SYSTEM MODELING AND ANALYSIS

CHAPTER 4
Response and Stability Analysis in Time Domain

Outline:

4.1 Introduction (1 hr)

- System category: order and type
- Poles, Zeros and System Response (pg 162-165)
- Standard input test signals

4.2 **First order system (1 hr)** (pg.166-167)

- Transient response
- Steady state response

4.3 **Second order system**

- Transient & Steady state response (2 hr) (pg 168-172)
- Response Specifications (2 hrs) (pg.173-184)

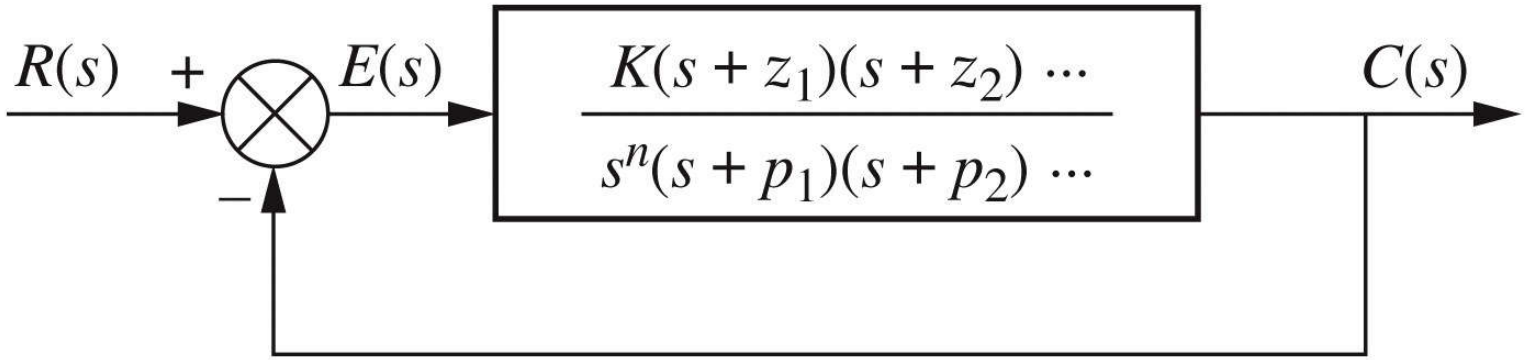
4.4 **System Response with Additional Poles (1 hr)** (pg.186-189)

4.5 **Steady State Error (2 hrs)** (pg.339-349)

- Introduction: Input signal, Evaluating steady-state error
- Steady state error for unity feedback

4.6 Stability Analysis (5 hrs)

4.1
System category
Order and Type



- **System Order** is the order of the denominator of the transfer function after cancellation of the common factors in the numerator.

- **System Type** is the value of *n* in the denominator or the number of pure integration in the forward path,

  n=0 : type 0

  n=1 : type 1 and so on

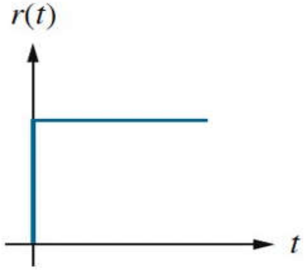
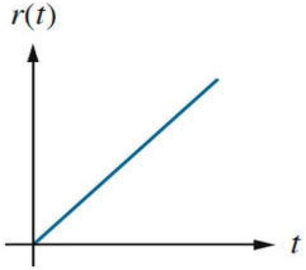
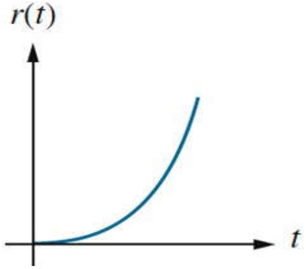
Poles, Zeros and System Response

Poles (x): the value of the Laplace transform variable, *s*, that cause the transfer function to become infinite

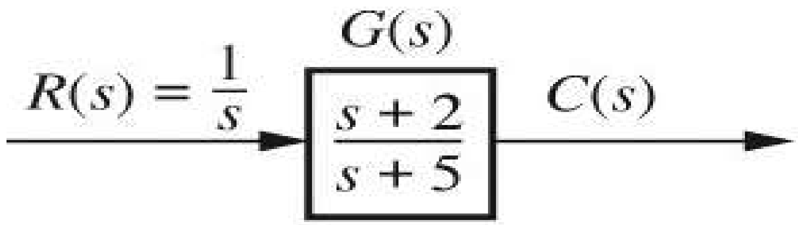
Zeros (o): the value of the Laplace transform variable, *s*, that cause the transfer function to become zero

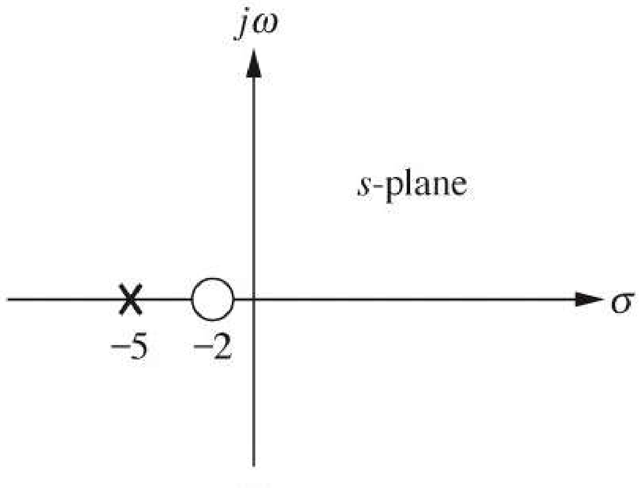
System Response : the sum of two response:
Forced response: steady state response
Natural response: homogeneous response solution

Standard input test signals

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Given a system below: find the step response of the system *c*(*t*)





$$C(s) = \frac{1}{s}\frac{(s+2)}{(s+5)}$$

$$C(s) = \frac{1}{s}\frac{(s+2)}{(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

(Use Partial Diff. Eq.)

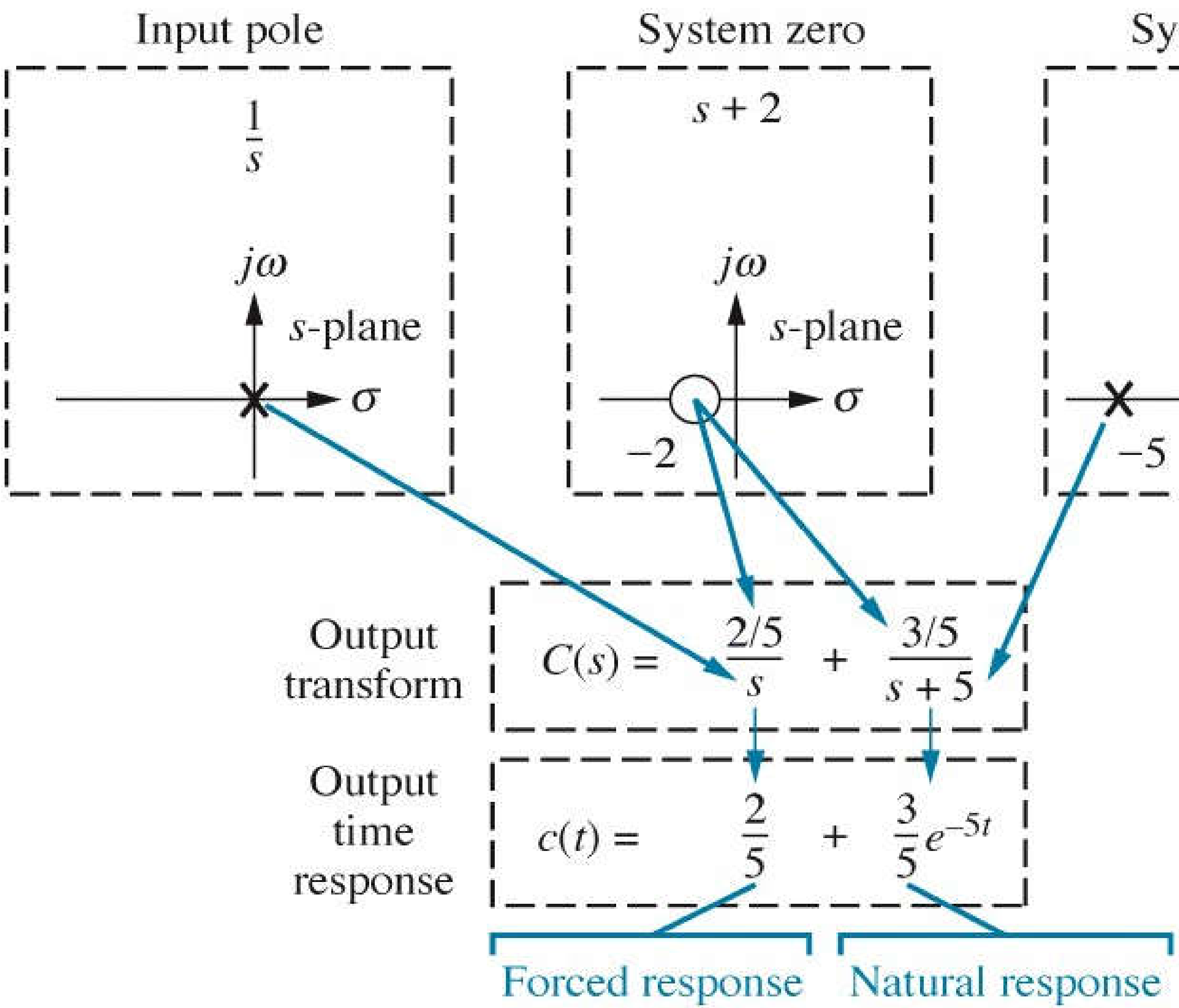
$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s\to 0} = \frac{2}{5}$$

$$C(s) = \frac{2/5}{s} + \frac{3/5}{s+5}$$

Inverse L.T

$$B = \left.\frac{1}{s}\frac{(s+2)}{1}\right|_{s\to -5} = \frac{3}{5}$$

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$


Input pole
$\frac{1}{s}$
$j\omega$
s-plane
σ
System zero
$s + 2$
$j\omega$
s-plane
σ
-2
System pole
$\frac{1}{s+5}$
$j\omega$
s-plane
σ
-5
Output transform
$C(s) = \frac{2/5}{s} + \frac{3/5}{s+5}$
Output time response
$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$
Forced response
Natural response

Find $c(t)$ if the input is a unit step input

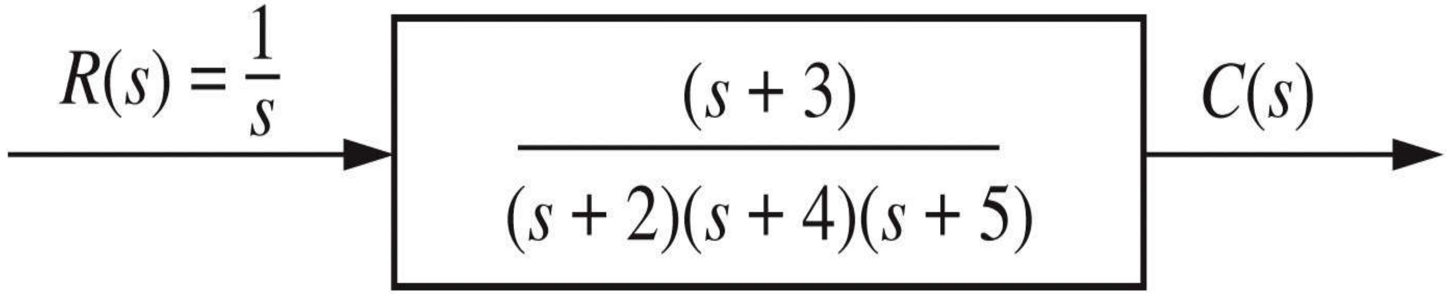
$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 9s + 9}$$

$$C(s) = \frac{1}{s} \cdot \frac{9}{s^2 + 9s + 9} = \frac{9}{s(s + 7.854)(s + 1.146)}$$

$$C(s) = \frac{9}{s(s + 7.854)(s + 1.146)} = \frac{A}{s} + \frac{B}{s + 7.854} + \frac{C}{s + 1.146}$$

Inverse L.T

$$c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}$$

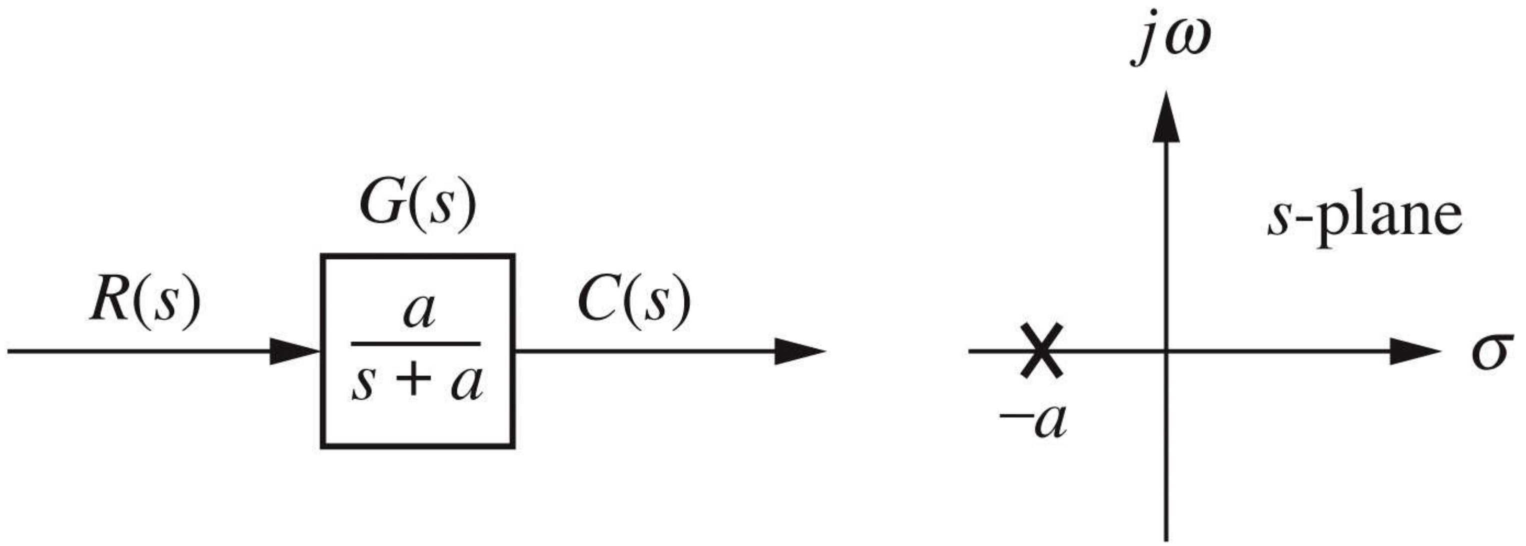


In general

$$c(t) = \underbrace{K_1}_{\text{Force response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}}$$

4.2
First order system

-Transient response and steady state response



$$C(s) = \frac{1}{s}.\frac{a}{s+a} \Rightarrow c(t) = c_f(t) + c_n(t) = 1 - e^{-at} \quad (*)$$

When $t = 1/a$: $c(1/a) = 1 - e^{-a(1/a)} = 1 - 0.37 = 0.63$

Plot $c(t)$ Vs t

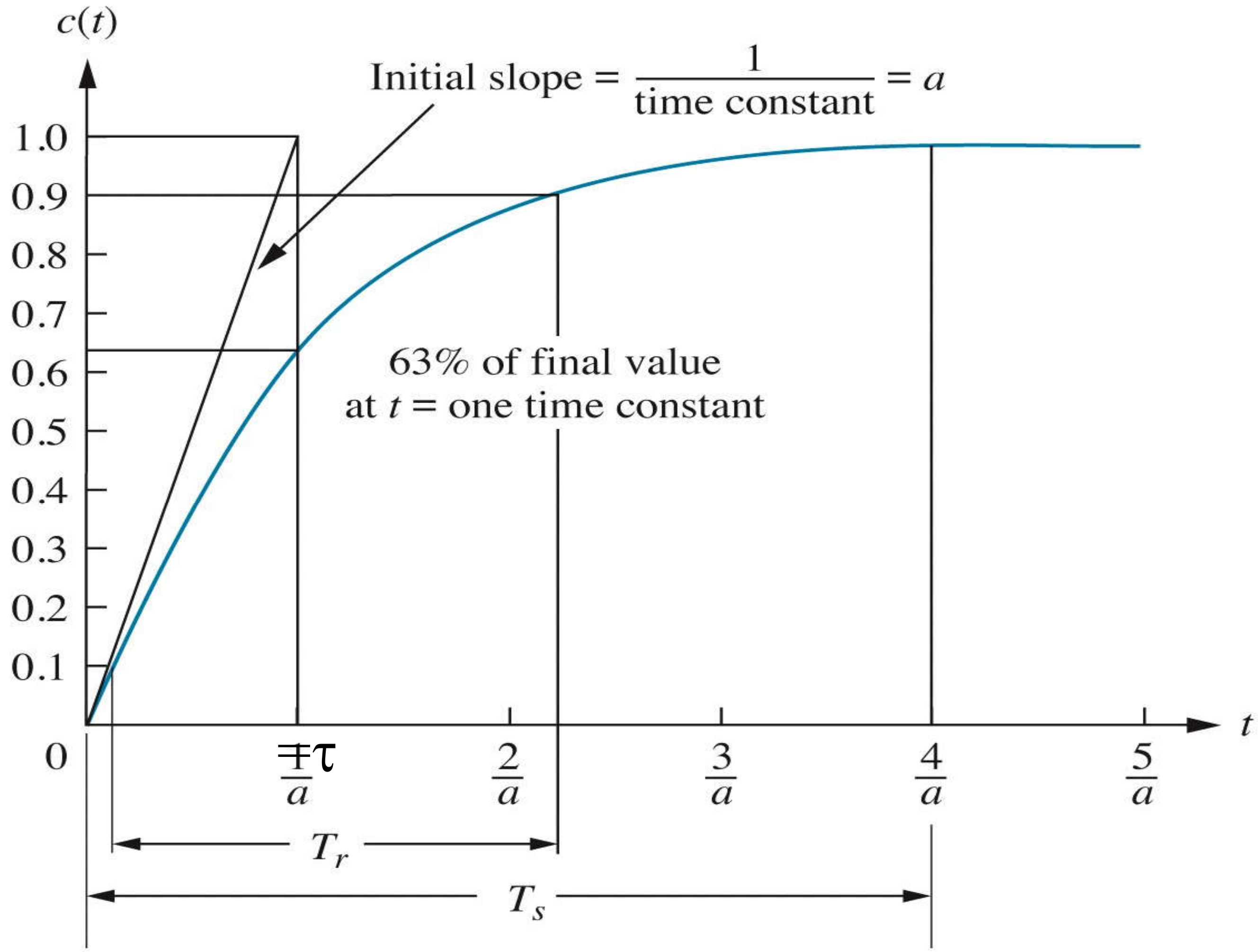

c(t)
Initial slope = 1 / time constant = a
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
63% of final value
at t = one time constant
t
1/a τ
2/a
3/a
4/a
5/a
Tr
Ts

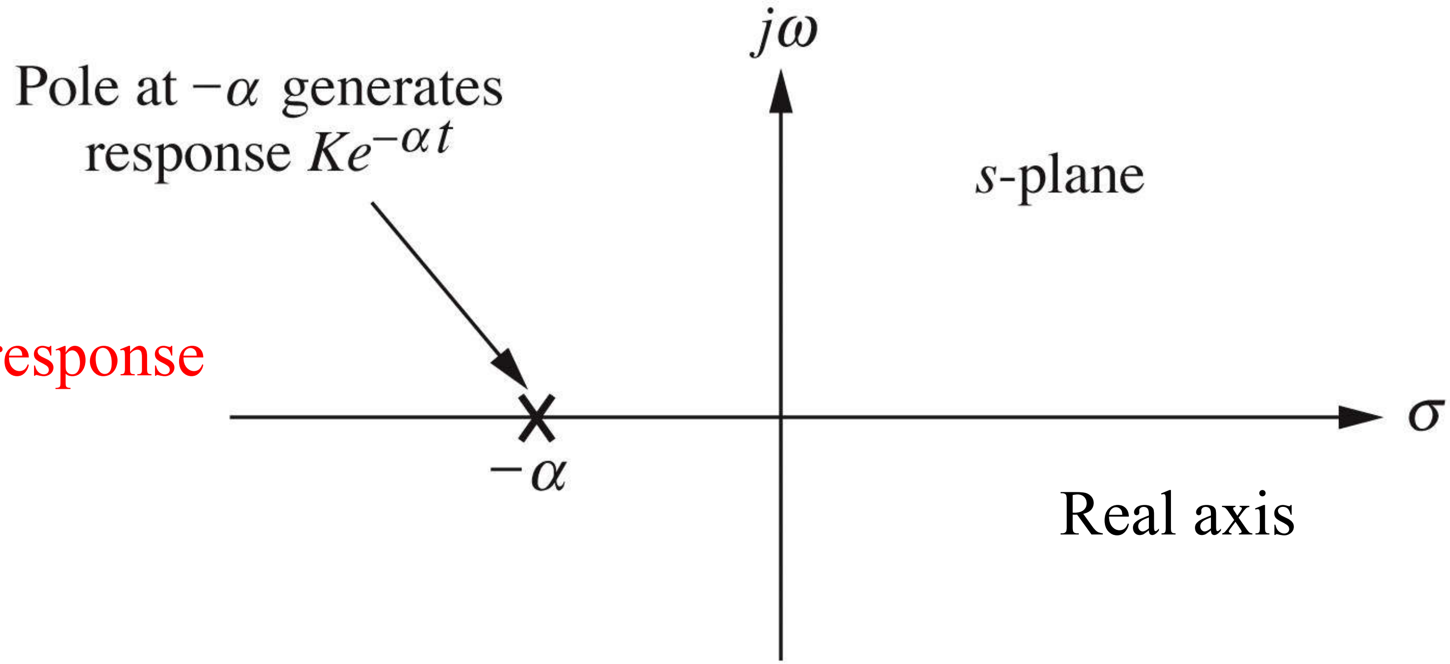
Figure 4.5

Time constant, τ

- Is the time for step response to rise to 63% of its final value
  - (speed of the system responds to the step input)

- Is the time for e^{-at} to decay to 37% of initial value.
- Parameter *a* is exponential frequency (unit 1/sec)

Exponential response



The further to the left a pole is on the –ve real axis, the faster the exponential transient response will decay to zero

Rise Time, *Tr*

•Is the time for the waveform to go from 10% to 90% of its final value

$$T_r = (\textbf{Time at } c(t) = 0.9) - \textbf{Time at } c(t) = 0.1) \quad \text{From } (*)$$

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

Settling Time, *Ts*

•Is the time for the response to reach and stay within ±2% (or ±5%) from of its final value

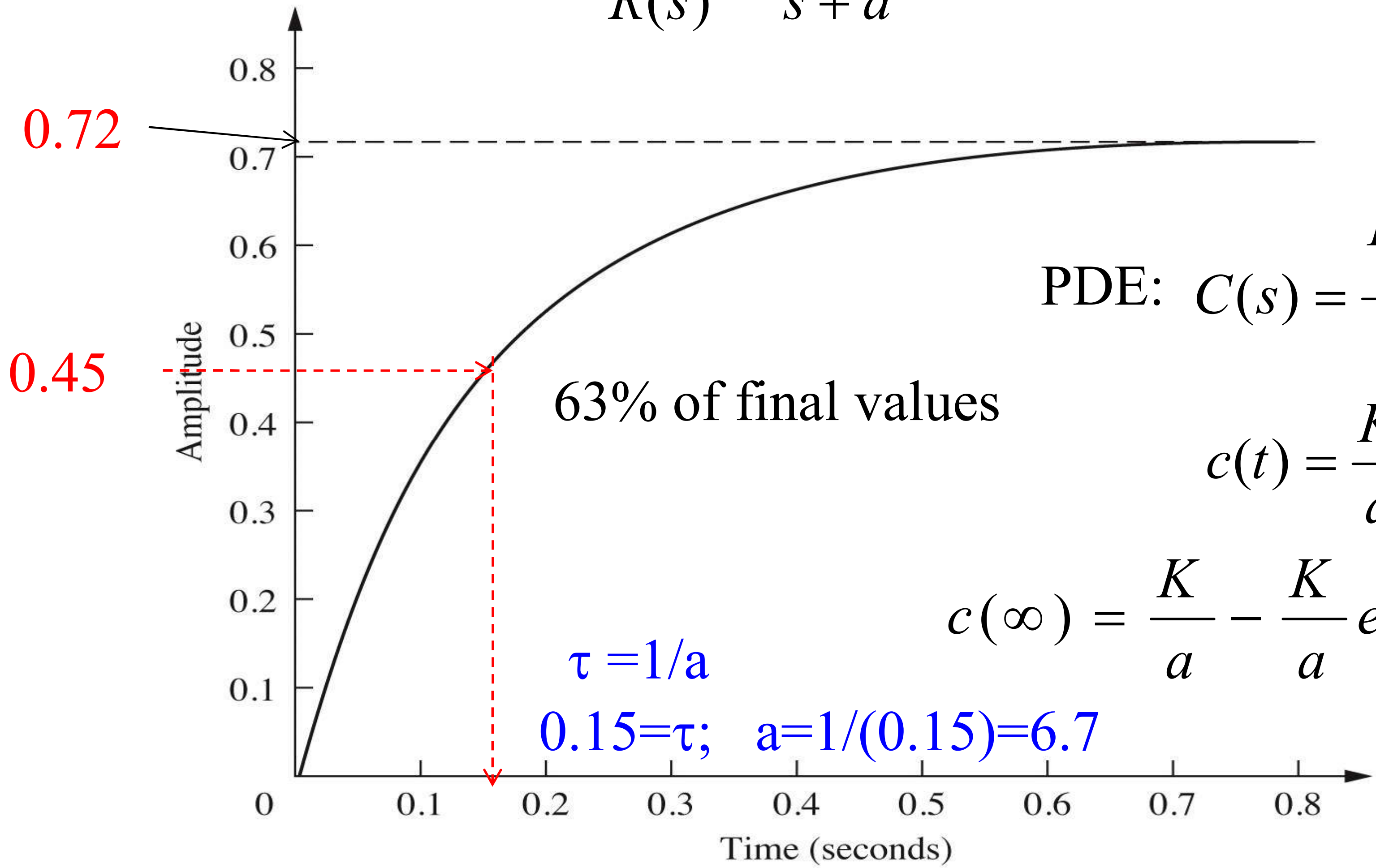
$$c(T_s) = 0.98 \qquad \text{From } (*)$$

$$T_s = \frac{4}{a}$$

Example:

First order unit step response is given below. Identify *K* and *a*

$$\frac{C(s)}{R(s)} = \frac{K}{s+a}$$



$\tau = 1/a$

$0.15 = \tau; \quad a = 1/(0.15) = 6.7$

$$\text{PDE:} \quad C(s) = \frac{K/a}{s} - \frac{K/a}{s+a}$$

$$c(t) = \frac{K}{a} - \frac{K}{a} e^{-at}$$

$$c(\infty) = \frac{K}{a} - \frac{K}{a} e^{-a\infty} = 0.72$$

K=4.8

Example:

First order unit step response is given below. Identify time constant (τ), settling time *Ts*, and rise time *Tr*

$$\frac{C(s)}{R(s)} = \frac{50}{s+50}$$

$\tau = 1/a$ $\tau = 1/50 = 0.02$ sec

$$T_s = \frac{4}{a}$$ $T_s = 4/50 = 0.08$ sec

$$T_r = \frac{2.2}{a}$$ $T_s = 2.2/50 = 0.044$ sec

4.3
Second order system
- Transient & Steady state response

-Transient response and steady state response

General 2nd order system

$$\frac{C(s)}{R(s)} = \frac{b}{s^2 + as + b}$$

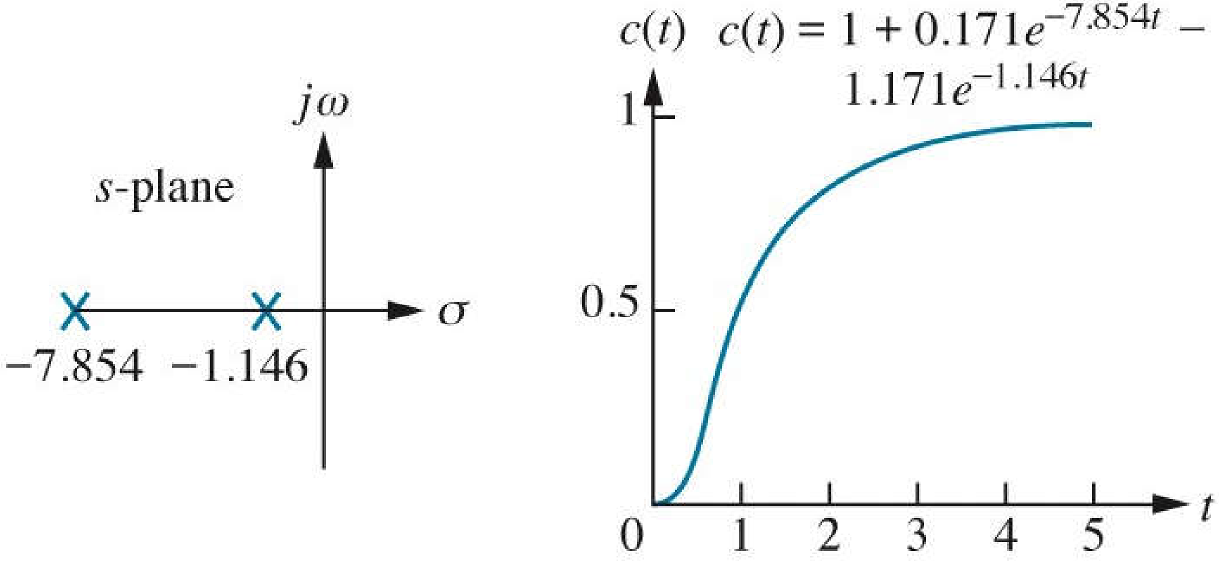
$$R(s) \longrightarrow \boxed{\frac{b}{s^2 + as + b}} \longrightarrow C(s)$$

Example 1: Plot unit step response

$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 9s + 9}$$

$$C(s) = \frac{1}{s} \cdot \frac{9}{s^2 + 9s + 9} = \frac{9}{s(s + 7.854)(s + 1.146)}$$

$$c(t) = 1 + 0.171 e^{-7.854 t} - 1.171 e^{-1.146 t}$$



OVERDAMPED RESPONSE

Poles: 2 poles at –ve real part : $-\sigma_1$, $-\sigma_2$

Natural response: two exponential with

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

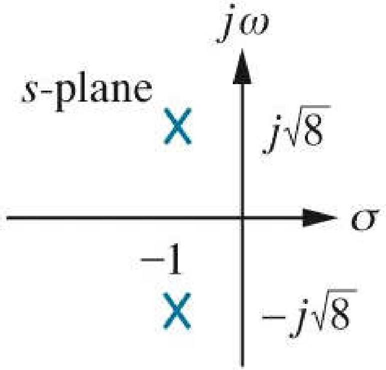
Example 2: Plot unit step response

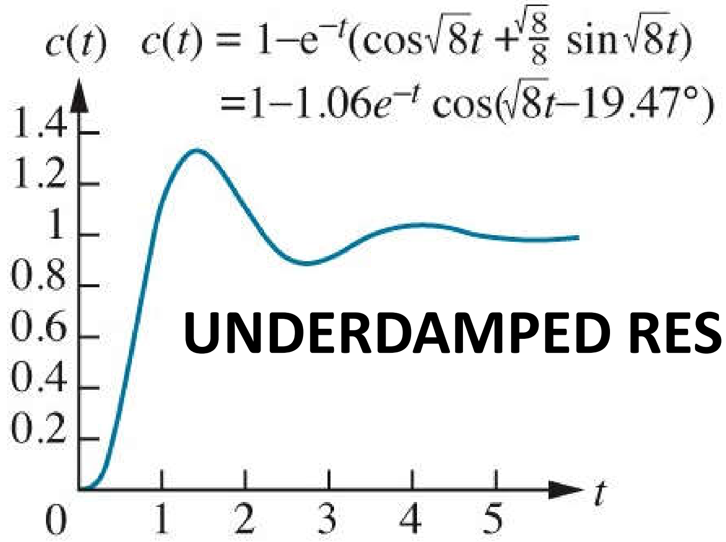
$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 2s - 9}$$

$$C(s) = \frac{1}{s} \cdot \frac{9}{s^2 + 2s - 9} = \frac{9}{s(s+1-j\sqrt{8})(s+1+j\sqrt{8})}$$

$$c(t) = 1 - e^{-t}(\cos\sqrt{8}t + \frac{\sqrt{8}}{8}\sin\sqrt{8}t)$$

$$= 1 - 1.06e^{-t}\cos(\sqrt{8}t - 19.47^o)$$





UNDERDAMPED RESPONSE

Poles: 2 complex poles at –ve real part : $-\sigma_d \pm j\omega_d$

$$(s+1-j\sqrt{8}) : s = -1 + j\sqrt{8}$$

$$(s+1+j\sqrt{8}) : s = -1 - j\sqrt{8}$$

Natural response: two exponential with

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$$

Damp freq oscillation- ω_d imaginary part of the poles

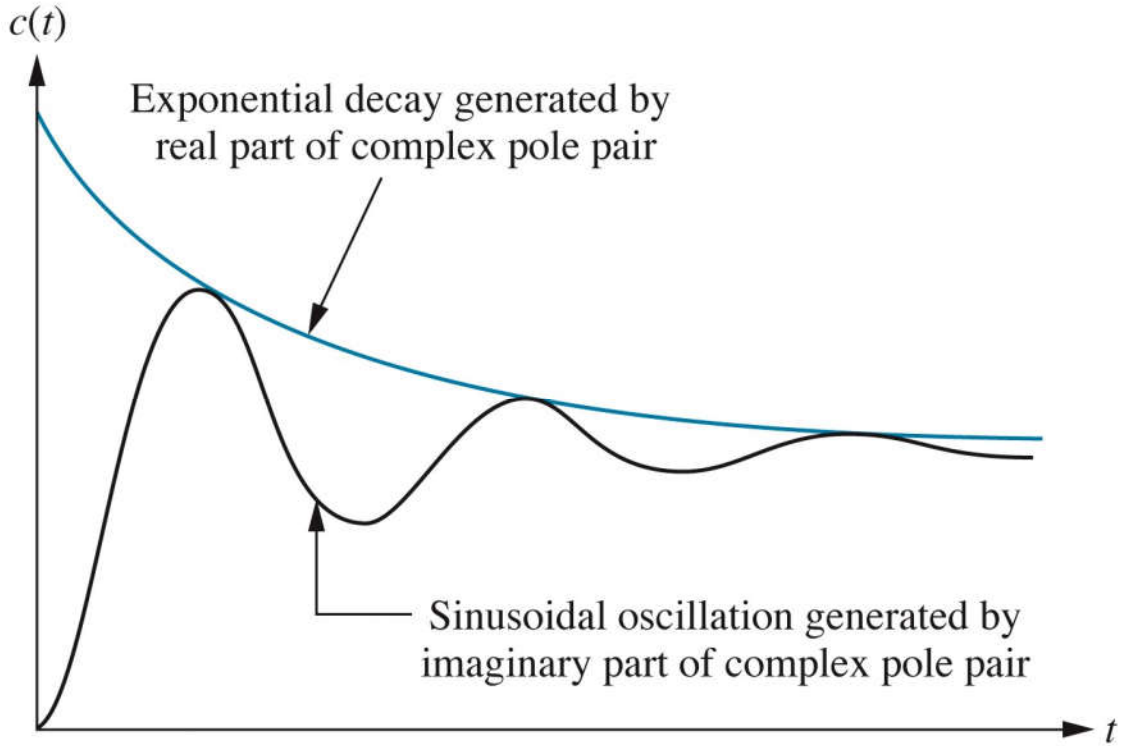
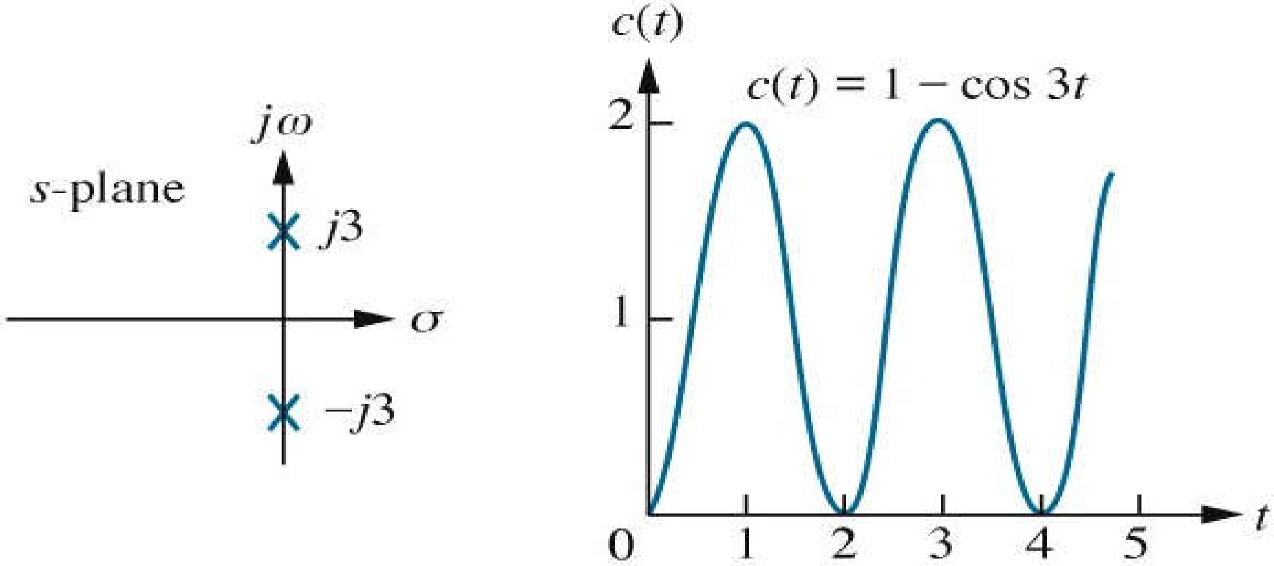

c(t)
Exponential decay generated by real part of complex pole pair
Sinusoidal oscillation generated by imaginary part of complex pole pair
t

Figure 4.8

Example 3: Plot unit step response

$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 9}$$

$$C(s) = \frac{1}{s}.\frac{9}{s^2 + 9} = \frac{9}{s(s - j3)(s + j3)}$$

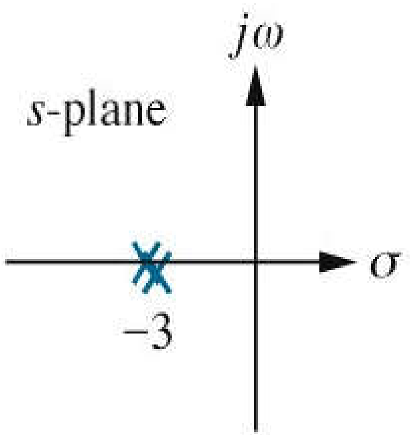


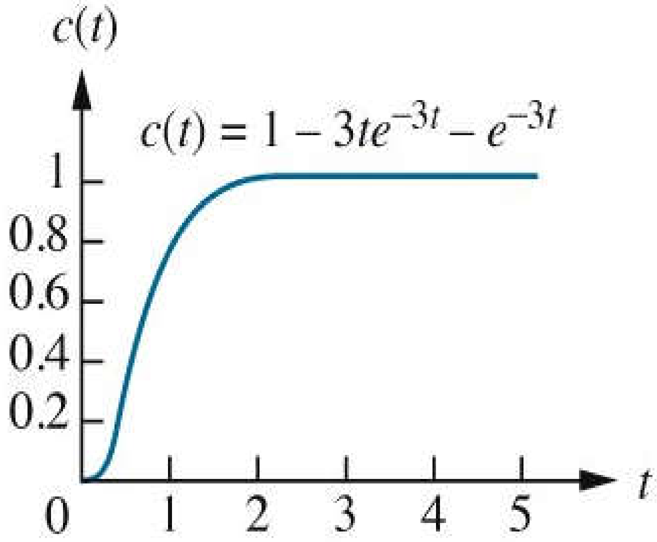
UNDAMPED RESPONSE

Example 4: Plot unit step response

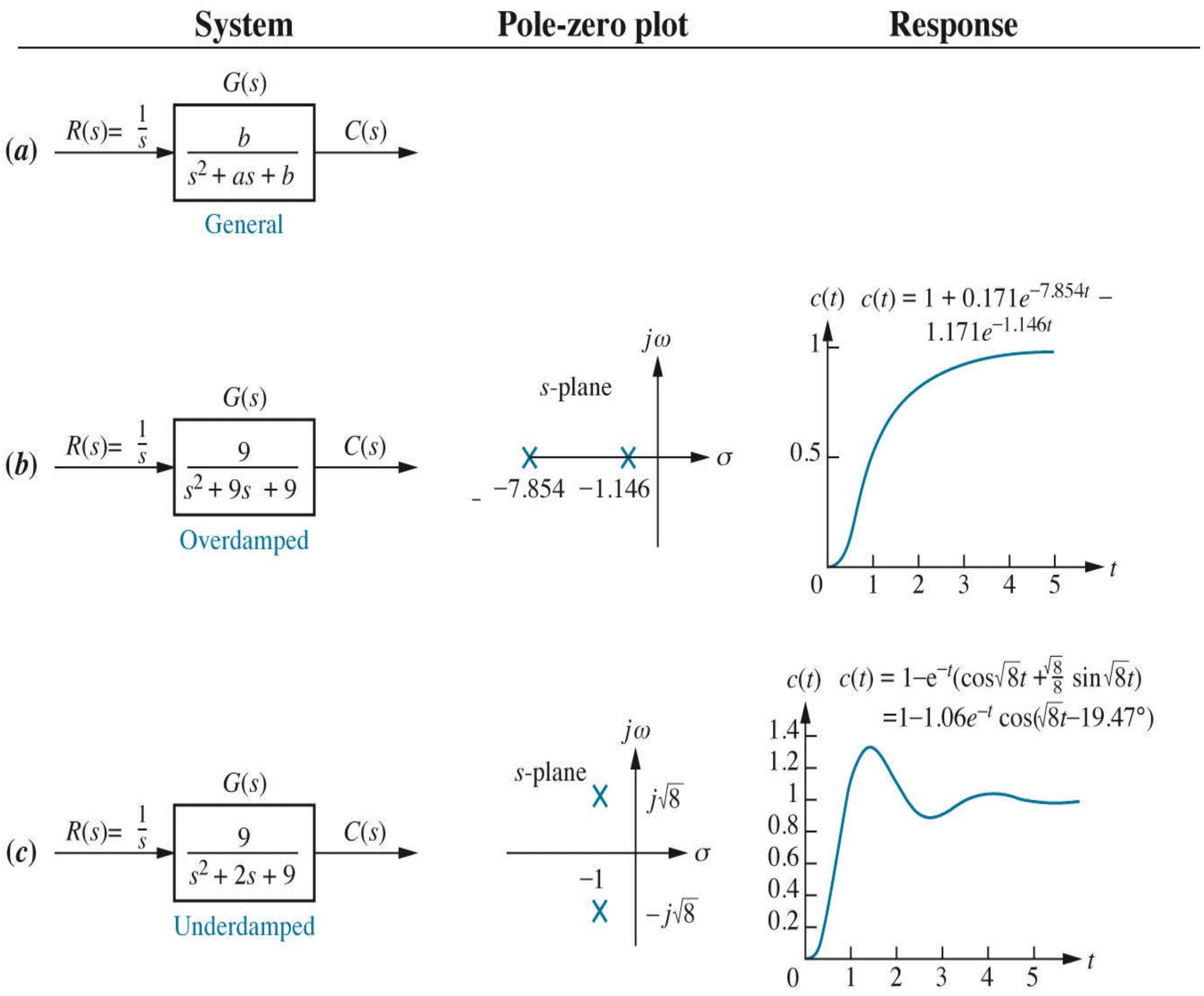
$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 6s + 9}$$

$$C(s) = \frac{1}{s} \cdot \frac{9}{s^2 + 6s + 9} = \frac{9}{s(s+3)(s+3)}$$





CRITICALLY DAMPED RESPONSE


System
Pole-zero plot
Response
(a)
$R(s)=\frac{1}{s}$
$G(s)$
$\frac{b}{s^2+as+b}$
$C(s)$
General
(b)
$R(s)=\frac{1}{s}$
$G(s)$
$\frac{9}{s^2+9s+9}$
$C(s)$
Overdamped
$j\omega$
s-plane
σ
-7.854 -1.146
$c(t)$ $c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}$
1
0.5
0 1 2 3 4 5 t
(c)
$R(s)=\frac{1}{s}$
$G(s)$
$\frac{9}{s^2+2s+9}$
$C(s)$
Underdamped
$j\omega$
s-plane
$j\sqrt{8}$
σ
-1
$-j\sqrt{8}$
$c(t)$ $c(t) = 1-e^{-t}(\cos\sqrt{8}t + \frac{\sqrt{8}}{8}\sin\sqrt{8}t)$
$=1-1.06e^{-t}\cos(\sqrt{8}t-19.47°)$
1.4
1.2
1
0.8
0.6
0.4
0.2
0 1 2 3 4 5 t

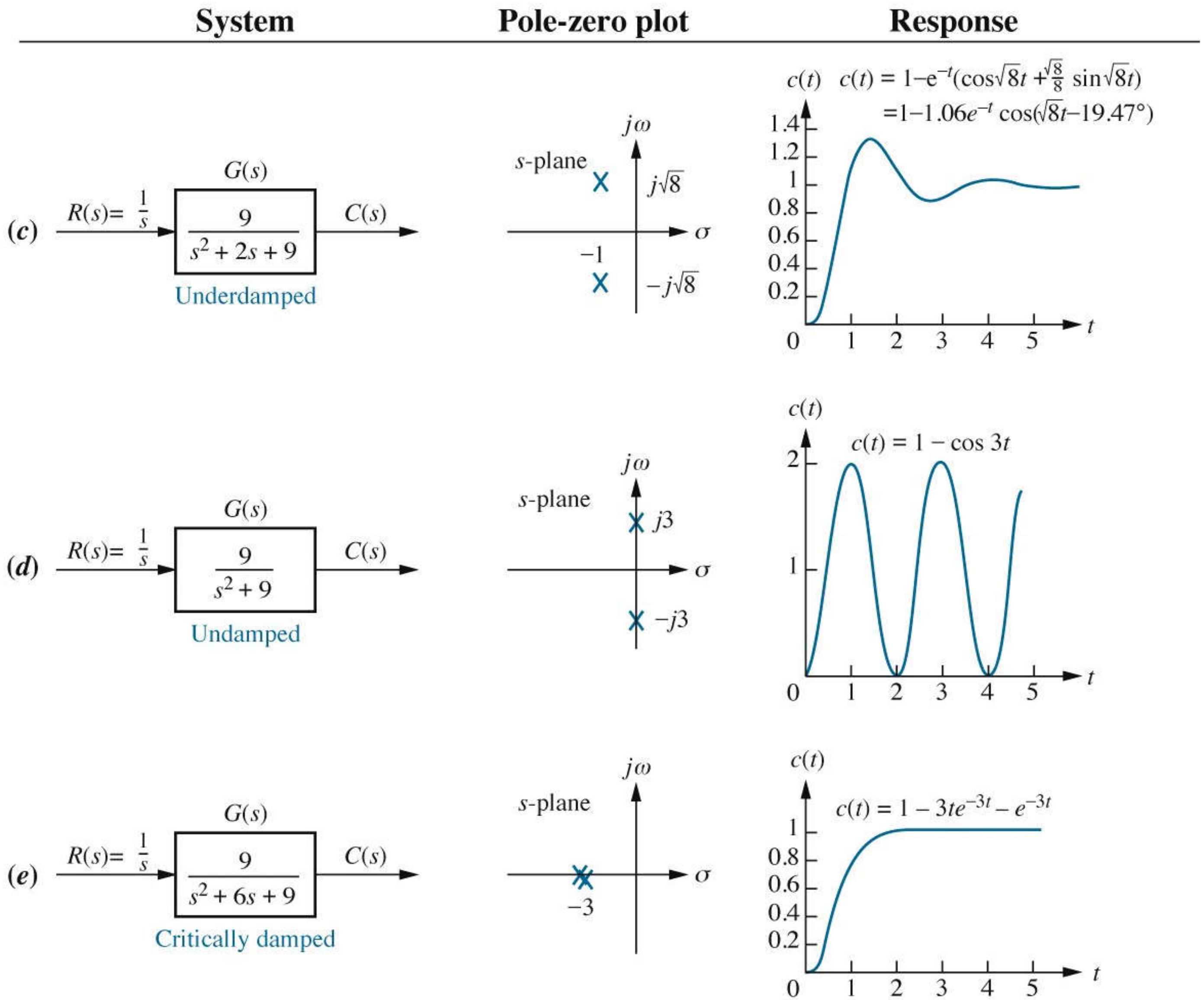


Figure 4.7cde

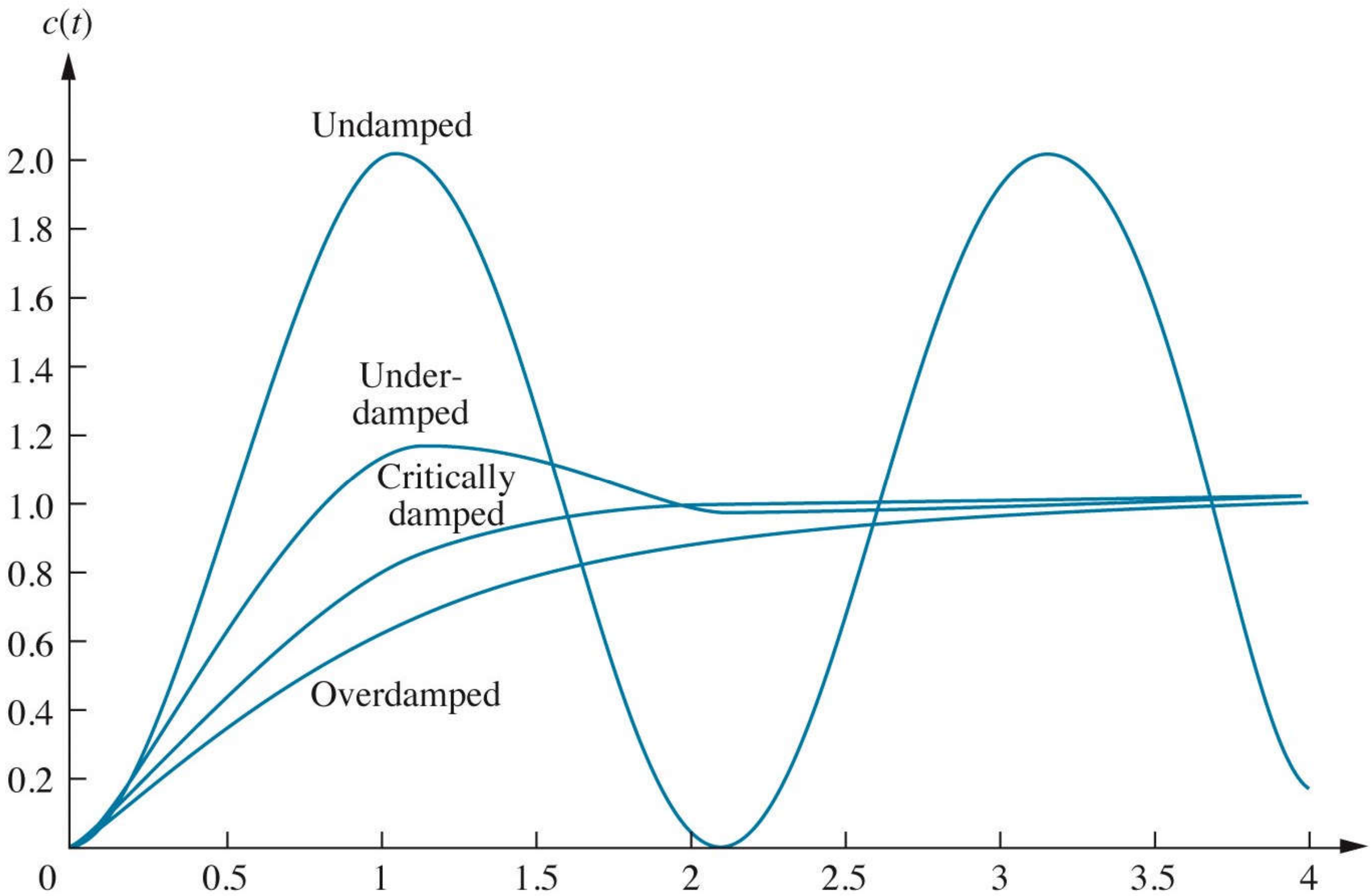

c(t)
Undamped
Under-
damped
Critically
damped
Overdamped
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.5
1
1.5
2
2.5
3
3.5
4
t

Figure 4.10