4.3
Second order system
- Response specifications

The general TF of second-order system:

1. Natural frequency. ω_n

   Frequency of oscillation of the system without damping

2. Damping Ratio. ζ

$$\varsigma = \frac{\textbf{exponential decay frequency}}{\textbf{natural frequency (rad/sec)}}$$

Second-order system can be transformed to show the quantities of ζ and ω_n. Consider the general system

$$\frac{C(s)}{R(s)} = G(s) = \frac{b}{s^2 + as + b}$$

For poles purely imaginary, *a* = 0, poles on the *jω* axis

$$\frac{C(s)}{R(s)} = G(s) = \frac{b}{s^2 + b}$$

$$as^2 + bs + c \Rightarrow \boxed{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$\omega_n = \sqrt{b}; \quad b = \omega_n^{\ 2}$$

For under-damped system, poles have real part,=-*a/2*

$$\varsigma = \frac{\textbf{exponential decay frequency}}{\textbf{natural frequency (rad/sec)}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n}$$

$$a = 2\varsigma\omega_n$$

General second-order system transfer function:

$$\frac{C(s)}{R(s)} = G(s) = \frac{\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

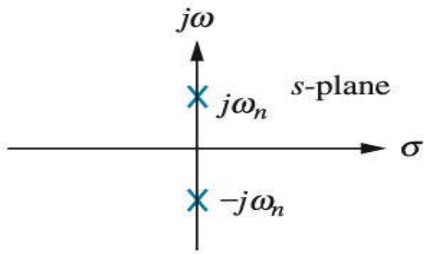
Poles:

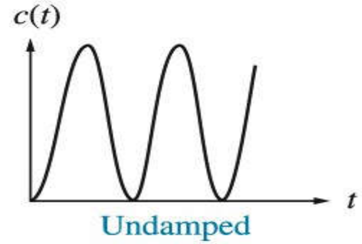
$$S_{1,2} = -\varsigma\omega_n \pm \omega_n\sqrt{\varsigma^2 - 1}$$

General response relationship

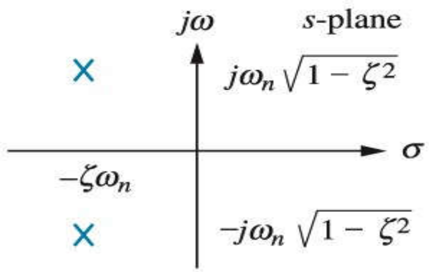
ζ | **Poles** | **Step response**

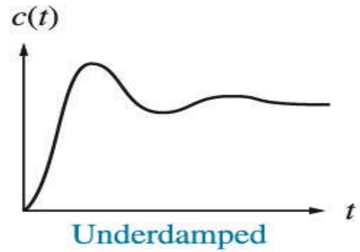
0



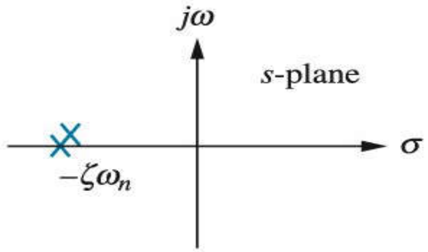


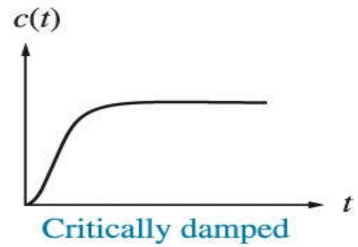
$0 < \zeta < 1$



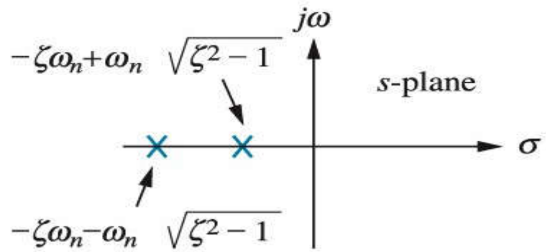


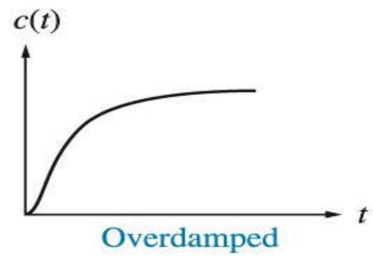
$\zeta = 1$





$\zeta > 1$





Example :

Find the value of ς and ω_n. State the kind of response expected

$$R(s) \longrightarrow \boxed{\frac{12}{s^2+8s+12}} \longrightarrow C(s)$$

(*a*)

$$R(s) \longrightarrow \boxed{\frac{16}{s^2+8s+16}} \longrightarrow C(s)$$

(*b*)

$$R(s) \longrightarrow \boxed{\frac{20}{s^2+8s+20}} \longrightarrow C(s)$$

(*c*)

$$\boxed{\frac{\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}}$$

Underdamped second-order system

$$\frac{C(s)}{R(s)} = G(s) = \frac{\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2} \quad ; \quad R(s) = \frac{1}{s}$$

Used PDE

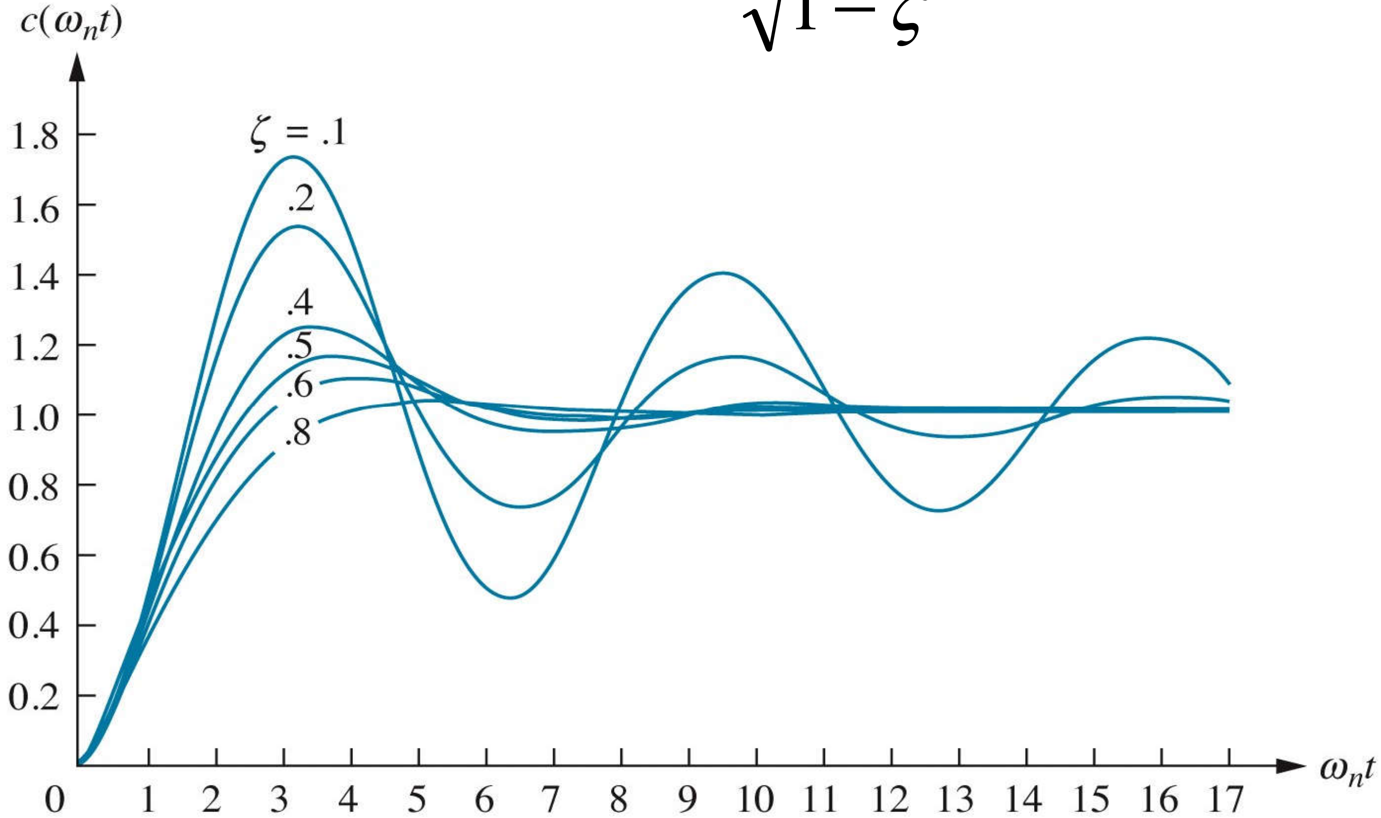
$$C(s) = \frac{1}{s}.\frac{\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

Assume that $\zeta<1$, case underdamped, obtain K_1, K_2 and K_3 after that taking the Laplace transform; produces:

$$\boxed{c(t) = 1 - e^{-\varsigma\omega_n t}\left(\cos\omega_n\sqrt{1-\varsigma^2}\,t + \frac{\varsigma}{\sqrt{1-\varsigma^2}}\sin\omega_n\sqrt{1-\varsigma^2}\,t\right)}$$

$$c(t) = 1 - \frac{1}{\sqrt{1-\varsigma^2}} e^{-\varsigma\omega_n t} \cos\left(\omega_n \sqrt{1-\varsigma^2}\, t - \phi\right)$$

$$\phi = \tan^{-1} \frac{\varsigma}{\sqrt{1-\varsigma^2}}$$



Rise time, *Tr*:
Time required for waveform to go from 0.1 (10%) of final value to 0.9 (90%) of the final value

Peak time, *Tp* :
Time required to reach the first/maximum peak

Percentage overshoot: *%OS*:
The amount that waveform overshoots from the steady-state or final value.

Settling time, *Ts* :
Time required for the transient's damped oscillations to reach and stay within ±2% (5%) of the steady-state value

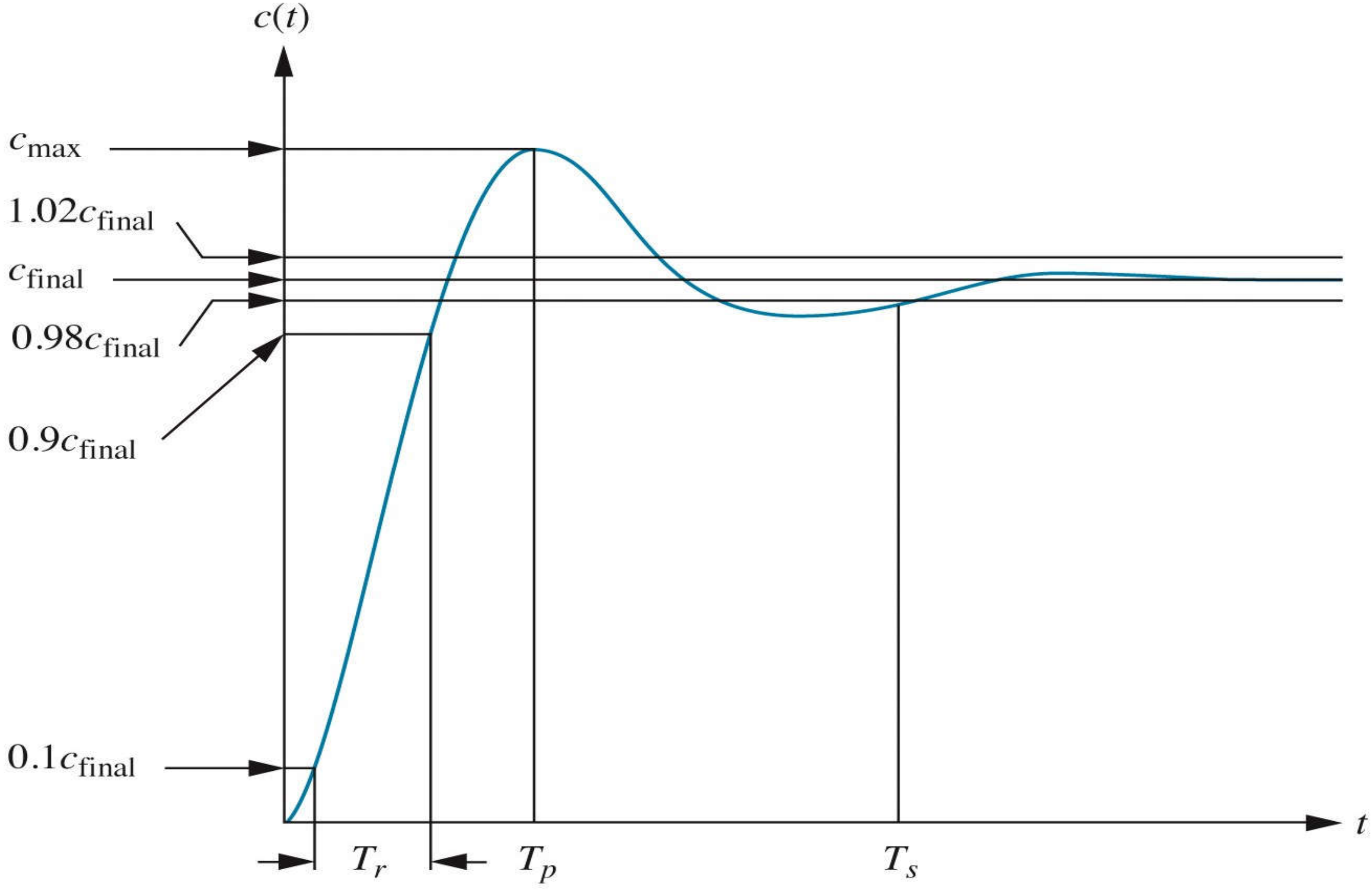

c(t)
cmax
1.02cfinal
cfinal
0.98cfinal
0.9cfinal
0.1cfinal
t
Tr
Tp
Ts

Figure 4.14

Evaluation of *Tp*

- Differentiating *c*(*t*)
- Find the first zero after *t*=0
- Assuming zero initial conditions

$$c(t) = 1 - \frac{1}{\sqrt{1-\varsigma^2}} e^{-\varsigma\omega_n t} \cos\left(\omega_n \sqrt{1-\varsigma^2}\, t - \phi\right)$$

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\varsigma^2}} e^{-\varsigma\omega_n t} \sin\left(\omega_n \sqrt{1-\varsigma^2}\, t\right)$$

$$\omega_n \sqrt{1-\varsigma^2}\, t = n\pi$$

$$\boxed{t = T_p = \frac{n\pi}{\omega_n \sqrt{1-\varsigma^2}}}$$

The 1st peak $n = 1$

Evaluation of $\%OS$

$$\%OS = \frac{C_{\max} - C_{final}}{C_{final}} x100\%$$

Cmax: from $c(t)$ at T_p

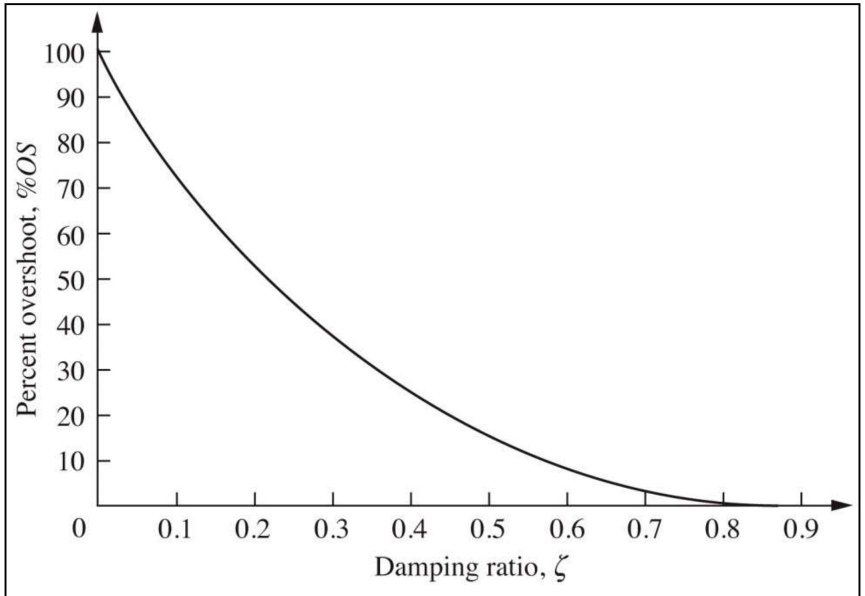


Figure 4.15

$$C_{\max} = c(T_p) = 1 - e^{-\varsigma\omega_n T_p} \cos\left(\omega_n \sqrt{1-\varsigma^2}\, t - \phi\right)$$

$$C_{\max} = c(T_p) = 1 - e^{-\varsigma\pi / \sqrt{1-\varsigma^2}}$$

$$\boxed{\%OS = e^{-\varsigma\pi / \sqrt{1-\varsigma^2}} \mathbf{x100}}$$

Evaluation of *Ts*

The time $c(t)$ reaches and stay within ±2% (±4%) of the steady state value C_{final}

$$c(t) = 1 - \frac{1}{\sqrt{1-\varsigma^2}} e^{-\varsigma\omega_n t} \cos\left(\omega_n \sqrt{1-\varsigma^2}\, t - \phi\right)$$

$$\cos\left(\omega_n \sqrt{1-\varsigma^2}\, t - \phi\right) = 1, \qquad t = T_s$$

$$\Rightarrow \frac{1}{\sqrt{1-\varsigma^2}} e^{-\varsigma\omega_n T_s} = 0.02$$

$$\Rightarrow T_s = \frac{-\ln(0.02\sqrt{1-\varsigma^2}}{\varsigma\omega_n} \boxed{\approx \frac{4}{\varsigma\omega_n}}$$

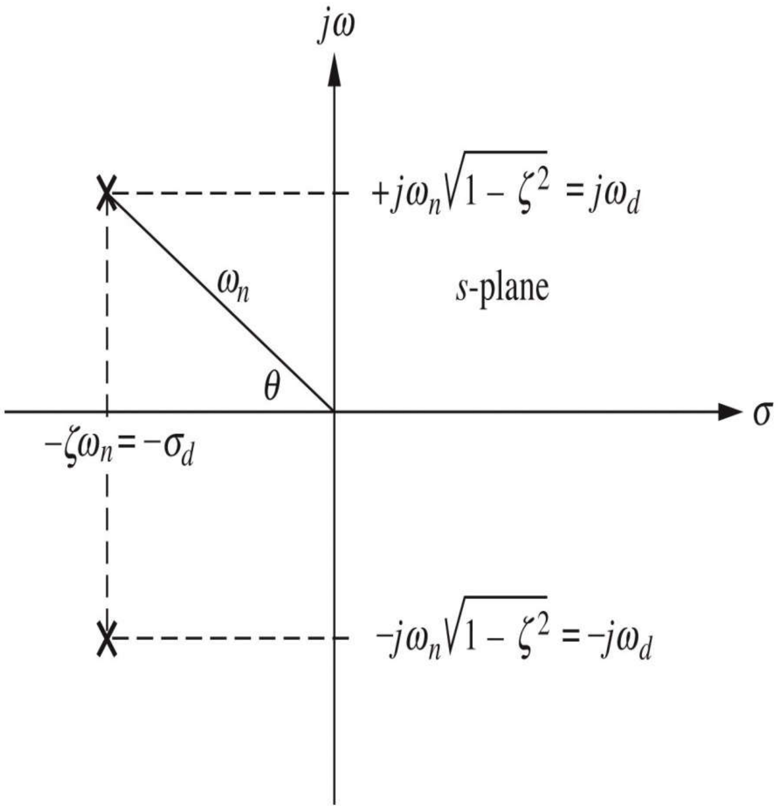


Figure 4.17

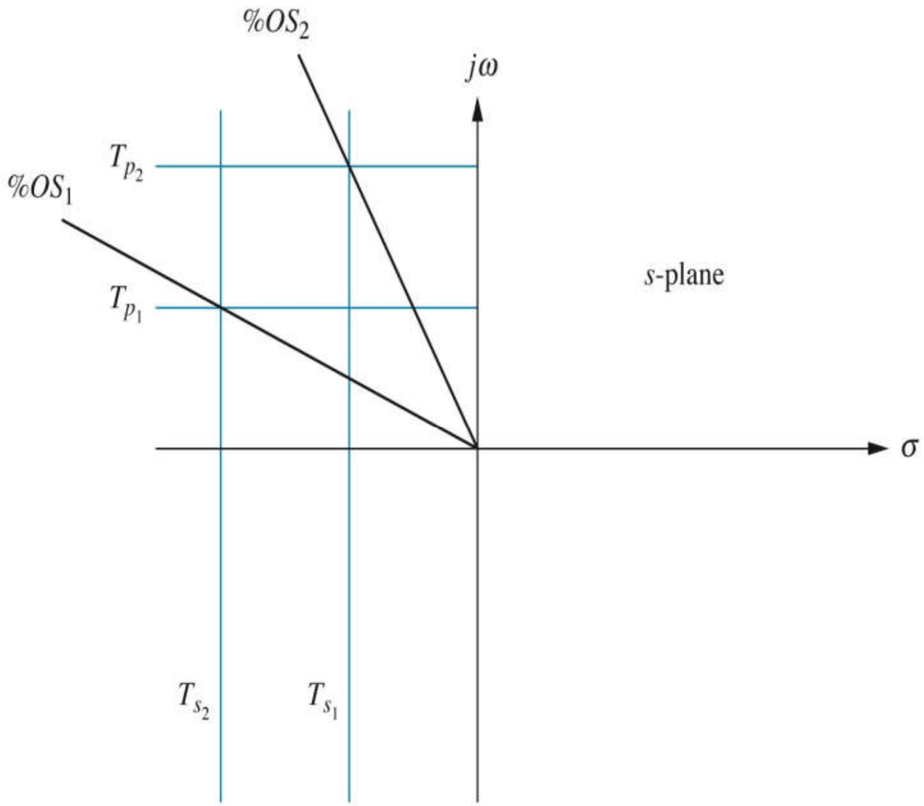


Figure 4.18

Example 4.6: Find T_p, %OS and T_s

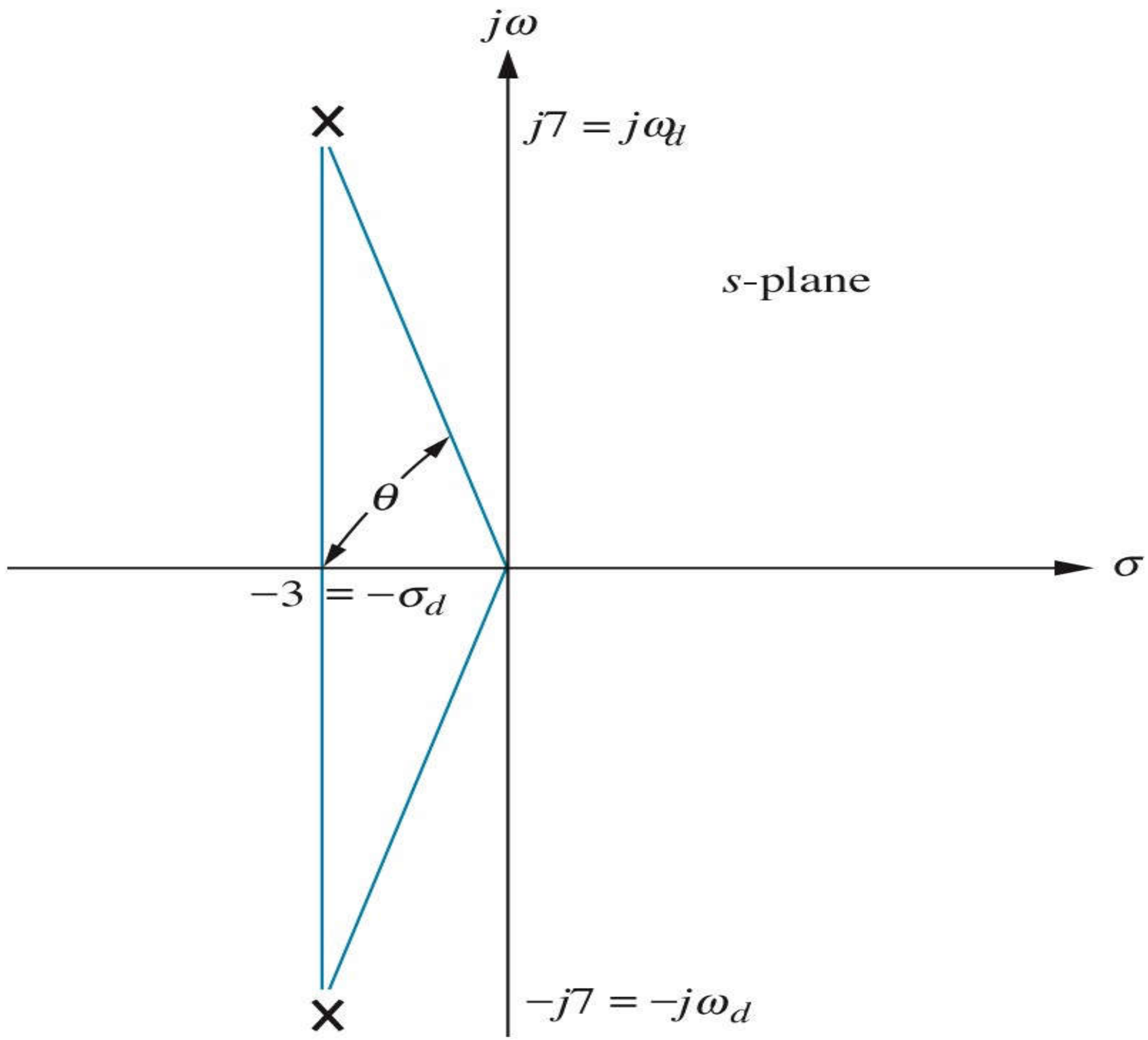
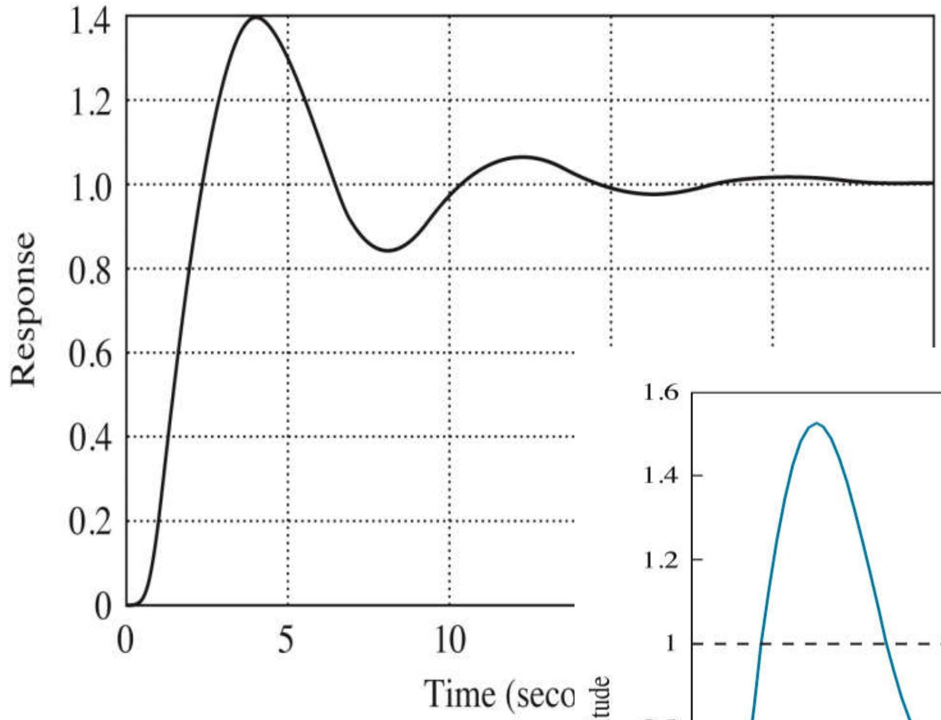


Figure 4.20



(c)

Figure P4.9c

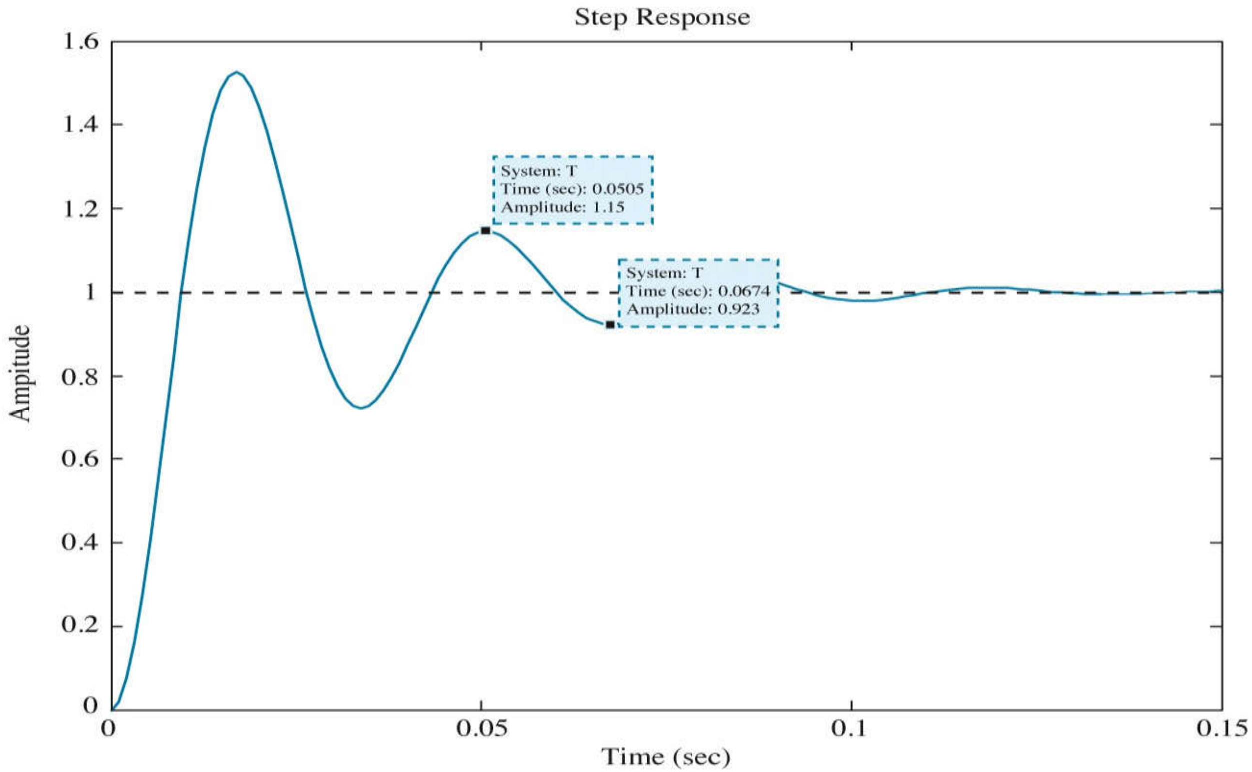
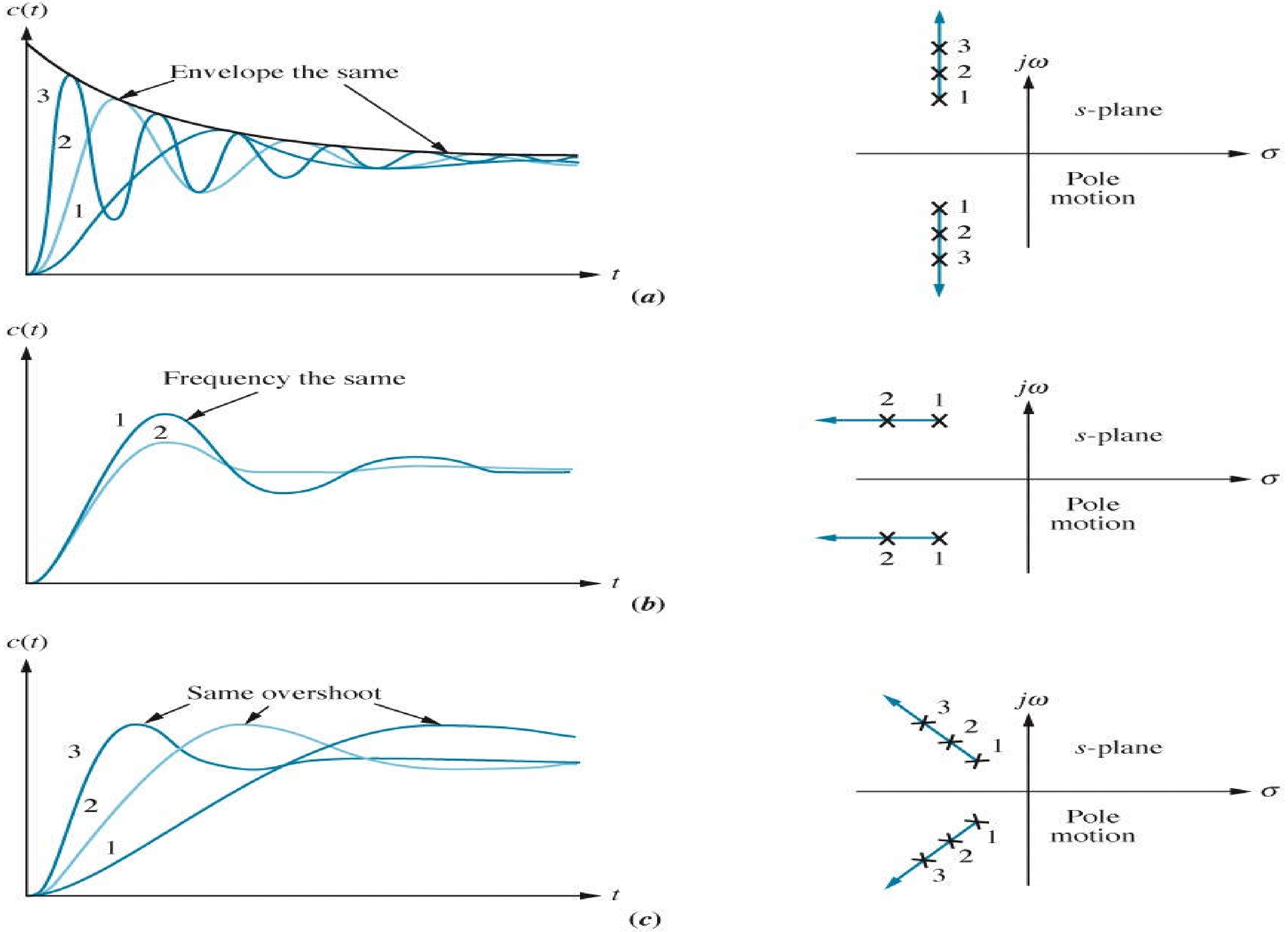
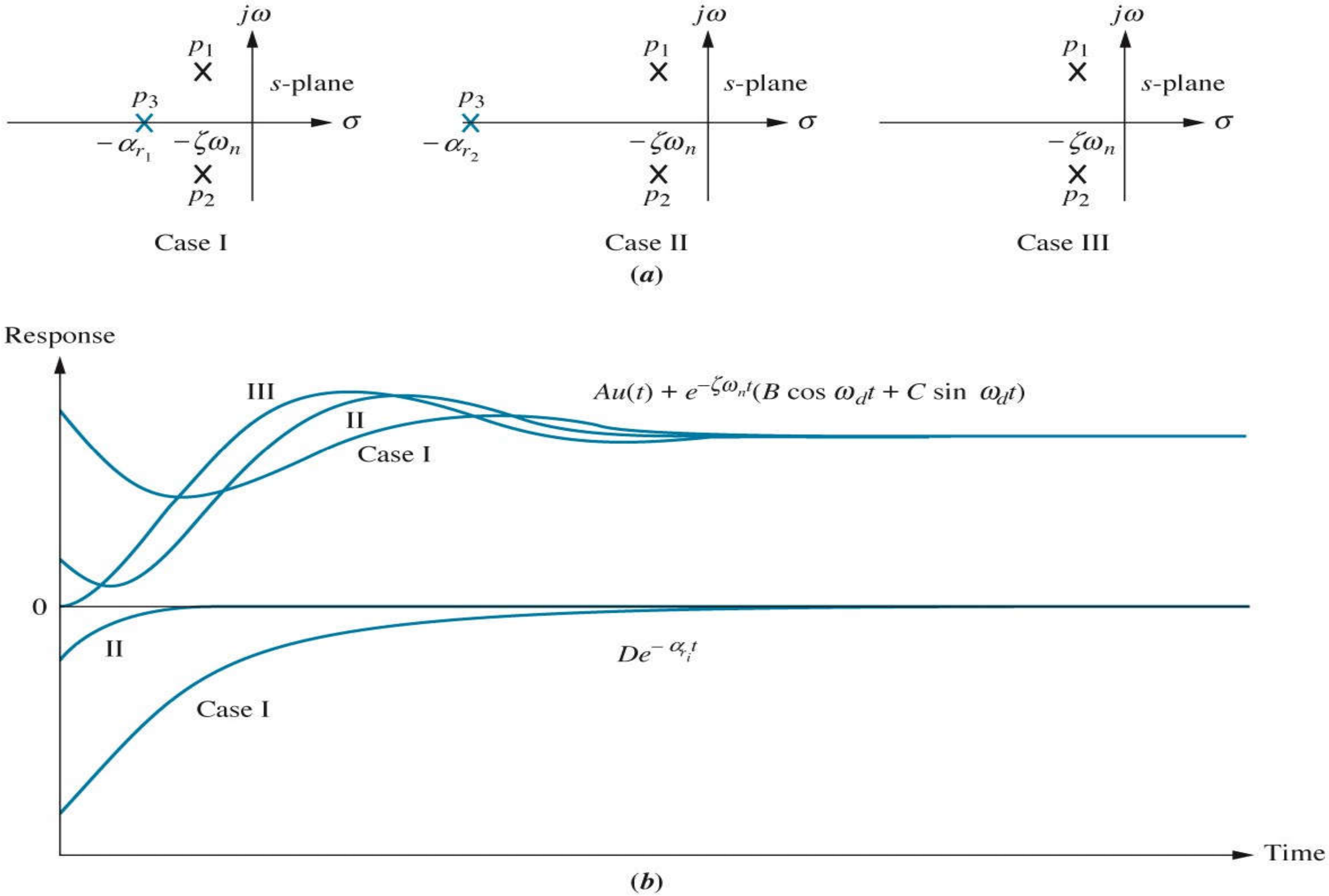


Figure P4.13

Conclude the variation of $\varsigma, \omega_n, \omega_d, T_p, T_s, \% OS$



4.5 System response with additional poles



$j\omega$
p_1
s-plane
p_3
σ
$-\alpha_{r_1}$
$-\zeta\omega_n$
p_2
Case I
$-\alpha_{r_2}$
Case II
Case III
(a)
Response
III
II
Case I
$Au(t) + e^{-\zeta\omega_n t}(B \cos \omega_d t + C \sin \omega_d t)$
0
$De^{-\alpha_{r_i} t}$
Time
(b)

Example 4.8: System response with additional poles

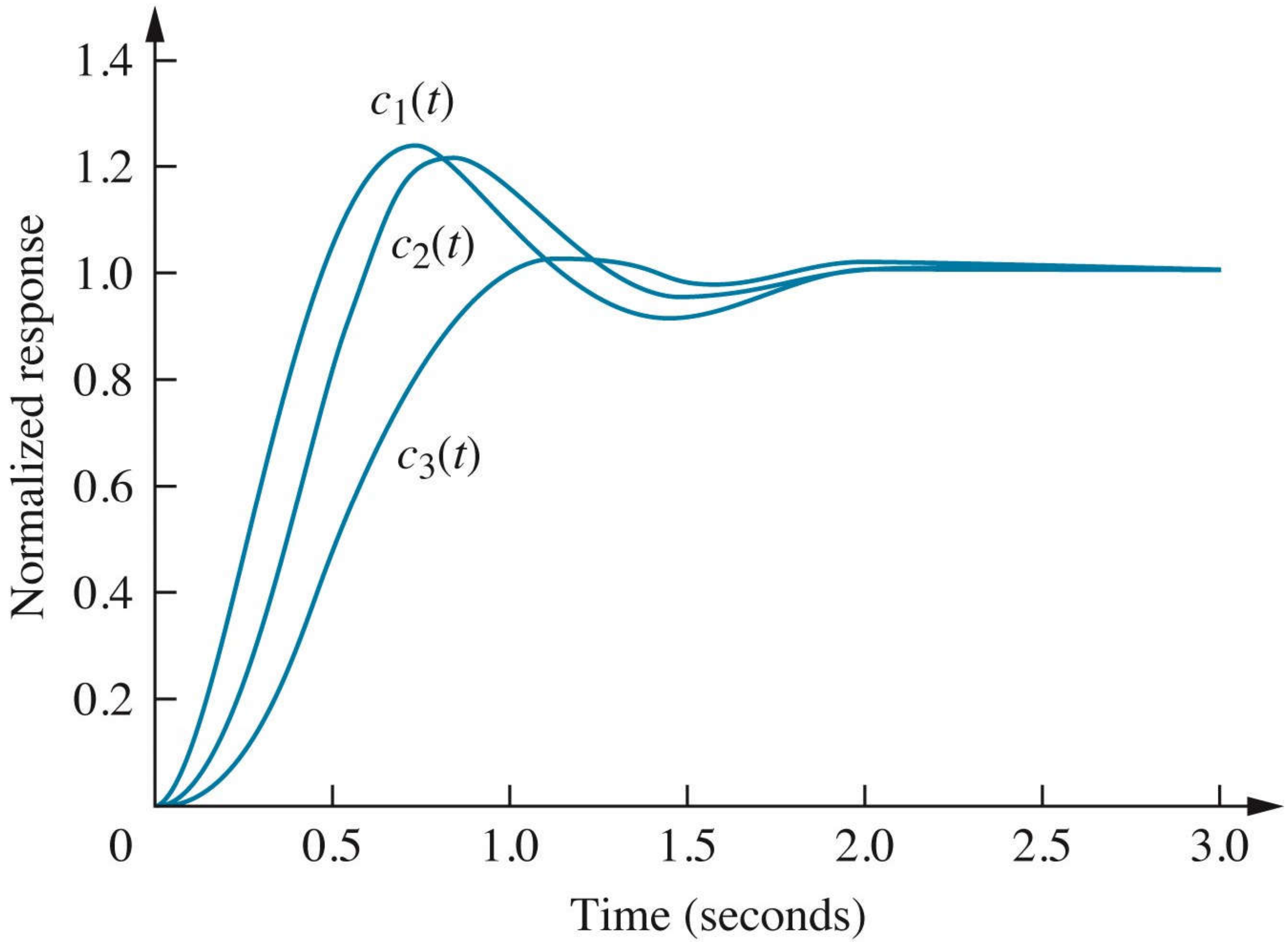
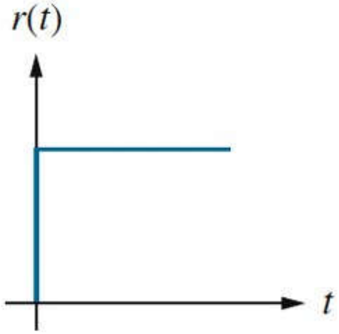
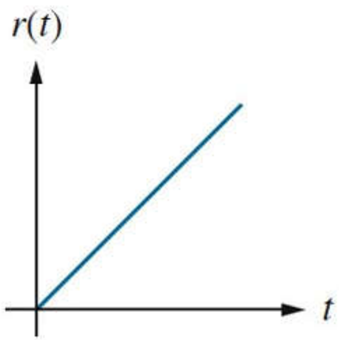
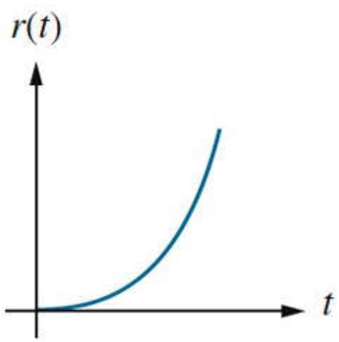


Figure 4.24

4.6
Steady State Error

TABLE 7.1 Test waveforms for evaluating steady-state errors of position control systems

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

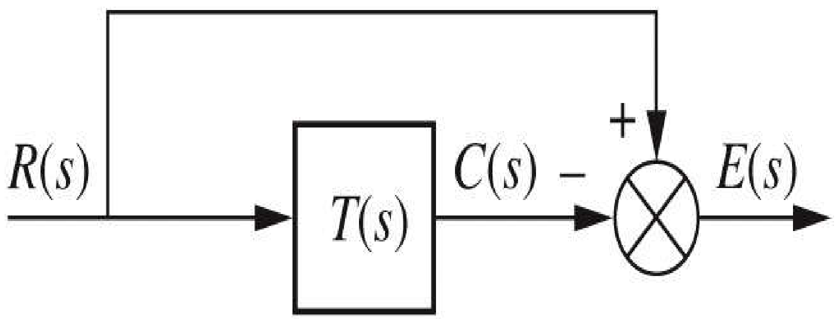
Introduction:

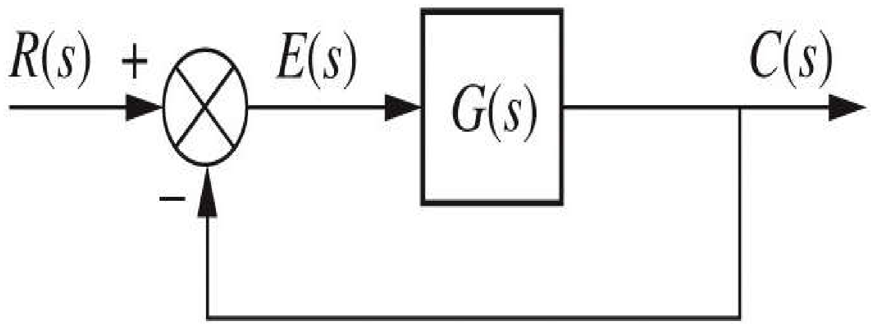
Steady state error (e_{ss}) is the difference between input and output for a prescribed test input as $t \to \infty$.

$$e_{ss}(\infty) \quad e_{ss} = \left| r(t) - c(t) \right|_{t \to \infty}$$

Case I: in terms of *T*(*s*)

Case II: in terms of *G*(*s*)



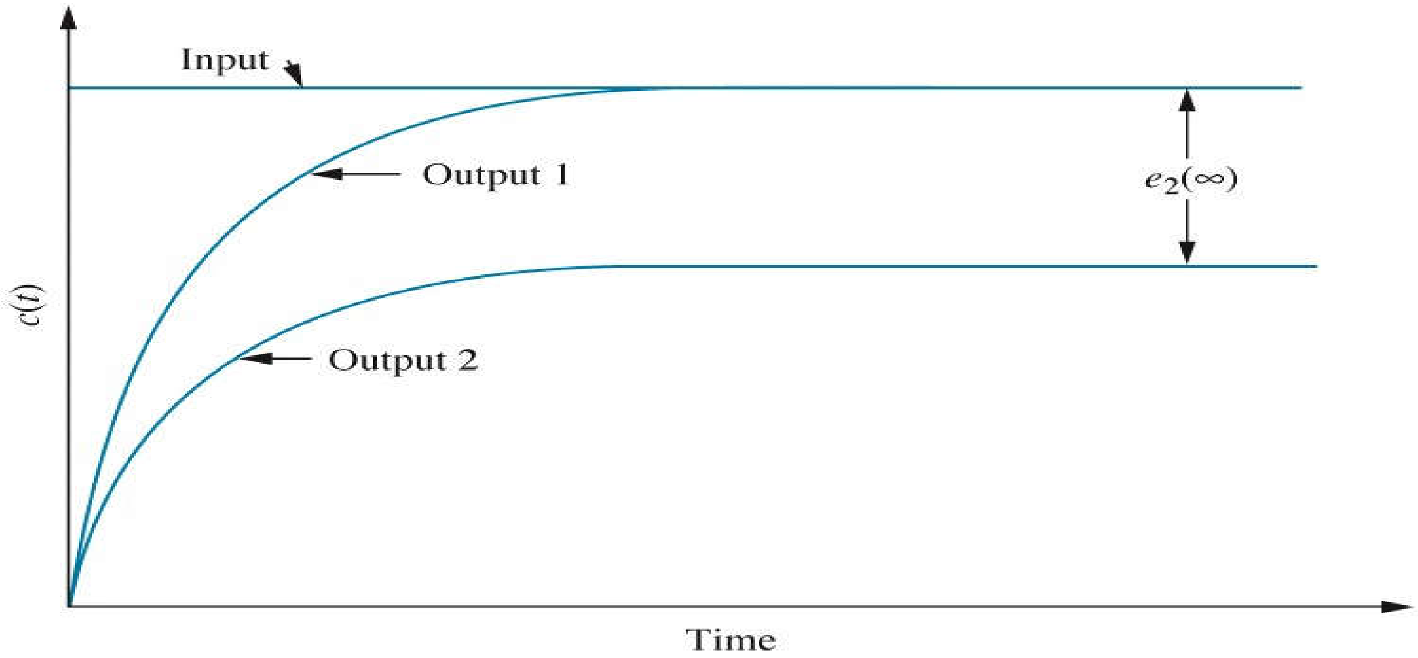


Closed-loop TF: *T*(*s*)

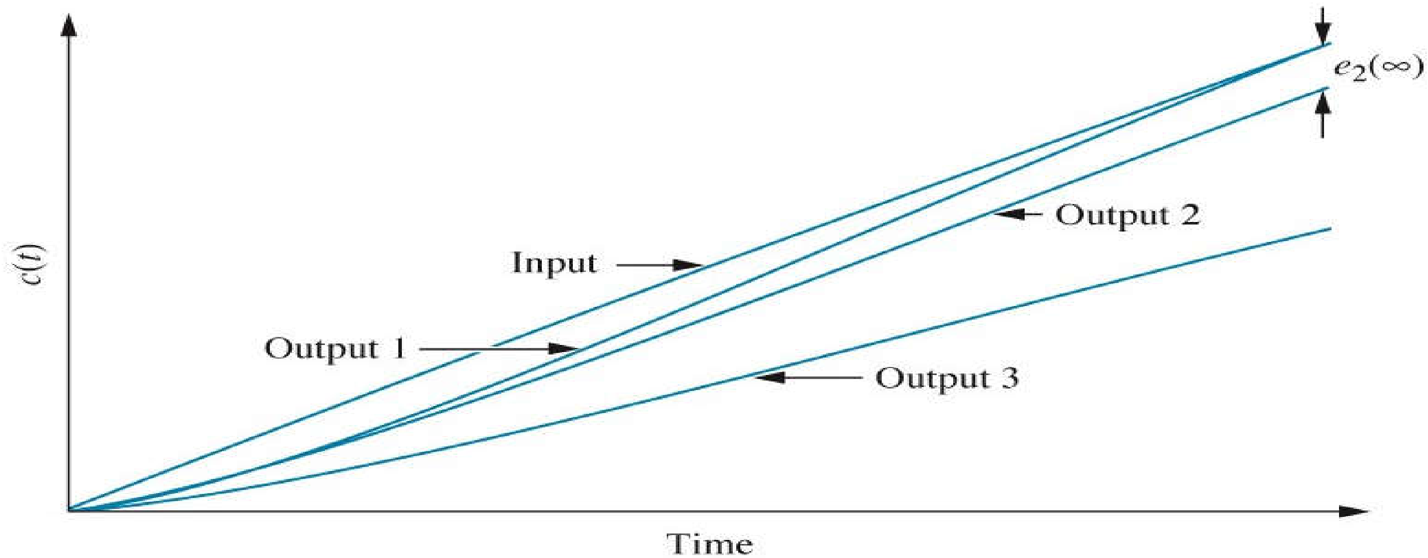
Forward TF: *G*(*s*)

Steady state error:

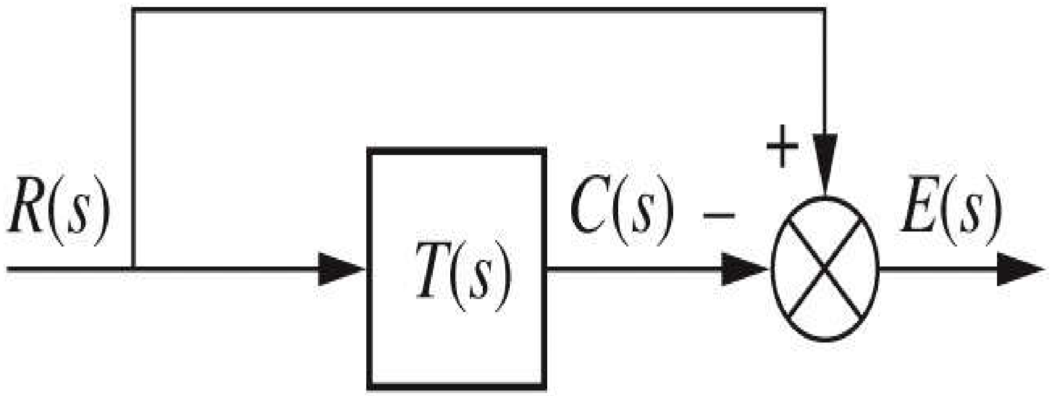
Step input



Ramp input



In terms of $T(s)$:



$$E(s) = input - output$$

$$= R(s) - C(s)$$

$$C(s) = R(s)\ T(s)$$

$$E(s) = R(s)\ [1 - T(s)]$$

Using Final Value Theorem:

$$\boxed{e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)}$$

$$e_{ss} = \lim_{t\to\infty} sR(s)\ [1 - T(s)]$$

Problem: Find the steady state error for a unit step input

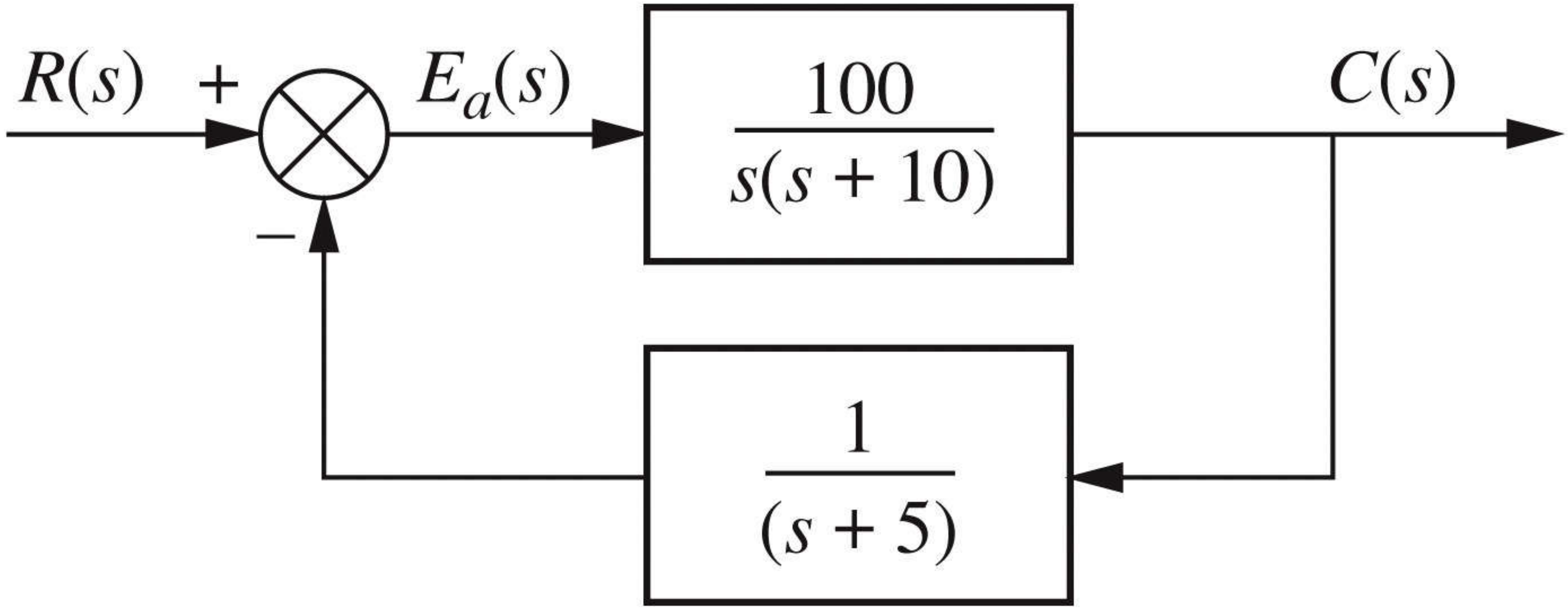
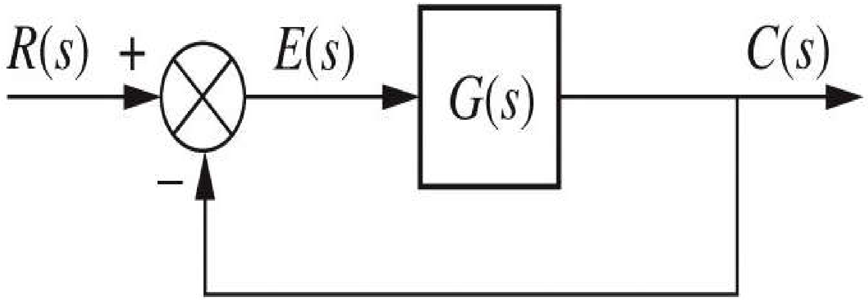


Figure 7.16

$$e_{ss}(\infty) = -4$$

(-ve = o/p > i/p)

In terms of $G(s)$:



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$E(s) = input - output$$

$$= R(s) - C(s)$$

$$= R(s) - \frac{G(s)}{1+G(s)}.R(s)$$

$$= R(s)\left[1 - \frac{G(s)}{1+G(s)}\right]$$

$$= R(s)\left[\frac{1+G(s)-G(s)}{1+G(s)}\right]$$

$$= \frac{1}{1+G(s)}.R(s)$$

Using Final Value Theorem:

$$e_{ss}(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$e_{ss}(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

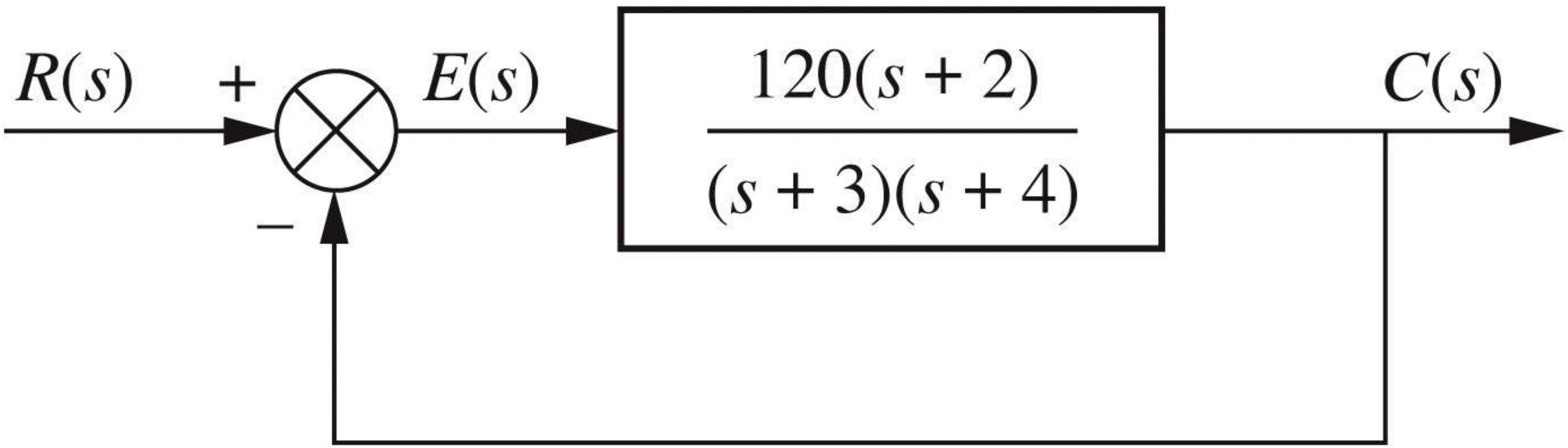


Figure 7.5

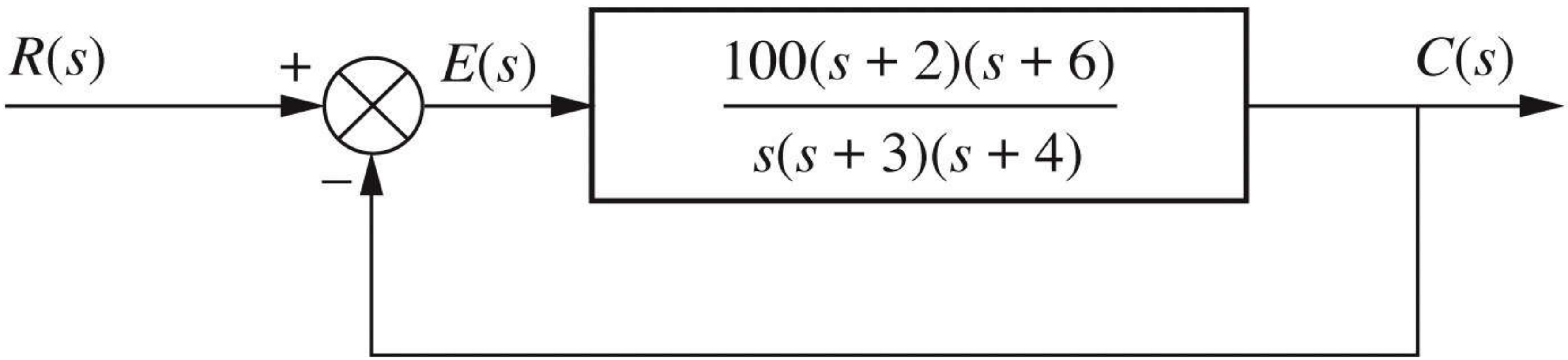


Figure 7.6

Non-unity feedback

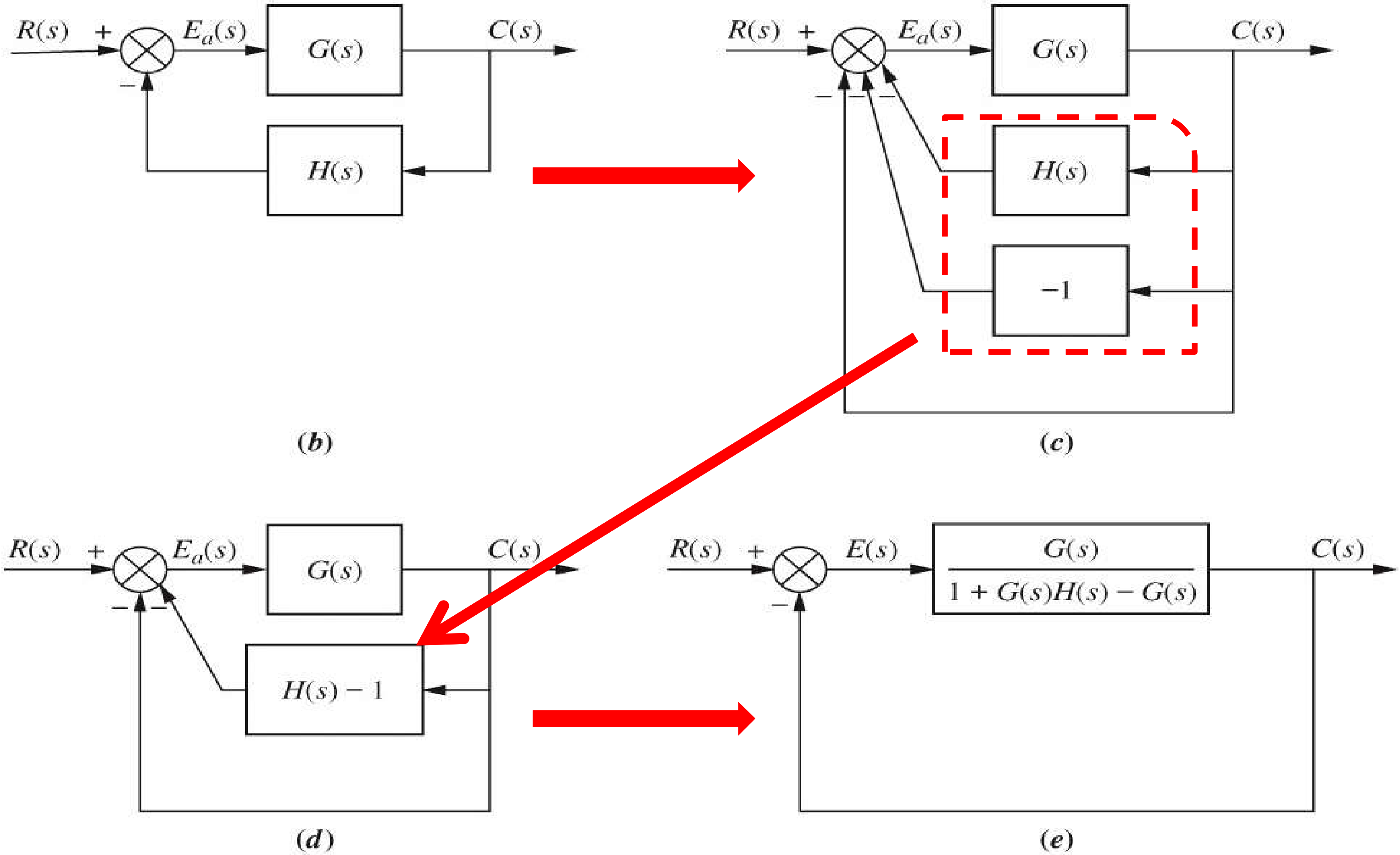


Figure 7.15

$$e_{ss} =$$

Problem: Find the steady state error for a unit step input

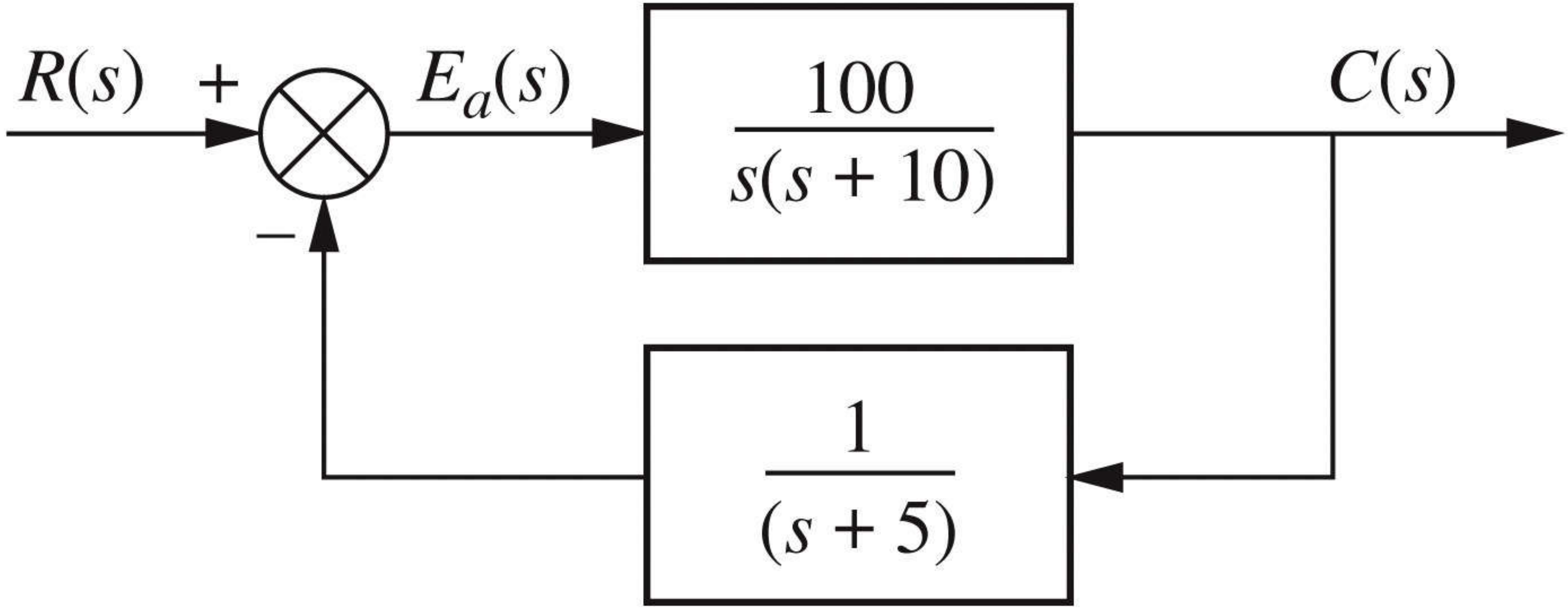


Figure 7.16

$$e_{ss}(\infty) = -4$$

(-ve = o/p > i/p)