

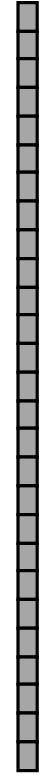
COMPUTER VISION LECTURE 16 – TWO-VIEW GEOMETRY

Prof. Dr. Francesco Maurelli
2018-10-30

Courtesy of Ioannis Gkioulekas, CMU

... and an object we like to photograph

real-world
object

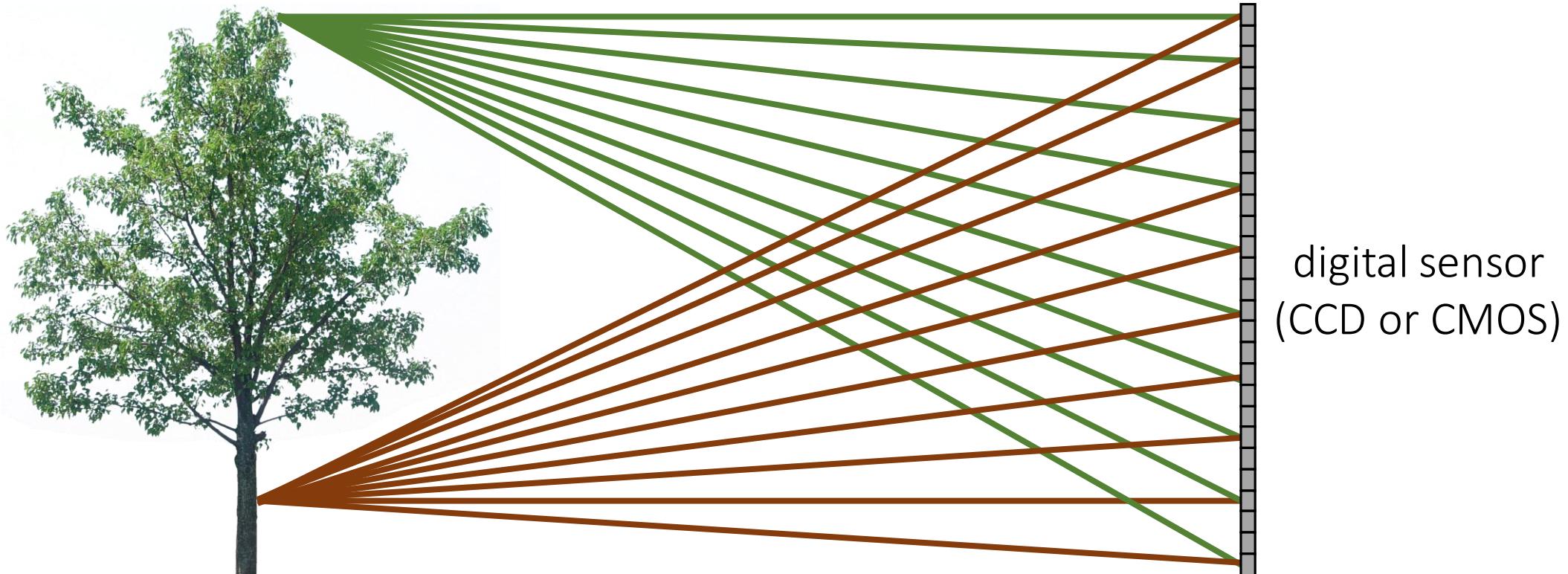


digital sensor
(CCD or CMOS)

What would an image taken like this look like?

Bare-sensor imaging

real-world
object



All scene points contribute to all sensor pixels

What does the
image on the
sensor look like?

Bare-sensor imaging



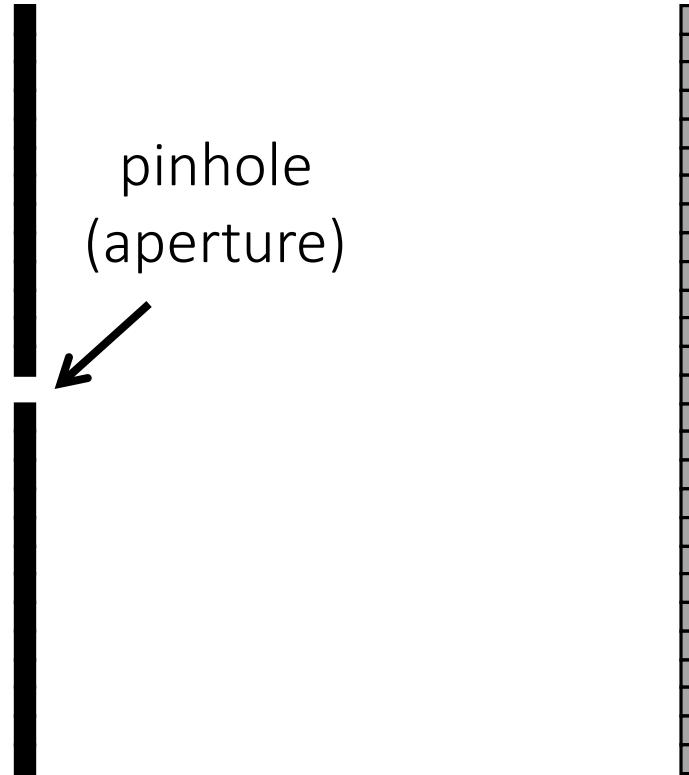
All scene points contribute to all sensor pixels

Let's add something to this scene

real-world
object



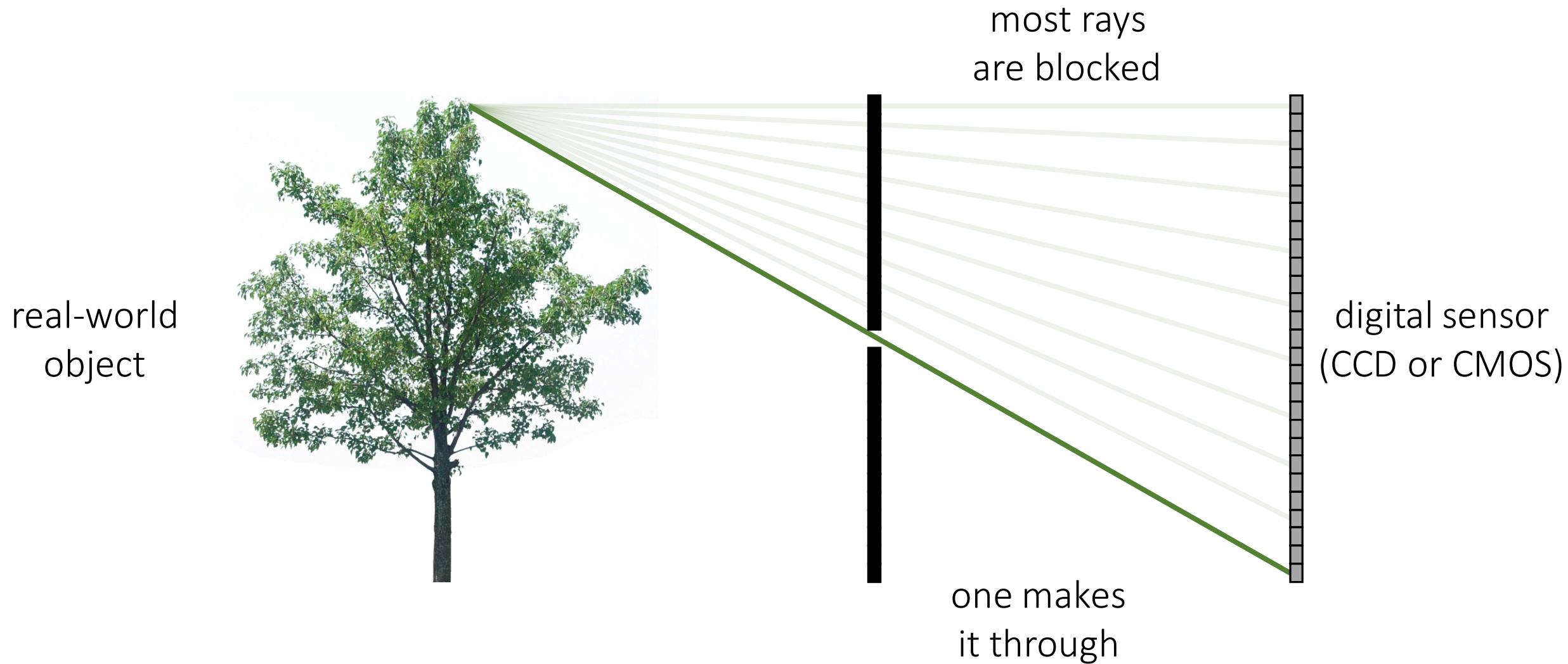
barrier (diaphragm)



digital sensor
(CCD or CMOS)

What would an image taken like this look like?

Pinhole imaging



Pinhole imaging



Pinhole camera terms

real-world
object



barrier (diaphragm)

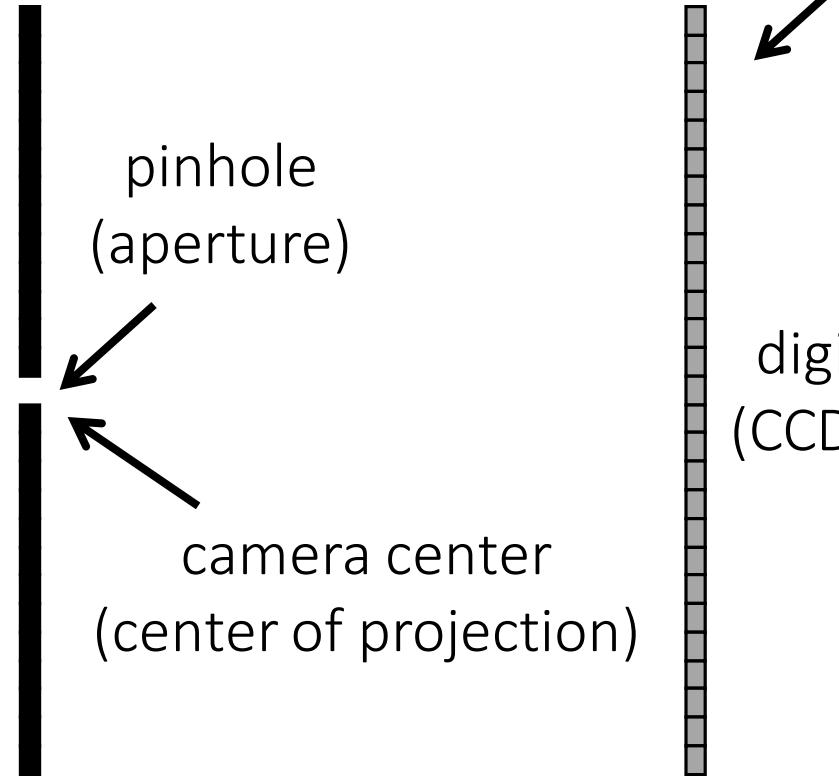
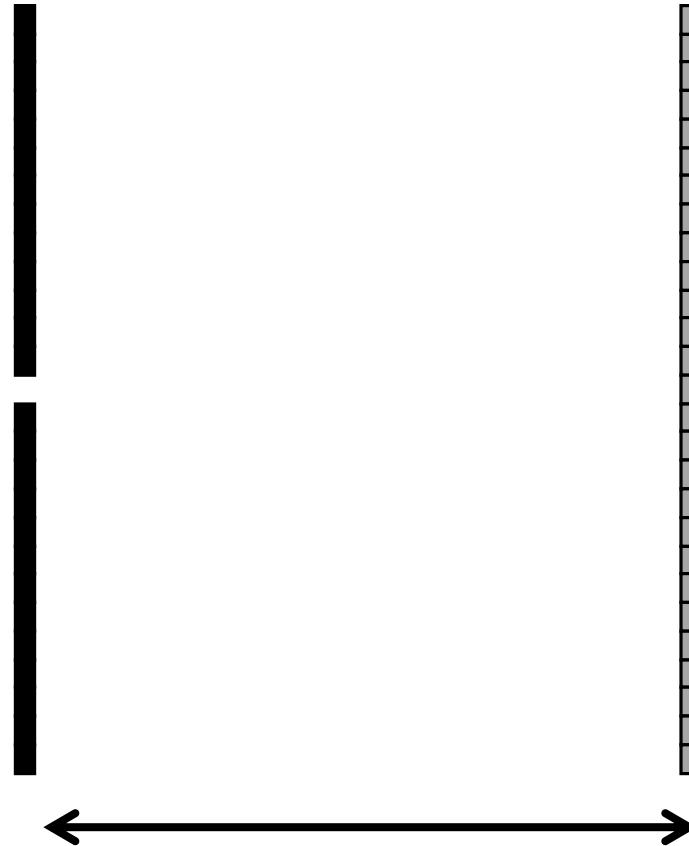


image plane

digital sensor
(CCD or CMOS)

Focal length

real-world
object



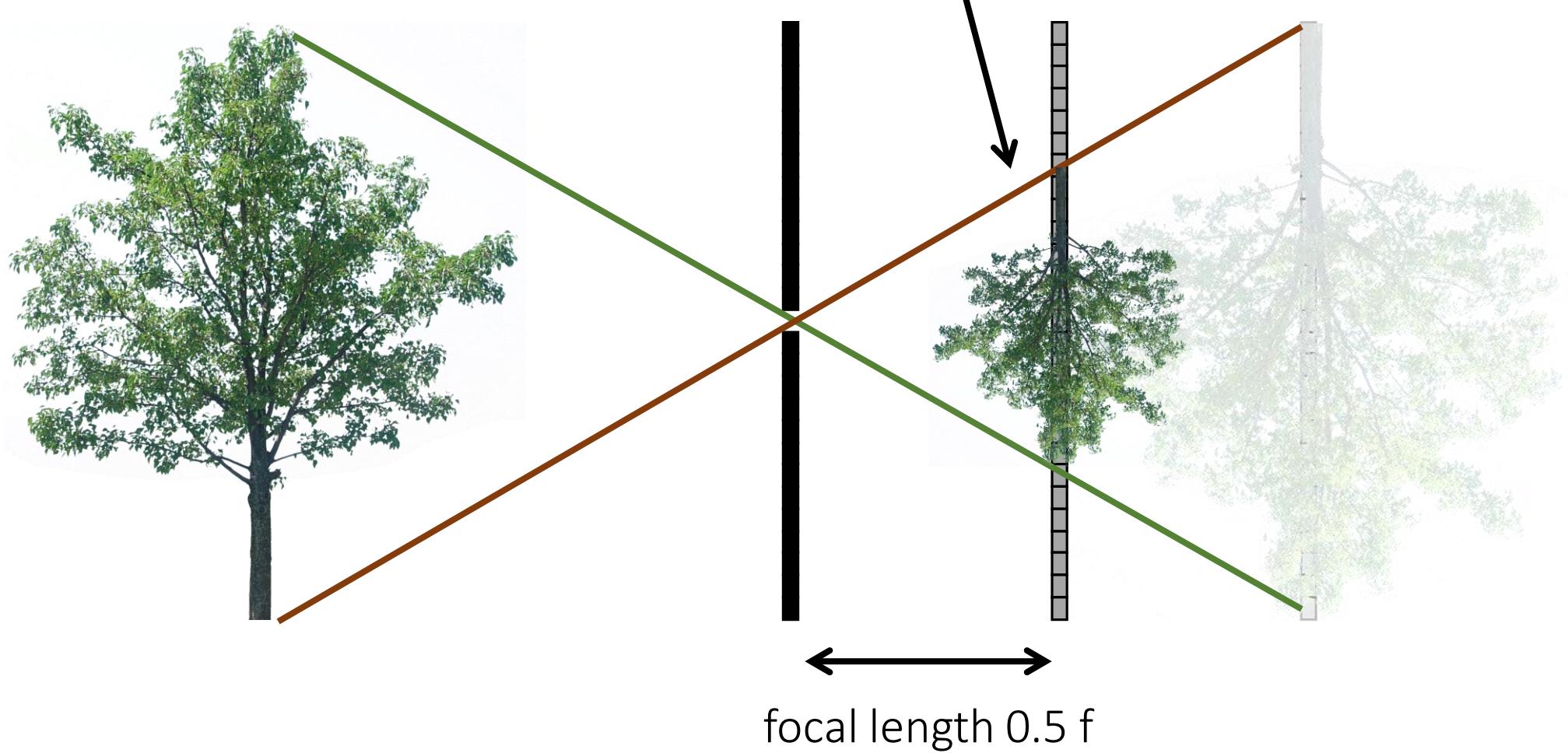
focal length f

Focal length

What happens as we change the focal length?

real-world
object

object projection is half the size



Pinhole size

What happens as we change the pinhole diameter?

real-world
object



pinhole
diameter

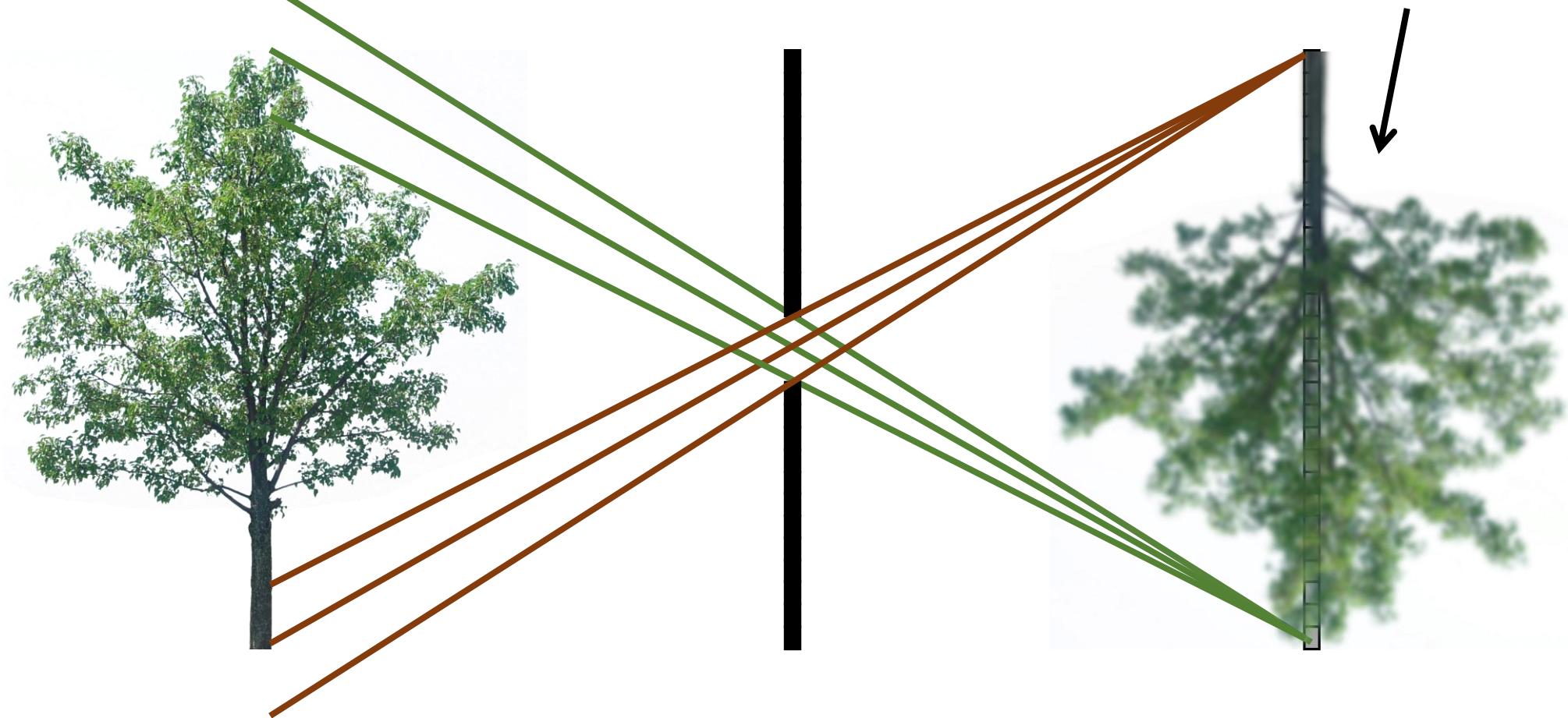


Pinhole size

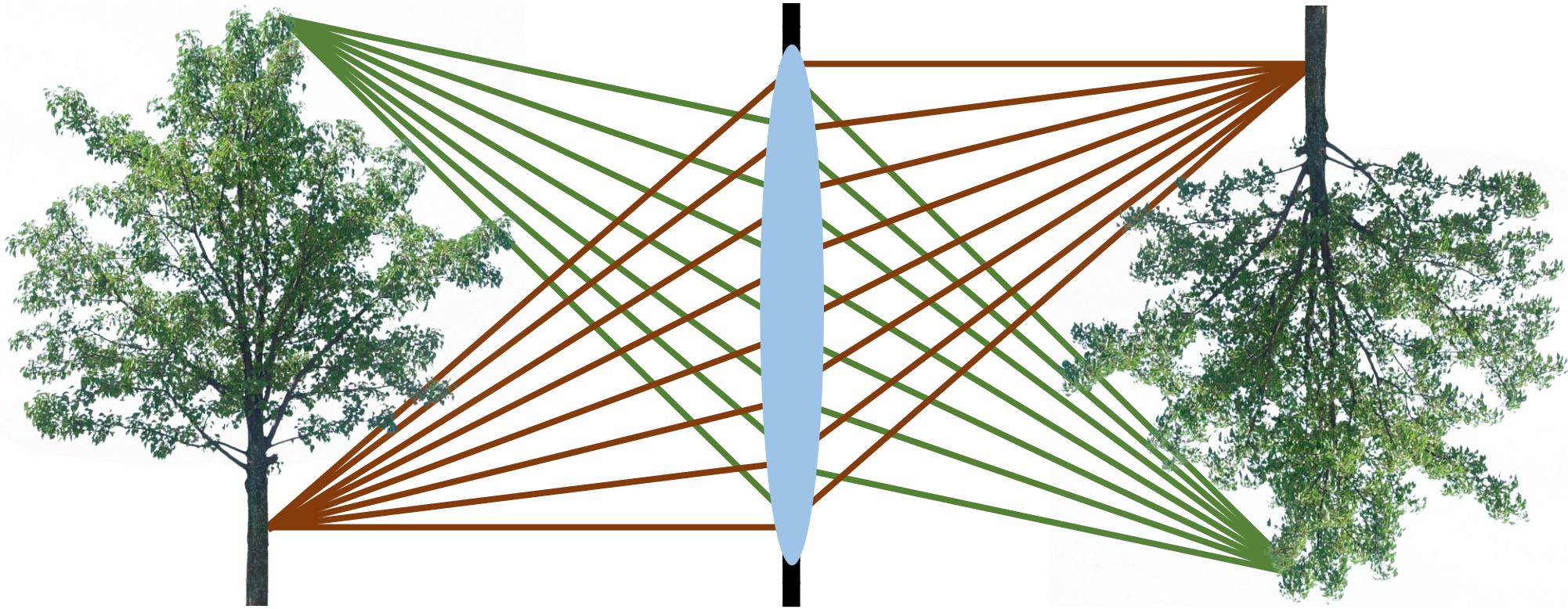
What happens as we change the pinhole diameter?

object projection becomes blurrier

real-world
object



In practice



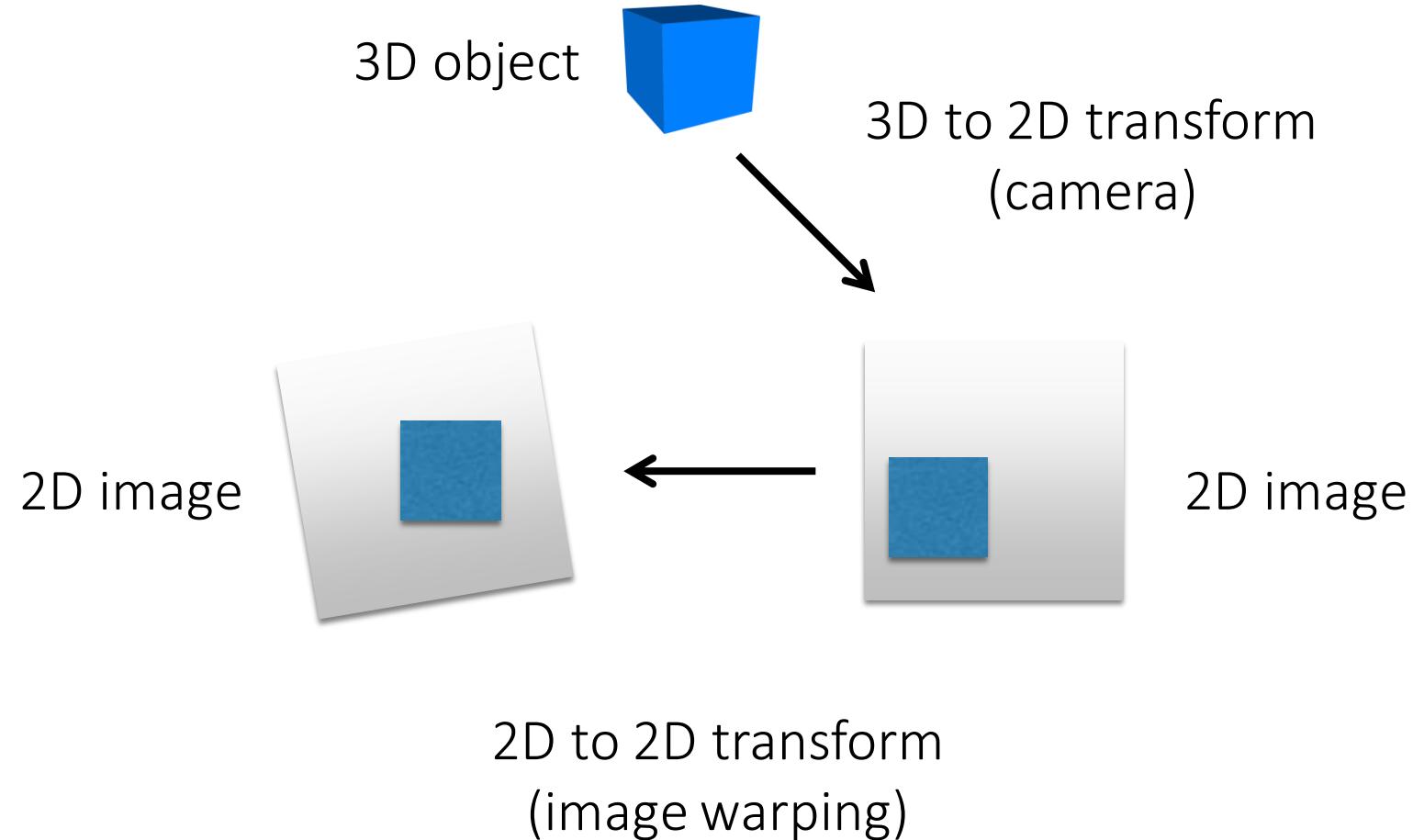
Lenses map “bundles” of rays from points on the scene to the sensor.

The camera as a coordinate transformation

A camera is a mapping from:

the 3D world
to:

a 2D image



The camera as a coordinate transformation

A camera is a mapping from:

the 3D world

to:

a 2D image

homogeneous coordinates

$$\mathbf{c} = \mathbf{P}\mathbf{x}$$

2D image
point

camera matrix 3D world point

What are the dimensions of each variable?

The camera as a coordinate transformation

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

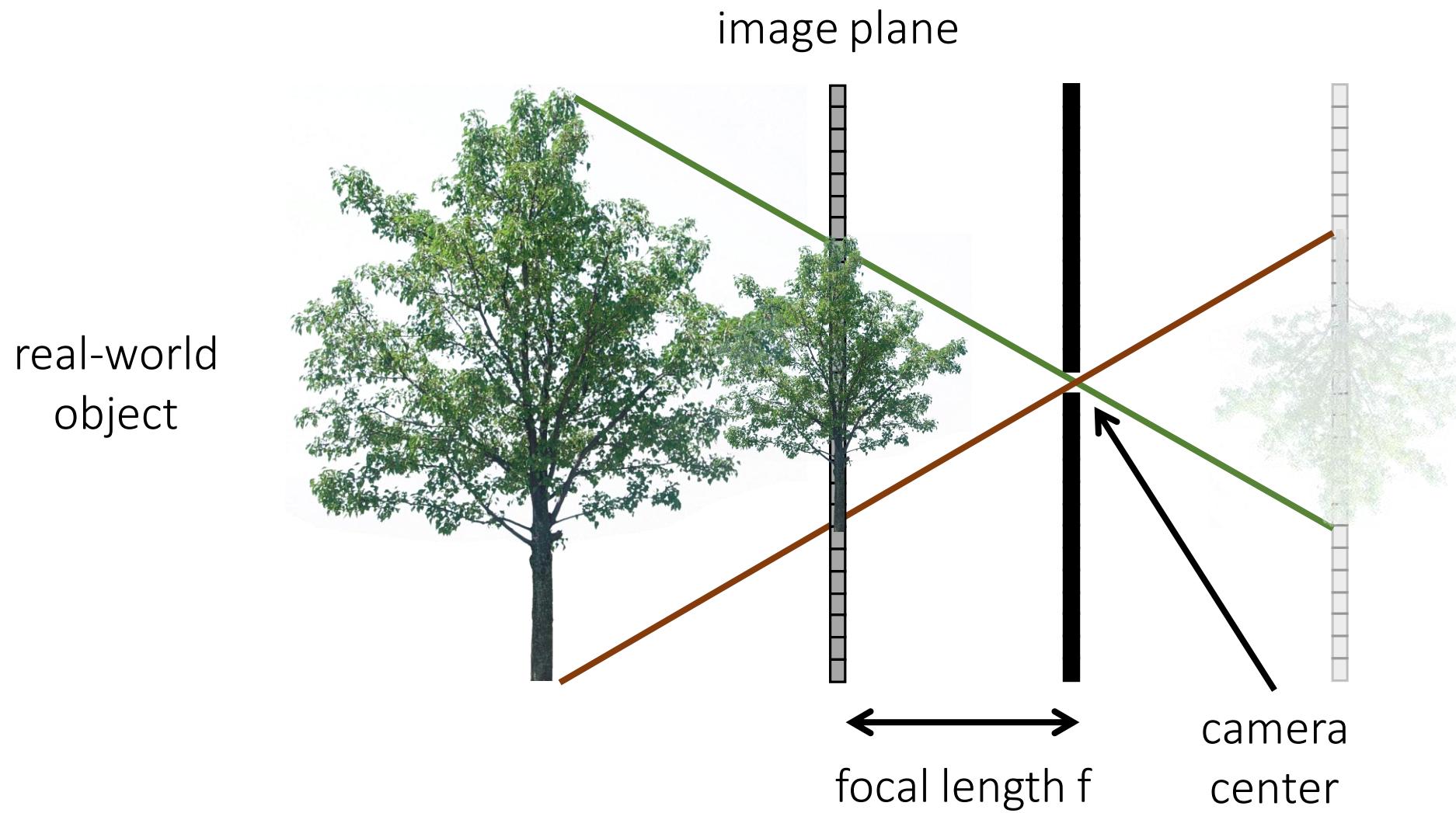
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

homogeneous
image coordinates
 3×1

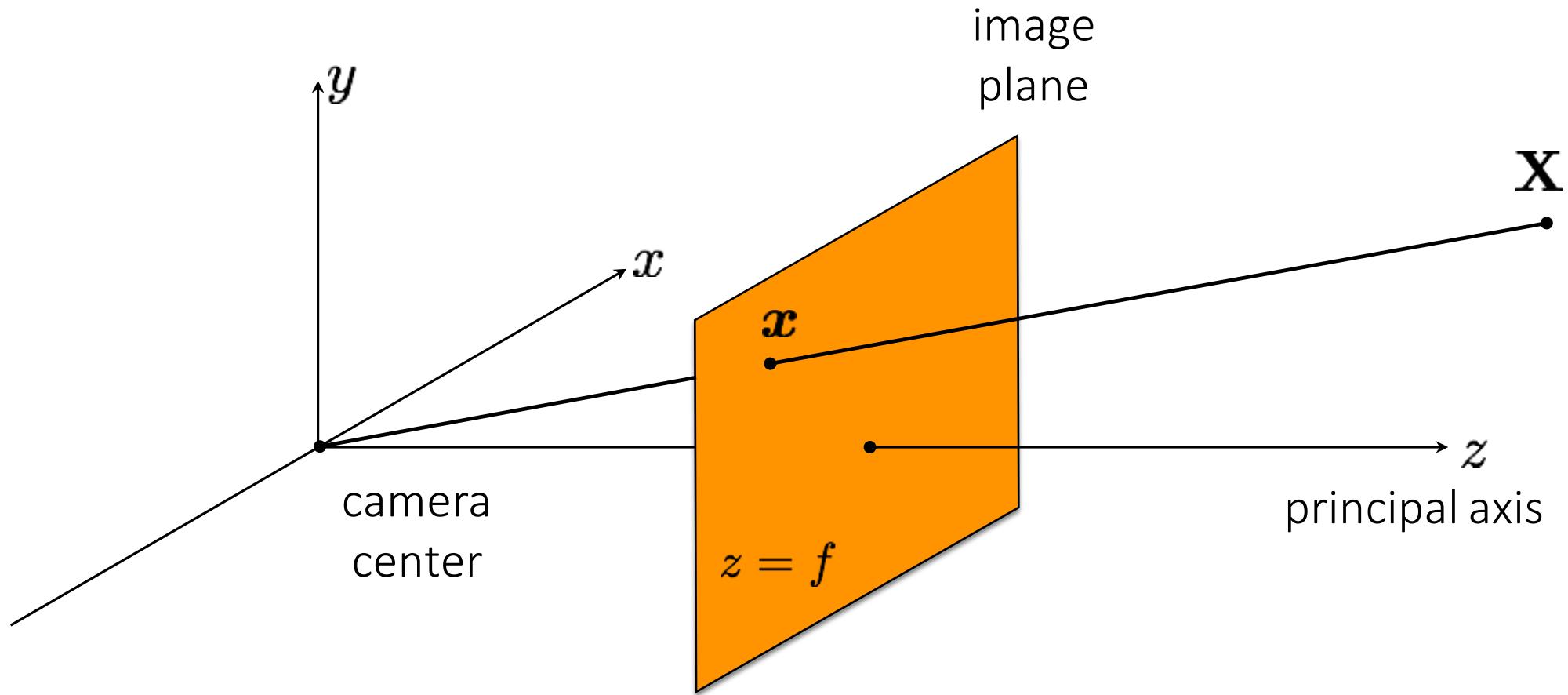
camera
matrix
 3×4

homogeneous
world coordinates
 4×1

The (rearranged) pinhole camera

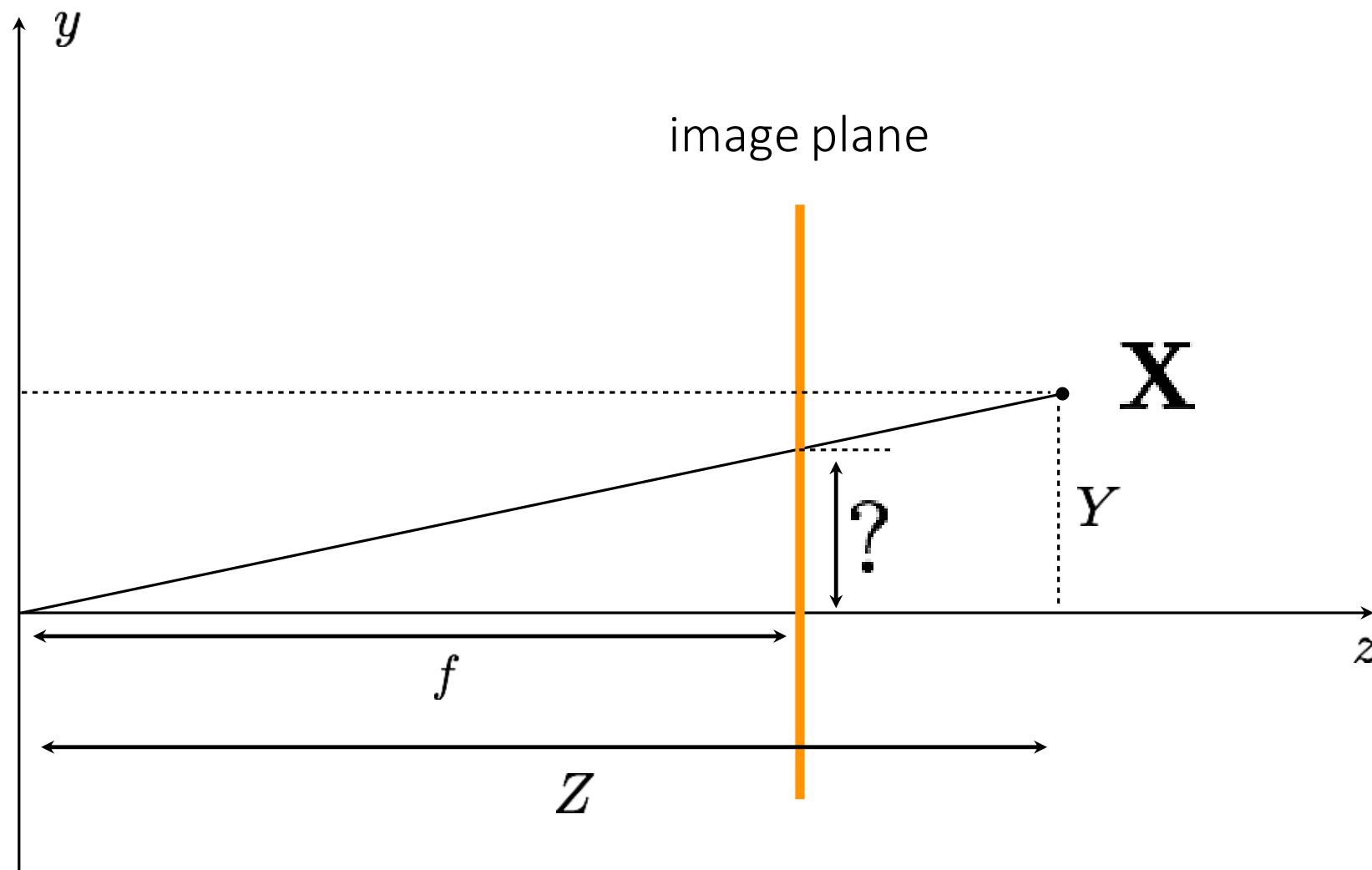


The (rearranged) pinhole camera



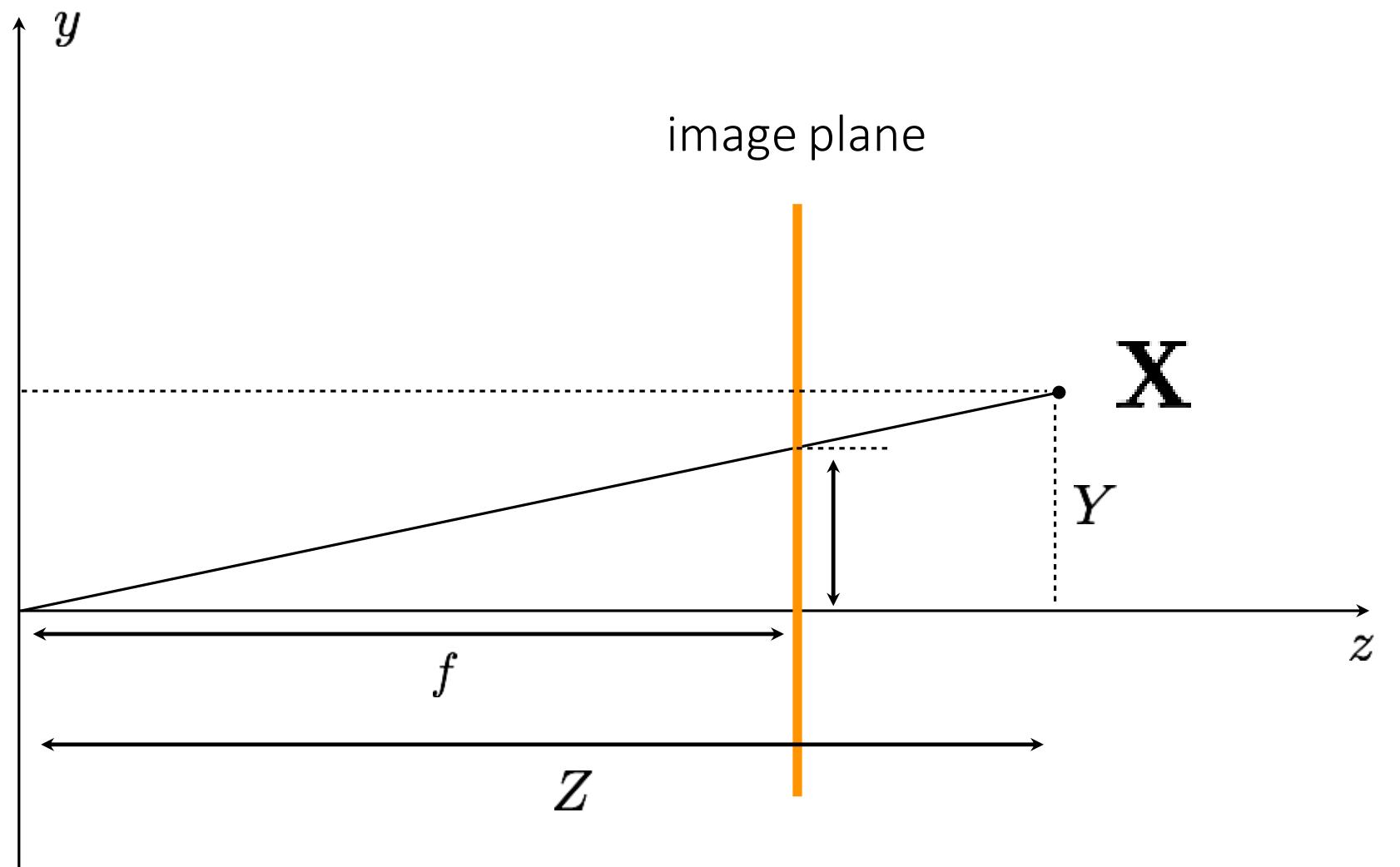
What is the equation for image coordinate \mathbf{x} in terms of \mathbf{X} ?

The 2D view of the (rearranged) pinhole camera



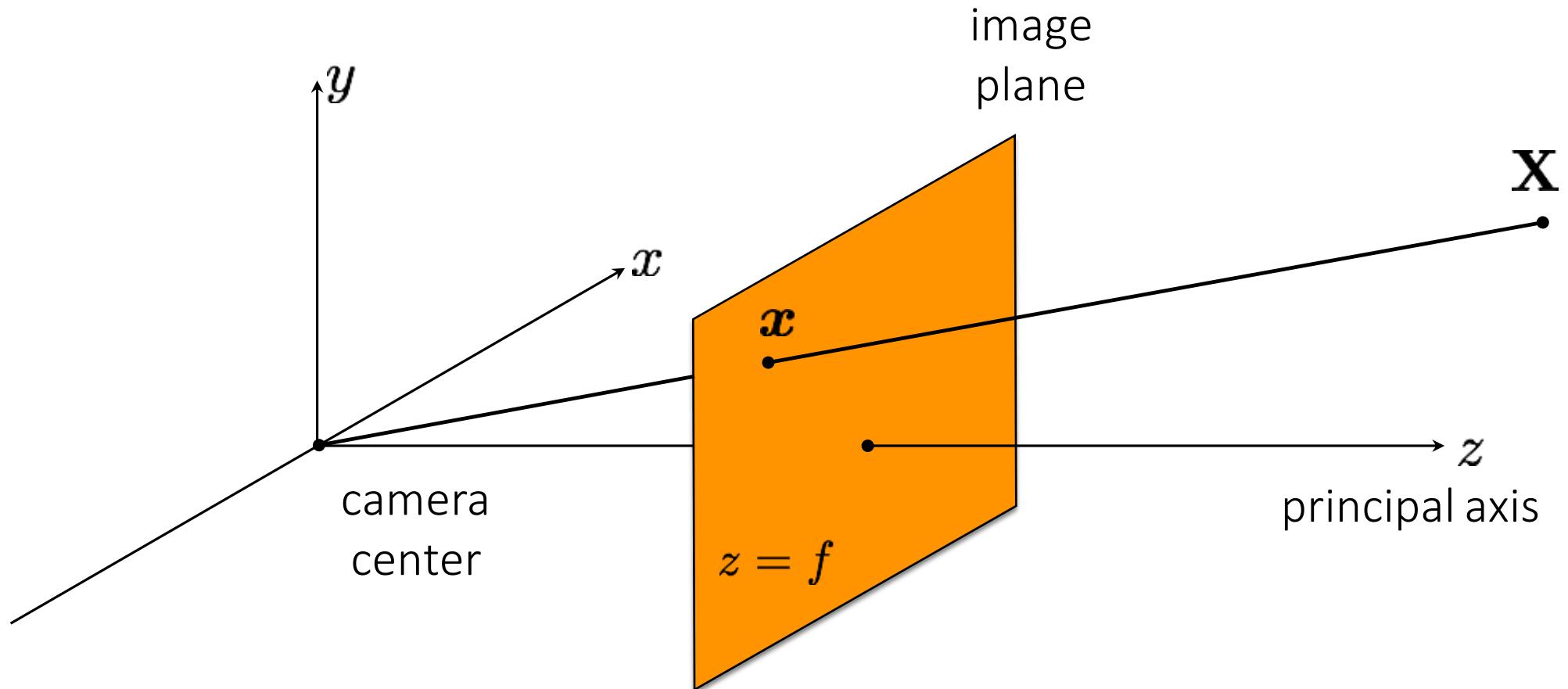
What is the equation for image coordinate x in terms of X ?

The 2D view of the (rearranged) pinhole camera



$$[X \ Y \ Z]^\top \mapsto [fX/Z \ fY/Z]^\top$$

The (rearranged) pinhole camera



What is the camera matrix \mathbf{P} for a pinhole camera?

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \ Y \ Z]^\top \mapsto [fX/Z \ fY/Z]^\top$$

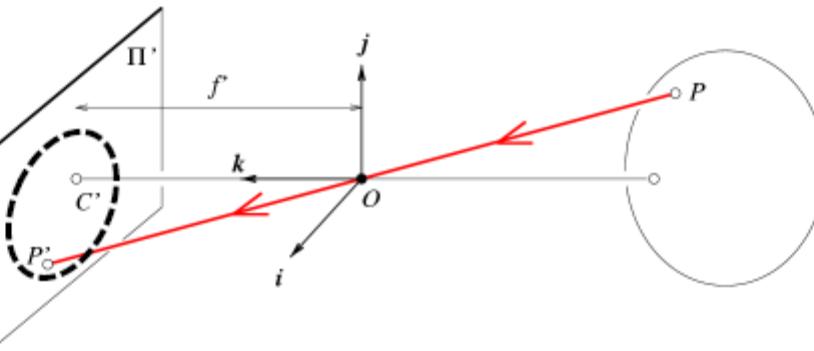
General camera model:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

$$\mathbf{P} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Relating a real-world point to a point on the camera



In homogeneous coordinates:

$$P' = \begin{bmatrix} f & x \\ f & y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_M \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

ideal world

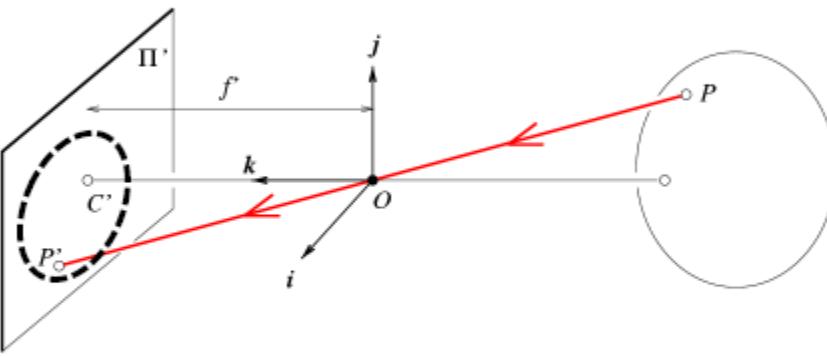
Intrinsic Assumptions

- Unit aspect ratio
- Optical center at (0,0)
- No skew

Extrinsic Assumptions

- No rotation
- Camera at (0,0,0)

Relating a real-world point to a point on the camera



In homogeneous coordinates:

$$P' = \begin{bmatrix} f x \\ f y \\ z \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = K[I \ 0]P$$

K

Intrinsic Assumptions

- Unit aspect ratio
- Optical center at $(0,0)$
- No skew

Extrinsic Assumptions

- No rotation
- Camera at $(0,0,0)$

Remove assumption: known optical center

Intrinsic Assumptions

- Optical center at $(0,0)$
- **Optical center at (u_0, v_0)**
- Square pixels
- No skew

Extrinsic Assumptions

- No rotation
- Camera at $(0,0,0)$

$$P' = K \begin{bmatrix} I & 0 \end{bmatrix} P \xrightarrow{\text{w}} w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Remove assumption: square pixels

Intrinsic Assumptions

- Optical center at (u_0, v_0)
- ~~Square pixels~~
- **Rectangular pixels**
- No skew

Extrinsic Assumptions

- No rotation
- Camera at $(0,0,0)$

$$P' = K \begin{bmatrix} I & 0 \end{bmatrix} P \rightarrow w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Remove assumption: non-skewed pixels

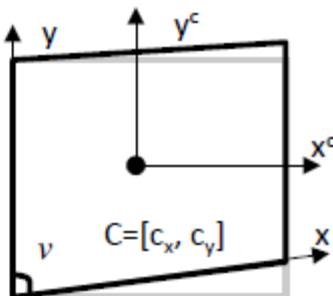
Intrinsic Assumptions

- Optical center at (u_0, v_0)
- Rectangular pixels
- ~~No skew~~
- Small skew

Extrinsic Assumptions

- No rotation
- Camera at $(0,0,0)$

$$P' = K \begin{bmatrix} I & 0 \end{bmatrix} P \rightarrow w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Remove assumption: non-skewed pixels

Intrinsic Assumptions

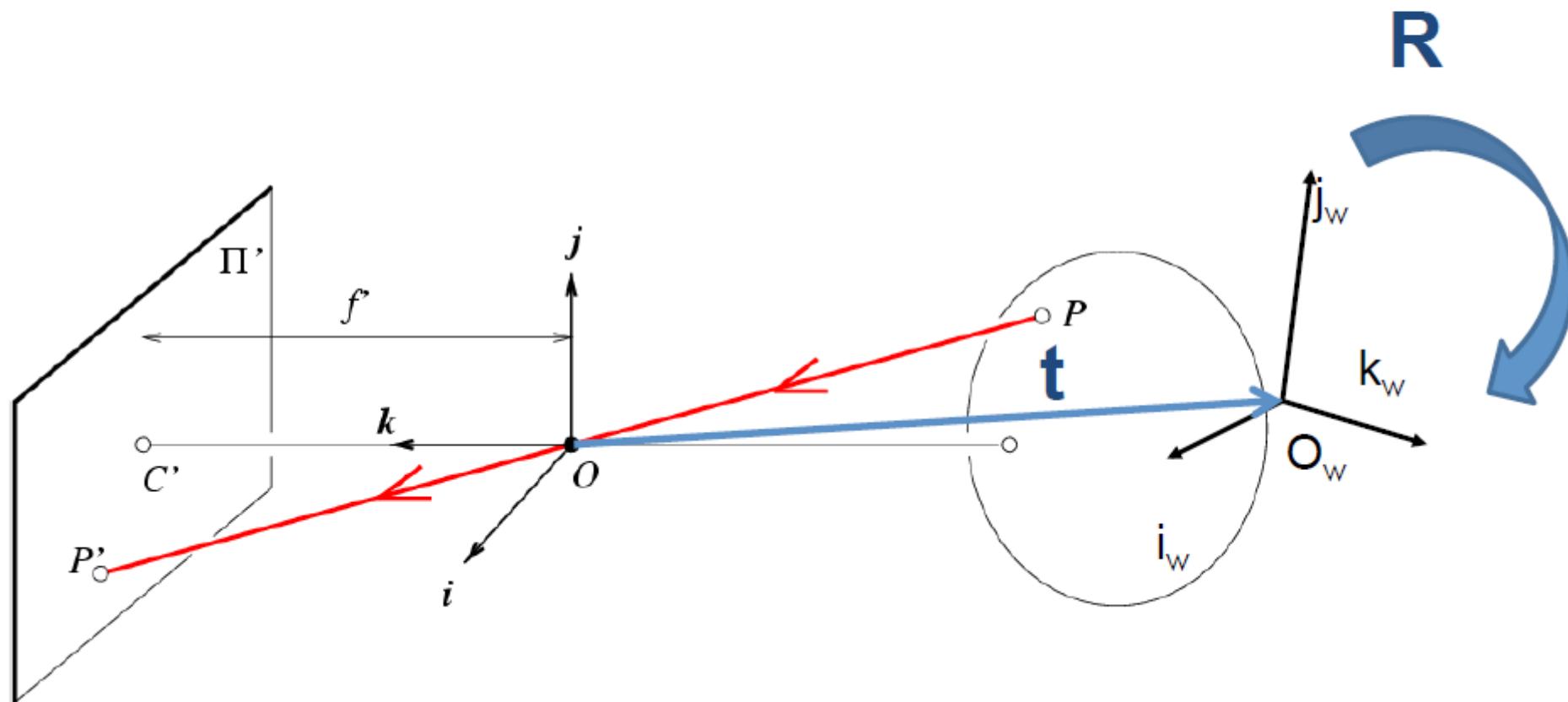
- Optical center at (u_0, v_0)
- Rectangular pixels
- Small skew

Extrinsic Assumptions

- No rotation
- Camera at $(0,0,0)$

$$P' = K \begin{bmatrix} I & 0 \end{bmatrix} P \xrightarrow{\text{Intrinsic parameters}} w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Real world camera:
Translate + Rotate



Remove assumption: allow translation

Intrinsic Assumptions

- Optical center at (u_0, v_0)
- Rectangular pixels
- Small skew

Extrinsic Assumptions

- No rotation
- Camera at $(0,0,0)$ $\rightarrow (t_x, t_y, t_z)$

$$P' = K \begin{bmatrix} I & \bar{t} \end{bmatrix} P \quad \xrightarrow{\text{blue arrow}} \quad w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

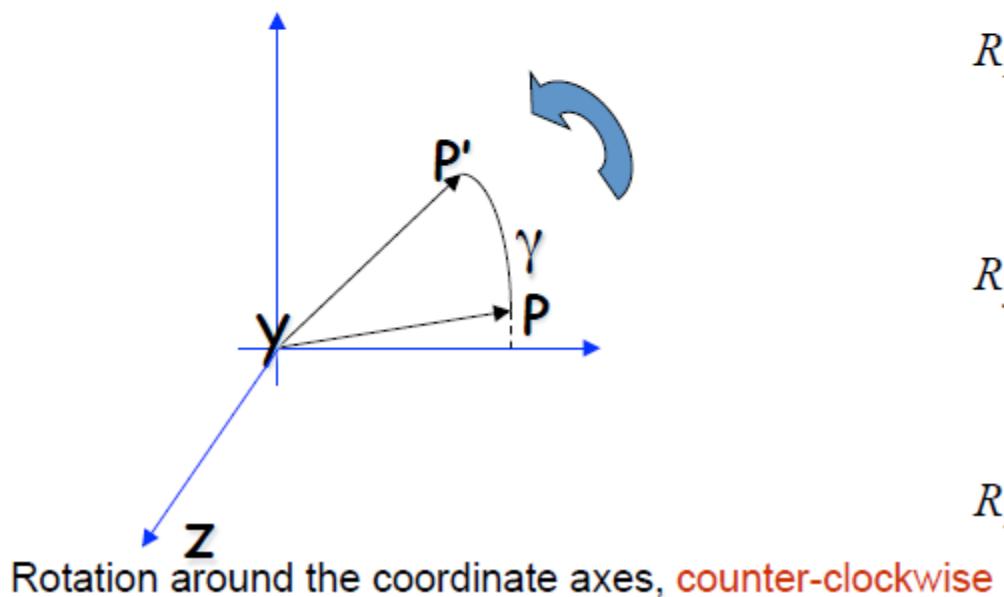
Remove assumption: allow rotation

Intrinsic Assumptions

- Optical center at (u_0, v_0)
- Rectangular pixels
- Small skew

Extrinsic Assumptions

- ~~No~~ rotation
- Camera at (t_x, t_y, t_z)



$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

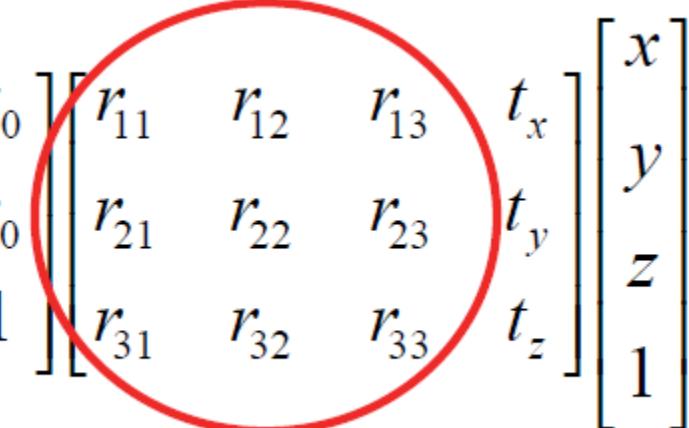
Remove assumption: allow rotation

Intrinsic Assumptions

- Optical center at (u_0, v_0)
- Rectangular pixels
- Small skew

Extrinsic Assumptions

- ~~No~~ rotation
- Camera at (t_x, t_y, t_z)

$$P' = K \begin{bmatrix} R & \bar{t} \end{bmatrix} P \rightarrow w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$


A generic projection matrix

Intrinsic Assumptions

- Optical center at (u_0, v_0)
- Rectangular pixels
- Small skew

Extrinsic Assumptions

- Allow rotation
- Camera at (t_x, t_y, t_z)

$$P' = K \begin{bmatrix} R & \bar{t} \end{bmatrix} P \xrightarrow{\text{w}} w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

A generic projection matrix

Intrinsic Assumptions

- Optical center at (u_0, v_0)
- Rectangular pixels
- Small skew

Extrinsic Assumptions

- Allow rotation
- Camera at (t_x, t_y, t_z)

$$P' = K \begin{bmatrix} R & \bar{t} \end{bmatrix} P \xrightarrow{\text{blue arrow}} w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Degrees of freedom??

A generic projection matrix

Intrinsic Assumptions

- Optical center at (u_0, v_0)
- Rectangular pixels
- Small skew

Extrinsic Assumptions

- Allow rotation
- Camera at (t_x, t_y, t_z)

$$P' = K \begin{bmatrix} R & \bar{t} \end{bmatrix} P \xrightarrow{\text{5}} w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & p & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Degrees of freedom??

Pose estimation

3D Pose Estimation

(Resectioning, Geometric Calibration, Perspective n-Point)

Given a set of matched points

$$\{\mathbf{X}_i, \mathbf{x}_i\}$$

point in 3D point in the
space image

and camera model

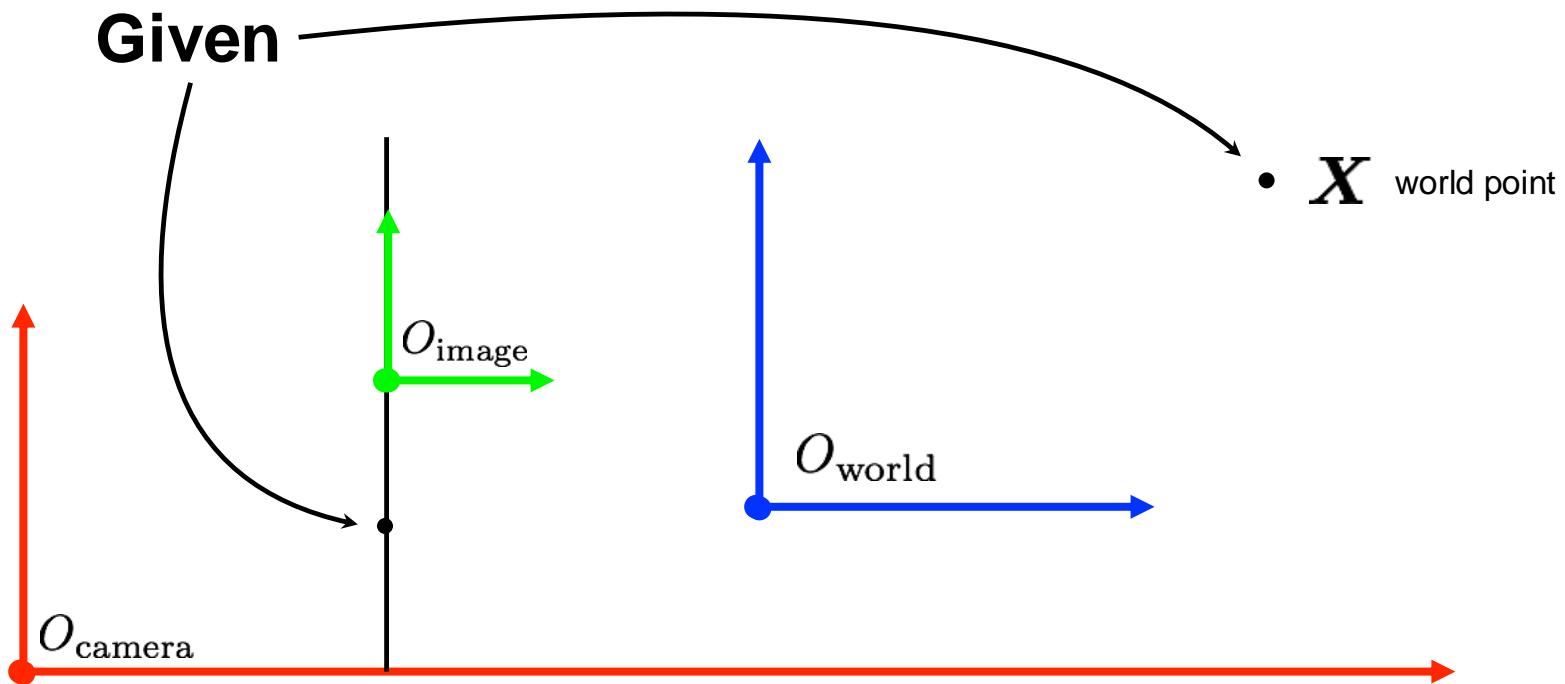
$$\mathbf{x} = f(\mathbf{X}; \mathbf{p}) = \mathbf{P}\mathbf{X}$$

projection parameters Camera
model matrix

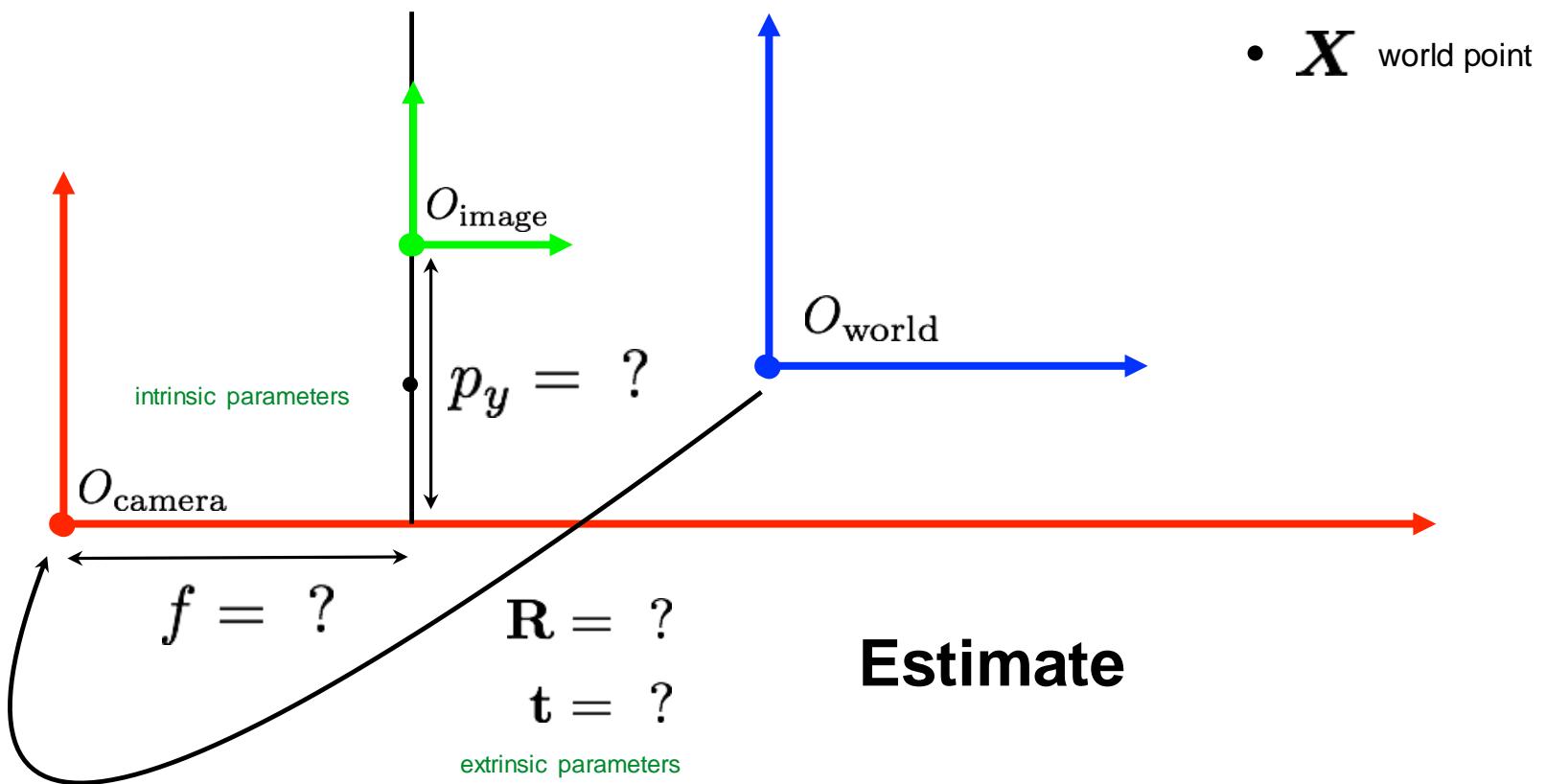
Find the (pose) estimate of

P

What is Pose Estimation?



What is Pose Estimation?



Mapping between 3D point and image points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What are the unknowns?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \text{---} & \mathbf{p}_1^\top & \text{---} \\ \text{---} & \mathbf{p}_2^\top & \text{---} \\ \text{---} & \mathbf{p}_3^\top & \text{---} \end{bmatrix} \begin{bmatrix} | \\ \mathbf{X} \\ | \end{bmatrix}$$

Heterogeneous coordinates

$$x' = \frac{\mathbf{p}_1^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}} \quad y' = \frac{\mathbf{p}_2^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}}$$

(non-linear correlation between coordinates)

How can we make these relations linear?

How can we make these relations linear?

$$x' = \frac{\mathbf{p}_1^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}} \quad y' = \frac{\mathbf{p}_2^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}}$$

Make them linear with algebraic manipulation...

$$\mathbf{p}_2^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} y' = 0$$

$$\mathbf{p}_1^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} x' = 0$$

Now you can setup a system of linear equations
with multiple point correspondences

$$\mathbf{p}_2^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} y' = 0$$

$$\mathbf{p}_1^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} x' = 0$$

In matrix form ...

$$\begin{bmatrix} \mathbf{X}^\top & \mathbf{0} & -x' \mathbf{X}^\top \\ \mathbf{0} & \mathbf{X}^\top & -y' \mathbf{X}^\top \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{0}$$

For N points ...

$$\begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -x' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -y' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -x' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -y' \mathbf{X}_N^\top \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{0}$$

Solve for camera matrix by

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \text{ subject to } \|\mathbf{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -\mathbf{y}' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -\mathbf{y}' \mathbf{X}_N^\top \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

SVD!

Solve for camera matrix by

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \text{ subject to } \|\mathbf{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -\mathbf{y}' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -\mathbf{y}' \mathbf{X}_N^\top \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Solution \mathbf{x} is the column of \mathbf{V}
corresponding to smallest singular
value of

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$$

Solve for camera matrix by

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \text{ subject to } \|\mathbf{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -\mathbf{y}' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -\mathbf{y}' \mathbf{X}_N^\top \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Equivalently, solution \mathbf{x} is the
Eigenvector corresponding to
smallest Eigenvalue of

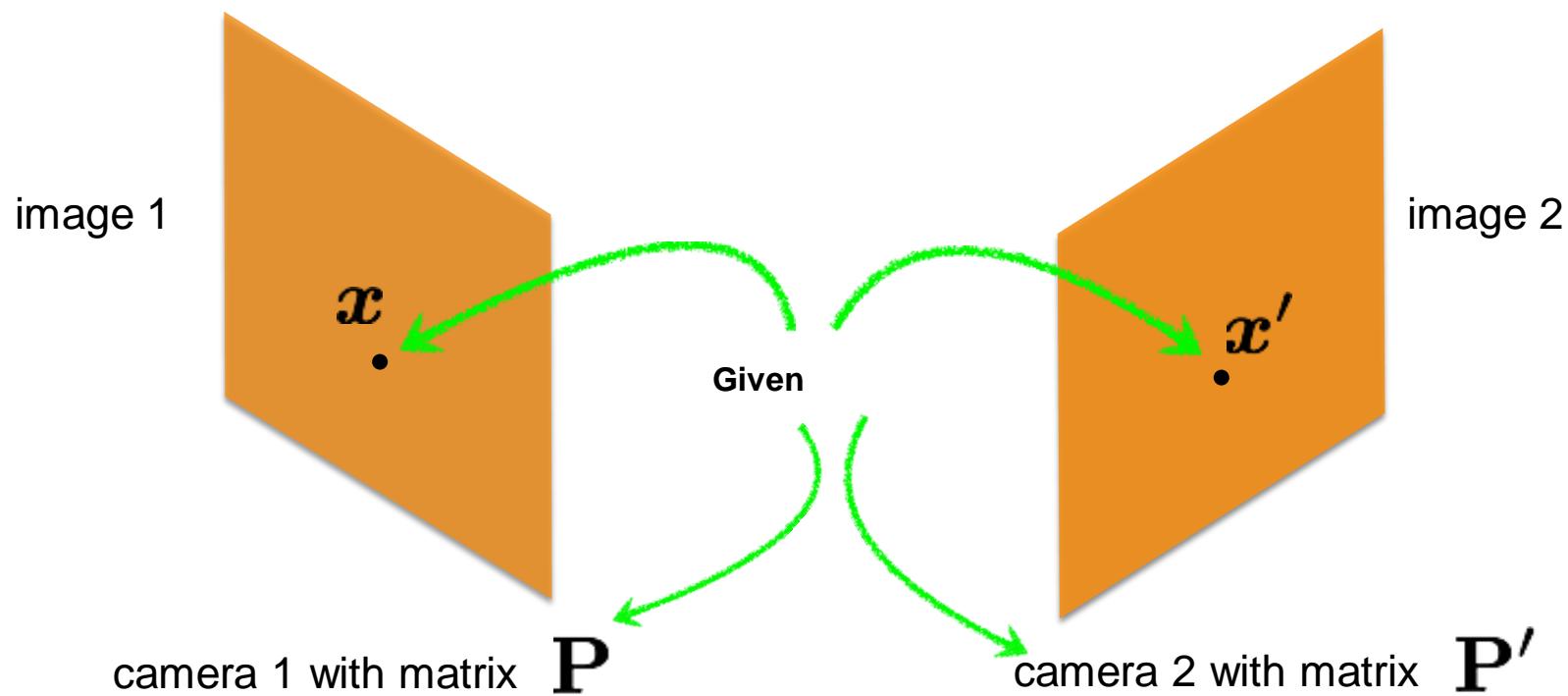
$$\mathbf{A}^\top \mathbf{A}$$

Overview of today's lecture

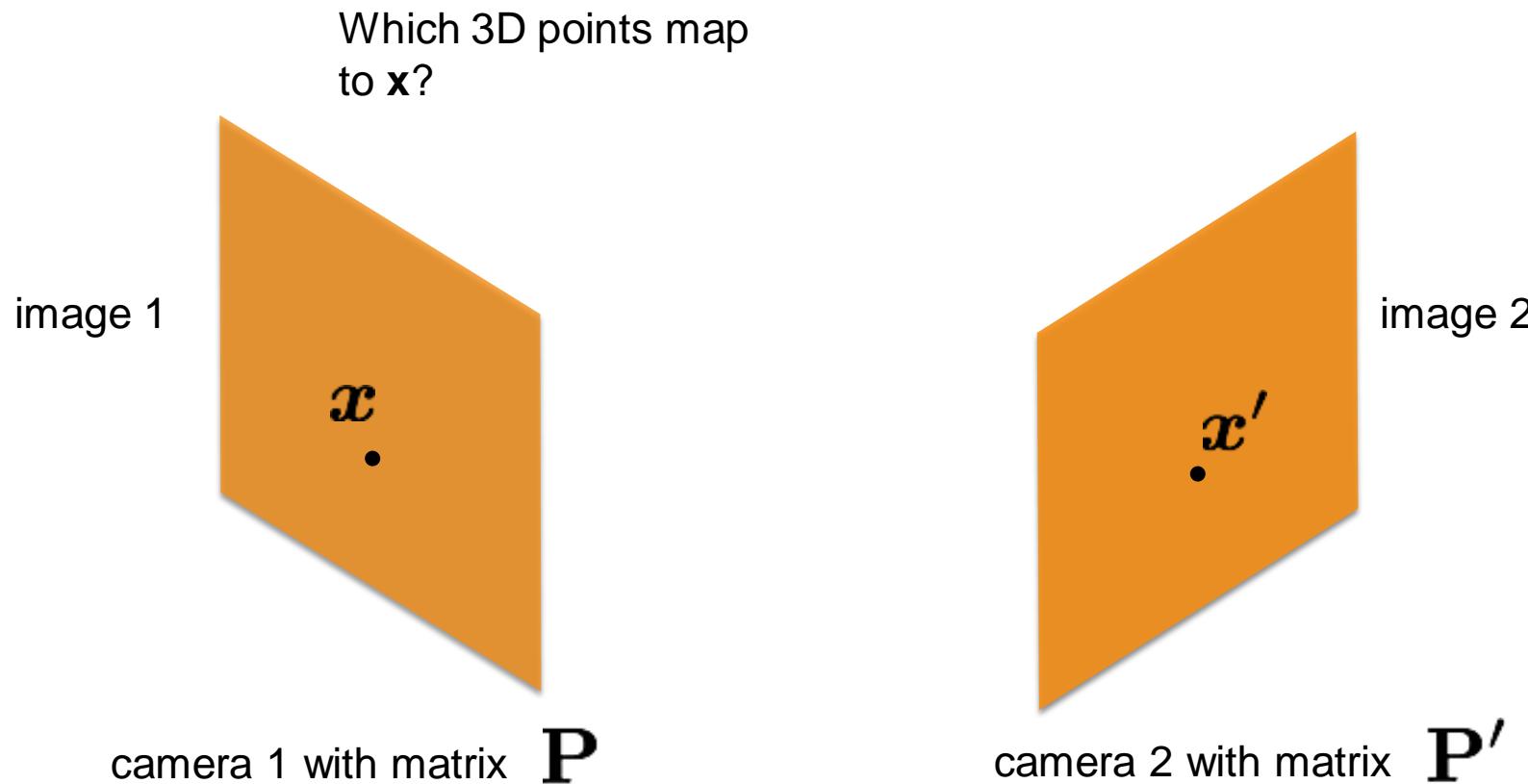
- Triangulation.
- Epipolar geometry.
- Essential matrix.
- Fundamental matrix.
- 8-point algorithm.

Triangulation

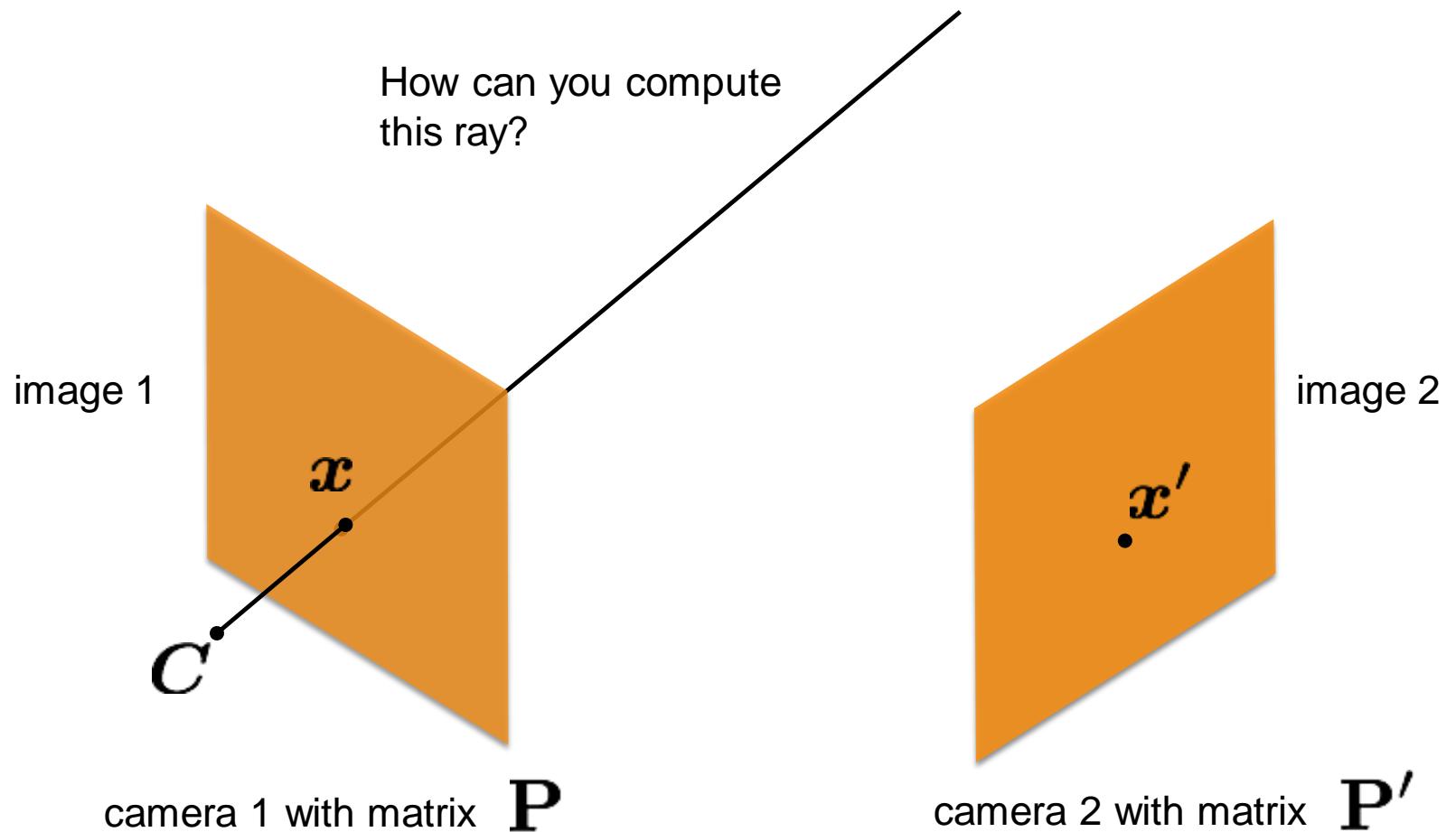
Triangulation



Triangulation



Triangulation

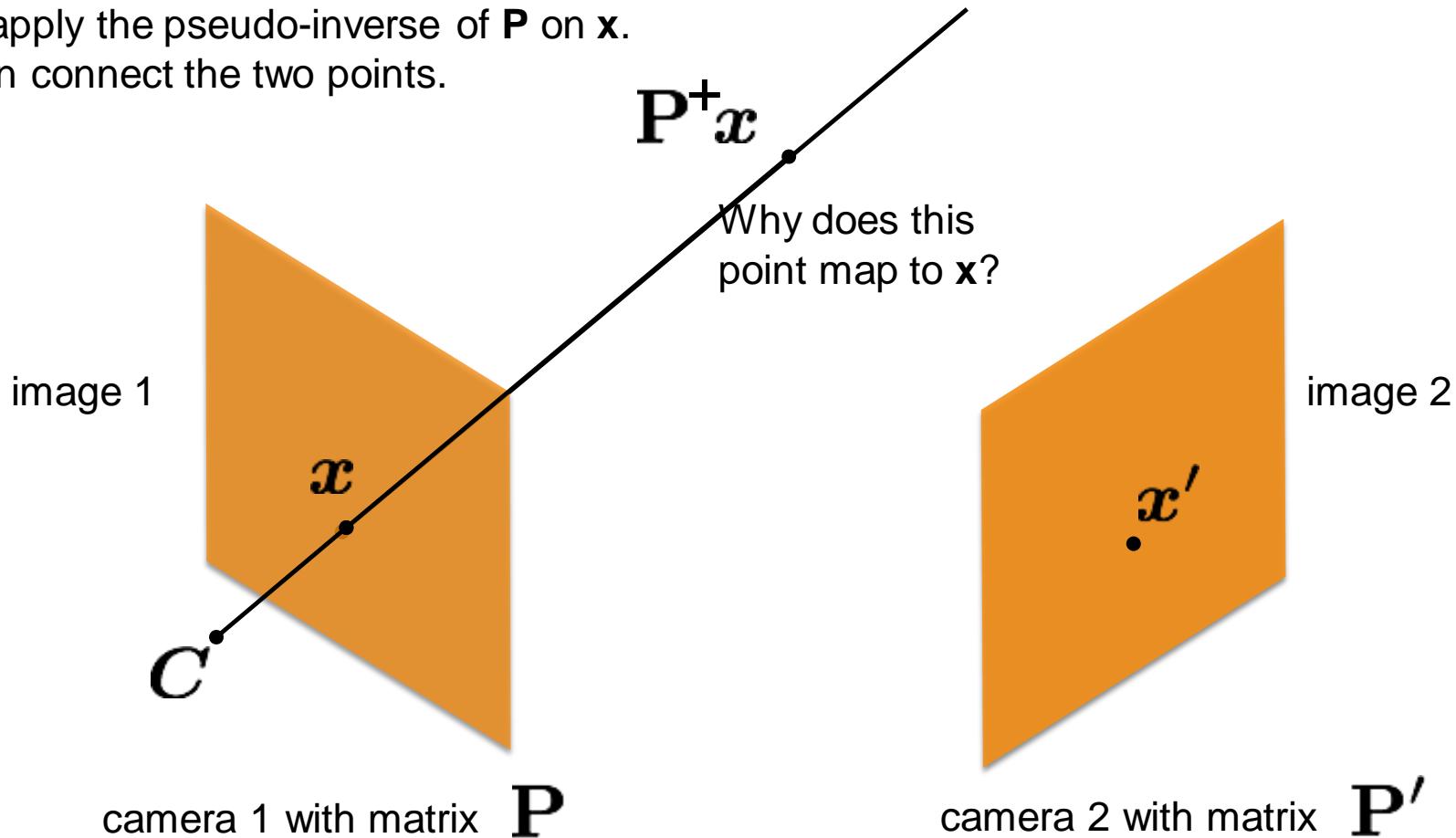


Triangulation

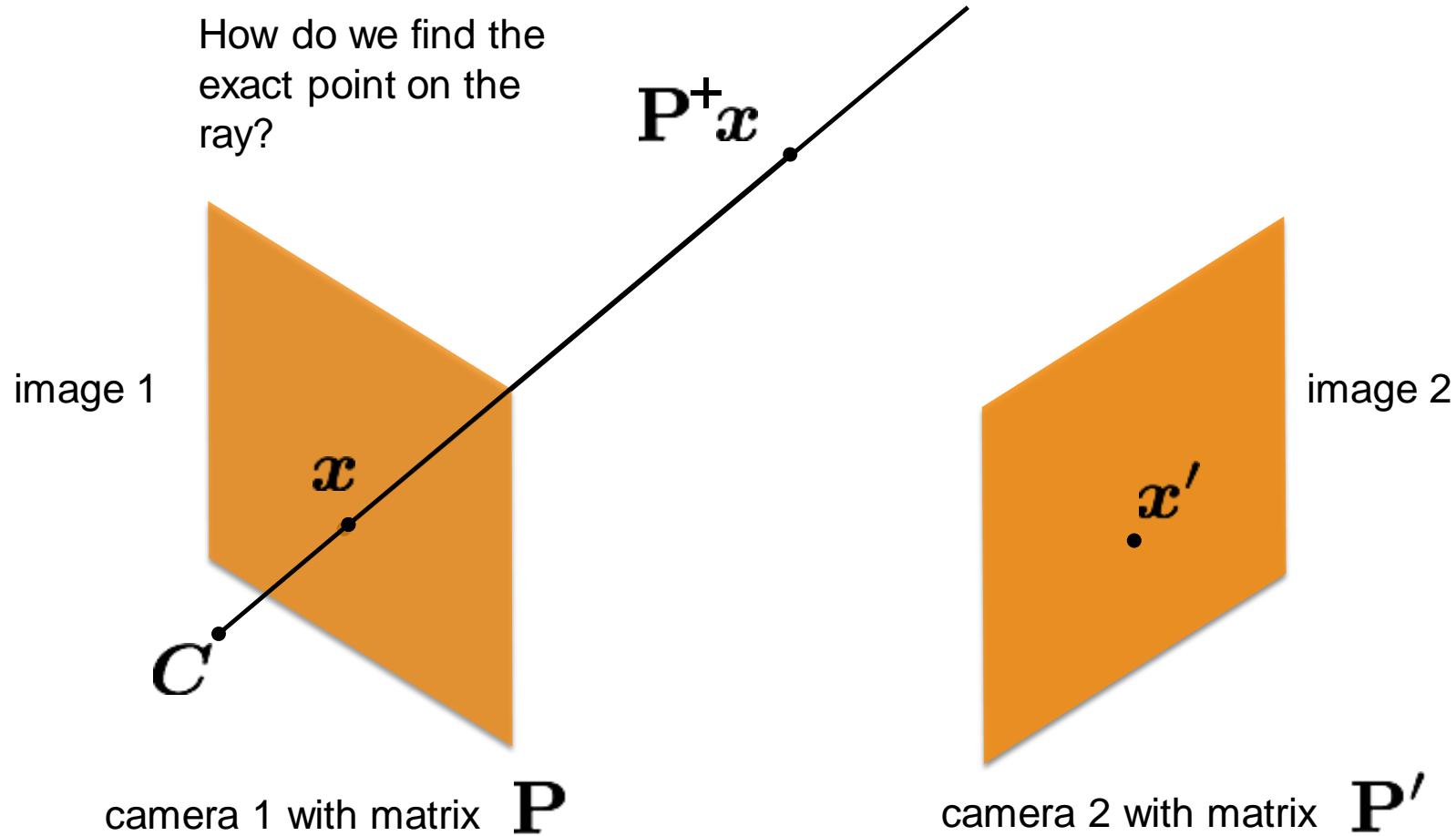
Create two points on the ray:

- 1) find the camera center; and
- 2) apply the pseudo-inverse of \mathbf{P} on \mathbf{x} .

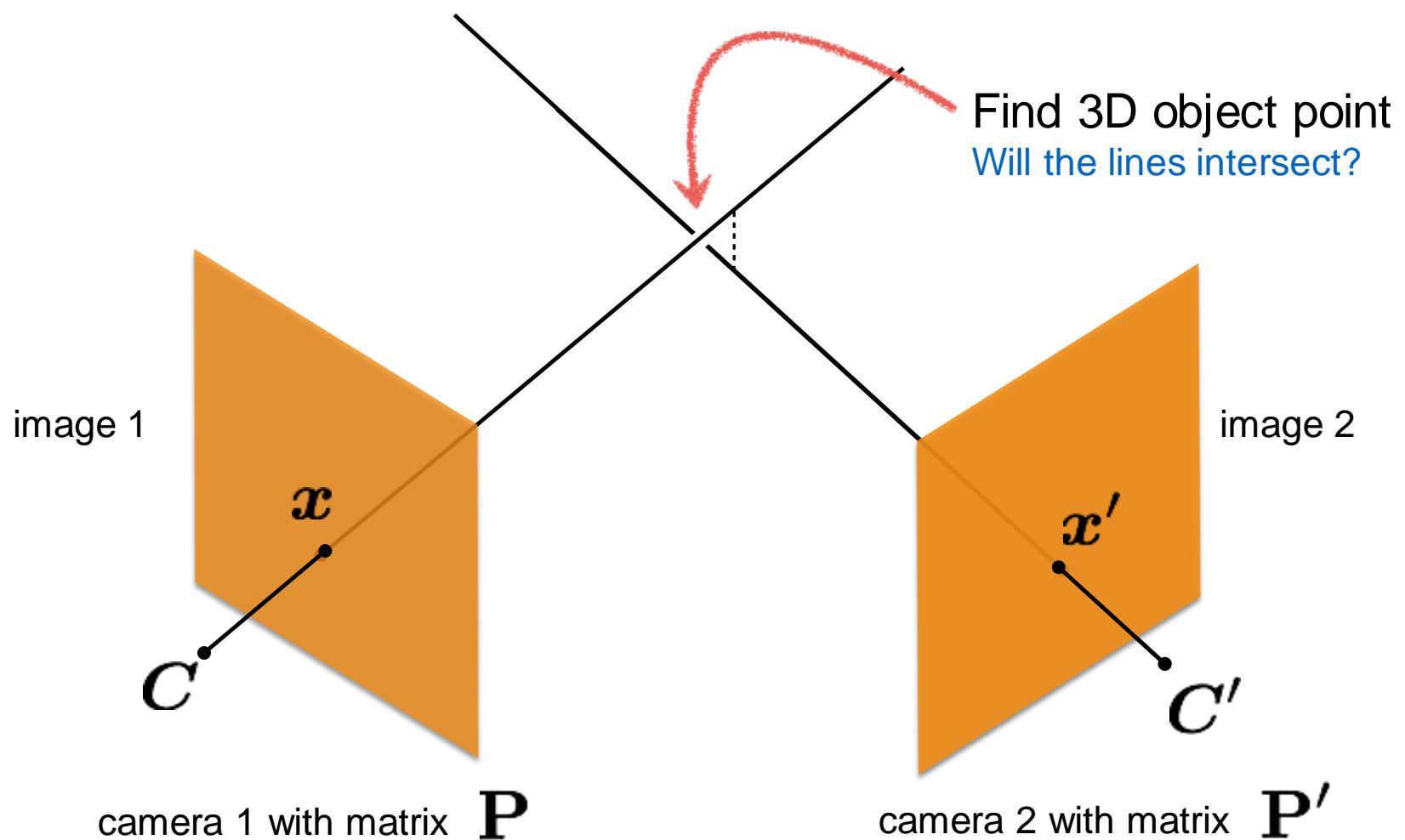
Then connect the two points.



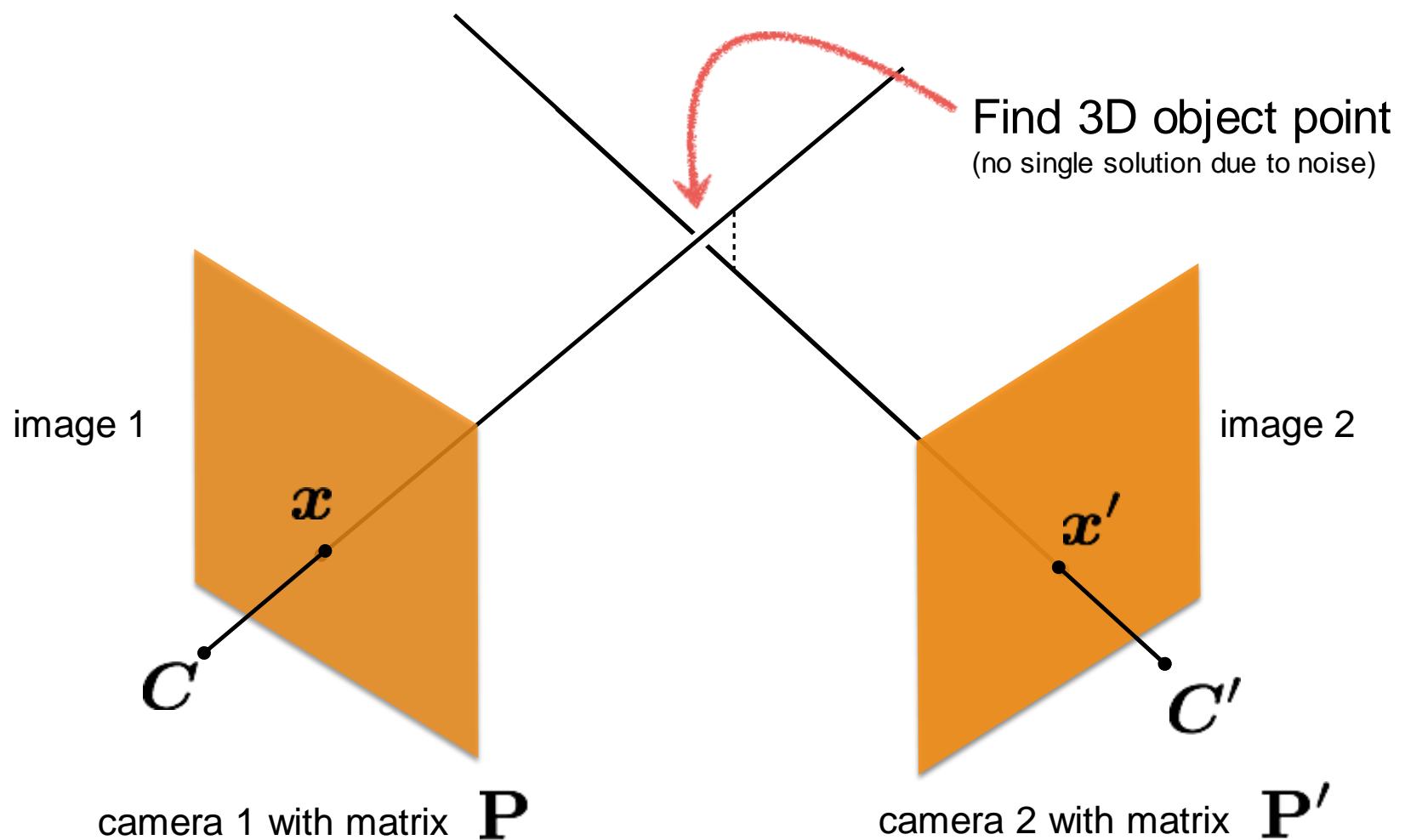
Triangulation



Triangulation



Triangulation



Triangulation

Given a set of (noisy) matched points

$$\{\mathbf{x}_i, \mathbf{x}'_i\}$$

and camera matrices

$$\mathbf{P}, \mathbf{P}'$$

Estimate the 3D point

$$\mathbf{\tilde{x}}$$

$$\mathbf{x} = \mathbf{P} \mathbf{X}$$

known known

Can we compute \mathbf{X} from a single correspondence \mathbf{x} ?

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

known known

Can we compute \mathbf{X} from two correspondences \mathbf{x} and \mathbf{x}' ?

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

known known

Can we compute \mathbf{X} from two correspondences \mathbf{x} and \mathbf{x}'' ?

yes if perfect measurements

$$\mathbf{x} = \mathbf{P} \mathbf{X}$$

known known

Can we compute \mathbf{X} from two correspondences \mathbf{x} and \mathbf{x}'' ?

yes if perfect measurements

There will not be a point that satisfies both constraints because the measurements are usually noisy

$$\mathbf{x}' = \mathbf{P}' \mathbf{X} \quad \mathbf{x} = \mathbf{P} \mathbf{X}$$

Need to find the **best fit**

$$\mathbf{x} = \mathbf{P}X$$

(homogeneous
coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha \mathbf{P}X$$

(homogeneous
coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How do we solve for unknowns in a similarity relation?

$$\mathbf{x} = \mathbf{P}X$$

(homogeneous
coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha \mathbf{P}X$$

(inhomogeneous
coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How do we solve for unknowns in a similarity relation?

Remove scale factor, convert to linear system and solve with 

$$\mathbf{x} = \mathbf{P}X$$

(homogeneous
coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha \mathbf{P}X$$

(inhomogeneous
coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How do we solve for unknowns in a similarity relation?

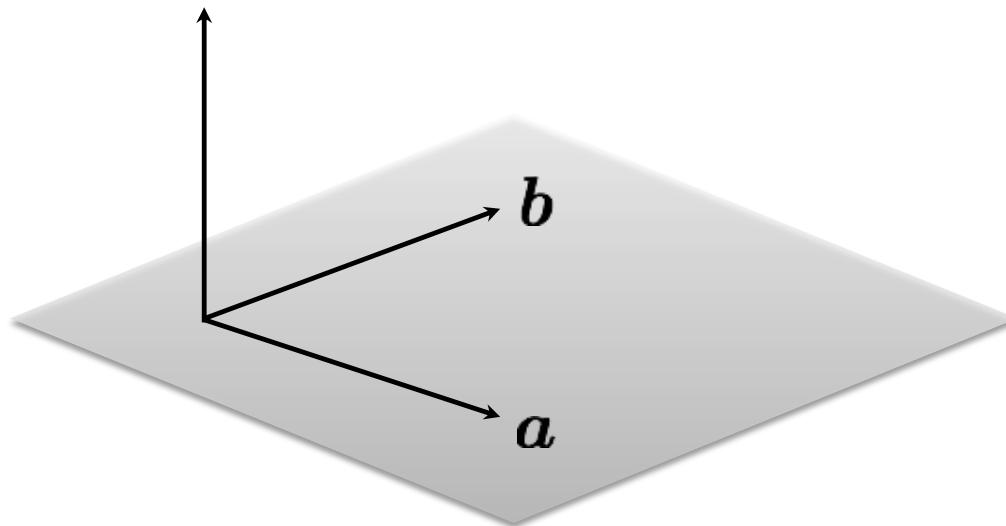
Remove scale factor, convert to linear system and solve with SVD!

Recall: Cross Product

Vector (cross) product

takes two vectors and returns a vector perpendicular to both

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}$$



$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

cross product of two vectors in the same direction is zero

$$\mathbf{a} \times \mathbf{a} = 0$$

remember this!!!

$$\mathbf{c} \cdot \mathbf{a} = 0$$

$$\mathbf{c} \cdot \mathbf{b} = 0$$

$$\mathbf{x} = \alpha \mathbf{P} \mathbf{X}$$

Same direction but differs by a scale factor

$$\mathbf{x} \times \mathbf{P} \mathbf{X} = \mathbf{0}$$

Cross product of two vectors of same direction is zero
(this equality removes the scale factor)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \text{---} & \mathbf{p}_1^\top & \text{---} \\ \text{---} & \mathbf{p}_2^\top & \text{---} \\ \text{---} & \mathbf{p}_3^\top & \text{---} \end{bmatrix} \begin{bmatrix} | \\ \mathbf{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \mathbf{p}_1^\top \mathbf{X} \\ \mathbf{p}_2^\top \mathbf{X} \\ \mathbf{p}_3^\top \mathbf{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \text{---} & \mathbf{p}_1^\top & \text{---} \\ \text{---} & \mathbf{p}_2^\top & \text{---} \\ \text{---} & \mathbf{p}_3^\top & \text{---} \end{bmatrix} \begin{bmatrix} | \\ \mathbf{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \mathbf{p}_1^\top \mathbf{X} \\ \mathbf{p}_2^\top \mathbf{X} \\ \mathbf{p}_3^\top \mathbf{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \mathbf{p}_1^\top \mathbf{X} \\ \mathbf{p}_2^\top \mathbf{X} \\ \mathbf{p}_3^\top \mathbf{X} \end{bmatrix} = \begin{bmatrix} y\mathbf{p}_3^\top \mathbf{X} - \mathbf{p}_2^\top \mathbf{X} \\ \mathbf{p}_1^\top \mathbf{X} - x\mathbf{p}_3^\top \mathbf{X} \\ x\mathbf{p}_2^\top \mathbf{X} - y\mathbf{p}_1^\top \mathbf{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using the fact that the cross product should be zero

$$\mathbf{x} \times \mathbf{P} \mathbf{X} = \mathbf{0}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \mathbf{p}_1^\top \mathbf{X} \\ \mathbf{p}_2^\top \mathbf{X} \\ \mathbf{p}_3^\top \mathbf{X} \end{bmatrix} = \begin{bmatrix} y\mathbf{p}_3^\top \mathbf{X} - \mathbf{p}_2^\top \mathbf{X} \\ \mathbf{p}_1^\top \mathbf{X} - x\mathbf{p}_3^\top \mathbf{X} \\ x\mathbf{p}_2^\top \mathbf{X} - y\mathbf{p}_1^\top \mathbf{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you equations

Using the fact that the cross product should be zero

$$\mathbf{x} \times \mathbf{P} \mathbf{X} = \mathbf{0}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \mathbf{p}_1^\top \mathbf{X} \\ \mathbf{p}_2^\top \mathbf{X} \\ \mathbf{p}_3^\top \mathbf{X} \end{bmatrix} = \begin{bmatrix} y\mathbf{p}_3^\top \mathbf{X} - \mathbf{p}_2^\top \mathbf{X} \\ \mathbf{p}_1^\top \mathbf{X} - x\mathbf{p}_3^\top \mathbf{X} \\ x\mathbf{p}_2^\top \mathbf{X} - y\mathbf{p}_1^\top \mathbf{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you 2 equations

$$\begin{bmatrix} y\mathbf{p}_3^\top \mathbf{X} - \mathbf{p}_2^\top \mathbf{X} \\ \mathbf{p}_1^\top \mathbf{X} - x\mathbf{p}_3^\top \mathbf{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y\mathbf{p}_3^\top - \mathbf{p}_2^\top \\ \mathbf{p}_1^\top - x\mathbf{p}_3^\top \end{bmatrix} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}_i \mathbf{X} = \mathbf{0}$$

Now we can make a system of linear equations
(two lines for each 2D point correspondence)

Concatenate the 2D points from both images

$$\begin{bmatrix} y\mathbf{p}_3^\top - \mathbf{p}_2^\top \\ \mathbf{p}_1^\top - x\mathbf{p}_3^\top \\ y'\mathbf{p}'_3^\top - \mathbf{p}'_2^\top \\ \mathbf{p}'_1^\top - x'\mathbf{p}'_3^\top \end{bmatrix} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

sanity check! dimensions?

$$\mathbf{A}\mathbf{X} = \mathbf{0}$$

How do we solve homogeneous linear system?

Concatenate the 2D points from both images

$$\begin{bmatrix} y\mathbf{p}_3^\top - \mathbf{p}_2^\top \\ \mathbf{p}_1^\top - x\mathbf{p}_3^\top \\ y'\mathbf{p}'_3^\top - \mathbf{p}'_2^\top \\ \mathbf{p}'_1^\top - x'\mathbf{p}'_3^\top \end{bmatrix} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}\mathbf{X} = \mathbf{0}$$

How do we solve homogeneous linear system?

S V D !

Recall: Total least squares

(Warning: change of notation. \mathbf{x} is a vector of parameters!)

$$\begin{aligned} E_{\text{TLS}} &= \sum_i (\mathbf{a}_i \mathbf{x})^2 \\ &= \|\mathbf{A}\mathbf{x}\|^2 \quad \text{(matrix form)} \end{aligned}$$

$$\|\mathbf{x}\|^2 = 1 \quad \text{constraint}$$

minimize

$$\|\mathbf{A}\mathbf{x}\|^2$$

subject to

$$\|\mathbf{x}\|^2 = 1$$



minimize

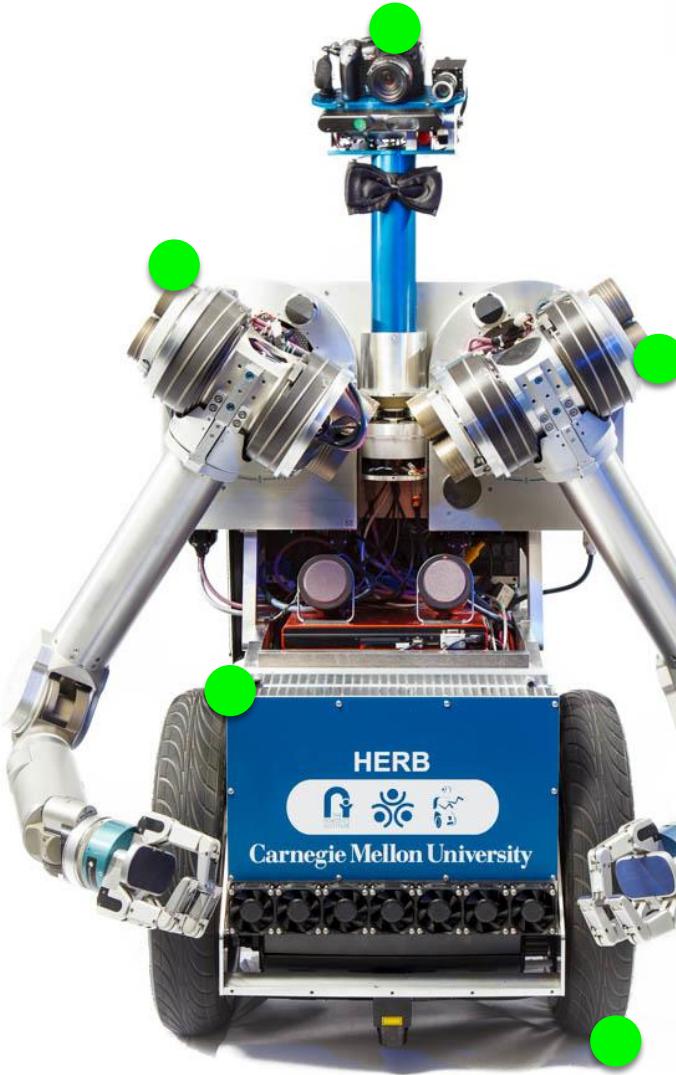
$$\frac{\|\mathbf{A}\mathbf{x}\|^2}{\|\mathbf{x}\|^2}$$

(Rayleigh quotient)

Solution is the eigenvector
corresponding to smallest eigenvalue of

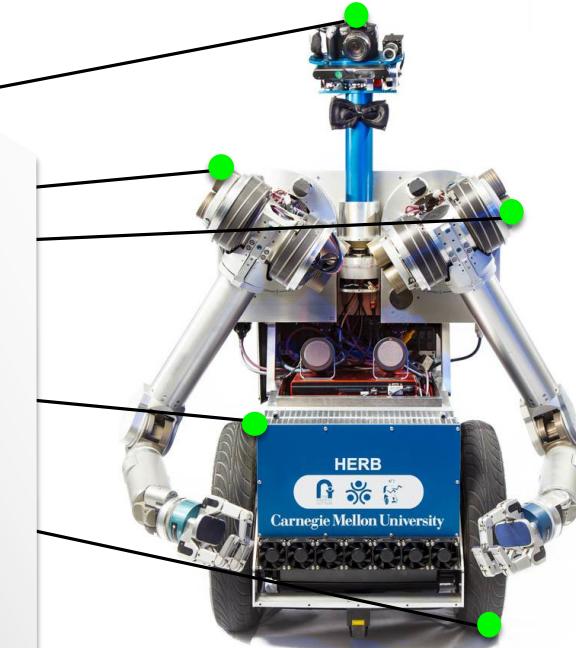
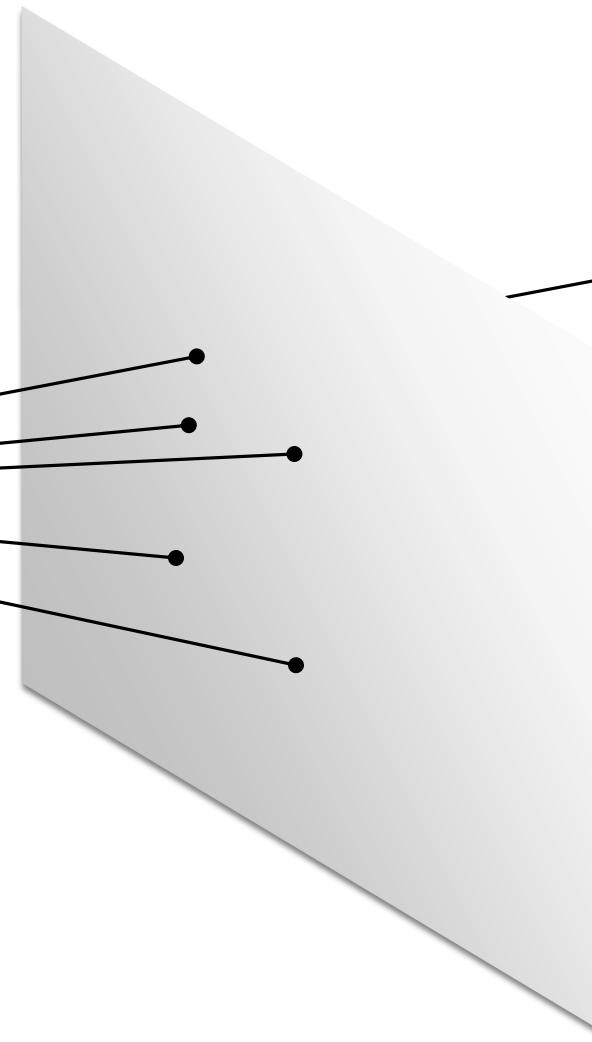
$$\mathbf{A}^\top \mathbf{A}$$

Epipolar geometry

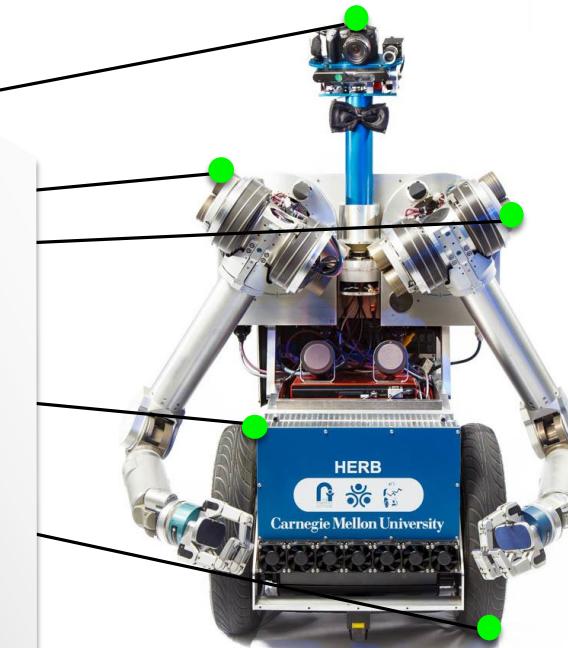
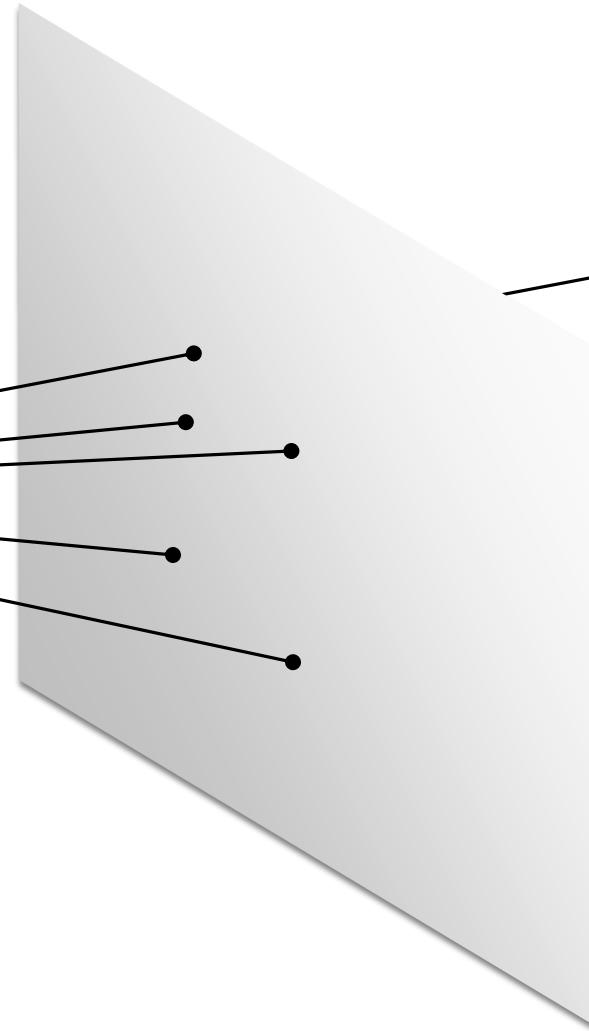


Tie tiny threads on HERB and pin them to your eyeball

[What would it look like?](#)

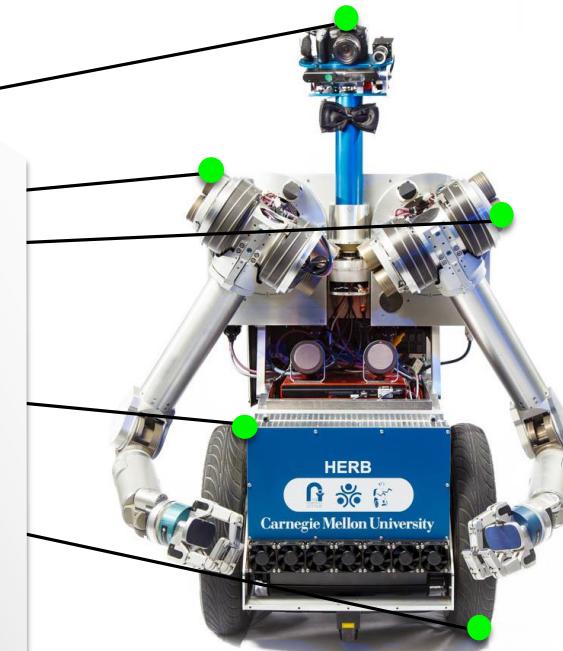
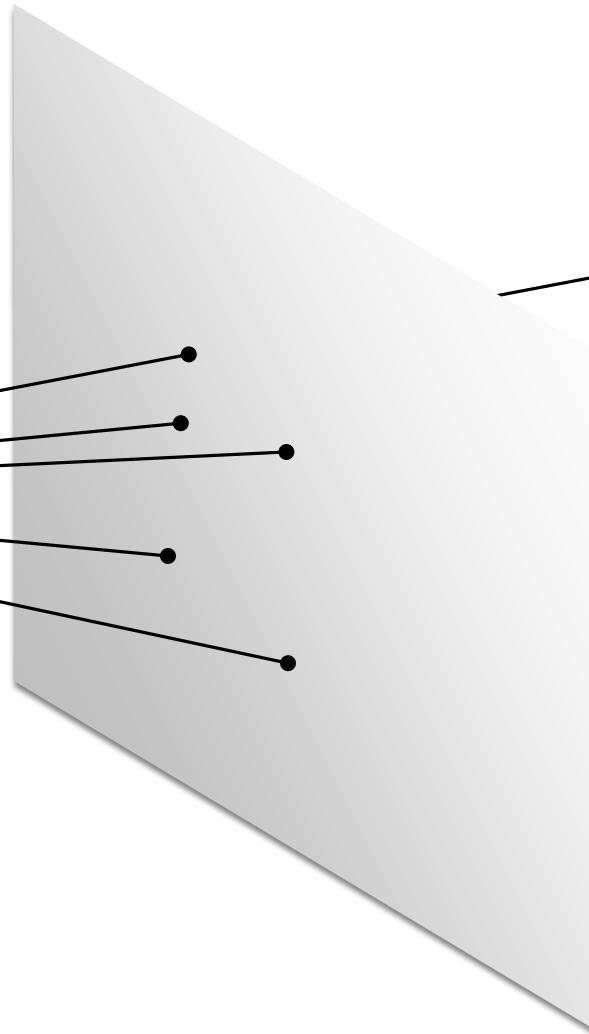


You see points on HERB



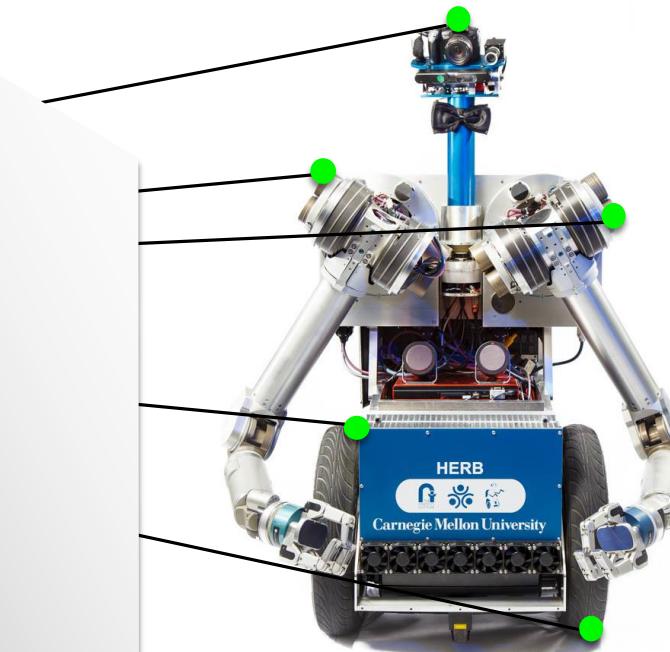
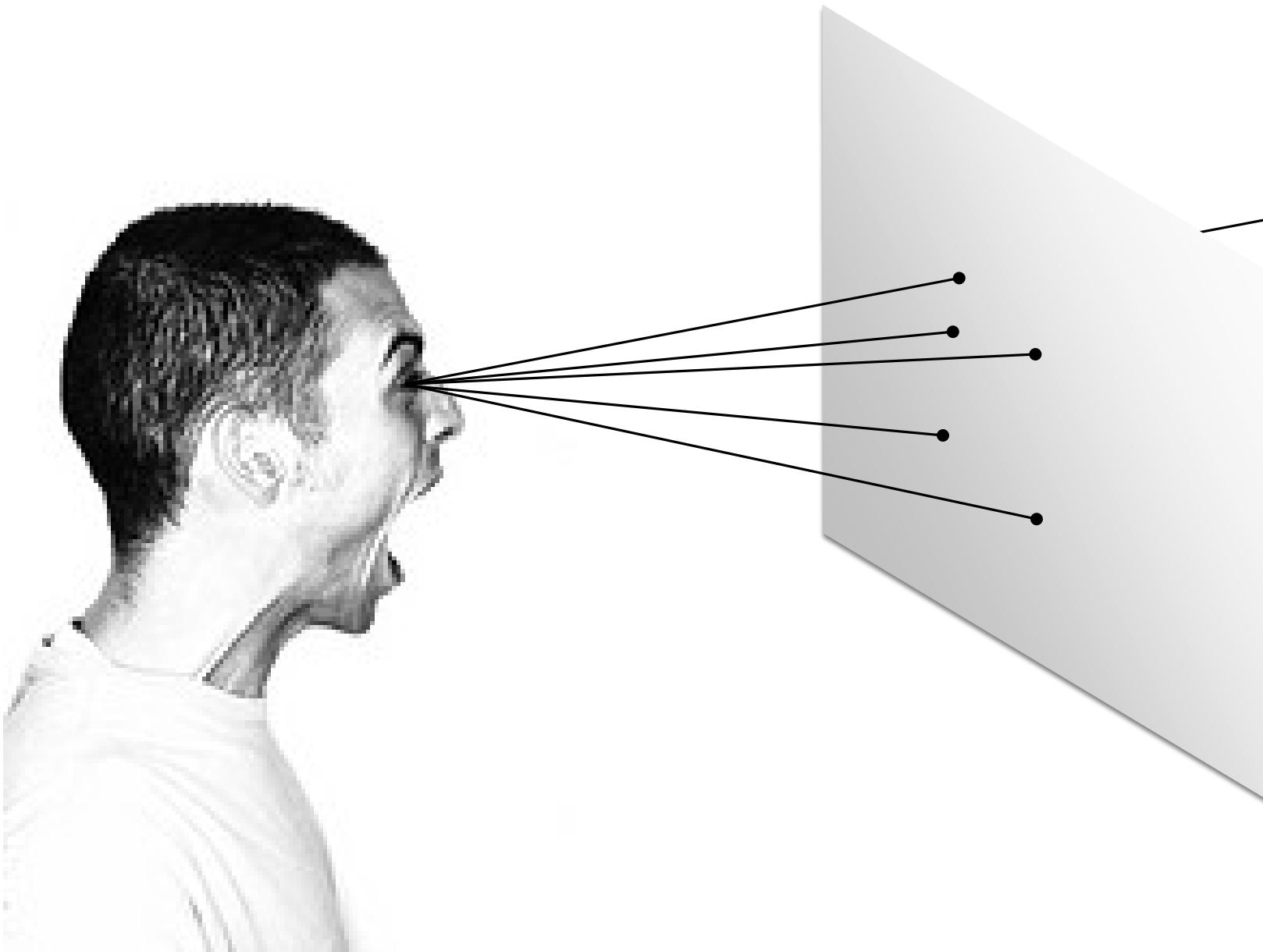
What does the second observer see?

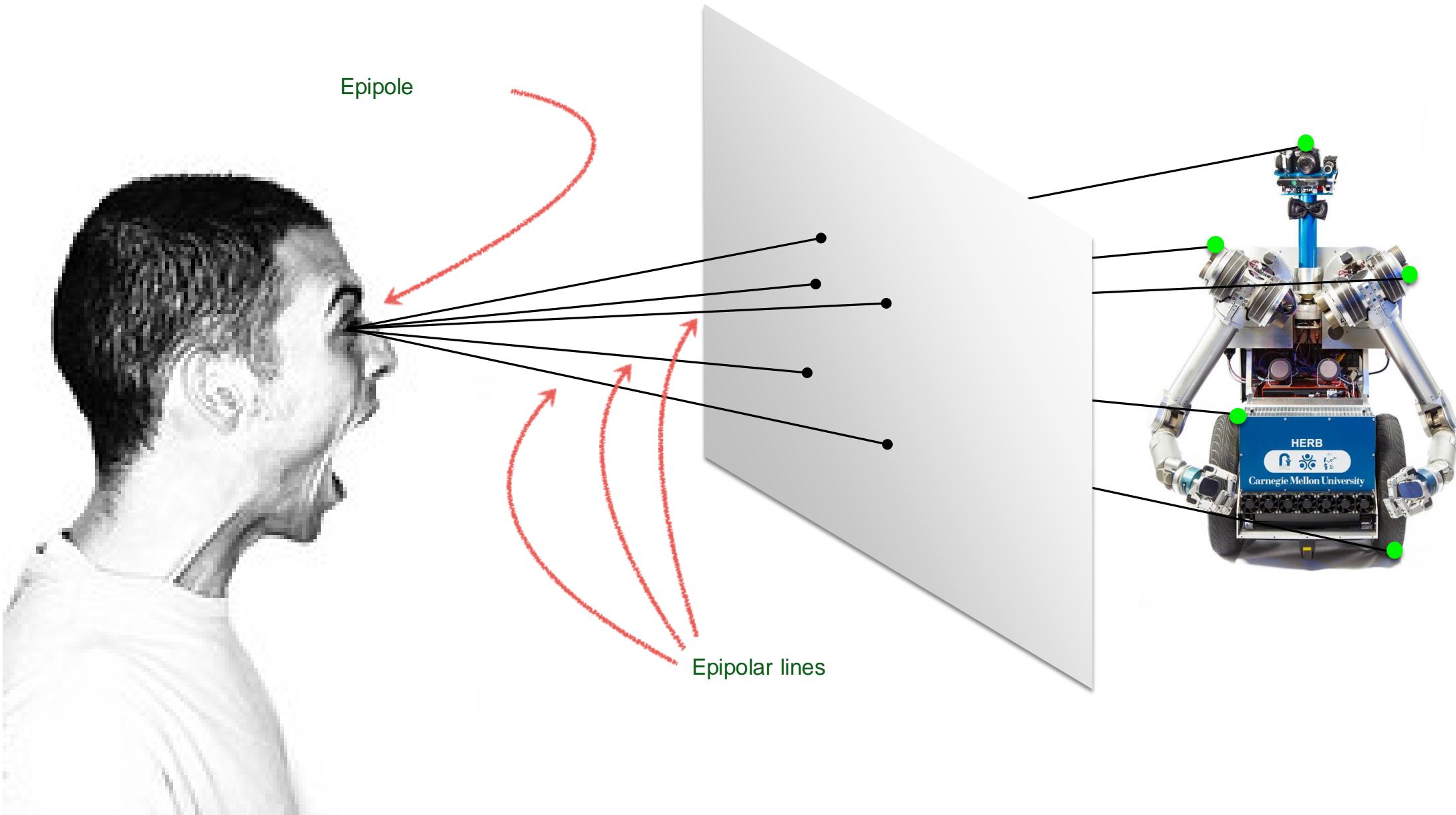
You see points on HERB



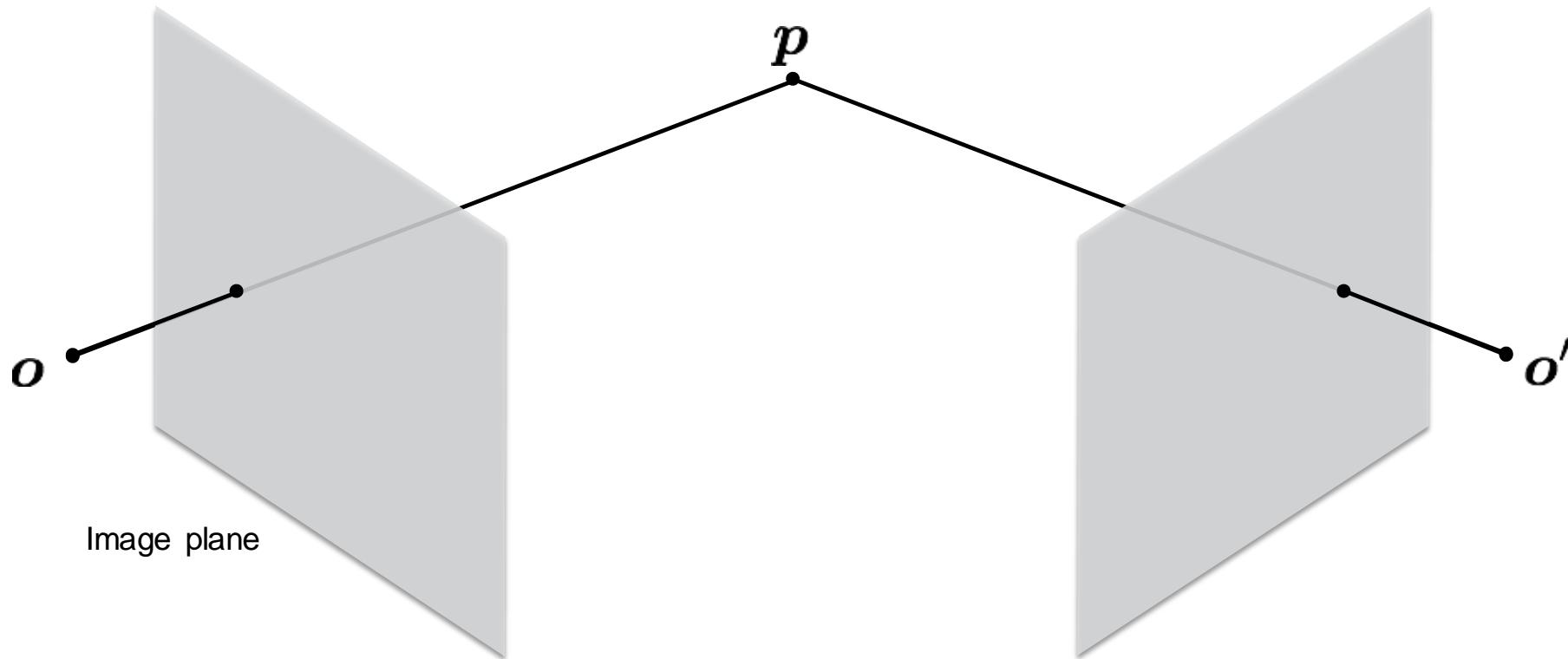
Second person sees lines

This is Epipolar Geometry

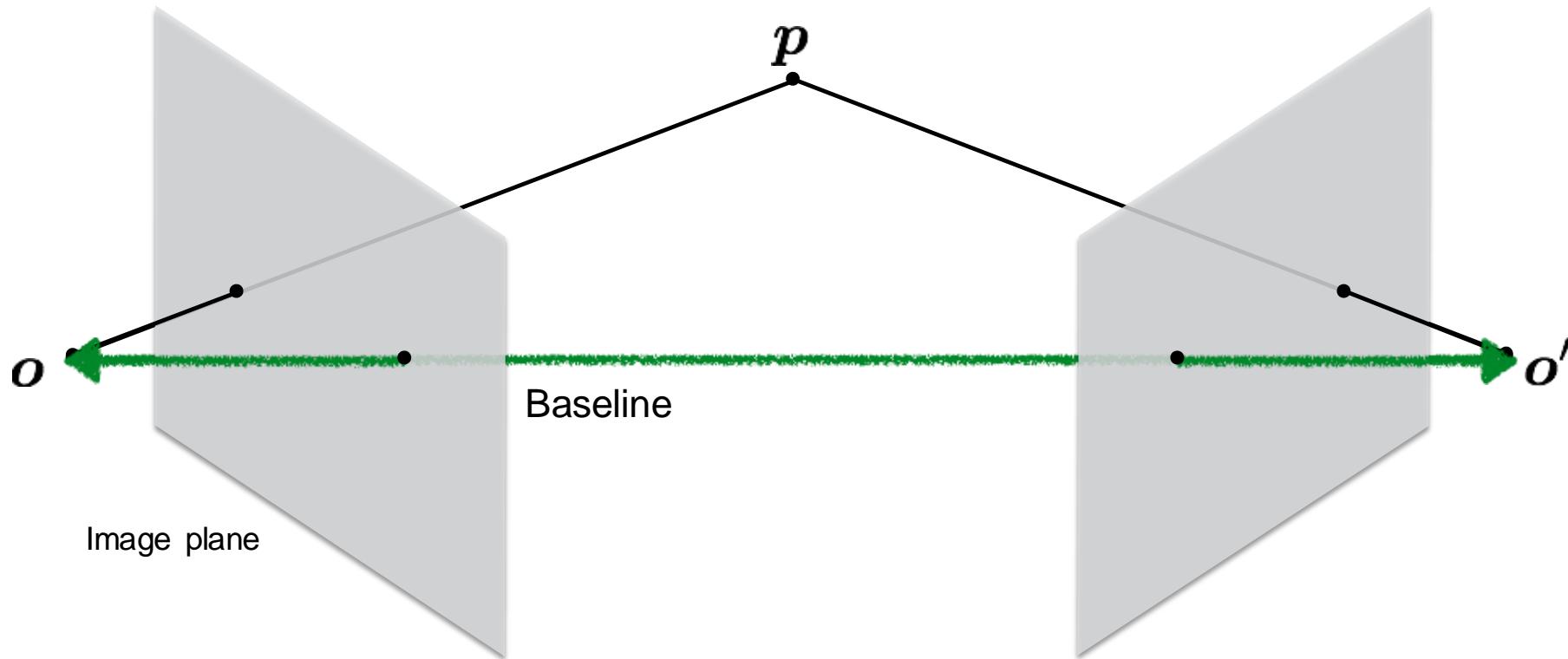




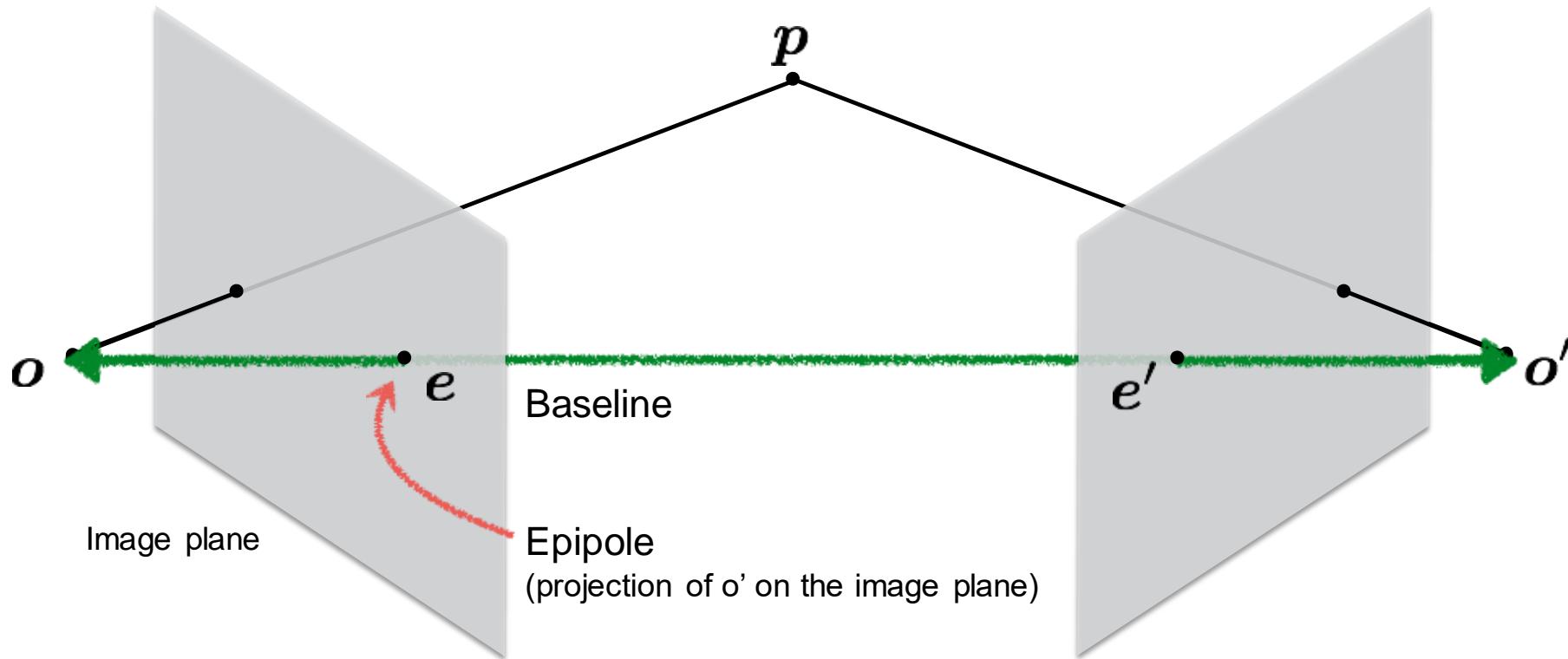
Epipolar geometry



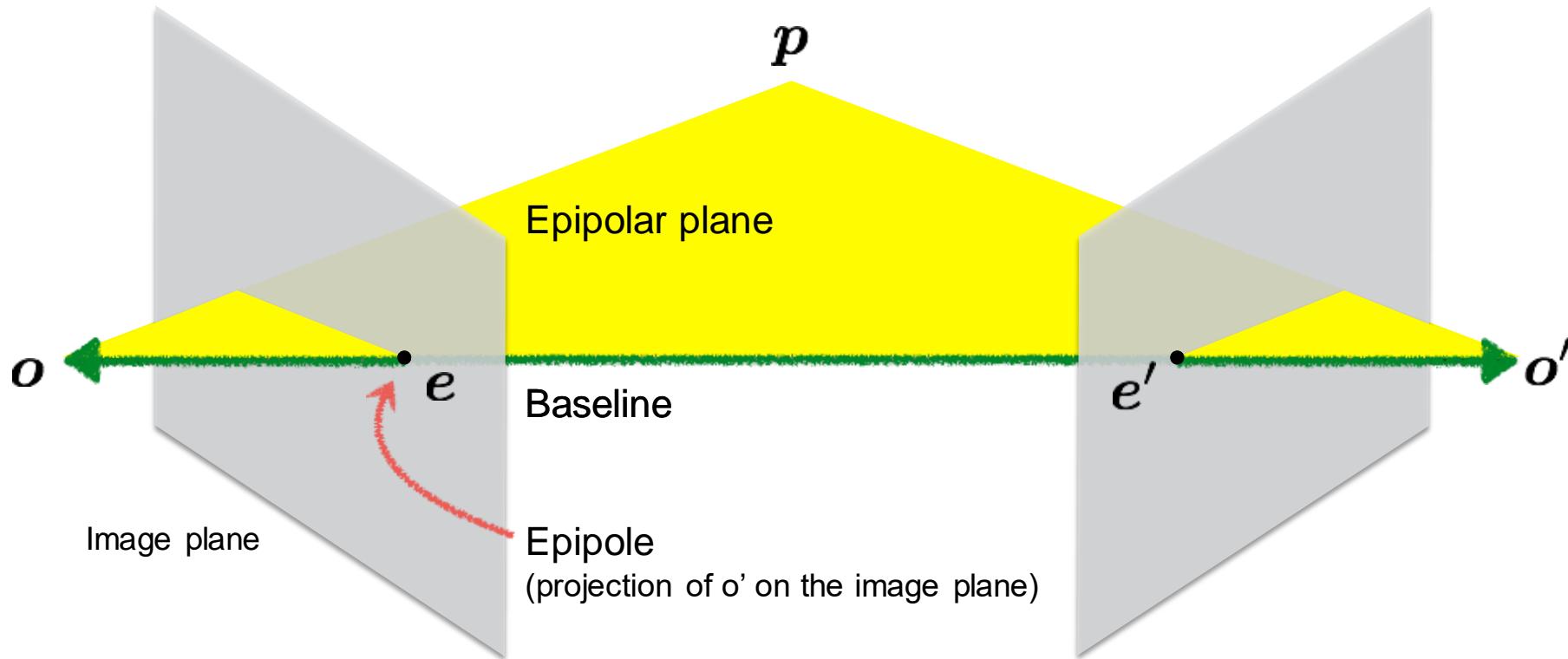
Epipolar geometry



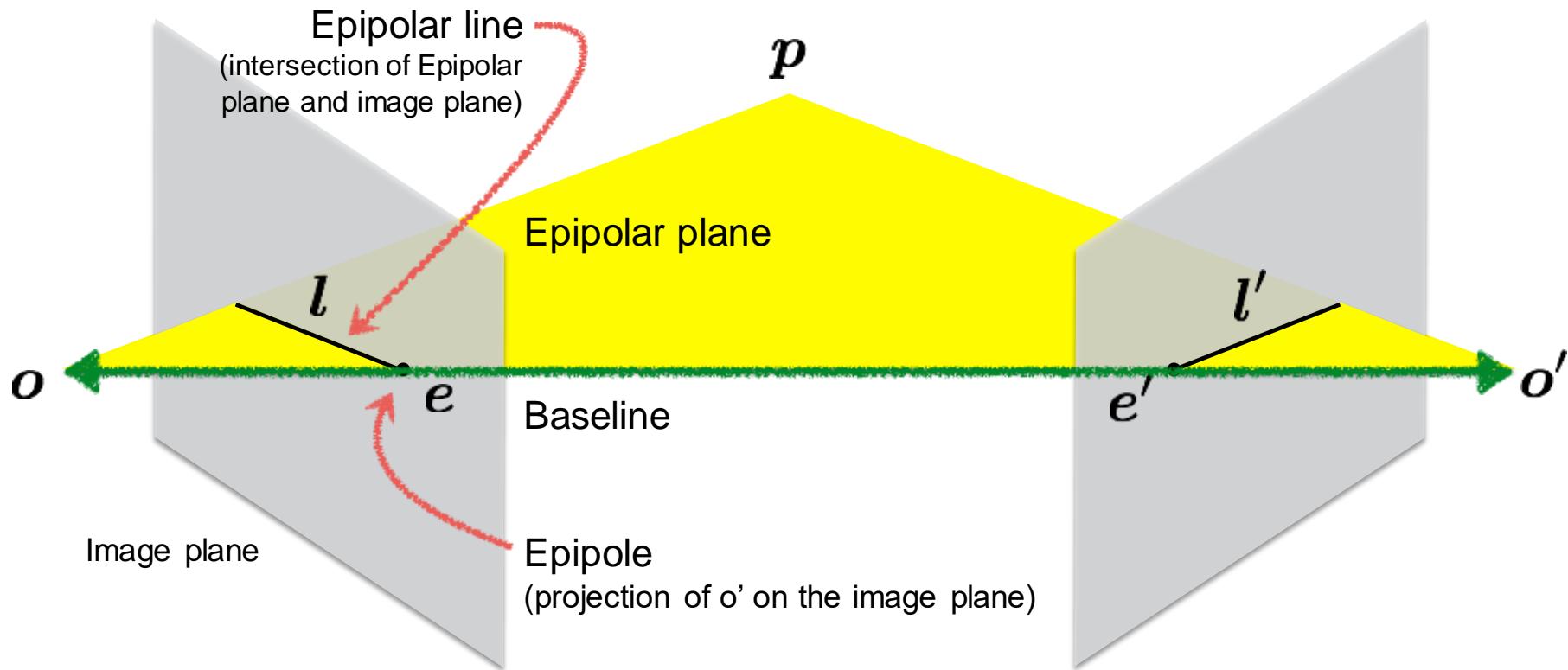
Epipolar geometry



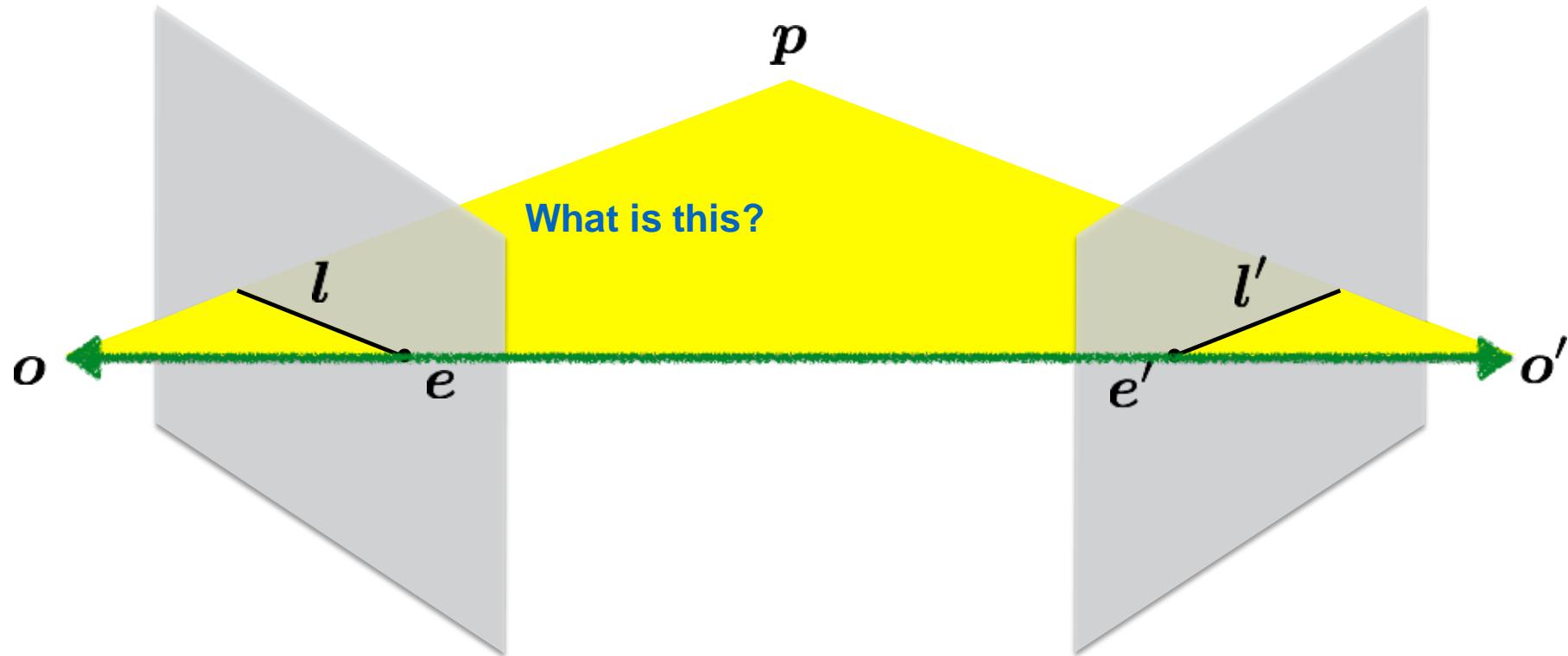
Epipolar geometry



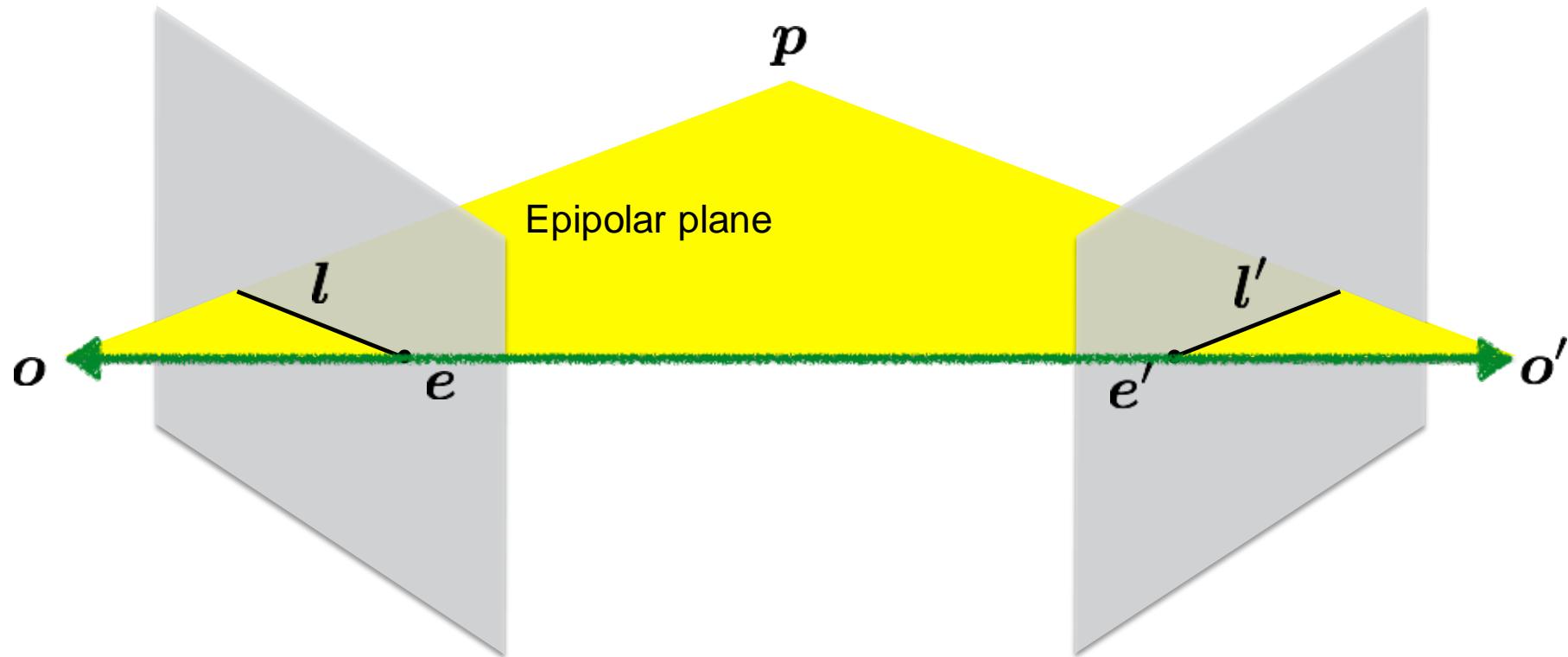
Epipolar geometry



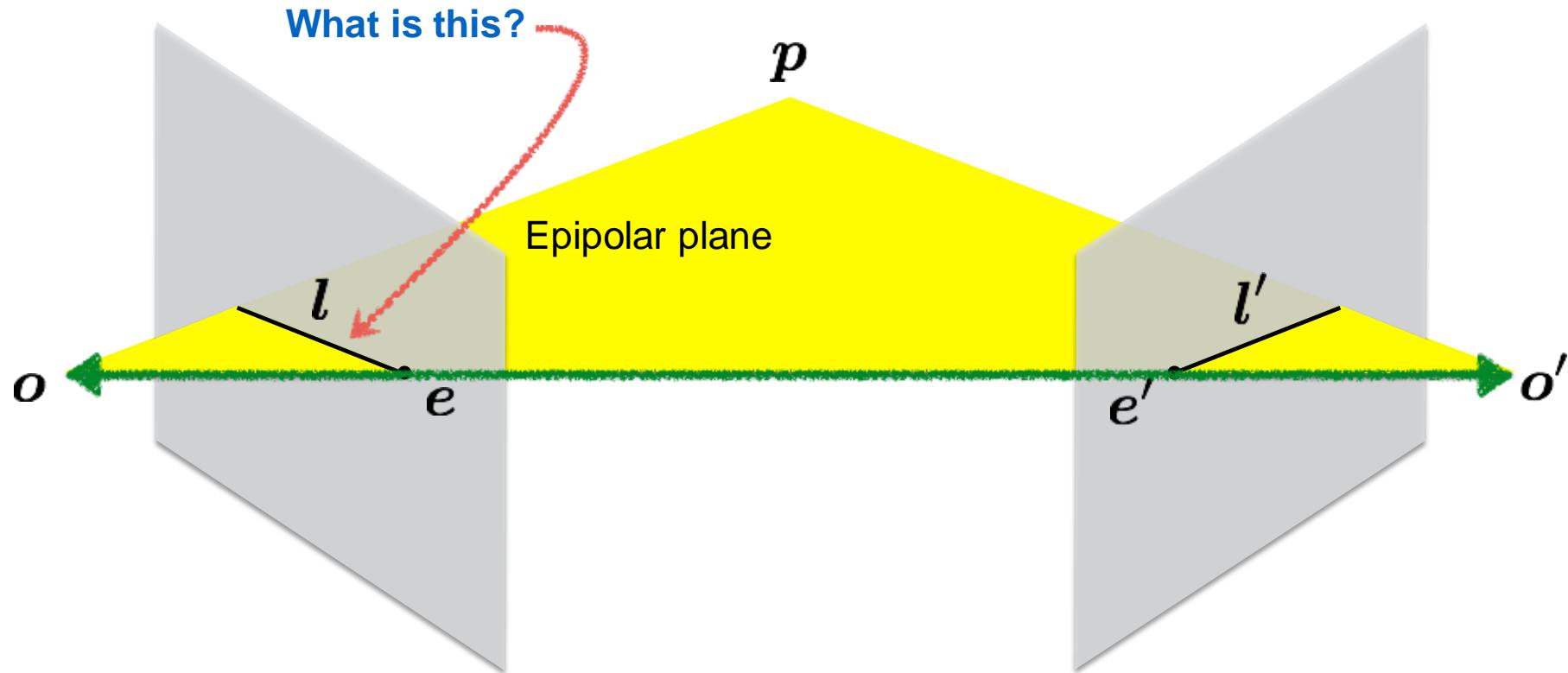
Quiz



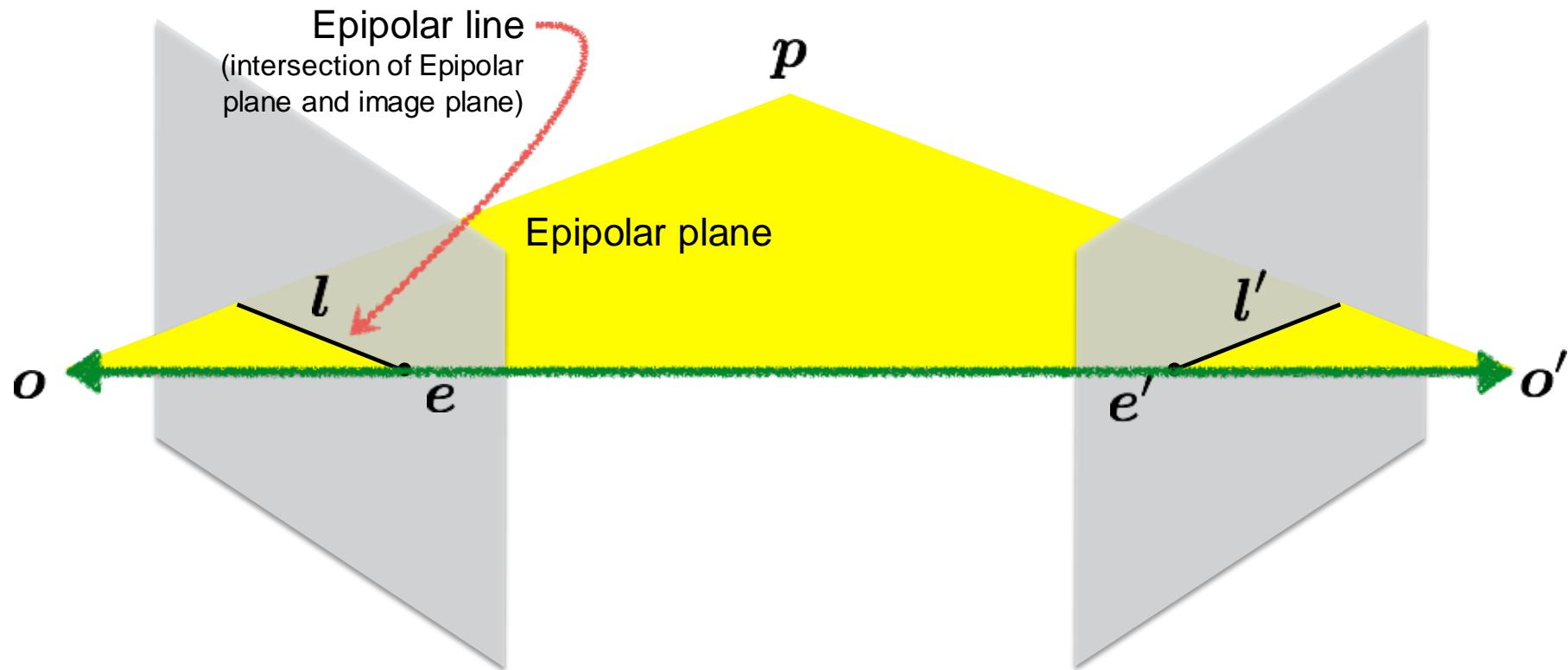
Quiz



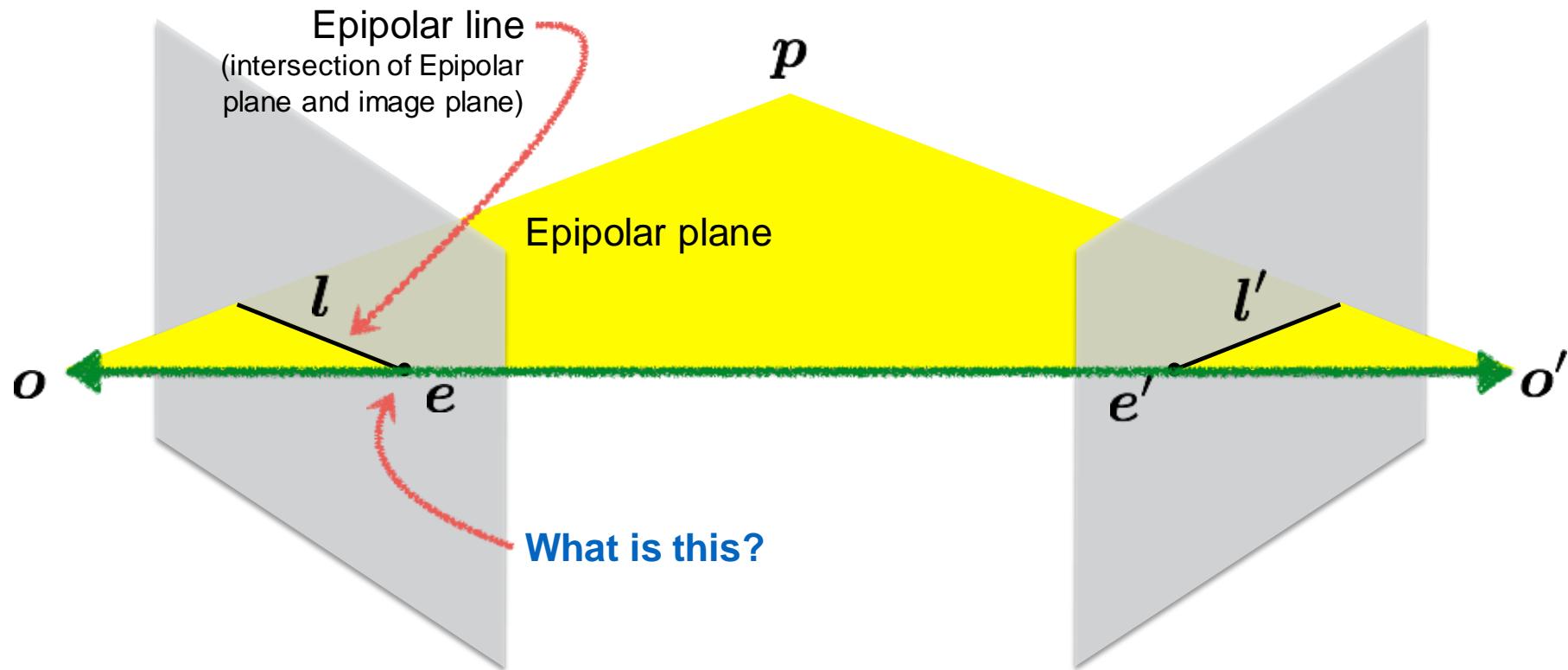
Quiz



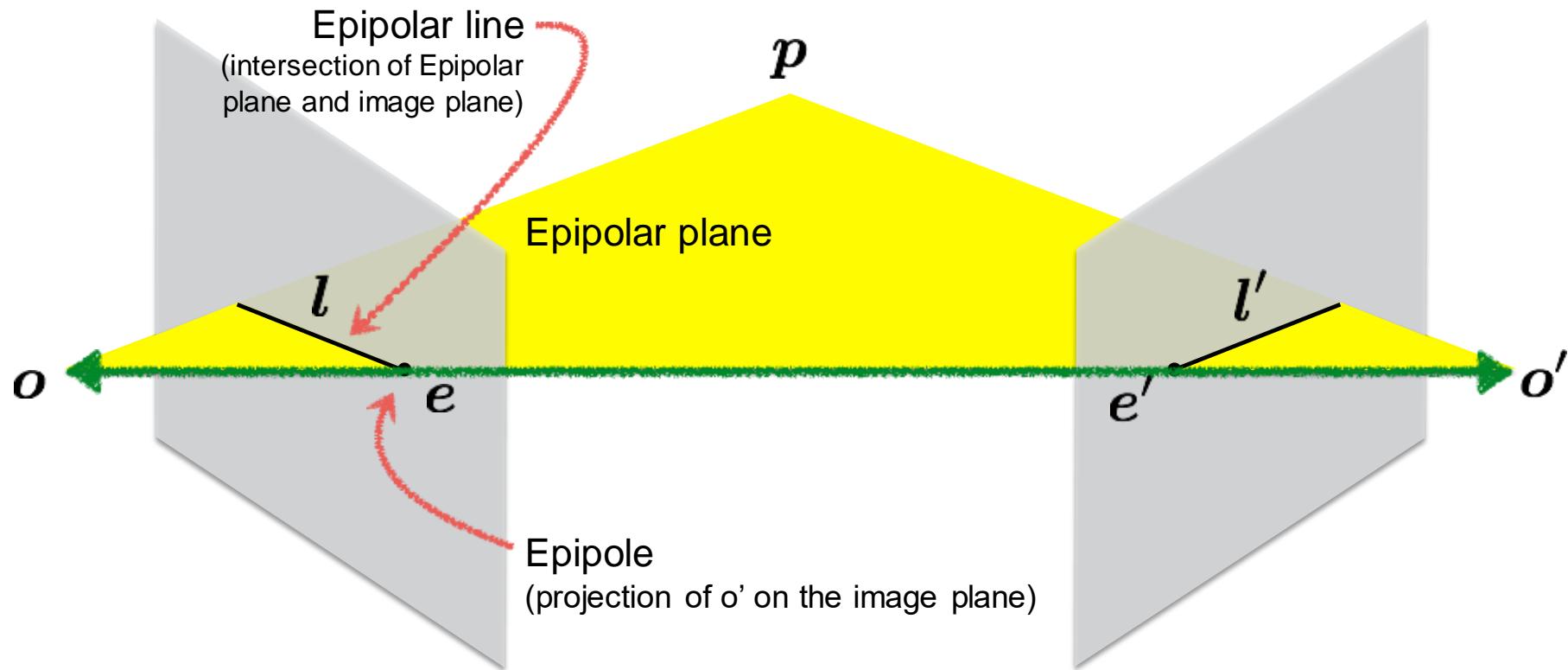
Quiz



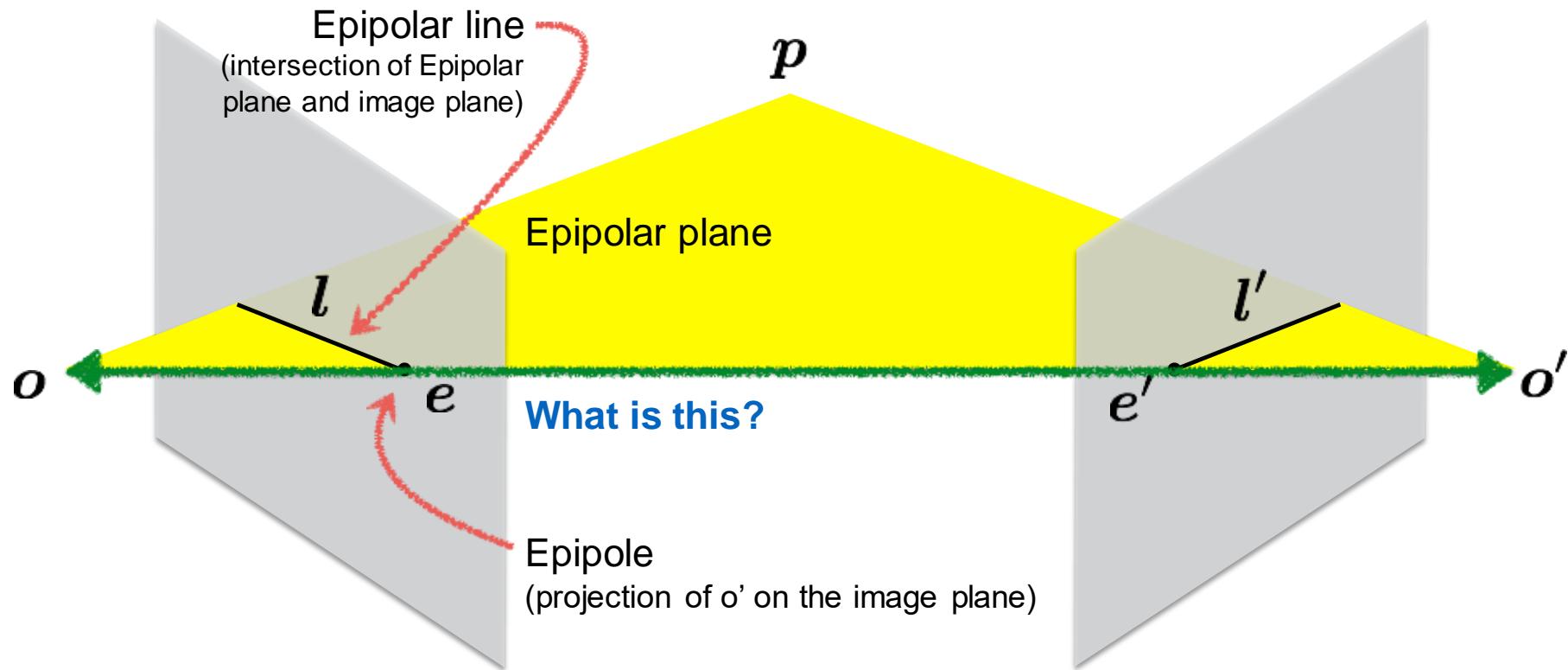
Quiz



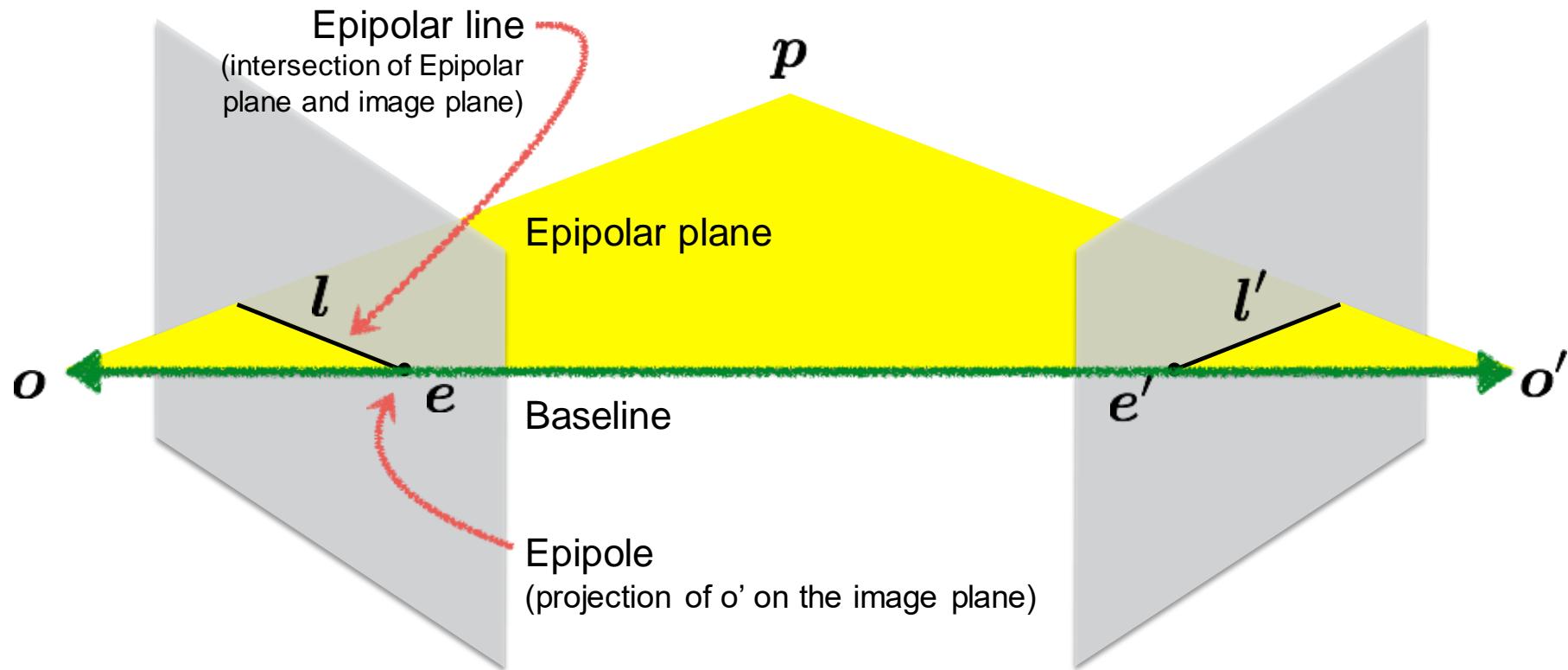
Quiz



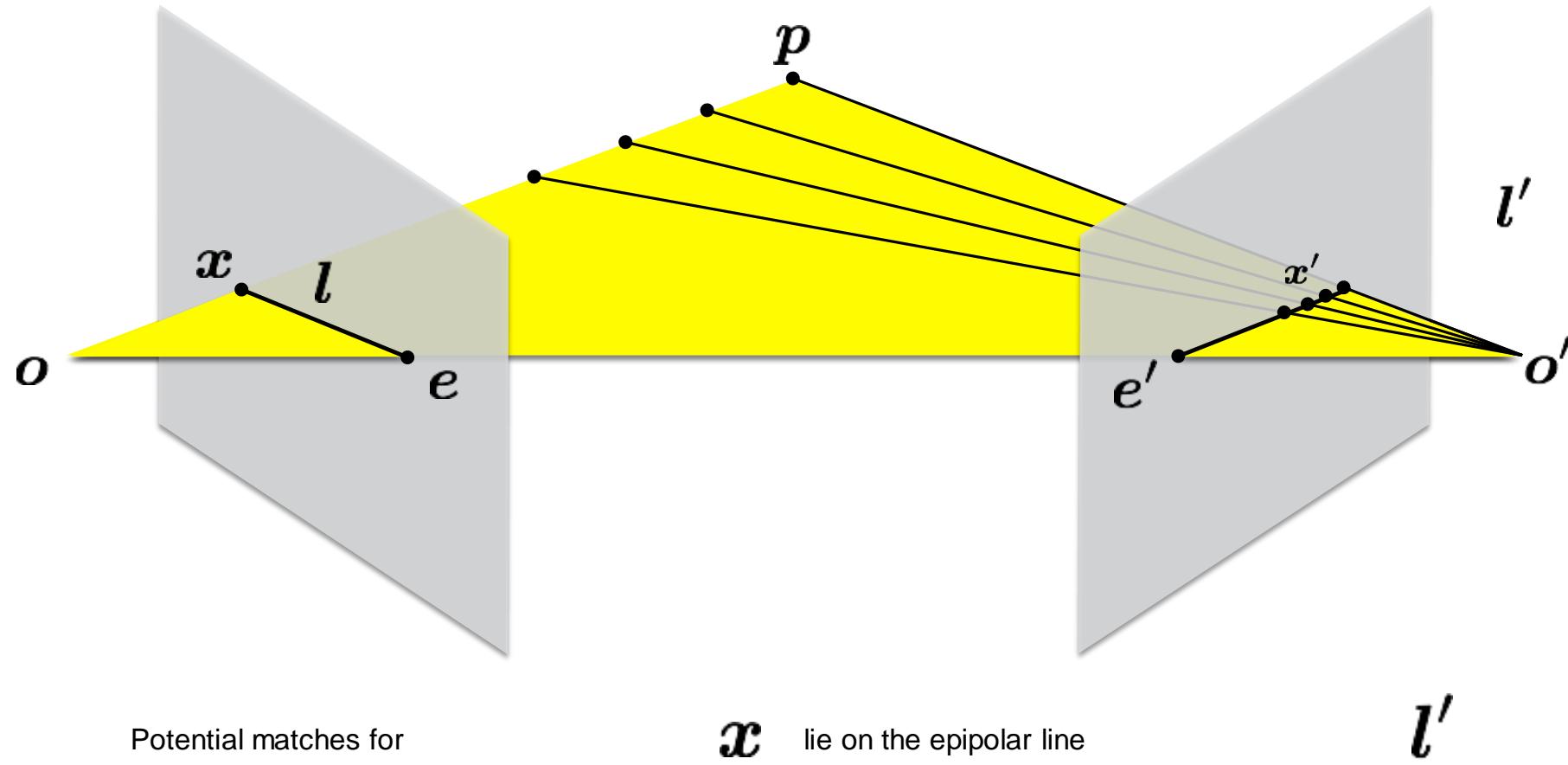
Quiz



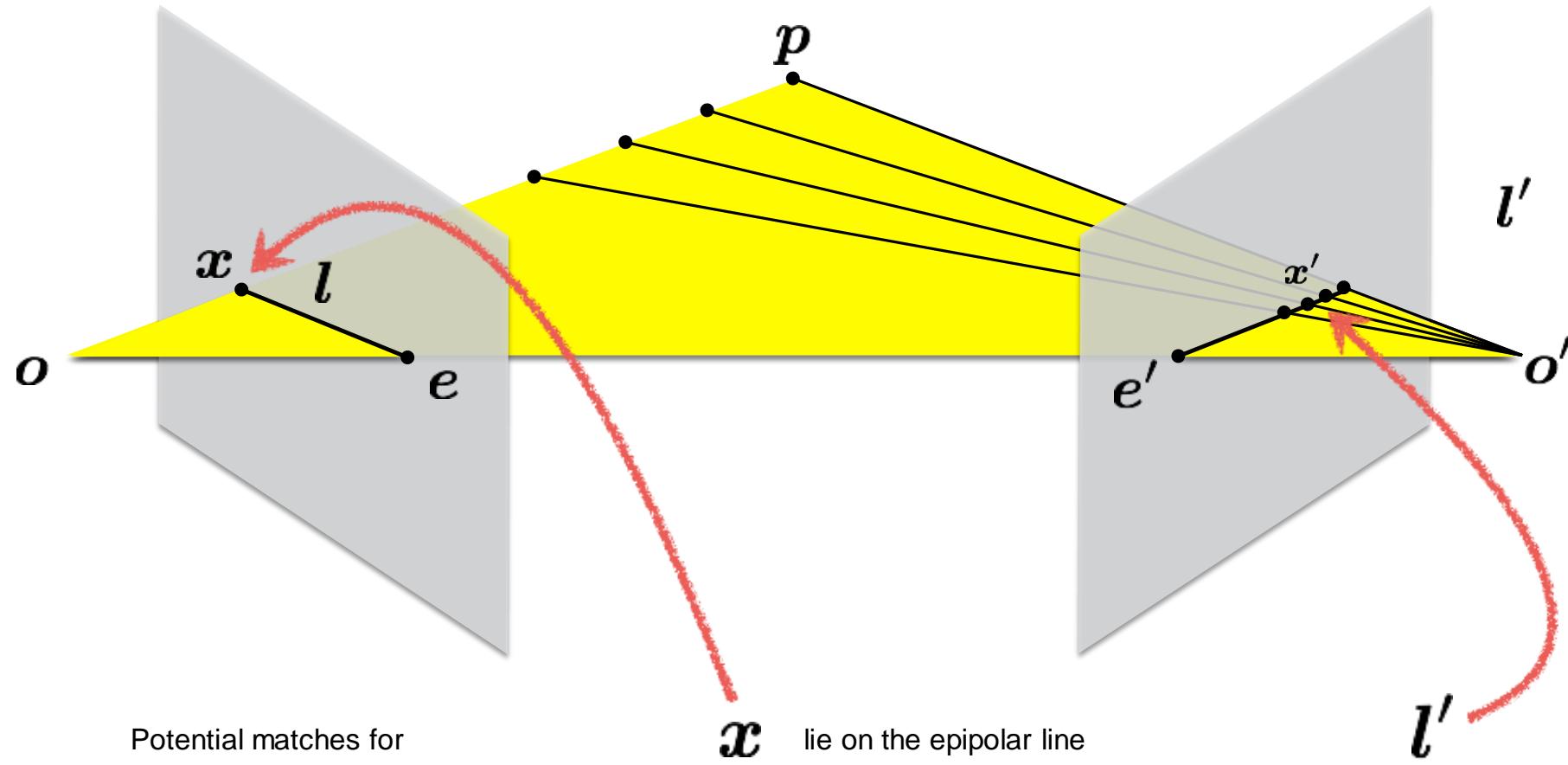
Quiz

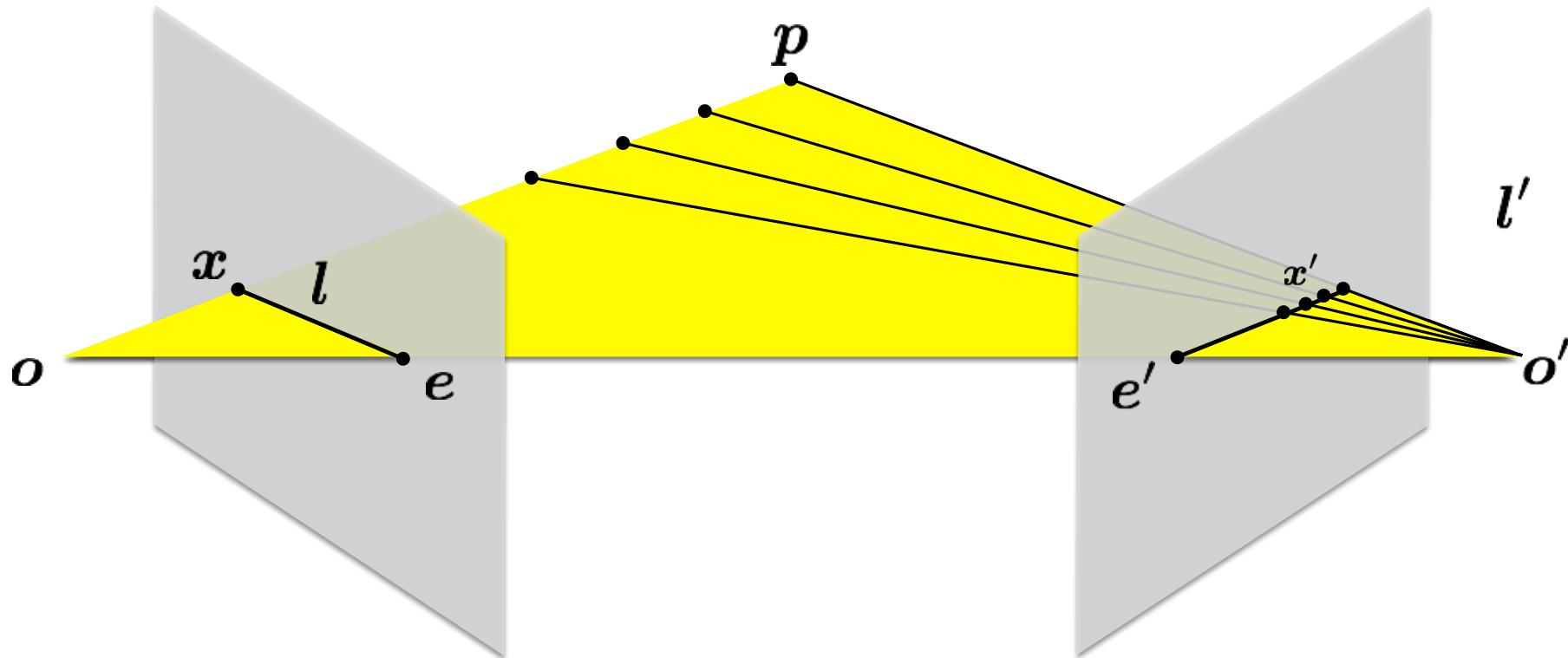


Epipolar constraint



Epipolar constraint





The point x (left image) maps to a _____ in the right image

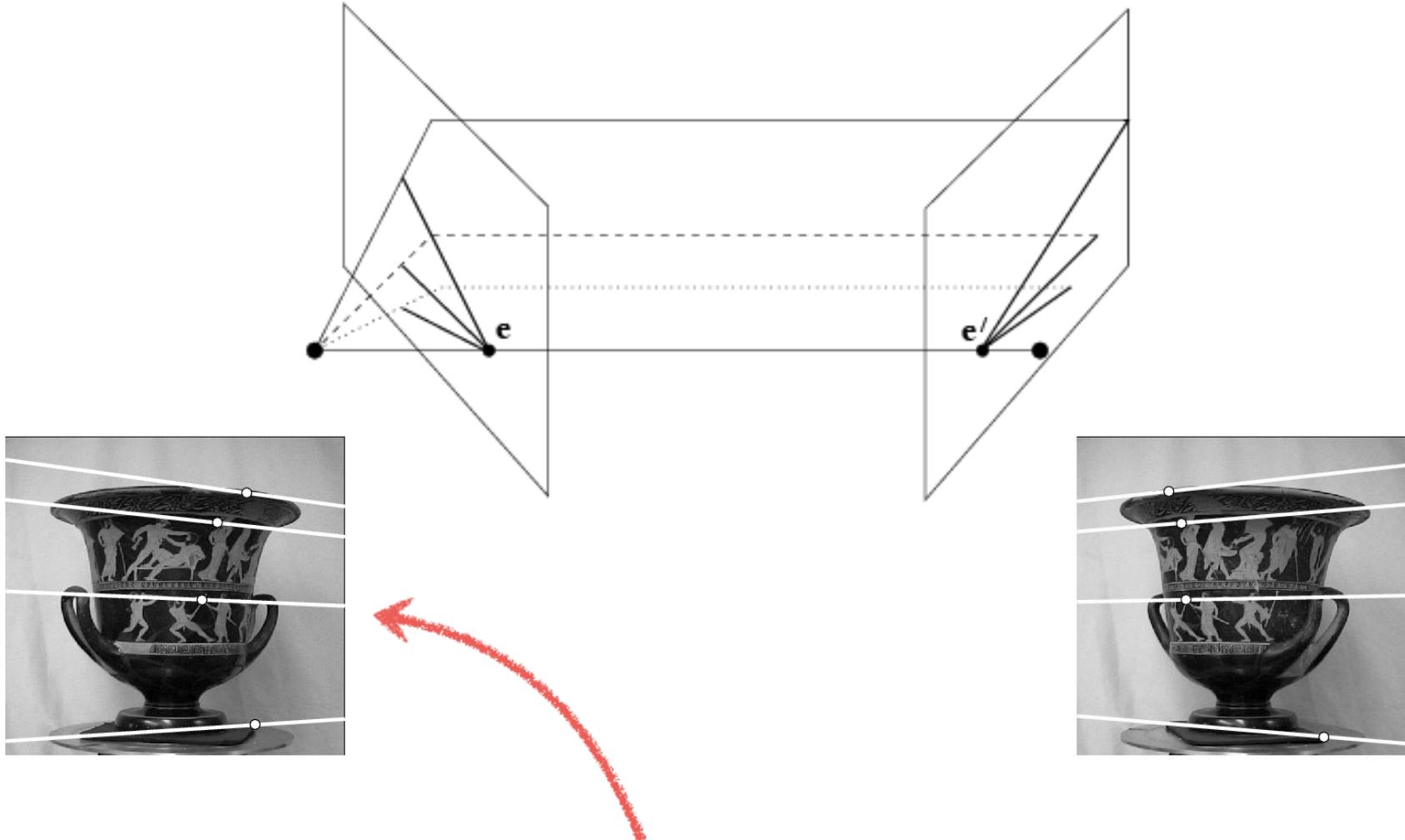
The baseline connects the _____ and _____

An epipolar line (left image) maps to a _____ in the right image

An epipole e is a projection of the _____ on the image plane

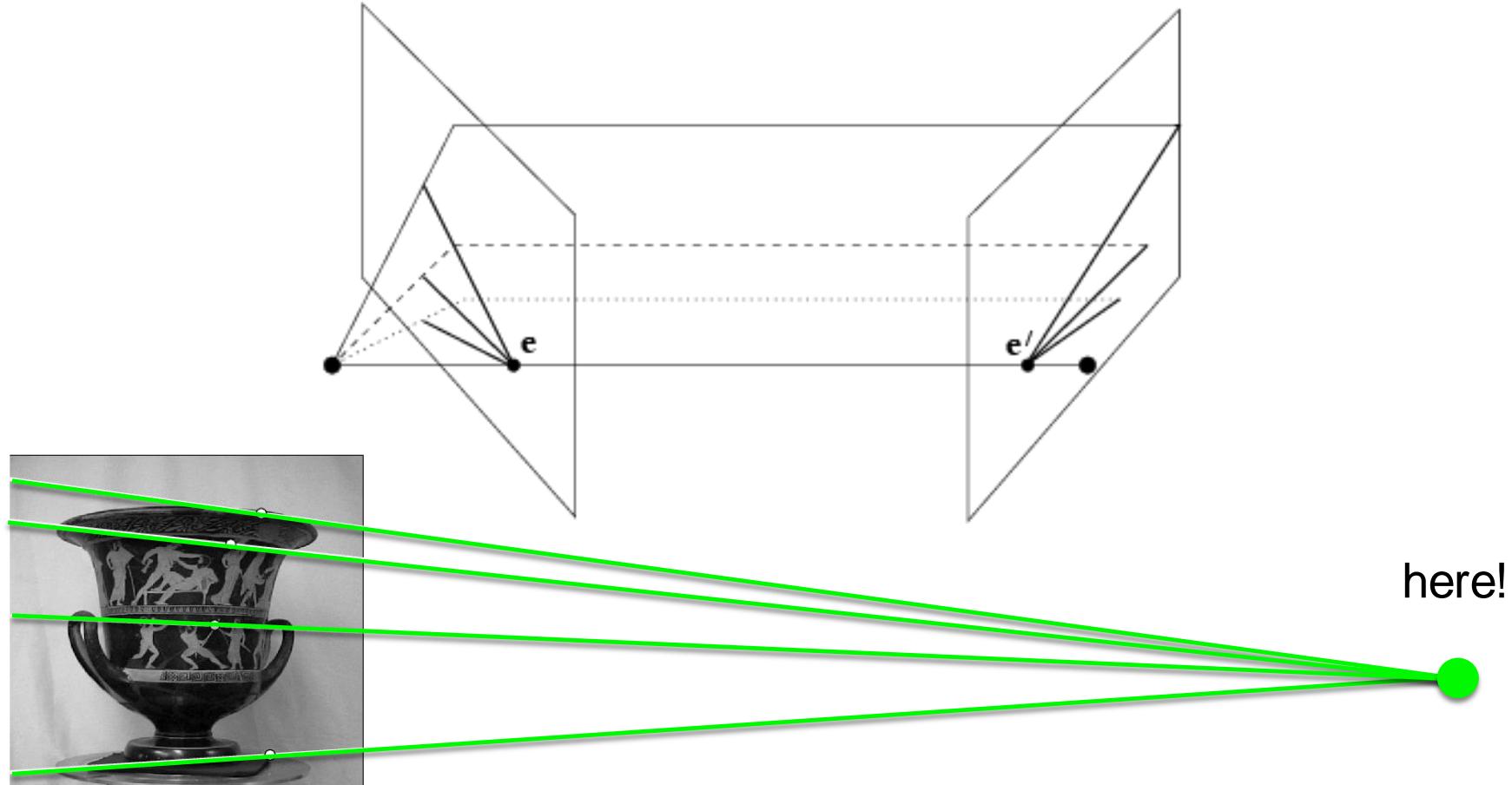
All epipolar lines in an image intersect at the _____

Converging cameras



Where is the epipole in this image?

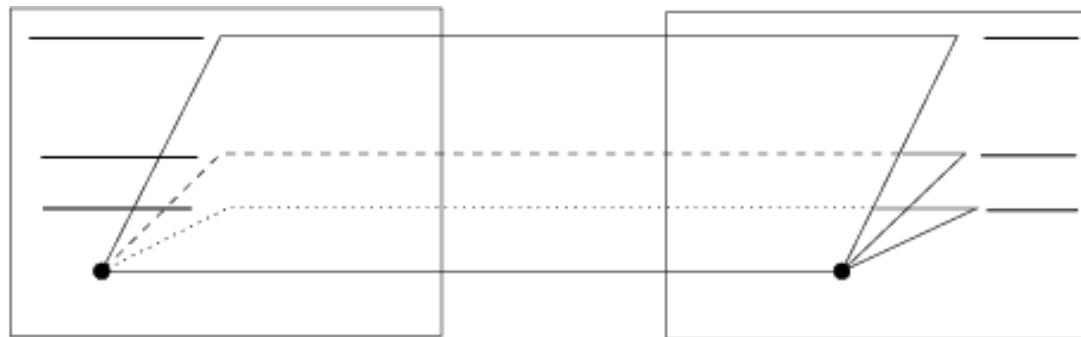
Converging cameras



Where is the epipole in this image?

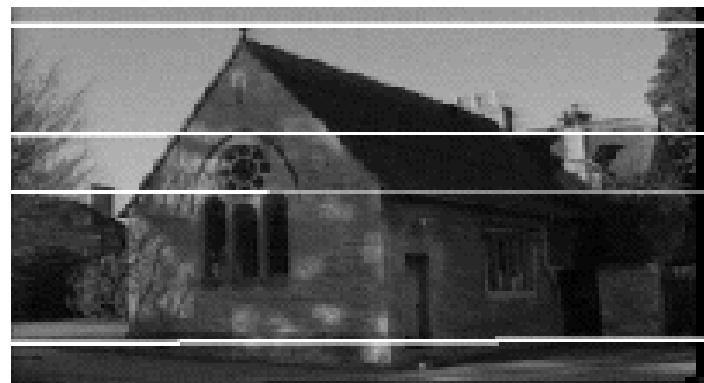
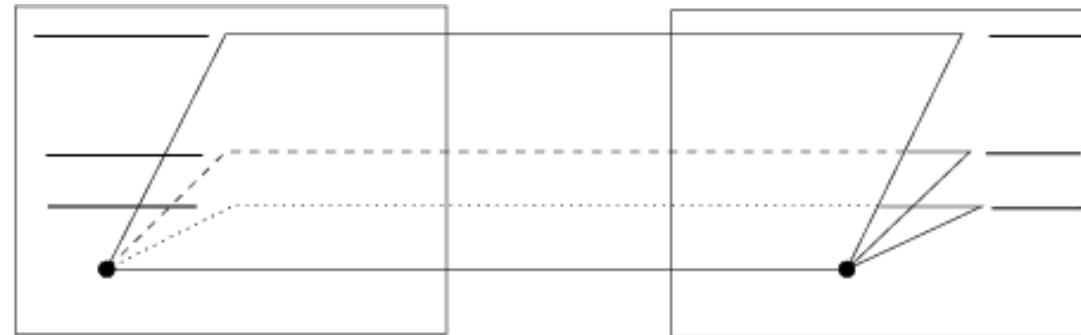
It's not always in the image

Parallel cameras



Where is the epipole?

Parallel cameras



epipole at infinity

Forward moving camera



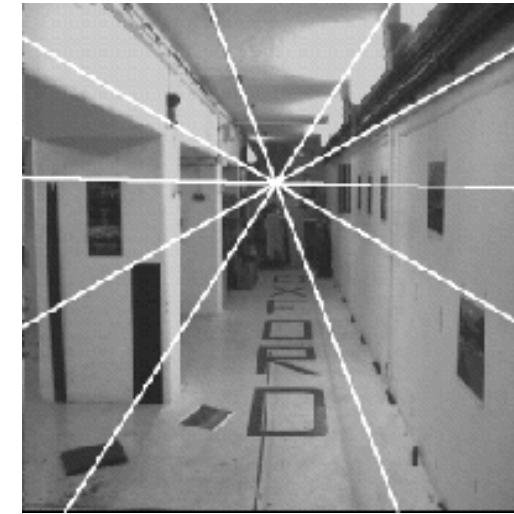
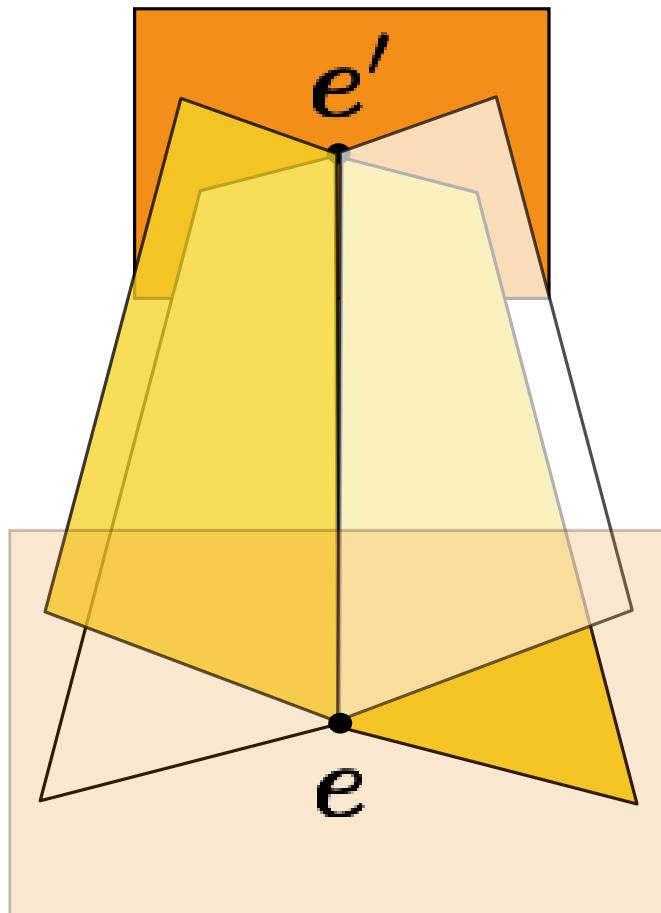
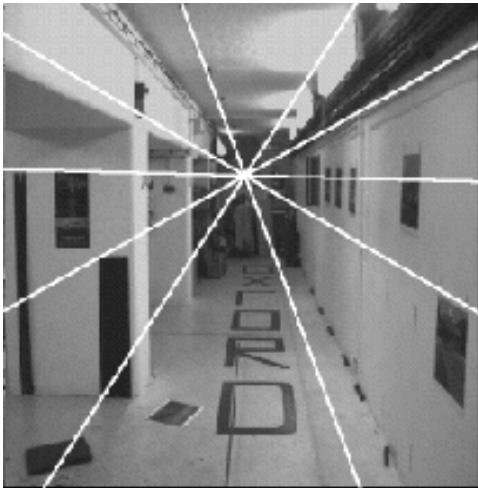
Forward moving camera



Where is the epipole?

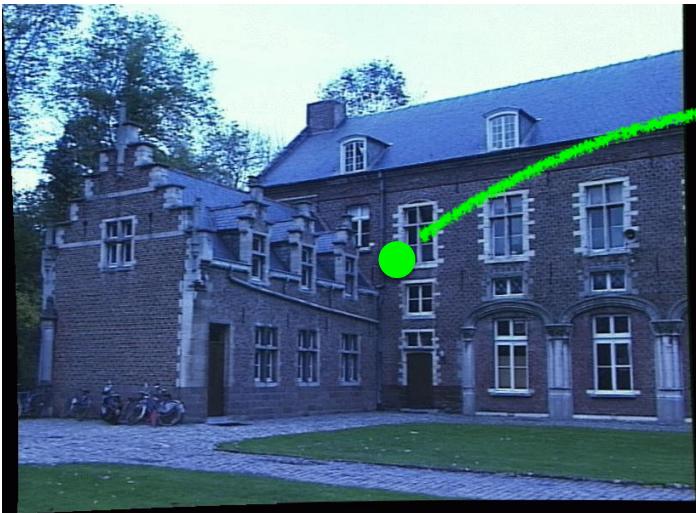
What do the epipolar lines look like?

Epipole has same coordinates in both images.
Points move along lines radiating from “Focus of expansion”

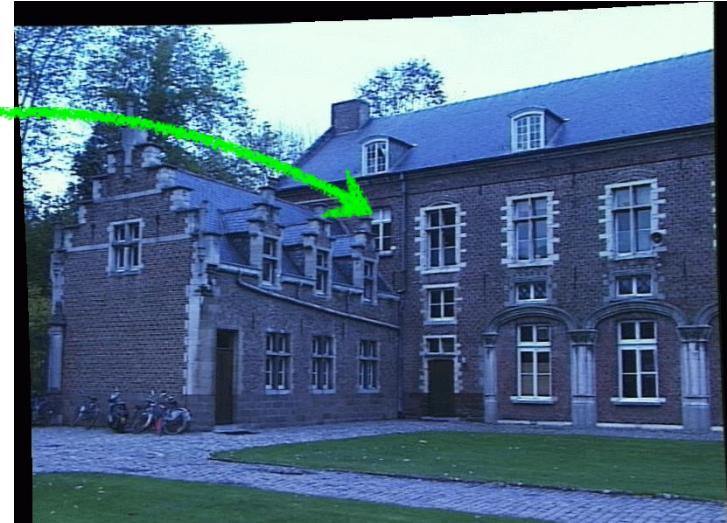


The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image



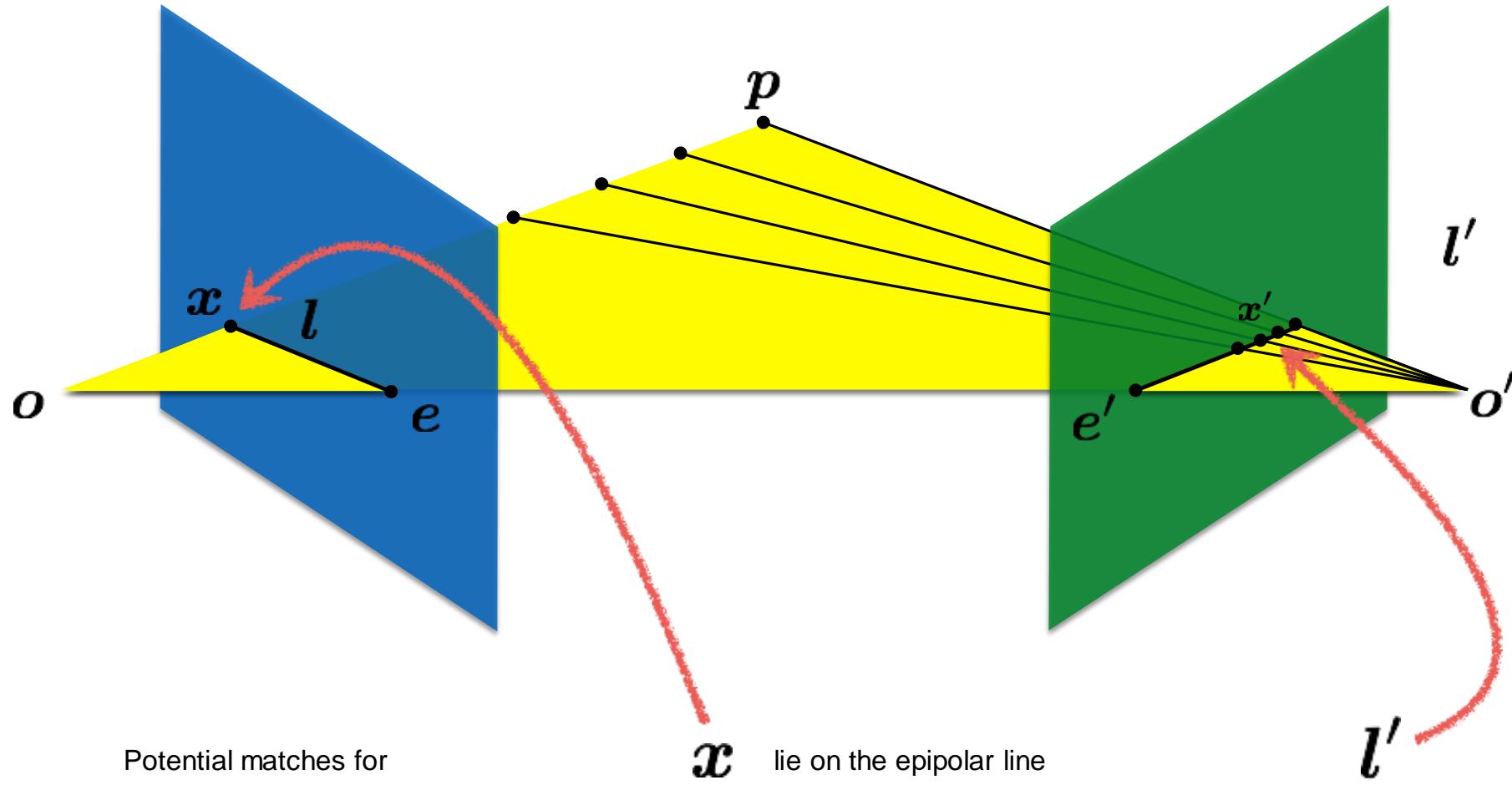
Left image



Right image

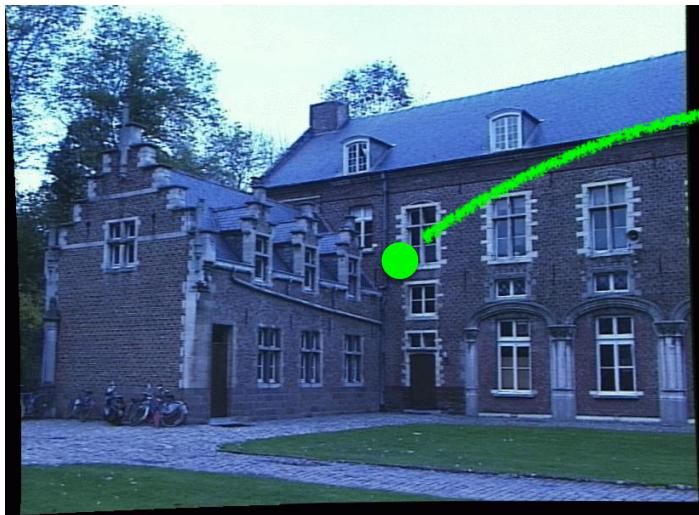
How would you do it?

Recall: Epipolar constraint

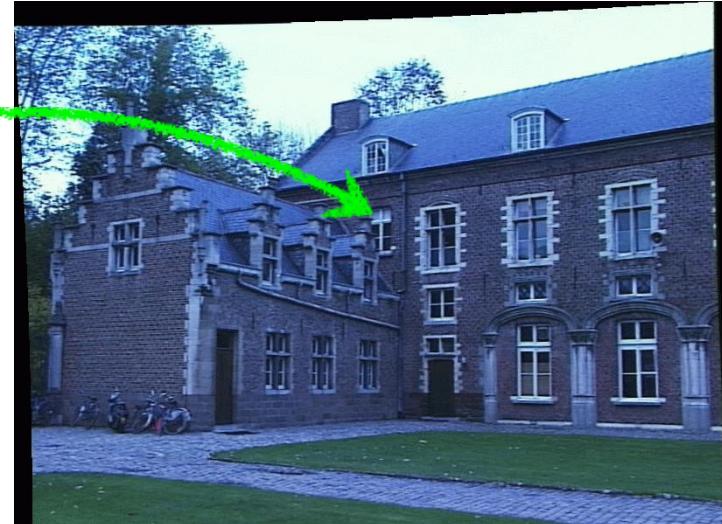


The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image



Left image



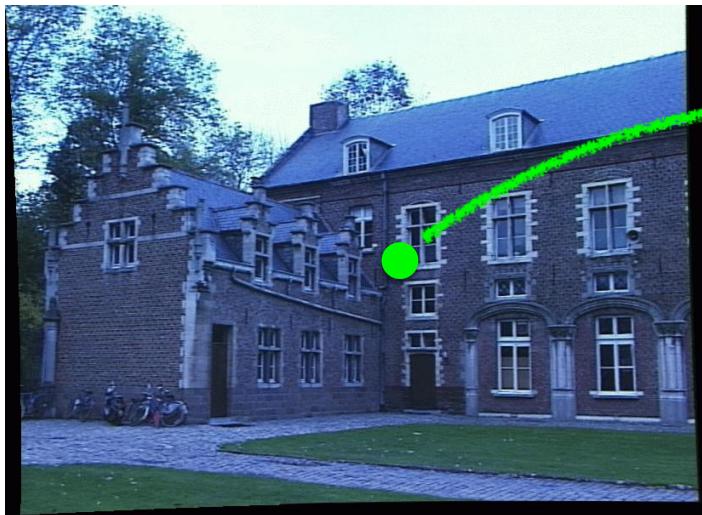
Right image

Want to avoid search over entire image

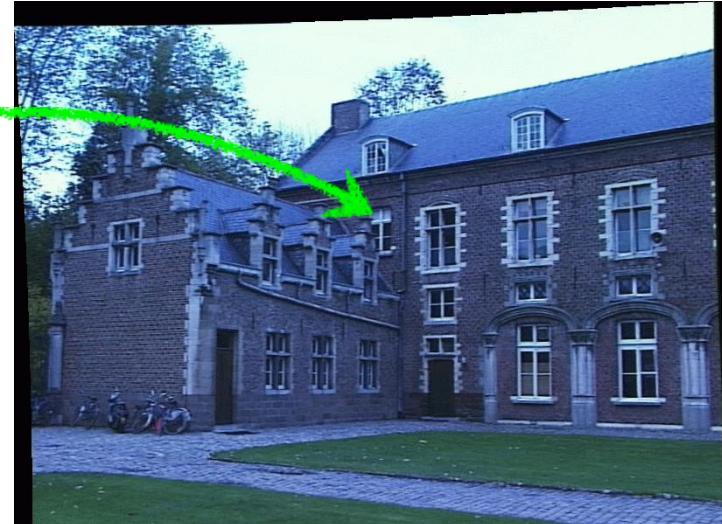
Epipolar constraint reduces search to a single line

The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image



Left image



Right image

Want to avoid search over entire image

Epipolar constraint reduces search to a single line

[How do you compute the epipolar line?](#)

The essential matrix
(next week)

References

Basic reading:

- Szeliski textbook, Sections 7.1, 7.2, 11.1.
- Hartley and Zisserman, Chapters 9, 11, 12.