Pf. Let $A = QR$ for $A, Q \in \mathbb{C}^{m\times m}$, $R \in \mathbb{C}^{m\times n}$, unitary Q, and upper triangular R. If A has bandwidth $2p+1$ then for $i - j > p$,

$$0 = a_{ij} = \sum_{k=1}^{m} q_{ik} r_{kj}$$

Since R is upper triangular, $r_{kj} = 0$ for $k > j$. Then when $i > j + p$,

$$0 = a_{ij} = \sum_{k=1}^{j} q_{ik} r_{kj}$$

so $q_{ij} = 0$. Hence

$$q_j = \begin{pmatrix} q_{1,j} \\ q_{2,j} \\ \vdots \\ q_{j+p,j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1}$$

From (1) when $i + p < j - p - 1$ (i.e., $j - i > 2p + 1$):

$$r_{ij} = q_i^* a_j = \begin{pmatrix} q_{1,i} & q_{2,i} & \dots & q_{i+p,i} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{j-p-1,j} \\ \vdots \\ a_{j+p+1,j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

Hence R is upper triangular with its only nonzero entries in the diagonal and $2p$ super diagonals. ■