Homework 4 - Math 525

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P. 1 Consider the simplex method applied to a standard form problem and assume that the rows of the matrix A are linearly independent. For each of the statements that follow, give either a proof or a counterexample.

a. An iteration of the simplex method may move the feasible solution by a positive distance while leaving the cost unchanged.

b. A variable that has just left the basis cannot reenter in the very next iteration.

c. A variable that has just entered the basis cannot leave in the very next iteration.

d. If there is a nondegenerate optimal basis, then there exists a unique optimal basis.

e. If x is an optimal solution found by the simplex method, no more than m of its components can be positive, where m is the number of equality constraints.

P. 3 Solve completely (i.e., both Phase I and Phase II) via the simplex method the following problem:

$$\begin{array}{rc} \text{minimize} & 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 \\ \text{subject to} & x_1 + 3x_2 + 4x_4 + x_5 = 2 \\ & x_1 + 2x_2 - 3x_4 + x_5 = 2 \\ & -x_1 - 4x_2 + 3x_3 = 1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

We see 4 iterations of Phase I. We construct the first tableau with the introduction of auxiliary variables:

-5	-1	-1	-3	-1	-2	0	0	0
2	1	3	0	4	1	1	0	0
2	1	2	0	-3	1	0	1	0
1	-1	-4	3	0	0	0	0	1

We get 3 as the pivot variable with x_3 entering the basis and x_6 exiting. Performing the necessary operations we get:

-4	-2	-5	0	1	-2	0	1
2	1	3	0	4	1	0	0
2	1	2	0	-3	1	1	0
$^1/_3$	$-^1/_3$	$-^4/_3$	1	0	0	0	$^1/_3$

Note that we completely removed x_6 from the tableau as it no longer serves any purpose. We now let x_2 enter the basis with x_7 exiting:

$-^2/_3$	$-^1/_3$	0	0	$^{17}/_3$	$-^1/_3$	0
$^2/_3$	$^1/_3$	1	0	$^4/_3$	$^1/_3$	0
$^2/_3$	$^1/_3$	0	0	$-^{17}/_3$	$^1/_3$	1
$^{11}/_9$	$^1/_9$	0	1	$^{16}/_9$	$^4/_9$	0

Now we let x_1 enter with x_7 exiting. Thus we get the final tableau:

0	0	0	0	0	0
0	0	1	0	0	0
2	1	0	0	-17	1
1	0	0	1	$^{11}/_3$	$^1/_3$

and the basic feasible solution is $x = (2, 0, 1, 0, 0)$.

Phase II

We copy over the same tableau and calculate the new reduced cost and cost to get :

7	2	3	3	-2	3
2	1	0	0	-17	0
0	0	1	0	7	0
1	0	0	1	$^{11}/_3$	$^1/_3$

We do one more iteration to get:

7	2	$^{23}/_{7}$	3	0	3
2	1		0	0	1
0	0		0	1	0
1	0		1	0	0

We now see that there are no more negative values in c so the optimal solution is $x = (2, 0, 1, 0, 0)$. Note that we can leave some values blank as calculating them does not give any information useful to the problem.

P. 4 While solving a standard form problem, we arrive at the following tableau, with x_3, x_4, and x_5 being the basic variables:

-10	δ	-2	0	0	0
4	-1	η	1	0	0
1	α	-4	0	1	0
β	γ	3	0	0	1

The entries $\alpha, \beta, \gamma, \delta, \eta$ in the tableau are unknown parameters. For each one of the following statements, find some parameter values that will make the statement true:

a. The current solution is optimal and there are multiple optimal solutions.

b. The optimal cost is $-\infty$.

c. The current solution is feasible but not optimal.

P. 6 Consider the following linear programming problem with a single constraint:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} c_i x_i \\
\text{subject to} \quad & \sum_{i=1}^{n} a_i x_i = b \\
& x_i \geq 0, \quad i = 1, \ldots, n.
\end{aligned}
$$

a. Derive a simple test for checking the feasibility of this problem. (Hint: Discuss when $b = 0, b > 0$ and $b < 0$)

b. Assuming that the optimal cost is finite, develop a simple method for obtaining an optimal solution directly.

P. 8 Consider the following optimization problem (P) : find a vector $x \in \mathbb{R}^n$ that satisfies $Ax = 0$ and $x \geq 0$, and such that the number of positive components of x is maximized. Let (P') be the linear program defined as:

$$\begin{aligned} \text{maximize } & \sum_{i=1}^{n} y_i \\ \text{subject to } & A(z+y) = 0 \\ & z, y \geq 0 \\ & y_i \leq 1, \quad i = 1, \ldots, n. \end{aligned}$$

Show that (P') can be used to solve (P). (Hint: You can show that (P) and (P') are equivalent - you must specify how to map a feasible solution of (P) to a feasible solution of (P') with greater or equal cost, and viceversa.)