Exercise Sheet 4: Interactive Proofs in the Predicate Calculus

1 Introduction

1.1 Purpose

The purpose of this exercise is to make you familiar with the practice of interactive proofs with the Rodin Platform on the predicate calculus.

The predicate calculus is an extension of the propositional calculus studied in the previous exercise. In this extension, the universal ($\forall$) and existential ($\exists$) quantifications are introduced.

1.2 Your Task

We distribute to you a Rodin development named "04_pred": it contains 8 contexts (from pred0 to pred7), each of which with one theorem. We also distribute to you 3 tactic profiles named "prd_1", "prd_2", and "prd_3". Each of them is a slight extension of the previous one.

Download this material.

You are asked to prove these theorems using the tactic profiles as follows: use tactic profile "prd_1" for the proofs of the theorems in contexts "pred0" and "pred1". Then do the same proofs with tactic profiles "prd_2" and "prd_3". For the other theorems (in contexts "pred2" to "pred7"), use directly profile "prd_3".

In section 4, we give some advices (also called hints) to perform the proofs of the proposed theorems in contexts "pred2" to "pred7".

2 Some Red Operator Buttons

In this section we present two more red operator buttons besides the ones we introduced in the previous exercise. We also present the very important notion of *instantiation* of existential goal and universal hypotheses.

2.1 Another Red Operator in the Goal

Forall Instantiation

$$\frac{\mathbf{H} \vdash \mathbf{P}(\mathbf{x})}{\mathbf{H} \vdash \forall \mathbf{x} \cdot \mathbf{P}(\mathbf{x})} \qquad \text{where } \mathbf{x} \text{ is not free in } \mathbf{H}$$

When **x** is free in **H**, a change of variable is automatically performed.

2.2 Another Red Operator in the Hypotheses

Free Existential Variables

$$\frac{\mathbf{H}, \mathbf{P}(\mathbf{x}) \vdash \mathbf{G}}{\mathbf{H}, \exists \mathbf{x} \cdot \mathbf{P}(\mathbf{x}) \vdash \mathbf{G}} \qquad \text{where } \mathbf{x} \text{ is not free in } \mathbf{H} \text{ and in } \mathbf{G}$$

When **x** is free in **H** or **G**, a change of variable is automatically performed.

2.3 Existential Instantiation in the Goal

$$\frac{\mathbf{H} \vdash \mathbf{P}(\mathbf{E})}{\mathbf{H} \vdash \exists \mathbf{x} \cdot \mathbf{P}(\mathbf{x})}$$

For doing this, write the instantiation **E** (there might be several of them in case there are several quantified variables) in the yellow box and then press the red operator $\exists$.

2.4 Universal Instantiation in the Hypotheses

$$\frac{\mathbf{H}, \mathbf{P}(\mathbf{E}) \vdash \mathbf{G}}{\mathbf{H}, \forall \mathbf{x} \cdot \mathbf{P}(\mathbf{x}) \vdash \mathbf{G}}$$

For doing this, write the instantiation **E** (there might be several of them in case there are several quantified variables) in the yellow box. Then there are two cases:

1. In the case where the hypothesis has the following shape:

   $$\forall x \cdot P(x) \Rightarrow Q(x)$$

   press the red operator $\Rightarrow$ and, in the coming menu, press the button "Instantiate universal followed by modus ponens" or the button "Instantiate universal followed by modus tollens".

2. In the case where the hypothesis has NOT the previous shape, press the red operator $\forall$.

3 Tactic Profiles

In this section, we give some information about the three tactic profiles "prd_1", "prd_2" and "prd_3".

3.1 Tactic Profile "prd_1"

The tactic profile "prd_1" is a slight extension of the tactic profile "prp_8" used in the previous exercise.

When using this tactic profile, the user will be asked to depress the additional red operator button in the goal (section 2.1) and then the additional red operator button in the hypotheses (section 2.2) . Some instantiations of existential goal (section section 2.3) or universal hypothesis (section 2.4) will also be necessary.

3.2 Tactic Profile "prd_2"

This profile extends profile "prd_1" by adding an elementary tactic for the goal. The added elementary tactic is the following:

Forall Goal (Forall Instantiation)

$$\frac{\mathbf{H} \vdash \mathbf{P}(\mathbf{x})}{\mathbf{H} \vdash \forall \mathbf{x} \cdot \mathbf{P}(\mathbf{x})} \qquad \text{where } \mathbf{x} \text{ is not free in } \mathbf{H}$$

When **x** is free in **H**, a change of variable is automatically performed.

3.3 Tactic Profile "prd_3"

This profile extends profile "prd_2" by adding an elementary tactic for the hypotheses. The added elementary tactic is the following:

Exists Hypothesis (Free Existential Variables)

$$\frac{\mathbf{H}, \mathbf{P}(\mathbf{x}) \vdash \mathbf{G}}{\mathbf{H}, \exists \mathbf{x} \cdot \mathbf{P}(\mathbf{x}) \vdash \mathbf{G}} \qquad \text{where } \mathbf{x} \text{ is not free in } \mathbf{H} \text{ and in } \mathbf{G}$$

When **x** is free in **H** or **G**, a change of variable is automatically performed.

When using this tactic profile, it is now not necessary any more to depress the red operator in the goal (section 2.1) and the red operator in the hypotheses (section 2.2). The user is still asked to perform some proof by cases and, more difficult, some instantiations in the goal (section 2.3) or in the hypotheses (section 2.4).

4 Hints

The given hints are very precise. Follow them carefully and you'll succeed. After that, try to redo the proof without looking at the hints.

4.1 Hint for the theorem in context "pred2"

1. Instantiate x with a in the existential goal.
2. Then always instantiate x with a in some universal hypotheses.

4.2 Hint for the theorem in context "pred3"

1. Instantiate x with x in the last universal hypothesis $\forall x \cdot \exists y \cdot x \mapsto y \in R$.
2. Instantiate x with x, y with y and z with x in the first universal hypothesis.
3. Instantiate x with x, y with y in the universal hypothesis

4.3 Hint for the theorem in context "pred4"

This is the most difficult theorem to prove. If you fail proving it, try the other ones.

1. Instantiate x with x, y with z and z with y in the hypothesis

   $$\forall x, y, z \cdot x \mapsto z \in R \land y \mapsto z \in R \Rightarrow x \mapsto y \in R$$

   WARNING: After doing this instantiation, you will see that the hypothesis

   $$\forall x, y, z \cdot x \mapsto z \in R \land y \mapsto z \in R \Rightarrow x \mapsto y \in R$$

   disappeared. You can get it back again by pressing the "Show cache hypotheses" button in the "Proof Control" pannel and then select this hypothesis on the window appearing on the right part of the screen.

2. Instantiate x with z, y with y and z with z in the hypothesis:

$$\forall x, y, z \cdot x \mapsto z \in R \land y \mapsto z \in R \Rightarrow x \mapsto y \in R$$

3. Instantiate x with z in the hypothesis:

$$\forall x \cdot \exists y \cdot x \mapsto y \in R$$

   Then instantiate x with z, y with z and z with $y0$ in the hypothesis

$$\forall x, y, z \cdot x \mapsto z \in R \land y \mapsto z \in R \Rightarrow x \mapsto y \in R$$

4.4 Hint for the theorem in context "pred5"

1. Instantiate x with x in the existential goal.
2. Instantiate x with x in the first universal hypothesis.
3. Instantiate x with x and y with y in the first universal hypothesis.
4. Instantiate y with y in the existential goal.
5. Instantiate x with x and y with z in the second universal hypothesis

4.5 Hint for the theorem in context "pred6"

1. Instantiate the first universal hypothesis with x
2. Instantiate the second universal hypothesis with x
3. Instantiate the universal hypothesis with y

4.6 Hint for the theorem in context "pred7"

1. Instantiate x with a in the existential goal
2. Instantiate x with y in hypothesis $\forall x \cdot x \in Q \lor x \in R$
3. Terminate the proof by cases