Estimating Seats–Votes Partisan Advantage

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Abstract

A new class of metrics is defined that measure whether redistricting plans favor one political party or the other: partisan advantage using a seats–votes benchmark. Ten prominent measures of partisan bias are evaluated against a cross section of nearly three dozen past, present, and hypothetical congressional plans. Only (dis)proportionality and the efficiency gap are shown to reliably measure this across a wide range of statewide vote shares. When winner's bonuses are between one and two inclusive, plans are shown to have acceptable bias with respect to the efficiency gap ideal of 2-proportionality.

Introduction

This paper proceeds as follows:

1. The many broad criteria for evaluating redistricting plans are identified, and the confusing depth of metrics in the bias subcategory are enumerated.

2. The relatively new concept of partisan advantage is explained and extended to use a seats–votes benchmark instead of the original jurisdictional one.

3. Three sets of plans are introduced: the 2011 congressional plans for 11 states using a composite of 2012 elections, the corresponding 2020 plans using a composite of 2016–2020 election results, and a dozen carefully curated hypothetical plans.

4. Ten prominent metrics are calculated for the sample plans: declination (δ), lopsided outcomes (LO), mean–median (MM), seats bias (α_S), votes bias (α_V), geometric seats bias (β), global symmetry (GS), proportional (PR), the efficiency gap (EG), and gamma (γ). Only (dis)proportionality and the efficiency gap are shown to reliably measure seats–votes partisan advantage across a wide range of statewide vote shares.

Buckle up!

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* I work on Dave's Redistricting in Seattle, Washington, USA. I would like to thank Gregory Warrington – his early feedback helped me frame these ideas – and Jon Eguia – his patient, gracious feedback helped me develop them.

1. Background

Redistricting technology has been substantially democratized this census cycle. Anyone with access to a web browser can draw block-level maps suitable for submission to redistricting officials and can quickly evaluate plans they and others draw,[1] all free of charge (Dave's Redistricting; District Builder; PlanScore).

People consider many factors when evaluating redistricting plans, including:[2]

- Basic requirements, like whether a plan is complete and contiguous[3]
- Straightforward formulas, like population deviation and measures of compactness[4]
- Much discussed partisan measures of bias & responsiveness[5]
- How much counties, cities, and communities are split by districts[6]
- Whether districts are VRA compliant—a complex analysis that requires experts,[7] and
- More qualitative considerations where there aren't established metrics, such as the effects on incumbents and preservation of district cores

Even just under the "bias" subset of the partisan dimension, there are ten prominent metrics that may yield seemingly conflicting results for any given map:[8]

- Declination (δ)
- Lopsided outcomes (LO)
- Mean–median (MM)
- Seats bias (α_S)
- Votes bias (α_V)
- Geometric seats bias (β)
- Global symmetry (GS)

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[1] While DRA uses one methodology for partisan metrics (see https://bit.ly/2ZlqVfl) and PlanScore uses another (see https://bit.ly/3or5IgK), they both use multiple elections and aggregate census block results up to district-by-district and statewide vote shares.

[2] DRA supports all these directly, except the qualitative considerations, of course. In this paper, I take an approach like (McGhee 2017): where he evaluates metrics against the Efficiency Principle, I assess them with respect to seats–votes partisan advantage.

[3] Even something as conceptually simple as contiguity is sometimes non-trivial in real life. See the discussion of 'operational contiguity' in "Contiguity" (https://bit.ly/2ZtmB0Q).

[4] DRA computes Reock, Polsby–Popper, and "know it when you see it" compactness (see https://bit.ly/3Gdm8PU).

[5] For example (Tufte 1973; Grofman 1983; Wang 2016; McGhee 2017, McDonald et al. 2018; Stephanopoulos and McGhee 2018; Wang et al. 2019; Warrington 2019; Katz et al. 2020; Nagle and Ramsay 2021).

[6] Appendix 6 of (Duchin 2018) and (Wang et al. 2021) are the bases for the county-splitting and community-splitting metrics in Dave's Redistricting, respectively.

[7] DRA incorporates some heuristics for evaluating the opportunity for minority representation from (PGP 2018).

[8] All of these metrics can be computed from only a statewide vote share and district-by-district vote shares. DRA reports them all, as well as Eguia's measure of geographic bias which requires precinct-by-precinct election data (Eguia 2021). PlanScore calculates the efficiency gap, partisan bias, mean–median, and declination.

- Proportional (PR)
- Efficiency gap (EG),[9] and
- Gamma (γ)

To complicate matters, different experts focus on different subsets of these metrics: There is no consensus about which of them to use under what circumstances. Moreover, analysts frequently report simple delegation splits in whole seats, e.g., 10–3 Republicans/Democrats instead of any of these formal metrics (Wasserman).

Despite this unsettled state, some states have added language like this to their state constitutions:

> (a) No apportionment plan or individual district shall be drawn with the intent to **favor or disfavor a political party** or an incumbent; and districts shall not be drawn with the intent or result of denying or abridging the equal opportunity of racial or language minorities to participate in the political process or to diminish their ability to elect representatives of their choice; and districts shall consist of contiguous territory. [emphasis added] (Florida Constitution)
>
> (d) Districts shall not provide a **disproportionate advantage to any political party**. A disproportionate advantage to a political party shall be determined using accepted measures of partisan fairness. [emphasis added] (Michigan Constitution)

With tens of thousands of people now able to evaluate hundreds of thousands of redistricting plans,[10] this begs the question "How do you know whether a plan favors one party or the other, gives it an advantage?"[11]

2. Seats–Votes Partisan Advantage

The problem sketched above stems from loose terminology and imprecise definitions:

- People tend to use the terms "bias" and "fairness" interchangeably and sometimes use the term "partisan bias" generically even though it has a specific meaning in the literature.[12]
- Some view what favors or harms a party as a combination of bias and responsiveness, even though the measures are independent.

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[9] I calculate the efficiency gap using the formula shown in Equation 13 and the actual statewide vote share. This is what DRA does as well.
[10] DRA alone has this level of usage.
[11] You have the same issue with automated redistricting programs using techniques such as Markov Chain Monte Carlo (MCMC) simulation.
[12] King's geometric seats bias (β) measure.

- People tend to treat all the measures of "bias" as though they measure the same thing – Instead, some metrics measure partisan gerrymandering via packing & cracking,[13] some measure the bias inherent in a state's political geography,[14] others measure aspects of seats–votes curves to assess partisan symmetry,[15] and still others measure "fairness" relative to some normative standard such as proportionality. These apples-and-oranges categories cannot be meaningfully compared or combined.

For clarity, I will extend the relatively new concept of partisan advantage[16] to define a class of measures in this last category.

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In our political system dominated by two parties, the number of seats that candidates of the two parties win determines control of Congress and state legislatures: seats won are political currency. Moreover, because district boundaries determine how a state's votes will likely get translated into seats, single-member redistricting is, in effect, the process of deciding how many seats each party will win.[17] Hence, to gauge whether a proposed or adopted plan will favor one party or the other, you simply compare how votes will *likely* get translated into seats under the plan to some normative ideal for how they *should* get translated into seats in our representative democracy.[18]

Simply put:

Partisan advantage is the difference between the ideal and actual seat shares.

Whereas Eguia used a jurisdictional benchmark to account for political geography when he introduced the concept of partisan advantage (2021) – call it $PA|J$ – I use a seats–votes benchmark which I dub $PA|SV$.

Specifically, outside a courtroom the more a plan will likely translate votes into a number of seats that closely matches the ideal seats–votes relationship, the more the plan is politically neutral. The more the map will likely translate votes into seats in a way that significantly deviates from that ideal relationship – whatever it may be! – the more the plan favors one party or the other, the more it confers a seats–votes partisan advantage.

[13] Declination (δ) does provably well on this (Warrington 2019).
[14] For example, (Eguia 2021).
[15] The principle that a plan should treat the two parties equally.
[16] (Eguia 2021) introduces this concept using a jurisdictional benchmark.
[17] Some districts may be competitive and "flip" between parties given actual election results, but that is increasingly rare, and the general point holds.
[18] One's notion of how votes *should* translate into seats won does not need to depend solely on an inferred seat-votes curve. It could depend on other factors such as the geographic or racial distribution of votes shares or of turnout across the state. I restrict my focus here to those that be computed from a statewide vote share and district-by-district vote shares.

2.1. General Formula & Units

To formalize this, call the ideal seats–votes relationship $S(V)$, where V and the resulting S are the two-party Democratic vote share and seat share, respectively.[19] As you will see below, there are many candidates for this expected seats-votes function, not just proportionality.

Similarly, call the actual the statewide vote share and the resulting seat share $\bar{V}$and $\bar{S}$, respectively.

Seats–votes partisan advantage is simply the difference between these two:[20]

$$PA|SV = S(\bar{V}) - \bar{S} \tag{1}$$

In words, it is the difference between the share of seats that *should* be won given a statewide vote share and the share of seats *actually* won given district vote shares.[21] The unit of measure is a difference of seat shares (ΔS).[22] Hence, metrics that don't compare the difference between actual (or likely) seat shares and some ideal measure some other aspect of a plan. That other quantity may be interesting, but those metrics don't measure partisan advantage directly.

You can use a composite of prior elections as a proxy for a not-yet-held election to infer a *likely* seats–votes curve and use that to estimate $\bar{V}$and $\bar{S}$.[23]

2.2. Ideal Seats–Votes Relationships

The concept of $PA|SV$ is agnostic to what you believe the ideal seats–votes relationship should be, and I won't advocate a specific one here.[24] Nonetheless, to give you a sense of the breadth of possibilities, I'll say a few words about some:

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[19] I use two-party Democratic vote shares by convention. Two-party Replication vote shares are simply $1 - V$.

[20] This is essentially the same formula as (Eguia 2021, 7) with different notation.

[21] Jon Eguia has pointed out: "This notion of the ideal number of seats requires that each vote count equally, regardless of who casts it and where. There are other notions of fairness leading to other 'ideals' about seat outcomes. For instance, under the VRA, or under the Northern Ireland Good Friday Accords, *who* casts each vote matters when determining whether an outcome is fair, since group representation of distinct ethnic groups is deemed essential to fairness. Similarly, under much of the US Democratic tradition – including most first State constitutions -- *where* a vote was cast also mattered, since representation to state assemblies was by county. (Where a vote is cast is still essential to the US Senate and the US Electoral College, of course.) Finally, SCOTUS has explicitly rejected this notion of fairness." Since my focus is on advantage *in the legislative body* as opposed to fairness, I use a seats–votes benchmark.

[22] In programming terms, we call this the return type of the function.

[23] DRA uses Nagle's fractional seat probabilities (2019a) and his method for inferring a seats–votes curve using proportional shift (2019b). PlanScore takes a directionally similar approach different in the details.

[24] I'm <u>not</u> making the argument that proportionality is the ideal seats–votes relationship!

- One is simple proportionality, the $45°$ line on a graph of votes (x-axis) and seats (y-axis). On purely little 'd' democratic principles one might say that this is the ideal seats–votes relationship:

$$S = V \tag{2}$$

Alternatively, in the most real-world terms – only "butts in seats" vote in legislatures! – any deviation from proportionality is an advantage for one party and disadvantage for the other.

- Another is the two times winner's bonus embedded in the formula for the efficiency gap (EG). One can argue that this comports better with how single-member districts actually perform in practice (Stephanopoulos and McGhee 2018). The is the functional form of the EG formula:

$$S = 2V - 0.5 \tag{2}$$

You will see these first two possibilities reflected in the metrics analyzed in §4. You can generalize them to any k-prop line such that $S = kV$, where k is the constant responsiveness (r).

- A variation of (2) is a modified version of the EG:

$$S = 2V^2 \tag{3}$$

where V is the *minority* vote share (Barton 2021).

- You might instead think the non-linear "cube law" is the ideal seats–votes (Tufte 1973; Grofman 1983):[25]

$$\frac{S}{1-S} = (\frac{V}{1-V})^3 \tag{4}$$

This reduces to:

$$S = V^3/(1 - 3V + 3V^2) \tag{5}$$

- You might argue for a winner-takes-all scenario in which the party that gets most of the votes wins *all* the seats, i.e., $S(V) = 1$ for any $V > 0.5$.

[25] Alternatively, the power of three can be replaced by any constant.

- Or you might advocate that how votes should translate into seats should reflect the underlying political geography of and jurisdictional boundaries within a state (Eguia 2021).

In short, there are many possible ideal seats–votes relationships.

2.3. Super-proportionality

To complete my definition of $PA|SV$, I add one minimal constraint:

- Super-proportional outcomes can't favor the minority party.

In graphical terms shown in Fig. 1, what this means is that a valid measure of $PA|SV$ can't classify points D1 or D2 as favoring Republicans or points R1 or R2 as favoring Democrats.[26]

In algebraic terms:

a) If $\bar{V} > 0.5$ and $\bar{S} > \bar{V}$, a valid measure of $PA|SV$ will not indicate that the plan favors Republicans.[27]
b) Similarly, if $1 - \bar{V} > 0.5$ and $1 - \bar{S} > 1 - \bar{V}$, a valid measure of $PA|SV$ will not indicate that the plan favors Democrats.

Plans where a party gets more than half the seats when they receive more than half the votes may or may not be considered biased in favor of that party *depending on which ideal $S(V)$ relationship you choose.* Moreover, this notion of partisan advantage is not like a ridgeline that always falls away to one side or the other. Instead, when an ideal seats–votes relationship—such as 2-proportionality—has S > V when V > 0.5, the region between that ideal and 1-proportionality is deemed to not favor one party or another.

Points D1 & D2 and R1 & R2 are super-proportional outcomes, D3 and R3 are sub-proportional outcomes, and D4 and R4 are anti-majoritarian outcomes.

Note that not all of the seats–votes relationships enumerated in §2.2 satisfy this constraint.

With $PA|SV$ defined, we can sort through the various metrics enumerated in §1.

[26] Some struggle with this idea. It is not an endorsement of 1-proportionality as the ideal seats–votes relationship. It simply says that *if* a plan will likely yield a seat share greater than the vote share for the majority party, then that plan can't be construed as favoring the *minority* party.
[27] To make formulas easier to write, I express percentages in the body text as [0–1] fractions. Except as noted, in tables and figures, I show them as percentages.

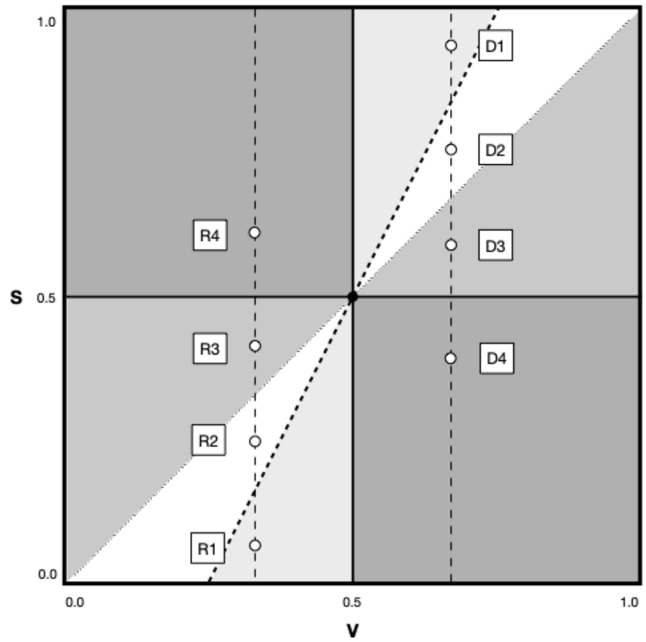

Fig. 1. S(V) space with 2-proportionality ideal (dashed line)

3. Sample Plans

This section presents three groups of partisan "profiles" that I will use to test the ten metrics enumerated above.[28] The 34 plans in them represent a broad cross section of partisan characteristics and, thus, provide a good test suite for these metrics.

3.1. Select 2012 Congressional Plans

The first set of maps are the congressional plans studied in (Nagle and Ramsay, 2021). In terms of their typical statewide two-party vote shares, four lean heavily Democratic (California, Illinois, Massachusetts, and Maryland), three lean heavily Republican (South Carolina, Tennessee, and Texas), and four are nearly balanced politically (Colorado, North Carolina, Ohio, and Pennsylvania).

The plans were drawn in 2011 and the profiles use a composite of 2012 election results.[29] The partisan profiles for these plans may be found in the Supplemental Material.

[28] A partisan profile consists of a statewide vote share and district-by-district vote shares, using Democratic two-party votes by convention.
[29] I call them "2012" plans hereafter.

	$\bar{V}$	$S(\bar{V})$	R	r	δ	LO	MM	α_S	α_V	β	GS	PR	EG	γ
CA	59.2%	72.9%	2.49	1.9	-3	9.2%	-1.0%	-1.8%	-0.6%	3.3%	-1.8%	-13.7%	-4.5%	-5.1%
IL	60.0%	74.7%	2.46	3.1	6	11.7%	2.2%	6.5%	1.8%	2.5%	2.1%	-14.6%	-4.6%	6.9%
MD	59.3%	85.0%	3.74	1.1	-33	4.1%	-1.1%	-5.5%	-1.0%	1.4%	-2.7%	-25.6%	-16.3%	-24.9%
MA	60.0%	95.8%	4.57	2.0	N/A	N/A	2.3%	6.7%	1.0%	-3.0%	2.0%	-35.8%	-25.8%	-26.0%
CO	50.6%	50.8%	1.38	3.8	0	0.3%	1.4%	1.4%	0.4%	1.4%	0.9%	-0.2%	0.4%	1.4%
NC	51.5%	32.3%	-11.81	4.4	37	11.1%	5.7%	21.7%	4.5%	21.0%	6.7%	19.3%	20.8%	24.3%
OH	51.3%	39.0%	-8.44	4.1	22	6.9%	4.2%	15.5%	3.3%	14.9%	4.4%	12.3%	13.6%	16.3%
PA	52.9%	41.7%	-2.83	3.4	24	8.7%	5.7%	16.9%	4.7%	15.5%	5.7%	11.3%	14.2%	18.4%
TN	41.6%	21.1%	3.44	0.5	35	0.6%	4.1%	18.4%	3.2%	-3.2%	6.5%	20.5%	12.1%	25.0%
TX	40.4%	28.9%	2.20	1.2	17	-3.0%	2.1%	5.5%	1.4%	-8.3%	5.4%	11.5%	1.9%	9.9%
SC	43.0%	15.8%	4.88	0.9	48	4.2%	4.7%	15.0%	2.4%	0.7%	5.1%	27.2%	20.2%	28.1%

Table 1 – Metrics for Select 2012 Congressional Plans

The measurements in Table 1 are grouped into five sets of columns:

- The 1st set shows the state, the statewide vote share ($\bar{V}$), and the corresponding likely seat share ($S(\bar{V})$ or simply $\bar{S}$).
- The 2nd set shows the overall responsiveness or winner's bonus for the plan (R) and the responsiveness at the typical statewide vote share (r).
- The 3rd set shows the measures of partisan gerrymandering via packing & cracking: declination (δ), lopsided outcomes (LO), and mean–median difference (MM).
- The 4th set shows the measures of partisan symmetry: seats bias (α_S), votes bias (α_V), geometric seats bias (β), and global symmetry (GS).
- The last set shows the deviation from proportionality (PR), the efficiency gap (EG), and gamma (γ).

All values are percentages, except δ which is an angle in degrees and the two measures of responsiveness, R and r. By convention, positive values indicate Republican advantage and negative values Democratic advantage. The highlighted values will be discussed in §4.

The seats–votes curves for the IL and CO plans above are examples of the D1 and D2 super-proportional outcomes in Fig. 1 below, respectively.

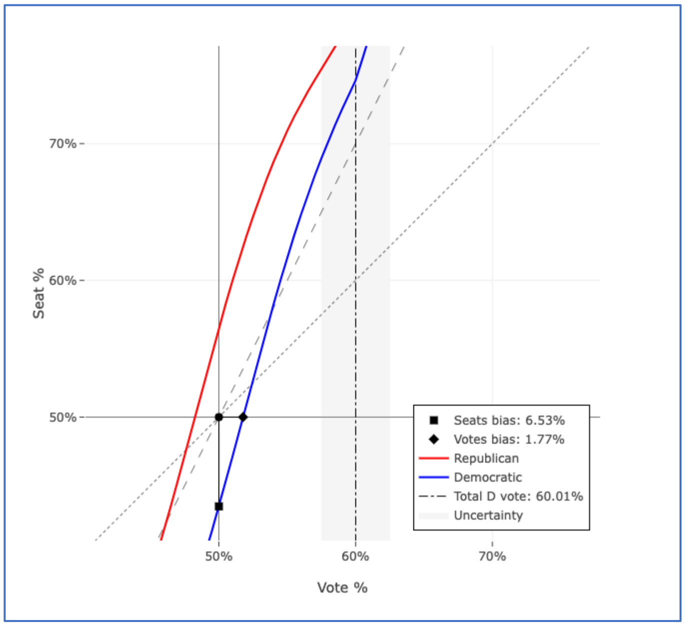

Fig. 2. IL 2012 Congressional

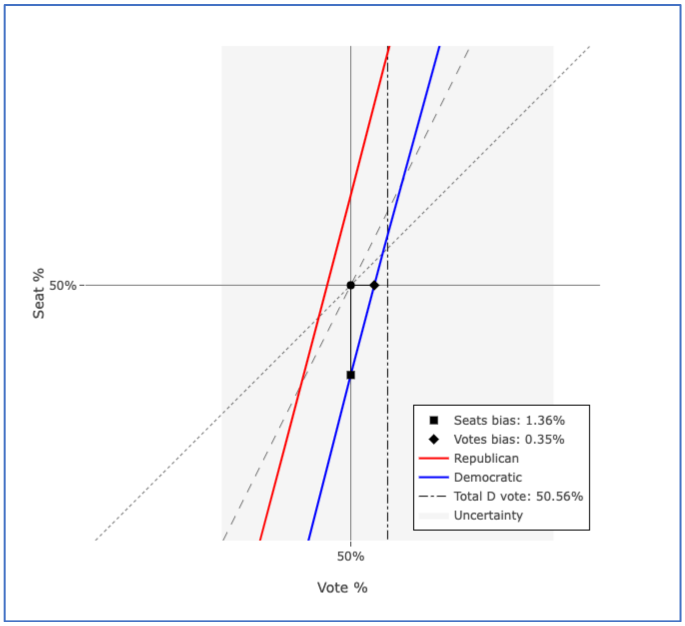

Fig. 3. CO 2012 Congressional

The NC, OH, and PA seats–votes curves (not shown) are all examples of D4 anti-majoritarian outcomes. Seats–votes curves for all the plans may be found in the Supplementary Material.

3.2. Corresponding 2020 Congressional Plans

The second set of plans are the corresponding 2020 maps for the same states but using a composite of 2016–2020 election.[30] The plans are otherwise the same as the 2012 plans, except in NC and PA where courts redrew them mid-decade.

	$\bar{V}$	$S(\bar{V})$	R	r	δ	LO	MM	α_S	α_V	β	GS	PR	EG	γ
CA	64.2%	83.1%	2.32	1.8	-7	13.4%	-2.8%	-2.6%	-0.8%	6.2%	-3.0%	-18.8%	-4.6%	-8.2%
IL	58.2%	66.6%	2.03	2.9	3	9.6%	0.0%	1.2%	0.5%	2.1%	1.4%	-8.4%	-0.3%	7.3%
MD	62.1%	87.0%	3.05	0.3	-44	2.6%	-1.9%	-9.1%	-1.6%	4.3%	-4.3%	-24.9%	-12.8%	-33.9%
MA	61.4%	96.3%	4.07	1.5	N/A	N/A	3.1%	8.4%	1.3%	-2.3%	2.0%	-35.0%	-23.6%	-28.8%
CO	54.5%	58.7%	1.95	1.5	2	4.6%	-2.0%	0.1%	0.1%	3.7%	1.9%	-4.2%	0.2%	-2.0%
NC	49.4%	41.3%	15.03	1.6	11	1.6%	3.2%	7.6%	3.0%	7.5%	1.9%	8.1%	7.5%	7.7%
OH	46.4%	28.2%	6.03	1.9	30	4.1%	3.0%	11.0%	2.4%	7.8%	4.3%	18.2%	14.6%	14.9%
PA	52.8%	51.1%	0.39	2.9	6	3.9%	0.2%	7.4%	2.4%	7.4%	3.7%	1.7%	4.5%	6.9%
TN	36.7%	22.1%	2.09	0.1	32	-1.7%	6.4%	19.1%	3.6%	-9.9%	8.8%	14.5%	1.2%	27.2%
TX	46.3%	40.7%	2.49	3.1	10	0.8%	0.9%	-2.6%	-0.8%	-2.6%	-2.9%	5.6%	1.8%	-2.1%
SC	43.2%	16.1%	4.98	1.0	48	4.7%	2.2%	10.6%	1.6%	1.1%	4.2%	27.1%	20.3%	27.1%

Table 2 – Metrics for Select 2020 Congressional Plans

Table 2 above shows the measurements for these metrics. The seats–votes curves for the TX and PA maps below are examples of the R1 super-proportional and D3 sub-proportional outcomes in Fig. 1, respectively.

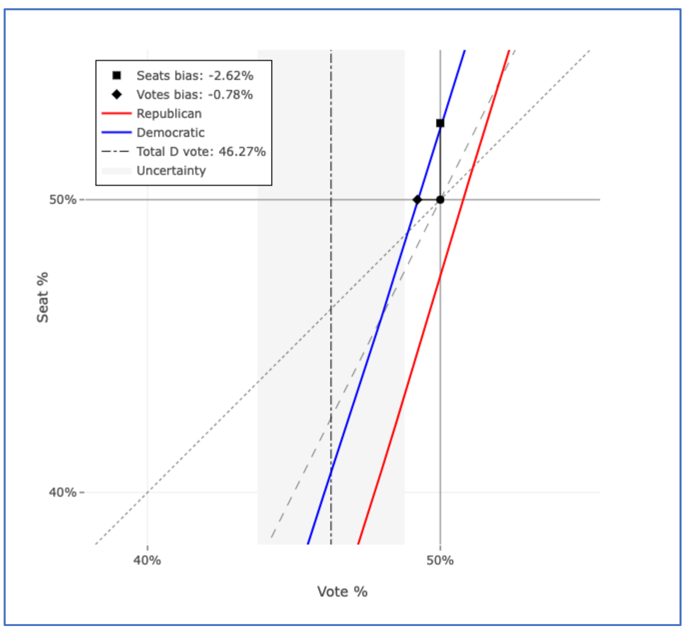

Fig. 4. TX 2020 Congressional

[30] These maps can be found in the Official Maps collection of DRA. The composite is described in "Election Composites" (http://bit.ly/2SeQoDV). The specific elections used in each state's composite are documented in the Supplementary Material.

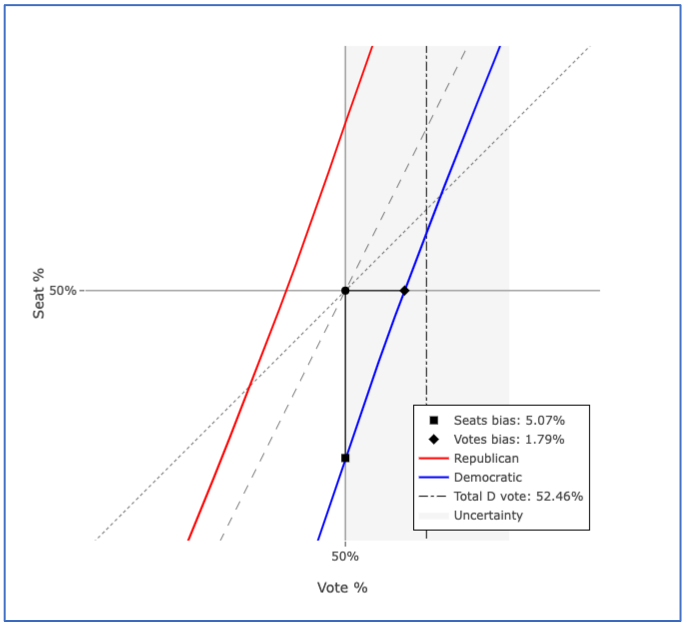

Fig. 5. PA 2020 Congressional

Again, seats–votes curves for all plans may be found in the Supplementary Material.

3.3. Warrington's Hypothetical Plans

Finally, Warrington created a set of 12 hypothetical plans so he could study how well various metrics detected partisan gerrymandering (2019). They are described in Appendix A.

The measurements for these hypothetical profiles are shown in Table 3.

	$\bar{V}$	$S(\bar{V})$	R	r	δ	LO	MM	α_S	α_V	β	GS	PR	EG	γ
1-proportionality	60.0%	60.9%	1.00	0.8	4.9	11.7%	0.0%	0.0%	0.0%	2.6%	-3.6%	0.0%	10.0%	-1.6%
2-proportionality	60.0%	68.0%	1.92	1.6	1.2	10.3%	0.0%	0.0%	0.0%	4.7%	-2.6%	-9.2%	0.8%	-3.1%
3-proportionality	60.0%	80.0%	2.98	2.4	5.9	10.9%	0.0%	0.0%	0.0%	5.2%	-1.7%	-19.8%	-9.8%	-5.5%
Sweep	64.0%	100%	3.46	0.7	N/A	N/A	0.0%	0.0%	0.0%	0.7%	-0.8%	-34.4%	-20.4%	-37.9%
Competitive	52.0%	83.3%	8.32	8.0	N/A	2.6%	1.3%	0.2%	0.0%	0.3%	0.1%	-14.7%	-12.7%	-0.6%
Competitive even	51.0%	70.0%	6.57	5.0	-0.7	0.8%	-0.5%	-1.5%	-0.3%	-1.3%	-0.4%	-5.6%	-4.6%	-1.6%
Uncompetitive	52.3%	60.0%	4.32	0.1	-15.3	-2.9%	-9.2%	-9.6%	-6.7%	-9.2%	-4.6%	-7.6%	-5.3%	-9.8%
Very uncompetitive	52.3%	60.0%	4.35	0.0	-18.1	-8.0%	-19.2%	-10.0%	-12.6%	-10.0%	-8.0%	-7.7%	-5.4%	-10.0%
Cubic	57.0%	80.0%	3.43	1.9	-30.0	-1.6%	0.0%	0.0%	0.0%	0.9%	-1.4%	-17.0%	-10.0%	-10.4%
Anti-majoritarian	44.3%	60.0%	-0.98	1.6	-29.0	-15.4%	-9.2%	-9.5%	-8.0%	-7.8%	-3.4%	-11.3%	-17.0%	-14.9%
Classic	50.0%	30.0%	N/A	1.3	31.2	8.8%	6.0%	16.1%	4.9%	16.1%	5.4%	16.1%	16.1%	16.1%
Inverted	30.0%	30.0%	1.60	5.0	-24.6	-23.6%	6.0%	13.2%	3.2%	-9.0%	8.3%	12.0%	-8.0%	-68.5%

Table 3 – Measurements for Hypothetical Plans

Warrington evaluated the plans using first-past-the-post accounting, as opposed to the fractional seat probabilities method that I use, so some of these scenarios may not report as crisply here. To simplify the values, I show the percentages for whole seats in the $S(\bar{V})$ column.

4. Analysis of Metrics

This section evaluates the ten metrics shown in Tables 1–3 above as potential measures of $PA|SV$, using the sample plans described in the previous section.

4.1. Measures of Partisan Gerrymandering

The first three metrics measure partisan gerrymandering via packing & cracking: declination (δ), lopsided outcomes (LO), and mean–median (MM) (Warrington 2019). While the packing & cracking is an interesting quantity, none of these metrics measure a difference in seat shares. Hence, they aren't measures of $PA|SV$ as I've defined it.[31]

These are their detailed definitions.

Given vote shares by district $(v = v_1\ v_2 \ldots v_N)$, declination ($\delta$) measures a difference in angles:

$$left = (\frac{180}{\pi})\tan^{-1}(S_B - R_B)/(0.5 - V_B)$$

$$right = (\frac{180}{\pi})\tan^{-1}(R_A - S_B)/(V_A - 0.5)$$

$$\delta = right - left \tag{6}$$

where:

$$\bar{S} = \textstyle\sum_1^N p(v_i)$$

$$S_B = \bar{S}/N$$

$$R_A = (1 + S_B)/2$$

$$R_B = S_B/2$$

$$V_A = (\textstyle\sum_1^n p(1-v) * (1-v))/(N - \bar{S})$$

$$V_B = 1 - (\textstyle\sum_1^n p(v) * v)/\bar{S}$$

$p(v) =$ the fractional seat probability for vote share v

Lopsided outcomes (LO) measures a difference in *vote* shares:

$$LO = (0.5 - V_B) - (V_A - 0.5) \tag{7}$$

[31] Even though their units of measure invalidate them as measures of $PA|SV$, their measurements sometimes also violate the constraint that super-proportional outcomes can't favor the minority party, e.g., suggesting that the IL 2012 plan favors Republicans.

Mean–median (MM) also measures a difference in vote shares:

$$MM = \text{mean}(v) - \text{median}(v) \tag{8}$$

These measure partisan gerrymandering via packing & cracking but not $PA|SV$.

4.2. Measures of Partisan Symmetry

The next four metrics measure some aspect of a seats–votes curve: seats bias (α_S), votes bias (α_V), geometric seats bias (β), and global symmetry (GS). Neither votes bias nor global symmetry measure a difference in seat shares, so again they aren't measures of $PA|SV$.

Both seats bias and geometric seats bias *do* measure differences in seat shares, but they sometimes violate the property that super-proportional outcomes can't favor the minority party. Among many others, two examples are illustrative: the IL 2012 and the TX 2020 plans shown in Fig. 2 and Fig. 4 above.

- In both cases, seats bias gets confounded because the seats–votes curves pass the $(0.5{,}0.5)$ center point of symmetry[32] on one side of the $45°$ line of proportionality where $S = V$ before crossing over it and reaching the statewide vote share where one party gets a large majority of the votes and an even bigger share of the seats.

- β also sends the wrong signal for these two plans, because the vote shares where the counterfactual minority seats–votes curves are evaluated – Republican (red) and Democratic (blue), respectively – are well outside the zone of uncertainty around the statewide vote share.

You can think of these failures as indicative of the *principle of locality* in physics.[33] While seats–votes curves show the likely seat share over all theoretically possible vote shares,[34] only a small range around the typical statewide vote share are likely in practice.[35] Absent some theory and some empirical evidence to support it, there is no reason to believe that any metric that measures a seats–votes curve far away from the likely statewide vote share measures anything related to partisan advantage close to it.

These are their detailed definitions of the partisan symmetry metrics.

Seats bias (α_S) measures a difference in seat shares:

$$\alpha_S = (N/2) - S(0.5) \tag{9}$$

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[32] All symmetric seats–votes curves pass through this point.
[33] See https://en.wikipedia.org/wiki/Principle_of_locality.
[34] Because statewide vote shares tend to not fall much outside the range [0.4, 0.6], I only infer the points of the seats–votes curve for the range [0.25, 0.75].
[35] I consider the zone of uncertainty to be the 5% range that brackets the statewide vote share, because the average uncertainty for the seats–votes curves in (Nagle and Ramsay, 2021) was roughly 2%.

Votes bias (α_V) measures a difference in vote shares, vote share required to win 50% of the seats, from the inferred seats–votes curve.

Geometric seats bias (β) measures a difference in seat shares at the statewide vote share $\bar{V}$:

$$\beta = 0.5 * (S(1 - \bar{V}) - S(\bar{V})) \quad (10)$$

Global symmetry (GS) measures the area of asymmetry between the Democratic (blue) and Republican (red) seats–votes curves – basically the geometric seats bias summed over the entire range of vote shares – normalized by the total seats–votes unit square.

4.3. Measures of Seats–Votes Partisan Advantage

The last three metrics share a common underlying functional form:

$$\Delta S = m\,(V - 0.5) - (S - 0.5) \quad (11)$$

where m is an actual or idealized value of responsiveness (r). They all yield differences in seat shares and – with one slight modification – none violate the constraint that super-proportional outcomes can't favor the minority party.

Proportional (PR) measures a difference between the actual (or likely) seat share ($\bar{S}$) and an ideal seat share that matches the statewide vote share ($\bar{V}$):

$$PR = \bar{V} - \bar{S} \quad (12)$$

PR is zero on the $45°$ line where $S = V$, i.e., a responsiveness (r) of one.

In contrast, the efficiency gap (EG) embodies a constant responsiveness (r) of two:

$$EG = 2\,(\bar{V} - 0.5) - (\bar{S} - 0.5) \quad (13)$$

The EG measurements for both CO plans are highlighted in Tables 1 & 2, as is the EG value for the 1-proportionality hypothetical plan in Table 3. Here's why:

The dashed 2-proportionality line in Fig. 1 is where $EG = 0$. Above that line, EG values are negative (indicating Democratic bias), and below that line they are positive (indicating Republican bias). But the EG formula formalizes the notion that a two-times winner's bonus is acceptable, so to say that a point just below the 2-proportionality line immediately favors Republicans is to, in some sense, to contradict the essential EG framework. Hence, I argue that when the winner's bonus (R) is between one and two inclusive – in the white regions in Fig. 1 – a map is not biased *with respect to the efficiency gap ideal.* Hence, I highlight these values.

You can see an example of this in the CO 2012 plan shown in Fig. 3.

The EG thus modified (or interpreted) is a valid measure of $PA|SV$.

Finally, gamma (γ) uses the responsiveness (r) measured at the statewide vote share:

$$\gamma = r\,(\bar{V} - 0.5) - (\bar{S} - 0.5) \tag{14}$$

This formula has the analogous "doesn't acknowledge acceptable bias" issue as EG, when the responsiveness (r) is very high. The IL and CO 2012 and IL and TX 2020 plans are examples which is why those values are highlighted.

Technically, an appropriately interpreted γ is a valid measure of $PA|SV$, but when the measured responsiveness (r) is large almost no plan can be judged as favoring the majority party. Hence, gamma (γ) is of limited practical value.

Conclusion

Most prominent measures of "bias" don't measure seats–votes partisan advantage ($PA|SV$) specifically or aren't reliable for states that are unbalanced politically. Two do measure it directly and are reliable across the full range of statewide vote shares: proportional (PR) and the efficiency gap (EG).

Appendix A. Warrington's Hypothetical Plans

Warrington created a set of 12 hypothetical plans so he could study how well various metrics detected partisan gerrymandering (2019). Each archetypal plan has an associated partisan profile. These are short descriptions of each:

1. A: 1-proportionality – Designed for seats won to track votes received.
2. B: 2-proportionality – Designed such that $EG = 0$ for all vote shares.
3. C: 3-proportionality – Designed with a responsiveness (r) of three for all vote shares.
4. D: Sweep – Designed so that Democrats win all the seats, even though the statewide vote share is only 64%, e.g., like Massachusetts congressional plans.
5. E: Competitive – Even though statewide vote share is nearly even (52%) and there are several very competitive races, they all lean slightly towards Democrats.
6. F: Competitive even – Again, the statewide vote share is nearly even (51%) with several competitive districts. Here though none of them fall "in the 'counterfactual window' (i.e., between the majority party's statewide support and 50%)" (Warrington 2019, 12) and they all still lean Democratic.
7. G: Uncompetitive – This profile models an "uncompetitive election as might arise from a bipartisan gerrymander." (Warrington 2019, 12). The average winning margins for both parties are large. The statewide vote share marginally favors Democrats (52.3%).
8. H: Very uncompetitive – This plan is like the previous example, except that the average winning margins are even more pronounced.
9. I: Cubic – This profile embodies the classic "cube law" seats–votes relationship.
10. J: Anti-majoritarian – Here Democrats get less than half the votes but win more than half the seats.

11. K: Classic – This profile models a classic partisan gerrymander: The statewide vote share is evenly split (50%), but “Republicans win a significant majority through having a number of narrow victories in contrast to their Democratic opponents whose few victories are overwhelming.” (Warrington 2019, 12).
12. L: Inverted – This profile is somewhat complementary to the 2-proportionality example, except the Democratic & Republican vote shares are switched and more extreme.

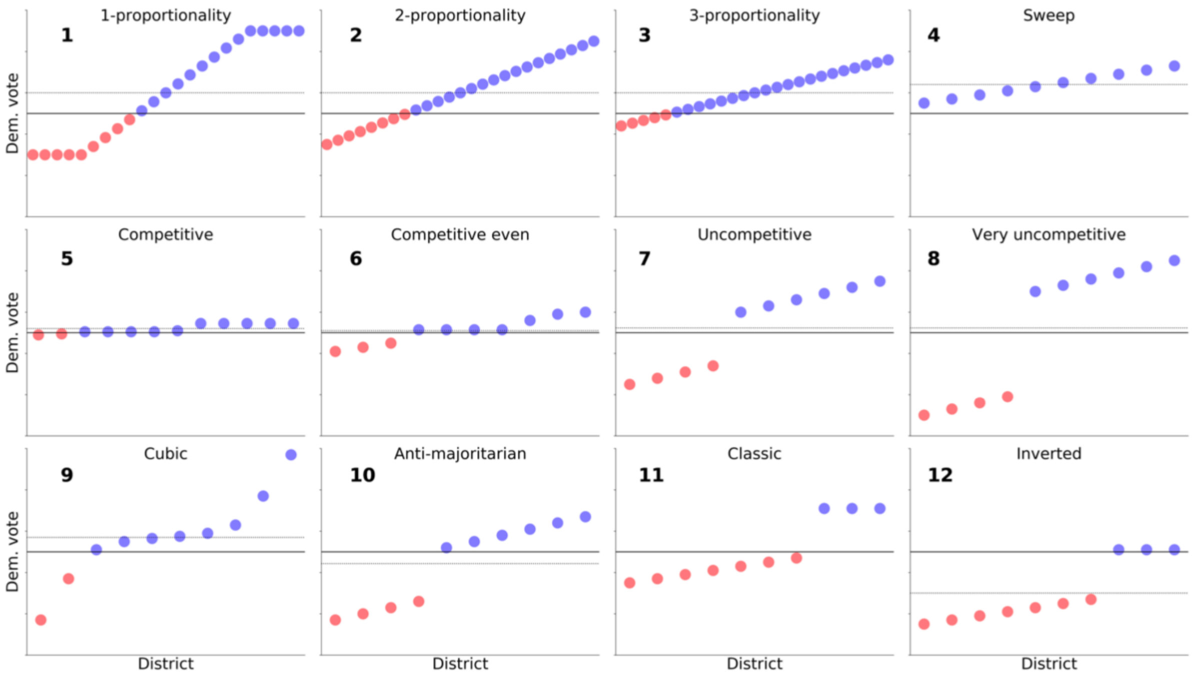

Fig. 6. Rank–Votes Graphs for Warrington’s Hypothetical Plans

The essence of each plan is illustrated by rank–votes graphs shown in Fig. 6. The partisan profiles and seats–votes curves for each may be found in the Supplementary Material.

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