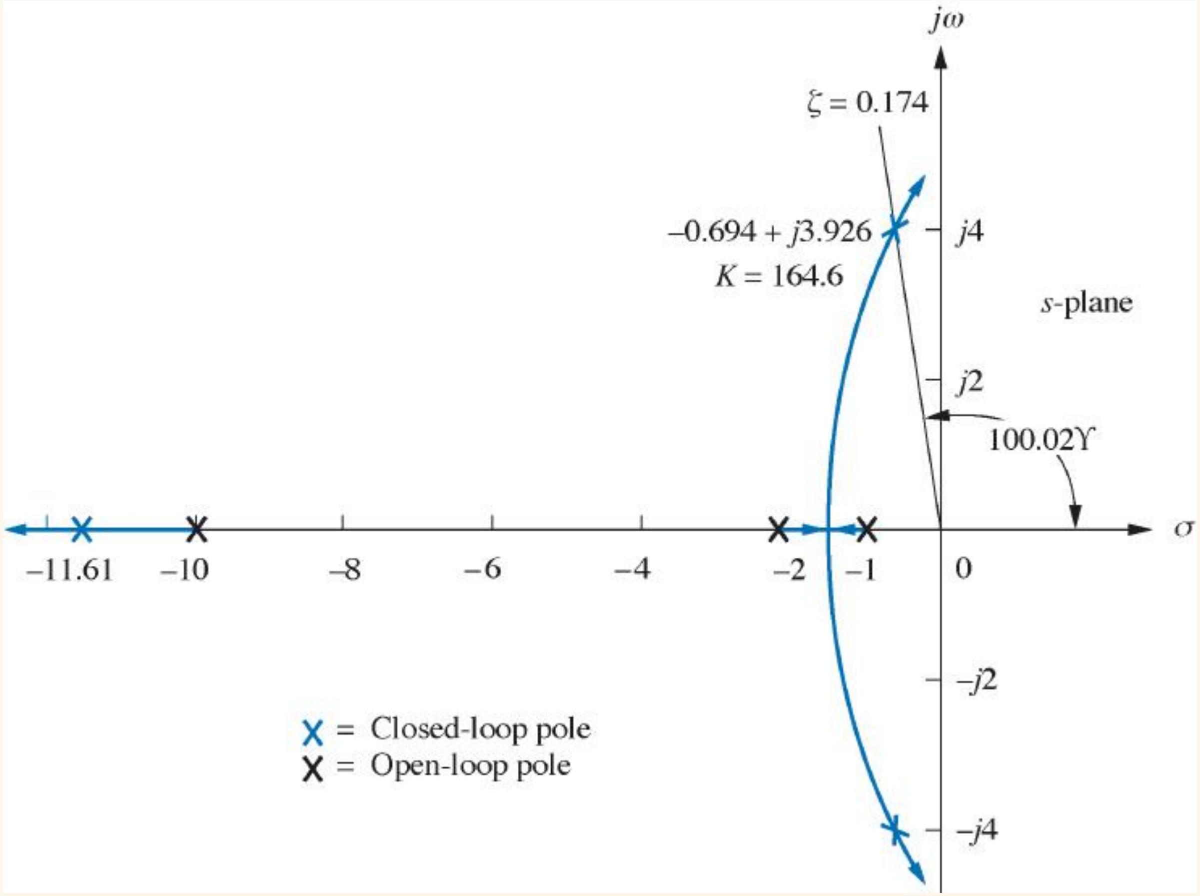


Before compensation



After compensation

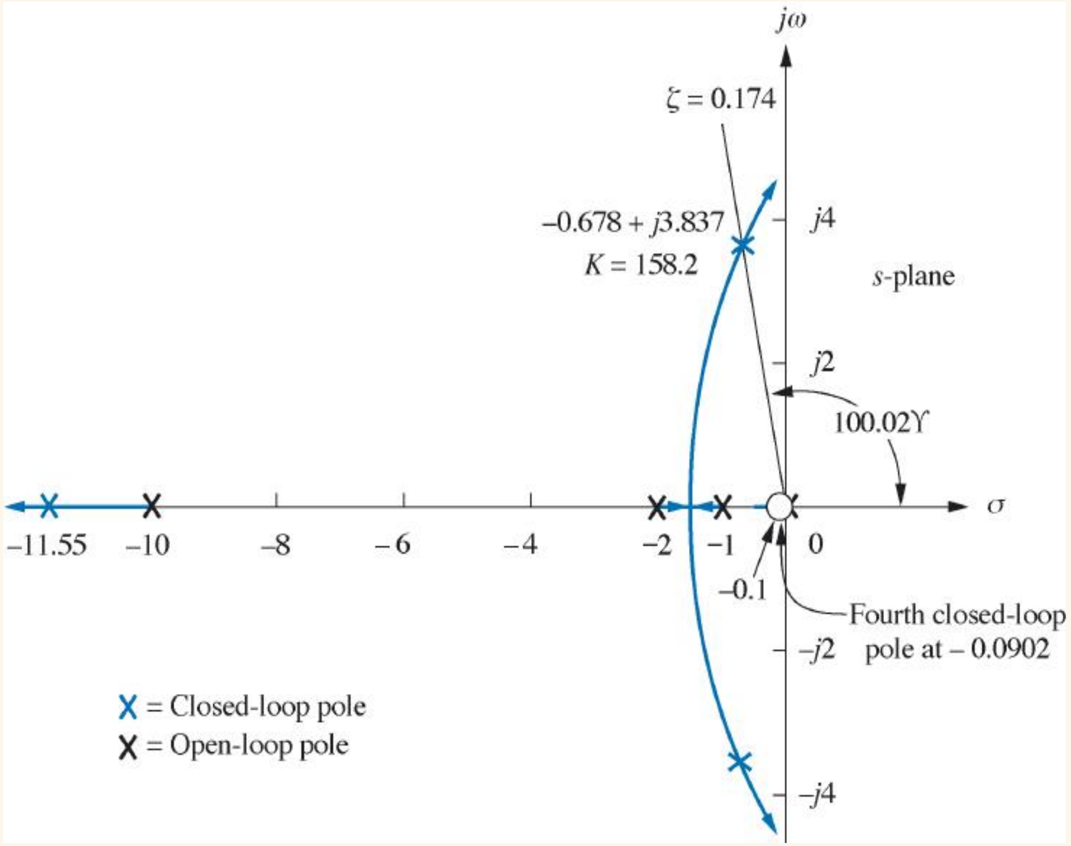
Solution 2



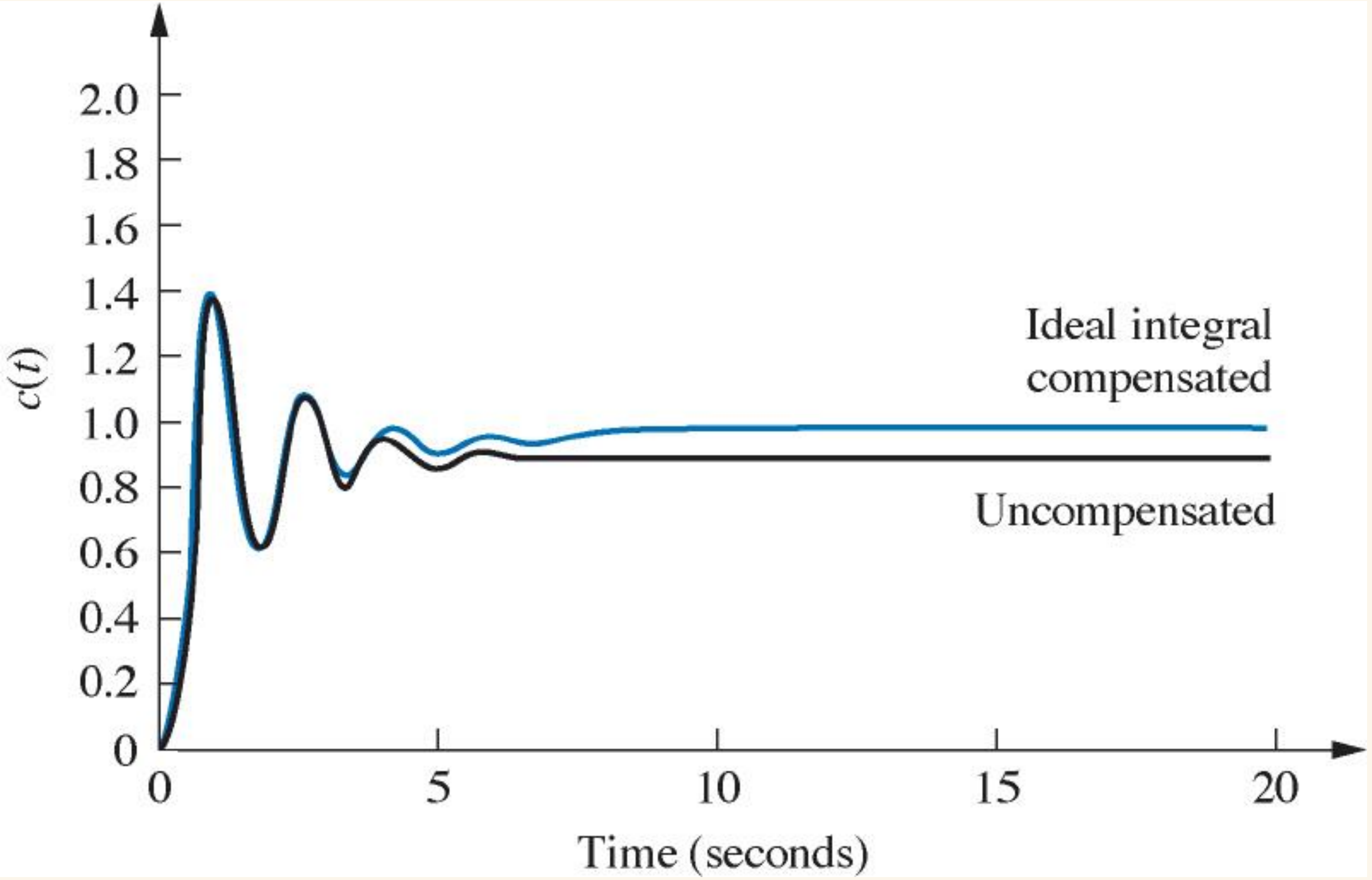
- For a damping ratio 0.174, the dominant poles: s = -0.694 ± j 3.962, K = 164.6.
- The third pole at -11.61 and second-order approximation is valid.
- K_p=8.23, e_{ss} = 0.108.

Solution 2

- Adding a pole at the origin and zero at -0.1.
- With the same damping ratio, dominant poles, s = -0.678 ± j.837, K = 158.2.
- Compensated CL poles and gain are approximately the same as the uncompensated system.
- Both gives the same transient response.
- However, with PI, system is type 1 and $e_{ss} = 0$.



Solution 2



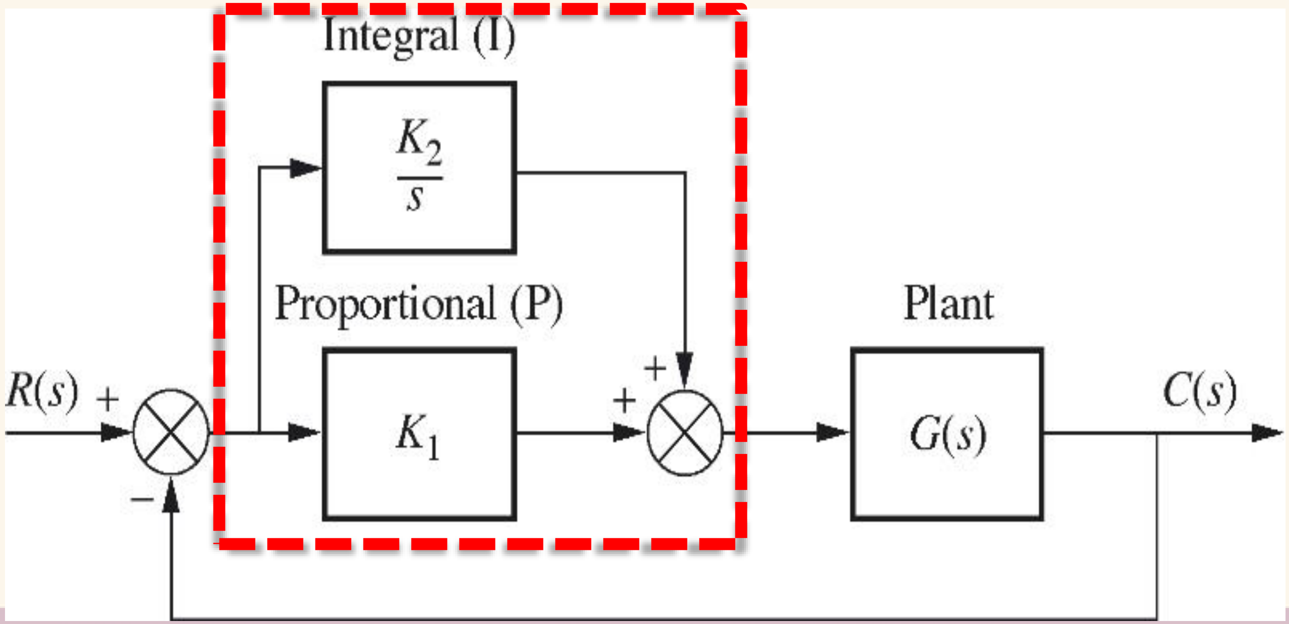
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
$c(t)$
Ideal integral compensated
Uncompensated
0
5
10
15
20
Time (seconds)

Proportional-Integral (PI) Controller/Compensator

- In general, transfer function of PI controller

$$G_{PI}(s) = K_P + \frac{K_I}{s} = \frac{K_P(s + {K_I}/{K_P})}{s} = \frac{K(s+z)}{s}$$

- Implementation



Proportional-Integral (PI) Controller/Compensator

- PI controller is designed by adding a pole at the origin and a zero at $s = -z$.
- The zero is chosen to satisfy transient response specifications.
- The value of the zero can be adjusted by varying K_I/K_P.
- If the same transient response as the uncompensated system is required, choose the zero close to the origin. Example: $s = -0.1$, $s = -0.01$.

$$G_{PI}(s) = K_P + \frac{K_I}{s} = \frac{K_P\left(s + {}^{K_I}\!/_{K_P}\right)}{s} = \frac{K(s+z)}{s}$$

Example 3

- For the unity feedback system with

$$G(s) = \frac{K}{(2s+1)(0.5s+1)}$$

- design a PI controller to achieve a transient response as the following:
  - OS = 10 %
  - Settling time 16/3 s
  - Zero steady-state error to unit step input

Solution 3

- Sketch the root locus of the system without the PI Controller.
- Find the CL poles for the required transient response
- Desired CL poles: s = -0.75 ± j1

$$T_s = \frac{4}{|\mathrm{Re}|}$$

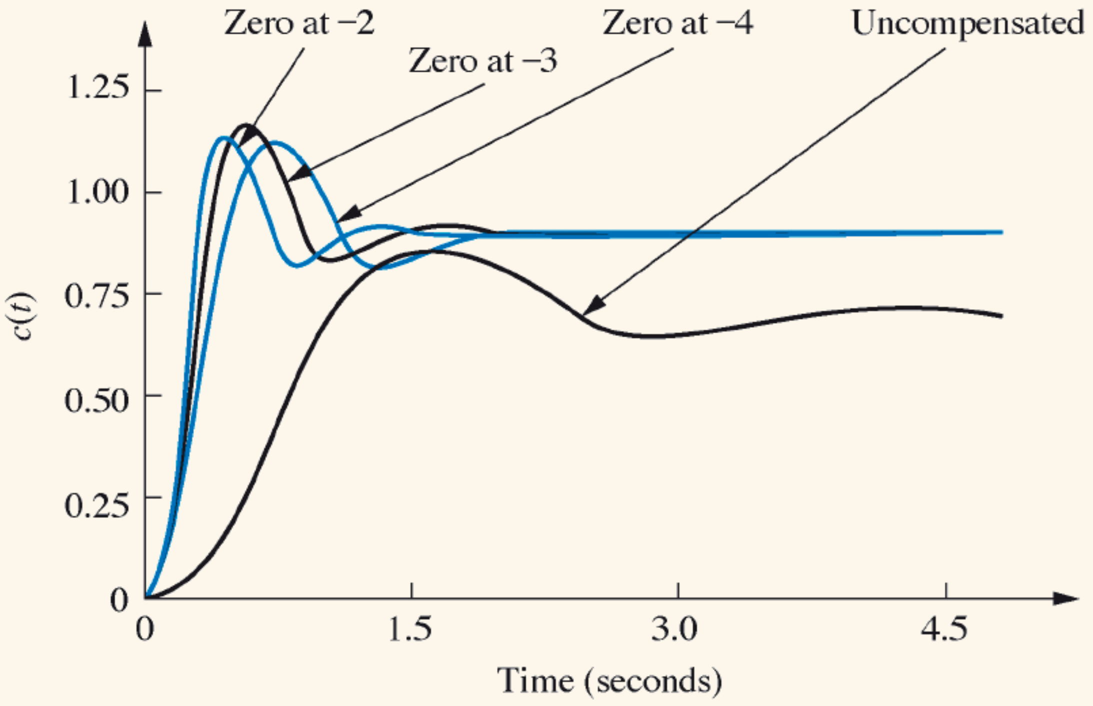
- To achieve zero e_{ss}, add a pole at the origin.
- Determine the location of the zero and let K_p = 1.

$$G_{PI}(s) = \frac{(s+0.01)}{s}$$

- Re-sketch the root locus of the system with the PI Controller
- Find the CL poles and the gain to achieve the transient and steady state responses.
- z = 0.75, K = 2.6

$$G_{PI}(s) = \frac{2.6(s+0.75)}{s}$$

➔we want the overall system with the same transient response but zero steady state error.



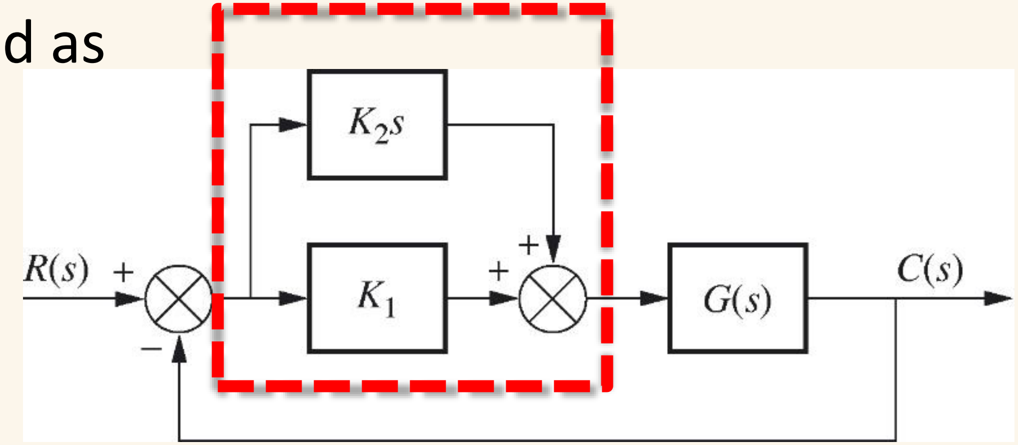
2.4 Improving the Transient Response (USE PD CONTROLLER)

Proportional-Derivative (PD) Controller/Compensator

- PD controller is used to improve transient response and maintaining the steady-state error.
- Transient response can be improved by adding a single zero to the forward path of the feedback control system.
- This zero can be represented as

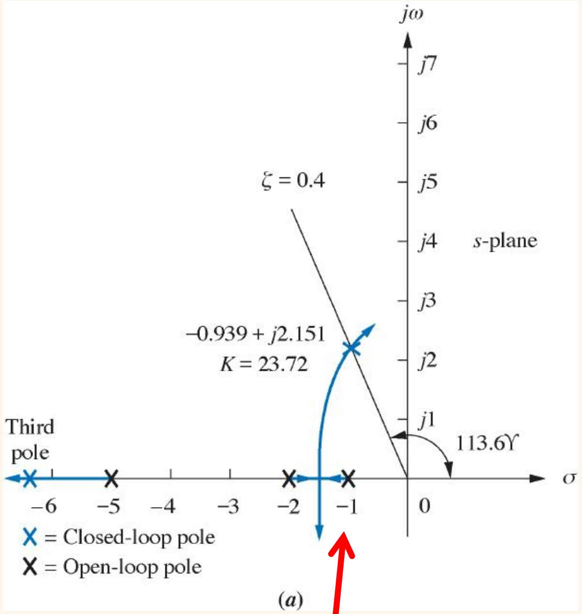
$$G_{PD}(s) = K_P + K_D s = K_D(s + \frac{K_P}{K_D})$$

$$G_{PD}(s) = K(s+z)$$



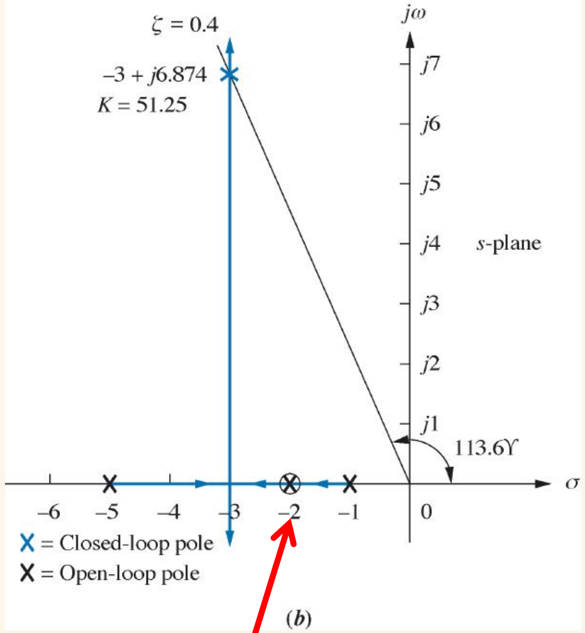
- A proper selection of the zero can improve the system's transient response.

Proportional-Derivative (PD) Controller/Compensator



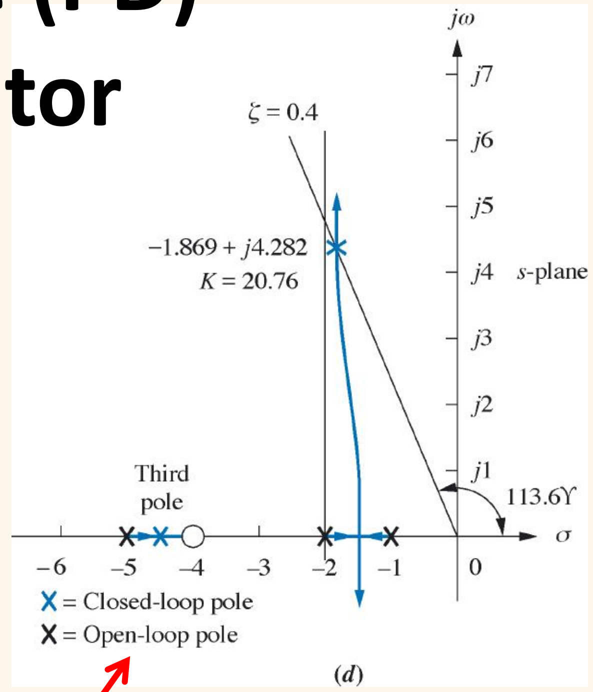
(a)

Uncompensated



(b)

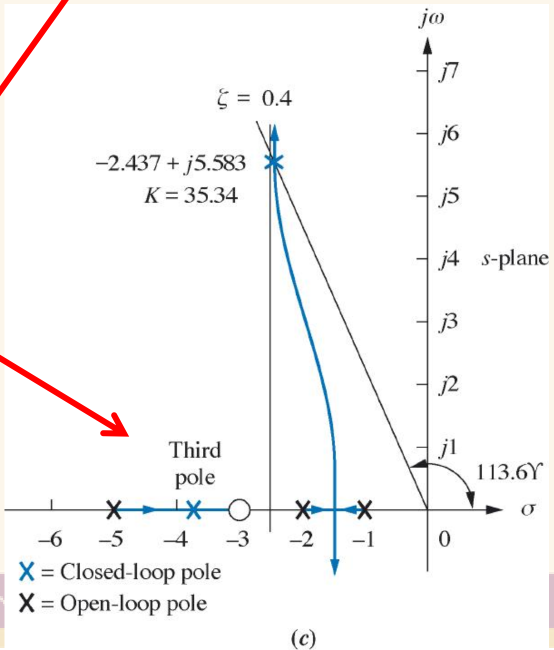
Zero, s = -2



(d)

Zero, s = -4

Zero, s = -3

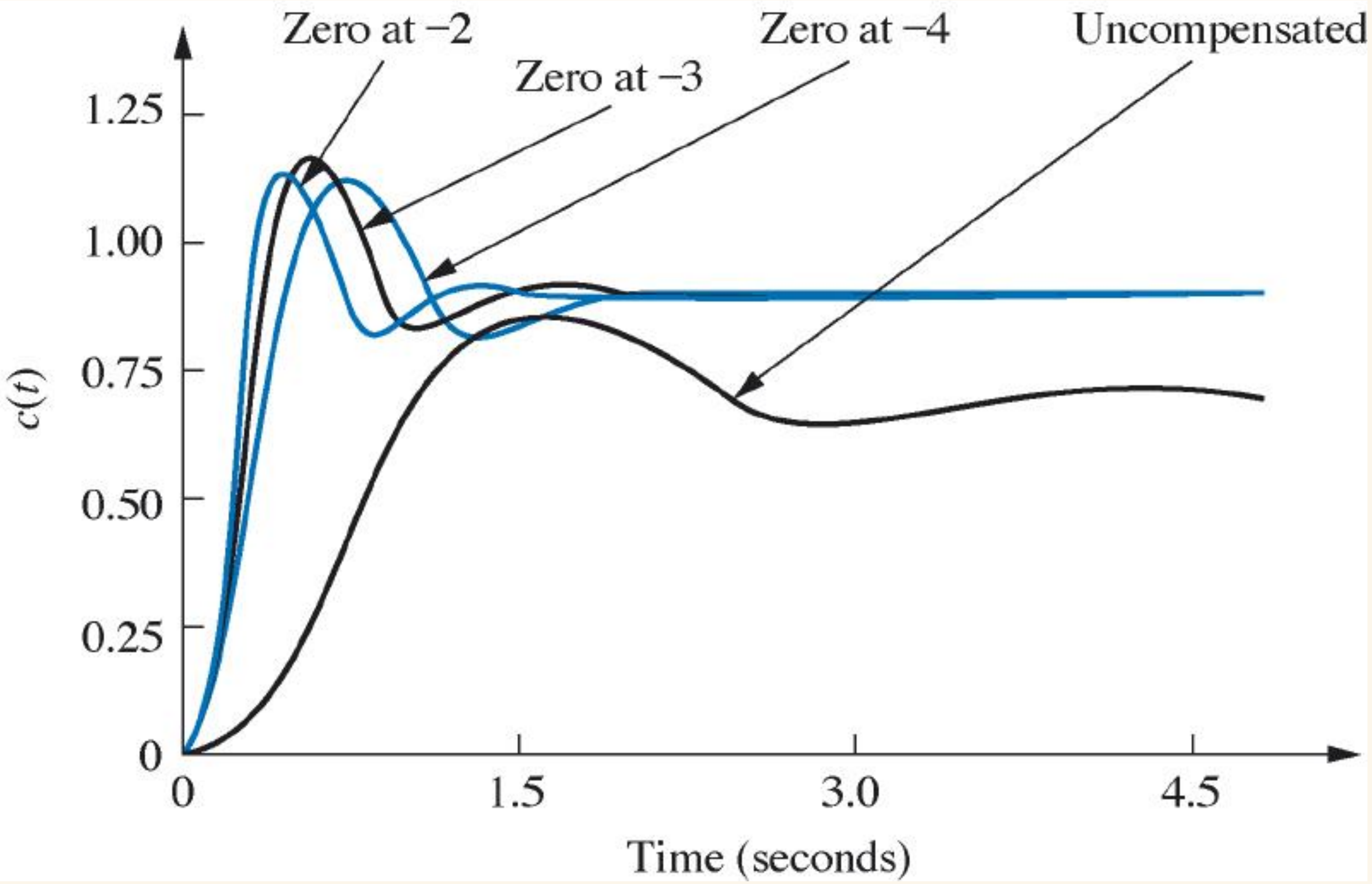


(c)

Proportional-Derivative (PD) Controller/Compensator

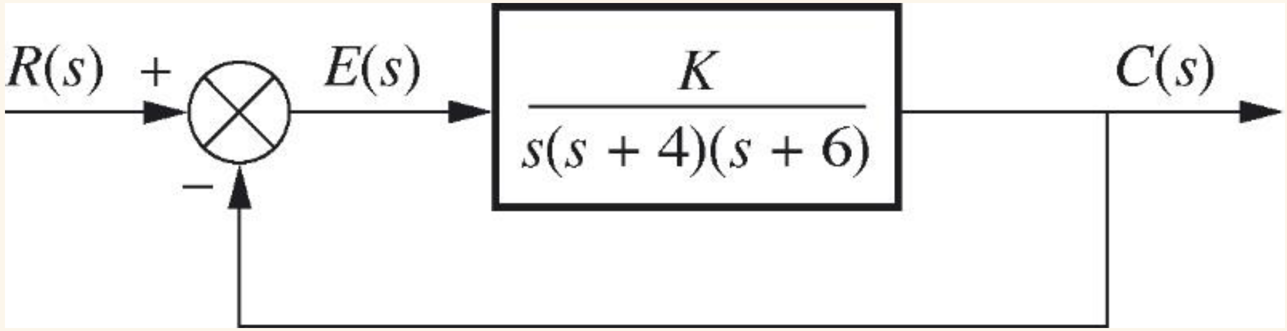
- All compensated systems operate at damping ratio, 0.4 as the uncompensated system.
- For all the compensated systems, the CL poles have more negative real and larger imaginary part as compared to the uncompensated system.
- Thus, the compensated systems operate with shorter settling time and peak time.
- System (b) gives the best transient response.
- In summary, adding a zero on the forward path improves the transient response.
- A proper selection of the value of the zero is required.

Proportional-Derivative (PD) Controller/Compensator



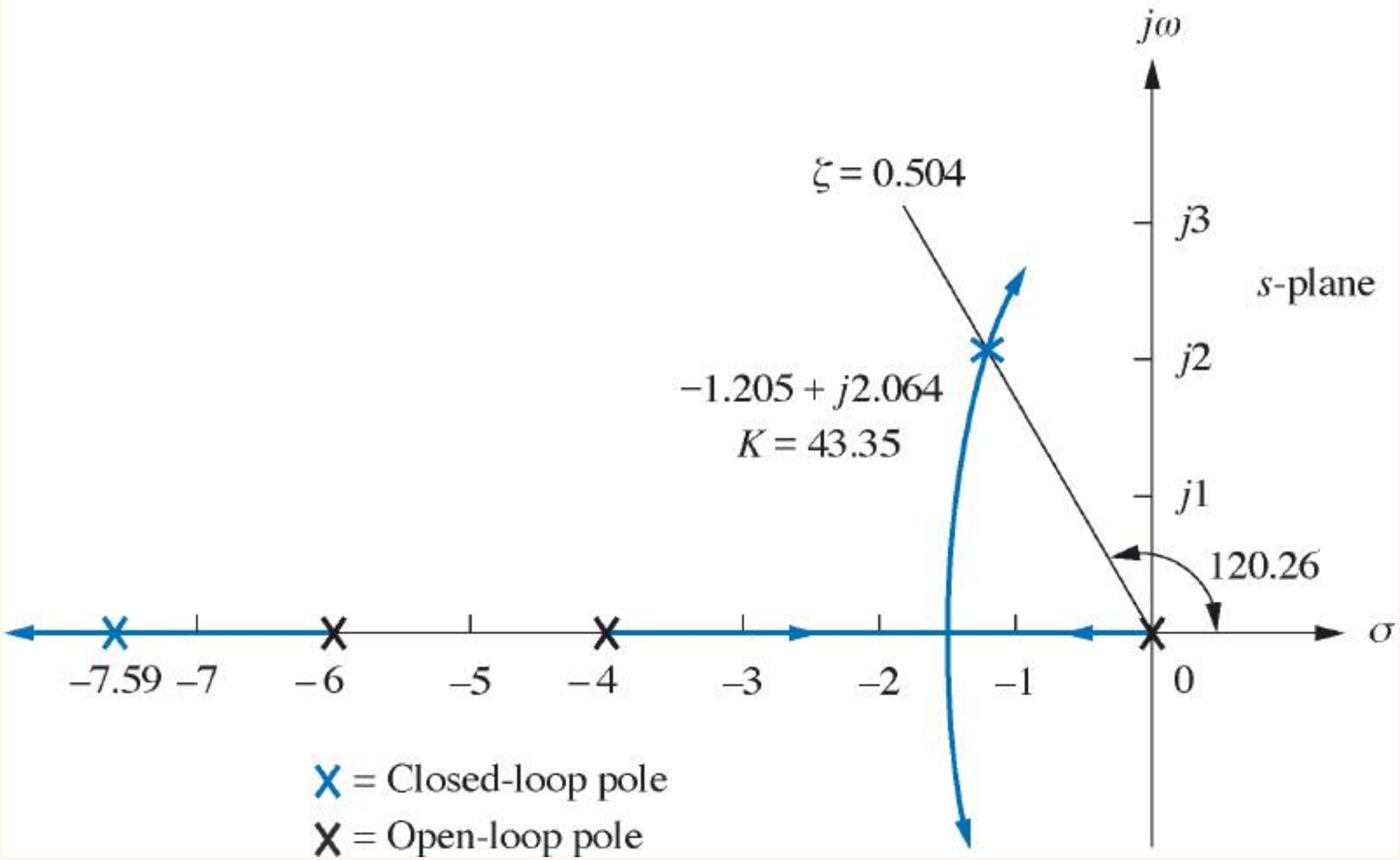
Example 4

- Given the system, design a PD controller to yield 16 % overshoot, with a threefold reduction in the settling time.



Solution 4

- Draw the root locus without PD Controller



Solution 4

- To get 16% *OS,* the CL poles can be found at with T_s = 3.32 s, T_p = 1.52 s
- e_{ss} = 1/K_v = 0.55 (by calculation when *K* = 43.35)
- Third pole at *s* = -7.59. Thus second order approximation is valid.

- Desired specifications:
  OS 16 %, T_s = 3.32/3 = 1.107 s

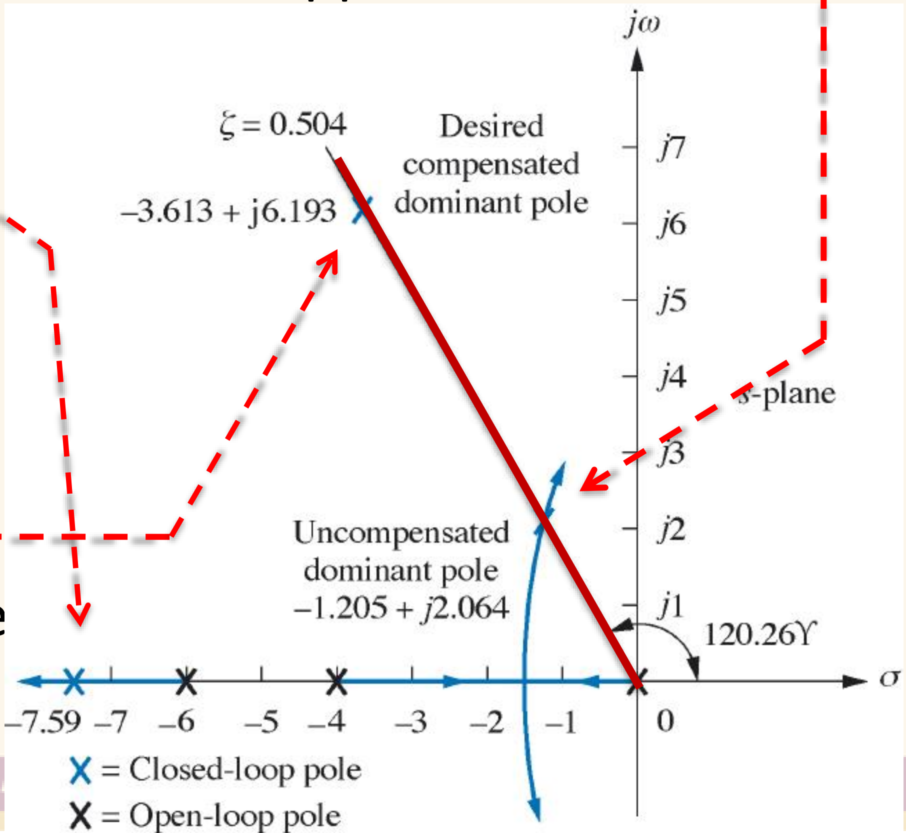
- Desired CL poles, (calculation)

$$T_s = \frac{4}{\sigma}$$

s = -3.61 ± j6.193

- How to calculate? - Use triangle

$$\omega = 3.613\tan(180° - 120.26°) = 6.193$$



Solution 4

- To locate the zero, use angle property:

  Total angles from poles to the root (275^0) – **total angles from zeros to the root** = 180^0.

- The angle = 95.6^0.

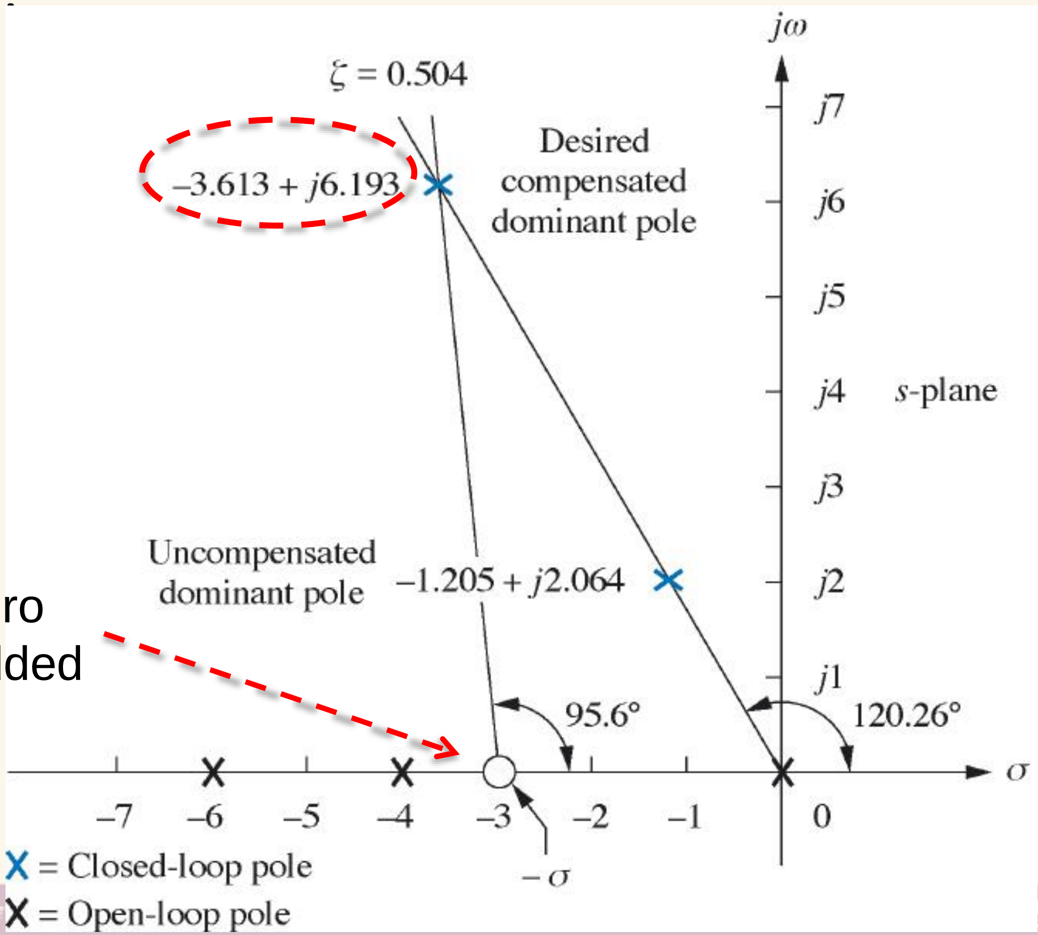
  How?

- Thus, z = 3= σ

  How? Get the triangle.

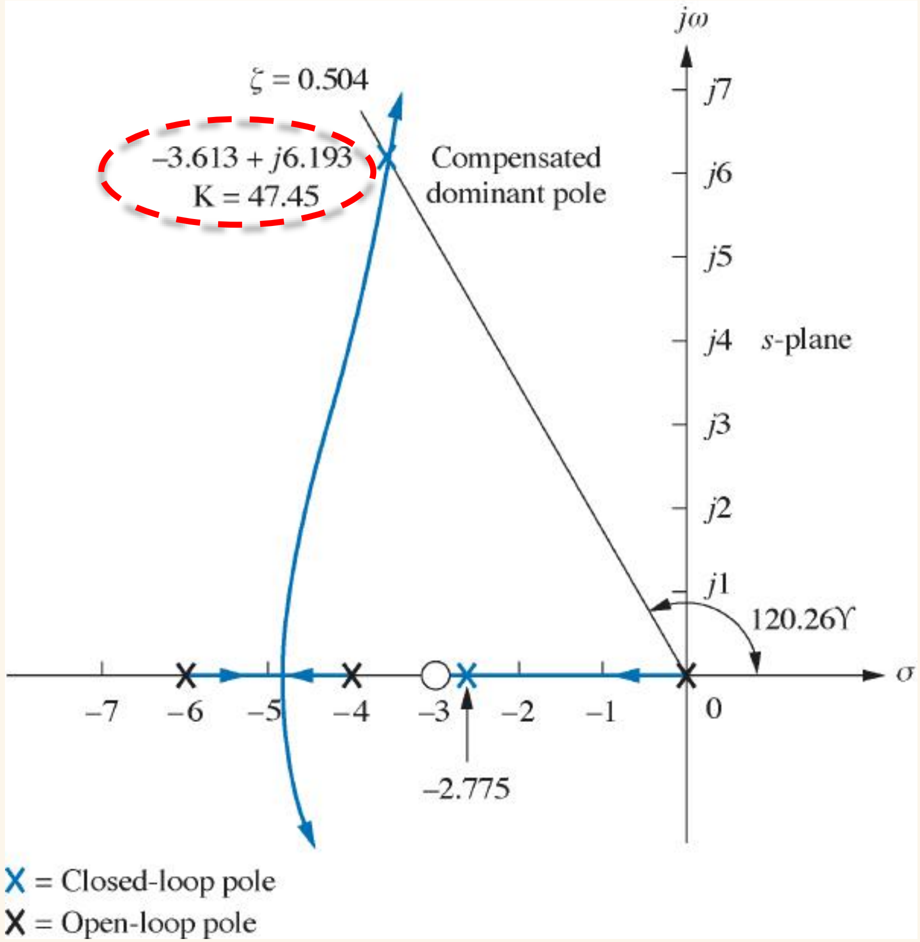
$$\frac{6.193}{3.613-\sigma} = \tan(180^\circ - 95.6^\circ)$$

The zero to be added

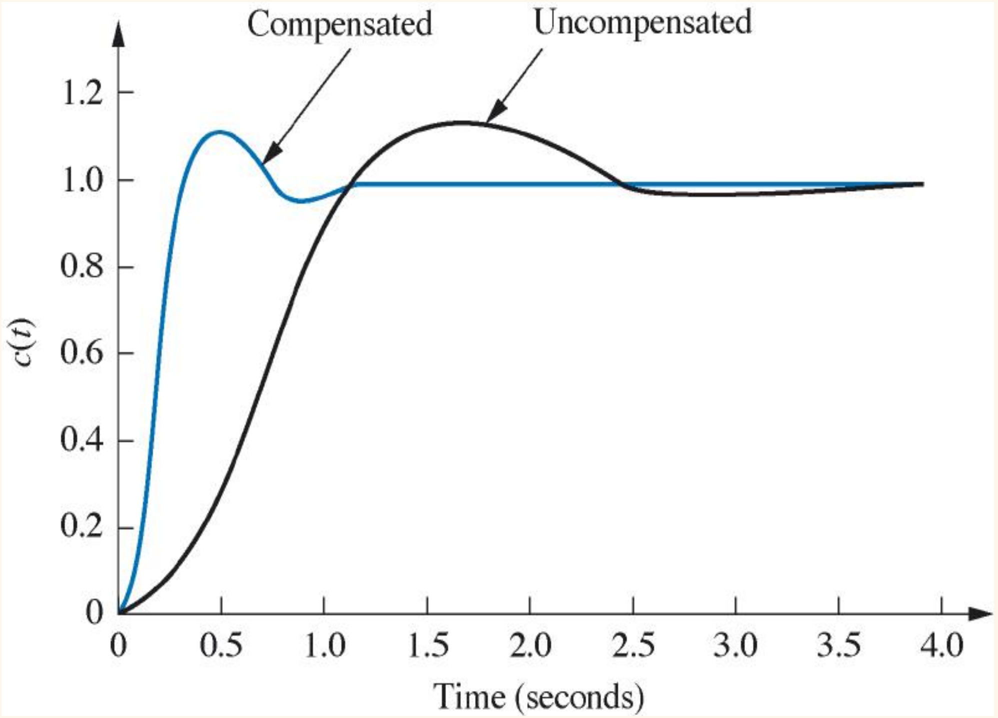


Solution 4

- The complete root locus.



Response



Solution 4

- PD controller

$$G_{PD}(s) = K_P + K_D s = K_D(s + \frac{K_P}{K_D})$$

- K_D = 47.45, K_P = 142.35.
- A complete system:

$$G_{PD}(s)G(s) = \frac{47.45(s+3)}{s(s+4)(s+6)}$$

Example 5

For a unity feedback system with

$$G(s) = \frac{50}{s(s+5)}$$

a) Determine percent *OS* and settling time of the system
b) Design a controller to have 50 % reduction in the *OS* and four fold improvement in the settling time.

Solution 5

- CLTF $$G(s)=\frac{50}{s^2+5s+50}=\frac{\omega_n{}^2}{s^2+2\xi\omega_n s+\omega_n{}^2}$$
- Do we need to draw the root locus of the system without controller? Can we find the specification needed?
  - No need to draw RL.
- *OS* = 30.45 %, T_s = 1.6 s.
- Desired specifications: *OS* = 15.23 %, *Ts* = 0.4 s.
- Desired CL poles: *s* = -10 ± j16.69
- Using angle property, *z* = 25.29
- *K* = 15 = K_D
- Transfer function: $$G_{PD}(s)G(s)=\frac{15(s+25.29)}{s(s+5)}$$

2.5
Improving Transient Response and Steady-State Error (USE PID CONTROLLER)

Proportional-Integral-Derivative (PID) Controller

- PID controller is used to improve steady-state error and transient response independently.
- PID controller is a combination of PI and PD controllers.
- We first design for transient response and then design for steady-state error.
- Transfer function:

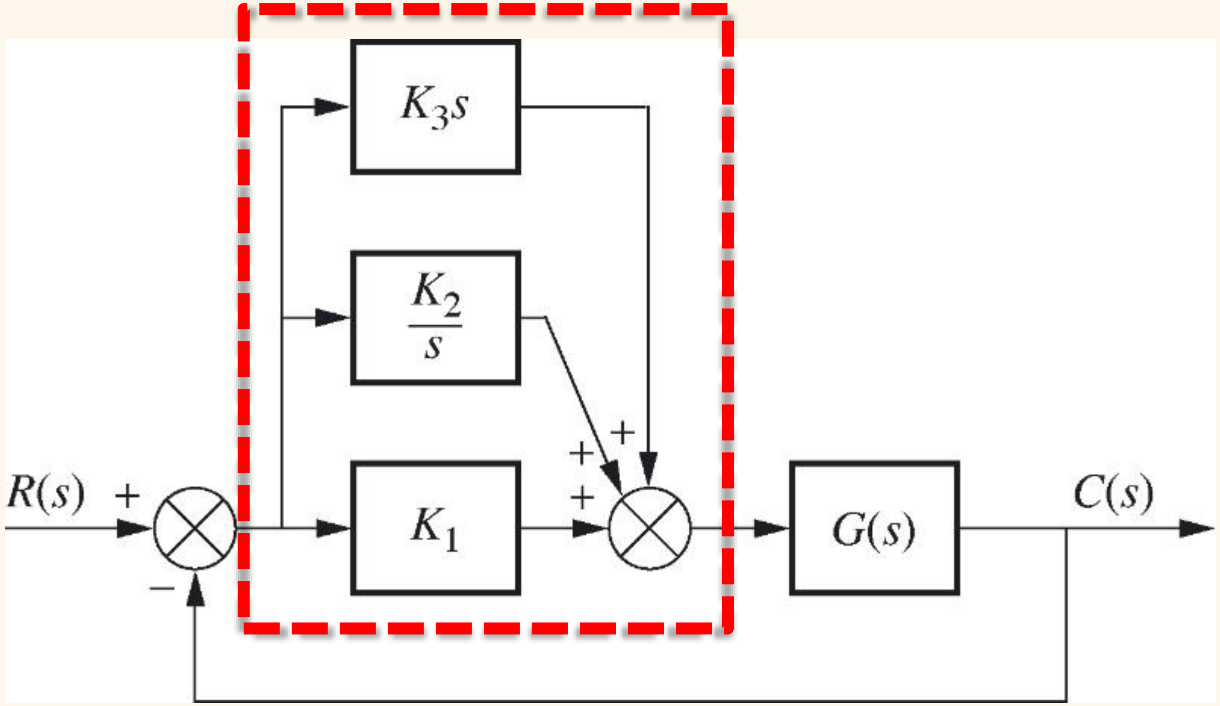
$$G_{PID}(s) = K_P + \frac{K_I}{s} + K_D s$$

$$= \frac{K_P s + K_I + K_D s^2}{s} = \frac{K_D\left(s^2 + \frac{K_P}{K_D}s + \frac{K_I}{K_D}\right)}{s}$$

or $$G_{PID}(s) = G_{PI}(s) * G_{PD}(s) \Rightarrow G_{PID}(s) = K\frac{(s+z_1)}{s} * (s+z_2)$$

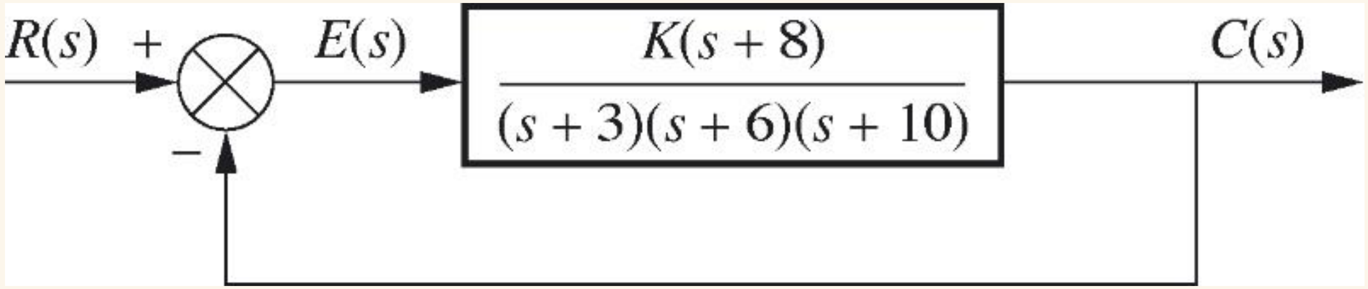
Proportional-Integral-Derivative (PID) Controller

- Implementation:
- PID controller consists of two zeros and a pole at the origin.



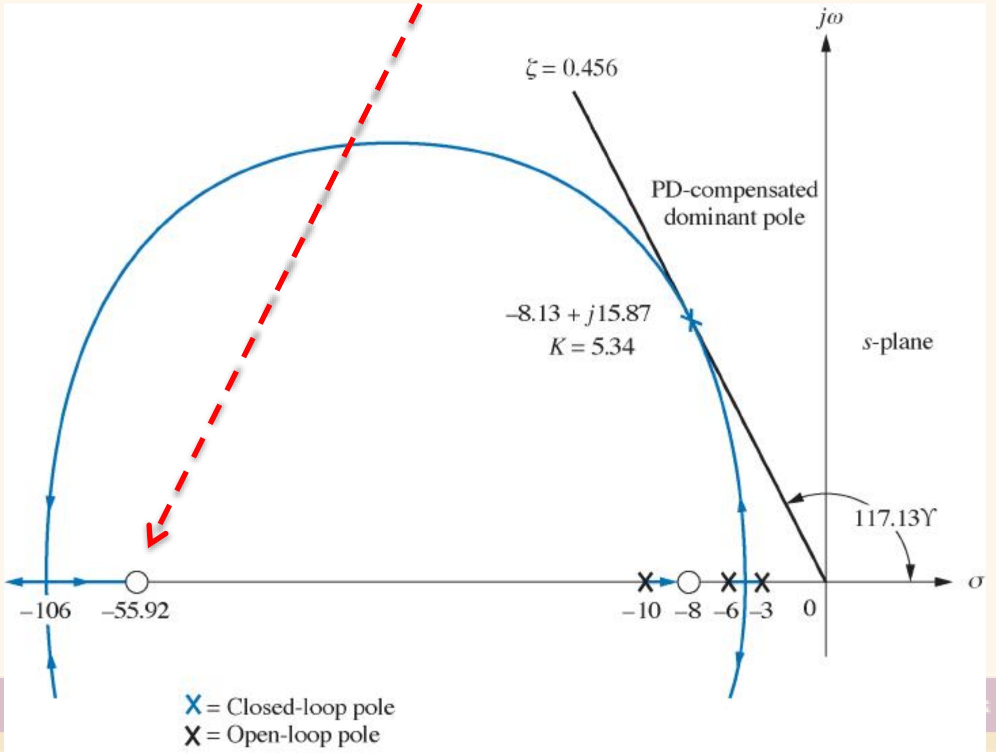
Example 6

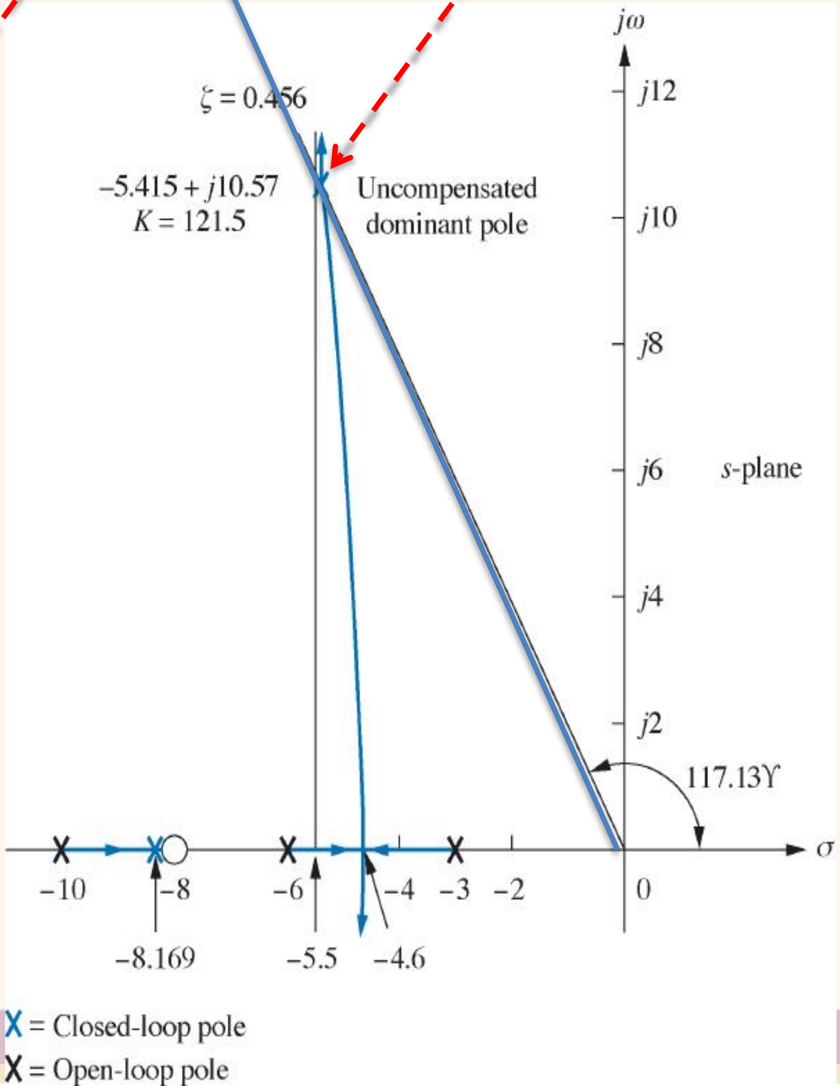
Given the system, design a PID controller so that the system can operate with a peak time that is two-thirds of the uncompensated system at 20 % overshoot, and with zero steady-state error for a step input.



Solution 6

- Do we need to draw the original root locus?
- Peak time = 0.297 s.
- Desired pole: *s* = -8.13 ± j15.87
- Angle of the zero = 18.37⁰.
- Thus *z* = 55.92. $G_{PD}(s) = s + 55.92$





Solution 6

- Design PI controller to reduce the error to zero. Choose

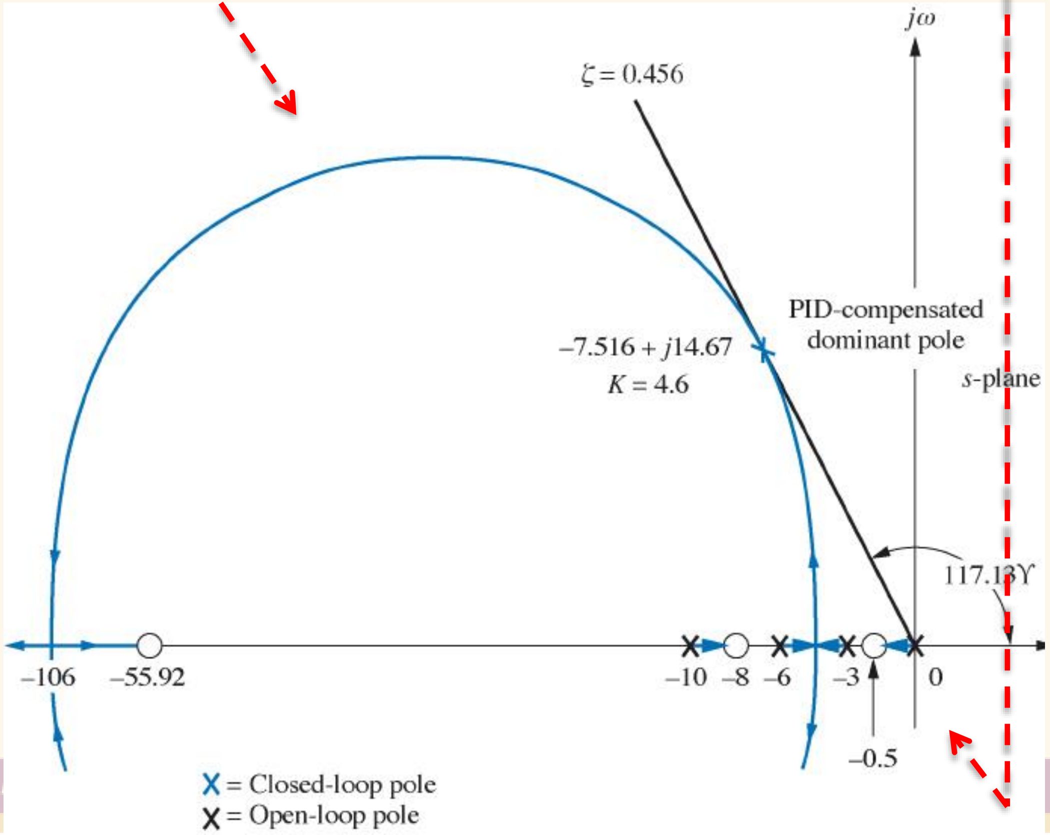
$$G_{PI}(s) = \frac{s+0.5}{s}$$

- The root locus with PID controller:

$$G_{PID}(s) = \frac{K(s+55.92)(s+0.5)}{s}$$

$$= \frac{4.6(s+55.92)(s+0.5)}{s}$$

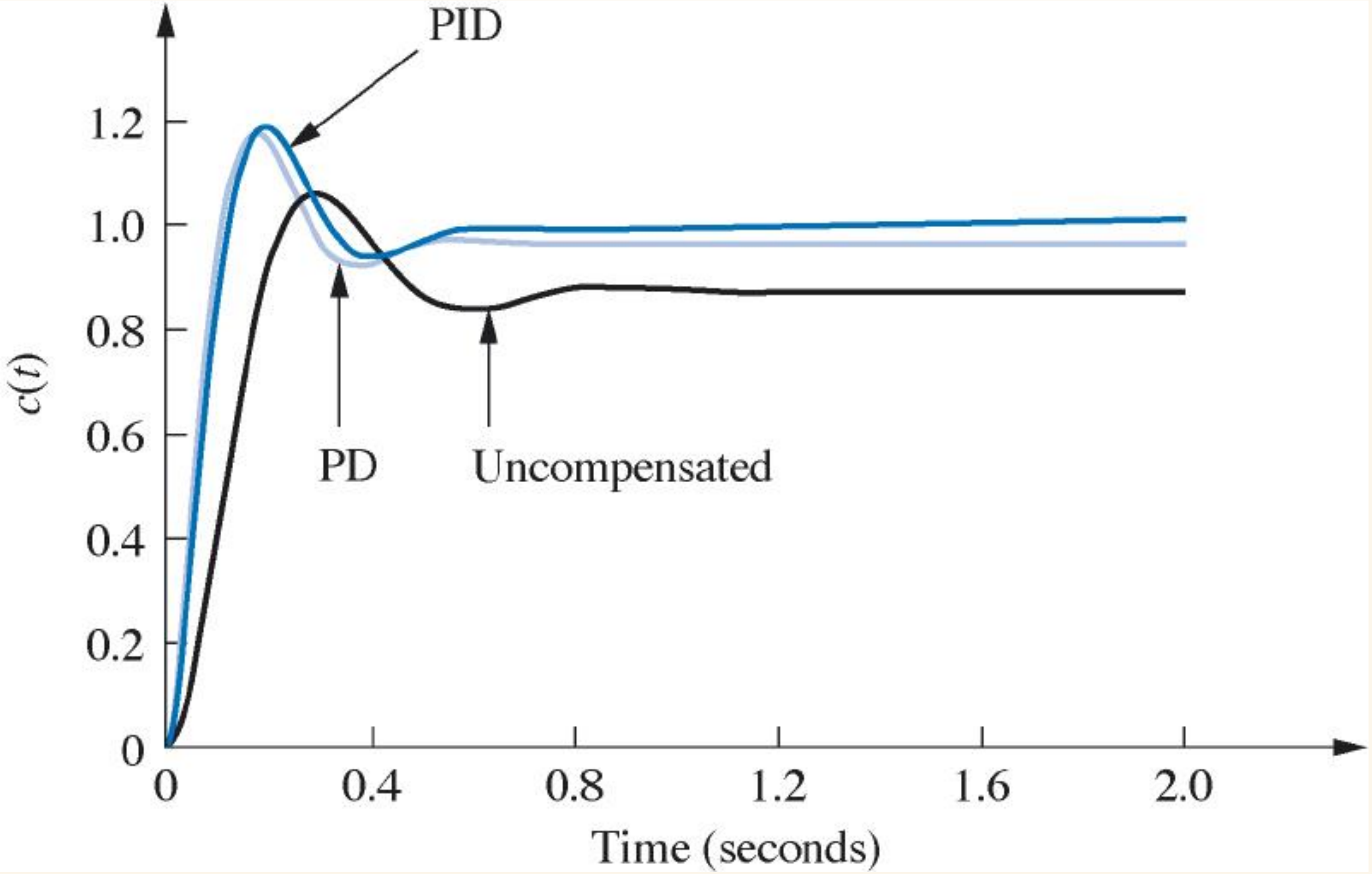
$$= \frac{4.6(s^2+56.42s+27.96)}{s}$$



- Thus, K_D = 4.6, K_P = 259.5, K_I = 128.6

Solution 6

- The response



2.6
Tuning the PID using Ziegler-Nichols Technique

UTM

Ziegler-Nichols Method

- This method is used when the systems' models cannot or difficult to be obtained either in the form of differential equation or transfer function.
- The PID controller is widely used in the industries in which the K_p, K_i and K_d parameters often can be easily adjusted.
- Ziegler-Nichols has introduced an effective method for the parameters adjustment by using the ***ultimate cycle method***.

Ziegler-Nichols steps

- The steps are as follows:
  1. Set $K_d = K_i = 0$ (to minimise the effect of derivation and integration)
  2. Increase the K_p gain until the system reach the critically stable and oscillate (i.e. when the closed-loop poles located at the imaginary axis). Obtain the gain, K_g and the oscillation frequency, ω_g on that time.
  3. Calculate the K_p gain that is supposedly needed using formula in the table.
  4. Based on the required type of controller, calculate the necessary gain using formulas in the table.

Ziegler-Nichols formulas

Controller	Optimum gain
a) Proportional: P	$K_p = 0.5K_g$
b) PI	$K_p = 0.45K_g \quad , \quad K_i = \dfrac{0.54K_g}{\omega_g}$
c) PID	$K_p = 0.6K_g \ , \ K_i = \dfrac{1.2K_g}{\omega_g} \ , \ K_d = \dfrac{0.3K_g\omega_g}{4}$

$$G_{pid}(s) = K_p + K_d s + \frac{K_i}{s}$$

Example 7

- Design a PID controller for unity feedback system with the open loop system which is given as:

$$G(s) = \frac{K300}{s(s^2 + 30s + 100)}$$

[K_p =6, K_i =1.2, K_d=7.5]

Conclusion

We have covered the design of

- ✓ Proportional (Gain) Controller
- ✓ PI Controller
- ✓ PD Controller
- ✓ PID Controller

THE END