Homework 4 - Math 525

Jack Brolin

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P. 1 Consider the simplex method applied to a standard form problem and assume that the rows of the matrix A are linearly independent. For each of the statements that follow, give either a proof or a counterexample.

a. An iteration of the simplex method may move the feasible solution by a positive distance while leaving the cost unchanged.

b. A variable that has just left the basis cannot reenter in the very next iteration.

c. A variable that has just entered the basis cannot leave in the very next iteration.

d. If there is a nondegenerate optimal basis, then there exists a unique optimal basis.

e. If x is an optimal solution found by the simplex method, no more than m of its components can be positive, where m is the number of equality constraints.

P. 3 Solve completely (i.e., both Phase I and Phase II) via the simplex method the following problem:

$$\begin{array}{rl} \text{minimize} & 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 \\ \text{subject to} & x_1 + 3x_2 + 4x_4 + x_5 = 2 \\ & x_1 + 2x_2 - 3x_4 + x_5 = 2 \\ & -x_1 - 4x_2 + 3x_3 = 1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

We see 4 iterations of Phase I. We construct the first tableau with the introduction of auxiliary variables:

-5	-1	-1	-3	-1	-2	0	0	0
2	1	3	0	4	1	1	0	0
2	1	2	0	-3	1	0	1	0
1	-1	-4	3	0	0	0	0	1

We get 3 as the pivot variable with x_3 entering the basis and x_6 exiting. Performing the necessary operations we get:

-4	-2	-5	0	1	-2	0	1
2	1	3	0	4	1	0	0
2	1	2	0	-3	1	1	0
$1/3$	$-1/3$	$-4/3$	1	0	0	0	$1/3$

Note that we completely removed x_6 from the tableau as it no longer serves any purpose. We now let x_2 enter the basis with x_7 exiting:

$-2/3$	$-1/3$	0	0	$17/3$	$-1/3$	0
$2/3$	$1/3$	1	0	$4/3$	$1/3$	0
$2/3$	$1/3$	0	0	$-17/3$	$1/3$	1
$11/9$	$1/9$	0	1	$16/9$	$4/9$	0

Now we let x_1 enter with x_7 exiting. Thus we get the final tableau:

0	0	0	0	0	0
0	0	1	0	0	0
2	1	0	0	-17	1
1	0	0	1	$11/3$	$1/3$

and the basic feasible solution is $x = (2, 0, 1, 0, 0)$.

Phase II

We copy over the same tableau and calculate the new reduced cost and cost to get :

7	2	3	3	-2	3
2	1	0	0	-17	0
0	0	1	0	7	0
1	0	0	1	$11/3$	$1/3$

We do one more iteration to get:

7	2	$^{23}/_{7}$	3	0	3
2	1		0	0	1
0	0		0	1	0
1	0		1	0	0

We now see that there are no more negative values in c so the optimal solution is $x = (2, 0, 1, 0, 0)$. Note that we can leave some values blank as calculating them does not give any information useful to the problem.

P. 4 While solving a standard form problem, we arrive at the following tableau, with x_3, x_4, and x_5 being the basic variables:

-10	δ	-2	0	0	0
4	-1	η	1	0	0
1	α	-4	0	1	0
β	γ	3	0	0	1

The entries $\alpha, \beta, \gamma, \delta, \eta$ in the tableau are unknown parameters. For each one of the following statements, find some parameter values that will make the statement true:

a. The current solution is optimal and there are multiple optimal solutions.

b. The optimal cost is $-\infty$.

c. The current solution is feasible but not optimal.

a.

b. Note that $\beta \geq 0$ for x to be feasible. We then have to find some u such that no component is positive. This, as seem from section 3.2, ensures that the optimal cost is $-\infty$. Thus, we choose $\delta \leq 0$ and $\alpha, \gamma < 0$. η is free. A possible tableau could be:

-10	-1	-2	0	0	0
4	-1	7	1	0	0
1	-1	-4	0	1	0
1	-1	3	0	0	1

c. To be feasible $\beta \geq 0$. The rest of the variables are free as long as at least one $\delta, \gamma, \alpha \geq 0$ to avoid an optimal cost of $-\infty$. We will require $\alpha, \beta, \gamma, \delta, \eta > 0$ giving a possible tableau of:

-10	5	-2	0	0	0
4	-1	1	1	0	0
1	1	-4	0	1	0
5	1	3	0	0	1

P. 6 Consider the following linear programming problem with a single constraint:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} c_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} a_i x_i = b \\ & x_i \geq 0, \quad i = 1, \ldots, n. \end{aligned}$$

a. Derive a simple test for checking the feasibility of this problem. (Hint: Discuss when $b = 0, b > 0$ and $b < 0$)

b. Assuming that the optimal cost is finite, develop a simple method for obtaining an optimal solution directly.

a. As instructed, we break b into 3 cases:

* $b = 0$: This is trivial as $x = \mathbf{0}$ is always a solution
* $b > 0$: We only require some $a_j > 0$ as we can choose x such that $x_i = 0 \quad \forall i \neq j$ and $x_j = \frac{b}{a_j}$.
* $b < 0$: Similarly, this is feasible of some $a_j < 0$. Again, we can always choose some x such that $x_i = 0 \forall i \neq j$ and $x_j = \frac{b}{a_j}$.

b. Noticing that we only have one constraint, we note that the solution will only have 1 non-zero value. To find this value, consider $\frac{b}{a_i}$. The smallest of these ratios is the optimal cost if we let all other values be 0 and $a = \mathbf{1}$. Because $a \neq \mathbf{1}$ in most cases, we account for this by multiplying c_i to the ratio and considering the smallest. That is, $\frac{b \cdot c_i}{a_i}$. The smallest of this gives an optimal solution. That is, $x_i = \frac{b}{a_j}$ and $x_j = 0 \forall j \neq i$ where i is the smallest of the $\frac{b \cdot c_i}{a_i}$.

P. 8 Consider the following optimization problem (P) : find a vector $x \in \mathbb{R}^n$ that satisfies $Ax = 0$ and $x \geq 0$, and such that the number of positive components of x is maximized. Let (P') be the linear program defined as:

$$\begin{aligned} \text{maximize } & \sum_{i=1}^{n} y_i \\ \text{subject to } & A(z+y) = 0 \\ & z, y \geq 0 \\ & y_i \leq 1, \quad i = 1, \ldots, n. \end{aligned}$$

Show that (P') can be used to solve (P). (Hint: You can show that (P) and (P') are equivalent - you must specify how to map a feasible solution of (P) to a feasible solution of (P') with greater or equal cost, and viceversa.)