Computational Math Project

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0.0 Introduction

0.1 Theory

Theorem 0.1.1. *If A is a tridiagonal matrix. Then R in the the product $A = QR$ is a upper triangular matrix with non zero entries only in the diagonal and the two super diagonals.*

$$A = \begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & \ddots & \ddots & \ddots \\ & & a_{mm-1} & a_{mm} \end{pmatrix}, \quad R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \\ & & & r_{mm} \end{pmatrix}$$

Pf. To prove the statement we will use the classical Gram-Schmidt method for the QR decomposition.

Step 1: we want to show that q_j has the form $q_j = \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \end{pmatrix} \leftarrow j+1\text{-th entry.}$

We prove this by induction:

Base step $j = 1$ the if we assume that $\|a_1\| = 1$ then $q_1 = a_1$ thus

$$q_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ 0 \\ \vdots \end{pmatrix}$$

Induction step Assume that the statement holds for $j-1$. Then

$$v_j = a_j - \sum_{k=1}^{j-1} (q_k^* a_j) q_k \quad \text{and } q_j = v_j / \|v\|_j$$

and by using the form of q_{j-1} we obtain

$$q_j = \begin{pmatrix} 0 \\ \vdots \\ a_{j-1,j} \\ a_{jj} \\ a_{j+1,j} \\ 0 \\ \vdots \end{pmatrix} - \sum_{k=1}^{j-1} \begin{pmatrix} * \\ \vdots \\ \vdots \\ * \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \leftarrow k+1\text{-th entry} = \begin{pmatrix} * \\ \vdots \\ \vdots \\ * \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \leftarrow j+1\text{-th entry}$$

Step 2: Compute r_{ij} in the CGS method

For $j = 1$ to n and for $i = 1$ to $j-1$: $r_{ij} = q_i^* a_j$. Then by step 1 we obtain that $r_{ij} = 0$ if $i \leq j-3$ since then by the form of the vectors q_{j-3} and a_j

$$0 = \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ * \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{-th enttry}$$

The above argument holds for all $i \leq j-3$. ■