SMJE 3153

SYSTEM MODELING AND ANALYSIS

Stability Analysis in Time Domain

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5.1 Introduction

- Stability is the most important specification.
- Unstable systems are harmful to the plant and may cause serious accidents.
- The total response of a system is the sum of the forced and natural responses

$$c(t) = c_{forced}(t) + c_{natural}(t)$$

Introduction

- A system is stable if the natural response approaches zero as time approaches infinity.
- A system is unstable if the natural response approaches infinity as time approaches infinity.
- A system is marginally stable if the natural response neither decays nor grows but remains constant or oscillates.

A Stable System

- Poles in the left half-plane (lhp) yield either pure exponential decay or damped sinusoidal natural responses.
- Thus, if the closed-loop system poles are in the lhp, the system is stable.
- Example:

$$G(s) = \frac{1}{s+1}$$

Unstable System

- Poles in the right half-plane (rhp) yield either pure exponentially increasing or exponentially increasing sinusoidal natural responses.
- These natural responses approach infinity as time approaches infinity
- Thus, if the closed-loop system poles are in the rhp, the system is unstable.

Unstable System

- Thus, unstable systems have closed-loop transfer functions with at least one pole in the rhp.
- Example:

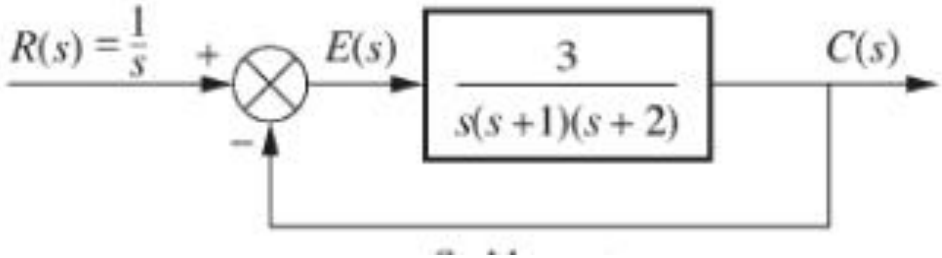
$$G_1(s) = \frac{1}{s-1}$$

Marginally Stable System

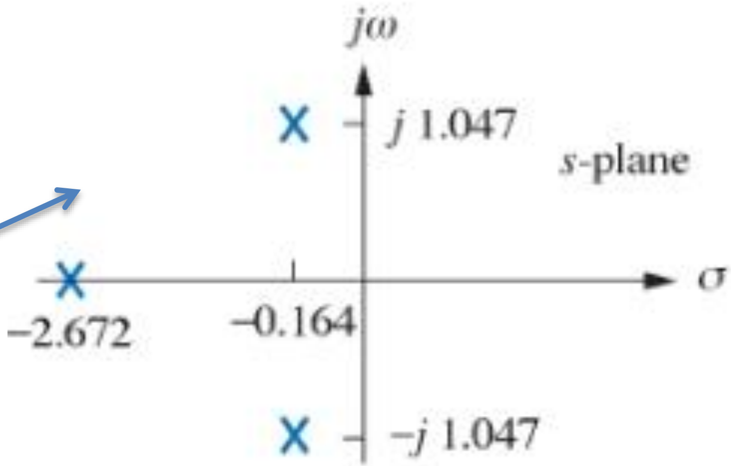
- A system that has imaginary axis poles yields pure sinusoidal oscillations as a natural response.
- Thus, marginally stable systems have closed-loop transfer functions with only imaginary axis poles or/and poles in the lhp.
- Example:

$$G_2(s) = \frac{1}{s^2 + 1}$$

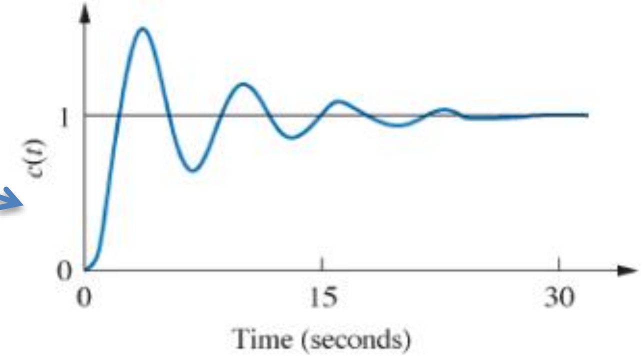
Example 1



- CLTF.



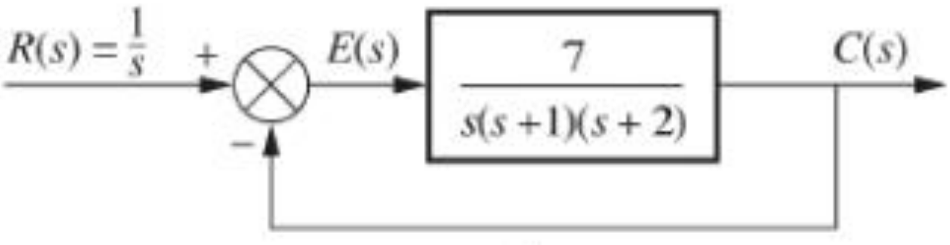
- Pole-zero map.
- Step response.

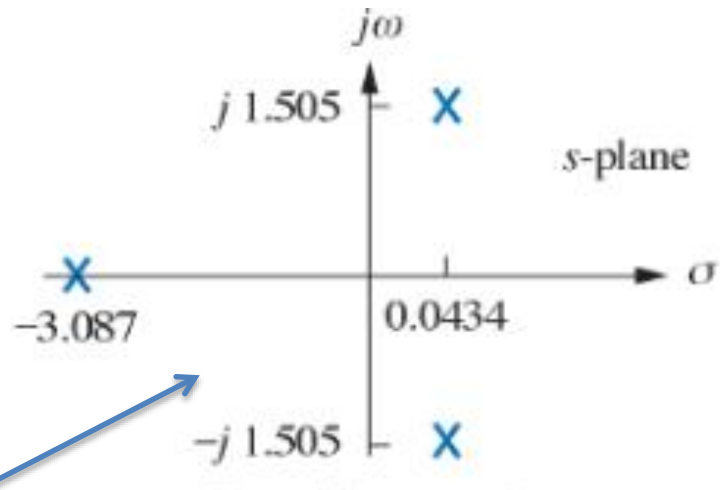


- All poles lie on the lhp.

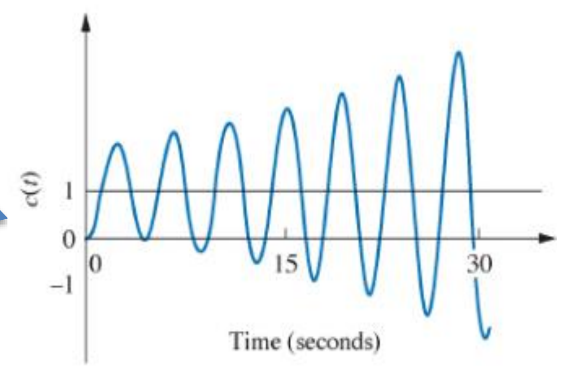
Thus, the system is stable.

Example 2





- CLTF.
- Pole-zero map.
- Step response.

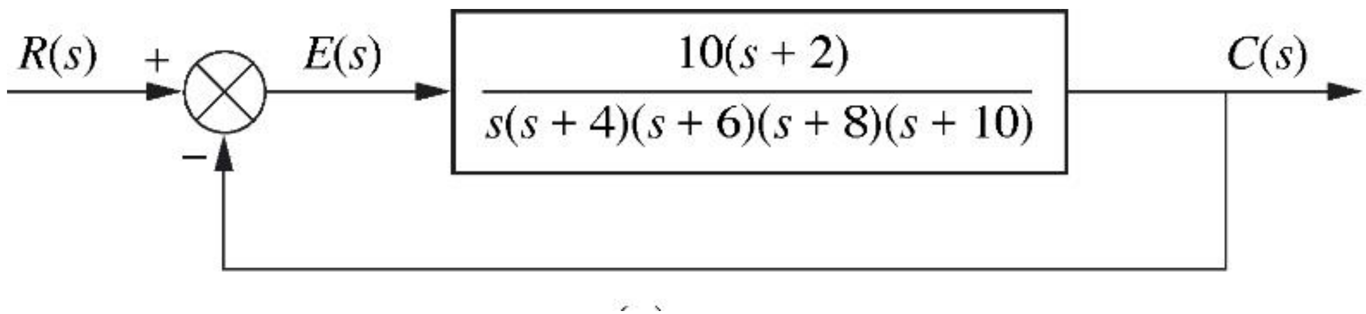


- Two poles lie on the rhp.

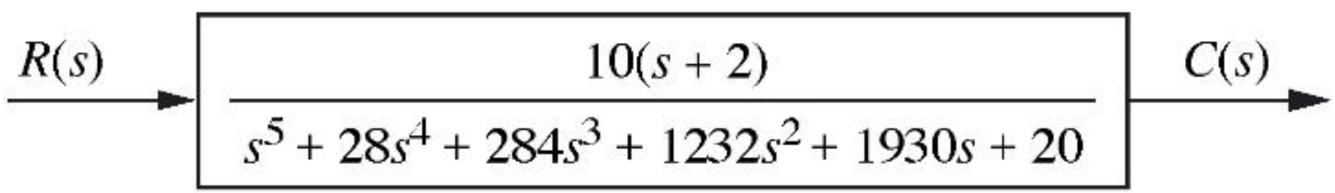
Thus, the system is unstable.

Methods to Test Stability

- The technique based on determining CL poles is difficult for a higher order systems.
- Example:



- CLTF



Methods to Test Stability

- Computers and advanced calculators can also be used to determine poles of the CL system.
- A method to test for stability without having to solve for the roots of the denominator.
- Routh-Hurwitz Criterion

R – H Criterion

- R-H criterion is a method that yields stability information without the need to solve for the closed-loop system poles.
- Using this method, we can tell how many closed-loop system poles are in the lhp, rhp, and on the imaginary axis BUT we cannot find their locations.
- 2 steps:
  - generate the Routh table
  - Interpret the table

Routh Table

- Consider the transfer function. We focus on the denominator that relates to the poles.

$$R(s) \longrightarrow \boxed{\dfrac{N(s)}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}} \longrightarrow C(s)$$

- Generate the Routh Table.
  - Labeling the rows
  - Coefficients in the first row.
  - Second row

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Routh Table

- The remaining entries:

1. Each entry is a negative determinant of entries in the previous two rows divided by the entry in the first column directly above the calculated row.
2. The left-hand column of the determinant is always the first column of the previous two rows, and the right-hand column is the elements of the column above and to the right.

Routh Table

3. The table is complete when all of the rows are completed down to s^0.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

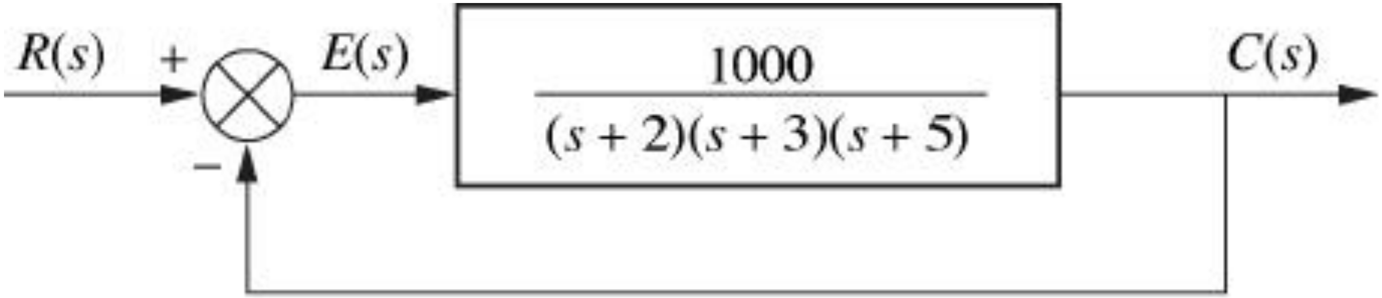
Interpreting the Routh Table

- The R-H criterion declares that *the number of roots of the polynomial that are in the rhp is equal to the number of sign changes in the first column*.

- A system is stable if all the CL poles lie on the lhp. Thus, a system is **stable** if there are **no sign changes** in the first column of the Routh table.

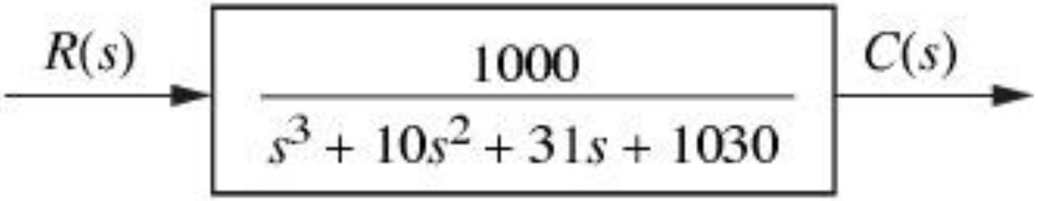
Example 3

- Make the Routh table for the system. Then determine stability of the system.



Example 3

- The CLTF:



- The completed Routh table:

s^3	1	31	0
s^2	$\cancel{10}\quad 1$	$\cancel{1030}\quad 103$	0
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

Example 3

s^3	1	31	0
s^2	$\cancel{10}\ \ 1$	$\cancel{1030}\ \ 103$	0
s^1	$\dfrac{-\begin{vmatrix}1 & 31\\ 1 & 103\end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix}1 & 103\\ -72 & 0\end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix}1 & 0\\ -72 & 0\end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\ -72 & 0\end{vmatrix}}{-72} = 0$

- Two sign changes, two poles exist in the rhp.
- Thus, the system is unstable.

Example 4

- Determine stability of a system with CLTF as

  $P(s) = 1000/(s^4 + 30s^3 + 300s^2 + 2000s + 8000)$

- Answer:

Ex4. solution

RHP-0

LHP-4

IMIGINARY AXIS-0

SYSTEM STABLE

$$\begin{bmatrix} 1 & 300 & 8000 & s^4 \\ 30 & 2000 & 0 & s^3 \\ \frac{700}{3} & 8000 & 0 & s^2 \\ \frac{6800}{7} & 0 & 0 & s \\ 8000 & 0 & 0 & 1 \end{bmatrix}$$

Example 5

- Make a Routh table and tell how many roots of the following CLTF are in the right half-plane and in the left half-plane.

$$T(s) = 100/(3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6)$$

- Answer: 4 rhp, 3 lhp.

Ex5. solution

RHP-4

LHP-3

IMIGINARY AXIS-0

SYSTEM UNSTABLE

$$\begin{array}{c} \begin{bmatrix} 3 & 6 & 7 & 2 \\ 9 & 4 & 8 & 6 \\ \frac{14}{3} & \frac{13}{3} & 0 & 0 \\ -\frac{61}{14} & 8 & 6 & 0 \\ \frac{787}{61} & \frac{392}{61} & 0 & 0 \\ \frac{8004}{787} & 6 & 0 & 0 \\ -\frac{1581}{1334} & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \\ s \\ 1 \end{array} \end{array}$$

R-H Criterion: Special Cases

- Two special cases can occur:
  - Row of zero
  - Zero only in the first column

- Row of Zero. Example: Determine the number of rhp poles in the CLTF

$$T(s)=\frac{10}{s^5+7s^4+6s^3+42s^2+8s+56}$$

Row of Zero (ROZ)

- Routh table:

s^5	1	6	8
s^4	$\cancel{7}\ 1$	$\cancel{42}\ 6$	$\cancel{56}\ 8$
s^3	$\cancel{0}\ \cancel{4}\ 1$	$\cancel{0}\ \cancel{12}\ 3$	$\cancel{0}\ \cancel{0}\ 0$

- ROZ occurs at row s^3.

Row of Zero (ROZ)

- Steps of solving ROZ:

  1. Return to the row immediately above the ROZ and form a polynomial using the entries in that row as coefficients.

     Example: $P(s) = 7s^4 + 42s^2 + 56$

  2. Differentiate the polynomial with respect to s.

     $dP(s)/ds = 28s^3 + 84s$

  3. Use the coefficients to replace the ROZ.
  4. Complete the Routh table.

Interpreting Row of Zero (ROZ)

- Routh table:

s^5	1	6	8
s^4	$\cancel{7}\ \ 1$	$\cancel{42}\ \ 6$	$\cancel{56}\ \ 8$
s^3	$\cancel{0}\ \ \cancel{4}\ \ 1$	$\cancel{0}\ \ \cancel{12}\ \ 3$	$\cancel{0}\ \ \cancel{0}\ \ 0$
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

- When there is ROZ, poles might be located on $j\omega$-axis (in this case – 4 poles since ROZ is at s^3.
- There is no sign change in the first column after the ROZ. Hence, there are no rhp poles. The system is marginally stable.

Interpreting Row of Zero (ROZ)

- So, where are the poles located?

ROZ

s^5	1	6	8
s^4	$\cancel{7}$ 1	$\cancel{42}$ 6	$\cancel{56}$ 8
s^3	$\cancel{0}$ $\cancel{4}$ 1	$\cancel{0}$ $\cancel{12}$ 3	$\cancel{0}$ $\cancel{0}$ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

- 4 poles might be on the jω-axis.
- No sign change from ROZ to s^0. Therefore – those 4 poles are on the jω-axis.
- Since in this example we should have a total 5 poles, the fifth pole is located on the lhp because no sign change from s^5 to ROZ.

Interpreting Row of Zero (ROZ)

- So, where are the poles located?

s^5	1	6	8
s^4	$\cancel{7}\ \ 1$	$\cancel{42}\ \ 6$	$\cancel{56}\ \ 8$
s^3	$\cancel{0}\ \ \cancel{4}\ \ 1$	$\cancel{0}\ \ \cancel{12}\ \ 3$	$\cancel{0}\ \ \cancel{0}\ \ 0$
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

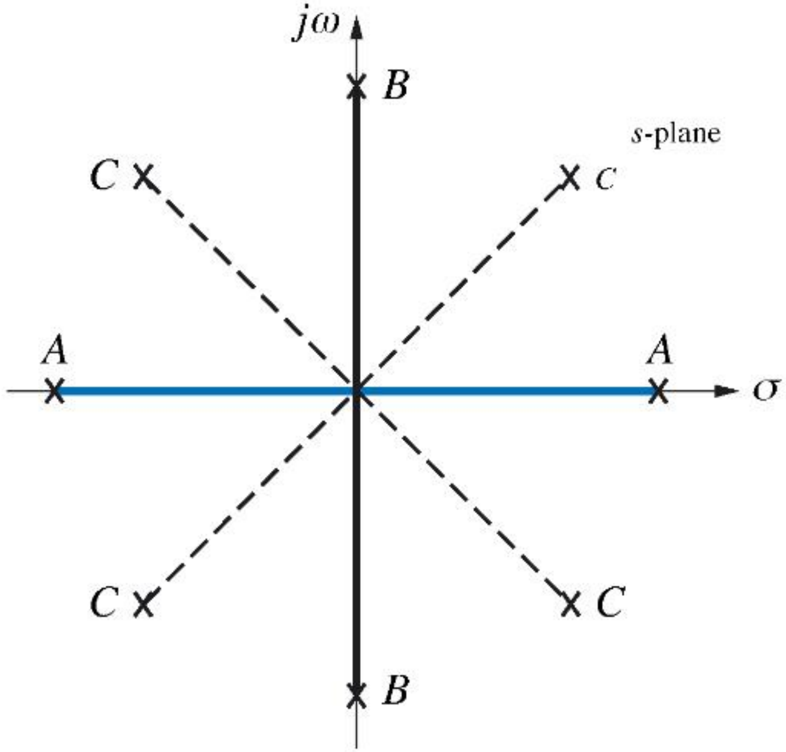
- Hence, 1 pole on lhp and 4 poles on the jω-axis.
- System is marginally stable.
- Note: every change of sign going down after the ROZ will imply that one pole is on lhp and another one is on rhp.

Row of Zero (ROZ)

- ROZ appears in the Routh table when purely even or odd polynomial is a factor of the original polynomial.
- Example:
  - Even polynomial: $s^4 + 4s^2 + 10$.
  - Odd polynomial: $s^5 + 6s^3 + 7s$. Odd polynomials are the product of even polynomial and odd power of s.
- Even polynomials have roots that is a symmetrical about the origin.

Row of Zero (ROZ)

- The symmetrical can occur in three conditions:



A: Real and symmetrical about the origin

B: Imaginary and symmetrical about the origin

C: Quadrantal and symmetrical about the origin

Row of Zero (ROZ)

- ROZ tells the existence of an even polynomial whose roots are symmetric about the origin.
- The row previous to the ROZ contains the even polynomial.
- From the even polynomial to the end of the Routh table is a test of the even polynomial only.

Example 6

For the transfer function

$$T(s)=\frac{20}{s^8+s^7+12s^6+22s^5+39s^4+59s^3+48s^2+38s+20}$$

Tell how many poles are in the rhp, lhp and on $j\omega$-axis.

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Example 6 (solution)

– The Routh table:

Even polynomial

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	$\cancel{-10}\ \ -1$	$\cancel{-20}\ \ -2$	$\cancel{10}\ \ 1$	$\cancel{20}\ \ 2$	0
s^5	$\cancel{20}\ \ 1$	$\cancel{60}\ \ 3$	$\cancel{40}\ \ 2$	0	0
s^4	1	3	2	0	0
s^3	$\cancel{0}\ \ \cancel{4}\ \ 2$	$\cancel{0}\ \ \cancel{6}\ \ 3$	$\cancel{0}\ \ \cancel{0}\ \ 0$	0	0
s^2	$\cancel{\frac{3}{2}}\ \ 3$	$\cancel{2}\ \ 4$	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

Row of zero

Ex6. solution

RHP-2

LHP-2

IMIGINARY AXIS-4

SYSTEM UNSTABLE

$$\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\left[\begin{array}{cccccc} 1 & 12 & 39 & 48 & 20 & s^8 \\ 1 & 22 & 59 & 38 & 0 & s^7 \\ -10 & -20 & 10 & 20 & 0 & s^6 \\ 20 & 60 & 40 & 0 & 0 & s^5 \\ 10 & 30 & 20 & 0 & 0 & s^4 \\ 40 & 60 & 0 & 0 & 0 & [s^3] \\ 15 & 20 & 0 & 0 & 0 & s^2 \\ \frac{20}{3} & 0 & 0 & 0 & 0 & s \\ 20 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Example 6 (solution)

- Auxiliary polynomial:
  - $P(s) = 10s^4 + 30s^2 + 20$
  - $dP(s)/ds = 40s^3 + 60s$
- Total number of poles = 8
- No sign change from ROZ onwards.
  - No rhp poles
  - No lhp poles due to requirement of symmetry
  - 4 poles on the $j\omega$-axis.
- From s^8 to s^4, 2 sign changes.
  - 2 rhp poles, 2 lhp poles.

Example 6 (solution)

- Summary of pole locations:

	Polynomial		
Location	Even (fourth-order)	Other (fourth-order)	Total (eighth-order)
Right half-plane	0	2	2
Left half-plane	0	2	2
$j\omega$	4	0	4

- The system is unstable due to the rhp poles.

Example 7

Use the R-H criterion to find how many poles of the following CL system, T(s), are in the rhp, lhp and imaginary axis.

$$T(s)=\frac{s^3+7s^2-21s+1020}{s^6+s^5-6s^4+0s^3-s^2-s+6}$$

– Answer: 2 rhp, 2 lhp, 2 $j\omega$.

Ex7. solution

RHP-2

LHP-2

IMIGINARY AXIS-2

SYSTEM UNSTABLE

$$\begin{bmatrix} 1 & -6 & -1 & 6 & s^6 \\ 1 & 0 & -1 & 0 & s^5 \\ -6 & 0 & 6 & 0 & s^4 \\ -24 & 0 & 0 & 0 & [s^3] \\ -6 & 6 & 0 & 0 & s^2 \\ -24 & 0 & 0 & 0 & s \\ 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Zero only in the First Column

- For this case, an epsilon, ε is assigned to replace the zero.
- The value is allowed to approach zero from positive and negative.
- The signs of entries of the first column is then analysed with the value, ε.

Example 8

- Determine the stability of the closed-loop transfer function

$$T(s)=\frac{10}{s^5+2s^4+3s^3+6s^2+5s+3}$$

Example 8 (solution)

- Zero in the first column occurs at s^3.
- Replace the zero with a small number, ε and complete the table.

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0}\quad \epsilon$	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

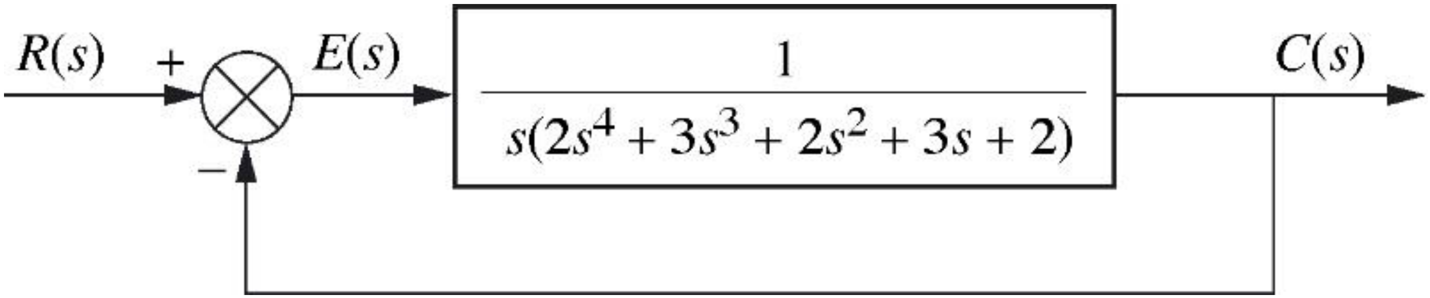
Example 8 (solution)

- Complete the table with positive or negative values of ε.
- 2 rhp poles, the system is unstable.

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	$+$	$+$
s^4	2	$+$	$+$
s^3	$\cancel{0}\ \ \epsilon$	$+$	$-$
s^2	$\dfrac{6\epsilon - 7}{\epsilon}$	$-$	$+$
s^1	$\dfrac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	$+$	$+$
s^0	3	$+$	$+$

Example 9

- Find the number of poles in the lhp, rhp and the $j\omega$ -axis for the system.



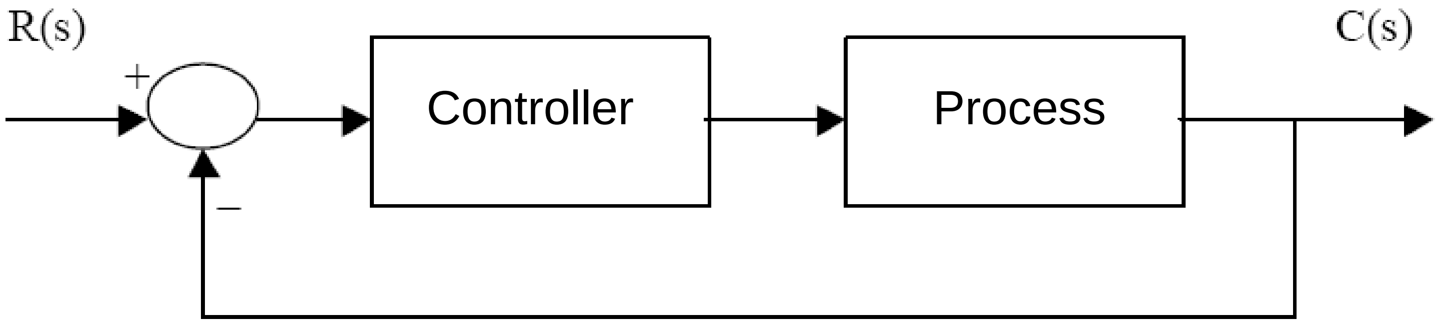
Example 9 (solution)

s^5	2	2	2
s^4	3	3	1
s^3	$\cancel{0}\quad\epsilon$	$\frac{4}{3}$	
s^2	$\frac{3\epsilon-4}{\epsilon}$	1	
s^1	$\frac{12\epsilon-16-3\epsilon^2}{9\epsilon-12}$		
s^0	1		

- 2 sign changes, 2 rhp poles, 3 lhp poles.

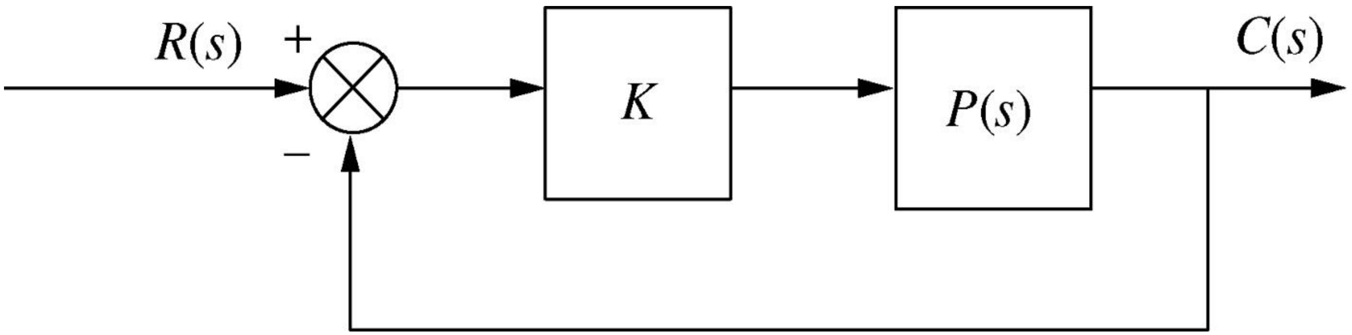
Proportional Controller, *K*

- Controller is used to control the performance of a system.



- The proportional controller is one example where it can change the location of poles of the overall system.

Proportional Controller, *K*

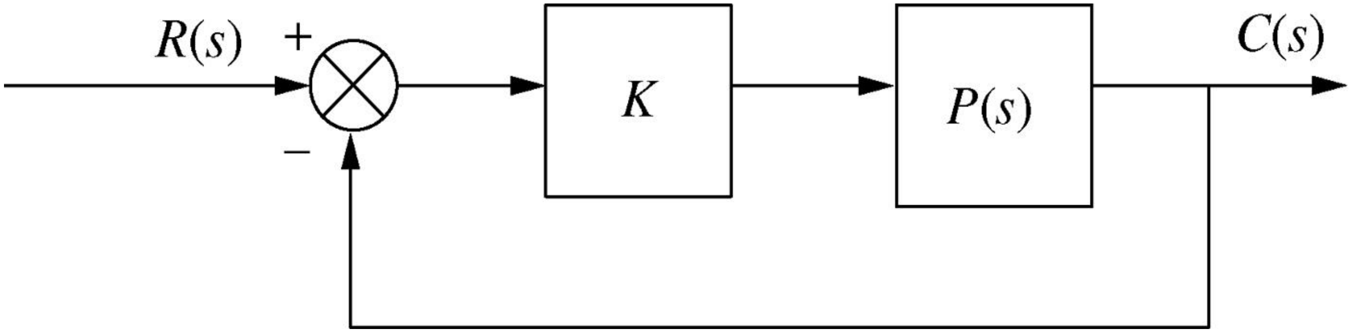


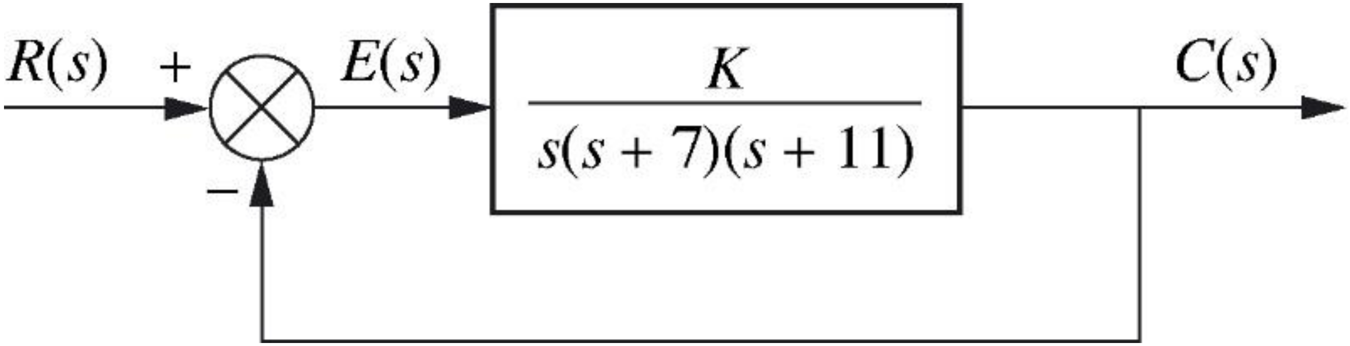
Proportional Controller, *K*

- This type of controller normally known as the Gain Controller, K
- By just changing the value of K, the system performance can be changed since the closed loop poles location also change.
- This will also affect the stability of the system and hence implying that K has certain range for a system to be stable.
- R-H criterion can be used to find the range of K for stability.

Example 10

- Find the range of K for the system that will cause the system to be stable, unstable and marginally stable.





Example 10 (solution)

- Routh table for the CLTF

s^3	1	77
s^2	18	K
s^1	$\dfrac{1386-K}{18}$	
s^0	K	

Example 10 (solution)

- If K < 1386:
  - No sign change, 3 lhp poles, stable.
- If K > 1386:
  - 2 sign changes, 2 rhp poles, unstable.
- If K = 1386, ROZ appears.
  - No sign change, 2 poles on the $j\omega$-axis, 1 lhp pole, marginally stable system.
- Simulation example.

Example 11

- Determine the stability of the CL system with CLTF

$$T(s) = \frac{126}{s^3 + 9s^2 + 14s + 126}$$

$$\begin{bmatrix} 1 & 14 & s^3 \\ 9 & 126 & s^2 \\ 18 & 0 & [s] \\ 126 & 0 & 1 \end{bmatrix}$$

Example 11 (solution)

- ROZ appears at s^1.
- 2 poles on the $j\omega$-axis.
- The system is marginally stable.
- Solving the auxiliary equation gives the values on the $j\omega$-axis or the natural/ oscillating frequency.

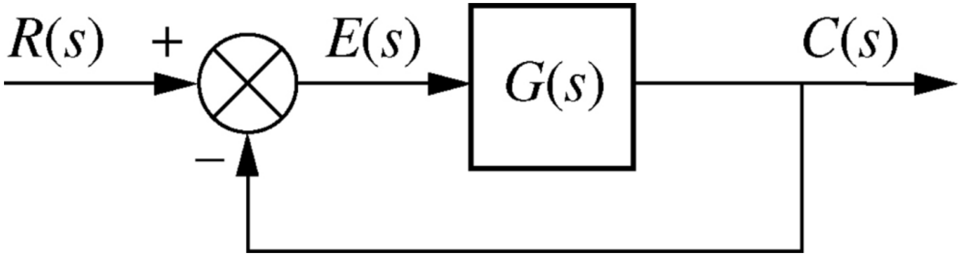
$$\begin{bmatrix} 1 & 14 & s^3 \\ 9 & 126 & s^2 \\ 18 & 0 & [s] \\ 126 & 0 & 1 \end{bmatrix}$$

- Solving:
  - » $9s^2 + 126 = 0$
  - » $S = \pm j3.74$

Example 12

- Given a unity feedback system with a transfer function

$$G(s) = \frac{K(s+1)}{s^3 + as^2 + 2s + 1}$$



- Find *K* and *a* if the closed-loop system is marginally stable and the natural frequency of the output is 2 rad/s with a unit step input.

$$
\begin{bmatrix}
1 & K+2 & s^3 \\
a & K+1 & s^2 \\
K+2-\dfrac{K+1}{a} & 0 & s \\
K+1 & 0 & 1
\end{bmatrix}
$$

ROZ $K+2-\dfrac{K+1}{a}=0$

a= (K+1)/(K+2)----(1)

Auxiliary equation $\left|as^2+(K+1)=0\right|_{s=j2}$

$s^2 + 4 = (s-j2)(s+j2)$

$s = \pm j2$

a = (K+1)/4-------------(2)

Hence, K =2 and a = 3/4