Homework 5

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P. 1 Consider the linear programming problem:

$$\min\ x_1 - x_2$$
$$\text{s.t.}\ 2x_1 + 3x_2 - x_3 + x_4 \leq 0$$
$$3x_1 + x_2 + 4x_3 - 2x_4 \geq 3$$
$$-x_1 - x_2 + 2x_3 + x_4 = 6$$
$$x_1 \leq 0$$
$$x_2, x_3 \geq 0.$$

Write down the corresponding dual problem.

P. 2 Consider a linear programming problem in standard form and assume that the rows of A are linearly independent. For each one of the following statements, provide either a proof or a counterexample.

    a. Let x^* be a basic feasible solution. Suppose that for every basis corresponding to x^*, the associated basic solution to the dual is infeasible. Then, the optimal cost must be strictly less than $c'x^*$.

    b. The dual of the auxiliary primal problem considered in Phase I of the simplex method is always feasible.

    c. Let p_i be the dual variable associated with the ith equality constraint in the primal. Eliminating the ith primal equality constraint is equivalent to introducing the additional constraint $p_i = 0$ in the dual problem.

    d. If the unboundedness criterion is satisfied when we solve the primal with the simplex method, then the dual problem is infeasible.

P. 3 Let A be a symmetric square matrix. Consider the linear programming problem:

$$\text{minimize } c'x$$
$$\text{subject to } Ax \geq c$$
$$x \geq 0$$

Prove that if x^* satisfies $Ax^* = c$ and $x^* \geq 0$, then x^* is an optimal solution.

P. 7 Consider the following pair of linear programming problems:

$$
\begin{array}{ll}
\text{minimize } c'x & \qquad \text{maximize } p'b \\
\text{subject to } Ax \geq b & \qquad \text{subject to } p'A \leq c' \\
\qquad\quad x \geq 0, & \qquad\qquad\quad p \geq 0.
\end{array}
$$

Suppose at least one of these two problems has a feasible solution. Prove that the set of feasible solutions to at least one of the two problems is unbounded.

P. 10 Consider the problem:

$$\min_{i=1,\ldots,m} \max \ (a_i'x - b_i)$$
$$\text{s.t. } x \in \mathbb{R}^n.$$

Let v be the optimal value of the above optimization problem, let A be the $m \times n$ matrix whose rows are $a_1', \ldots, a_m'$, and let b be the vector whose components are $b_1, \ldots, b_m$.

   a. Consider any vector $p \in \mathbb{R}^m$ such that $p'A = 0', p \geq 0$, and $\sum_{i=1}^m p_i = 1$. Show that $-p'b \leq v$.

   b. In order to obtain the best possible lower bound of the form obtained in part (a), we consider the linear program:

$$\max -p'b$$
$$\text{s.t. } p'A = 0'$$
$$\sum_{i=1}^m p_i = 1$$
$$p \geq 0.$$

   Show that the optimal cost of the above linear program is v.