Mathematics and Computational Methods for Complex Systems

Assignment 1

1

A. Derivitives and Their Use

A.1. Compute the derivative with respect to t of

$$\boldsymbol{y = -\frac{1}{2}gt^2 + ut + c}$$

$$\frac{dy}{dt} = -1gt + u$$

$$\frac{dy}{dt} = u - gt$$

At the maximum height, $\frac{dy}{dt} = 0$, substitute that along with $u = 5$, $g = 10$ to find the time t at which the maximum height is reached.

$$5 - 10t = 0$$

$$\therefore t = 0.5 \text{ seconds}$$

Substitute all values in to original equation to calculate the maximum height reached

$$y = -\tfrac{1}{2} \times 10 \times 0.5^2 + 5 \times 0.5 + 1 = 2.25 \text{ metres}$$

A.2.

$$\boldsymbol{f(x) = 2x^3 - 3x^2 - 36x + 2}$$

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 36 \\ &= 6(x^2 - x - 6) \\ &= 6(x - 3)(x + 2) \end{aligned}$$

$\therefore$ gradient $= 0$ at $x = 3$ and $x = -2$

$$f''(x) = 12x - 6$$

At $x = 3$, $f''(x) = 30$ $\therefore$ Minimum point at $(3, -79)$
At $x = -2$, $f''(x) = -30$ $\therefore$ Maximum point at $(-2, 46)$

A.3.

$$\boldsymbol{f(x) = \frac{1}{1+e^{-m(x-x_0)}}}$$

A.3.1.

as $x \to \infty$
$$f(x) = \tfrac{1}{1+e^{-m(\infty)}}$$
$$= \tfrac{1}{1+e^{-\infty}}$$
$$= \tfrac{1}{1+0}$$

as $x \to -\infty$
$$f(x) = \tfrac{1}{1+e^{-m(-\infty)}}$$
$$= \tfrac{1}{1+e^{\infty}}$$
$$= \tfrac{1}{1+\infty}$$

$\therefore \lim_{x\to\infty} f(x) = 1$ assuming m is positive, $\frac{1}{2}$ if $m = 0$, or 0 if m is negative.
$\therefore \lim_{x\to-\infty} f(x) = 0$ assuming m is positive, $\frac{1}{2}$ if $m = 0$, or 1 if m is negative.

A.3.2.

$$f(x) = (1+e^{-m(x-x_0)})^{-1}$$
$$f'(x) = -1 \times (1+e^{m(x-x_0)})^{-2} \times (0+e^{-m(x-x_0)}) \times -m$$
$$= -\frac{-me^{-m(x-x_0)}}{(1+e^{-m(x-x_0)})^2}$$
$$\boldsymbol{= \frac{me^{-m(x-x_0)}}{(1+e^{-m(x-x_0)})^2}}$$

To calculate the second derivative, bring on to one line to allow the use of product rule

$$f(x) = uv \qquad\qquad f'(x) = uv' + vu'$$

$$f'(x) = me^{-m(x-x_0)} \cdot (1+e^{-m(x-x_0)})^{-2}$$

$$u = me^{-m(x-x_0)} \qquad\qquad u' = me^{-m(x-x_0)} \cdot -m$$
$$v = (1+e^{-m(x-x_0)})^{-2} \qquad\qquad v' = -2(1+e^{-m(x-x_0)})^{-3} \cdot (-me^{-m(x-x_0)})$$

let $k = e^{-m(x-x_0)}$

$$f''(x) = mk \cdot -2(1+k)^{-3} \cdot (-mk) + (1+k)^{-2} \cdot mk \cdot -m$$
$$= 2\frac{(mk)^2}{(k+1)^3} - \frac{mk}{(k+1)^2}$$
$$\boldsymbol{= 2\frac{(me^{-m(x-x_0)})^2}{(e^{-m(x-x_0)}+1)^3} - \frac{me^{-m(x-x_0)}}{(e^{-m(x-x_0)}+1)^2}}$$

At the stationary point of the drivative of $f(x)$, $f''(x) = 0$

$$\therefore \frac{mk}{(k+1)^2} = \frac{2m^2k^2}{(k+1)^3}$$

$$mk = \frac{2m^2k^2}{k+1}$$

$$k + 1 = 2mk$$

$$k(2m - 1) - 1 = 0$$

Substitute k value back in

$$0 = e^{-m(x-x_0)}(2m - 1) - 1$$

$$\frac{1}{2m-1} = e^{-m(x-x_0)}$$

$$ln(\frac{1}{2m-1}) = ln(1) - ln(2m - 1) = -m(x - x_0)$$

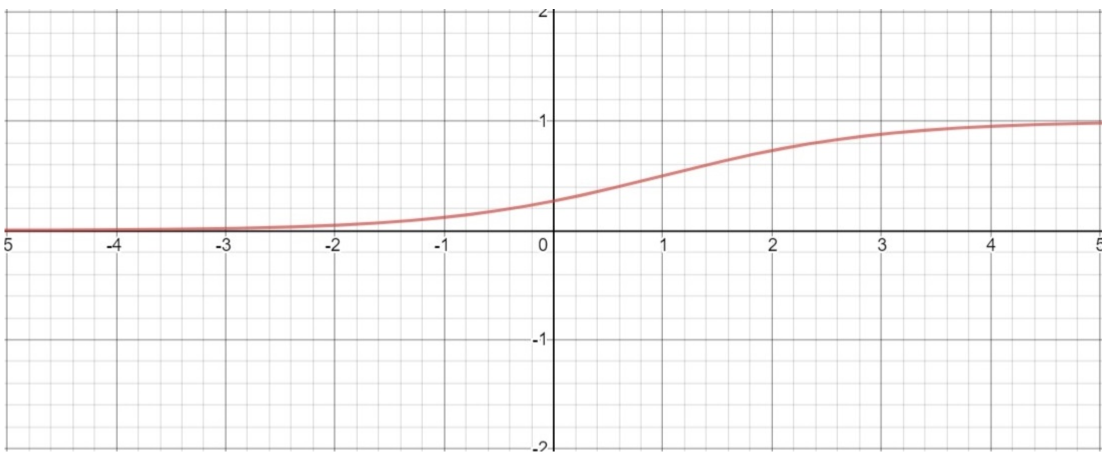
$$-ln(2m - 1) = -m(x - x_0)$$

$$ln(2m - 1) = mx - mx_0$$

$$x = \frac{ln(2m - 1) + mx_0}{m}$$

x_0 describes the x coordinate of the point of sigmoid function's point of inflection
m describes the gradient of the line at that point

A.3.3.



As you can see in the plot above (where m and $x_0 = 1$), our findings made in the previous section are confirmed

A.4.

$$E = -\sum_{i=1}^{2}\sum_{j=1}^{2} w_{ij}x_i x_j$$

$$= -(w_{11}x_1x_1 + w_{12}x_1x_2 + w_{21}x_2x_1 + w_{22}x_2x_2)$$

$$\frac{\partial E}{\partial x_1} = -(2w_{11} + w_{12}x_2 + w_{21}x_2$$

$$\frac{\partial E}{\partial w_{12}} = -(x_1x_2)$$