CSC317:Data Structures and Algorithm Design

Homework 2

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Problem 1

a

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + \Theta(n) & n > 1 \end{cases} \tag{1}$$

Which is the same as

$$T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n \tag{2}$$

b

By using induction, we can easily just prove that the time complexity for our recurrence is equal to O(nlogn) and therefore we say that T(n)=O(n log(n)) which implies that $T(n) \leq cnlog(n)$

$$\begin{aligned}
T(n) \leq& 2T(\lfloor \frac{n}{2} \rfloor) + n && \text{Substitute nlog(n)} && (3)\\
T(n) \leq& 2(c(\frac{n}{2})(log(\frac{n}{2}))) + n && \text{Next we simplify} && (4)\\
T(n) \leq& cnlog(\frac{n}{2}) + n && \text{We expand the logarithm} && (5)\\
T(n) \leq& cnlog(n) - cnlog(2) + n && \text{Simply the log(2) since it is equal to 1} && (6)\\
T(n) \leq& cnlog(n) - cn + n && \text{If n} \geq 1 \text{ then n cancels out} && (7)\\
T(n) \leq& cnlog(n) && \text{If n} \geq 1 \text{ then n cancels out} && (8)\\
& && && (9)
\end{aligned}$$

Now we need to check for which values of n is our solution correct therefore we must evaluate for a few cases to examine if our solution might be correct

$$\begin{aligned}
T(1) =& 1! & T(1) = c_1 log(1) & & (10)\\
T(1) =& 1! & T(1) = 0 & \quad \therefore T(1) \neq T(1) & (11)\\
T(2) =& 2T(1) + 2 & T(2) \leq c(2)log(2) & & (12)\\
T(2) =& 4 & 4 \leq 2c & \quad c = 2 & (13)\\
T(3) =& 2T(1) + 3 & T(2) \leq c(3)log(3) & & (14)\\
T(3) =& 5 & 5 \leq 3log(3)c & \quad c = 2 & (15)\\
& & & & (16)
\end{aligned}$$

Therefore T(n)=O(n log(n)) is correct as long as $c \geq 2$

Problem 2

a) $O(f_3(x)) = f_1(x)$

$$
\begin{aligned}
O(f_3(x)) =& f_1(x) && \text{Substituting the actual values} && (17)\\
O(x^3 - 3x^2) =& 2x^2 + 3x && \text{Convert to } 0 \leq f(x) \leq cg(x) && (18)\\
0 \leq 2x^2 + 3x \leq& c(x^3 - 3x^2) && 2x^2 + 3x \text{ is always greater than } 0 && (19)\\
2x^2 + 3x \leq& c(x^3 - 3x^2) && \text{Factor out} && (20)\\
2x^2 + 3x \leq& cx^2(x-3) && \text{Move to the other side} && (21)\\
\frac{2x^2+3x}{x^2(x-3)} \leq& c && \text{Simplify} && (22)\\
\frac{2x+3}{x(x-3)} \leq& c && \text{Simplify} && (23)\\
\frac{2x+3}{x(x-3)} \leq& c && \therefore x \geq 4 \implies n_0 = 4 && (24)\\
& && && (25)
\end{aligned}
$$

Therefore if a constant c exist for a value greater than or equal to n_0 then $O(f_3(x)) = f_1(x)$

b) $O(g(x)) = f_2(x)$

$$
\begin{aligned}
O(g(x)) = f_2(x) & && \text{Substituting the actual values} && (26)\\
O(x^3 - 0.5x^2 + 3x + 1) = 0.5x^2 + 1 & && \text{Convert to } 0 \leq f(x) \leq cg(x) && (27)\\
0 \leq 0.5x^2 + 1 \leq& c(x^3 - 0.5x^2 + 3x + 1) && 0.5x^2 + 1 \text{ is always greater than } 0 && (28)\\
0.5x^2 + 1 \leq& c(x^3 - 0.5x^2 + 3x + 1) && \text{Move to the other side} && (29)\\
\frac{0.5x^2+1}{x^3 - 0.5x^2 + 3x + 1} \leq c & && \therefore c \geq 0 \text{ for } x^3 - 0.5x^2 + 3x + 1 > 0 && (30)\\
& && && (31)
\end{aligned}
$$

Now we need to solve for x

$$
\begin{aligned}
0.5x^2 + 1 \leq& x^3 - 0.5x^2 + 3x + 1 && \text{Simplify it towards one side} && (32)\\
x^2 - 3x - x^3 \leq& 0 && \text{Factor out -x} && (33)\\
-x(x^2 - x + 3) \leq& 0 && \text{Multiply both sides by -1} && (34)\\
x(x^2 - x + 3) \geq& 0 && \therefore x = 0 \text{ and } x > 0 && (35)
\end{aligned}
$$

If we test x=0 we $(0)^3 - 0.5(0)^2 + 3(0) + 1 > 0 \therefore 1 > 0$ which is true meaning
Therefore a constant c exist for a value greater than or equal to n_0 then $O(g_(x)) = f_2(x)$

c) $\Omega(f_2(x)) = f_3(x)$

$$\begin{aligned}
\Omega(f_2(x)) &= f_3(x) && \text{Substituting the actual values} && (36)\\
\Omega(0.5x^2+1) &= x^3-3x^2 && \text{Convert to } 0 \le cg(n) \le f(n) && (37)\\
0 \le c(0.5x^2+1) &\le x^3-3x^2 && 0.5x^2+1 \text{ is always greater than } 0 && (38)\\
c(0.5x^2+1) &\le x^3-3x^2 && \text{Move to the other side} && (39)\\
0 \le c &\le \frac{x^3-3x^2}{0.5x^2+1} && \text{c is true for all values of x} && (40)
\end{aligned}$$

$\therefore \Omega(f_2(x)) = f_3(x)$ is true

d) $\Omega(g(x)) = f_3(x)$

$$\begin{aligned}
\Omega(g(x)) &= f_3(x) && \text{Substituting the actual values} && (41)\\
\Omega(x^3-0.5x^2+3x+1) &= x^3-3x^2 && \text{Convert to } 0 \le cg(n) \le f(n) && (42)\\
0 \le c(x^3-0.5x^2+3x+1) &\le x^3-3x^2 && \text{Move to the other side} && (43)\\
0 \le c &\le \frac{x^3-3x^2}{x^3-0.5x^2+3x+1} && \text{c is true for } x^3-0.5x^2+3x+1>0 && (44)
\end{aligned}$$

Now we need to find the x values where c is true

$$\begin{aligned}
x^3-0.5x^2+3x+1 &= x^3-3x^2 && \text{Multiply by 10} && (45)\\
10x^3-5x^2+30x+10 &= 10x^3-30x^2 && \text{Simplify} && (46)\\
25x^2+30x+10 &= 0 && \text{Use quadratic equation} && (47)\\
x &= -\frac{3}{5} \pm i\frac{1}{5} && \therefore \text{ there is no x values where c is true} && (48)
\end{aligned}$$

$\therefore \Omega(g(x)) \neq f_3(x)$

e) $\Theta(g(x)) = f_3(x)$

$$\Theta(g(x)) = f_3(x) \quad \text{Substituting the actual values} \tag{49}$$

$$\Theta(x^3 - 0.5x^2 + 3x + 1) = x^3 - 3x^2 \quad \text{Convert to } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \tag{50}$$

$$0 \le c_1(x^3 - 0.5x^2 + 3x + 1) \le x^3 - 3x^2 \le c_2(x^3 - 0.5x^2 + 3x + 1) \quad \text{Test the lower bounds first} \tag{51}$$

$$0 \le c_1(x^3 - 0.5x^2 + 3x + 1) \le x^3 - 3x^2 \quad \text{Same result as in the previous question} \tag{52}$$

$$\tag{53}$$

Therefore $\Theta(g(x)) \neq f_3(x)$ since $\Omega(g(x)) \neq f_3(x)$