Homework 5

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P. 1 Consider the linear programming problem:

$$
\begin{aligned}
\min\ & x_1 - x_2 \\
\text{s.t. } & 2x_1 + 3x_2 - x_3 + x_4 \leq 0 \\
& 3x_1 + x_2 + 4x_3 - 2x_4 \geq 3 \\
& -x_1 - x_2 + 2x_3 + x_4 = 6 \\
& x_1 \leq 0 \\
& x_2, x_3 \geq 0.
\end{aligned}
$$

Write down the corresponding dual problem.

Following the table from section 4.2 of the text, the corresponding dual problem is as follows:

$$
\begin{aligned}
\text{Max } & 3p_2 + 6p_3 \\
\text{s.t. } & 2p_1 + 3p_2 - p_3 \geq 1 \\
& p_1 + p_2 - p_3 \leq -1 \\
& -p_1 + 4p_2 + 2p_3 \leq 0 \\
& p_1 - 2p_2 - p_3 = 0 \\
& p_1 \leq 0 \\
& p_2 \geq 0 \\
& p_3 \text{ is free}
\end{aligned}
$$

P. 2 Consider a linear programming problem in standard form and assume that the rows of A are linearly independent. For each one of the following statements, provide either a proof or a counterexample.

a. Let x^* be a basic feasible solution. Suppose that for every basis corresponding to x^*, the associated basic solution to the dual is infeasible. Then, the optimal cost must be strictly less than $c'x^*$.

b. The dual of the auxiliary primal problem considered in Phase I of the simplex method is always feasible.

c. Let p_i be the dual variable associated with the i^{th} equality constraint in the primal. Eliminating the i^{th} primal equality constraint is equivalent to introducing the additional constraint $p_i = 0$ in the dual problem.

d. If the unboundedness criterion is satisfied when we solve the primal with the simplex method, then the dual problem is infeasible.

a. True. We know that the optimal cost of the dual is less than or equal to the optimal most of the primal. If $x^\star$ is a basic feasible solution of the primal but infeasible for the dial, the its cost me be greater than the cost of the dual.

b. True. Since the primal auxiliary problem is always feasible, the dual is feasible too.

c. True. Constraints are of the form $\sum p_i a_i$, so setting $p_i = 0$ effectively eliminates it.

d. True. If the primal cost goes to $-\infty$ than the dual is infeasible.

P. 3 Let A be a symmetric square matrix. Consider the linear programming problem:

$$\begin{aligned} \text{minimize } & c'x \\ \text{subject to } & Ax \geq c \\ & x \geq 0 \end{aligned}$$

Prove that if x^* satisfies $Ax^* = c$ and $x^* \geq 0$, then x^* is an optimal solution.

Proof. Consider the dual problem:

$$\begin{aligned} \max\ & p'b \\ \text{s.t. } & p'A \leq c' \\ & p \geq 0 \end{aligned}$$

Let $x^\star$ satisfy $Ax^\star = c$ and $x^\star \geq 0$. Since $A \in GL_n(\mathbb{R})$, basic linear algebra tells us $p'A \leq c' = A'p \leq c$. But A is a symmetric matrix, so $A' = A$ and thus $A'p \leq c = Ap \leq c$ with $p \geq 0$. Now if we let $p^\star = x^\star$ we see $p^{\star\prime} = c^\star x$ so by corollary 4.2 $x^\star$ and $p^\star$ are optimal solutions. ∎

P. 7 Consider the following pair of linear programming problems:

$$\begin{array}{rlcrl} \text{minimize} & c'x & \qquad & \text{maximize} & p'b \\ \text{subject to} & Ax \geq b & & \text{subject to} & p'A \leq c' \\ & x \geq 0, & & & p \geq 0. \end{array}$$

Suppose at least one of these two problems has a feasible solution. Prove that the set of feasible solutions to at least one of the two problems is unbounded.

Proof. We are proving Clark's Lemma.

1. If the primal set us unbounded, the dual can be either bounded or unbounded.
2. If the primal set is bounded, consider the problem:

$$\begin{array}{rlcrl} \min & -c'x & \qquad & \max & p'b \\ \text{s.t.} & Ax \geq b & & \text{s.t.} & p'A \leq -c' \\ & x \geq 0, & & & p \geq 0. \end{array}$$

Say w is a solution to this problem, so $w'A \leq c'$ and $w \geq 0$. Say d is a solution to the original dual problem and let $\theta \geq 0$. If we take $d + \theta w$ and show that this is a feasible solution to the original dual problem, we get that the dual is unbounded. Indeed, $d + \theta w \geq 0$ as $d, \theta, w \geq 0$ and $d + \theta w' A \leq c'$ since $(\theta w)A \leq -c' \Rightarrow (d + \theta w)'A \leq c'$.

■

P. 10 Consider the problem:

$$\begin{aligned} &\min \max_{i=1,\ldots,m} (a_i'x - b_i) \\ &\text{s.t. } x \in \mathbb{R}^n. \end{aligned}$$

Let v be the optimal value of the above optimization problem, let A be the $m \times n$ matrix whose rows are $a_1', \ldots, a_m'$, and let b be the vector whose components are $b_1, \ldots, b_m$.

a. Consider any vector $p \in \mathbb{R}^m$ such that $p'A = 0', p \geq 0$, and $\sum_{i=1}^m p_i = 1$. Show that $-p'b \leq v$.

b. In order to obtain the best possible lower bound of the form obtained in part (a), we consider the linear program:

$$\begin{aligned} \max\ & -p'b \\ \text{s.t. } & p'A = 0' \\ & \sum_{i=1}^m p_i = 1 \\ & p \geq 0. \end{aligned}$$

Show that the optimal cost of the above linear program is v.

a. *Proof.* We first convert the problem into a linear programming problem following the same instructions we used in chapter 1. Thus, the given problem is equivalent to

$$\begin{aligned} \min\ & z \\ \text{s.t. } & z \geq a_1'x - b_1 \\ & \ldots \\ & z \geq a_m'x - b_m \end{aligned}$$

Then we can get the corresponding dual:

$$\begin{aligned} \max\ & -p'b \\ \text{s.t. } & p'A = c' = 0 \\ & \sum_{i=1}^m p_1 = 1 \\ & p \geq 0 \end{aligned}$$

By weak duality, $p'b \leq c'x = v$. ■

b. *Proof.* From the above construction we can see that we have arrived at the dual, and thus string duality implies that the optimal costs are the same. i.e. v for the given problem. ■