Homework 3 - Jack Brolin, Abhiram Nallamalli

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P. 2 Let $P = \left\{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1, x \geq 0\right\}$ and consider the vector $x = (0, 0, 1)$. Find the set of feasible directions at x.

Proof. Let $d = (a, b, c); a, b, c \in \mathbb{R}$. Then

$$\begin{aligned} & (0+\theta a) + (0+\theta b) + (1+\theta c) = 1 \\ \Rightarrow \quad & (a+b+c)\theta = 0 \\ \Rightarrow \quad & a+b+c = 0 \end{aligned}$$

and thus $c = -a - b$. Since $x = (0, 0, 1)$, we must require $a, b > 0$ so the shift meets the condition at $x \geq 0$. Furthermore, we must also require $1 + \theta c \leq 0 \Rightarrow 1 \geq \theta(a+b) \Rightarrow \frac{1}{a+b} \geq \theta$. This is always satisfied by $a > 0$ or $b > 0$. We get our final set of feasible directions:

$$d = \{(a, b, -a-b) : a, b \geq 0\}$$

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P. 3 Consider the problem of minimizing $c'x$ over a polyhedron P. Prove the following:

a. A feasible solution x is optimal if and only if $c'd \geq \mathbf{0}$ for every feasible direction d at x.

b. A feasible solution x is the unique optimal solution if and only if $c'd > \mathbf{0}$ for every nonzero feasible direction d at x.

a. *Proof.* Assume $x^\star$ is an optimal solution and let d be any arbitrary feasible direction vector at $x^\star$. Then $x^\star + \theta d \in P$ for some $\theta > 0$. For this new vector, the optimality of $x^\star$ implies

$$\begin{aligned} & c'x^\star \leq c'(x^\star + \theta d) \\ \Rightarrow\ & c'x^\star \leq c'x^\star + \theta c'd \\ \Rightarrow\ & \mathbf{0} \leq \theta c'd \\ \Rightarrow\ & \mathbf{0} \leq c'd \end{aligned}$$

Now assume $c'd \geq \mathbf{0}$ for all feasible directions d at x. Let $y := x + \theta d$ for any d. Then $\theta d = y - x$ and we see

$$\mathbf{0} \leq \theta c'd = \theta c'(y - x) = \theta c'y - \theta c'x \Rightarrow \theta c'y \geq \theta c'x \Rightarrow c'y \geq c'x$$

Showing that x is indeed optimal. ■

b. *Proof.* Assume $x^\star$ is the unique optimal solution and let d be any arbitrary feasible direction vector at $x^\star$. Then $x^\star + \theta d \in P$ for some $\theta > 0$. For this new vector, the optimality of $x^\star$ implies

$$\begin{aligned} & c'x^\star < c'(x^\star + \theta d) \\ \Rightarrow\ & c'x^\star < c'x^\star + \theta c'd \\ \Rightarrow\ & \mathbf{0} < \theta c'd \\ \Rightarrow\ & \mathbf{0} < c'd \end{aligned}$$

Now assume $c'd > \mathbf{0}$ for all nonzero feasible directions d at x. Let $y := x + \theta d$ for any d. Then $\theta d = y - x$ and we see

$$\mathbf{0} < \theta c'd = \theta c'(y - x) = \theta c'y - \theta c'x \Rightarrow \theta c'y > \theta c'x \Rightarrow c'y > c'x$$

Showing that x is indeed the unique optimal solution. ■

P. 4 Let x be an element of the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$. Prove that a vector $d \in \mathbb{R}^n$ is a feasible direction at x if and only if $Ad = 0$ and $d_i \geq 0$ for every i such that $x_i = 0$.

Proof. ($\Rightarrow$) Assume d is a feasible direction at x. We are only concerned with the edges of the P so it must hold that if $x + \theta d \in P; A(x + \theta d) = Ax + \theta Ad = b \Rightarrow Ad = 0$ as $Ax = b, \theta > 0$. Furthermore, from feasibilty of d we get $x + \theta d \geq 0$ and we simply look component wise to get $x_i + \theta d_i \geq 0$ but where $x_i = 0$, it must hold that $d_i \geq 0$ as $\theta > 0$.
($\Leftarrow$) Assume $Ad = \mathbf{0}$ and $d_i \geq 0$ when $x_i = 0$. Note that $Ad = \mathbf{0}$ ensures $A(x + \theta d) = b$ never gets violated so we construct θ such that $x + \theta d \geq \mathbf{0}$ i.e. $x_i + \theta d_i \geq 0$. We have three cases:

- If $x_i = 0$ then $d_i \geq 0$ by assumption
- If $x_i > 0$ and $d_i > 0$, there is no restriction on θ
- If $x_i > 0$ and $d_i < 0$, define $\theta := \min\limits_{i:d_i<0} \left\{ \dfrac{-x_i}{d_i} \right\}$ so $\theta \leq \dfrac{-x_i}{d_i} \Rightarrow \theta d_i \geq -x_i \Rightarrow x_i + \theta d_i \geq 0$ for all relavant i

In each case, $x + \theta d \geq \mathbf{0}$ and thus d is a feasible direction at x. ■

P. 7 Consider a feasible solution x to the standard form problem

$$\begin{aligned} \text{minimize } & c'x \\ \text{subject to } & Ax = b \\ & x \geq 0, \end{aligned}$$

and let $Z = \{i : x_i = 0\}$. Show that x is an optimal solution if and only if the linear programming problem

$$\begin{aligned} \text{minimize } & c'd \\ \text{subject to } & Ad = 0 \\ & d_i \geq 0, \quad i \in Z, \end{aligned}$$

has an optimal cost of zero.

Proof. Suppose x is an optimal solution. **P. 3** tells us that $c' \geq 0$ for all feasible direction vectors d at x. Clearly, the min of this is 0. For the sufficient condition, suppose $c'd$ has an optimal cost of zero. From **P. 4** we know d is a feasible direction at x. Furthermore, $c'd \leq 0$ as the optimal cost is 0. Again, **P. 3** comes to the rescue with the result thatthis condition is sufficent for x to be optimal. ■

P. 9 Consider the problem

$$\begin{aligned} \text{minimize} \quad & -2x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 2 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We first find the equivalent standard form representation:

$$\begin{aligned} \min\ & -2x_1 - x_2 \\ \text{s.t. } & x_1 - x_2 + x_3 = 2 \\ & x_1 + x_2 + x_4 = 6 \\ & x \geq 0 \end{aligned}$$

and we can immediately see that $x = (0, 0, 2, 6)$ is a basic feasible solution and so we let $B(1) = 3$ and $B(2) = 4$ giving a basis matrix of I. Filling in the first tableau is trivial, as the necessary conditions arise very naturally:

0	-2	-1	0	0
2	1	-1	1	0
6	1	1	0	1

Both relevant values in the 0^{th} are negative so we make an arbitrary choice of the second column x_2 and get $u = (-1, 1)$. We only need to consider the ratio between $\frac{x_{B(2)}}{u_2}$ as $u_1 < 0$ to get $\ell = 2$. This gives our pivot element and choice of vector to enter the basis. Computing our version of Echelon Reduction we get the next tableau:

6	-1	0	0	1
8	2	0	1	1
6	1	1	0	1

We only have one choice for our pivot column and we get $u = (2, 1)$ with ratios

$$\begin{aligned} \frac{x_{B(1)}}{u_1} &= \frac{8}{2} = 4 \\ \frac{x_{(2)}}{u_2} &= \frac{6}{1} = 6 \end{aligned}$$

So we let $\ell = 1$ which corresponds to x_3. Now, with 2 as our pivot variable, we can run the next round of reductions and get the last tableau:

10	0	0	$1/2$	$3/2$
4	1	0	$1/2$	$1/2$
2	0	1	$1/2$	$3/2$

With no more negative values in the reduced cost vector, we are sure that we are done and have achieved the optimal solution. At every step we can do a confidence check and compute the values of the needed vector and make sure that each is a basic feasible solution. We know at all times what row corresponds to what component by considering the components we choice to enter the basis, but in practice we just look at the unit vector in specific columns to get the corresponding components. This allows us to see our final answer of $x = (4, 2)$ which is chosen to minimize the above cost vector.