COMPUTER VISION
LECTURE 3 – LINEAR ALGEBRA REVIEW

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OUTLINE

- Vectors and matrices
  - Basic Matrix Operations
  - Determinants, norms, trace
  - Special Matrices
- Transformation Matrices
  - Homogeneous coordinates
  - Translation
- Matrix inverse
- Matrix rank
- Singular Value Decomposition
  - Image Compression

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Vectors and matrices are just **collections of ordered numbers** that represent something: movements in space, scaling factors, pixel brightness, etc.

We'll define some common uses and standard operations on them.

VECTOR

- A column vector $\mathbf{v} \in \mathbb{R}^{n \times 1}$ where $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

- A row vector $\mathbf{v}^T \in \mathbb{R}^{1 \times n}$ where $\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$

T denotes the transpose operation

VECTOR

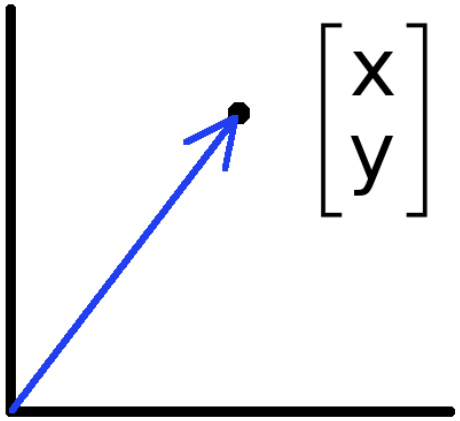
- We will only consider column vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- Keep track of the orientation of your vectors when programming in MATLAB
- You can transpose a vector V in MATLAB by writing **V'**

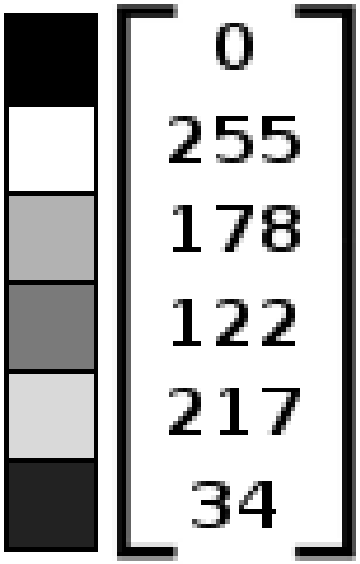
(But in class materials, we will always use V^T to indicate transpose)

VECTORS HAVE TWO MAIN USES



- Vectors can represent an offset in 2D or 3D space
- Points are just vectors from the origin

- Data (pixels, gradients at an image keypoint, etc) can also be treated as a vector
- Such vectors don't have a geometric interpretation, but calculations like "distance" can still have value



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- A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is an array of numbers with size by *m* rows and *n* columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

- If $m = n$, we say that $\mathbf{A}$ is square.

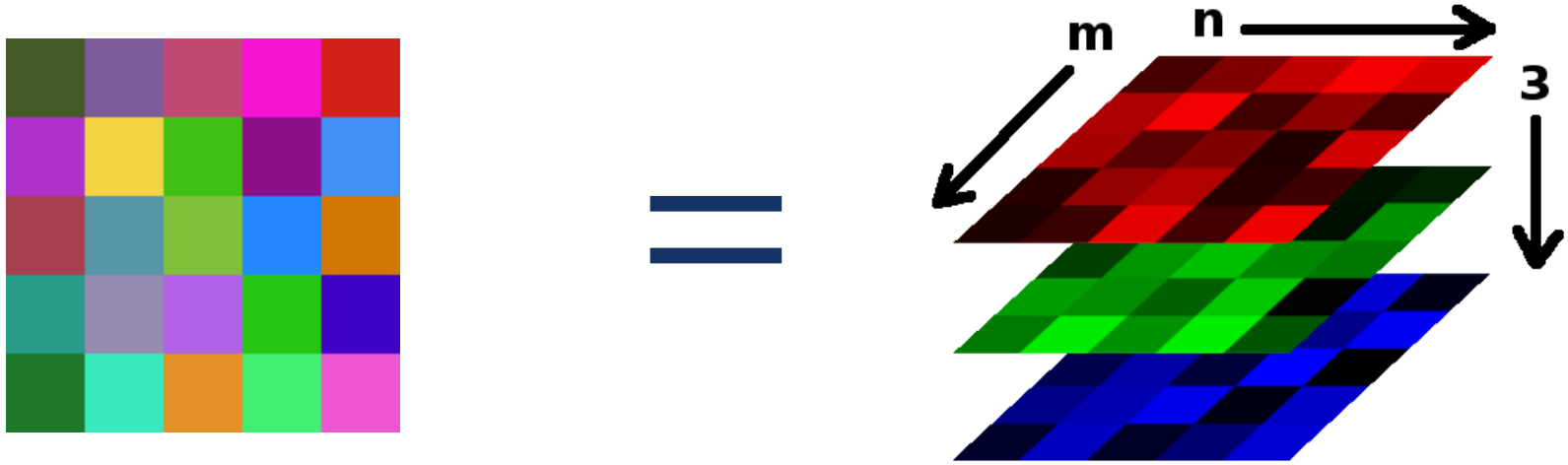
$$=$$

$$\begin{bmatrix} 193 & 180 & 210 & 112 & 125 \\ 189 & 8 & 177 & 97 & 114 \\ 100 & 71 & 81 & 195 & 165 \\ 167 & 12 & 242 & 203 & 181 \\ 44 & 25 & 9 & 48 & 192 \end{bmatrix}$$

- MATLAB represents an image as a matrix of pixel brightnesses
- Note that the upper left corner is [y,x] = (1,1)

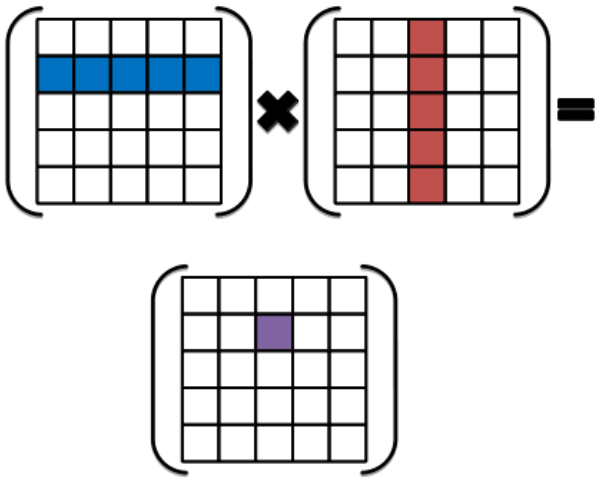
COLOR IMAGES

- Grayscale images have one number per pixel, and are stored as an $m \times n$ matrix.
- Color images have 3 numbers per pixel – red, green, and blue brightnesses (RGB)
- Stored as an $m \times n \times 3$ matrix



BASIC MATRIX OPERATIONS

- We will discuss:
  - Addition
  - Scaling
  - Dot product
  - Multiplication
  - Transpose
  - Inverse / pseudoinverse
  - Determinant / trace



MATRIX OPERATIONS

- **Addition**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+1 & b+2 \\ c+3 & d+4 \end{bmatrix}$$

  - Can only add a matrix with matching dimensions, or a scalar.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 7 = \begin{bmatrix} a+7 & b+7 \\ c+7 & d+7 \end{bmatrix}$$

- **Scaling**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times 3 = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

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- **Vector Norm**

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

$$f : \mathbb{R}^n \to \mathbb{R}$$

More formally, a norm is any function that satisfies 4 properties:

- **Non-negativity:** For all $x \in \mathbb{R}^n,\ f(x) \geq 0$
- **Definiteness:** f(x) = 0 if and only if x = 0.
- **Homogeneity:** For all $x \in \mathbb{R}^n,\ t \in \mathbb{R},\ f(tx) = |t| f(x)$
- **Triangle inequality:** For all $x, y \in \mathbb{R}^n,\ f(x + y) \leq f(x) + f(y)$

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- **Example Norms**

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_\infty = \max_i |x_i|$$

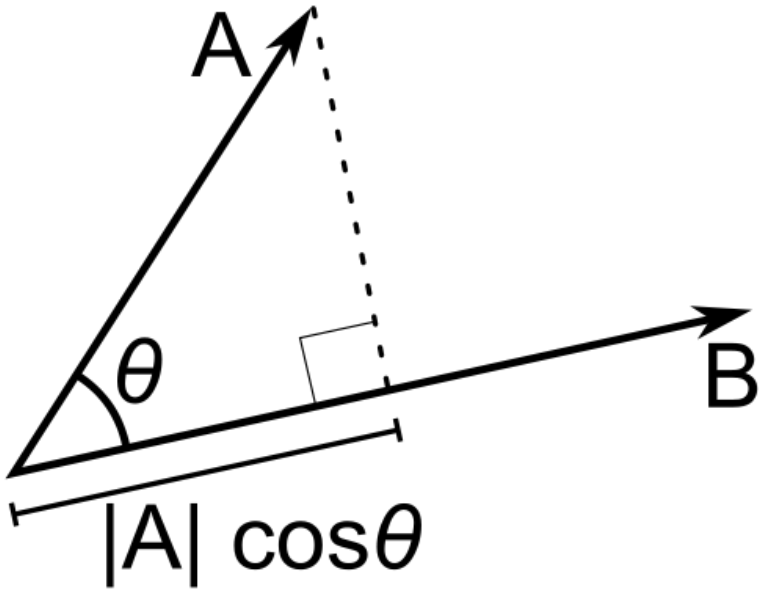
- General ℓ_p norms:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

- Inner product (dot product) of vectors
  - Multiply corresponding entries of two vectors and add up the result
  - x·y is also |x| |y| cos (*the angle between x and y*)

$$\mathbf{x}^T\mathbf{y} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i \qquad (\text{scalar})$$

- Inner product (dot product) of vectors
  - If B is a unit vector, then A·B gives the length of A which lies in the direction of B



MATRIX OPERATIONS

- The product of two matrices

$$A \in \mathbb{R}^{m \times n}$$

$$B \in \mathbb{R}^{n \times p}$$

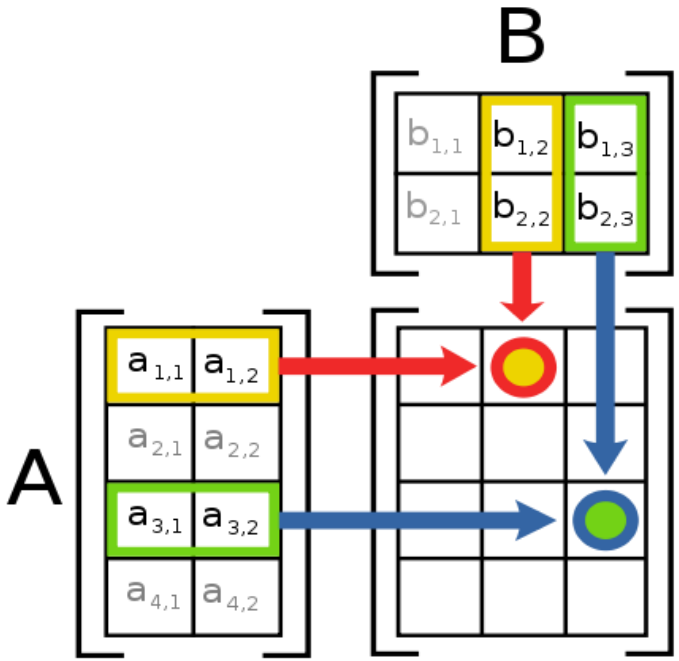
$$C = AB \in \mathbb{R}^{m \times p}$$

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

$$C = AB = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p \end{bmatrix}$$

MATRIX OPERATIONS

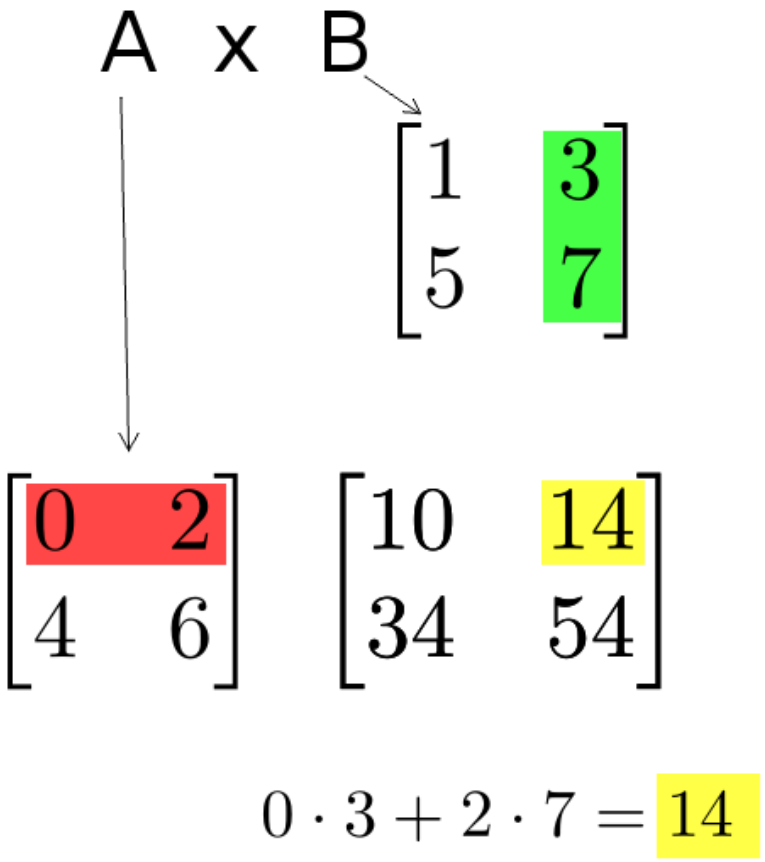
- Multiplication
- The product AB is:



- Each entry in the result is (that row of A) dot product with (that column of B)
- Many uses, which will be covered later

MATRIX OPERATIONS

- Multiplication example:



Each entry of the matrix product is made by taking the dot product of the corresponding row in the left matrix, with the corresponding column in the right one.

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- The product of two matrices - properties

Matrix multiplication is associative: $(AB)C = A(BC)$.

Matrix multiplication is distributive: $A(B + C) = AB + AC$.

Matrix multiplication is, in general, *not* commutative; that is, it can be the case that $AB \neq BA$. (For example, if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times q}$, the matrix product BA does not even exist if m and q are not equal!)

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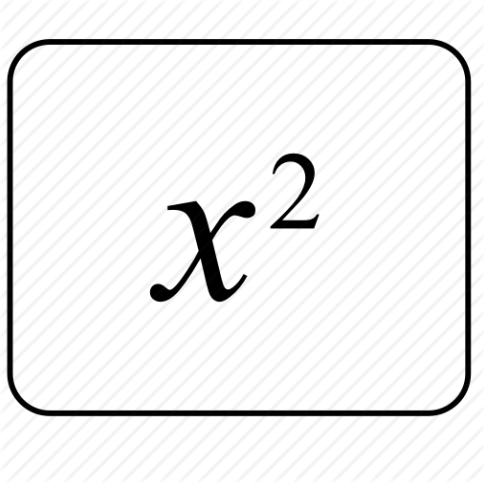
Powers

- By convention, we can refer to the matrix product AA as A^2, and AAA as A^3, etc.
- Obviously only square matrices can be multiplied that way



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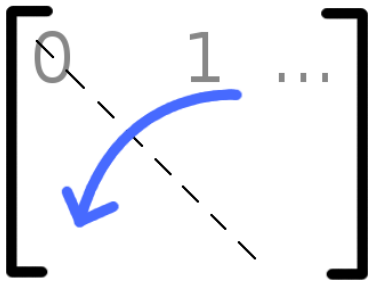
- **Transpose** – flip matrix, so row 1 becomes column 1



$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

- A useful identity:

$$(ABC)^T = C^T B^T A^T$$

- Determinant

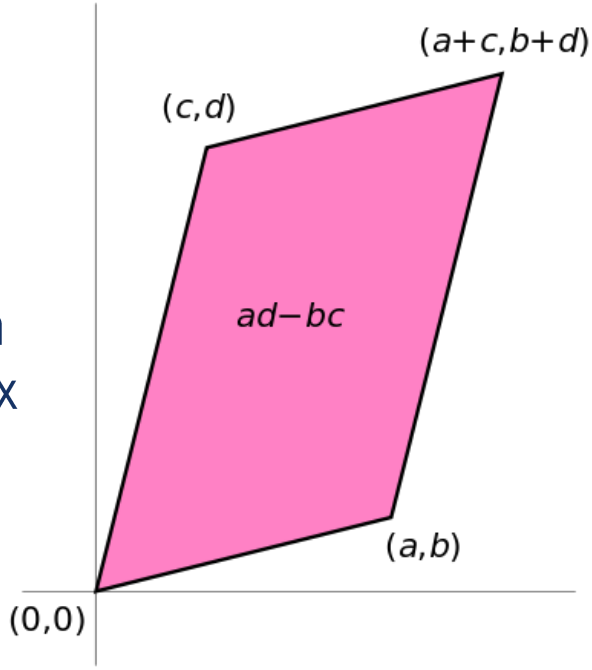
  - $\det(\mathbf{A})$ returns a scalar
  - Represents area (or volume) of the parallelogram described by the vectors in the rows of the matrix
  - $$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(\mathbf{A}) = ad - bc$$
  - Properties:

$$\det(\mathbf{AB}) = \det(\mathbf{BA})$$

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

$$\det(\mathbf{A}^T) = \det(\mathbf{A})$$

$$\det(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A} \text{ is singular}$$



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- **Trace**

$$\mathrm{tr}(\mathbf{A}) = \text{sum of diagonal elements} \qquad \mathbf{tr}(\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}) = 1 + 7 = 8$$

  - Invariant to a lot of transformations, so it's used sometimes in proofs. *(rarely in this class though)*

  - Properties:

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$$
$$\mathrm{tr}(\mathbf{A} + \mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$$

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- **Vector Norms**

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_\infty = \max_i |x_i|$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}. \qquad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

- **Matrix norms**: Norms can also be defined for matrices, such as the Frobenius norm:

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2} = \sqrt{\mathrm{tr}(A^T A)}$$

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- **Identity matrix `I`**
  - Square matrix, 1's along diagonal, 0's elsewhere
  - `I` · [another matrix] = [that matrix]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Diagonal matrix**
  - Square matrix with numbers along diagonal, 0's elsewhere
  - A diagonal · [another matrix] scales the rows of that matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

SPECIAL MATRICES

- Symmetric matrix

$$\mathbf{A}^T = \mathbf{A} \qquad \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 5 & 7 & 1 \end{bmatrix}$$

- Skew-symmetric matrix

$$\mathbf{A}^T = -\mathbf{A} \qquad \begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & -7 \\ 5 & 7 & 0 \end{bmatrix}$$

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Matrix multiplication can be used to transform vectors. A matrix used in this way is called a transformation matrix.

TRANSFORMATION

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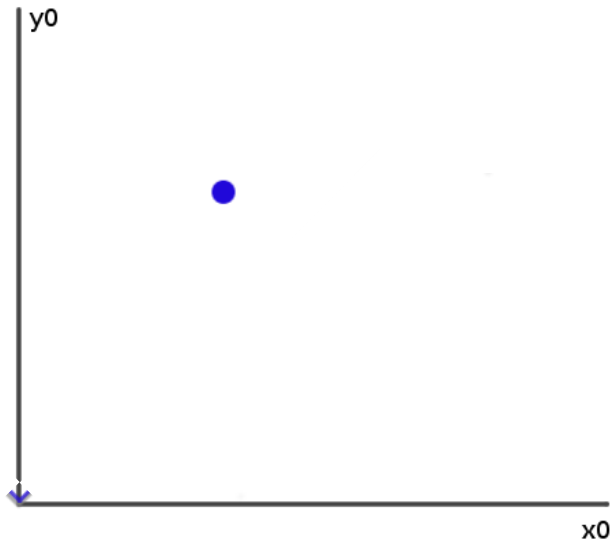
- Matrices can be used to transform vectors in useful ways, through multiplication: x'= Ax

- Simplest is scaling:

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

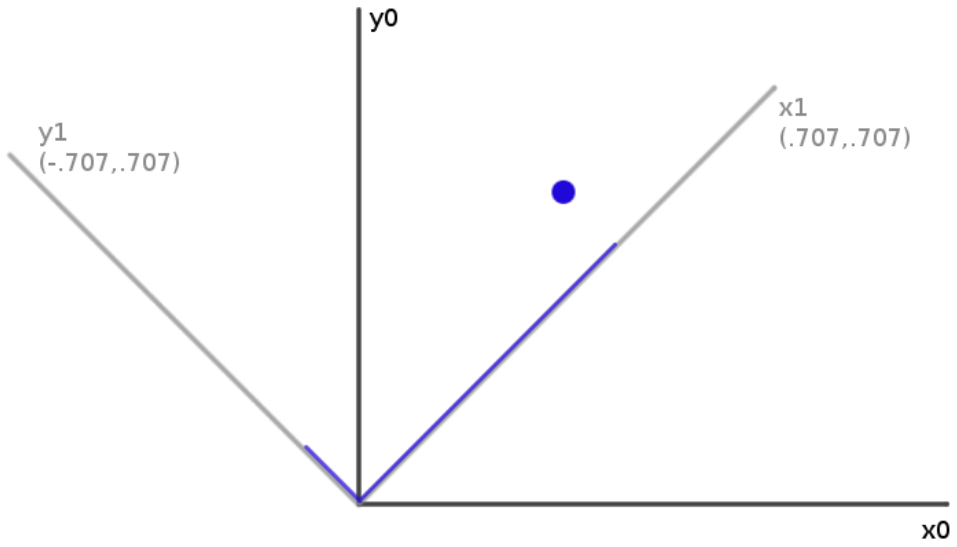
ROTATION

- How can you convert a vector represented in frame “0” to a new, rotated coordinate frame “1”?



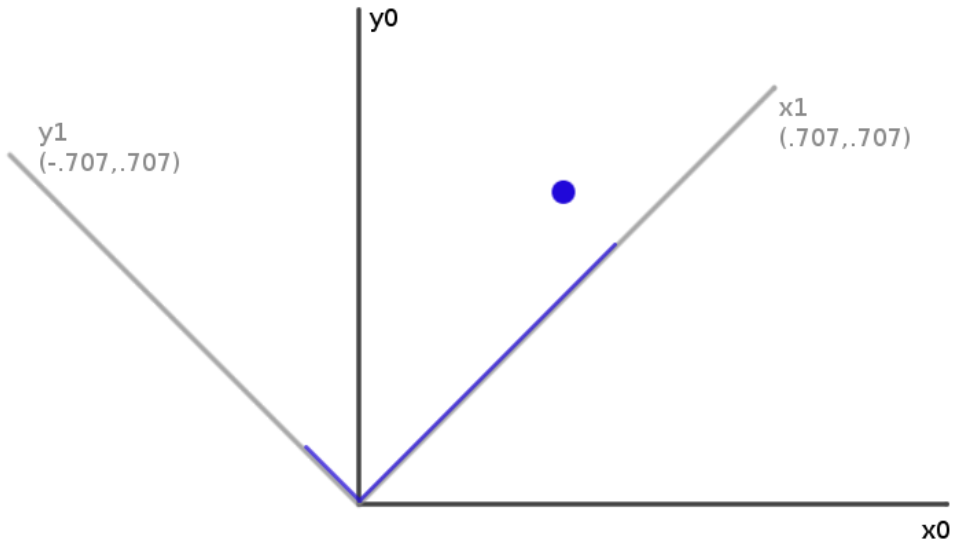
ROTATION

- How can you convert a vector represented in frame “0” to a new, rotated coordinate frame “1”?
- Remember what a vector is:
  [component in direction of the frame’s x axis, component in direction of y axis]



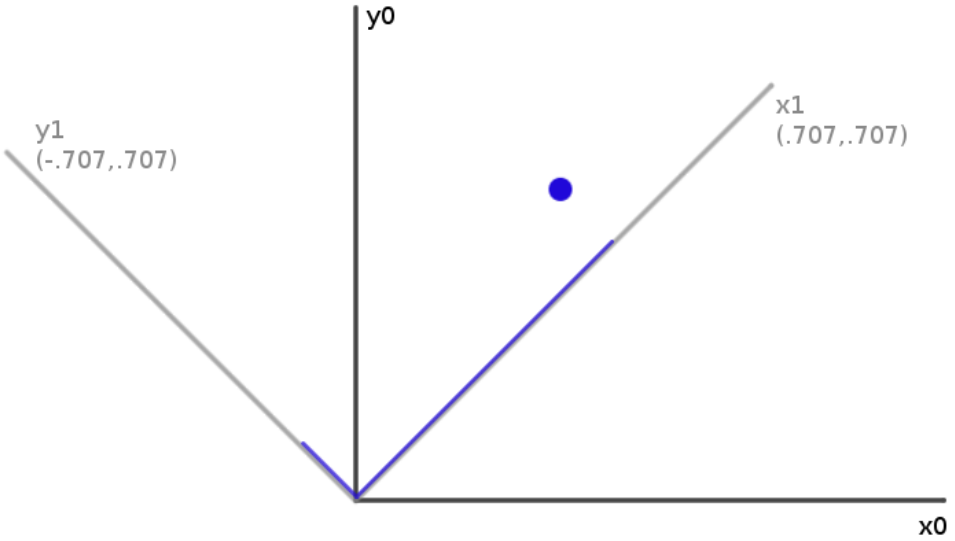
ROTATION

- So to rotate it we must produce this vector:
  [component in direction of **new** x axis, component in direction of **new** y axis]
- We can do this easily with dot products!
- New x coordinate is [original vector] **dot** [the new x axis]
- New y coordinate is [original vector] **dot** [the new y axis]



ROTATION

- Insight: this is what happens in a matrix*vector multiplication
  - Result x coordinate is:
    [original vector] **dot** [matrix row 1]
  - So matrix multiplication can rotate a vector p:



$$R \times p = \text{rotated } p'$$

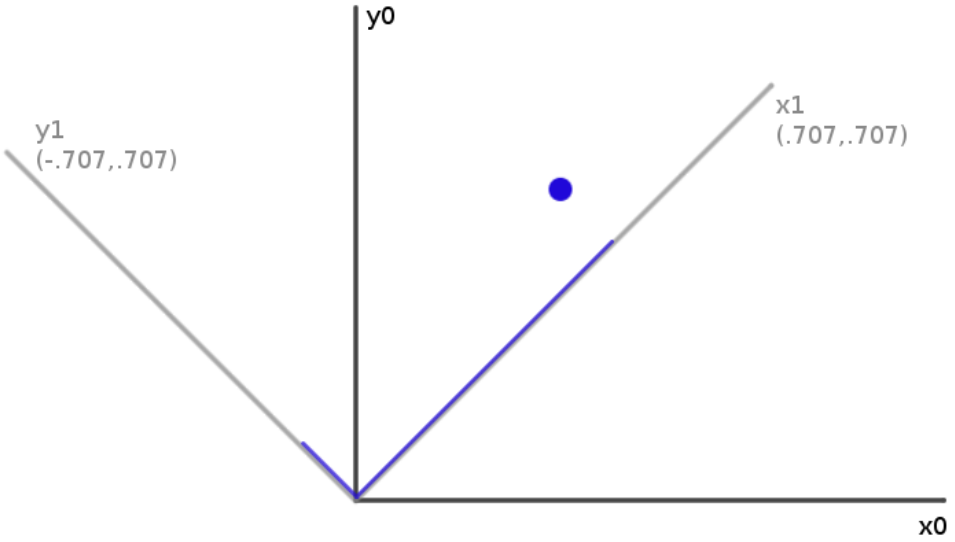
$$p = \begin{bmatrix} px \\ py \end{bmatrix}$$

$$R = \begin{bmatrix} .707 & .707 \\ -.707 & .707 \end{bmatrix}$$

$$\begin{bmatrix} px' \\ py' \end{bmatrix}$$

ROTATION

- Suppose we express a point in the new coordinate system which is rotated left
- If we plot the result in the **original** coordinate system, we have rotated the point right



Thus, rotation matrices can be used to rotate vectors. We'll usually think of them in that sense-- as operators to rotate vectors

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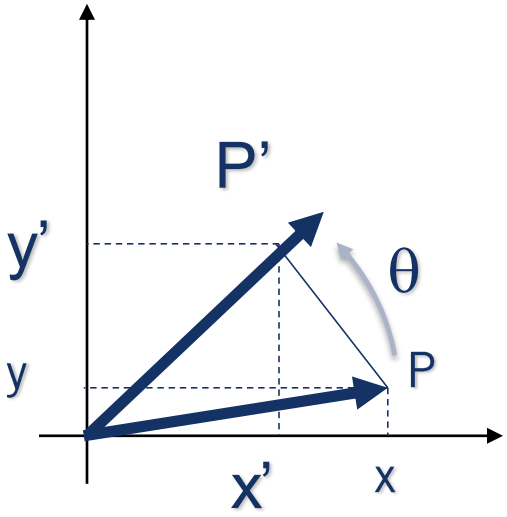
Counter-clockwise rotation by an angle θ



$$x' = \cos\theta\ x - \sin\theta\ y$$

$$y' = \cos\theta\ y + \sin\theta\ x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R}\,\mathbf{P}$$

TRANSFORMATION MATRICES

- Multiple transformation matrices can be used to transform a point: $p' = R_2 R_1 S p$

TRANSFORMATION MATRICES

- Multiple transformation matrices can be used to transform a point:
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- The effect of this is to apply their transformations one after the other, from right to left.
- In the example above, the result is
  $(R_2 (R_1 (S p)))$

TRANSFORMATION MATRICES

- Multiple transformation matrices can be used to transform a point:
  $p' = R_2 R_1 S p$
- The effect of this is to apply their transformations one after the other, from **right to left**.
- In the example above, the result is
  $(R_2 (R_1 (S p)))$
- The result is exactly the same if we multiply the matrices first, to form a single transformation matrix:
  $p' = (R_2 R_1 S) p$

HOMOGENEOUS SYSTEM

- In general, a matrix multiplication lets us linearly combine components of a vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

- This is sufficient for scale, rotate, skew transformations.
- But notice, we can't add a constant! ☹

HOMOGENEOUS SYSTEM

- The (somewhat hacky) solution? Stick a “1” at the end of every vector:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- Now we can rotate, scale, and skew like before, AND translate (note how the multiplication works out, above)
- This is called “homogeneous coordinates”

HOMOGENEOUS SYSTEM

- In homogeneous coordinates, the multiplication works out so the rightmost column of the matrix is a vector that gets added.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- Generally, a homogeneous transformation matrix will have a bottom row of [0 0 1], so that the result has a "1" at the bottom too.

HOMOGENEOUS SYSTEM

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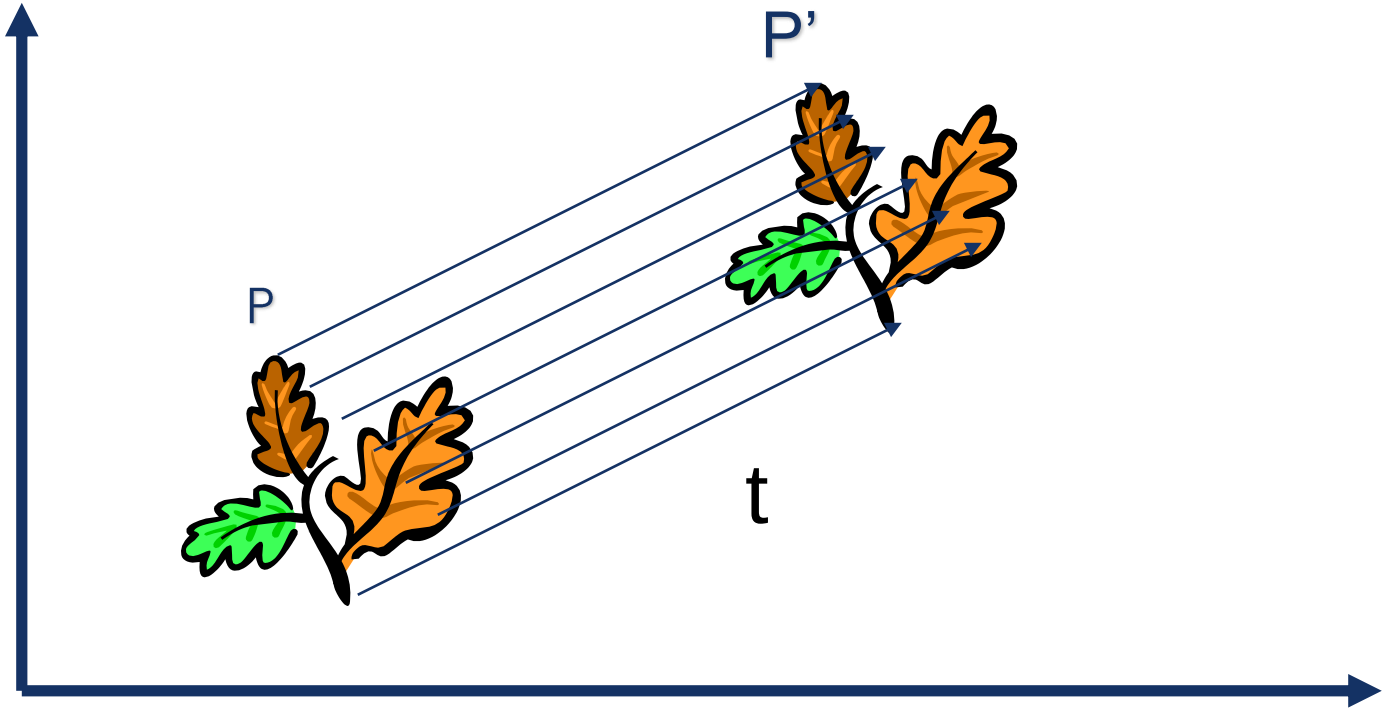
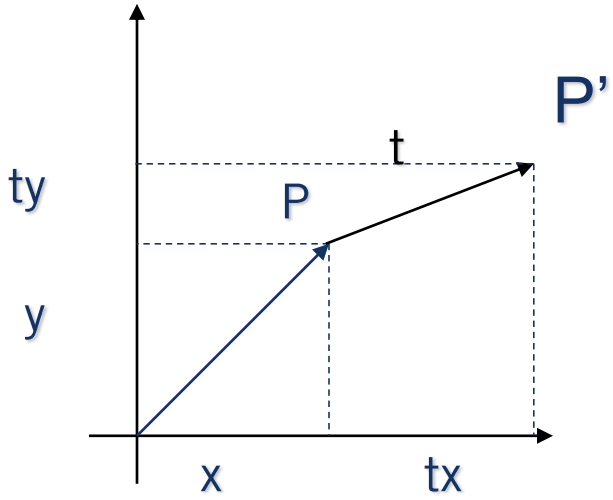
- One more thing we might want: to divide the result by something
  - For example, we may want to divide by a coordinate, to make things scale down as they get farther away in a camera image
  - Matrix multiplication can't actually divide
  - So, **by convention**, in homogeneous coordinates, we'll divide the result by its last coordinate after doing a matrix multiplication

$$\begin{bmatrix} x \\ y \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} x/7 \\ y/7 \\ 1 \end{bmatrix}$$

2D TRANSLATION



2D TRANSLATION USING HOMOGENEOUS COORDINATES



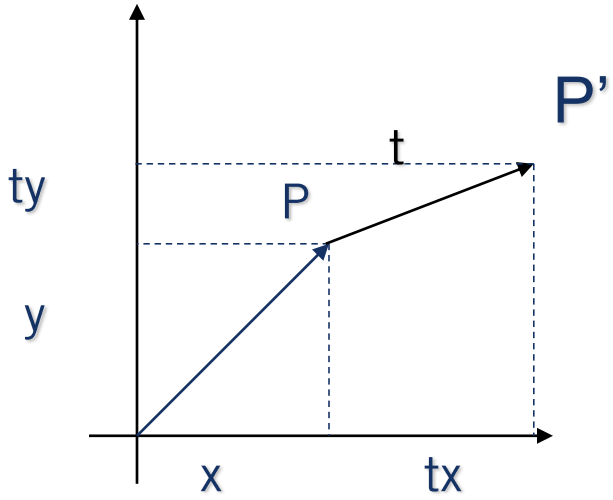
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$\mathbf{P'} \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

P

2D TRANSLATION USING HOMOGENEOUS COORDINATES



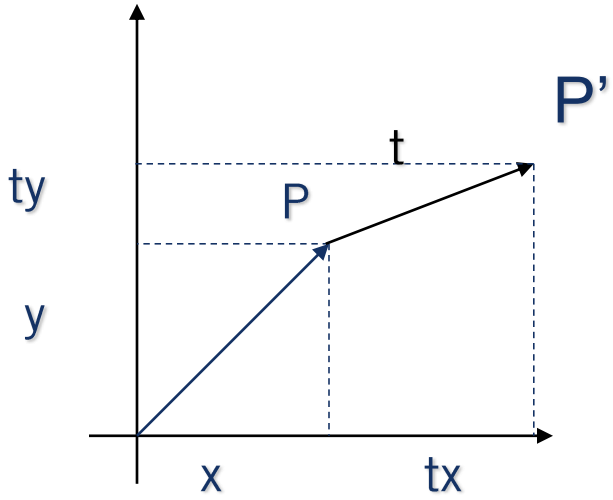
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$\mathbf{P'} \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

P

2D TRANSLATION USING HOMOGENEOUS COORDINATES



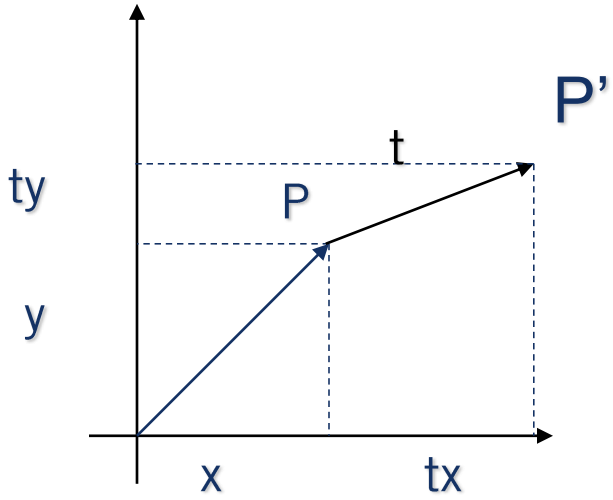
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$\mathbf{P'} \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

P

2D TRANSLATION USING HOMOGENEOUS COORDINATES



$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$\mathbf{P'} \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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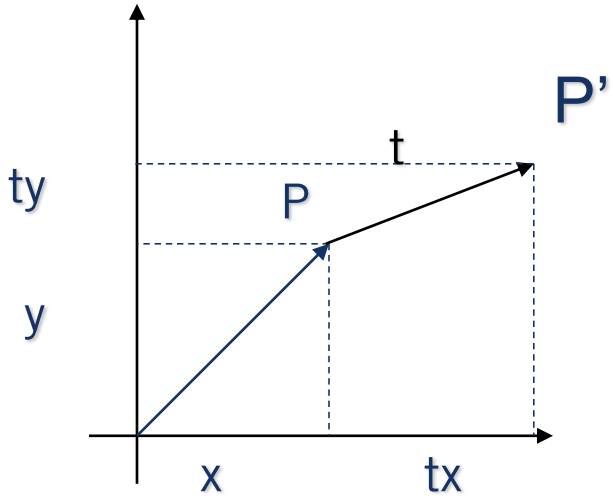

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t

P

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}$$

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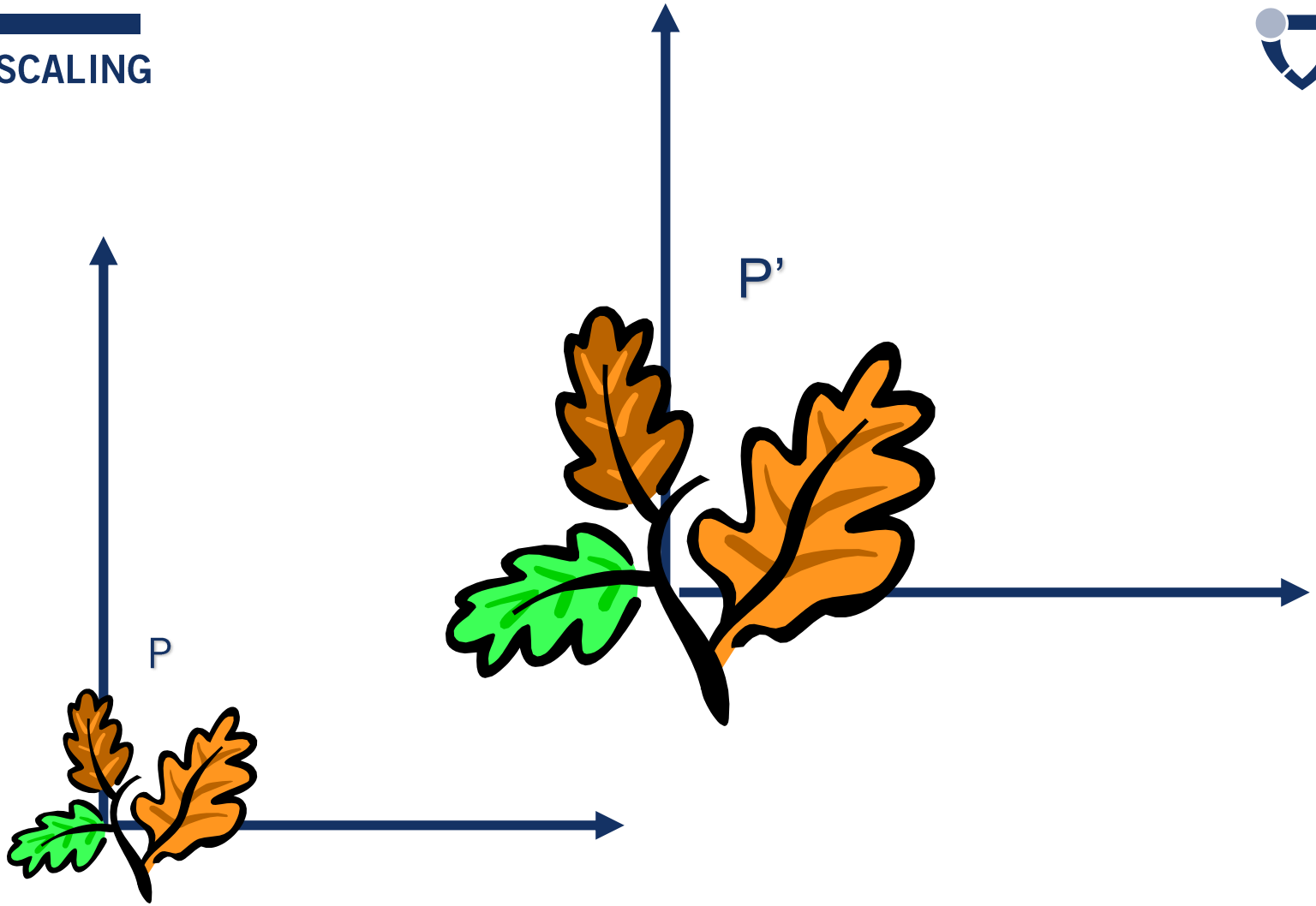
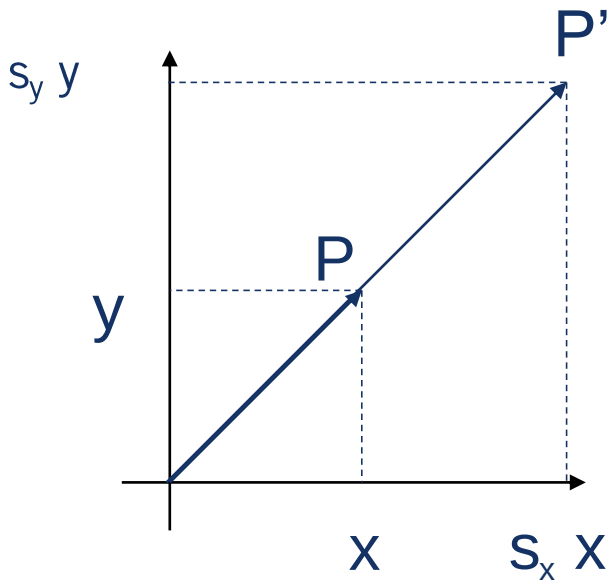


SCALING EQUATION



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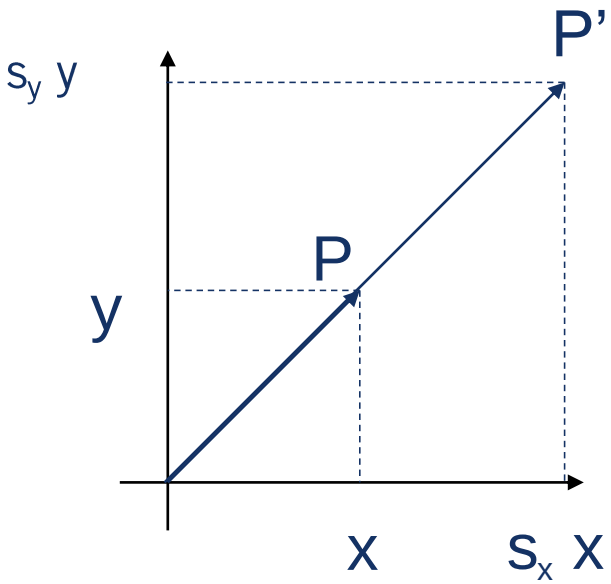

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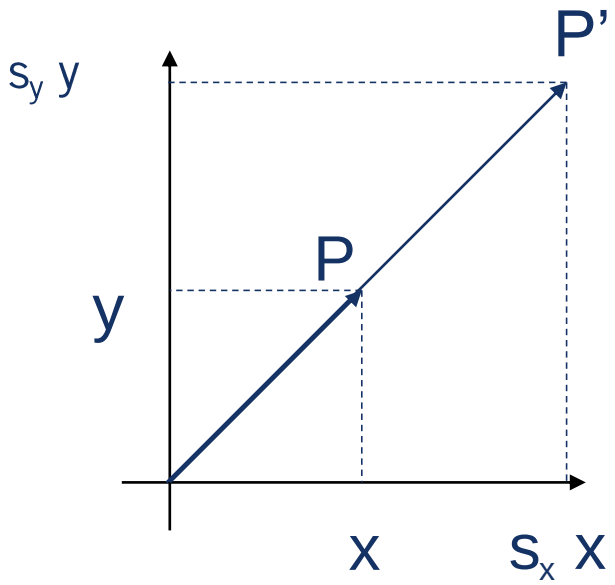
SCALING EQUATION



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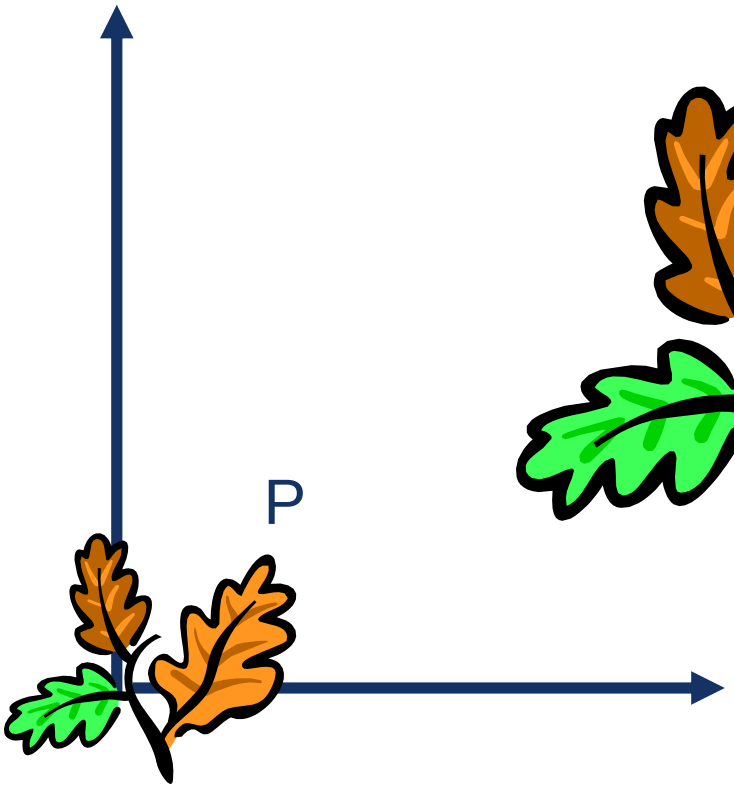

$P' = S \cdot P$

$P'' = T \cdot P'$

$$P'' = T \cdot P' = T \cdot (S \cdot P) = T \cdot S \cdot P$$

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$$\mathbf{P''} = \mathbf{T} \times \mathbf{S} \times \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

SCALING & TRANSLATING

$$\mathbf{P''} = \mathbf{T} \times \mathbf{S} \times \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

TRANSLATING & SCALING VERSUS SCALING & TRANSLATING

$$\mathbf{P'''} = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

TRANSLATING & SCALING != SCALING & TRANSLATING

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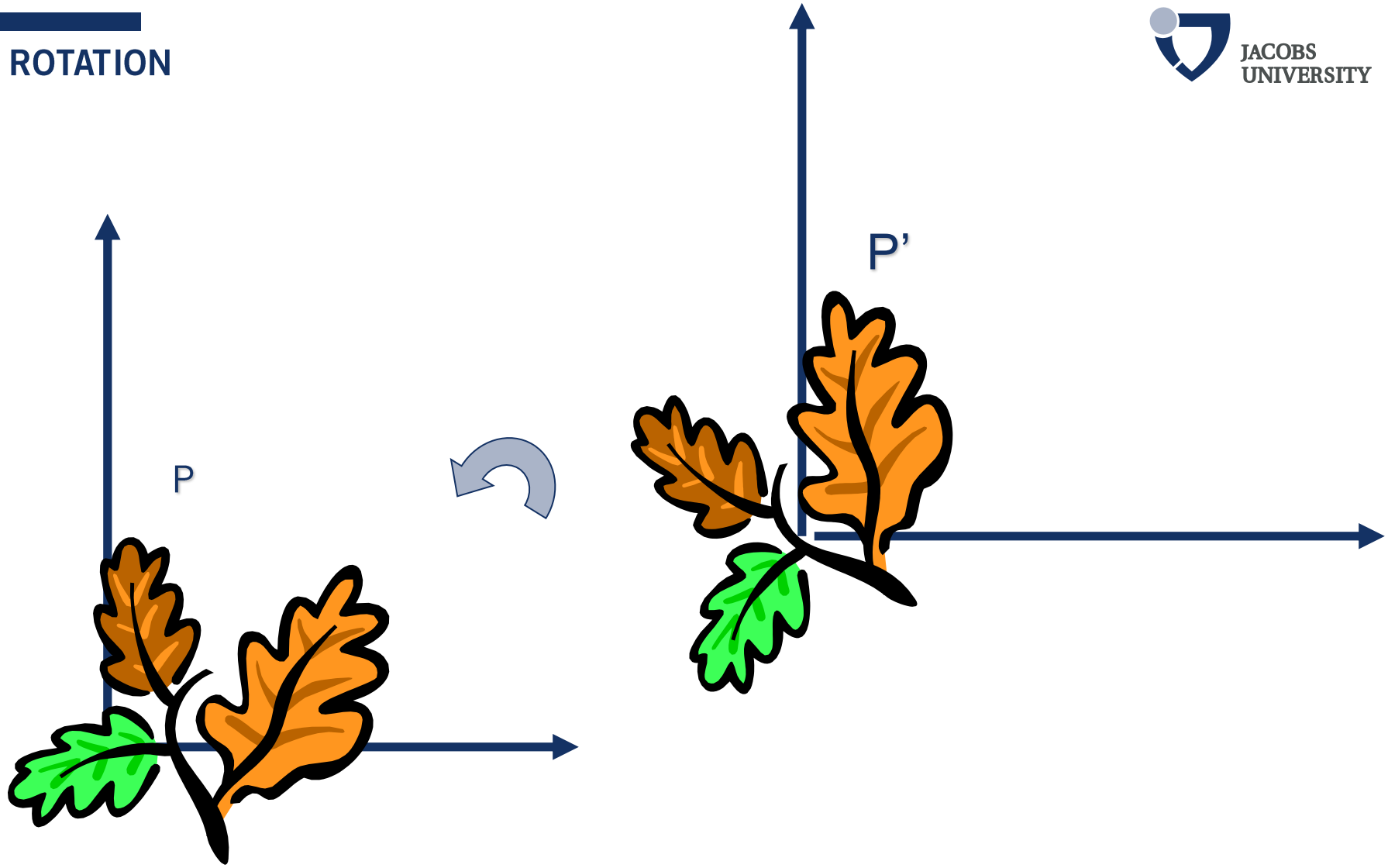
$$= \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

ROTATION



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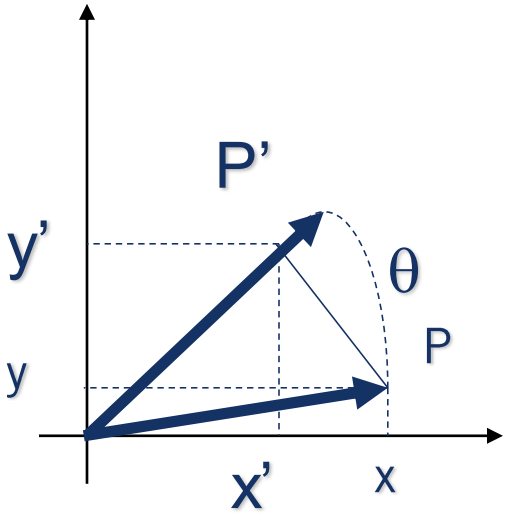
Counter-clockwise rotation by an angle θ



$$x' = \cos\theta\, x - \sin\theta\, y$$

$$y' = \cos\theta\, y + \sin\theta\, x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R}\,\mathbf{P}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A 2D rotation matrix is 2x2

Note: R belongs to the category of *normal* matrices and satisfies many interesting properties:

$$\mathbf{R} \cdot \mathbf{R}^{\mathbf{T}} = \mathbf{R}^{\mathbf{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

ROTATION MATRIX PROPERTIES

- Transpose of a rotation matrix produces a rotation in the opposite direction

$$\mathbf{R} \cdot \mathbf{R}^{\mathbf{T}} = \mathbf{R}^{\mathbf{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

- The rows of a rotation matrix are always mutually perpendicular (a.k.a. orthogonal) unit vectors
- *(and so are its columns)*

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P'= (T R S) P

$$\mathbf{P'} = \mathbf{T} \cdot R \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} R\,S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This is the form of the general-purpose transformation matrix

OUTLINE

- Vectors and matrices
  - Basic Matrix Operations
  - Determinants, norms, trace
  - Special Matrices
- Transformation Matrices
  - Homogeneous coordinates
  - Translation
- Matrix inverse
- Matrix rank
- Singular Value Decomposition
  - Image Compression

The inverse of a transformation matrix reverses its effect

INVERSE

- Given a matrix **A**, its inverse $\mathbf{A}^{-1}$ is a matrix such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- E.g. $$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$
- Inverse does not always exist. If $\mathbf{A}^{-1}$ exists, **A** is *invertible* or *non-singular*. Otherwise, it's *singular*.
- Useful identities, for matrices that are invertible:

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$
$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$
$$\mathbf{A}^{-T} \triangleq (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$$

- **Pseudoinverse**
  - Say you have the matrix equation AX=B, where A and B are known, and you want to solve for X

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  - MATLAB command would be **inv(A)*B**
  - But calculating the inverse for large matrices often brings problems with computer floating-point resolution (because it involves working with very small and very large numbers together).
  - Or, your matrix might not even have an inverse.

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- **Pseudoinverse**
  - Fortunately, there are workarounds to solve AX=B in these situations. And MATLAB can do them!
  - Instead of taking an inverse, directly ask MATLAB to solve for X in AX=B, by typing **A \ B**
  - MATLAB will try several appropriate numerical methods (including the pseudoinverse if the inverse doesn't exist)
  - MATLAB will return the value of X which solves the equation
    - If there is no exact solution, it will return the closest one
    - If there are many solutions, it will return the smallest one

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- MATLAB example:

$$AX = B$$

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
>> x = A\B
x =
    1.0000
   -0.5000
```

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The rank of a transformation matrix tells you how many dimensions it transforms a vector to.

LINEAR INDEPENDENCE

- Suppose we have a set of vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$
- If we can express $\mathbf{v}_1$ as a linear combination of the other vectors $\mathbf{v}_2 \ldots \mathbf{v}_n$, then $\mathbf{v}_1$ is linearly *dependent* on the other vectors.
  - The direction $\mathbf{v}_1$ can be expressed as a combination of the directions $\mathbf{v}_2 \ldots \mathbf{v}_n$. (E.g. $\mathbf{v}_1 = .7\ \mathbf{v}_2 - .7\ \mathbf{v}_4$)

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- If no vector is linearly dependent on the rest of the set, the set is linearly *independent*.
  - Common case: a set of vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ is always linearly independent if each vector is perpendicular to every other vector (and non-zero)

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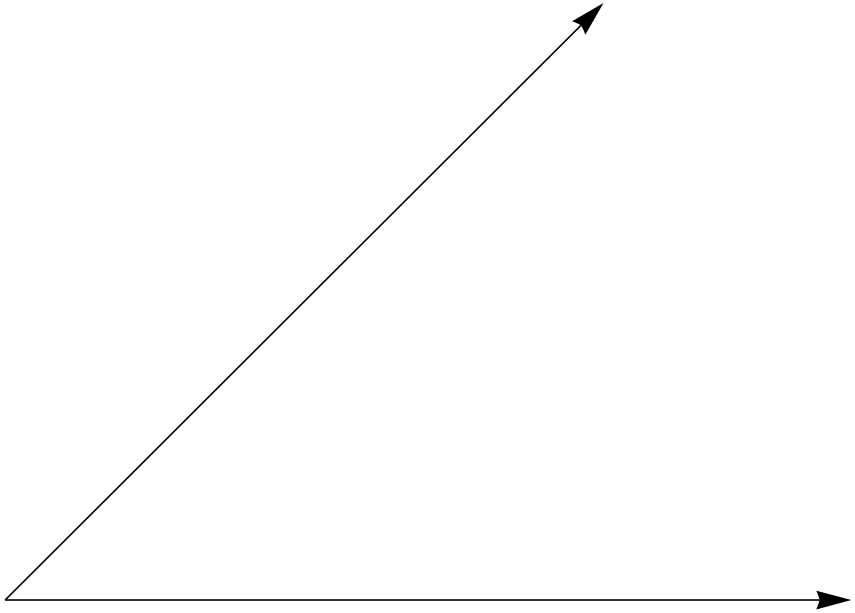
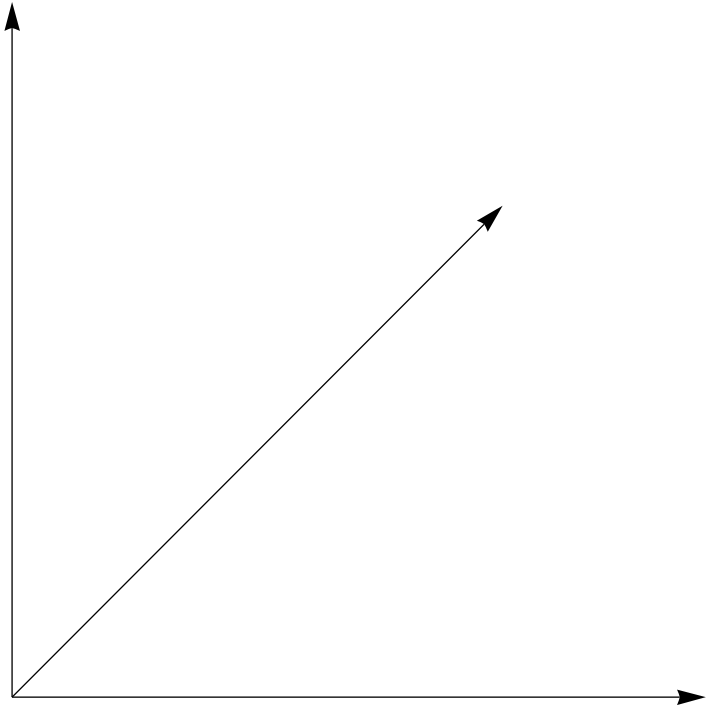
Linearly independent set



Not linearly independent



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- Column/row rank

$$\begin{aligned}\text{col-rank}(\mathbf{A}) &= \text{ the maximum number of linearly independent column vectors of } \mathbf{A}\\ \text{row-rank}(\mathbf{A}) &= \text{ the maximum number of linearly independent row vectors of } \mathbf{A}\end{aligned}$$

  - Column rank always equals row rank

- Matrix rank

$$\text{rank}(\mathbf{A}) \triangleq \text{col-rank}(\mathbf{A}) = \text{row-rank}(\mathbf{A})$$

MATRIX RANK

- For transformation matrices, the rank tells you the dimensions of the output
- E.g. if rank of **A** is 1, then the transformation

$$\mathbf{p'=Ap}$$

maps points onto a line.

- Here's a matrix with rank 1:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x+2y \end{bmatrix}$$

All points get mapped to the line y=2x

MATRIX RANK

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- If an *m* x *m* matrix is rank *m*, we say it's "full rank"
  - Maps an *m* x 1 vector uniquely to another *m* x 1 vector
  - An inverse matrix can be found
- If rank < *m*, we say it's "singular"
  - At least one dimension is getting collapsed. No way to look at the result and tell what the input was
  - Inverse does not exist
- Inverse also doesn't exist for non-square matrices

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SVD is an algorithm that represents any matrix as the product of 3 matrices. It is used to discover interesting structure in a matrix

SINGULAR VALUE DECOMPOSITION (SVD)

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- There are several computer algorithms that can “factorize” a matrix, representing it as the product of some other matrices
- The most useful of these is the Singular Value Decomposition.
- Represents any matrix **A** as a product of three matrices: $\mathbf{U\Sigma V^T}$
- MATLAB command: [U,S,V]=svd(A)

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$A = U\Sigma V^T$

- Where **U** and **V** are rotation matrices and **Σ** is a scaling matrix.

- For example:

$$\underset{U}{\begin{bmatrix} -.40 & .916 \\ .916 & .40 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 5.39 & 0 \\ 0 & 3.154 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.05 & .999 \\ .999 & .05 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}}$$

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- Beyond 2D:
  - In general, if **A** is *m* x *n*, then **U** will be *m* x *m*, **Σ** will be *m* x *n*, and $\mathbf{V^T}$ will be *n* x *n*.
  - (Note the dimensions work out to produce *m* x *n* after multiplication)

$$\underset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

SINGULAR VALUE DECOMPOSITION (SVD)

- **U** and **V** are always rotation matrices.
  - Geometric rotation may not be an applicable concept, depending on the matrix. So we call them "unitary" matrices – each column is a unit vector.

- **Σ** is a diagonal matrix
  - The number of nonzero entries = rank of **A**
  - The algorithm always sorts the entries high to low

$$\underset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

- Let's look at a non-geometric interpretation

SINGULAR VALUE DECOMPOSITION (SVD) - APPLICATIONS

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- Column 1 of **U** gets scaled by the first value from **Σ**.

$$\underset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \qquad \underset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

- The resulting vector gets scaled by row 1 of $\mathbf{V^T}$ to produce a contribution to the columns of **A**

SINGULAR VALUE DECOMPOSITION (SVD) - APPLICATIONS

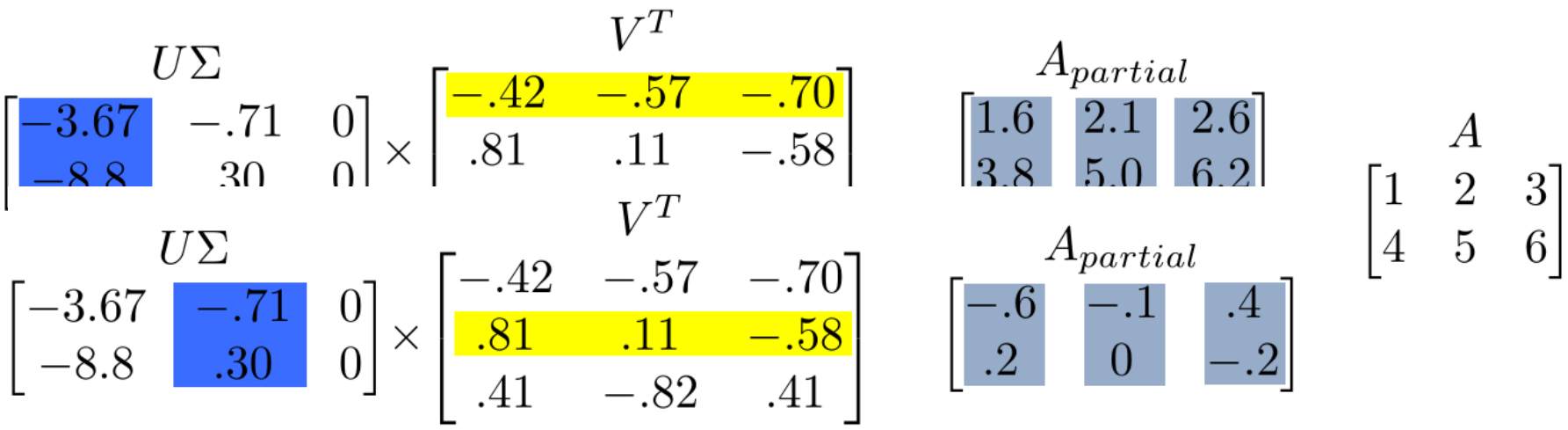
$$\underset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \underset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

$$+ \quad \underset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \underset{A_{partial}}{\begin{bmatrix} -.6 & -.1 & .4 \\ .2 & 0 & -.2 \end{bmatrix}}$$

$$= \quad \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

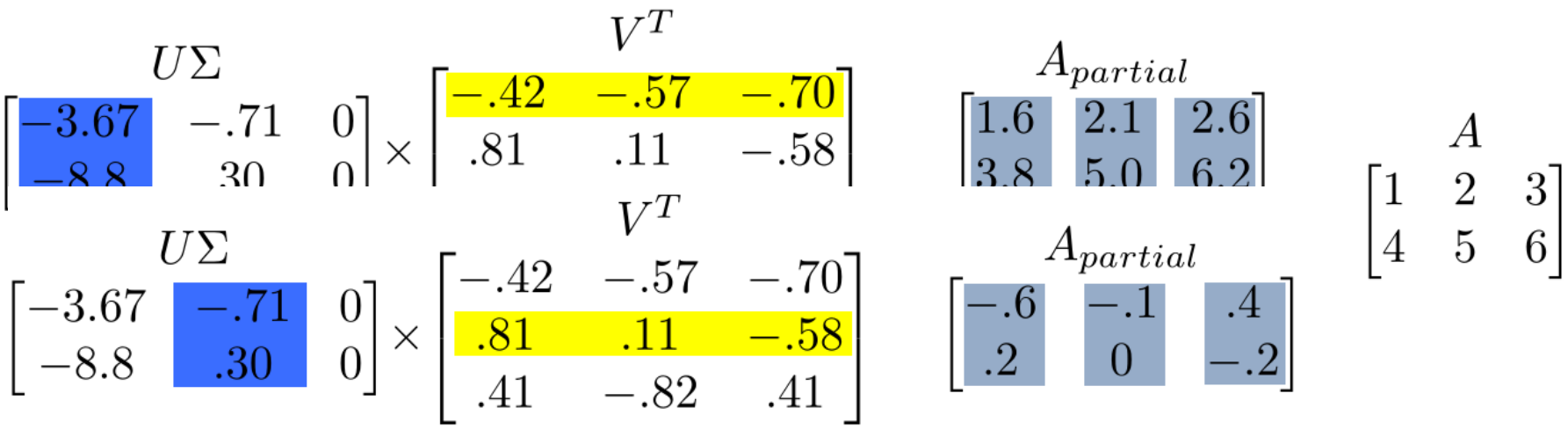
- The resulting vector gets scaled by row 1 of V^T to produce a contribution to the columns of A
- Each product of (*column i of* U)·(*value i from* Σ)·(*row i of* V^T) produces a component of the final **A**.

SINGULAR VALUE DECOMPOSITION (SVD) - APPLICATIONS



- We're building **A** as a linear combination of the columns of ***U***
- Using all columns of ***U***, we'll rebuild the original matrix perfectly
- But, in real-world data, often we can just use the first few columns of ***U*** and we'll get something close (e.g. the first $\boldsymbol{A_{partial}}$, above)

SINGULAR VALUE DECOMPOSITION (SVD) - APPLICATIONS



- We can call those first few columns of ***U*** the *Principal Components* of the data
- They show the major patterns that can be added to produce the columns of the original matrix
- The rows of $\boldsymbol{V^T}$ show how the *principal components* are mixed to produce the columns of the matrix

SINGULAR VALUE DECOMPOSITION (SVD) - APPLICATIONS

$$\Sigma$$
$$\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}$$

- First column large effect
- Second column much smaller effect

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- Second column much smaller effect

SINGULAR VALUE DECOMPOSITION (SVD) - APPLICATIONS

- A geometric explanation of SVD is here:

http://www.ams.org/publicoutreach/feature-column/fcarc-svd