Event-B Course

5. Sequential Program Development

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Purpose of this Lecture

- To present a formal approach for developing sequential programs

- To present a large number of examples:
    - array programs
    - pointer programs
    - numerical programs

- A typical sequential program is made of :
    - a number of MULTIPLE ASSIGNMENTS ($:=$)
    - scheduled by means of some :
        - CONDITIONAL operators (**if**)
        - ITERATIVE operators (**while**)
        - SEQUENTIAL operators (**;**)

An Example


```
while  j ≠ m  do
   if  g(j + 1) > x  then
      j := j + 1
   elsif  k = j  then
      k, j := k + 1, j + 1
   else
      k, j, g := k + 1, j + 1, swap (g, k + 1, j + 1)
   end
end ;
p := k
```

while $condition$ **do** $statement$ **end**

if $condition$ **then** $statement$ **else** $statement$ **end**

if $condition$ **then** $statement$ **elsif** $\ldots$ **else** $statement$ **end**

$statement\,;statement$

$variable\_list\ :=\ expression\_list$

An Event Approach (1)

- Separating completely in the design:
    - the individual assignments
    - from their scheduling

- This approach favors:
    - the distribution of computation
    - over its centralization

An Event Approach (2)

- Each assignment is formalized by a guarded event made of:
    - A firing condition: the guard,
    - An action: the multiple assignment.

- These events are scheduled implicitly.

```
while  j ≠ m  do
   if  g(j + 1) > x  then
      j := j + 1
   elsif  k = j  then
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```

```
when
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   g(j + 1) > x
then
   j := j + 1
end
```

```
while  j ≠ m  do
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   else
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   end
end ;
p := k
```

```
when
   j ≠ m
   g(j + 1) ≤ x
   k = j
then
   k, j := k + 1, j + 1
end
```

```
while  j ≠ m  do
   if  g(j + 1) > x  then
      j := j + 1
   elsif  k = j  then
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   end
end ;
p := k
```

```
when
   j ≠ m
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then
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```

```
while  j ≠ m  do
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   else
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   end
end ;
p := k
```

```
when
   j = m
then
   p := k
end
```

when
$j \neq m$
$g(j+1) > x$
then
$j := j + 1$
end

when
$j \neq m$
$g(j+1) \leq x$
$k = j$
then
$k, j := k+1, j+1$
end

when
$j \neq m$
$g(j+1) \leq x$
$k \neq j$
then
$k, j, g := \ldots$
end

when
$j = m$
then
$p := k$
end

- We have just decomposed a program into separate events

- Our approach will consists in doing the reverse operation

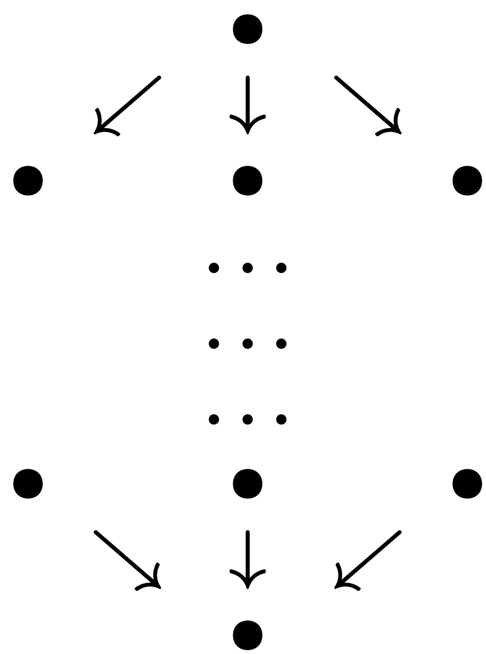
- We shall construct the events first

- And then compose our program from these events

Specification Phase — initial event: Specification

Design Phase — new events: Refinements

Composing Phase — final event: Program



- Sequential Programs are usually specified by means of:
  - A pre-condition
  - and a post-condition

- It is expressed by means of a Hoare-triple

$$\{Pre\} \quad \mathbf{P} \quad \{Post\}$$

Encoding a Hoare triple in Event-B

$$\{Pre\} \quad \textbf{P} \quad \{Post\}$$

- The parameters are constants.

- The pre-conditions are the axioms of these constants.

- The results are variables.

- The post-conditions are the guards of an event with a skip action.

Example 1: The search Program

- We are given (Pre-condition)

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    - an array f of n elements built on a set S: $f \in 1..n \rightarrow S$

Example 1: The search Program

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  - a value v known to be in the array: $v \in \mathrm{ran}(f)$

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    - an index r in the domain of the array: $r \in \mathrm{dom}(f)$

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$$\left\{\begin{array}{l} n \in \mathbb{N} \\ 0 < n \\ f \in 1\,..\,n \to S \\ v \in \mathrm{ran}(f) \end{array}\right\} \quad \textbf{search} \quad \left\{\begin{array}{l} r \in \mathrm{dom}(f) \\ f(r) = v \end{array}\right\}$$

Encoding the Hoare-triple in an Event System

$$\left\{\begin{array}{l} n \in \mathbb{N} \\ 0 < n \\ f \in 1\,..\,n \rightarrow S \\ v \in \mathrm{ran}(f) \end{array}\right\} \quad \textsf{search} \quad \left\{\begin{array}{l} r \in \mathrm{dom}(f) \\ f(r) = v \end{array}\right\}$$

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sets: S

constants: n
f
v

axm0_1: $n \in \mathbb{N}$

axm0_2: $0 < n$

axm0_3: $f \in 1\,..\,n \rightarrow S$

axm0_4: $v \in \mathrm{ran}(f)$

Encoding the Hoare-triple in an Event System

$$\left\{ \begin{array}{l} n \in \mathbb{N} \\ 0 < n \\ f \in 1\,..\,n \rightarrow S \\ v \in \operatorname{ran}(f) \end{array} \right\} \quad \textsf{search} \quad \left\{ \begin{array}{l} r \in \operatorname{dom}(f) \\ f(r) = v \end{array} \right\}$$

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axm0_2: $0 < n$

axm0_3: $f \in 1\,..\,n \rightarrow S$

axm0_4: $v \in \operatorname{ran}(f)$

variables: r

inv0_1: $r \in \mathbb{N}$

init
  $r :\in \mathbb{N}$

final
  when
    $r \in 1\,..\,n$
    $f(r) = v$
  then
    skip
  end

Introducing an additional anticipated Event

```
progress
   status
      anticipated
   then
      r :∈ ℕ
   end
```

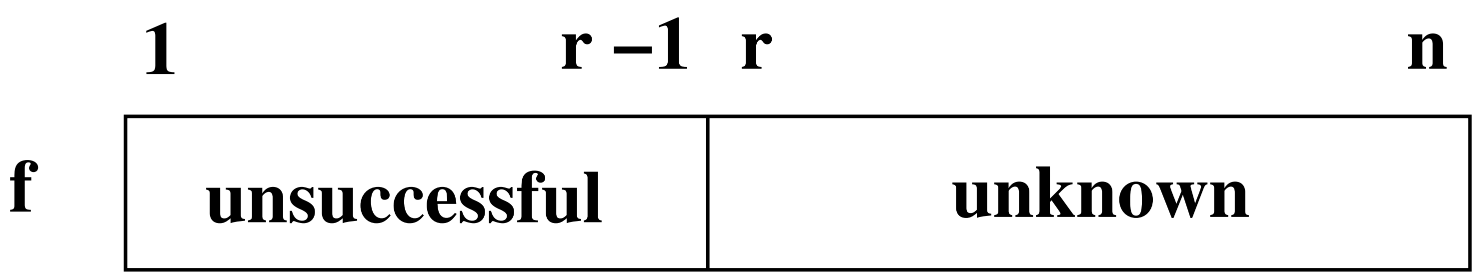
- This event modifies r non-deterministically

We introduce more invariants for the result r

inv1_1: $r \in 1 \,..\, n$

inv1_2: $v \notin f[1 \,..\, r-1]$

- This can be illustrated in the following figure:



init
  $r := 1$

progress
  status
    convergent
  when
    $f(r) \neq v$
  then
    $r := r + 1$
  end

final
  when
    $f(r) = v$
  then
    skip
  end

- The event progress is now made convergent

- We thus propose a variant:

variant1: $n - r$

To be Proved (as usual)

- Events search and init refine their abstractions

- The exhibited variant is a natural number

- "New" event progress decreases the variant

- The system is deadlock free

We are using some Merging Rules to build the final program

init
$r := 1$

progress
when
$f(r) \neq v$
then
$r := r + 1$
end

final
when
$f(r) = v$
then
skip
end

Merging Rule M_WHILE

when P Q **then** S **end**	**when** P $\neg Q$ **then** T **end**	$\leadsto$	**when** P **then** **while** Q **do** S **end**; T **end**	M_WHILE

- Side Conditions:
  - P must be invariant under S
  - The first event introduced at one level below the second one.
- The resulting level is that of the second event
- Special Case: If P is missing the resulting "event" has no guard

Merging Rule M_IF

when
P
Q
then
S
end
	when
	P
	$\neg\, Q$
	then
	T
	end
		$\leadsto$	**when**
			P
			then
			if Q **then**
			S
			else
			T
			end
			end
				M_IF

- Side Conditions:
    - The two events introduced at the same refinement level
- The resulting level is the same
- Special Case: If P is missing the resulting "event" has no guard

Applying Rule M_WHILE (special case)

```
progress
  when
    f(r) ≠ v
  then
    r := r + 1
  end
```

```
final
  when
    f(r) = v
  then
    skip
  end
```

```
progress_final
  while f(r) ≠ v do
    r := r + 1
  end
```

Final Rule M_INIT

- Once we have obtained an “event” without guard

- We add to it the event init by sequential composition

- We then obtain the final “program”

Applying Rule M_INIT

init
$r := 1$

progress_final
while $f(r) \neq v$ **do**
$r := r + 1$
end

$$\left\{ \begin{array}{l} n \in \mathbb{N} \\ 0 < n \\ f \in 1 \,..\, n \to S \\ v \in \mathrm{ran}(f) \end{array} \right\}$$

search_program
$r := 1;$
while $f(r) \neq v$ **do**
$r := r + 1$
end

$$\left\{ \begin{array}{l} r \in \mathrm{dom}(f) \\ f(r) = v \end{array} \right\}$$

Example 2: The Very Classical Binary Search

- Almost the same specification as in Example 1

- It will show the usage of more merging rules

- We are given (Pre-condition)

- We are given (Pre-condition)
    - a natural number n: $n \in \mathbb{N}$

- We are given (Pre-condition)
    - a natural number $\boldsymbol{n}$: $\boldsymbol{n} \in \mathbb{N}$
    - $\boldsymbol{n}$ is positive: 0<n

- We are given (Pre-condition)
  - a natural number n: $n \in \mathbb{N}$
  - n is positive: 0<n
  - a sorted array f of n elements built on a set $\mathbb{N}$: $f \in 1\,..\,n \rightarrow \mathbb{N}$

- We are given (Pre-condition)
  - a natural number n: $n \in \mathbb{N}$
  - n is positive: 0<n
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  - a natural number n: $n \in \mathbb{N}$
  - n is positive: 0<n
  - a sorted array f of n elements built on a set $\mathbb{N}$: $f \in 1\,..\,n \rightarrow \mathbb{N}$
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- We are looking for (Post-condition)

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    - a natural number n: $n \in \mathbb{N}$
    - n is positive: 0<n
    - a sorted array f of n elements built on a set $\mathbb{N}$: $f \in 1\,..\,n \rightarrow \mathbb{N}$
    - a value v known to be in the array: $v \in \mathrm{ran}(f)$

- We are looking for (Post-condition)
    - an index r in the domain of the array: $r \in \mathrm{dom}(f)$

- We are given (Pre-condition)
  - a natural number n: $n \in \mathbb{N}$
  - n is positive: 0<n
  - a sorted array f of n elements built on a set $\mathbb{N}$: $f \in 1\,..\,n \rightarrow \mathbb{N}$
  - a value v known to be in the array: $v \in \mathrm{ran}(f)$

- We are looking for (Post-condition)
  - an index r in the domain of the array: $r \in \mathrm{dom}(f)$
  - such that $f(r) = v$

Binary Search: Encoding of the pre-condition

constants: n
f
v

axm0_1: $n \in \mathbb{N}$

axm0_2: $f \in 1 \,..\, n \rightarrow \mathbb{N}$

axm0_3: $v \in \mathrm{ran}(f)$

thm0_1: $n \geq 1$

axm0_4: $$\forall i, j \cdot \begin{array}{l} i \in 1 \,..\, n \\ j \in 1 \,..\, n \\ i \leq j \\ \Rightarrow \\ f(i) \leq f(j) \end{array}$$

Binary Search: Encoding of the post-condition

variables: r

inv0_1: $r \in \mathbb{N}$

init
    $r :\in \mathbb{N}$

final
    when
        $r \in 1 \,..\, n$
        $f(r) = v$
    then
        skip
    end

- We have also an anticipated event:

progress
    status
        anticipated
    then
        $r :\in \mathbb{N}$
    end

First Refinement: the State

- We introduce two new variables p and q

variables: r, p, q

inv1_1: $p \in 1 \,..\, n$

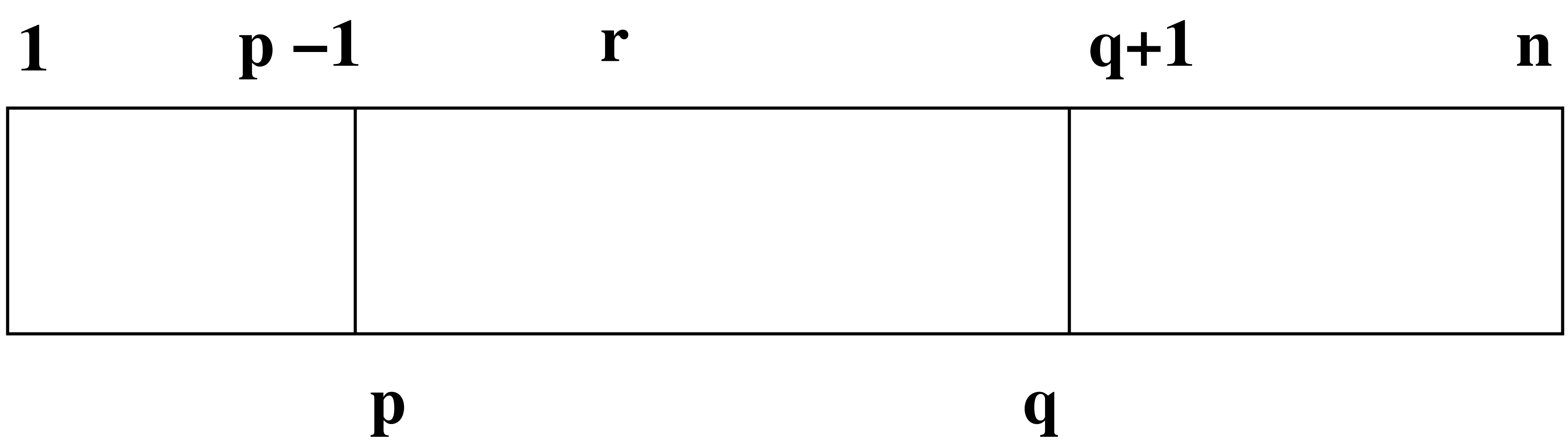
inv1_2: $q \in 1 \,..\, n$

inv1_3: $r \in p \,..\, q$

inv1_4: $v \in f[p \,..\, q]$

variant1: $q - p$

- The current situation is illustrated in the following figure:



First Refinement: the Events

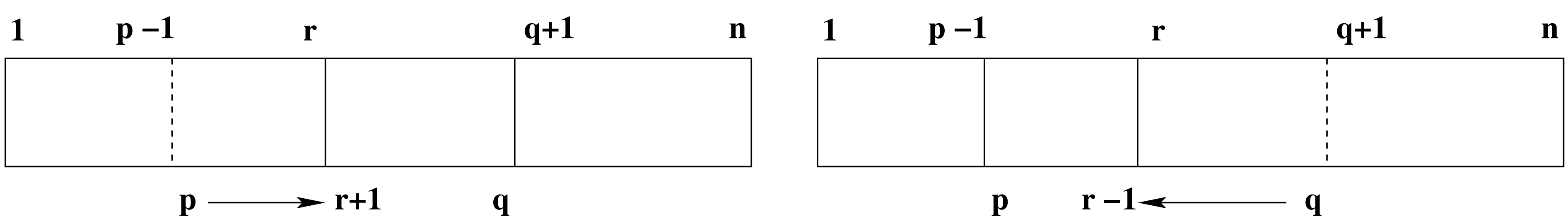
init
$p := 1$
$q := n$
$r :\in 1 \,..\, n$

inc
refines
progress
status
convergent
when
$f(r) < v$
then
$p := r + 1$
$r :\in r + 1 \,..\, q$
end

dec
refines
progress
status
convergent
when
$v < f(r)$
then
$q := r - 1$
$r :\in p \,..\, r - 1$
end

final
when
$f(r) = v$
then
skip
end

The following figure illustrates the situation encountered by events inc (left) and dec (right)



Proofs

- Proofs of inc

- Feasibility of inc

- Proofs and feasibility for dec (similar to those for inc)

- Proofs for final (obvious)

- Proofs of non-divergence of inc and dec (variant: $q - p$)

- Proof of dealock freeness (easy)

Second Refinement

- At the previous stage, inc and dec were non-deterministic

- r was chosen arbitrarily within the interval $p \mathinner{..} q$

- We now remove the non-determinacy in inc and dec

- r is chosen to be the middle of the interval $p \mathinner{..} q$

Reducing non-determincy

- r is chosen in the "middle" of the intervals $r+1\,..\,q$ or $p\,..\,r-1$.

```
init
   p := 1
   q := n
   r := (1 + n)/2
```

```
inc
   when
      f(r) < v
   then
      p := r + 1
      r := (r + 1 + q)/2
   end
```

```
dec
   when
      v < f(r)
   then
      q := r − 1
      r := (p + r − 1)/2
   end
```

```
final
   when
      f(r) = v
   then
      skip
   end
```

Merging Rule M_IF

when
P
Q
then
S
end
	when
	P
	$\neg Q$
	then
	T
	end
		$\leadsto$	**when**
			P
			then
			if Q **then**
			S
			else
			T
			end
			end
				M_IF

```
inc
when
  f(r) ≠ v
  f(r) < v
then
  p := r + 1
  r := (r + 1 + q)/2
end
```

```
dec
when
  f(r) ≠ v
  v ≤ f(r)
then
  q := r − 1
  s := (p + r − 1)/2
end
```

```
inc_dec
when
  f(r) ≠ v
then
  if f(s) < v then
    p, r := r + 1, (r + 1 + q)/2
  else
    q, r := r − 1, (p + r − 1)/2
  end
end
```

```
final
  when
    f(r) = v
  then
    skip
  end
```

Merging Rule M_WHILE

when P Q **then** S **end**	**when** P $\neg\, Q$ **then** T **end**	$\rightsquigarrow$	**when** P **then** **while** Q **do** S **end**; T **end**	**M_WHILE**

- Side Conditions:
  - P must be invariant under S
  - The first event must have been introduced at one refinement step below the second one.

- Special Case: If P is missing the resulting "event" has no guard

```
inc_dec
when
   f(r) ≠ v
then
   if  f(r) < v  then
      p, r := r + 1, (r + 1 + q)/2
   else
      q, r := r − 1, (p + r − 1)/2
   end
end
```

```
inc_dec_final
   while  f(r) ≠ v  do
      if  f(r) < v  then
         p, r := r + 1, (r + 1 + q)/2
      else
         q, r := r − 1, (p + r − 1)/2
      end
   end
```

```
final
   when
      f(r) = v
   then
      skip
   end
```

```
init
   p, q := 1, n
   r := (1 + n)/2
```

inc_dec_final
while $f(r) \neq v$ **do**
  if $f(r) < v$ **then**
    $p, r := r + 1, (r + 1 + q)/2$
  else
    $q, r := r - 1, (p + r - 1)/2$
  end
end

init
  $p, q := 1, n$
  $r := (1 + n)/2$

bin_search_program
  $p, q, r := 1, n, (1 + n)/2;$
  while $f(r) \neq v$ **do**
    if $f(r) < v$ **then**
      $p, r := r + 1, (r + 1 + q)/2$
    else
      $q, r := r - 1, (p + r - 1)/2$
    end
  end

Example 3: Array Partitioning

- Given a numerical array f with n distinct elements

- Given a number x

- We construct another numerical array g with some constraints.

- g has the same elements as f

- there exists a number k in $0 \,..\, n$ such that elements of g are:
  - not greater than x in interval $1 \,..\, k$
  - greater than x in interval $k+1 \,..\, n$

1	$\leq x$	k	$k+1$	$> x$	n

Example

- Let the array f be the following:

$\mathbf{3}$	$\mathbf{7}$	$\mathbf{2}$	$\mathbf{5}$	$\mathbf{8}$	$\mathbf{9}$	$\mathbf{4}$	$\mathbf{1}$

- Let x be equal to 5

- The result g can be the following with k being set to 5

$\mathbf{3}$	$\mathbf{2}$	$\mathbf{5}$	$\mathbf{4}$	$\mathbf{1}$	$\mathbf{9}$	$\mathbf{7}$	$\mathbf{8}$

$\boldsymbol{k}$

Array Partitioning: first special case

- Let the array f be the following:

$\mathbf{3}$	$\mathbf{7}$	$\mathbf{2}$	$\mathbf{5}$	$\mathbf{8}$	$\mathbf{9}$	$\mathbf{4}$	$\mathbf{1}$

- Let x be equal to 0

- The result g can be the following with k being set to 0

$\mathbf{3}$	$\mathbf{7}$	$\mathbf{2}$	$\mathbf{5}$	$\mathbf{8}$	$\mathbf{9}$	$\mathbf{4}$	$\mathbf{1}$

$\boldsymbol{k}$

Array Partitioning: second special case

- Let the array f be the following:

$\mathbf{3}$	$\mathbf{7}$	$\mathbf{2}$	$\mathbf{5}$	$\mathbf{8}$	$\mathbf{9}$	$\mathbf{4}$	$\mathbf{1}$

- Let x be equal to 10

- The result g can be the following with k being set to 8

$\mathbf{3}$	$\mathbf{7}$	$\mathbf{2}$	$\mathbf{5}$	$\mathbf{8}$	$\mathbf{9}$	$\mathbf{4}$	$\mathbf{1}$
							$\boldsymbol{k}$

Initial Model

constants: n
f
x

axm0_1: $n \in \mathbb{N}$

axm0_2: $f \in 1 \,..\, n \rightarrowtail \mathbb{N}$

axm0_3: $x \in \mathbb{N}$

variables: k
g

inv0_1: $k \in \mathbb{N}$

inv0_2: $g \in \mathbb{N} \leftrightarrow \mathbb{N}$

init
$k :\in \mathbb{N}$
$g :\in \mathbb{N} \leftrightarrow \mathbb{N}$

final
when
$k \in 0 \,..\, n$
$g \in 1 \,..\, n \rightarrowtail \mathbb{N}$
$\mathrm{ran}\,(g) = \mathrm{ran}\,(f)$
$\forall m \cdot m \in 1 \,..\, k \Rightarrow g(m) \leq x$
$\forall m \cdot m \in k+1 \,..\, n \Rightarrow g(m) > x$
then
skip
end

progress
status
anticipated
then
$k :\in \mathbb{N}$
$g :\in \mathbb{N} \leftrightarrow \mathbb{N}$
end

Introducing a new variable j ranging from 0 to n

Current situation: array g is partitioned from 1 to j

1	$\leq x$	k	$k+1$	$> x$	j	$j+1$	$?$	n

Invariant

$$k \leq j$$

$$\forall l \cdot (\, l \in 1 \,..\, k \;\Rightarrow\; g(l) \leq x \,)$$

$$\forall l \cdot (\, l \in k+1 \,..\, j \;\Rightarrow\; g(l) > x \,)$$

Array Partitioning : First Refinement: the State

constants: n, f, x

variables: k, g, j

inv1_1: $j \in 0 \,..\, n$

inv1_2: $k \leq j$

inv1_3: $\forall l \cdot (\, l \in 1 \,..\, k \;\Rightarrow\; g(l) \leq x \,)$

inv1_4: $\forall l \cdot (\, l \in k+1 \,..\, j \;\Rightarrow\; g(l) > x \,)$

Partitioning with 5

3	7	2	5	8	9	4	1

Partitioning with 5

3	7	2	5	8	9	4	1

Partitioning with 5

3	7	2	5	8	9	4	1

Partitioning with 5

3	2	7	5	8	9	4	1

Partitioning with 5

3	2	5	7	8	9	4	1

Partitioning with 5

3	2	5	7	8	9	4	1

Partitioning with 5

3	2	5	7	8	9	4	1

Partitioning with 5

3	2	5	4	8	9	7	1

Partitioning with 5

3	2	5	4	1	9	7	8

init
$$\boldsymbol{g, j, k := f, 0, 0}$$

partition
when
$$\boldsymbol{j = n}$$
then
skip
end

$1 \quad\quad \leq x \quad\quad k$	$k+1 \quad\quad > x \quad\quad j$	$j+1 \quad\quad ? \quad\quad n$

progress_1
  when
    $j \neq n$
    $g(j+1) > x$
  then
    ?
  end

1	$\leq x$	k	$k+1$	$> x$	j	$j+1$	?	n

progress_1
  when
    $j \neq n$
    $g(j+1) > x$
  then
    $j := j+1$
  end

Partitioning with 5

3	2	5	7	8	9	4	1

Partitioning with 5

3	2	5	7	8	9	4	1

Array Partitioning : New Event

1	$\leq x$	k, j	$j+1$	**?**	n

progress_2
when
$j \neq n$
$g(j+1) \leq x$
$k = j$
then
?
end

1	$\leq x$	k, j	$j+1$	**?**	n

progress_2
when
$j \neq n$
$g(j+1) \leq x$
$k = j$
then
$k, j := k+1, j+1$
end

1	$\leq x$	k	$k+1$	$> x$	j	$j+1$	**?**	n

progress_3
when
$j \neq n$
$g(j+1) \leq x$
$k \neq j$
then
?
end

Array Partitioning : New Event

1	$\leq x$	k	$k+1$	$> x$	j	$j+1$	?	n

progress_3
when
$j \neq n$
$g(j+1) \leq x$
$k \neq j$
then
$k, j, g := k+1, j+1, \mathsf{swap}\,(g, k+1, j+1)$
end

$$\mathsf{swap}\,(g, k, j) \;=\; g \Leftarrow \{k \mapsto g(j)\} \Leftarrow \{j \mapsto g(k)\}$$

Partitioning with 5

3	2	5	7	8	9	4	1

Partitioning with 5

3	2	5	4	8	9	7	1

Putting together progress_2 and progress_3

progress_2
  when
    $j \neq n$
    $g(j+1) \leq x$
    $k = j$
  then
    $k, j := k+1, j+1$
  end

progress_3
  when
    $j \neq n$
    $g(j+1) \leq x$
    $k \neq j$
  then
    $k, j, g := k+1, j+1,$
      swap $(g, k+1, j+1)$
  end

Merging Rule (2)

when
P
Q
then
S
end
	when
	P
	$\neg\, Q$
	then
	T
	end
		$\leadsto$	**when**
			P
			then
			if Q **then**
			S
			else
			T
			end
			end
				M_IF

Applying Rule M_IF to progress_2 and progress_3

progress_23
  when
    $j \neq n$
    $g(j+1) \leq x$
  then
    if $k = j$ **then**
      $k, j := k+1, j+1$
    else
      $k, j, g := k+1, j+1, \mathsf{swap}\,(g, k+1, j+1)$
    end
  end

Putting together progress_1 and progress_23

```
progress_1
  when
    j ≠ n
    g(j + 1) > x
  then
    j := j + 1
  end
```

```
progress_23
  when
    j ≠ n
    g(j + 1) ≤ x
  then
    if k = j then
      k, j := k + 1, j + 1
    else
      k, j, g := k + 1, j + 1,
                 swap (g, k + 1, j + 1)
    end
  end
```

Merging Rule (3)

when
P
Q
then
S
end
 	
when
P
$\neg Q$
then
if R **then**
T
else
U
end
end
 	
$\rightsquigarrow$
 	
when
P
then
if Q **then**
S
elsif R **then**
T
else
U
end
end
	M_ELSIF

Applying M_ELSIF to progress_1 and progress_23

partition
when
  $j = n$
then
  skip
end

progress_123
when $j \neq n$ **then**
  if $g(j+1) > x$ **then**
    $j := j + 1$
  elsif $k = j$ **then**
    $k, j := k+1, j+1$
  else
    $k, j, g := k+1, j+1,$ **swap** $(g, k+1, j+1)$
  end
end

Merging Rule M_WHILE (special case)

when Q **then** S **end** **when** $\neg Q$ **then** skip **end** $\leadsto$ **while** Q **do** S **end**	M_WHILE

Array Partitioning : Final Merging (6)

Applying M_WHILE4 to partition and progress_123

init
$g := f$
$j := 0$
$k := 0$

progress_123_partition
while $j \neq n$ **do**
  if $g(j+1) > x$ **then**
    $j := j+1$
  elsif $k = j$ **then**
    $k, j := k+1, j+1$
  else
    $k, j, g := k+1, j+1, \text{swap}\,(g, k+1, j+1)$
  end
end

Applying Rule M_INIT to init and progress_123_partition yields

partition_program
$\boxed{g, k, j := f, 0, 0}$; init
while $j \neq m$ **do**
  if $g(j+1) > x$ **then**
    $\boxed{j := j+1}$ progress_1
  elsif $k = j$ **then**
    $\boxed{k, j := k+1, j+1}$ progress_2
  else
    $\boxed{k, j, g := k+1, j+1, \text{swap}\,(g, k+1, j+1)}$ progress_3
  end
end

Array Partitioning: Concluding Remarks

- The complete development requires 18 proofs.

- Among which 6 were interactive

Example 4: Array Sorting

- Given:
  - A numerical array f
- Result is:
  - Another numerical array g
- Such that:
  - g has the same elements as f
  - g is sorted in ascending order

Sorting

3	7	2	5	8	9	4	1

1	2	3	4	5	7	8	9

Sorting Initial State

constants: n
f

axm0_1: $0 < n$

axm0_2: $f \in 1\,..\,n \rightarrowtail \mathbb{N}$

variables: g

inv0_1: $g \in \mathbb{N} \leftrightarrow \mathbb{N}$

init
$g :\in \mathbb{N} \leftrightarrow \mathbb{N}$

final
when
$g \in 1\,..\,n \rightarrowtail \mathbb{N}$
$\mathrm{ran}\,(g) = \mathrm{ran}\,(f)$
$\forall\, i, j \cdot \begin{array}{l} i \in 1\,..\,n-1 \\ j \in i+1\,..\,n \\ \Rightarrow \\ g(i) < g(j) \end{array}$
then
skip
end

progress
status
anticipated
then
$g :\in \mathbb{N} \leftrightarrow \mathbb{N}$
end

Introducing a new variable k ranging form 1 to n

Current situation: array g is sorted from 1 to $k - 1$

1	**sorted**	**and**	$\leq$	$k-1$	k	**?**	n

Array Sorting First Refinement: the State

variables: g
k
l

inv1_1: $g \in 1\,..\,n \rightarrowtail \mathbb{N}$

inv1_2: $\mathrm{ran}(g) = \mathrm{ran}(f)$

inv1_3: $k \in 1\,..\,n$

inv1_4: $\forall\, i, j \cdot \begin{array}{l} i \in 1\,..\,k-1 \\ j \in i+1\,..\,n \\ \Rightarrow \\ g(i) < g(j) \end{array}$

inv1_5: $l \in \mathbb{N}$

Array Sorting First Refinement: the Events

init
  $g := f$
  $k := 1$
  $l :\in \mathbb{N}$

final
  when
    $k = n$
  then
    skip
  end

progress
  status
    convergent
  when
    $k \neq n$
    $l \in k \,..\, n$
    $g(l) = \min(g[k..n])$
  then
    $g := g \mathbin{\lhd\!\!-} \{k \mapsto g(l)\} \mathbin{\lhd\!\!-} \{l \mapsto g(k)\}$
    $k := k + 1$
    $l :\in \mathbb{N}$
  end

prog
  status
    anticipated
  then
    $l :\in \mathbb{N}$
  end

Sorting

3	7	2	5	8	9	4	1

Sorting

1	7	2	5	8	9	4	3

Sorting

1	2	7	5	8	9	4	3

Sorting

1	2	3	5	8	9	4	7

Sorting

1	2	3	4	8	9	5	7

Sorting

1	2	3	4	5	9	8	7

Sorting

1	2	3	4	5	7	8	9

Sorting

1	2	3	4	5	7	8	9

Sorting

1	2	3	4	5	7	8	9

- Introducing the variable j

variables:	g k l j

inv2_1:	$j \in k \,..\, n$
inv2_2:	$l \in k \,..\, j$
inv2_3:	$g(l) = \min(g[k \,..\, j])$

- Invariant **inv2_3** can be illustrated on the next diagram:

1 **sorted and smaller** $k-1$	k $g(l)$ **is the minimum** j	n

- Next are the refinements of the abstract events.

init
$g := f$
$k := 1$
$l := 1$
$j := 1$

final
when
$k = n$
then
skip
end

progress
when
$k \neq n$
$j = n$
then
$g := g \mathbin{\lhd\!\!\!-} \{k \mapsto g(l)\} \mathbin{\lhd\!\!\!-} \{l \mapsto g(k)\}$
$k := k + 1$
$j := k + 1$
$l := k + 1$
end

Second Refinement: the events (2)

prog1
refines
prog
status
convergent
when
$k \neq n$
$j \neq n$
$g(l) \leq g(j+1)$
then
$j := j+1$
end

prog2
refines
prog
status
convergent
when
$k \neq n$
$j \neq n$
$g(j+1) < g(l)$
then
$j := j+1$
$l := j+1$
end

Sorting

3	7	2	5	8	9	4	1

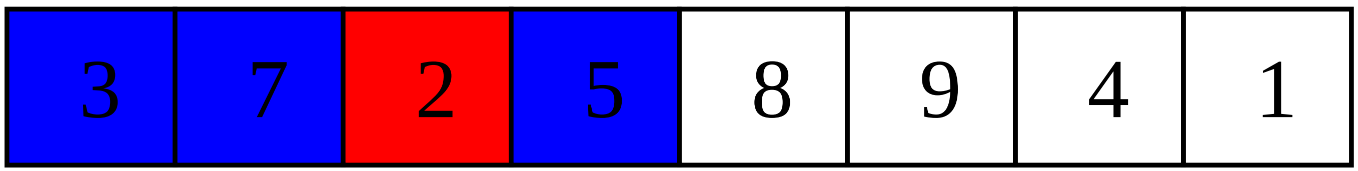
Sorting

3	7	2	5	8	9	4	1

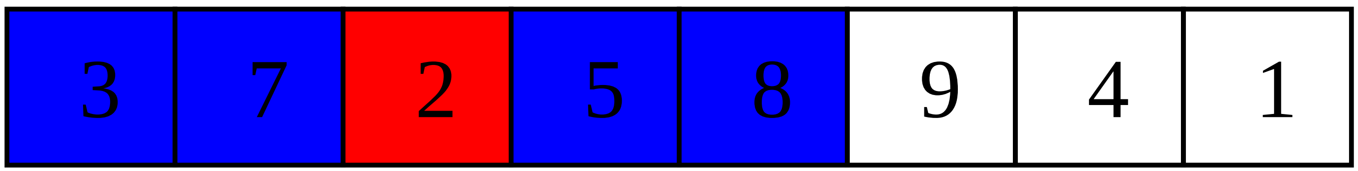
Sorting

3	7	2	5	8	9	4	1

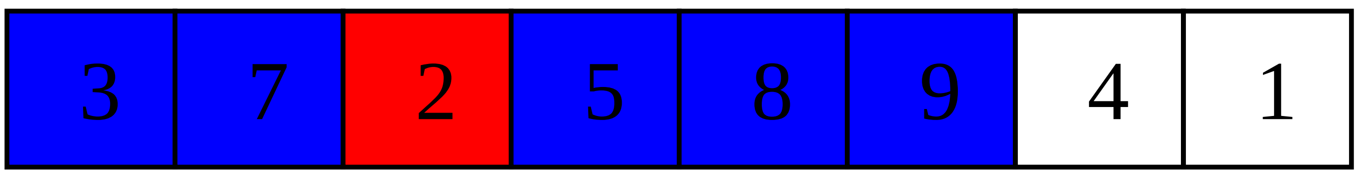
Sorting



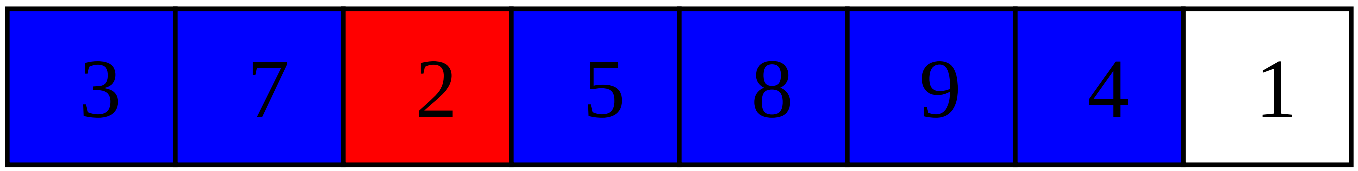
Sorting



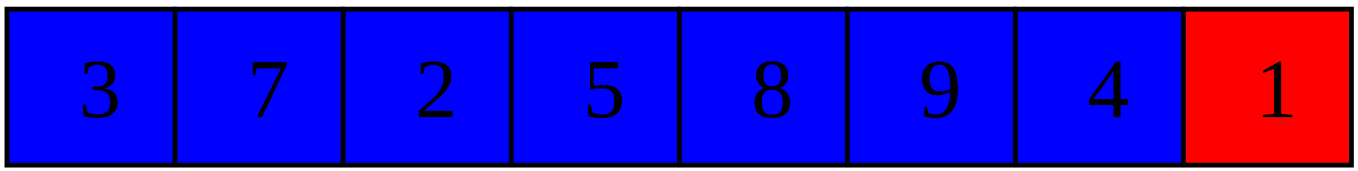
Sorting



Sorting



Sorting



Sorting

1	7	2	5	8	9	4	3

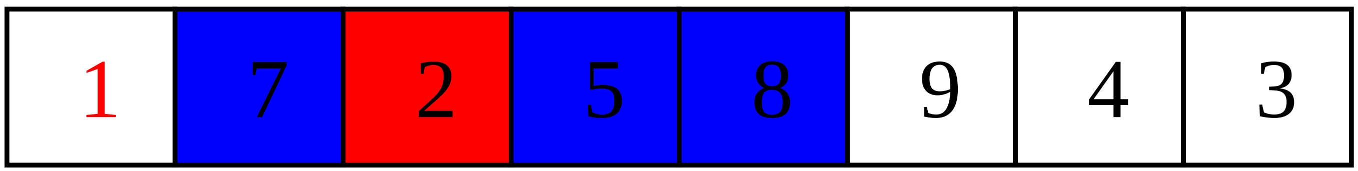
Sorting

1	7	2	5	8	9	4	3

Sorting

1	7	2	5	8	9	4	3

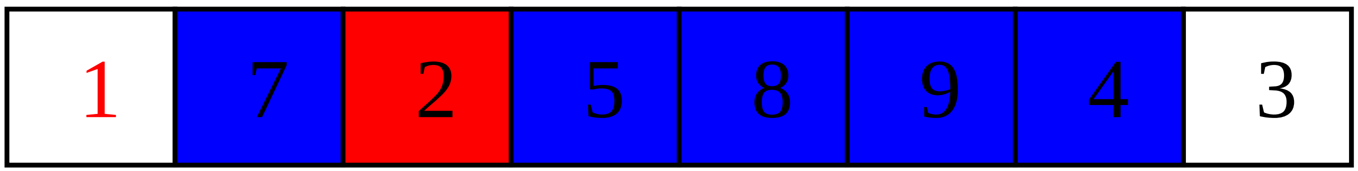
Sorting



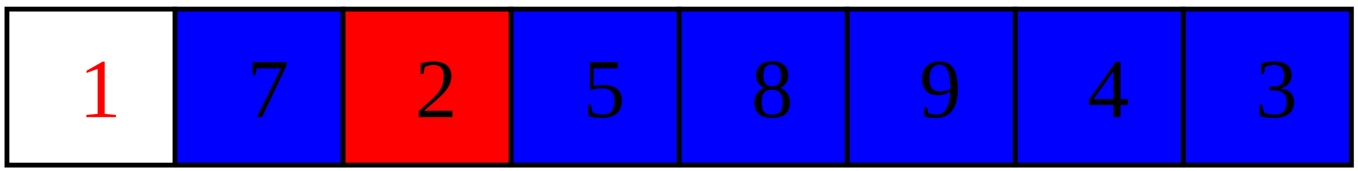
Sorting

1	7	2	5	8	9	4	3

Sorting



Sorting



Sorting

1	2	7	5	8	9	4	3

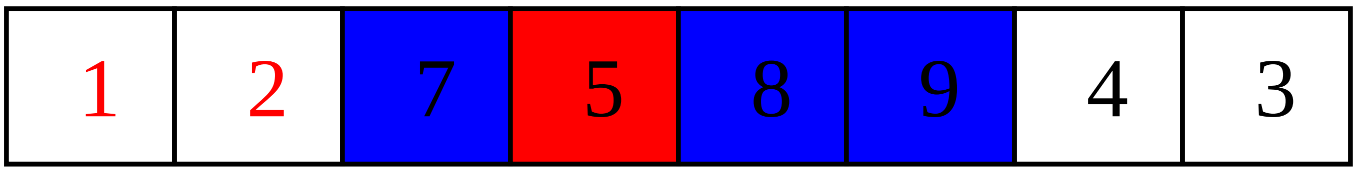
Sorting

1	2	7	5	8	9	4	3

Sorting

1	2	7	5	8	9	4	3

Sorting



Sorting

1	2	7	5	8	9	4	3

Sorting

1	2	7	5	8	9	4	3

Sorting

1	2	3	5	8	9	4	7

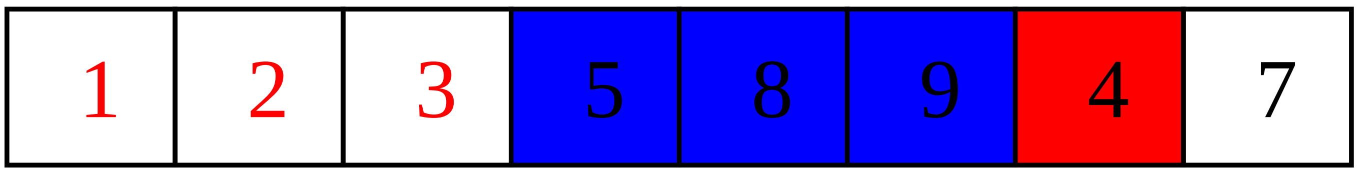
Sorting

1	2	3	5	8	9	4	7

Sorting

1	2	3	5	8	9	4	7

Sorting



Sorting

1	2	3	5	8	9	4	7

Sorting

1	2	3	4	8	9	5	7

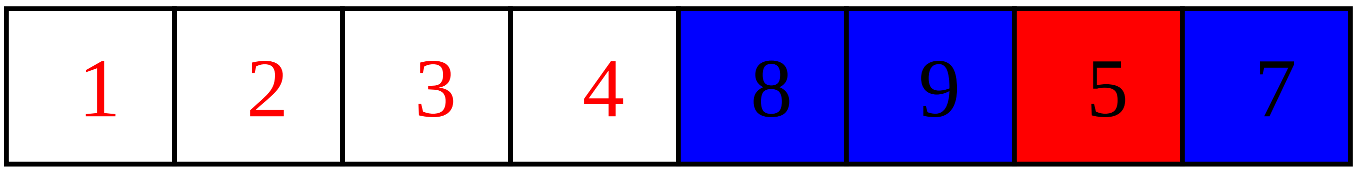
Sorting

1	2	3	4	8	9	5	7

Sorting

1	2	3	4	8	9	5	7

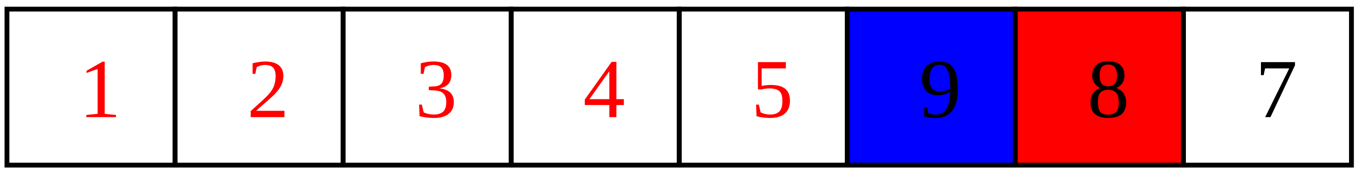
Sorting



Sorting

1	2	3	4	5	9	8	7

Sorting



Sorting

1	2	3	4	5	9	8	7

Sorting

1	2	3	4	5	7	8	9

Sorting

1	2	3	4	5	7	8	9

Sorting

1	2	3	4	5	7	8	9

Sorting

1	2	3	4	5	7	8	9

Applying M_IF to progr1 and progr2

```
progr_12
   when
      k < n
      j < n
   then
      if g(l) ≤ g(j + 1) then
         j := j + 1
      else
         j, l := j + 1, j + 1
      end
   end
```

```
progr
  when
    k < n
    j = n
  then
    k := k + 1
    j := k + 1
    l := k + 1
    g := swap (g, k, l)
  end
```

```
progr_12
  when
    k < n
    j < n
  then
    if g(l) ≤ g(j + 1) then
      j := j + 1
    else
      j, l := j + 1, j + 1
    end
  end
```

inv2_1: $j \in k \,..\, n$

Applying Rule M_WHILE to progr and progr_12

```
progr_progr_12
   when
      k < n
   then
      while j < n do
         if g(l) ≤ g(j+1) then
            j := j + 1
         else
            j, l := j + 1, j + 1
         end
      end;
      k, j, l, g := k + 1, k + 1, k + 1, swap (g, k, l)
   end
```

```
sort
when
    k = n
then
    skip
end
```

```
progr_progr_12
   when
      k < n
   then
      while  j < n  do
         if  g(l) ≤ g(j + 1)  then
            j := j + 1
         else
            j, l := j + 1, j + 1
         end
      end;
      k, j, l, g := k + 1, k + 1, k + 1, swap (g, k, l)
   end
```

inv1_3: $k \in 1\,..\,n$

Applying Rule M_WHILE to sort and progr_progr_12

```
sort_progr_progr_12
   while  k < n  do
      while j < n do
         if  h(l) ≤ h(j + 1)  then
            j := j + 1
         else
            j, l := j + 1, j + 1
         end
      end;
      k, j, l, g := k + 1, k + 1, k + 1, swap (g, k, l)
   end
```

Merging init and sort_progr_progr_12

init
$g := f$
$k := 1$
$j := 1$
$l := 1$

sort_progr_progr_12
while $k < n$ **do**
  while $j < n$ **do**
    if $g(l) \leq g(j+1)$ **then**
      $j := j + 1$
    else
      $j, l := j + 1, j + 1$
    end
  end;
  $k, j, l, g := k + 1, k + 1, k + 1,$ **swap** (g, k, l)
end

```
sort_program
  begin
    [g, k, j, l := f, 1, 1, 1] ;                                  init
    while k < n do
      while j < n do
        if g(l) ≤ g(j + 1) then
          [j := j + 1]                                            progr_1
        else
          [j, l := j + 1, j + 1]                                  progr_2
        end
      end;
      [k, j, l, g := k + 1, k + 1, k + 1, swap (g, k, l)]         progr
    end
  end
```

- The overall development requires 28 proofs.

- Among which 7 were interactive

Example 5: In Place Reversing of an Array

sets: S

constants: n, f

axm0_1: $n \in \mathbb{N}$

axm0_2: $0 < n$

axm0_3: $f \in 1\,..\,n \rightarrow S$

variables: g

inv0_1: $g \in \mathbb{N} \leftrightarrow S$

In Place Reversing of an Array: Example

Here is an array

$\mathbf{3}$	$\mathbf{2}$	$\mathbf{5}$	$\mathbf{4}$	$\mathbf{1}$	$\mathbf{9}$	$\mathbf{7}$	$\mathbf{8}$

Here is the reverse array

$\mathbf{8}$	$\mathbf{7}$	$\mathbf{9}$	$\mathbf{1}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{2}$	$\mathbf{3}$

An element which was at index $\boldsymbol{i}$ is now at index $\mathbf{8} - \boldsymbol{i} + \mathbf{1}$

init
  $g :\in \mathbb{N} \leftrightarrow S$

final
  when
    $g \in 1 \,..\, n \rightarrow S$
    $\forall k \cdot k \in 1 \,..\, n \;\Rightarrow\; g(k) = f(n - k + 1)$
  then
    skip
  end

progress
  status
    anticipated
  then
    $g :\in \mathbb{N} \leftrightarrow S$
  end

- We introduce two additional variables i and j, both in $1 \,..\, n$

- Initially i is equal to 1 and j is equal to n

- Here is the current situation:

1 **reversed**	i **unchanged** j	**reversed** n

- A new event is going to exchange elements in i and j.

variables: g, i, j

inv1_1: $g \in 1 \,..\, n \rightarrow S$

inv1_2: $i \in 1 \,..\, n$

inv1_3: $j \in 1 \,..\, n$

inv1_4: $i + j = n + 1$

inv1_5: $i \leq j + 1$

inv1_6: $\forall k \cdot k \in 1 \,..\, i - 1 \;\Rightarrow\; g(k) = f(n - k + 1)$

inv1_7: $\forall k \cdot k \in i \,..\, j \;\Rightarrow\; g(k) = f(k)$

inv1_8: $\forall k \cdot k \in j + 1 \,..\, n \;\Rightarrow\; g(k) = f(n - k + 1)$

init
$i := 1$
$j := n$
$g := f$

final
when
$j \leq i$
then
skip
end

progress
status
convergent
when
$i < j$
then
$g := g \mathbin{\lhd\!\!\!-} \{i \mapsto g(j)\} \mathbin{\lhd\!\!\!-} \{j \mapsto g(i)\}$
$i := i + 1$
$j := j - 1$
end

- All this leads to the following final program:

reverse_program
$\quad i, j, g := 1, n, f;$
$\quad$ **while** $i < j$ **do**
$\quad\quad i, j, g := i + 1, j - 1, \mathbf{swap}(g, i, j)$
$\quad$ **end**

Example 6: Reversing a Linear Chain

- So far, all our examples were dealing with arrays.

- This new example deals with pointers

- We want to reverse a linear chain

- A linear chain is made of nodes

- The nodes are pointing to each other by means of pointers

- To simplify, the nodes have no information fields

A Linear Chain

- Here is a linear chain:

$$\boxed{\quad f \quad} \to \boxed{\qquad} \to \ldots \to \boxed{\qquad} \to \boxed{\quad l \quad}$$

- The first node of the chain is denoted by f

- The last node is a special node denoted by l

- We suppose that f and l are distinct

- The nodes of the chain a taken in a set S

sets: S

constants: d, f, l, c

axm0_1: $d \subseteq S$

axm0_2: $f \in d$

axm0_3: $l \in d$

axm0_4: $f \neq l$

axm0_5: $c \in d \setminus \{l\} \rightarrowtail\!\!\!\!\twoheadrightarrow d \setminus \{f\}$

axm0_6: $\forall T \cdot T \subseteq c[T] \Rightarrow T = \varnothing$

Reversing the Chain

- Given the following initial chain

$$\boxed{\quad f \quad} \rightarrow \boxed{\quad x \quad} \rightarrow \ldots \rightarrow \boxed{\quad z \quad} \rightarrow \boxed{\quad l \quad}$$

- Then the transformed chain should look like this:

$$\boxed{\quad f \quad} \leftarrow \boxed{\quad x \quad} \leftarrow \ldots \leftarrow \boxed{\quad z \quad} \leftarrow \boxed{\quad l \quad}$$

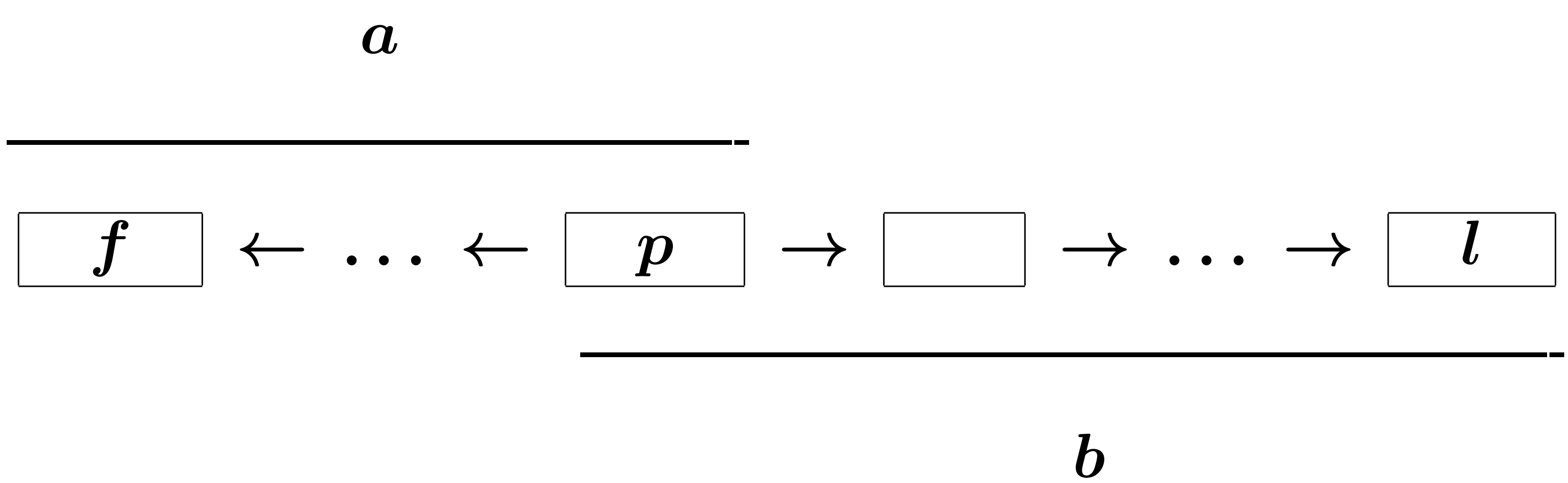
inv0_1: $r \in S \leftrightarrow S$

init
$r :\in S \leftrightarrow S$

reverse
$r := c^{-1}$

First Refinement

We introduce two additional chains a and b and a pointer p



- Node p starts both chains

- Main invariant: $a \cup b^{-1} = c^{-1}$

Initial and Final Situation

- At the beginning, p is equal to f, a is empty, and b is equal to c:

$$\boxed{\;p\;} \rightarrow \boxed{\;x\;} \rightarrow \ldots \rightarrow \boxed{} \rightarrow \boxed{} \rightarrow \ldots \rightarrow \boxed{\;z\;} \rightarrow \boxed{\;l\;}$$

$$b$$

- At the end, p is equal to l, a is the reversed chain, and b is empty:

$$a$$

$$\boxed{\;f\;} \leftarrow \boxed{\;x\;} \leftarrow \ldots \leftarrow \boxed{} \leftarrow \boxed{} \leftarrow \ldots \leftarrow \boxed{\;z\;} \leftarrow \boxed{\;p\;}$$

$$a$$

$$\boxed{f} \leftarrow \;\dots\; \leftarrow \boxed{p} \rightarrow \boxed{} \rightarrow \;\dots\; \rightarrow \boxed{l}$$

$$b$$

$$a$$

$$\boxed{f} \leftarrow \;\dots\; \leftarrow \boxed{} \leftarrow \boxed{p} \rightarrow \;\dots\; \rightarrow \boxed{l}$$

$$b$$

variables: r
a
b
p

inv1_1: $p \in d$

inv1_2: $a \in (\mathsf{cl}(c^{-1})[\{p\}] \cup \{p\}) \setminus \{f\} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathsf{cl}(c^{-1})[\{p\}]$

inv1_3: $b \in (\mathsf{cl}(c)[\{p\}] \cup \{p\}) \setminus \{l\} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathsf{cl}(c)[\{p\}]$

inv1_4: $c = a^{-1} \cup b$

progress
when
$p \in \mathrm{dom}(b)$
then
$p := b(p)$
$a(b(p)) := p$
$b := \{p\} \lhd b$
end

reverse
when
$b = \varnothing$
then
$r := a$
end

init
$r :\in S \leftrightarrow S$
$a, b, p := \varnothing, c, f$

Second Refinement: the State (1)

- We introduce a new constant nil

- We replace the chain b by the chain bn

- And we introduce a new pointer q

a

$$\boxed{f} \leftarrow \boxed{x} \leftarrow \ldots \leftarrow \boxed{p} \rightarrow \boxed{q} \rightarrow \ldots \rightarrow \boxed{z} \rightarrow \boxed{l} \rightarrow \boxed{nil}$$

bn

- Here is the new state:

constants: f, l, c, nil

variables: r, a, bn, p, q

axm2_1: $nil \in S$

axm2_2: $nil \notin d$

inv2_1: $bn = b \cup \{l \mapsto nil\}$

inv2_2: $q = bn(p)$

progress
when
$q \neq nil$
then
$p := q$
$a(q) := p$
$q := bn(q)$
$bn := \{p\} \lhd bn$
end

reverse
when
$q = nil$
then
$r := a$
end

init
$r :\in S \leftrightarrow S$
$a, bn := \varnothing, c \cup \{l \mapsto nil\}$
$p, q := f, c(f)$

- The previous situation with two chains a and bn

$$\overbrace{\boxed{f} \leftarrow \ldots \leftarrow \boxed{p}}^{a} \underbrace{\rightarrow \boxed{q} \rightarrow \ldots \rightarrow \boxed{l} \rightarrow \boxed{nil}}_{bn}$$

- The new situation with a single chain d

$$\underbrace{\boxed{f} \leftarrow \ldots \leftarrow \boxed{p} \qquad \boxed{q} \rightarrow \ldots \rightarrow \boxed{l} \rightarrow \boxed{nil}}_{d}$$

Third Refinement: the State

carrier set: S

constants: f, l, c

variables: r, p, q, d

inv3_1: $d \in S \nrightarrow S$

inv3_2: $d = (\{f\} \lhd bn) \lhd\!\!\!- a$

Third Refinement: the Events

progress
  when
    $q \neq nil$
  then
    $p := q$
    $d(q) := p$
    $q := d(q)$
  end

reverse
  when
    $q = nil$
  then
    $r := d \rhd \{nil\}$
  end

init
  $r :\in S \leftrightarrow S$
  $d := \{f\} \lhd (c \cup \{l \mapsto nil\}$
  $p, q := f, c(f)$

reverse_program

$$
\begin{array}{l}
\quad p, q, d := f, c(f), \{f\} \lhd (c \cup \{l \mapsto nil\}); \\
\quad \textbf{while } q \neq nil \textbf{ do} \\
\qquad p := q \\
\qquad d(q) := p \\
\qquad q := d(q) \\
\quad \textbf{end}; \\
\quad r := d \rhd \{nil\}
\end{array}
$$

Example 7: Integer Square root

- The squaring function is defined on all natural numbers

- And it is injective

- Therefore the inverse function, the square root function, exists

- But is is not defined for all natural number

- We want to make it total

- The integer square root of n by defect is a number r such that

$$r^2 \leq n < (r+1)^2$$

- The integer square root of 17, is 4 since we have

$$\mathbf{4^2 \leq 17 < 5^2}$$

- The integer square root of 16, is 4 since we have

$$\mathbf{4^2 \leq 16 < 5^2}$$

- The integer square root of 15, is 3 since we have

$$\mathbf{3^2 \leq 15 < 4^2}$$

Integer Square Root: Initial State and Events

constants: n

axm0_1: $n \in \mathbb{N}$

variables: r

inv0_1: $r \in \mathbb{N}$

init
  $r :\in \mathbb{N}$

final
  when
    $r^2 \leq n$
    $n < (r+1)^2$
  then
    skip
  end

progress
  status
    anticipated
  then
    $r :\in \mathbb{N}$
  end

variables: r

inv1_1: $r^2 \leq n$

init
$r := 0$

final
when
$n < (r+1)^2$
then
skip
end

progress
status
convergent
when
$(r+1)^2 \leq n$
then
$r := r + 1$
end

```
square_root_program
   r := 0;
   while (r + 1)^2 ≤ n do
      r := r + 1
   end
```

- We do not want to compute $(r+1)^2$ at each step

- We observe the following

$$\begin{aligned} ((r+1)+1)^2 &= (r+1)^2 + (2r+3) \\ 2(r+1)+3 &= (2r+3)+2 \end{aligned}$$

- We introduce two numbers a and b such that

$$\begin{aligned} a &= (r+1)^2 \\ b &= 2r+3 \end{aligned}$$

constants: n

variables: r, a, b

inv2_1: $a = (r+1)^2$

inv2_2: $b = 2r + 3$

init
$r := 0$
$a := 1$
$b := 3$

final
when
$n < a$
then
skip
end

progress
when
$a \leq n$
then
$r := r + 1$
$a := a + b$
$b := b + 2$
end

Program after Second Refinement

We obtain the following program:

square_root_program
$\boldsymbol{r, a, b := 0, 1, 3;}$
while $\boldsymbol{a \le n}$ **do**
$\boldsymbol{r, a, b := r + 1, a + b, b + 2}$
end

Example 8: Inverse of an Injective Numerical Function

- Same problem as in previous example but more general

- We are given a total numerical function f

- The function f is supposed to be strictly increasing

- Hence it is injective

- We want to compute its inverse by defect

- We shall borrow ideas form the binary search development

Inverse of an Injective Numerical Function: the State

constants: n
f

axm0_1: $f \in \mathbb{N} \rightarrow \mathbb{N}$

axm0_2: $\forall i, j \cdot \begin{array}{l} i \in \mathbb{N} \\ j \in \mathbb{N} \\ i < j \\ \Rightarrow \\ f(i) < f(j) \end{array}$

axm0_3: $n \in \mathbb{N}$

thm0_1: $f \in \mathbb{N} \rightarrowtail \mathbb{N}$

variables: r

inv0_1: $r \in \mathbb{N}$

init
  $r :\in \mathbb{N}$

final
  when
    $f(r) \leq n$
    $n < f(r+1)$
  then
    skip
  end

progress
  status
    anticipated
  then
    $r :\in \mathbb{N}$
  end

- We are supposedly given two constant numbers a and b such that

$$f(a) \;\leq\; n \;<\; f(b+1)$$

- We are thus certain that our result is within the interval $a \mathrel{..} b$

- We try to make this interval narrower

- We introduce a constant q in $a \mathrel{..} b$ and such that

$$f(r) \;\leq\; n \;<\; f(q+1)$$

First Refinement: the State

constants: f, n, a, b

axm1_1: $a \in \mathbb{N}$

axm1_2: $b \in \mathbb{N}$

axm1_3: $f(a) \leq n$

axm1_4: $n < f(b+1)$

variables: r, p, q

inv1_1: $q \in \mathbb{N}$

inv1_2: $r \leq q$

inv1_3: $f(r) \leq n$

inv1_4: $n < f(q+1)$

First Refinement: the Events (1)

init
$r := a$
$q := b$

final
when
$r = q$
then
skip
end

dec
refines
progress
status
convergent
any x **where**
$r \neq q$
$x \in r+1 \,..\, q$
$n < f(x)$
then
$q := x - 1$
end

inc
refines
progress
status
convergent
any x **where**
$r \neq q$
$x \in r+1 \,..\, q$
$f(x) \leq n$
then
$r := x$
end

What we Have to Prove

- Event init refines its abstraction

- Event inverse refines its abstraction

- Events inc and dec refine skip

- Events inc and dec decrease a variant

- The system is deadlock-free

- We reduce the non-determinacy

dec **when** $r \neq q$ $n < f((r+1+q)/2)$ **with** $x = (r+1+q)/2$ **then** $q := (r+1+q)/2 - 1$ **end**	inc **when** $r \neq q$ $f((r+1+q)/2) \leq n$ **with** $x = (r+1+q)/2$ **then** $r := (r+1+q)/2$ **end**

- In order to prove this refinement the following theorem can be useful:

thm2_1:

$$\forall\, x, y \cdot \begin{array}{l} x \in \mathbb{N} \\ y \in \mathbb{N} \\ x \leq y \\ \Rightarrow \\ (x+y)/2 \in x\,..\,y \end{array}$$

```
inverse_program
  r, q := a, b;
  while r ≠ q do
    if n < f((r + 1 + q)/2) then
      q := (r + 1 + q)/2 − 1
    else
      r := (r + 1 + q)/2
    end
  end
```

- The development made in this example is generic

- We can consider that the constants f, a, and b are parameters

- By instantiating them we obtain some new programs almost for free

- But we have to prove the properties of the instantiated constants:

  In our case we have to prove:

  - **axm0_1**: f is a total function
  - **axm0_2**: f is increasing
  - **axm1_3** and **axm1_4**: $f(a) \leq n < f(b+1)$

First Instantiation (1)

- f is instantiated to the squaring function

- a and b are instantiated to 0 and n since we have

$$0^2 \ \leq \ n \ < \ (n+1)^2$$

- We shall obtain an integer square root program

First Instantiation (2)

```
square_root_program
  r, q := 0, n;
  while r ≠ q do
    if n < ((r + 1 + q)/2)^2 then
      q := (r + 1 + q)/2 − 1
    else
      r := (r + 1 + q)/2
    end
  end;
  r := p
```

Second Instantiation (1)

- f is instantiated to the function which "multiply by m"

- a and b are instantiated to 0 and n since we have

$$m \times 0 \ \leq \ n \ < \ m \times (n+1)$$

- We shall obtain an integer division program: n/m

```
integer_division_program
   r, q := 0, n;
   while r ≠ q do
      if n < m × (r + 1 + q)/2) then
         q := (r + 1 + q)/2 − 1
      else
         r := (r + 1 + q)/2
      end
   end;
   r := p
```