Nelineární optimalizace a numerické metody

(MI–NON)

Magisterský program: Informatika

Obor: Teoretická informatika

Katedra: 18101 Katedra teoretické informatiky

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Evropský sociální fond

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Řešení nelineárních úloh

Porovnání nelineárních a lineárních soustav rovnic

- absence věty o řešitelnosti - obecně neznámý počet řešení
- absence univerzálního algoritmu řešení - neexistuje obdoba Gaussova eliminačního algoritmu

Metody řešení soustav nelineárních rovnic

- metoda půlení intervalu
- metoda sečen
- metoda regula falsi
- metoda prosté iterace
- Newtonova-Raphsonova metoda

Newtonova-Raphsonova metoda

Jedna funkce jedné proměnné

$$f(x) = 0$$

$$f\left(x^{(k+1)}\right) \approx f\left(x^{(k)}\right) + \frac{\mathrm{d}f\left(x^{(k)}\right)}{\mathrm{d}x}\left(x^{(k+1)} - x^{(k)}\right) = 0$$

$$x^{(k+1)} = x^{(k)} - \frac{f\left(x^{(k)}\right)}{\dfrac{\mathrm{d}f\left(x^{(k)}\right)}{\mathrm{d}x}}$$

Příklad - kvadratická funkce jedné proměnné

obecně

$$
\begin{aligned}
f(x) &= ax^2 + bx + c = 0 \\
f'(x) &= 2ax + b \\
x^{(k+1)} &= x^{(k)} - \frac{a(x^{(k)})^2 + bx^{(k)} + c}{2ax^{(k)} + b}
\end{aligned}
$$

konkrétní funkce

$$
\begin{aligned}
f(x) &= x^2 - 11x + 10 = 0 \\
f'(x) &= 2x - 11 \\
f(1) &= 0\,, \quad f(10) = 0
\end{aligned}
$$

k	$x^{(k)}$	$f(x^{(k)})$
0	0.000000000000e+00	1.000000000000e+01
1	9.090909090909e-01	8.264462809917e-01
2	9.990999099910e-01	8.101620243033e-03
3	9.999999100000e-01	8.100000161671e-07
4	1.000000000000e+00	7.993605777301e-15

Příklad - goniometrická funkce

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ x^{(k+1)} &= x^{(k)} - \frac{\sin x^{(k)}}{\cos x^{(k)}} \end{aligned}$$

k	$x^{(k)}$	$f(x^{(k)})$
0	8.000000000000e-01	7.173560908995e-01
1	-2.296385570504e-01	-2.276255837975e-01
2	4.123579169748e-03	4.123567483600e-03
3	-2.337247535615e-08	-2.337247535615e-08
4	3.308722450212e-24	3.308722450212e-24

k	$x^{(k)}$	$f(x^{(k)})$
0	2.200000000000e+00	8.084964038196e-01
1	3.573823056769e+00	-4.188971239432e-01
2	3.112499733480e+00	2.908881625187e-02
3	3.141600864433e+00	-8.210843004404e-06
4	3.141592653590e+00	1.224606353822e-16

k	$x^{(k)}$	$f(x^{(k)})$
0	1.700000000000e+00	9.916648104525e-01
1	9.396602139459e+00	2.817209343589e-02
2	9.424785419182e+00	-7.458413052948e-06
3	9.424777960769e+00	3.673819061467e-16

Vektorová funkce mnoha proměnných

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{pmatrix}$$

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$$

$$\boldsymbol{f}(\boldsymbol{x}^{(k+1)}) \approx \boldsymbol{f}(\boldsymbol{x}^{(k)}) + \frac{\mathrm{d}\boldsymbol{f}(\boldsymbol{x}^{(k)})}{\mathrm{d}\boldsymbol{x}} \left(\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right) = \boldsymbol{0}$$

$$f_i(\boldsymbol{x}^{(k+1)}) \approx f_i(\boldsymbol{x}^{(k)}) + \sum_{j=1}^{n} \frac{\partial f_i(\boldsymbol{x}^{(k)})}{\partial x_j} \left(x_j^{(k+1)} - x_j^{(k)} \right)$$

$$\boldsymbol{J}(\boldsymbol{x}) = \begin{pmatrix} \dfrac{\partial f_1(\boldsymbol{x})}{\partial x_1} & \dfrac{\partial f_1(\boldsymbol{x})}{\partial x_2} & \cdots & \dfrac{\partial f_1(\boldsymbol{x})}{\partial x_n} \\ \dfrac{\partial f_2(\boldsymbol{x})}{\partial x_1} & \dfrac{\partial f_2(\boldsymbol{x})}{\partial x_2} & \cdots & \dfrac{\partial f_2(\boldsymbol{x})}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \dfrac{\partial f_n(\boldsymbol{x})}{\partial x_1} & \dfrac{\partial f_n(\boldsymbol{x})}{\partial x_2} & \cdots & \dfrac{\partial f_n(\boldsymbol{x})}{\partial x_n} \end{pmatrix}$$

$$\begin{aligned}\boldsymbol{J}^{(k)} &= \boldsymbol{J}(\boldsymbol{x}^{(k)}) \\ \boldsymbol{f}^{(k)} &= \boldsymbol{f}(\boldsymbol{x}^{(k)})\end{aligned}$$

$$\boldsymbol{f}^{(k+1)} \approx \boldsymbol{f}^{(k)} + \boldsymbol{J}^{(k)} \left(\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right) = \mathbf{0}$$

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \left(\boldsymbol{J}^{(k)}\right)^{-1} \boldsymbol{f}^{(k)}$$

$$x^{(k+1)} = x^{(k)} - \frac{1}{\frac{\mathrm{d} f(x^{(k)})}{\mathrm{d} x}} f(x^{(k)})$$

Příklad - kvadratická funkce dvou proměnných

vektorová funkce

$$
\begin{aligned}
f_x(x,y) &= x^2 + 3x - y^2 + 3y - 10 \\
f_y(x,y) &= -x^2 - 4x + y^2 + y
\end{aligned}
$$

řešení

$$
\begin{aligned}
f_x(2,3) &= 0 \\
f_y(2,3) &= 0
\end{aligned}
$$

$$
\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} x^2 + 3x - y^2 + 3y - 10 \\ -x^2 - 4x + y^2 + y \end{pmatrix}
$$

$$\boldsymbol{J}(\boldsymbol{x}) = \begin{pmatrix} 2x+3 & -2y+3 \\ -2x-4 & -2y+1 \end{pmatrix}$$

k	$x^{(k)}$	$y^{(k)}$	$f_x(x^{(k)}, y^{(k)})$	$f_y(x^{(k)}, y^{(k)})$
0	0.000000000000e+00	0.000000000000e+00	-1.000000000000e+01	0.000000000000e+00
1	6.666666666667e-01	2.666666666667e+00	-6.666666666667e+00	6.666666666667e+00
2	2.444444444444e+00	3.111111111111e+00	2.962962962963e+00	-2.962962962963e+00
3	2.026143790850e+00	3.006535947712e+00	1.640394719979e-01	-1.640394719979e-01
4	2.000101726813e+00	3.000025431703e+00	6.358022806019e-04	-6.358022806019e-04
5	2.000000001552e+00	3.000000000388e+00	9.701276229394e-09	-9.701276673484e-09

Modifikovaná Newtonova-Raphsonova metoda

Jedna funkce jedné proměnné

$$f(x) = 0$$

$$f(x^{(k+1)}) \approx f(x^{(k)}) + \frac{\mathrm{d}f(x^{(0)})}{\mathrm{d}x} \left(x^{(k+1)} - x^{(k)}\right) = 0$$

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{\dfrac{\mathrm{d}f(x^{(0)})}{\mathrm{d}x}}$$

Příklad - kvadratická funkce jedné proměnné

obecně

$$\begin{aligned} f(x) &= ax^2 + bx + c = 0 \\ f'(x) &= 2ax + b \\ x^{(k+1)} &= x^{(k)} - \frac{a(x^{(k)})^2 + bx^{(k)} + c}{2ax^{(0)} + b} \end{aligned}$$

konkrétní funkce

$$\begin{aligned} f(x) &= x^2 - 11x + 10 = 0 \\ f'(x) &= 2x - 11 \quad \Rightarrow \quad f'(x^{(0)}) = 2x^{(0)} - 11 \\ f(1) &= 0\,, \quad f(10) = 0 \end{aligned}$$

k	$x^{(k)}$	$f(x^{(k)})$
0	0.000000000000e+00	1.000000000000e+01
1	9.090909090909e-01	8.264462809917e-01
2	9.842223891811e-01	1.422474303736e-01
3	9.971539737605e-01	2.562233602103e-02
4	9.994832770351e-01	4.650773686545e-03
5	9.999060746430e-01	8.453370350859e-04
6	9.999829234644e-01	1.536891123779e-04
7	9.999968952018e-01	2.794319300995e-05
8	9.999994354921e-01	5.080571226647e-06
9	9.999998973622e-01	9.237399151022e-07
10	9.999999813386e-01	1.679527020949e-07
11	9.999999966070e-01	3.053685505589e-08
12	9.999999993831e-01	5.552155735167e-09

Vektorová funkce mnoha proměnných

$$\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$$

$$\boldsymbol{f}(\boldsymbol{x}^{(k+1)}) \approx \boldsymbol{f}(\boldsymbol{x}^{(k)}) + \frac{\mathrm{d}\boldsymbol{f}(\boldsymbol{x}^{(0)})}{\mathrm{d}\boldsymbol{x}} \left(\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right) = \mathbf{0}$$

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \left(\boldsymbol{J}^{(0)}\right)^{-1} \boldsymbol{f}^{(k)}$$

Příklad

$$\begin{aligned} f_x(x,y) &= x^2 + 3x - y^2 + 3y - 10 \\ f_y(x,y) &= -x^2 - 4x + y^2 + y \end{aligned}$$

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} x^2 + 3x - y^2 + 3y - 10 \\ -x^2 - 4x + y^2 + y \end{pmatrix}$$

$$\boldsymbol{J}(\boldsymbol{x}^{(0)}) = \begin{pmatrix} 2x^{(0)} + 3 & -2y^{(0)} + 3 \\ -2x^{(0)} - 4 & -2y^{(0)} + 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -4 & 1 \end{pmatrix}$$

k	$x^{(k)}$	$y^{(k)}$	$f_x(x^{(k)}, y^{(k)})$	$f_y(x^{(k)}, y^{(k)})$
0	0.000000000000e+00	0.000000000000e+00	-1.000000000000e+01	0.000000000000e+00
1	6.666666666667e-01	2.666666666667e+00	-6.666666666667e+00	6.666666666667e+00
2	2.444444444444e+00	3.111111111111e+00	2.962962962963e+00	-2.962962962963e+00
3	1.654320987654e+00	2.913580246914e+00	-2.048468221308e+00	2.048468221308e+00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	2.000000001650e+00	3.000000000413e+00	1.031398611222e-08	-1.031398655631e-08
51	1.999999998900e+00	2.999999999725e+00	-6.875991774222e-09	6.875992218311e-09