SYSTEM MODELING AND ANALYSIS

CHAPTER 2
Mathematical Modeling in Transfer Function Form

Content

2.1

- Introduction to Laplace Transform and Transfer Function **(1 hour)**
  - 2.1.1 Laplace Transform
  - 2.1.2 Transfer function

2.2

- Modeling of Electrical Systems **(2 hours)**

2.3

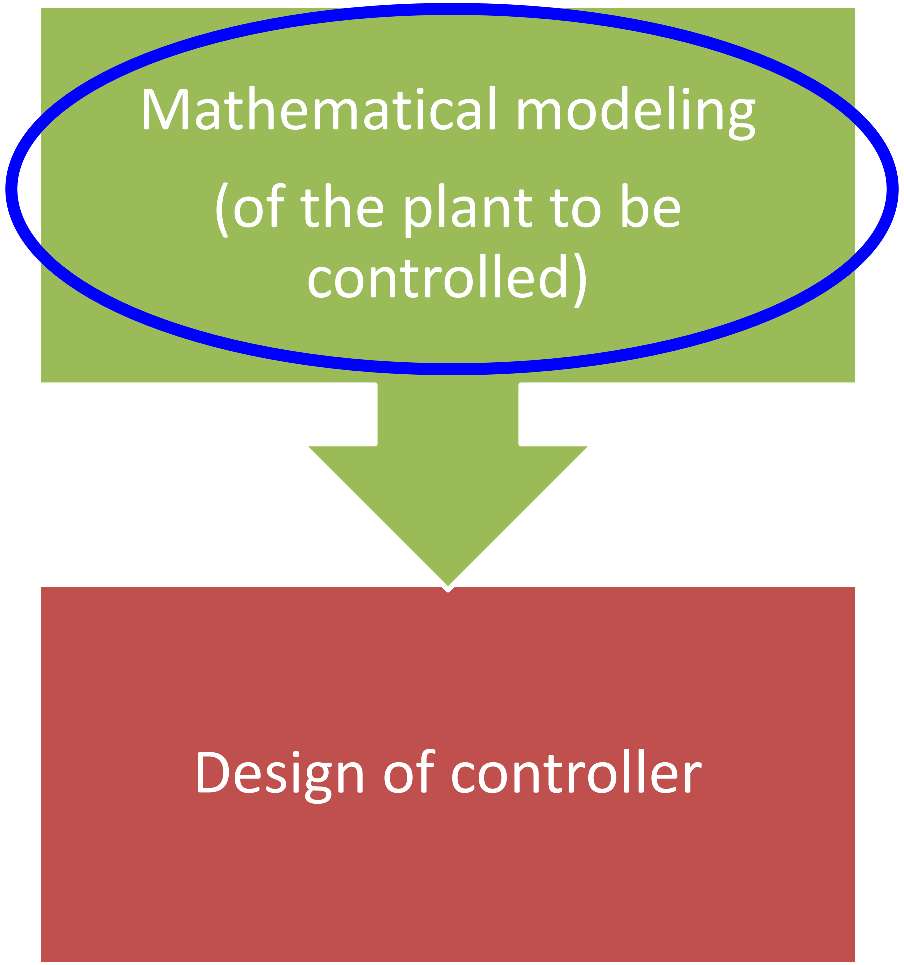
- Modeling of Mechanical Systems **(2 hours)**
  - Translational system
  - Rotational system
  - Rotational system with gears

2.4

- Modeling of Electromechanical Systems **(1 hour)**

2.1 Introduction to Laplace Transform and Transfer Function

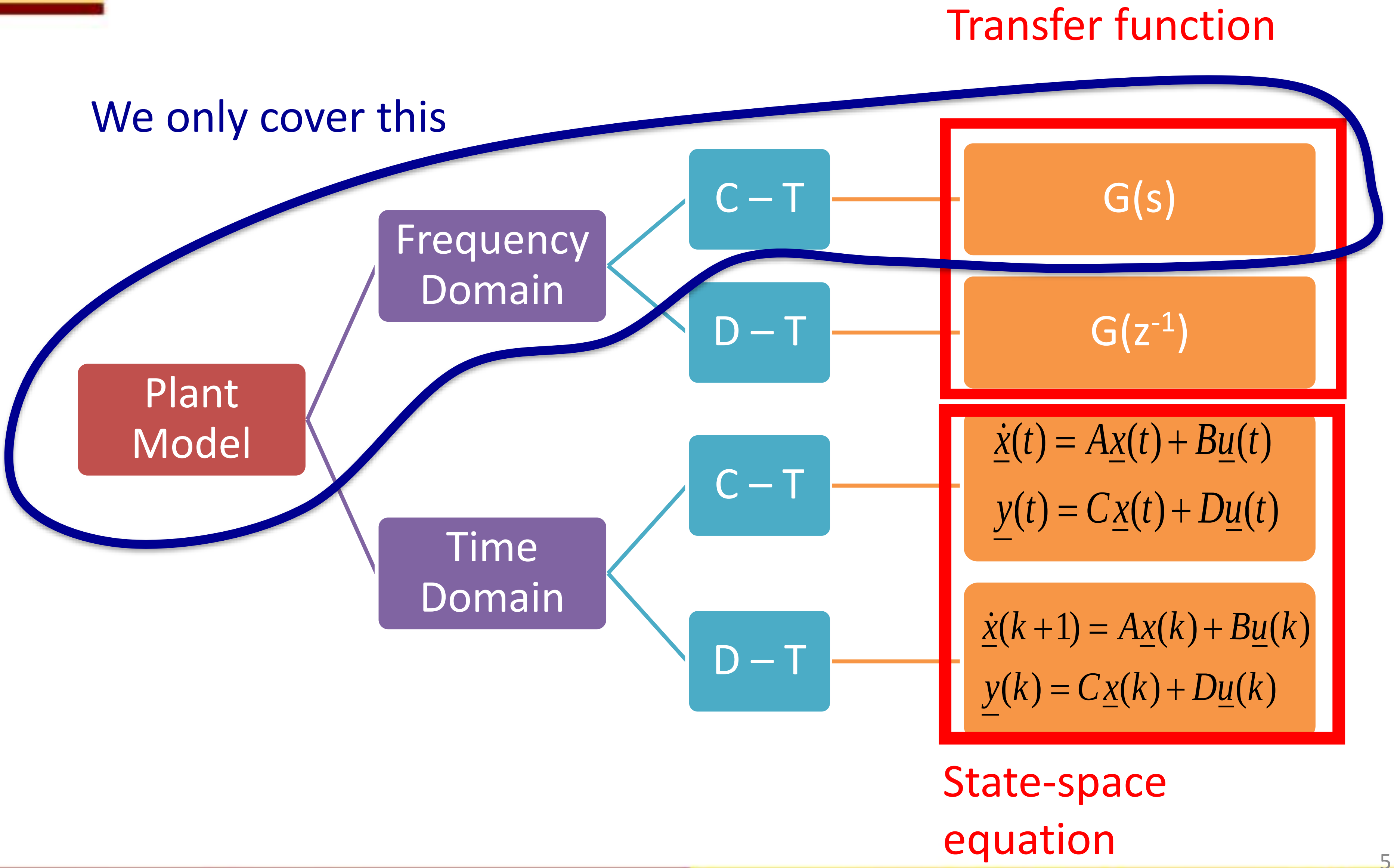
The Need for a Mathematical Model



Mathematical model of a dynamical system:

- May be obtained from the schematics of the physical systems,
- Based on physical laws of engineering
  - Newton's Laws of motion
  - Kirchoff's Laws of electrical network
  - Ohm's Law

Modeling of Control System Plants



2.1.1 Laplace Transform

Time-domain signals ⟷ **Frequency-domain** signals

Equations:

Laplace Transform: $$\boldsymbol{L}\left[f(t)\right] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Inverse Laplace Transform: $$\boldsymbol{L}^{-1}\left[F(s)\right] = f(t)u(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

$$f(t) \underset{\boldsymbol{L}^{-1}}{\overset{\boldsymbol{L}}{\rightleftarrows}} F(s)$$

$$\begin{aligned} u(t) &= 1, \quad t > 0 \\ &= 0, \quad t < 0 \end{aligned}$$

Laplace Transform Table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Given ***f (t)***, what is ***F(s)***?

Laplace Transform Theorem

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) \quad = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] \quad = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] \quad = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] \quad = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] \quad = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] \quad = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] \quad = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] \quad = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) \quad = \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) \quad = \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

Example 1:

- Find the Laplace Transform of $y(t)$, assuming zero initial condition $\frac{d^2y(t)}{dt^2} + 12\frac{dy(t)}{dt} + 32y(t) = 32u(t)$

  where $u(t)$ is a unit step.

Example 1:

- Find the Laplace Transform of $y(t)$, assuming zero initial condition $\frac{d^2y(t)}{dt^2}+12\frac{dy(t)}{dt}+32y(t)=32u(t)$

  where $u(t)$ is a unit step.

- Solution:

$$Y(s)=\frac{32}{s\left(s^2+12s+32\right)}$$

$$y(t)=\left(1-2e^{-4t}+e^{-8t}\right)u(t)$$

Inverse Laplace Transform

- Recall:

$$f(t) \underset{L^{-1}}{\overset{L}{\rightleftarrows}} F(s)$$

- Therefore, for Inverse Laplace Transform,

Given $F(s)$, what is $f(t)$?

- Refer to Laplace Transform Table on p8.

Example 2:

- Find the inverse Laplace Transform of

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)}$$

- Solution:

```
syms s;
g=32/(s*(s^2+12*s+32));
ilaplace(g)
```

Inverse Laplace Transform

$$F(s) = \frac{N(s)}{D(s)}$$

numerator

denominator

- 3 situations:

  i. Roots of ***D*(*s*)** are real & distinct, e.g.

  $$F(s) = \frac{2}{(s+1)(s+2)}$$

  ii. Roots of ***D*(*s*)** are real & repeated, e.g.

  $$F(s) = \frac{2}{(s+1)(s+2)^2}$$

  iii. Roots of ***D*(*s*)** are complex, e.g.

  $$F(s) = \frac{2}{s(s^2+2s+5)}$$

- Hint: Use '**Partial Fraction Expansion**'

Example 3:

- Find the inverse Laplace Transform of

$$F(s) = \frac{2}{(s+1)(s+2)}$$

- Solution:

Example 4:

- Find the inverse Laplace Transform of

$$F(s)=\frac{2}{(s+1)(s+2)^2}$$

- Solution:

Example 5:

- Find the inverse Laplace Transform of

$$F(s) = \frac{2}{s(s^2 + 2s + 5)}$$

- Solution:

Using MATLAB

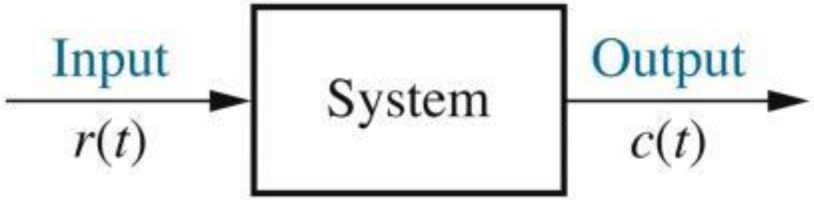
- Represent polynomials.
- Find roots of polynomials.
- Multiply polynomials.
- Find partial fraction expansion
- Solve Examples 1 – 5.

TRY

2.1.2
Transfer Function

2.1.2 Transfer Function, $G(s)$

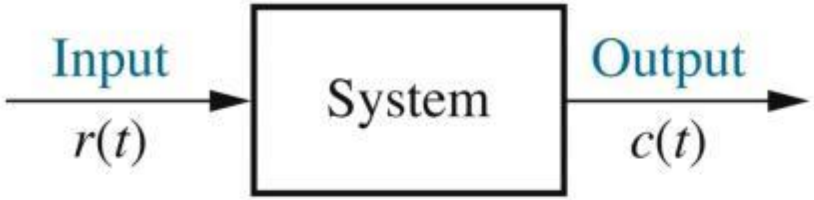
- Definition:



(a)

$$G(s) = \frac{\text{Laplace transform of output signal, c}(t)}{\text{Laplace transform of input signal, r}(t)}$$

$$= \frac{C(s)}{R(s)}$$



(a)

- Differential equation model:

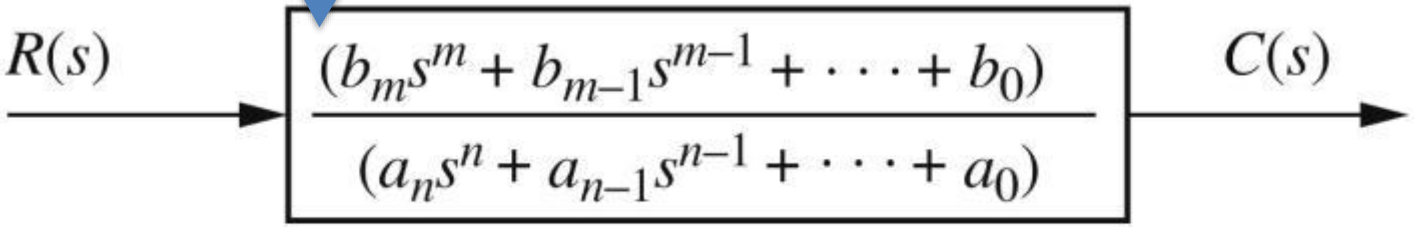
$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \ldots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \ldots + b_0 r(t)$$

- Laplace transform both sides (‘**Differentiation** condition:

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \ldots + a_0 C(s) = b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \ldots + b_0 R(s)$$

$$\frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0} = G(s)$$

Transfer function



Example 6:

- Find the transfer function represented by:

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

- Use MATLAB to create the above transfer function.

- Find the response, $c(t)$, to an input $r(t) = u(t)$, a unit step input, assuming zero initial condition.

Solution:

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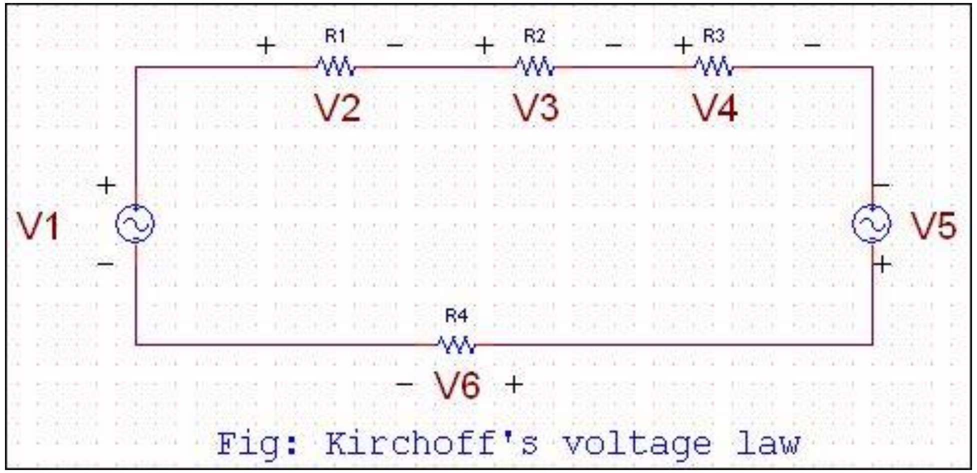
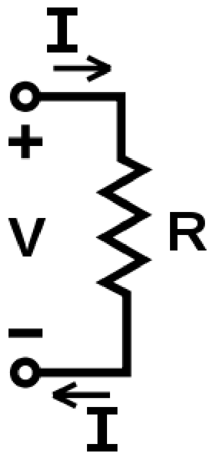
2.2
Modeling of
Electrical System

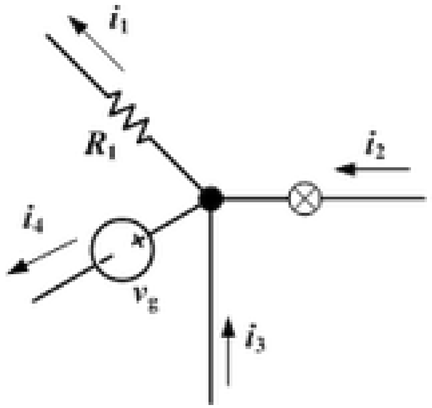
Review on Electrical Circuit Analysis

- Ohm's Law
- Kirchoff's Voltage Law
- Kirchoff's Current Law
- Mesh & Nodal Analysis





Ohm's Law



Kirchoff's Current Law

Scope

- Passive linear components
  - i. Capacitor (C) – **store** energy
  - ii. Resistor (R) – **dissipate** energy
  - iii. Inductor (L) – **store** energy
- Relationships:

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Summary of Relationship

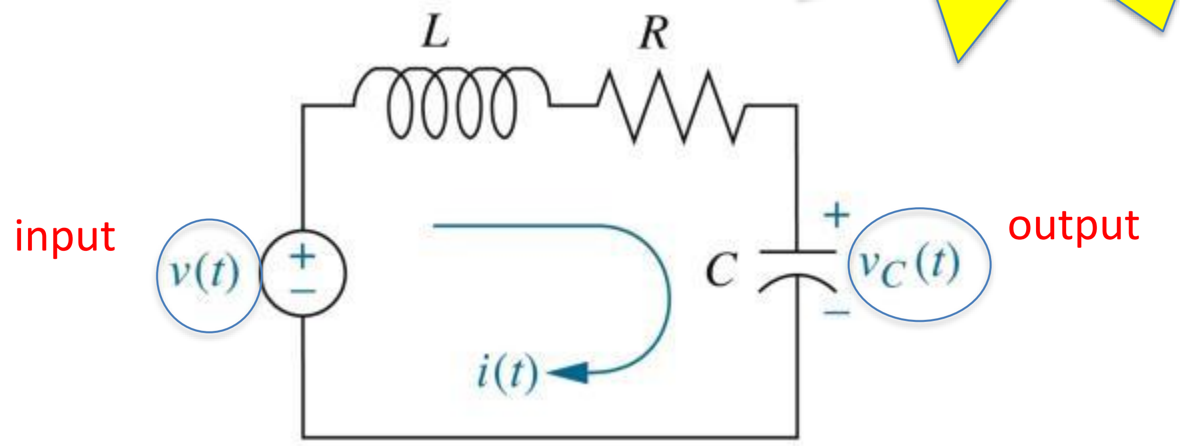
Comp.	$i(t)=\frac{dq(t)}{dt}$	$I(s)$	$v(t)$	$V(s)$	$Z(s)=\frac{V(s)}{I(s)}$
R	$i(t)=\frac{1}{R}v(t)$	$I(s)=\frac{1}{R}V(s)$	$v(t)=Ri(t)$	$V(s)=RI(s)$	R
L			$v(t)=L\frac{di(t)}{dt}$ $=L\frac{d^2q(t)}{dt^2}$	$V(s)=LsI(s)$ $=Ls^2Q(s)$	Ls
C	$i(t)=C\frac{dv(t)}{dt}$	$I(s)=CsV(s)$	$v(t)=\frac{1}{C}q(t)$	$V(s)=\frac{1}{C}Q(s)$	$\frac{1}{Cs}$

Laplace Tx.

Example 7: Single-loop network

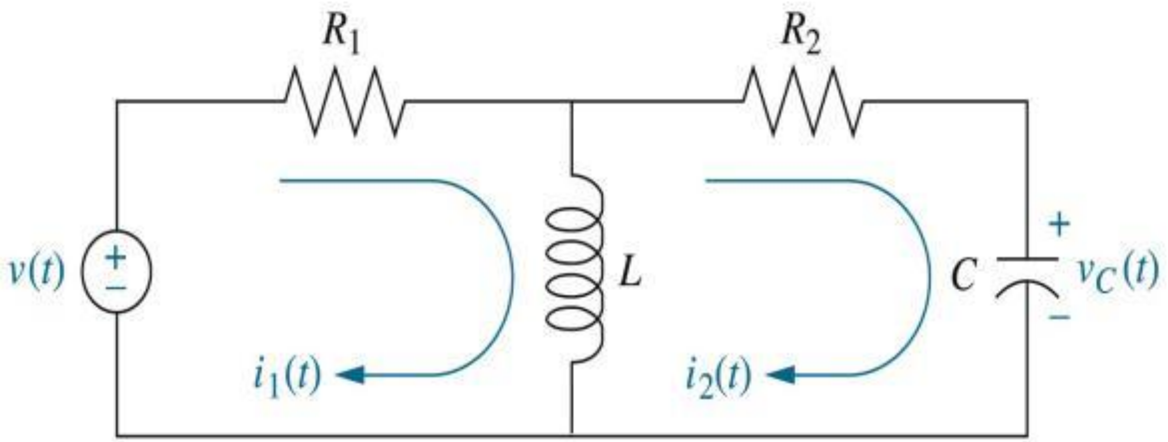
- Find the transfer function of the circuit using
  - Differential Equation Method
  - Mesh Analysis
  - Nodal Analysis

Kirchoff’s Law



Example 8: Multiple-loop network

- Find the transfer function $\dfrac{I_2(s)}{V(s)}$ of the circuit using
  - Differential Equation Method
  - Mesh Analysis
  - Nodal Analysis



2.3
Modeling of **Mechanical System**

- Translational
- Rotational
- Rotational with Gears

2.3.1 Translational

- Newton's Laws of Motion:

i. **First law**: The velocity of a body remains constant unless the body is acted upon by an external force.

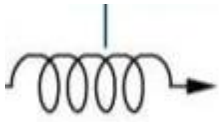
ii. **Second law**: The acceleration **a** of a body is parallel and directly proportional to the net force **F** and inversely proportional to the mass *m*, i.e., $\mathbf{F} = m\mathbf{a}$.

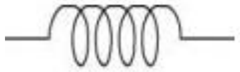
iii. **Third law**: The mutual forces of action and reaction between two bodies are equal, opposite and collinear.

Translational

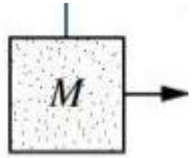
❖ 3 passive and linear components in mechanical system:

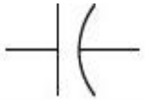
- **Spring** - energy storage element ⟷ inductor





- **Mass** - energy storage element ⟷ capacitor





- **Viscous damper** - energy-dissipative element ⟷ resistor

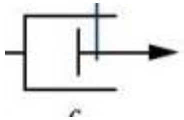
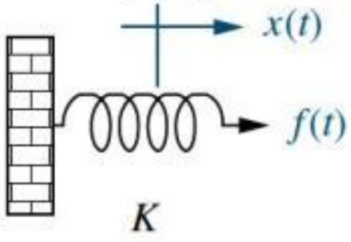
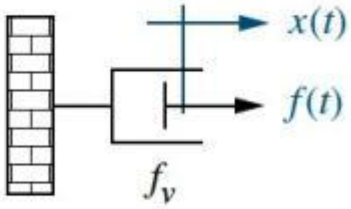
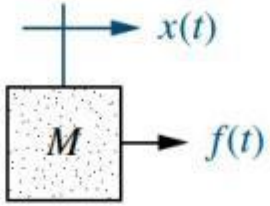
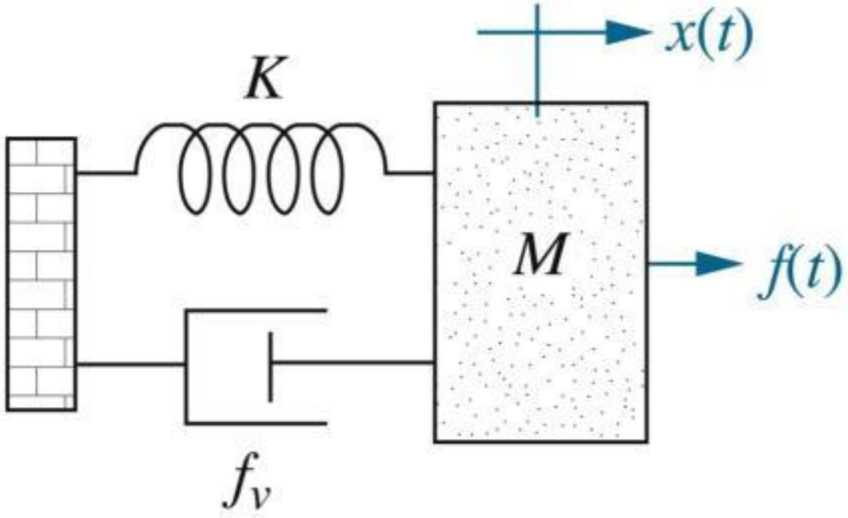


TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m(newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Spring, Mass & Damper in action

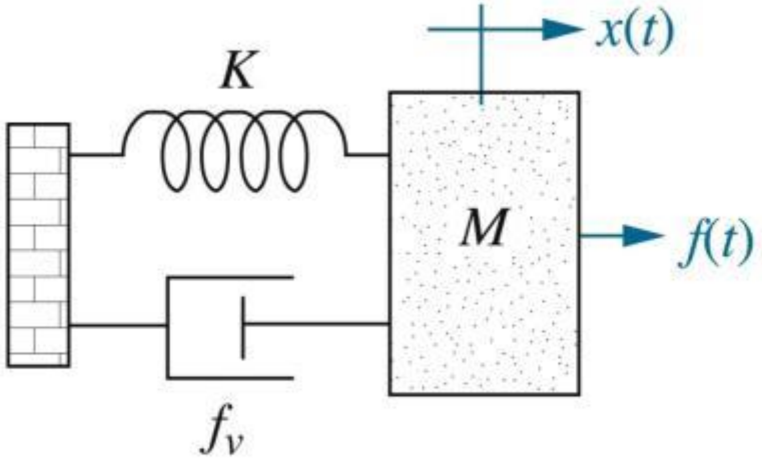


- Applied force *f*(*t*) points to the right
- Mass is traveling toward the right
- All other forces impede the motion and act to opposite direction
- Single input single output (SISO) system

Example 9:

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

- Find the transfer function $X(s)/F(s)$, for the following mechanical system.

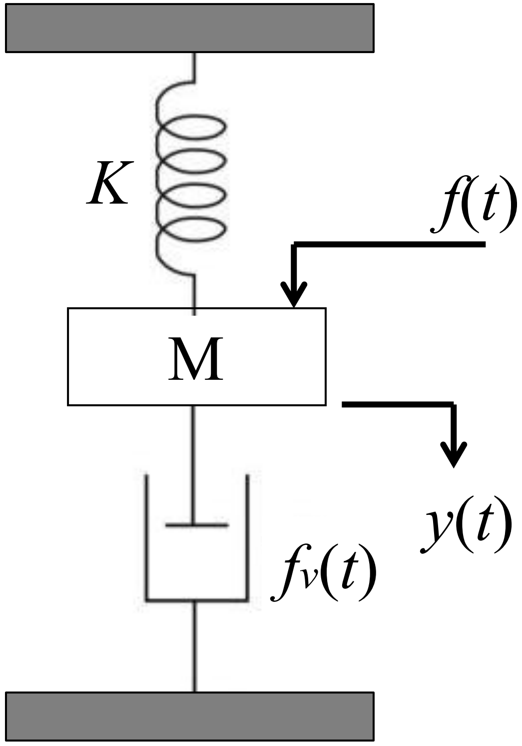


Example 10:

$$\frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

The external force ***f(t)*** is the input to the system, and the displacement ***y(t)*** of the mass is the output. The displacement ***y(t)*** is measured from the equilibrium position in the absence of external force. Find Y(s)/F(s).

(Assume that the system is linear and all initial conditions = 0).



Example 11:

- Find the transfer function $G(s) = \dfrac{X_2(s)}{F(s)}$

[Hint: Place a zero mass at x_2]

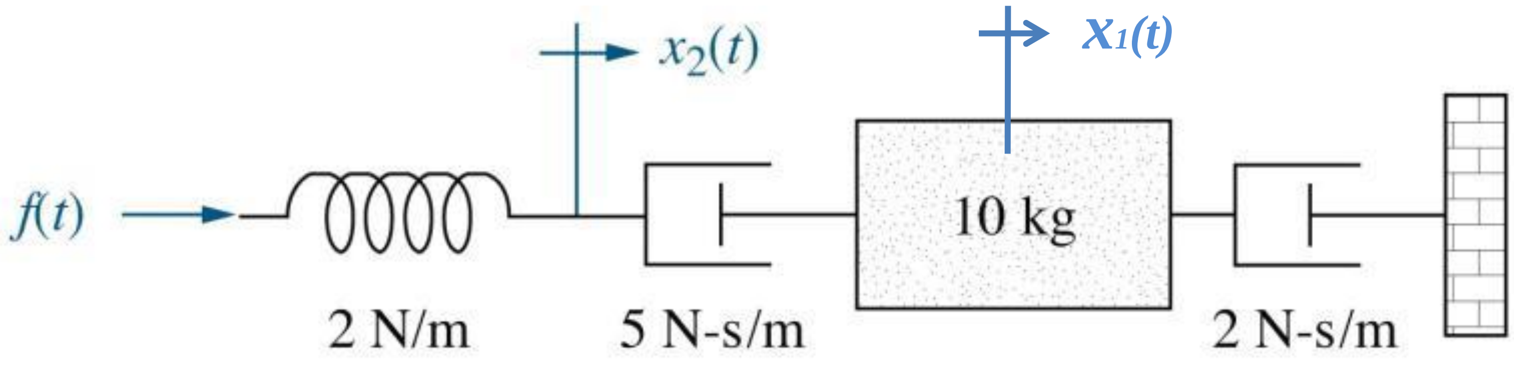
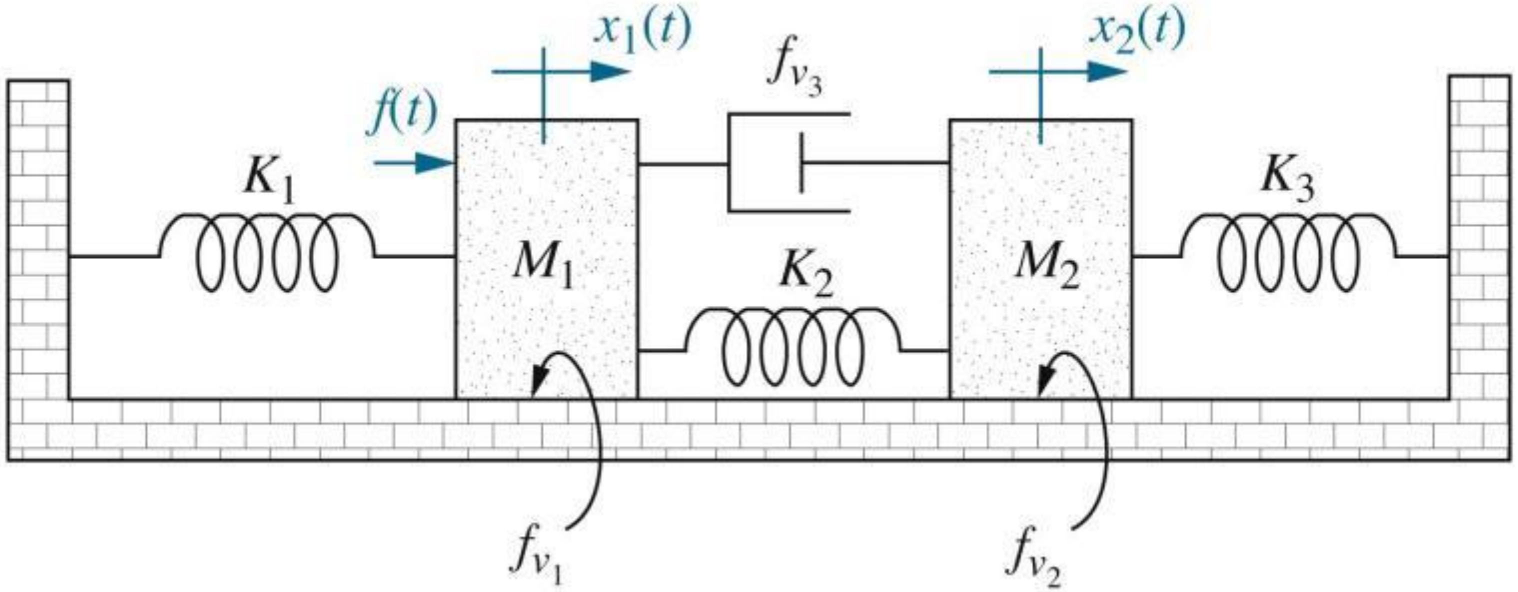


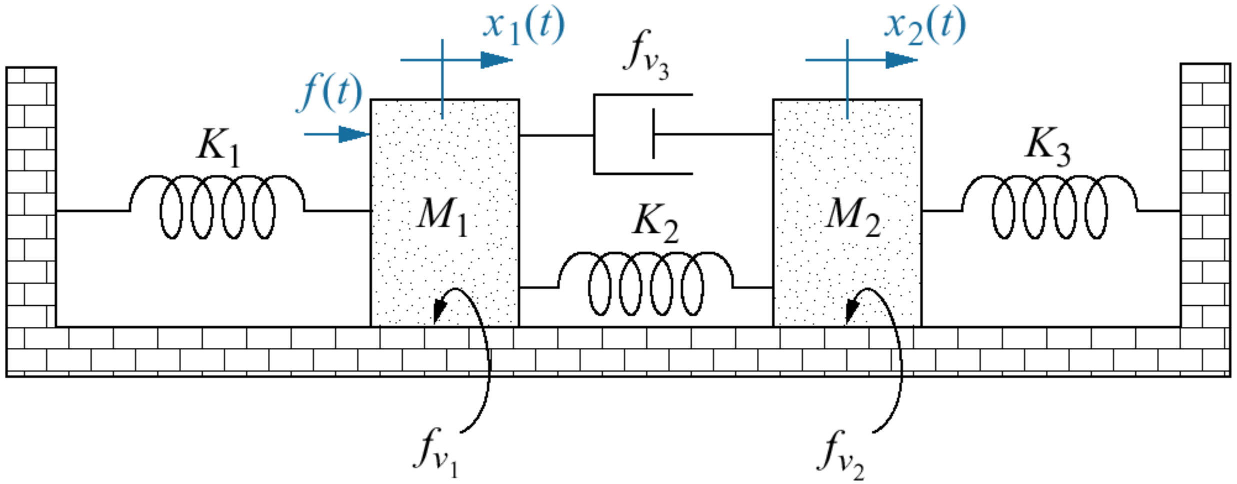
Figure P2.11

Example 12:

- Find the transfer function $X_2(s)/F(s)$, for the following mechanical system.

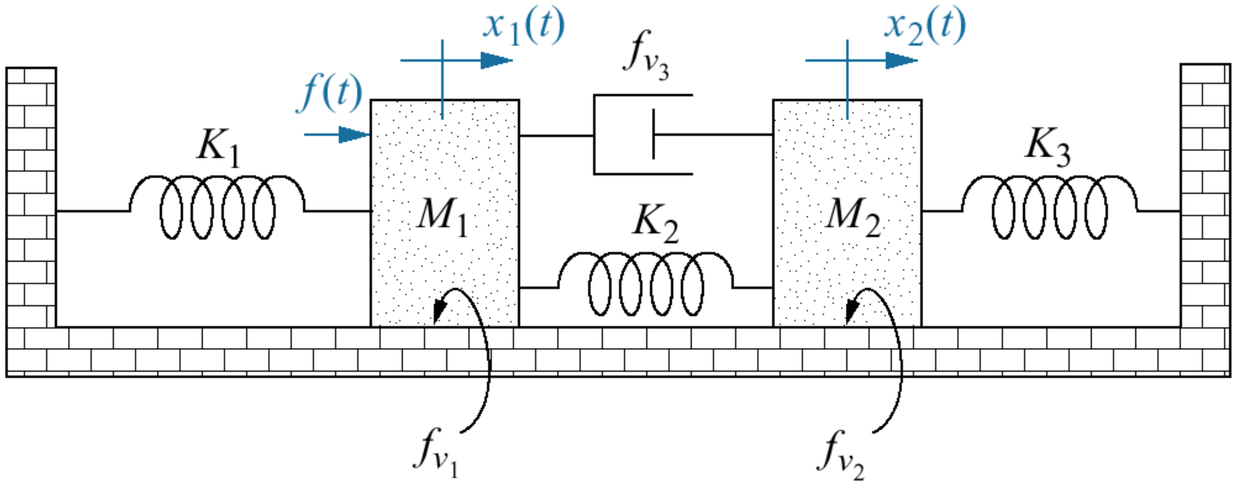


Example – two degrees of freedom

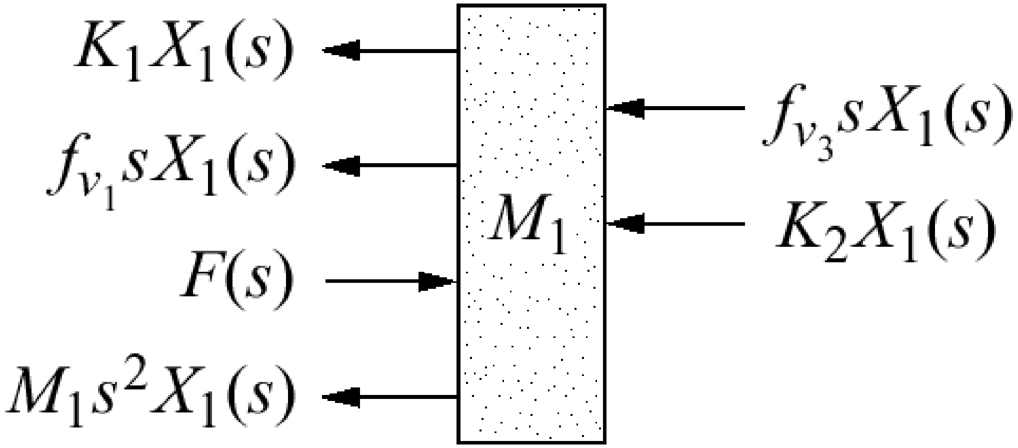


- Solution consists of 2 phases:
  - Forces on M1 due to its own motion and due to the motion of M2 transmitted to M1 through the system
  - Forces on M2 due to its own motion and due to the motion of M1 transmitted to M2 through the system

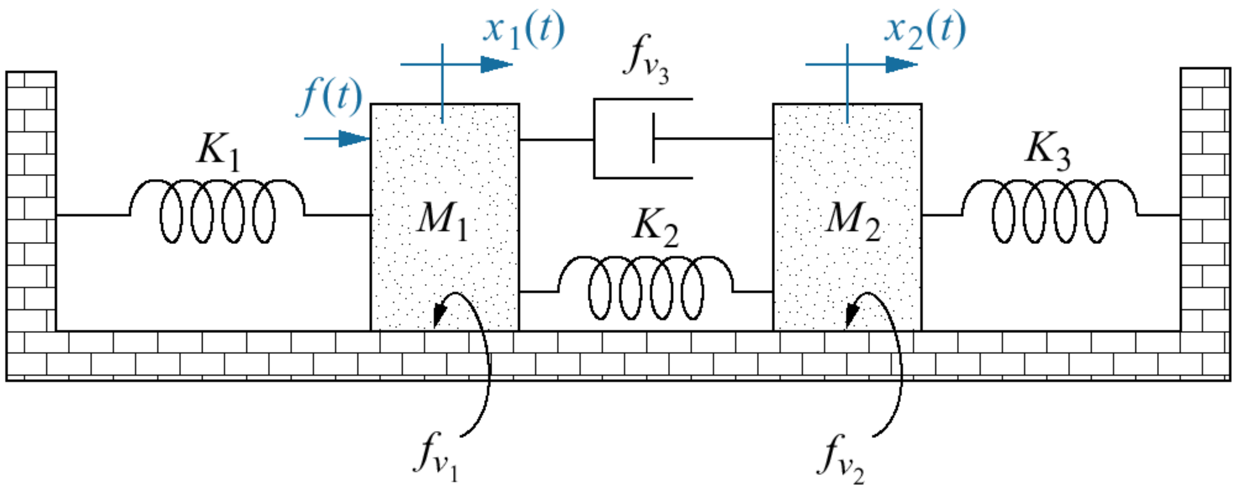
Example – two degrees of freedom

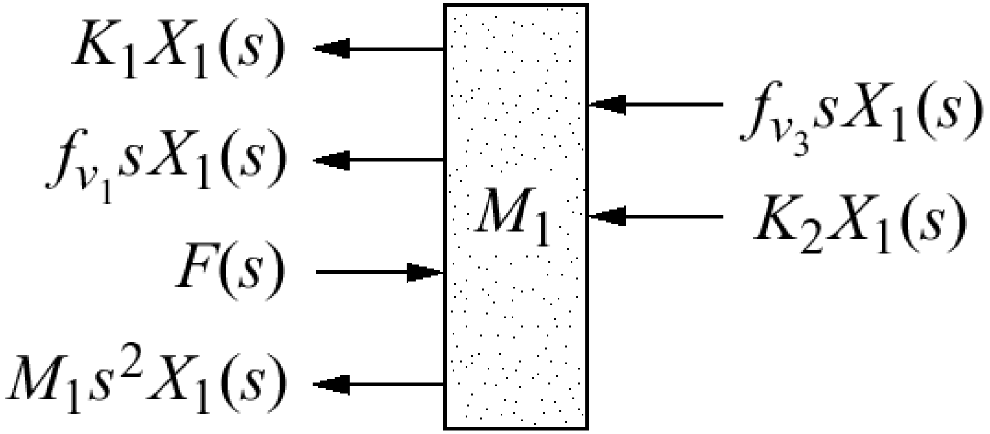


- If we hold M2 still and move M1 to the right



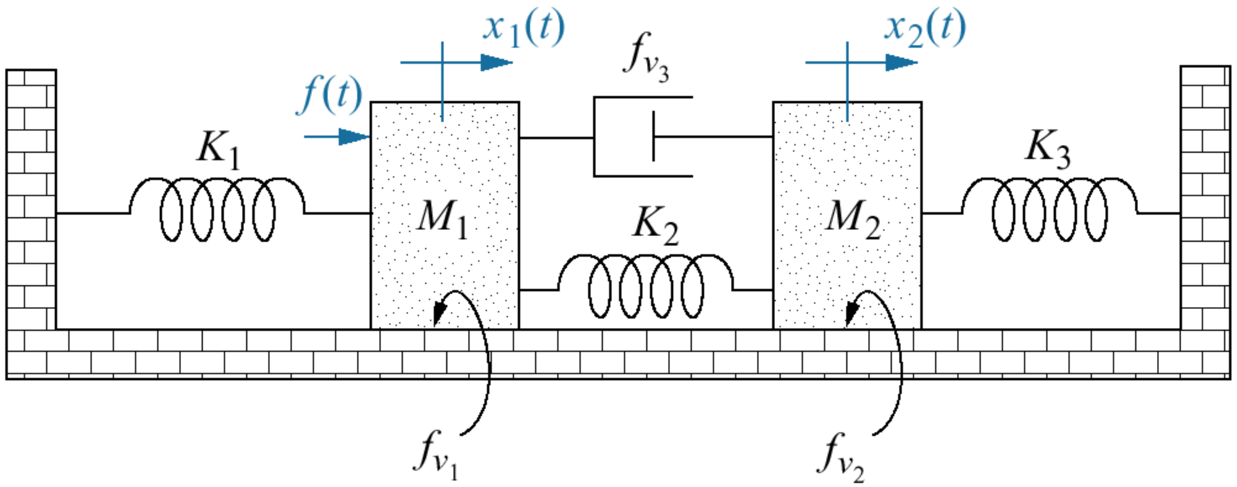
Exa



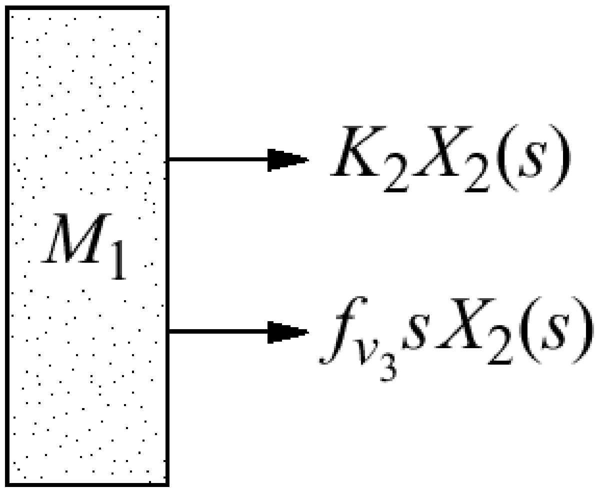


Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring ($x(t)$, $f(t)$, K)	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper ($x(t)$, $f(t)$, f_v)	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass ($x(t)$, M, $f(t)$)	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

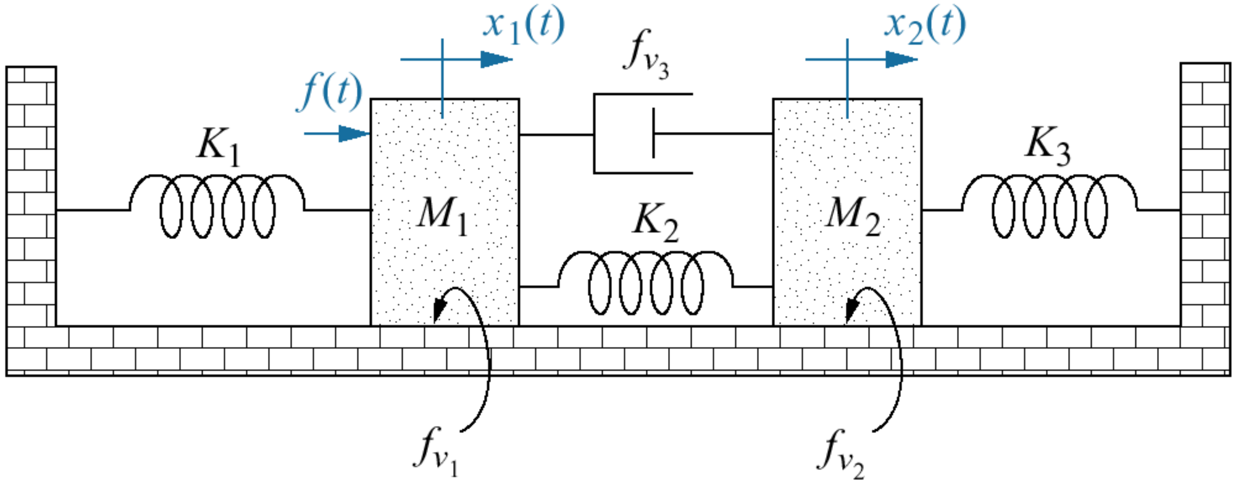
Example – two degrees of freedom

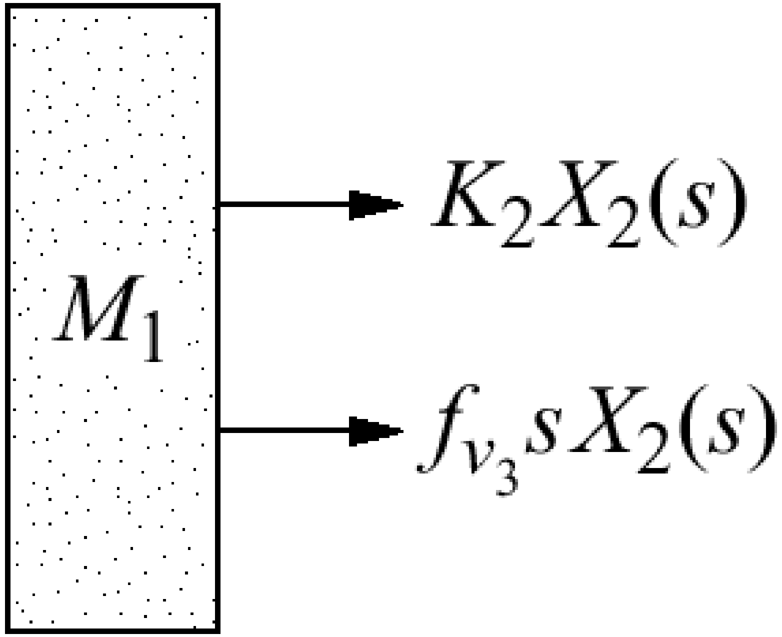


- If we hold M1 still and move M2 to the right



Exa





Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring (K; $x(t)$, $f(t)$)	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper (f_v; $x(t)$, $f(t)$)	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass (M; $x(t)$, $f(t)$)	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

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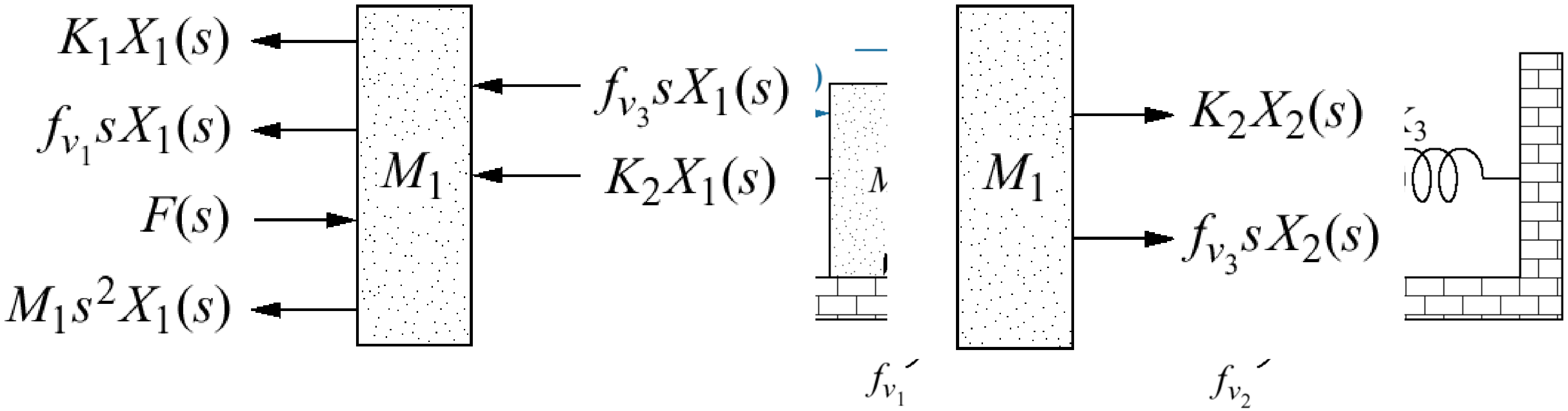
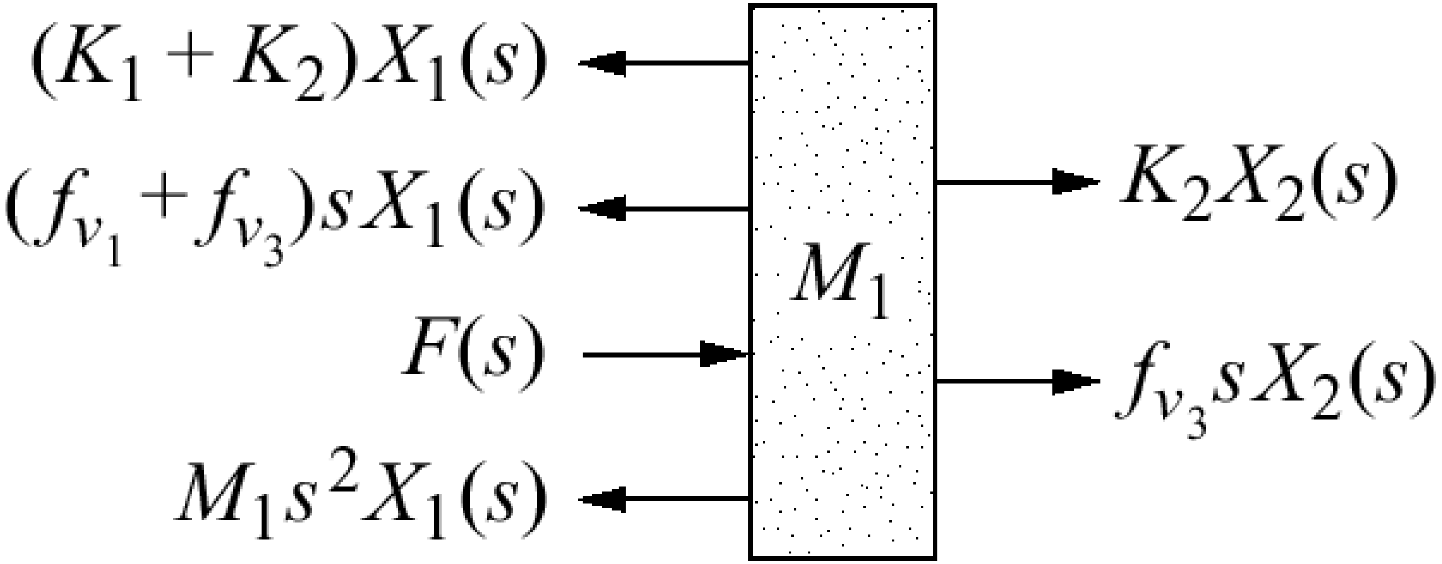
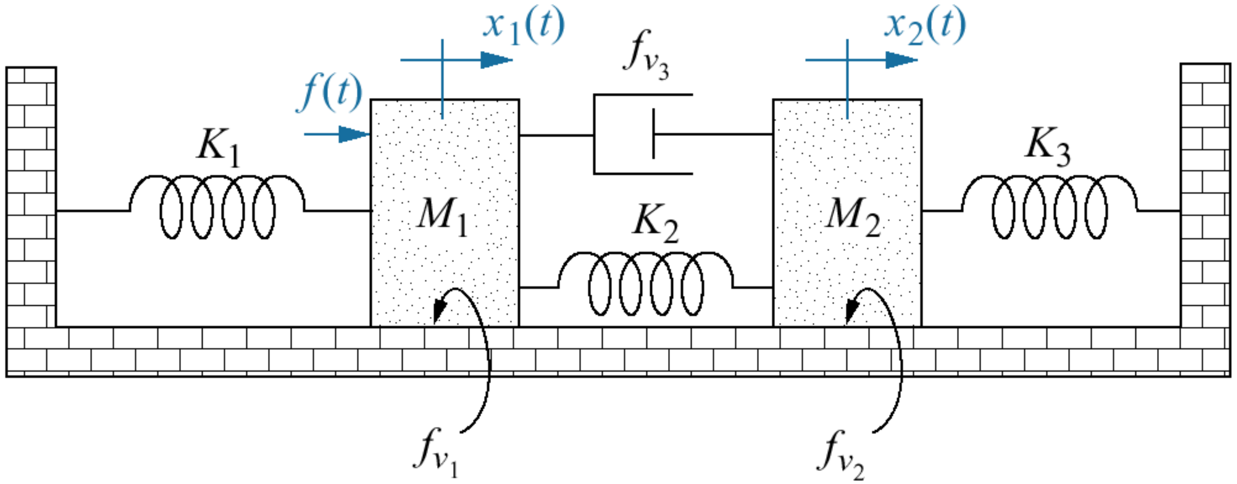


The total force on M1 is the superposition, or sum of forces

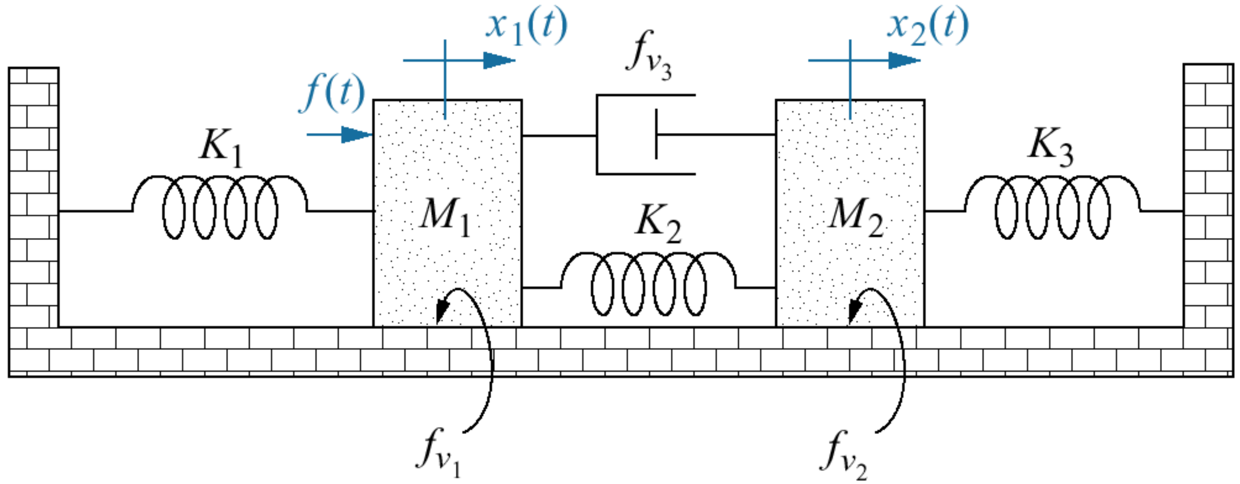


Example – two degrees of freedom

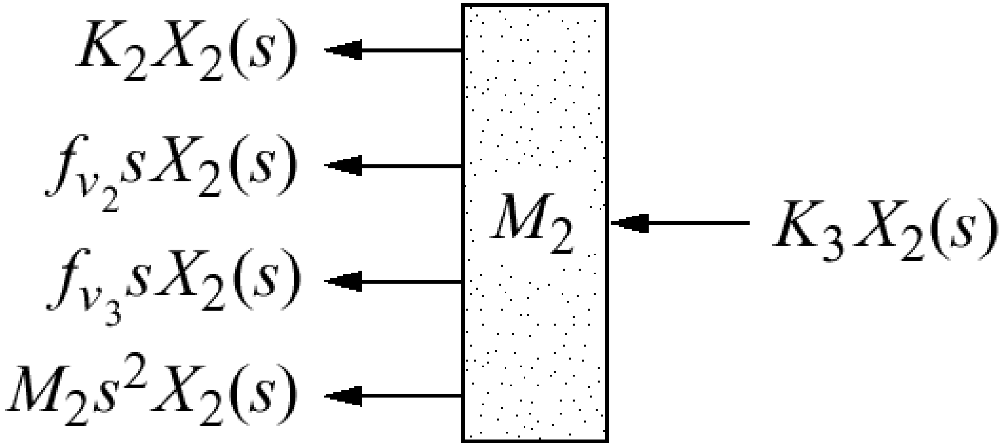


- Solution consists of 2 phases:
  - Forces on M1 due to its own motion and due to the motion of M2 transmitted to M1 through the system
  - Forces on M2 due to its own motion and due to the motion of M1 transmitted to M2 through the system

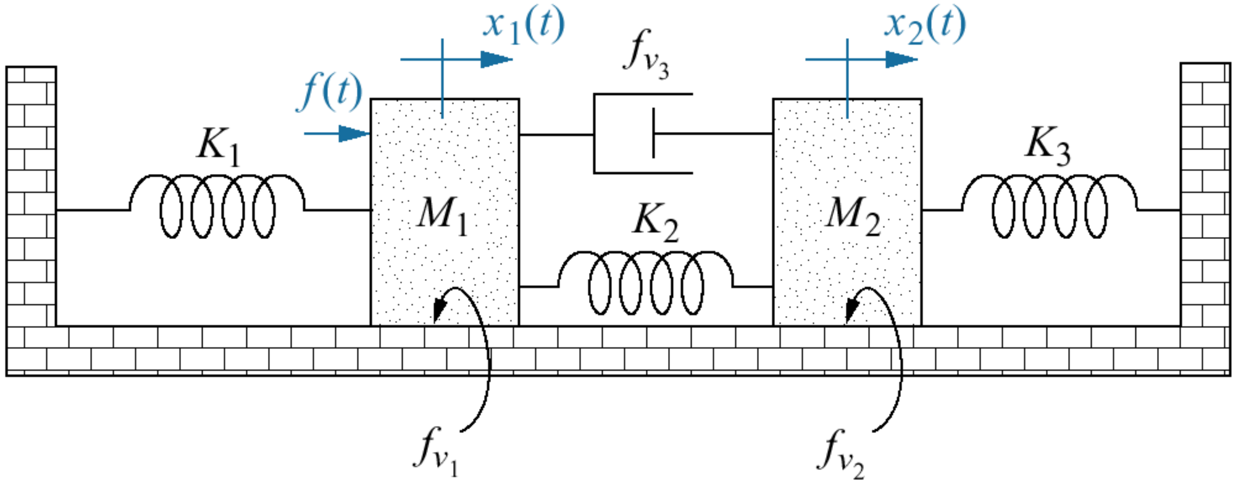
Example – two degrees of freedom



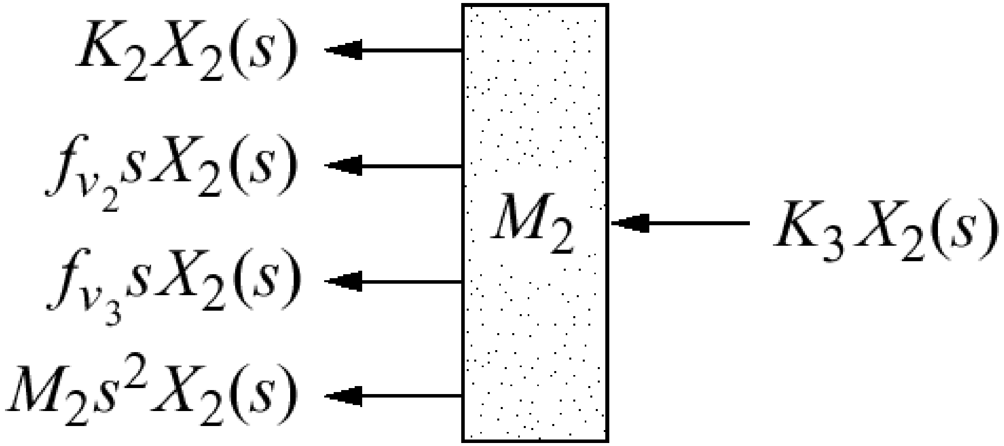
- If we hold M1 still and move M2 to the right

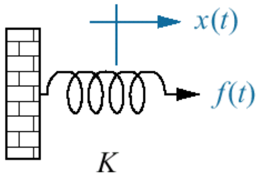
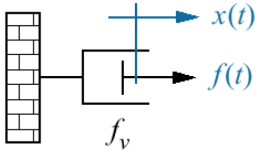
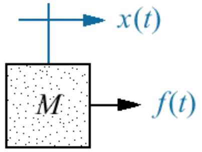


E m

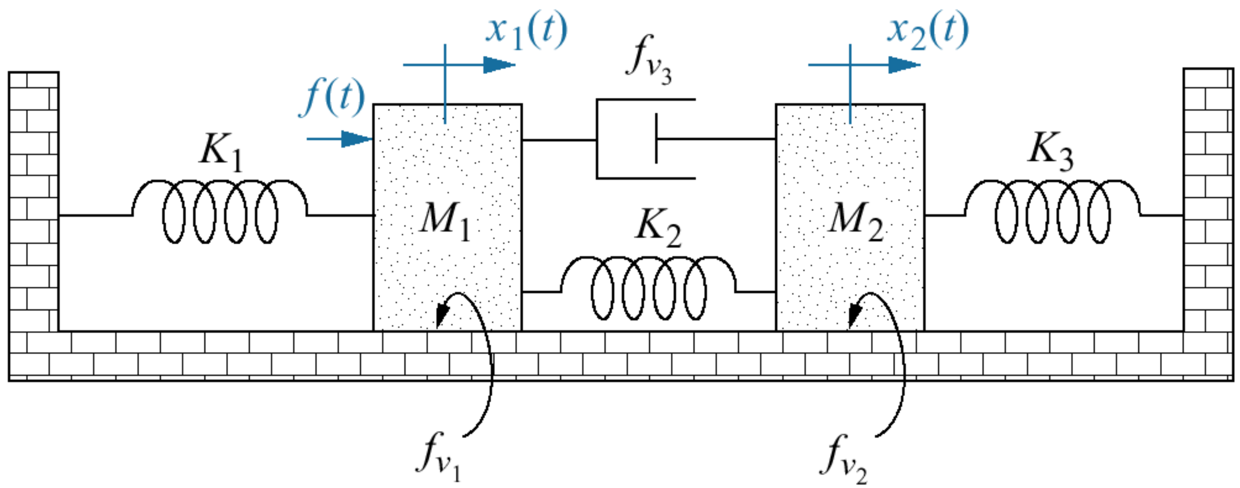


- If we hold M1 still a

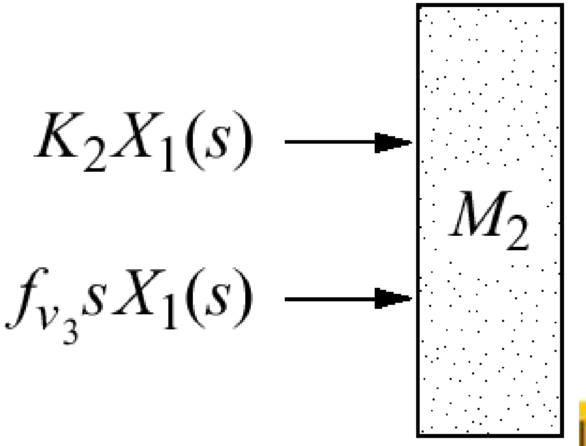


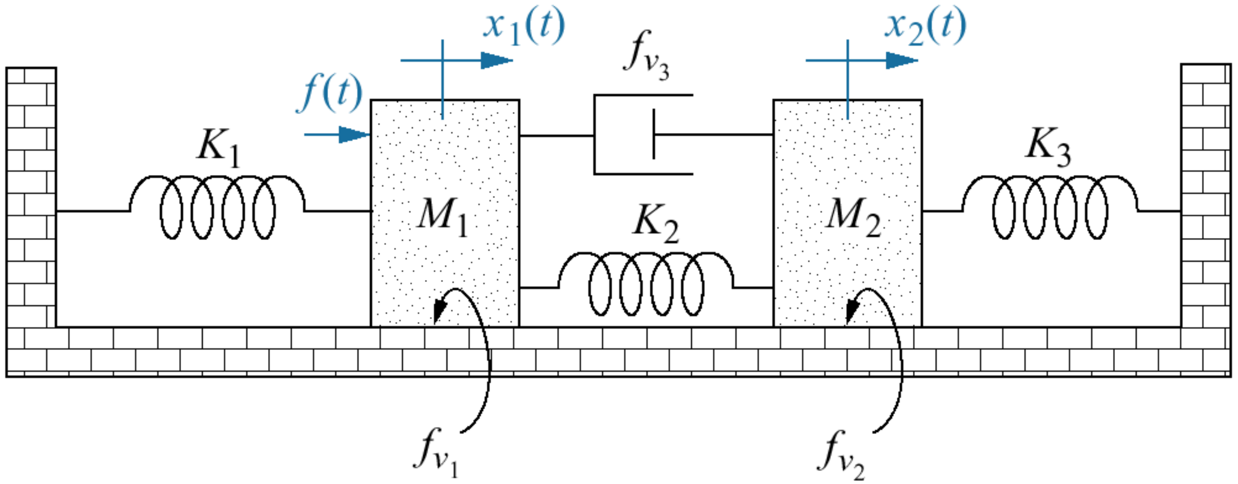
Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Example – two degrees of freedom

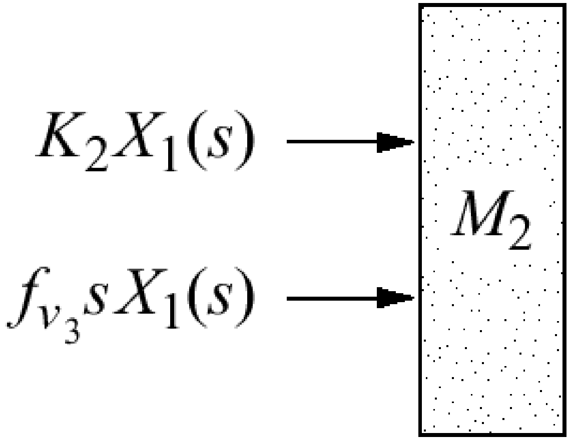


- If we hold M2 still and move M1 to the right

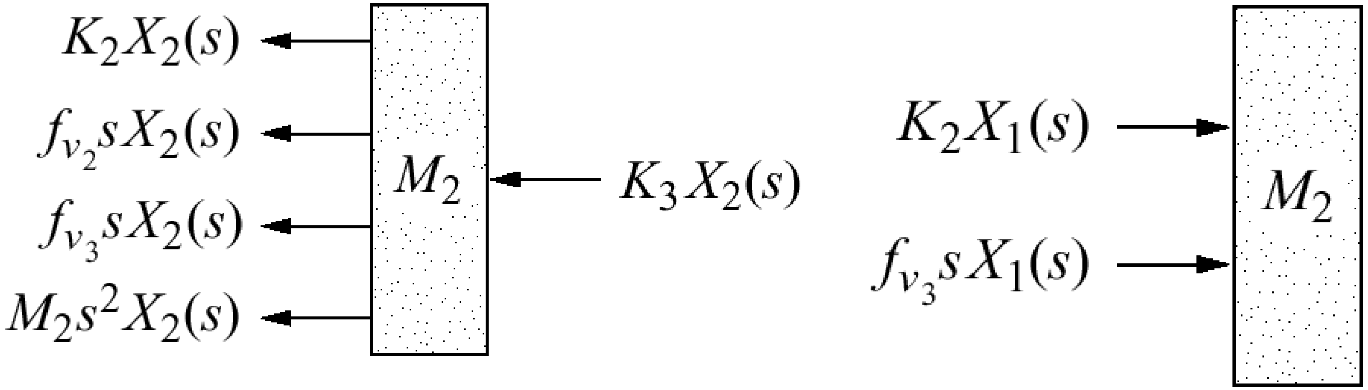




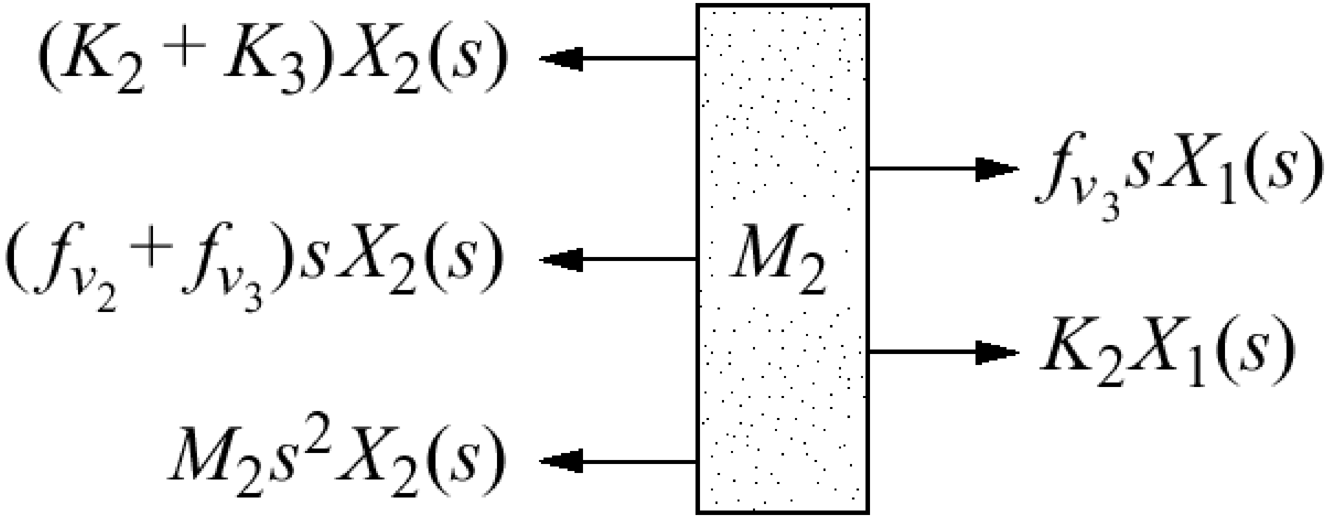
- If we hold M2 still and move M1 to the right

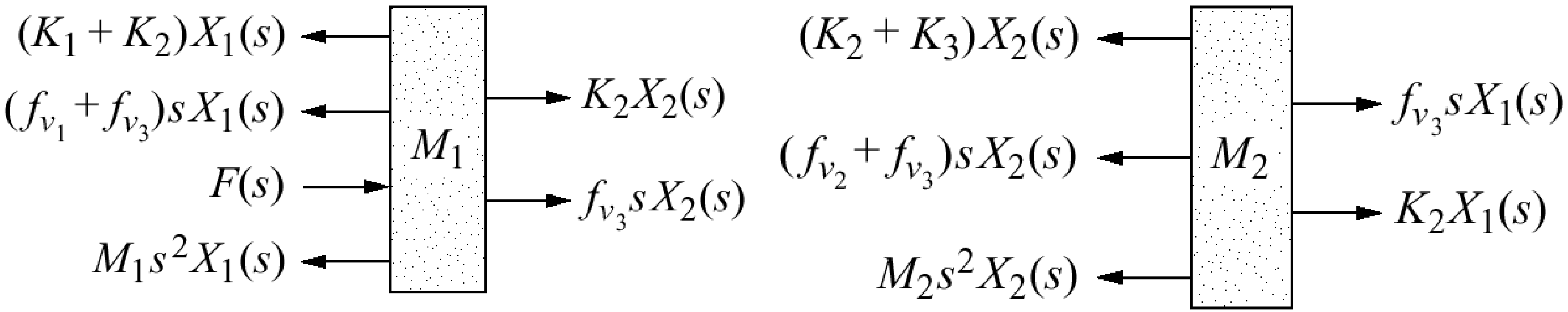


Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring (K)	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper (f_v)	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass (M)	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2



The total force on M2 is the superposition, or sum of forces





The Laplace transform of the equations of motion:

$$[M1s^2+(f_{v1}+f_{v3})s+(K1+K2)]X1(s)-(f_{v3}s+K2)X2(s)=F(s)$$

$$-(f_{v3}s+K2)X1(s)+[M2s^2+(f_{v2}+f_{v3})s+(K2+K3)]X2(s)=0$$

Recall – Cramer's rule

$$[M1s^2+(f_{v1}+f_{v3})s+(K1+K2)]X1(s)-(f_{v3}s+K2)X2(s)=F(s)$$

$$-(f_{v3}s+K2)X1(s)+[M2s^2+(f_{v2}+f_{v3})s+(K2+K3)]X2(s)=0$$

Recall – Cramer's rule

$$ax + by = e \text{ and}$$
$$cx + dy = f,$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - bf}{ad - bc}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ec}{ad - bc}$$

Recall – Cramer's rule

$$\begin{bmatrix} M1s^2+(f_{v1}+f_{v3})s+(K1+K2) & -(f_{v3}s+K2) \\ -(f_{v3}s+K2) & +[M2s^2+(f_{v2}+f_{v3})s+(K2+K3)] \end{bmatrix} \begin{bmatrix} X1(s) \\ X2(s) \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ec}{ad - bc}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ec}{ad - bc}$$

$$\begin{bmatrix} M1s^2+(f_{v1}+f_{v3})s+(K1+K2) & -(f_{v3}s+K2) \\ -(f_{v3}s+K2) & +[M2s^2+(f_{v2}+f_{v3})s+(K2+K3)] \end{bmatrix} \begin{bmatrix} X1(s) \\ X2(s) \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

$$X2(s) = \frac{\begin{vmatrix} M1s^2+(f_{v1}+f_{v3})s+(K1+K2) & F(s) \\ -(f_{v3}s+K2) & 0 \end{vmatrix}}{\begin{vmatrix} M1s^2+(f_{v1}+f_{v3})s+(K1+K2) & -(f_{v3}s+K2) \\ -(f_{v3}s+K2) & +[M2s^2+(f_{v2}+f_{v3})s+(K2+K3)] \end{vmatrix}}$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ec}{ad - bc}$$

$$\begin{bmatrix} M1s^2+(f_{v1}+f_{v3})s+(K1+K2) & -(f_{v3}s+K2) \\ -(f_{v3}s+K2) & +[M2s^2+(f_{v2}+f_{v3})s+(K2+K3)] \end{bmatrix} \begin{bmatrix} X1(s) \\ X2(s) \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

$$X2(s) = \frac{(f_{v3}s+K2) \quad F(s)}{\begin{vmatrix} M1s^2+(f_{v1}+f_{v3})s+(K1+K2) & -(f_{v3}s+K2) \\ -(f_{v3}s+K2) & +[M2s^2+(f_{v2}+f_{v3})s+(K2+K3)] \end{vmatrix}}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

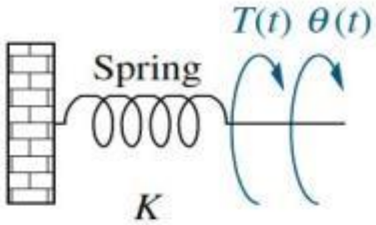
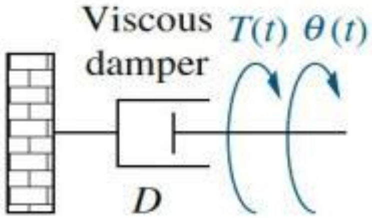
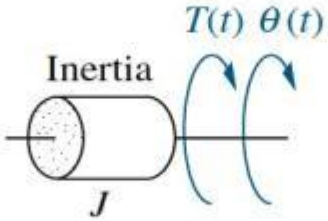
$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ec}{ad - bc}$$

$$\begin{bmatrix} M1s^2+(f_{v1}+f_{v3})s+(K1+K2) & -(f_{v3}s+K2) \\ -(f_{v3}s+K2) & +[M2s^2+(f_{v2}+f_{v3})s+(K2+K3)] \end{bmatrix} \begin{bmatrix} X1(s) \\ X2(s) \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

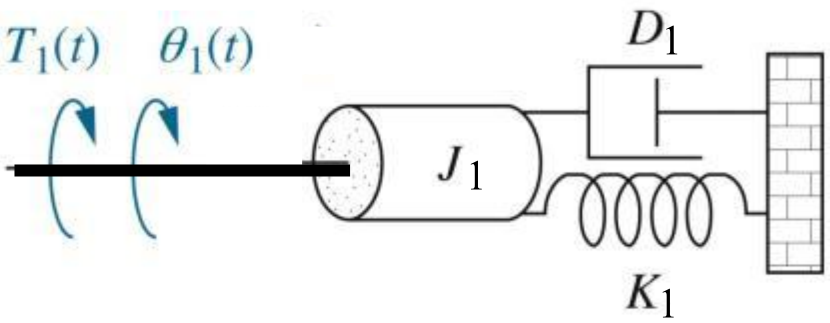
$$X2(s)\,/F(s) = \frac{(f_{v3}s+K2)}{\begin{vmatrix} M1s^2+(f_{v1}+f_{v3})s+(K1+K2) & -(f_{v3}s+K2) \\ -(f_{v3}s+K2) & +[M2s^2+(f_{v2}+f_{v3})s+(K2+K3)] \end{vmatrix}}$$

2.3.2 Rotational

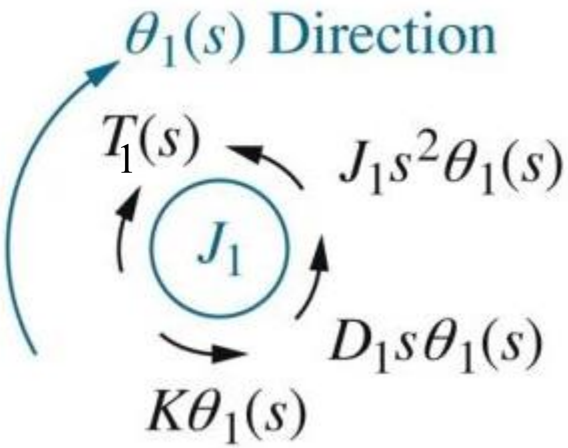
TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton- meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).



$T_1(t)$
$\theta_1(t)$
D_1
J_1
K_1

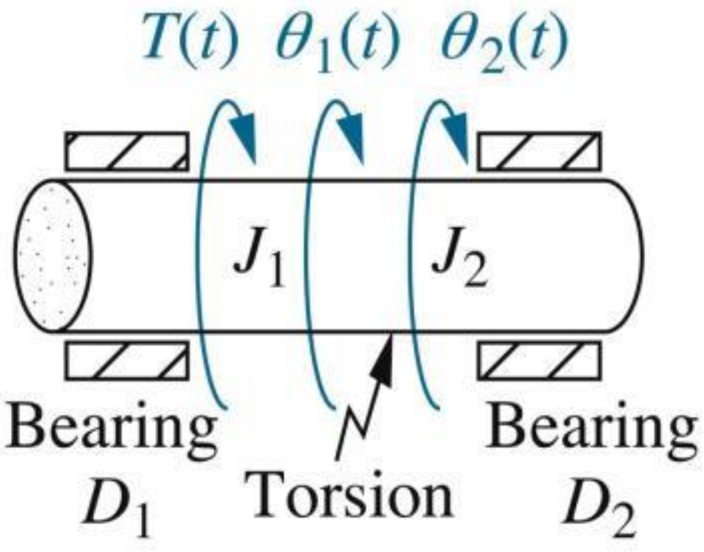


$\theta_1(s)$ Direction
$T_1(s)$
$J_1s^2\theta_1(s)$
J_1
$D_1s\theta_1(s)$
$K\theta_1(s)$

Example 13:

Find the transfer function $\theta_2(s) / T(s)$.

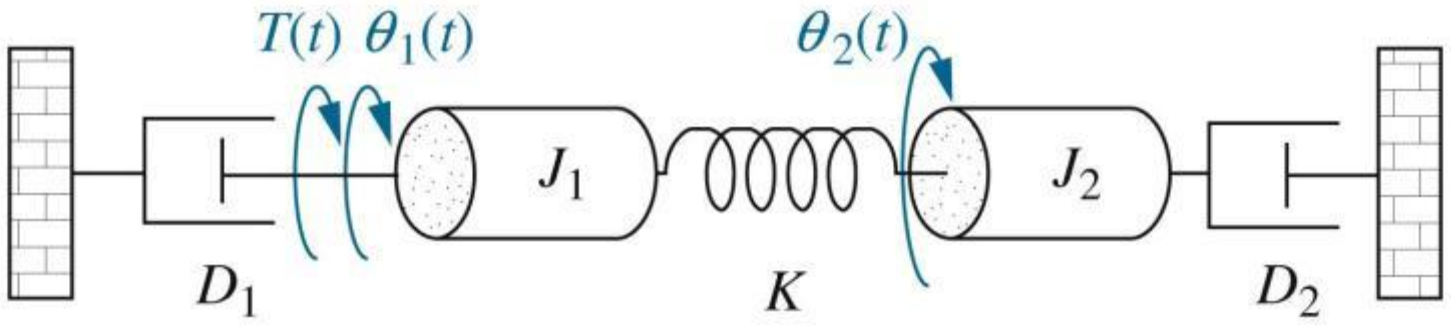
The rod is supported by bearing and at either end is undergoing torsion. A torque is applied at the left and the displacement is measured at the right.



$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\begin{vmatrix} (J_1 s^2 + D_1 s + K) & -K \\ -K & (J_2 s^2 + D_2 s + K) \end{vmatrix}}$$

Solution:

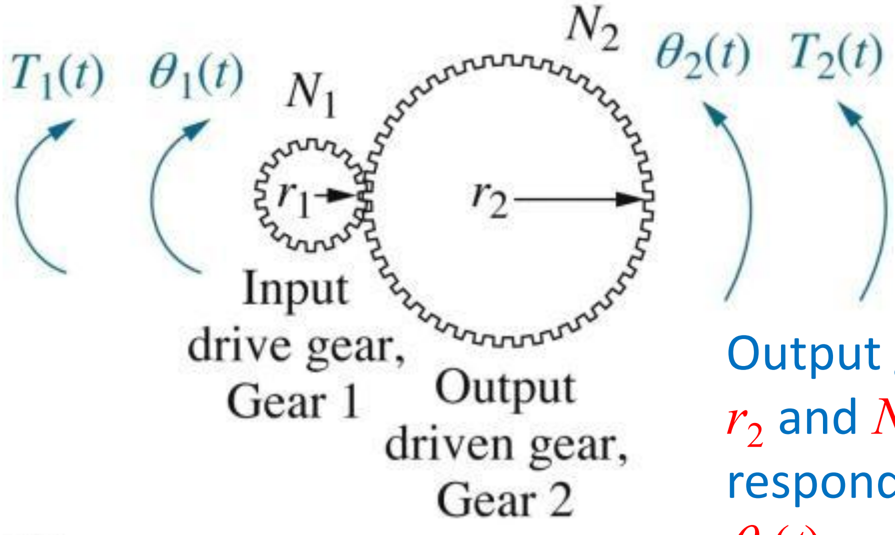
- Obtain the schematic from the physical system
- Assume:
    - The torsion acts like a spring, concentrated at one particular point in the rod
    - Inertia $J1$ to the left and $J2$ to the right
    - The damping inside the flexible shaft is negligible



2.3.3 Rotational with gears

- Gears
  - Used with rotational systems (esp. those driven by motors).
  - Match driving systems with loads.
  - E.g. Bicycles with gearing systems
    - **Uphill**: shift gear for more torque & less speed
    - **Level road**: shift gear for more speed & less torque

Input gear with radius r_1 and N_1 teeth rotated through angle $\theta_1(t)$ due to torque $T_1(t)$

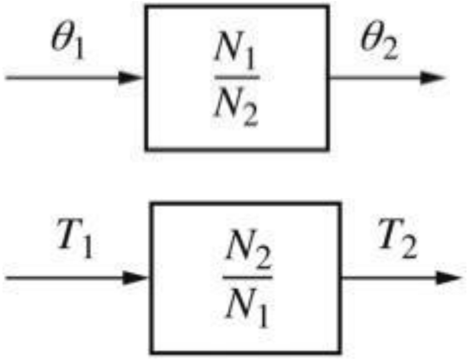


Output gear with radius r_2 and N_2 teeth responds through angle $\theta_2(t)$ and delivering a torque $T_2(t)$

$$\text{ratio number of teeth} \propto \text{ratio of radius} \qquad \frac{N_1}{N_2} = \frac{r_1}{r_2}$$

$$\text{ratio angular disp} \propto \frac{1}{\text{ratio of number of teeth}} \qquad \frac{q_1}{q_2} = \frac{N_2}{N_1}$$

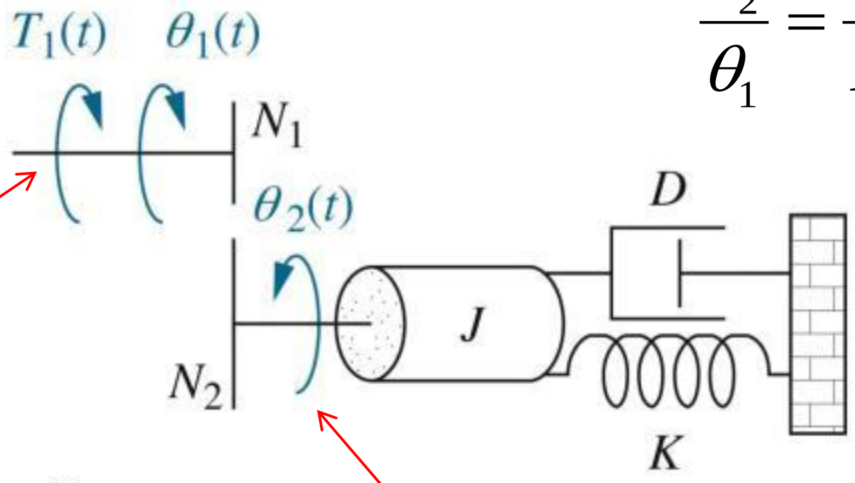
$$\Rightarrow \quad \frac{q_2}{q_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$



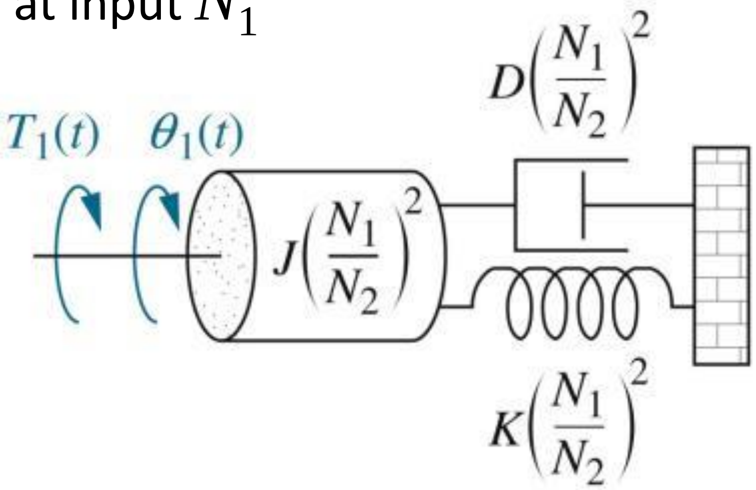
Just like translational motion, energy = force x disp.

$$T_1\theta_1 = T_2\theta_2 \quad \Rightarrow \quad \frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$

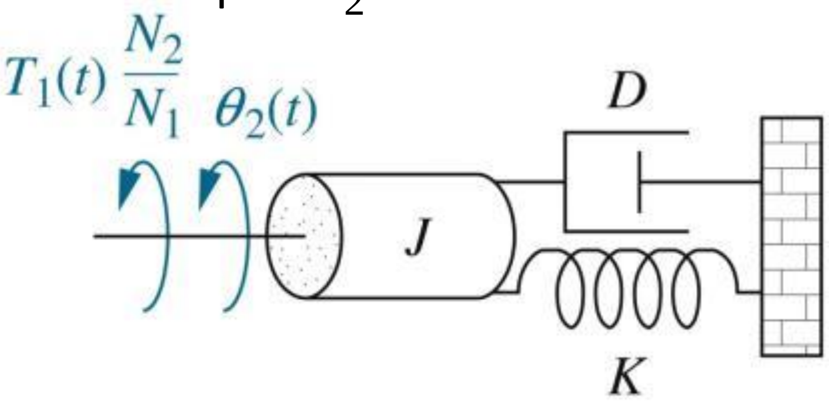
$$\frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{T_1}{T_2}$$



Equivalent system at θ_1 at input N_1



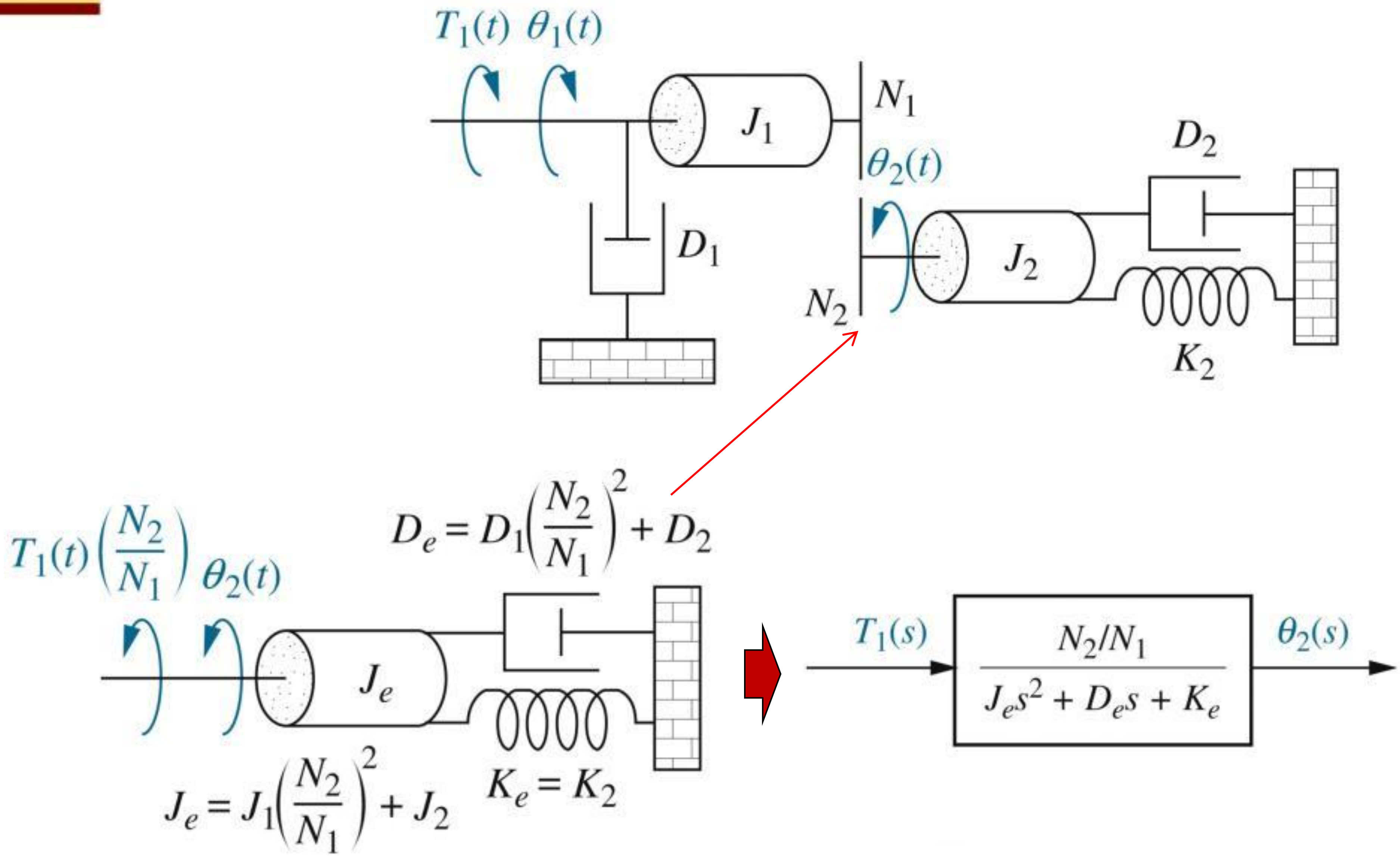
Equivalent system at θ_2 at output N_2



$$T_1 = J\frac{N_1^2}{N_2^2}s^2 q_1(s) + D\frac{N_1^2}{N_2^2}s q_1(s) + K\frac{N_1^2}{N_2^2}q_1(s)$$

$$T_1\frac{N_2}{N_1} = Js^2\theta_2(s) + Ds\theta_2(s) + K\theta_2(s)$$

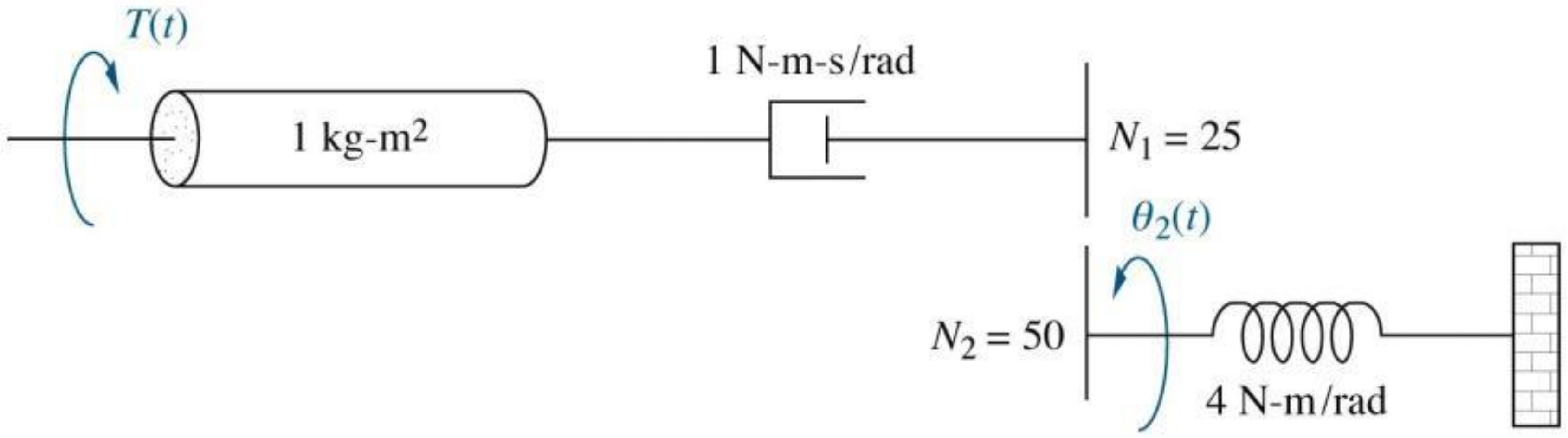
Example 14: Find the transfer function $\theta_2(s)/T_1(s)$



Example 15:

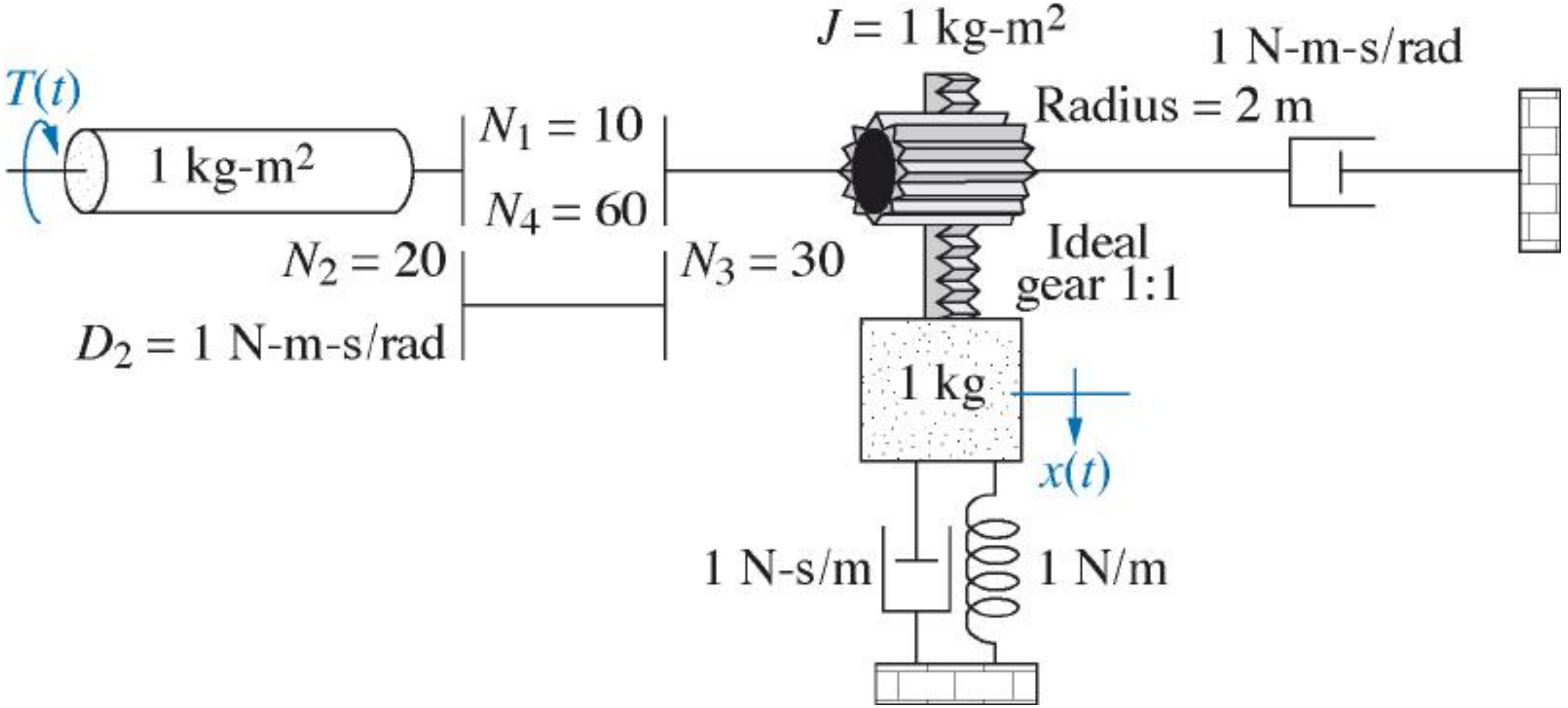
$$G(s) = \frac{1/2}{s^2 + s + 1}$$

- Find the transfer function $\theta_2(s) / T(s)$.



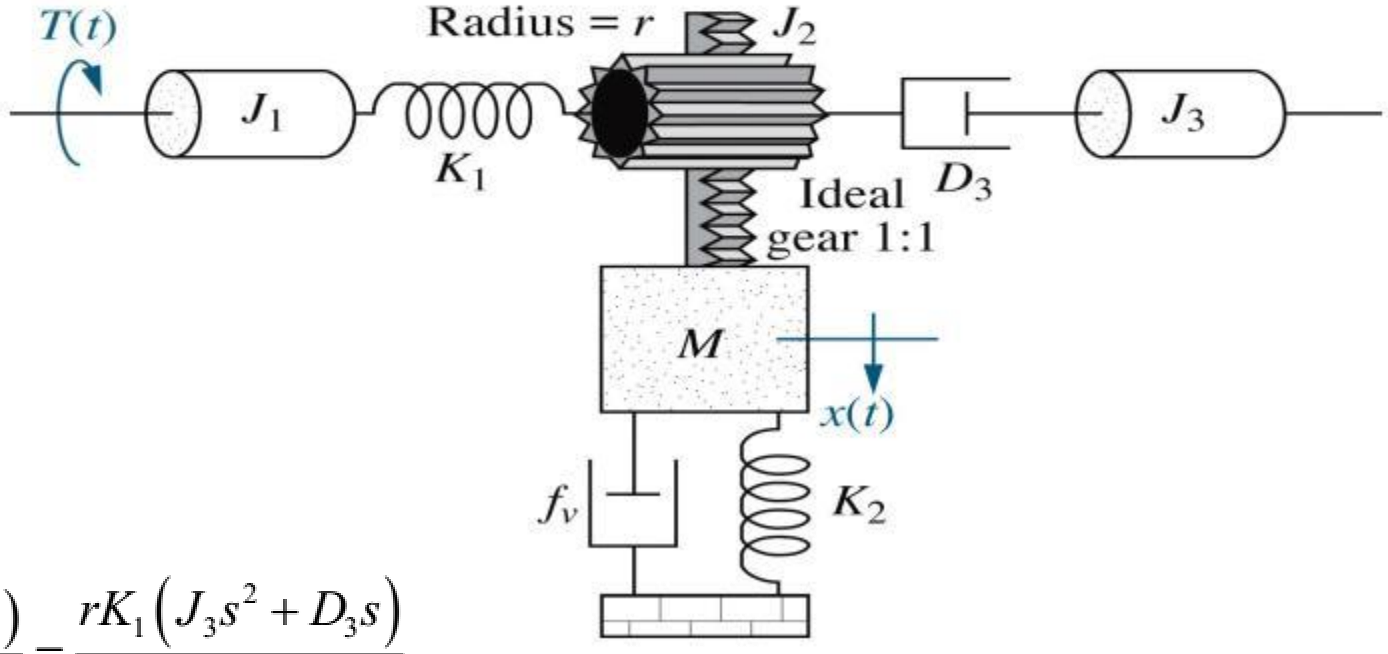
Example 16(a):

- Find $X(s) / T(s)$.



Example 16(b):

Find $X(s) / T(s)$.



Solution:

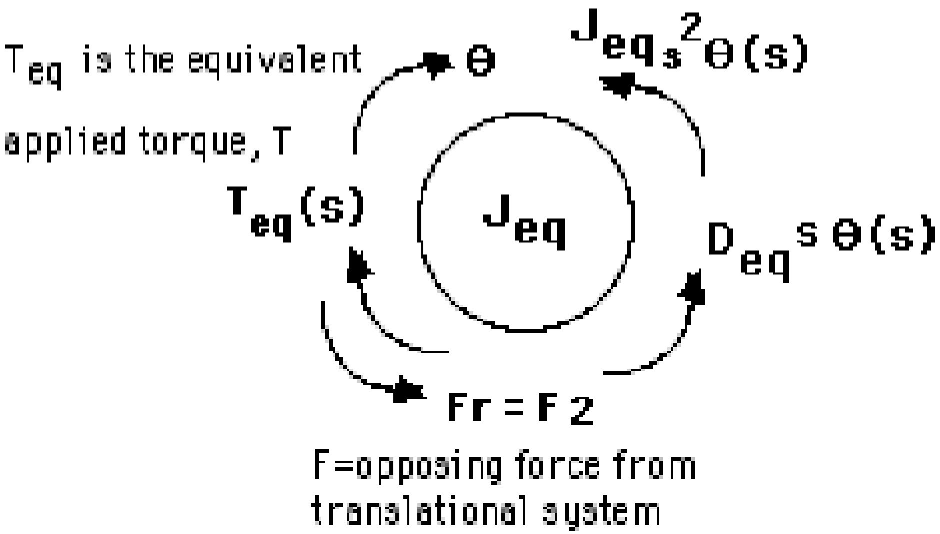
$$X(s) = r\theta_2(s), \frac{X(s)}{T(s)} = \frac{rK_1\left(J_3 s^2 + D_3 s\right)}{\Delta}$$

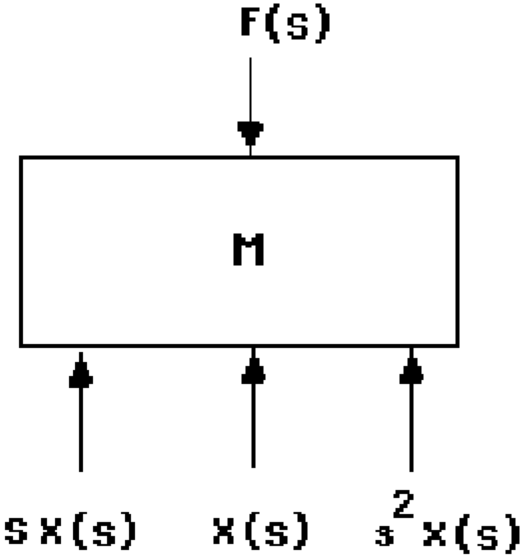
$$\left(J_1 s^2 + K_1\right)\theta_1(s) - K_1\theta_2(s) = T(s)$$

$$-K_1\theta_1(s) + \left[\left(J_2 + Mr^2\right)s^2 + \left(D_3 + f_v r^2\right)s + \left(K_1 + K_2 r^2\right)\right]\theta_2(s) - D_3 s\theta_3(s) = 0$$

$$-D_3 s\theta_2(s) + \left(J_2 s^2 + D_3 s\right)\theta_3(s) = 0$$

Solution 16(a)





$$J_{eq}=1+1(4)^2=17, D_{eq}=1(2)^2+1=5 \text{, and } T_{eq}(s)=4T(s)$$

$$F(s)=(s^2+s+1)X(s)$$

$$(J_{eq}s^2+D_{eq}s)\theta(s)+F(s)2=T_{eq}(s)$$

$$(J_{eq}s^2+D_{eq}s)\theta(s)+(2s^2+2s+2)X(s)=T_{eq}(s)$$

$$T_{eq}=\left[\left(\frac{J_{eq}}{2}+2\right)s^2+\left(\frac{D_{eq}}{2}+2\right)s+2\right]X(s)$$

$$\frac{X(s)}{T(s)}=\frac{\frac{8}{21}}{s^2+\frac{9}{21}s+\frac{4}{21}}$$

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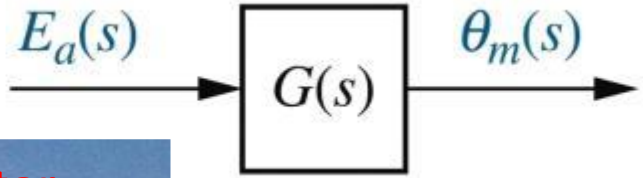
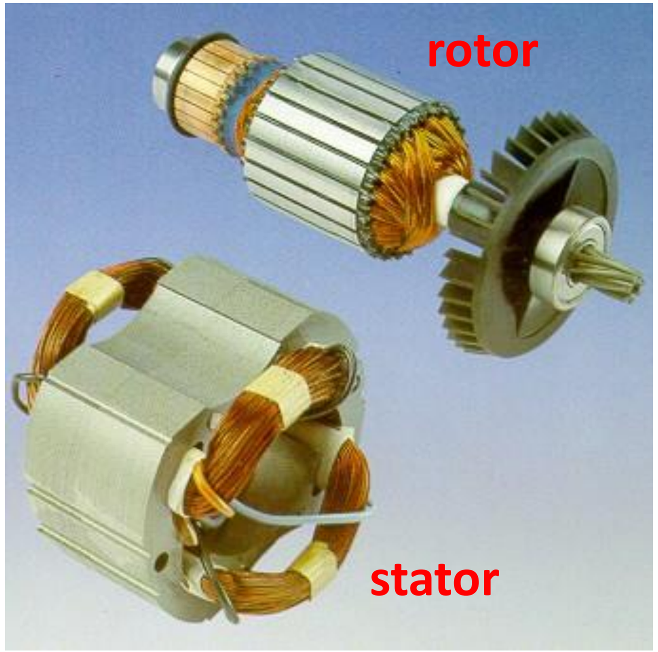
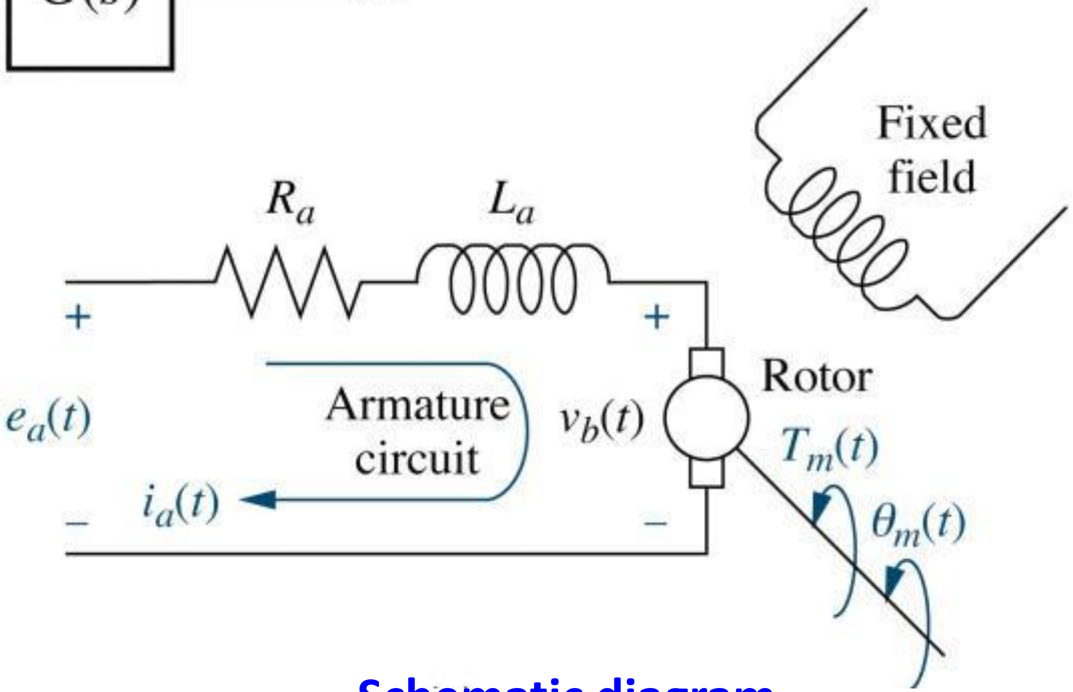
2.4
Modeling of
Electromechanical System

Electromechanical System

- Electromechanical system: **electrical + mechanical** components that generates a mechanical output by an electrical input (Motor)



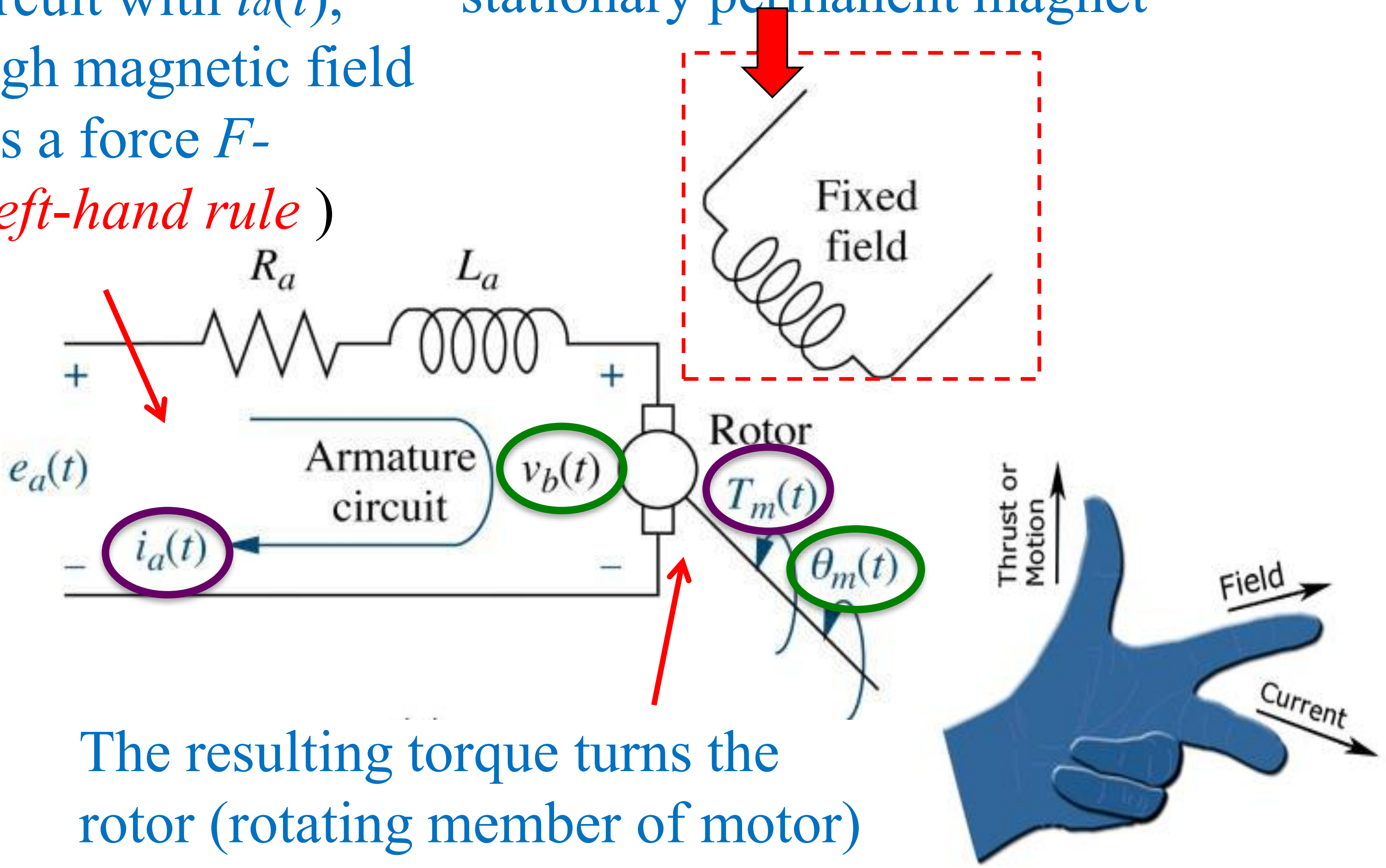




Schematic diagram

Develop magnetic field B by stationary permanent magnet

Armature circuit with $i_a(t)$, passes through magnetic field and produces a force F- (*Fleming's left-hand rule*)



The resulting torque turns the rotor (rotating member of motor)

Since the current-carrying armature is rotating in a magnetic field, its **voltage** *vb* (back electromagnetic force EMF) is proportional to **angular** velocity.

$$v_b(t) = K_b \frac{d\theta_m(t)}{dt} \qquad (1)$$

Taking the *L.T*: $$V_b(s) = K_b s\theta_m(s) \qquad (2)$$

KVL around the armature circuit

$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s) \qquad (3)$$

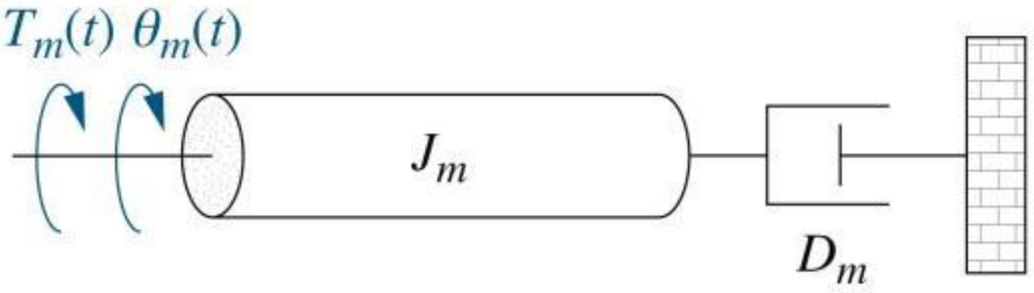
Torque developed by the motor (*Tm*) is proportional to the armature **current** (*ia*).

$$T_m(s) = K_t I_a(s) \quad \textbf{or} \quad I_a(s) = \frac{1}{K_t} T_m(s) \qquad (4)$$

Substitute (2) and (4) into (3)

$$\frac{(R_a + L_a s)T_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s) \qquad (5)$$

Equivalent mechanical loading on motor



$$T_m(s) = (J_m s^2 + D_m s)\theta_m(s) \qquad (6)$$

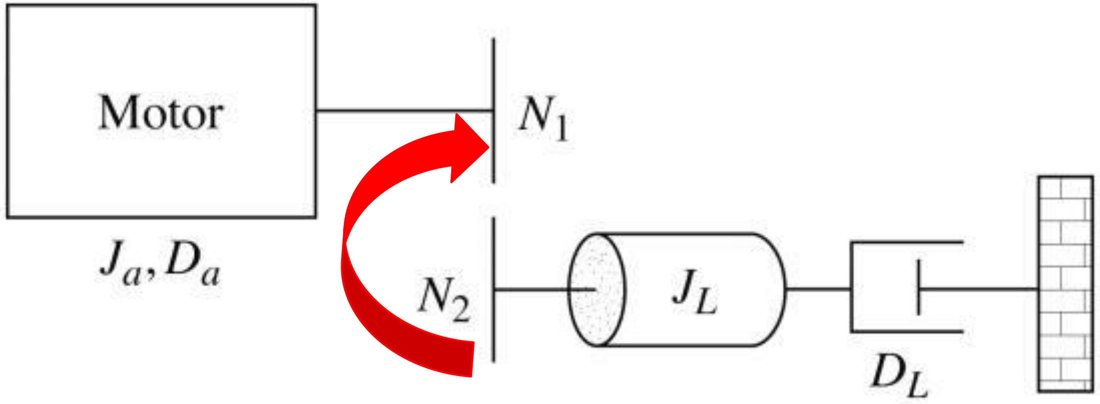
Substitute (6) into (5) yields

$$\frac{(R_a + L_a s)(J_m s^2 + D_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s) \qquad (7)$$

Assume La <<<Ra , then (7) becomes

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t / R_a J_m}{s\left(s + \frac{1}{J_m}\left(D_m + \frac{K_b K_t}{R_a}\right)\right)} \Rightarrow \frac{K}{s(s+\alpha)} \qquad (8)$$

DC motor driving a rotational mechanical load



$$J_m(s) = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 \quad ; \quad D_m(s) = D_a + D_L\left(\frac{N_1}{N_2}\right)^2$$

From (5), with $L_a = 0$

$$\frac{R_a T_m(s)}{K_t} + K_b s q_m(s) = E_a(s)$$ taking inverse Laplace transform

$$\frac{R_a T_m(t)}{K_t} + K_b \frac{d\theta_m(t)}{dt} = e_a(t)$$

$$T_m(t) = -\frac{K_b K_t}{R_a}\frac{d\theta_m(t)}{dt} + \frac{K_t}{R_a} e_a(t) = -\frac{K_b K_t}{R_a}\omega_m + \frac{K_t}{R_a} e_a(t)$$

At steady state:

$$T_m = -\frac{K_b K_t}{R_a}\omega_m + \frac{K_t}{R_a} e_a$$

$$\omega = 0 \;\Rightarrow\; T_m = \frac{K_t}{R_a} e_a$$

$$T = 0 \;\Rightarrow\; \omega = \frac{e_a}{K_b}$$

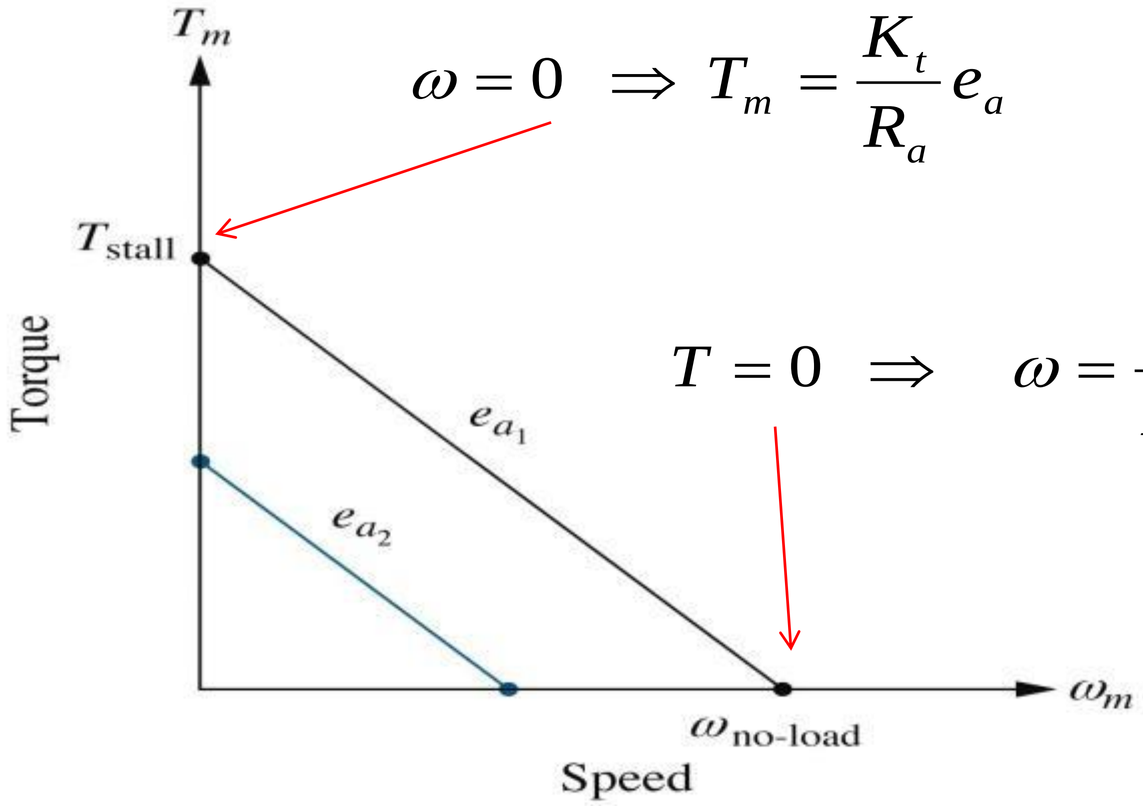
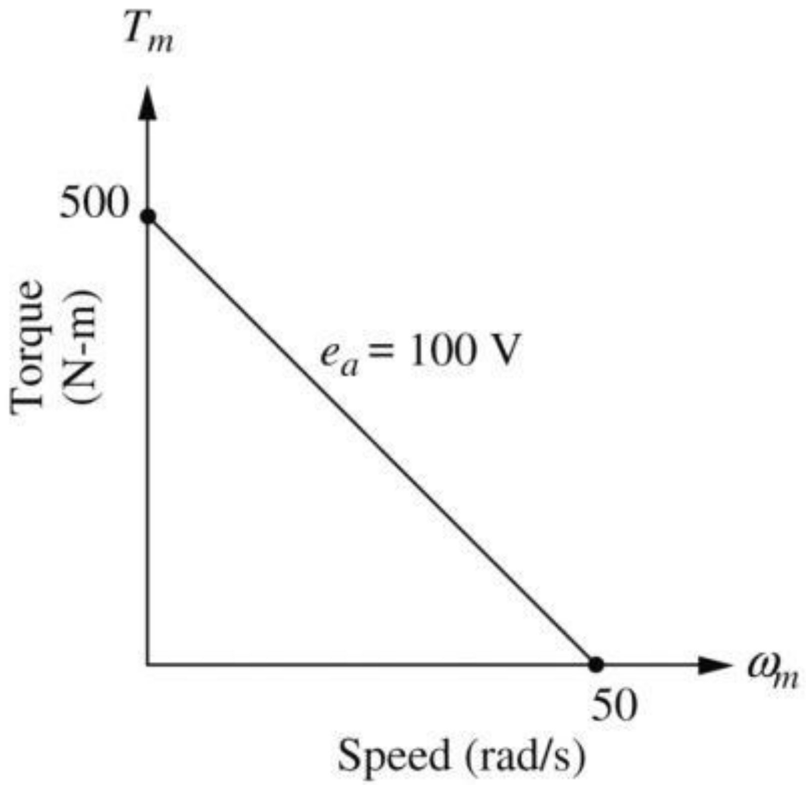


Figure 2.38

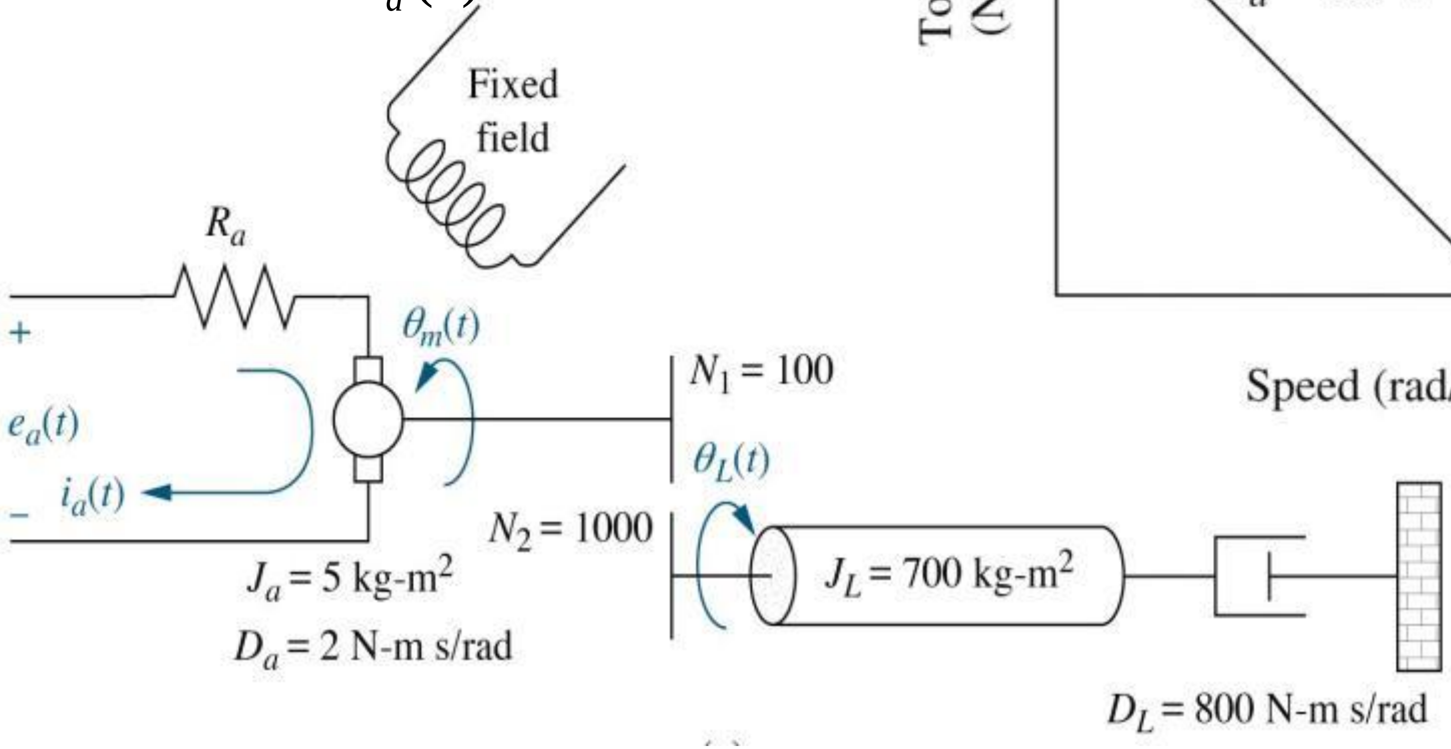
$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a}$$

$$K_b = \frac{e_a}{\omega}$$

Example 17:



Find $\dfrac{\theta_L(s)}{E_a(s)}$



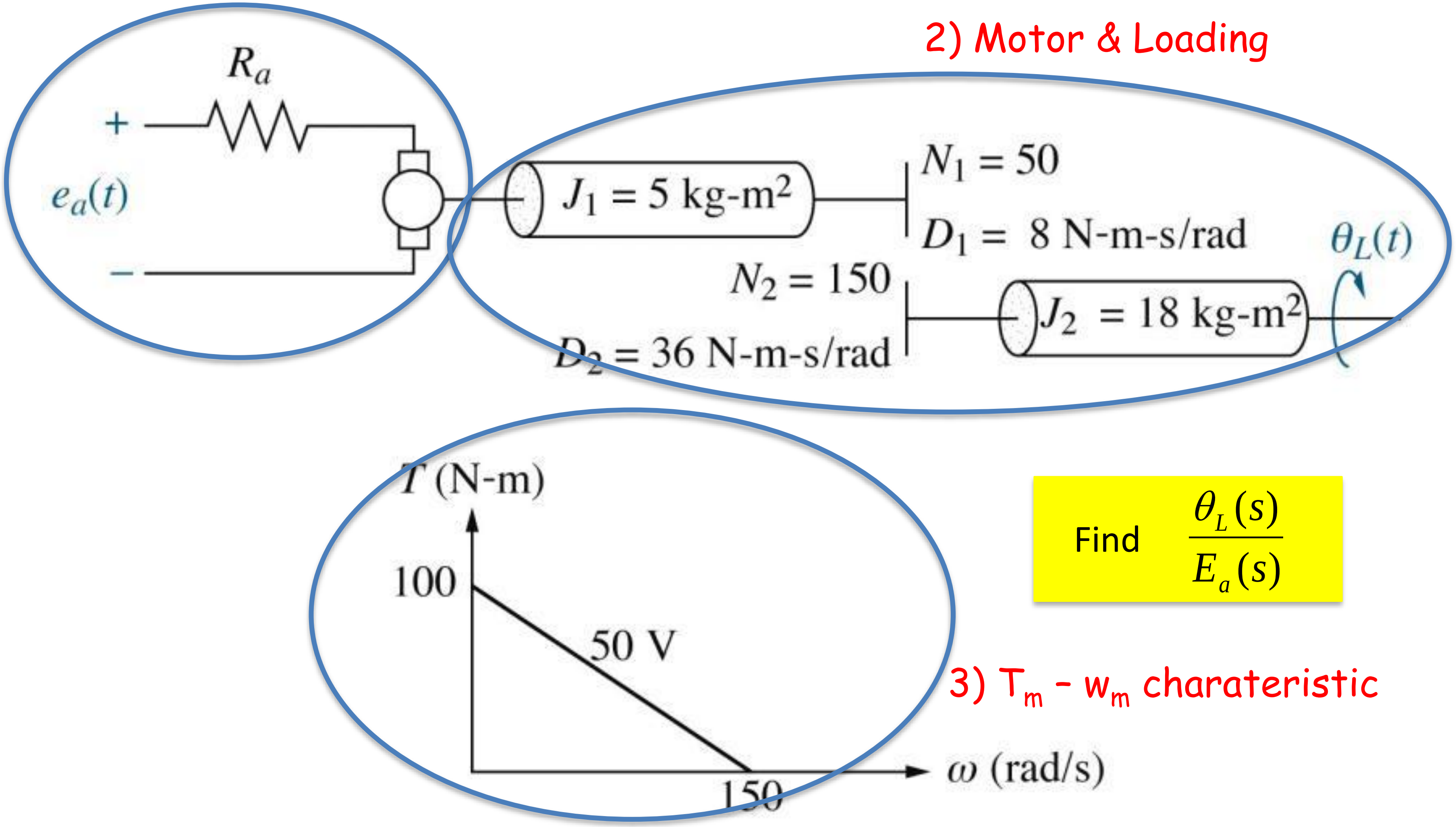
Answer: $\dfrac{q_L(s)}{E_a(s)} = \dfrac{1}{24s^2 + 40s}$

Example 18:

For question like this, you need equations on:

1) Amature circuit

2) Motor & Loading



Find $\dfrac{\theta_L(s)}{E_a(s)}$

3) T_m – w_m charateristic

Figure P2.28

Example 19:

Find $\dfrac{X(s)}{E_a(s)}$

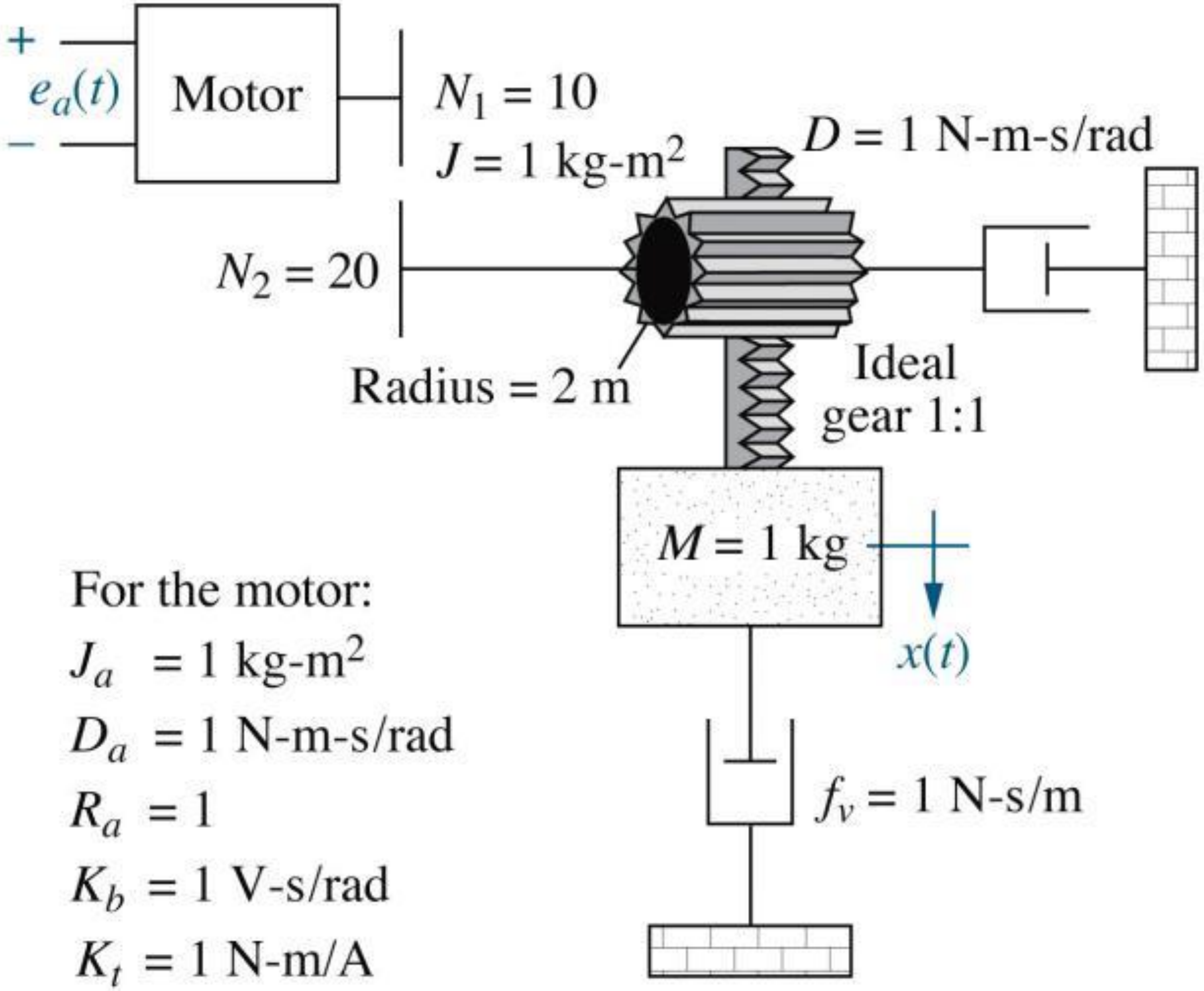


Figure P2.31

Answer:

$$\frac{X(s)}{E_a(s)} = \frac{\frac{4}{9}}{s\left(s+\frac{13}{9}\right)}.$$

Solution

Rotational

$$\left(J_{eqL}s^2 + D_{eqL}s\right)\theta_L(s) + F(s)r = T_{eq}(s)$$

$$J_{eqL} = 1(2)^2 + 1 = 5$$

$$D_{eqL} = 1(2)^2 + 1 = 5$$

$$\left(5s^2 + 5s\right)\theta_L(s) + F(s)r = T_{eq}(s) \qquad (1)$$

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2}$$

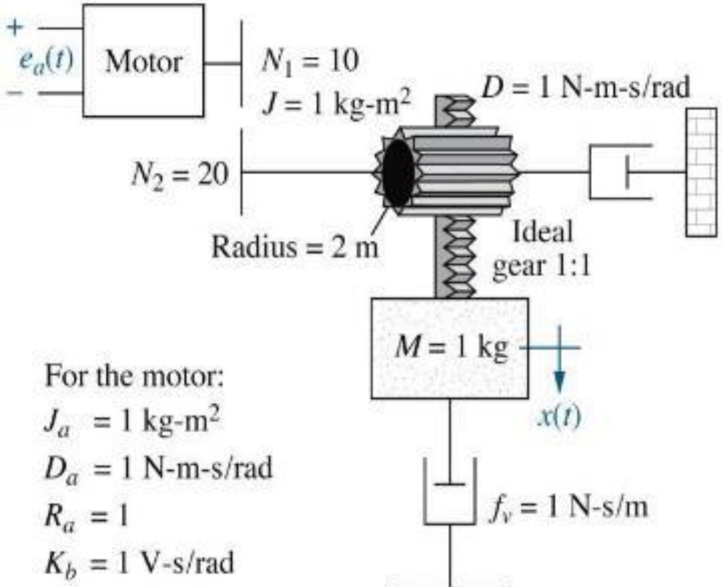
Translational

$$X(s) = 2\theta_L(s) \qquad F(s) = \left(s^2 + s\right)2\theta_L(s) \qquad (2)$$

Substitute (1) into (2) yields

$$\left(9s^2 + 9s\right)\theta_L(s) = T_{eq}(s)$$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{4}{9}}{s\left(s + \frac{4}{9}\left(\frac{9}{4} + 1\right)\right)} = \frac{\frac{4}{9}}{s\left(s + \frac{13}{9}\right)}$$



Since

$$\theta_L(s) = \frac{1}{2}\theta_m(s), \frac{\theta_L(s)}{E_a(s)} = \frac{\frac{2}{9}}{s\left(s+\frac{13}{9}\right)} \qquad (3)$$

But

$$X(s) = r\theta_L(s) = 2\theta_L(s) \qquad (4)$$

Therefore

$$\frac{X(s)}{E_a(s)} = \frac{\frac{4}{9}}{s\left(s+\frac{13}{9}\right)} \qquad (5)$$

END OF CHAPTER 2