# Computational Math Project

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November 7, 2018

# Contents



## 0.0 Introduction

## 0.1 Theory

**Theorem 0.1.1.** *If $A$ is a tridiagonal matrix. Then $R$ in the the product $A = QR$ is a upper triangular matrix with non zero entries only in the diagonal and the two super diagonals.*

$$A = \begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & \ddots & \ddots & \ddots \\ & & a_{mm-1} & a_{mm} \end{pmatrix}, \ R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \\ & & & r_{mm} \end{pmatrix}$$

*Pf.* To prove the statement we will use the classical Gram-Schmidt method for the QR decomposition.

Step 1: we want to show that $q_j$ has the form $q_j = \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \end{pmatrix} \leftarrow j+1\text{-th entry.}$

We prove this by induction:

**Base step** $j = 1$ the if we assume that $\|a_1\| = 1$ then $q_1 = a_1$ thus

$$q_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ 0 \\ \vdots \end{pmatrix}$$

**Induction step** Assume that the statement holds for $j-1$. Then

$$v_j = a_j - \sum_{k=1}^{j-1} (q_k^* a_j) q_k \quad \text{and } q_j = v_j / \|v\|_j$$

and by using the form of $q_{j-1}$ we obtain

$$q_j = \begin{pmatrix} 0 \\ \vdots \\ a_{j-1,j} \\ a_{jj} \\ a_{j+1,j} \\ 0 \\ \vdots \end{pmatrix} - \sum_{k=1}^{j-1} \begin{pmatrix} * \\ \vdots \\ \vdots \\ * \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \leftarrow k+1\text{-th entry} = \begin{pmatrix} * \\ \vdots \\ \vdots \\ * \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \leftarrow j+1\text{-th entry}$$

Step 2: Compute $r_{ij}$ in the CGS method

For $j = 1$ to $n$ and for $i = 1$ to $j-1$: $r_{ij} = q_i^* a_j$. Then by step 1 we obtain that $r_{ij} = 0$ if $i \leq j-3$ since then by the form of the vectors $q_{j-3}$ and $a_j$

$$0 = \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ * \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{-th enttry}$$

The above argument holds for all $i \leq j-3$. ■

Now we want to generalize the above ideas to the case where A still has only three non-zero bands. But now, the lower band has the distance $k-1$ from the diagonal and the upper band has the distance $l-1$ from the diagonal.

Consider the following example for A:

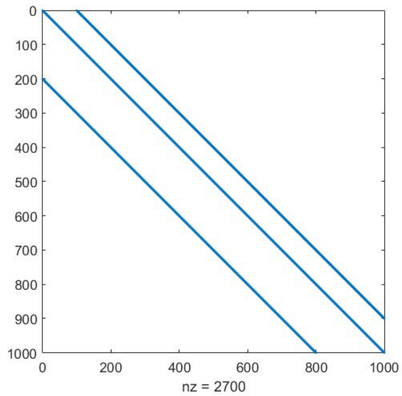
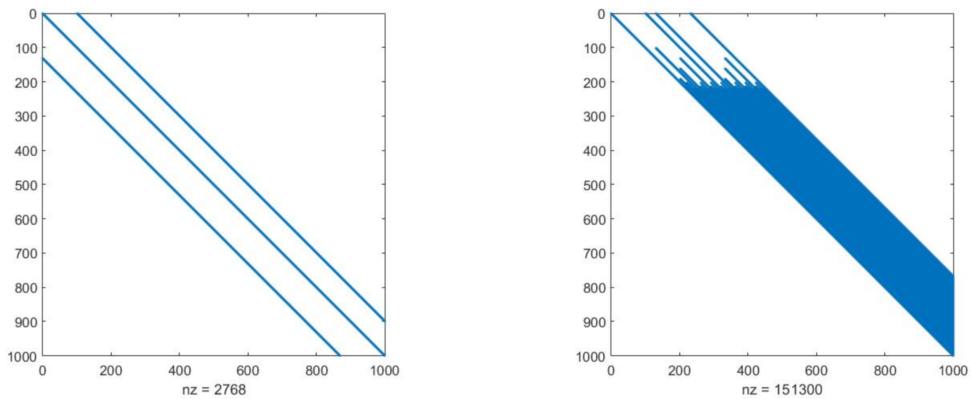



Figure 1: $k = 200$ and $l = 100$

**Theorem 0.1.2** (General case)**.** *The upper triangular matrix $R$ in the $QR$ decomposition of $A$ has a $k+l$-band structure.*

*Pf.* From the classical Gram Schmidt method we immediately see, that in the worst case, the first $j+k$ entries are non zero. Therefore, the inner product in the computation of the entries $r_{ij}$ is only zero if $i < j - l - k + 2$. ■

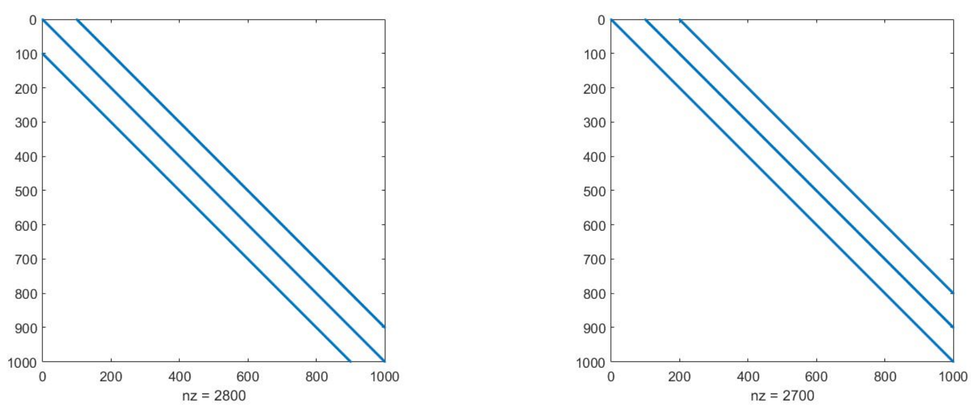
Example of a matrix close to the worst case, where the number of non-zero entries (nz) increases by order 50:




(a) A (b) R

Figure 2: $k = 131$ and $l = 101$

A special case occurs when $k = l$. Then again $R$ has only three non-zero band, i.e the diagonal, the band on the upper diagonal that has a distance $k$ to the diagonal and the upper diagonal that has a distance $2k$ to the diagonal.
Again we have an example for this case:




(a) A (b) R

Figure 3: $k = l = 100$