Exercise Sheet 3: Interactive Proofs in the Propositional Calculus

1 Introduction

1.1 Purpose

The purpose of this exercise is to make you familiar with the practice of interactive proofs with the Rodin Platform on the Proposition Calculus.

1.2 Your Task

We distribute to you a Rodin development named `03_prop1.zip`: it contains 12 contexts, each of which with one theorem. You will be asked to prove these theorems by using 8 successive, so-called, *tactic profile*: `prp_1`, ...,`prp_8`. Each of them is a more powerful extension of the previous one. These tactic profiles are also distributed to you. Load them as indicated in the class.

A tactic profile is made of a sequence of *elementary tactics*. Each elementary tactic is made of one (or more) *inference rule* or *re-writing rule*: these are described in corresponding sections where we present the various profiles.

Try first to prove each theorem by *using the first tactic profile* `prp_1` only. Once this is done, then try more powerful tactic profiles.

2 Some Red Operator Buttons

When a tactic profile cannot discharge a theorem, you are asked to use the *red operator buttons* that are visible on the prover user interface. A red operator button is usually made of several elementary tactics (defined in a menu) that you can invoke interactively by depressing the corresponding button in the menu. Some of these elementary tactics are described in the following subsections.

2.1 Goal Red Operators (as labeled in the Rodin Platform)

Deduction

$$\frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{G}}{\mathbf{H} \vdash \mathbf{P} \Rightarrow \mathbf{G}}$$

Conjunction Introduction

$$\frac{\mathbf{H} \vdash \mathbf{P} \qquad \mathbf{H} \vdash \mathbf{Q}}{\mathbf{H} \vdash \mathbf{P} \wedge \mathbf{Q}}$$

Disjunction to Implication

$$\frac{\mathbf{H} \vdash \neg \mathbf{P} \Rightarrow \mathbf{Q}}{\mathbf{H} \vdash \mathbf{P} \vee \mathbf{Q}}$$

Remove Negation in Goal

$$\frac{\mathbf{H} \vdash \neg \mathbf{P} \vee \neg \mathbf{Q}}{\mathbf{H} \vdash \neg (\mathbf{P} \wedge \mathbf{Q})} \qquad \frac{\mathbf{H} \vdash \neg \mathbf{P} \ \wedge \ \neg \mathbf{Q}}{\mathbf{H} \vdash \neg (\mathbf{P} \vee \mathbf{Q})} \qquad \frac{\mathbf{H} \vdash \mathbf{P} \ \wedge \ \neg \mathbf{Q}}{\mathbf{H} \vdash \neg (\mathbf{P} \Rightarrow \mathbf{Q})} \qquad \frac{\mathbf{H} \vdash \mathbf{P}}{\mathbf{H} \vdash \neg \neg \mathbf{P}}$$

Equivalence Rewrite

$$\frac{\mathbf{H} \vdash \mathbf{P} \Rightarrow \mathbf{Q} \qquad \mathbf{H} \vdash \mathbf{Q} \Rightarrow \mathbf{P}}{\mathbf{H} \vdash \mathbf{P} \Leftrightarrow \mathbf{Q}}$$

2.2 Hypotheses Red Operators (as labeled in the Rodin Platform)

Proof by Cases

$$\frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{G} \qquad \mathbf{H}, \mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \mathbf{P} \lor \mathbf{Q} \vdash \mathbf{G}}$$

Contradict (special case). It is not a red button. It is situated on the left of each hypothesis.

$$\frac{\mathbf{H}, \mathbf{P}, \neg\, \mathbf{Q} \vdash \mathbf{P}}{\mathbf{H}, \mathbf{P}, \neg\, \mathbf{P} \vdash \mathbf{Q}}$$

Do Case Distinction on this Implication

$$\frac{\mathbf{H}, \neg\, \mathbf{P} \vdash \mathbf{G} \qquad \mathbf{H}, \mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \mathbf{P} \Rightarrow \mathbf{Q} \vdash \mathbf{G}}$$

Remove Negation in Hypotheses

$$\frac{\mathbf{H}, \neg\, \mathbf{P} \lor \neg\, \mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \neg\, (\mathbf{P} \land \mathbf{Q}) \vdash \mathbf{G}} \qquad \frac{\mathbf{H}, \neg\, \mathbf{P}, \neg\, \mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \neg\, (\mathbf{P} \lor \mathbf{Q}) \vdash \mathbf{G}} \qquad \frac{\mathbf{H}, \mathbf{P}, \neg\, \mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \neg\, (\mathbf{P} \Rightarrow \mathbf{Q}) \vdash \mathbf{G}} \qquad \frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{G}}{\mathbf{H}, \neg\, \neg\, \mathbf{P} \vdash \mathbf{G}}$$

Equivalence Rewrite

$$\frac{\mathbf{H}, \mathbf{P} \Rightarrow \mathbf{Q}, \mathbf{Q} \Rightarrow \mathbf{P} \vdash \mathbf{G}}{\mathbf{H}, \mathbf{P} \Leftrightarrow \mathbf{Q} \vdash \mathbf{G}}$$

3 Tactic Profiles

3.1 Tactic Profile prp_1

It contains the following unique elementary tactic:

Goal in Hypotheses

$$\frac{}{\mathbf{H}, \mathbf{P} \vdash \mathbf{P}}$$

With this elementary tactic profile, you will have to depress many red operator buttons of the prover user interface. Always use FIRST the goal buttons (section 2.1). When you cannot do anything anymore on the goal buttons start to use the hypotheses buttons (section 2.2). If some red buttons reappears in the goal while treating the hypothesis buttons, then stop using these hypothesis buttons until you cannot do anything again on the goal buttons, and so on.

3.2 Tactic Profile prp_2

This profile extends Profile prp_1. The added elementary tactic is the following:

Implicative Goal (deduction)

$$\frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{G}}{\mathbf{H} \vdash \mathbf{P} \Rightarrow \mathbf{G}}$$

3.3 Tactic Profile prp_3

This profile extends Profile prp_2. The added elementary tactic is the following:

Conjunctive Goal (conjunction introduction)

$$\frac{\mathbf{H} \vdash \mathbf{P} \qquad \mathbf{H} \vdash \mathbf{Q}}{\mathbf{H} \vdash \mathbf{P} \land \mathbf{Q}}$$

3.4 Tactic Profile prp_4

This profile extends Profile prp_3. The added elementary tactic is the following:

Find Contradictory Hypotheses (Contradict)

$$\frac{}{\mathbf{H}, \mathbf{P}, \neg\, \mathbf{P} \vdash \mathbf{G}}$$

3.5 Tactic Profile prp_5

This profile extends Profile prp_4. The added elementary tactics are the following:

Generalized Modus Ponens

$$\frac{\mathbf{H}(\top) \vdash \mathbf{G}(\top)}{\mathbf{H}(\mathbf{P}), \mathbf{P} \vdash \mathbf{G}(\mathbf{P})} \qquad \frac{\mathbf{H}(\neg\, \top) \vdash \mathbf{G}(\neg\, \top)}{\mathbf{H}(\mathbf{P}), \neg\, \mathbf{P} \vdash \mathbf{G}(\mathbf{P})}$$

Simplification Rewriter

$$\mathbf{P} \wedge \top == \mathbf{P} \qquad \top \wedge \mathbf{P} == \mathbf{P}$$

$$\mathbf{P} \wedge \neg\, \top == \neg\, \top \qquad \neg\, \top \wedge \mathbf{P} == \neg\, \top$$

$$\mathbf{P} \vee \top == \top \qquad \top \vee \mathbf{P} == \top$$

$$\mathbf{P} \vee \neg\, \top == \mathbf{P} \qquad \neg\, \top \vee \mathbf{P} == \mathbf{P}$$

$$\mathbf{P} \Rightarrow \top == \top \qquad \top \Rightarrow \mathbf{P} == \mathbf{P}$$

$$\mathbf{P} \Rightarrow \neg\, \top == \neg\, \mathbf{P} \qquad \neg\, \top \Rightarrow \mathbf{P} == \top$$

True Goal

$$\frac{}{\mathbf{H} \vdash \top}$$

False Hypotheses

$$\frac{}{\mathbf{H}, \bot \vdash \mathbf{G}}$$

3.6 Tactic Profile prp_6

This profile extends Profile prp_5. The added elementary tactic is the following:

Remove Disjunction in a Disjunctive Goal (Disjunction to Implication)

$$\frac{\mathbf{H} \vdash \neg\, \mathbf{P} \Rightarrow \mathbf{Q}}{\mathbf{H} \vdash \mathbf{P} \vee \mathbf{Q}}$$

3.7 Tactic Profile prp_7

This profile extends Profile prp_6. The added elementary tactic is the following:

Put in negative normal form in goal and hypotheses (Remove negation)

$$\neg\,(\mathbf{P} \wedge \mathbf{Q}) == \neg\,\mathbf{P} \vee \neg\,\mathbf{Q}$$

$$\neg\,(\mathbf{P} \vee \mathbf{Q}) == \neg\,\mathbf{P} \wedge \neg\,\mathbf{Q}$$

$$\neg\,(\mathbf{P} \Rightarrow \mathbf{Q}) == \mathbf{P} \wedge \neg\,\mathbf{Q}$$

$$\neg\,\neg\,\mathbf{P} == \mathbf{P}$$

3.8 Tactic Profile prp_8

This profile extends Profile prp_7. The added elementary tactic is the following:

Remove all Equivalences in Goal and Hypotheses (Equivalence Rewrite)

$$\mathbf{P} \Leftrightarrow \mathbf{Q} == (\mathbf{P} \Rightarrow \mathbf{Q}) \wedge (\mathbf{Q} \Rightarrow \mathbf{P})$$

As a consequence, the only red operators one has to depress in the hypotheses are the following:

Proof by Cases

$$\frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{G} \qquad \mathbf{H}, \mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \mathbf{P} \vee \mathbf{Q} \vdash \mathbf{G}}$$

Do Case Distinction on this Implication

$$\frac{\mathbf{H}, \neg\,\mathbf{P} \vdash \mathbf{G} \qquad \mathbf{H}, \mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \mathbf{P} \Rightarrow \mathbf{Q} \vdash \mathbf{G}}$$