

# Computational Math Project

Illinois Institute of Technology

Miles Bakenhus

Ahmed Lodhika

Gunjan Sharma

Quinn Stratton

Jan-Eric Sulzbach

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## 0 Introduction

## 1 Origin of the problem and its applications

In this part of the project we explain the motivation behind our ideas and consider the applications of the results.

First, we consider the standard elliptic equation in two dimensions

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then the discretised equation on a grid is

$$\begin{aligned} -\Delta_h u_{ij} &= f_{ij} \quad \forall (x_i, y_i) \in \Omega_h, \quad f_{ij} = f(x_i, y_j) \\ u_{ij} &= 0 \quad \forall (x_i, y_i) \in \partial\Omega_h, \end{aligned}$$

where we use the second order central differencing to represent the Laplace operator

$$-\Delta_h u_{ij} = \frac{1}{h^2} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix} u_{ij} = \frac{1}{h^2} (-u_{ij+1} - u_{i-1j} + 4u_{ij} - u_{i+1j} - u_{ij1})$$

If we assume that the domain  $\Omega$  is a square and ordering the grid points from left to right and bottom to top yields the following matrix of the system  $A \in \mathbb{R}^{(h-1)^{-2} \times (h-1)^{-2}}$ :

$$A = h^2 \begin{pmatrix} 4 & -1 & 0 & \dots & 0 & -1 & & \\ -1 & 4 & -1 & & & & \ddots & \\ 0 & -1 & 4 & \ddots & & & & -1 \\ \vdots & & \ddots & \ddots & & & & 0 \\ 0 & & & & & & & \vdots \\ -1 & & & & & \ddots & \ddots & 0 \\ & \ddots & & & & \ddots & 4 & -1 \\ & & -1 & 0 & \dots & 0 & -1 & 4 \end{pmatrix}$$

And the problem we need to solve is the linear system  $Au = f$ .

Therefore it is important to understand how the structure of  $A$  affects the structure of the  $QR$  decomposition. Note that in this case the highest and lowest off-diagonal has a distance of order  $h$  from the diagonal.

Another example where these banded matrices show up is in the following: Consider the parabolic equation in two dimensions

$$\frac{\partial u}{\partial t} = \sigma \Delta u, \quad 0 \leq x \leq X, \quad 0 \leq y \leq Y \quad 0 \leq t \leq T$$

with Dirichlet boundary condition and a given initial data  $u(x, y, 0) = U^0(x, y)$ .

For the numerical implementation we consider the implicit Crank-Nicolson scheme

$$\begin{aligned} & -\frac{\mu_x}{2} (U_{j-1,l}^{n+1} + U_{j+1,l}^{n+1}) - \frac{\mu_y}{2} (U_{j,l-1}^{n+1} + U_{j,l+1}^{n+1}) + (1 + \mu_x + \mu_y) U_{j,l}^{n+1} \\ & = \frac{\mu_x}{2} (U_{j-1,l}^n + U_{j+1,l}^n) + \frac{\mu_y}{2} (U_{j,l-1}^n + U_{j,l+1}^n) + (1 - \mu_x - \mu_y) U_{j,l}^n, \end{aligned}$$

for  $0 \leq j \leq J_x$ ,  $0 \leq l \leq J_y$  and  $n > 0$ . Again we can rewrite this as a linear system  $AU^{n+1} = U^n$ , where

$$A = \begin{pmatrix} 1 + \mu_x + \mu_y & -\frac{\mu_x}{2} & 0 & \dots & 0 & -\frac{\mu_y}{2} & & \\ -\frac{\mu_x}{2} & 1 + \mu_x + \mu_y & -\frac{\mu_x}{2} & & & & \ddots & \\ 0 & -\frac{\mu_x}{2} & 1 + \mu_x + \mu_y & \ddots & & & & -\frac{\mu_y}{2} \\ \vdots & & & \ddots & \ddots & & & 0 \\ 0 & & & & & & & \vdots \\ -\frac{\mu_y}{2} & & & & & \ddots & \ddots & 0 \\ & & \ddots & & & \ddots & 1 + \mu_x + \mu_y & -\frac{\mu_x}{2} \\ & & & -\frac{\mu_y}{2} & 0 & \dots & 0 & -\frac{\mu_x}{2} & 1 + \mu_x + \mu_y \end{pmatrix}$$

and  $A \in \mathbb{R}^{(J_x-1)(J_y-1) \times (J_x-1)(J_y-1)}$  where the highest and lowest off-diagonal band have the distance  $J_x - 1$  from the diagonal.

*Remark 1.* In the case of three dimension, we would obtain one more non-zero sub/super-diagonal, now with a distance of  $\mathcal{O}(J_x * J_y)$  from the diagonal.

## 2 Matrix Properties

### 2.1 Tridiagonal Matrices

**Theorem 2.1.** *If  $A$  is a tridiagonal matrix. Then  $R$  in the the product  $A = QR$  is a upper triangular matrix with non zero entries only in the diagonal and the two super diagonals.*

$$A = \begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{mm-1} & a_{mm} \end{pmatrix}, R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & r_{mm} \end{pmatrix}$$

*Pf.* To prove the statement we will use the classical Gram-Schmidt method for the QR decomposition.

Step 1: we want to show that  $q_j$  has the form  $q_j = \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \end{pmatrix} \leftarrow j+1\text{-th entry}.$

We prove this by induction:

**Base step**  $j = 1$  the if we assume that  $\|a_1\| = 1$  then  $q_1 = a_1$  thus

$$q_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ 0 \\ \vdots \end{pmatrix}$$

**Induction step** Assume that the statement holds for  $j-1$ . Then

$$v_j = a_j - \sum_{k=1}^{j-1} (q_k^* a_j) q_k \quad \text{and} \quad q_j = v_j / \|v\|_j$$

and by using the form of  $q_{j-1}$  we obtain

$$q_j = \begin{pmatrix} 0 \\ \vdots \\ a_{j-1,j} \\ a_{jj} \\ a_{j+1,j} \\ 0 \\ \vdots \end{pmatrix} - \sum_{k=1}^{j-1} \begin{pmatrix} * \\ \vdots \\ \vdots \\ * \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \leftarrow k+1\text{-th entry} = \begin{pmatrix} * \\ \vdots \\ \vdots \\ * \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \leftarrow j+1\text{-th entry}$$

Step 2: Compute  $r_{ij}$  in the CGS method

For  $j = 1$  to  $n$  and for  $i = 1$  to  $j-1$ :  $r_{ij} = q_i^* a_j$ . Then by step 1 we obtain that  $r_{ij} = 0$  if  $i \leq j-3$

since then by the form of the vectors  $q_{j-3}$  and  $a_j$

$$0 = \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ * \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{-th entry}$$

The above argument holds for all  $i \leq j - 3$ . ■

## 2.2 General Banded Matrices

*Pf.* Let  $A = QR$  for  $A, Q \in \mathbb{C}^{m \times m}$ ,  $R \in \mathbb{C}^{m \times n}$ , unitary  $Q$ , and upper triangular  $R$ . If  $A$  has bandwidth  $2p + 1$  then for  $i - j > p$ ,

$$0 = a_{ij} = \sum_{k=1}^m q_{ik} r_{kj}$$

Since  $R$  is upper triangular,  $r_{kj} = 0$  for  $k > j$ . Then when  $i > j + p$ ,

$$0 = a_{ij} = \sum_{k=1}^j q_{ik} r_{kj}$$

so  $q_{ij} = 0$ . Hence

$$q_j = \begin{pmatrix} q_{1,j} \\ q_{2,j} \\ \vdots \\ q_{j+p,j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1}$$

From (1) when  $i + p < j - p - 1$  (i.e.,  $j - i > 2p + 1$ ):

$$r_{ij} = q_i^* a_j = \begin{pmatrix} q_{1,i} & q_{2,i} & \dots & q_{i+p,i} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{j-p-1,j} \\ \vdots \\ a_{j+p+1,j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

Hence  $R$  is upper triangular with its only nonzero entries in the diagonal and  $2p$  super diagonals. ■

## 2.3 Sparse Diagonal Matrices

Now we want to generalize the above ideas to the case where  $A$  still has only three non-zero bands. But now, the lower band has the distance  $k - 1$  from the diagonal and the upper band has the distance  $l - 1$  from the diagonal.

Consider the following example for  $A$ :

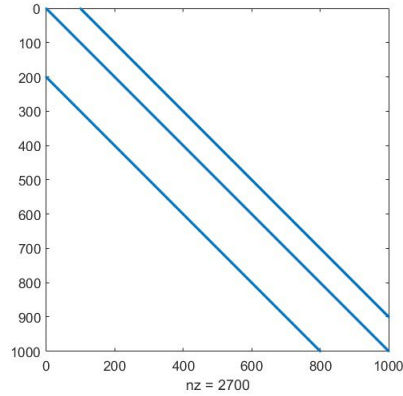


Figure 1:  $k = 200$  and  $l = 100$

**Theorem 2.2** (General case). *The upper triangular matrix  $R$  in the QR decomposition of  $A$  has*

a  $k + l$ -band structure.

*Pf.* From the classical Gram Schmidt method we immediately see, that in the worst case, the first  $j + k$  entries are non zero. Therefore, the inner product in the computation of the entries  $r_{ij}$  is only zero if  $i < j - l - k + 2$ . ■

Example of a matrix close to the worst case, where the number of non-zero entries (nz) increases by order 50:

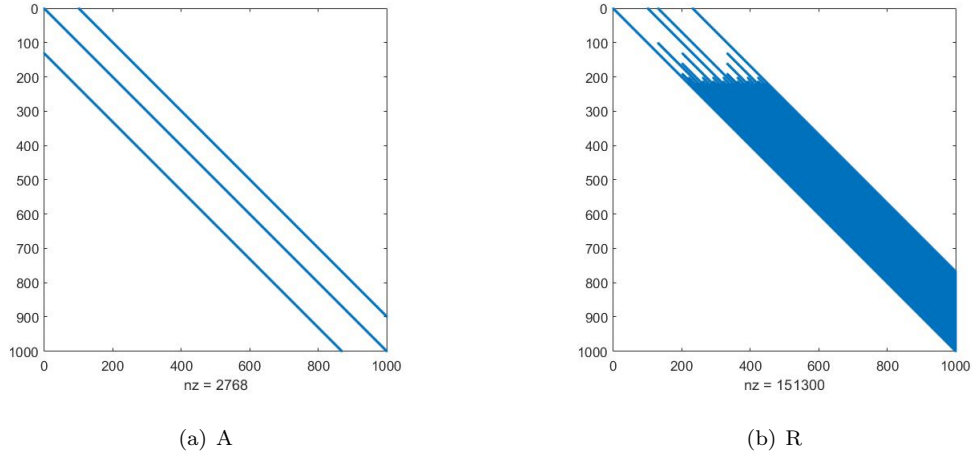
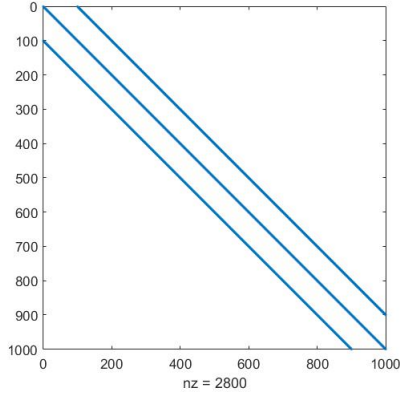


Figure 2:  $k = 131$  and  $l = 101$

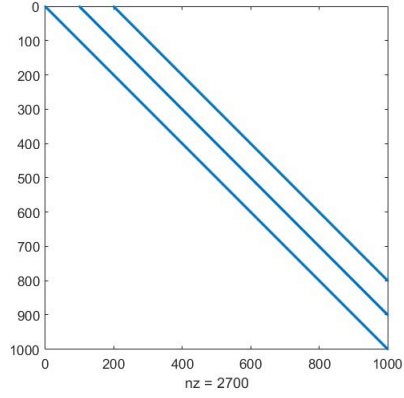
A special case occurs when  $k = l$ . Then again  $R$  has only three non-zero band, i.e the diagonal, the band on the upper diagonal that has a distance  $k$  to the diagonal and the upper diagonal that has a distance  $2k$  to the diagonal.

Again we have an example for this case:





(a) A



(b) R

Figure 3:  $k = l = 100$

### 3 Algorithms and Results

Based on the properties proved in the last section, it is fairly straightforward to modify existing algorithms for finding a QR-factorization of a matrix to exploit sparsity patterns.

#### 3.1 General Banded Matrices

Suppose  $A \in \mathbb{C}^{m \times n}$  with bandwidth  $p$ . Consider some QR-factorization of  $A$ ,  $A = QR$ . Then by **THEOREM**, we know that if  $j > i + 2p$ ,  $r_{ij} = 0$ . We can alter the well-known *Modified Gram-Schmidt* algorithm to take advantage of the above fact in the following way. Below are some

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**Algorithm 1** MGS for Banded Matrices

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- 1: **for**  $i = 1$  **to**  $n$  **do**
- 2:    $r_{ii} \leftarrow \|\mathbf{a}_i\|_2$
- 3:    $\mathbf{q}_i \leftarrow \mathbf{a}_i / r_{ii}$
- 4:   **for**  $j = i + 1$  **to**  $\min\{i + 2p, n\}$  **do**
- 5:      $r_{ij} \leftarrow \mathbf{q}_i^* \mathbf{a}_j$
- 6:      $\mathbf{a}_j \leftarrow \mathbf{a}_j - r_{ij} \mathbf{q}_i$

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[Note that this is based on the *Modified Gram-Schmidt* algorithm as described in [1]]

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results for the performance of **MGS for Banded Matrices** and the performance of the standard MGS algorithm applied to the same random banded matrices.

	Performance of <i>Modified Gram-Schmidt</i> <b>NOT COMPLETE!</b>					
$\mathbb{C}^{10 \times 10}$	0.0024461	0.0023396	0.0023416	0.0023403	0.0023373	0.0023707
$\mathbb{C}^{500 \times 500}$	5.1399053	5.0569402	5.0526341	5.0476353	5.0430921	5.3553525
$\mathbb{C}^{750 \times 750}$	12.0286885	12.1555834	12.3704160	12.0967193	12.5868144	12.2757503
$\mathbb{C}^{1000 \times 1000}$	21.6673735	21.1107465	21.1824376	21.3300323	21.0178360	22.8876292

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## 4 Further Study

## References

- [1] Lloyd N. Trefethen and III David Bau. *Numerical Linear Algebra*. SIAM, Philadelphia, Pennsylvania, 1997.