Algorithms and Data Structures

Homework 5

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Problem 1

a) My algorithm is using the first and last elements as pivots, and because the assignment says that I should use the first two elements that's why I am swapping the second element as last. My algorithm can be found in "quick.py"

b) • The worst case would be if the array is already sorted, it doesn't matter whether ascending or descending order.

$$T(n) = T(n-2) + \underbrace{T(0) + T(0)}_{\text{because the two subarrays will have 0 elements in them}} + \theta(n)$$

And this will be repeated $n/2$ times because in every step we are taking two pivots.

• And the best case would be if the three subarrays are of equal or similar length

If we say that x is the length of the middle subarray and y to the right subarray

$\Rightarrow T(n) = T(n - \cancel{x} - y - 2) + T(x) + T(y) + \theta(n)$

And like I said the best case would be if the subarrays are of equal length so $x = y = \frac{n}{3}$

$$\Rightarrow T(x) = T\left(\frac{n}{3}\right), \quad T(y) = T\left(\frac{n}{3}\right)$$

$$T(n-x-y-2) = T\left(n - \frac{n}{3} - \frac{n}{3} - 2\right) = T(n/3 - 2)$$

And we would have:

$$T(n) = T(n/3 - 2) + 2T(n/3) + \theta(n)$$

c) My algorithm in c) works in the same way as a) but now the condition is that we use two random pivots, so I am ~~[illegible]~~ swapping the first random element with the start of the array and the second one with the end element of the array.

My algorithm can be found in "modified.py".

Problem 2

a) $\sum_{k=2}^{n-1} k \lg k \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$

For the base case $n=3$ because it can't be lower

Base case:

$$2 \lg 2 \leq \frac{1}{2} 9 \lg 3 - \frac{1}{8} 9$$

$$0.6 \leq 2.147 - 1.125$$

$$0.6 \leq 1.022 \quad \checkmark$$

Induction hypothesis:

Let's suppose that for n $\sum_{k=2}^{n-1} k \lg k \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$

$$2 \lg 2 + \dots + (n-1) \lg (n-1) \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2 \quad (*)$$

Induction step:

And if we try it for $n+1$ we get:

$$\sum_{k=2}^{n} k \lg k \leq \frac{1}{2} (n+1)^2 \lg (n+1) - \frac{1}{8} (n+1)^2$$

$$2 \lg 2 + \dots + n \lg n \leq \frac{1}{2} (n^2 + 2n + 1) \lg (n+1) - \frac{1}{8} (n^2 + 2n + 1)$$

If we subtract with $(*)$ we get:

$$n \lg n \leq \frac{1}{2} (n^2 + 2n + 1) \lg (n+1) - \frac{1}{8} (n^2 + 2n + 1) - \left(\frac{1}{2} n^2 \lg n - \frac{1}{8} n^2\right)$$

$$n \lg n \leq \frac{n^2}{8} + \frac{1}{8} (-n^2 - 2n - 1) - \frac{1}{2} n^2 \lg (n) + \frac{1}{2} (n^2 + 2n + 1) \lg (n+1)$$

Problem 3

Prooving $\lg n! = \Theta(n \lg n)$

In order to proove this we need to find the upper and lower bound

Upper bound:

$$\lg(n!) = \Theta(n \lg n)$$

$$\lg\left(\prod_{i=1}^{n} n\right) = \lg n(n-1)(n-2)\dots(1)$$

$$\Rightarrow \lg(n) + \lg(n-1) + \dots + \lg(1)$$

$$\leq n \cdot \lg(n)$$

$$\Rightarrow O(n \lg(n))$$

Lower bound:

$$\lg n! \in \Omega(n \lg n)$$

$$\lg(n!) = \lg(n) + \lg(n-1) + \dots + \lg(1)$$

$$\geq \lg \frac{n}{2} + \lg\left(\frac{n}{2}+1\right) + \lg\left(\frac{n}{2}+2\right) \dots + \lg(n)$$

$$\geq \frac{n}{2} \lg\left(\frac{n}{2}\right) = \frac{n}{2}\left(\lg(n) - \lg(2)\right) \Rightarrow \Theta(n \lg(n))$$