Exercise Sheet 5: Interactive Proofs in Set Theory

1 Introduction

1.1 Purpose

The purpose of this exercise is to make you familiar with the practice of interactive proofs with the Rodin Platform on Set Theory.

1.2 Your Task

We distribute to you a Rodin development named "05_set": it contains 7 contexts (named set_0 to set_6), each of which with one theorem. You will be asked to prove these theorems by using 3 successive *tactic profile*: "set_1", "set_2", and "set_3". Each of them is a slight extension of the previous one. Download this material. Use the tactic profiles as follows:

1. Use profile "set_1" for proving theorems in contexts set_0 and set_1
2. Use profile "set_2" for proving the theorem in context set_2
3. Use profile "set_3" for proving theorems in the remaining contexts set_3 to set_6.

In section 4, we give some hints to help discharging some of the proofs in contexts set_3 to set_6.

2 Some Red Operator Buttons

In this section, we present more red operator buttons besides the ones introduced in previous exercises.

2.1 Other Red Operators in the Goal or in the Hypotheses (as labeled in the Rodin Platform)

Remove membership (incomplete)

$$E \in A \cup B \qquad == \quad E \in A \ \lor \ E \in B$$

$$E \in A \cap B \qquad == \quad E \in A \ \land \ E \in B$$

$$E \in A \setminus B \qquad == \quad E \in A \ \land \ E \notin B$$

$$E \in \varnothing \qquad == \quad \bot$$

$$r \in S \leftrightarrow T \qquad == \quad r \subseteq S \times T$$

$$E \mapsto F \in S \times T \quad == \quad E \in S \ \land F \in T$$

$$E \mapsto F \in r^{-1} \qquad == \quad F \mapsto E \in r$$

$$E \in r[A] \qquad == \quad \exists x \cdot x \in A \ \land \ x \mapsto E \in r$$

$$E \mapsto F \in f \,;\, g \quad == \quad \exists x \cdot E \mapsto x \in f \ \land \ x \mapsto F \in g$$

$$E \mapsto F \in \mathrm{id} \qquad == \quad E = F$$

Remove inclusion

$$A \subseteq B \ == \ \forall x \cdot x \in A \Rightarrow x \in B$$

3 Tactic Profiles

3.1 Profile "set_1"

The tactic profile "set_profile_1" is a slight extension of the tactic profile "prd_3" used in the previous exercise.

3.2 Profile "set_2"

This profile extends "set_profile_1" by adding an elementary tactic for the goal and an elementary tactic for the hypotheses. It corresponds to the removing of membership and the removing of inclusion. Their names are "Remove all Membership/Inclusion in goal" and "Remove all Membership/Inclusion in hypotheses".

3.3 Profile "set_3"

This profile extends "set_profile_2" by taking account of equalities

4 Hints

In all cases, prune the tree, then push the green "Run auto provers" button. Then you are left to perform a predicate calculus proof as in previous exercise. The hints are given to perform these predicate calculus proofs. These hints are very precise. Follow them carefully and you'll succeed. After that, try to REDO each proof without looking at the hints.

4.1 Hint for the Theorem in the Context "set_3"

1. Instantiate x with y and $x0$ with $x0$ in the second universal hypothesis.
2. Replace $x1$ by x in the hypothesis $x1 \mapsto x0 \in r$. For this push the red button $x1$.
3. Instantiate x with x in the existential goal.
4. Instantiate x with x and $x0$ with y in the universal hypothesis.

4.2 Hint for the Theorem in the Context "set_4"

1. Instantiate x with x and $x0$ with y in the second universal hypothesis.
2. Instantiate $x0$ with x in the existential goal.
3. Instantiate x with x in the last universal hypothesis.
4. Instantiate $x0$ with x in the second universal hypothesis.
5. Instantiate x with x in the last universal hypothesis.
6. Replace $x0$ by y in the hypothesis $x0 \mapsto x \in r$. For this push the red button $x0$.
7. Instantiate x with y and $x0$ with x in the first universal hypothesis.

4.3 Hint for the theorem in context "set_5"

1. Use profile "set_profile_3"
2. Push the blue button "ct" on the hypothesis $\neg\, y \mapsto z \in r$.
3. Instantiate x with y and $x0$ with z in the second universal hypothesis.
4. Replace $x3$ by x, $x1$ by y and $x2$ by x in three hypotheses. Push corresponding red buttons.
5. Get universal hypothesis by pushing "Cash Hypothesis" button on the right of the screen. Then instantiate x with y and $x0$ with x in the last Cache universal hypothesis.
6. Instantiate $x0$ with $x0$ in the existential goal.
7. Instantiate x with x and $x0$ with $x0$ in the universal hypothesis.
8. Instantiate x with x in the existential goal.

4.4 Hint for the theorem in context "set_6"

1. Instantiate x with $x2$ and $x0$ with $x1$ and $x1$ with $x3$ in the universal hypothesis.
2. Replace $x3$ by $x1$ in the hypothesis $x0 \mapsto x3 \in r$. For this push the red button $x3$.
3. Instantiate $x1$ with $x1$ in the existential goal.