0.0 Origin of the problem and its applications

In this part of the project we explain the motivation behind our ideas and consider the applications of the results.

First, we consider the standard elliptic equation in two dimensions

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Then the discretised equation on a grid is

$$-\Delta_h u_{ij} = f_{ij} \; \forall\, (x_i, y_i) \in \Omega_h, \;\; f_{ij} = f(x_i, y_j)$$
$$u_{ij} = 0 \; \forall\, (x_i, y_i) \in \partial\Omega_h,$$

where we use the second order central differencing to represent the Laplace operator

$$-\Delta_h u_{ij} = \frac{1}{h^2} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix} u_{ij} = \frac{1}{h^2} \big(- u_{ij+1} - u_{i-1j} + 4u_{ij} - u_{i+1j} - u_{ij_1} \big)$$

If we assume that the domain Ω is a square and ordering the grid points from left to right and bottom to top yields the following matrix of the system $A \in \mathbb{R}^{(h-1)^{-2} \times (h-1)^{-2}}$:

$$A = h^2 \begin{pmatrix} 4 & -1 & 0 & \dots & 0 & -1 & & \\ -1 & 4 & -1 & & & & \ddots & \\ 0 & -1 & 4 & \ddots & & & & -1 \\ \vdots & & \ddots & \ddots & & & & 0 \\ 0 & & & & & & & \vdots \\ -1 & & & & & \ddots & \ddots & 0 \\ & \ddots & & & & \ddots & 4 & -1 \\ & & -1 & 0 & \dots & 0 & -1 & 4 \end{pmatrix}$$

And the problem we need to solve is the linear system $Au = f$.

Therefore it is important to understand how the structure of A affects the structure of the QR decomposition. Note that in this case the highest and lowest off-diagonal has a distance of order h

from the diagonal.

Another example where these banded matrices show up is in the following: Consider the parabolic equation in two dimensions

$$\frac{\partial u}{\partial t} = \sigma \Delta u,\ 0 \le x \le X,\, 0 \le y \le Y\, 0 \le t \le T$$

with Dirichlet boundary condition and a given initial data $u(x, y, 0) = U^0(x, y)$.
For the numerical implementation we consider the implicit Crank-Nicolson scheme

$$\begin{aligned} &-\frac{\mu_x}{2}\big(U^{n+1}_{j-1,l} + U^{n+1}_{j+1,l}\big) - \frac{\mu_y}{2}\big(U^{n+1}_{j,l-1} + U^{n+1}_{j,l+1}\big) + \big(1 + \mu_x + \mu_y\big)U^{n+1}_{j,l} \\ =&\frac{\mu_x}{2}\big(U^{n}_{j-1,l} + U^{n}_{j+1,l}\big) + \frac{\mu_y}{2}\big(U^{n}_{j,l-1} + U^{n}_{j,l+1}\big) + \big(1 - \mu_x - \mu_y\big)U^{n}_{j,l}, \end{aligned}$$

for $0 \le j \le J_x$, $0 \le l \le J_y$ and $n > 0$. Again we can rewrite this as a linear system $AU^{n+1} = U^n$, where

$$A = \begin{pmatrix} 1+\mu_x+\mu_y & -\frac{\mu_x}{2} & 0 & \dots & 0 & -\frac{\mu_y}{2} & & \\ -\frac{\mu_x}{2} & 1+\mu_x+\mu_y & -\frac{\mu_x}{2} & & & & \ddots & \\ 0 & -\frac{\mu_x}{2} & 1+\mu_x+\mu_y & \ddots & & & & -\frac{\mu_y}{2} \\ \vdots & & \ddots & \ddots & & & & 0 \\ 0 & & & & & & & \vdots \\ -\frac{\mu_y}{2} & & & & & \ddots & \ddots & 0 \\ & \ddots & & & & \ddots & 1+\mu_x+\mu_y & -\frac{\mu_x}{2} \\ & & -\frac{\mu_y}{2} & 0 & \dots & 0 & -\frac{\mu_x}{2} & 1+\mu_x+\mu_y \end{pmatrix}$$

and $A \in \mathbb{R}^{(J_x-1)(J_y-1)\times(J_x-1)(J_y-1)}$ where the highest and lowest off-diagonal band have the distance $J_x - 1$ from the diagonal.

Remark 1. In the case of three dimension, we would obtain one more non-zero sub/super-diagonal, now with a distance of $\mathcal{O}(J_x * J_y)$ from the diagonal.