

Introduction to Spectral Graph Theory

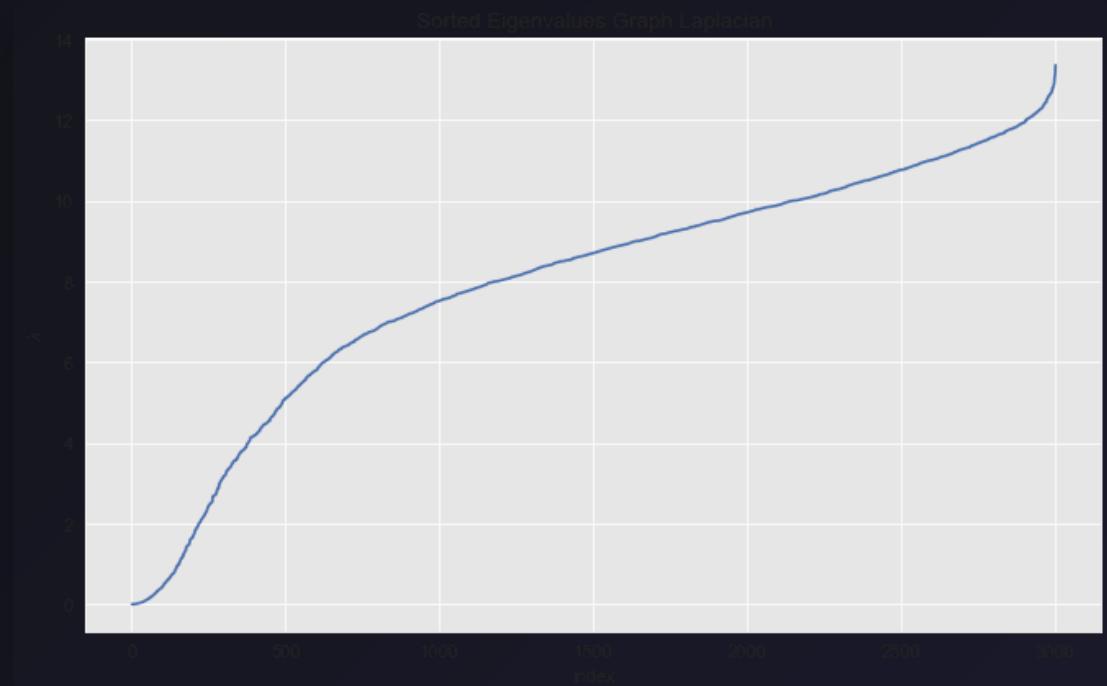
💡 What is Spectral Graph Theory?

Study of graphs through eigenvalues and eigenvectors of matrices associated with graphs

Σ Graph Laplacian & Eigenvalues

$$L = D - A$$

- ▶ D: **Degree matrix** (diagonal)
- ▶ A: **Adjacency matrix**
- ▶ $\lambda_1, \lambda_2, \dots, \lambda_n$: **Eigenvalues** of L



◼ Applications

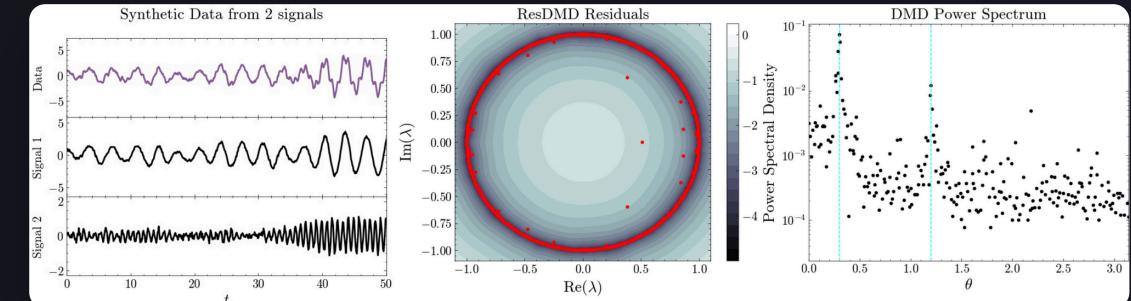
- ▶ Graph partitioning & clustering
- ▶ Network analysis
- ▶ Data mining & machine learning

The Ω -Core: Spectral Complexity

Complexity Measure

$$\Omega = \sum \lambda_i^2$$

Sum of squared eigenvalues of the graph Laplacian



Waveform

Eigenvalues = **resonant frequencies** of graph structure

Bar

Ω = total **vibrational energy** across all modes

Checkmark

Low Ω : simple dynamics • High Ω : **complex dynamics**

Mathematical Foundations

Graph Laplacian Matrix

$$L = D - A$$

D = diagonal degree matrix • A = adjacency matrix

Normalized Laplacian

$$L_{\text{norm}} = I - D^{-1/2}AD^{-1/2}$$

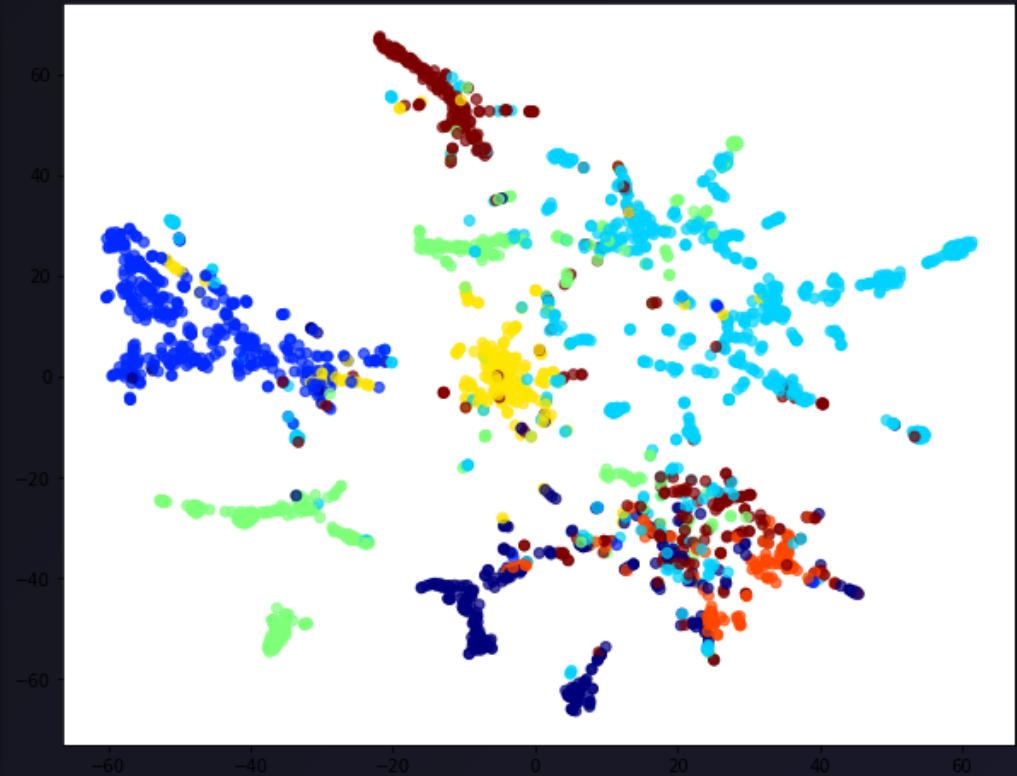
Symmetric Positive Semi-definite

All eigenvalues are **non-negative**

Eigenvalue Decomposition

$$L = \Phi \Lambda \Phi^T$$

Φ = eigenvector matrix • Λ = diagonal eigenvalue matrix



Eigenvalue Properties

① Zero Eigenvalue

$\lambda_n = 0$ with multiplicity = number of connected components

② Bounded Spectrum

$0 \leq \lambda_i \leq 2$ for normalized Laplacian

③ Trace Property

$\sum \lambda_i = 2m$ (m = number of edges)

④ Spectral Gap

λ_1 measures connectivity

Complexity Detection

Low Ω

$\Omega < 10$

Simple dynamics

High Ω

$\Omega > 100$

Complex dynamics

Complexity Interpretation

Energy Distribution

High Ω = concentrated energy in higher modes

Connectivity

Complex graphs have richer connectivity patterns

Spectral Gap

Small spectral gap + many eigenvalues = high Ω

Graph Examples

Path Graph

$\Omega \approx 8$

Linear structure

Complete Graph

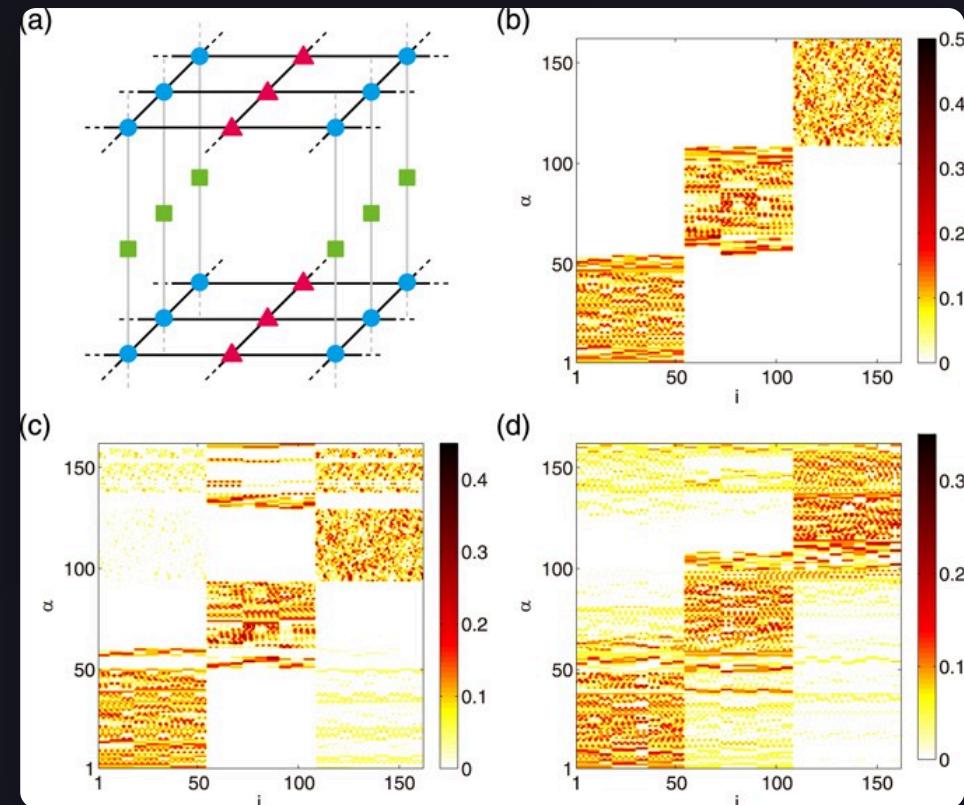
$\Omega \approx n^2$

Maximum connectivity

Scale-Free Network

$\Omega \approx 150+$

Heterogeneous structure



Applications

Network Analysis

- ▶ Complexity ranking of real-world networks
 - ▶ Detecting structural changes in evolving networks
 - ▶ Comparing network robustness and resilience

Graph Classification

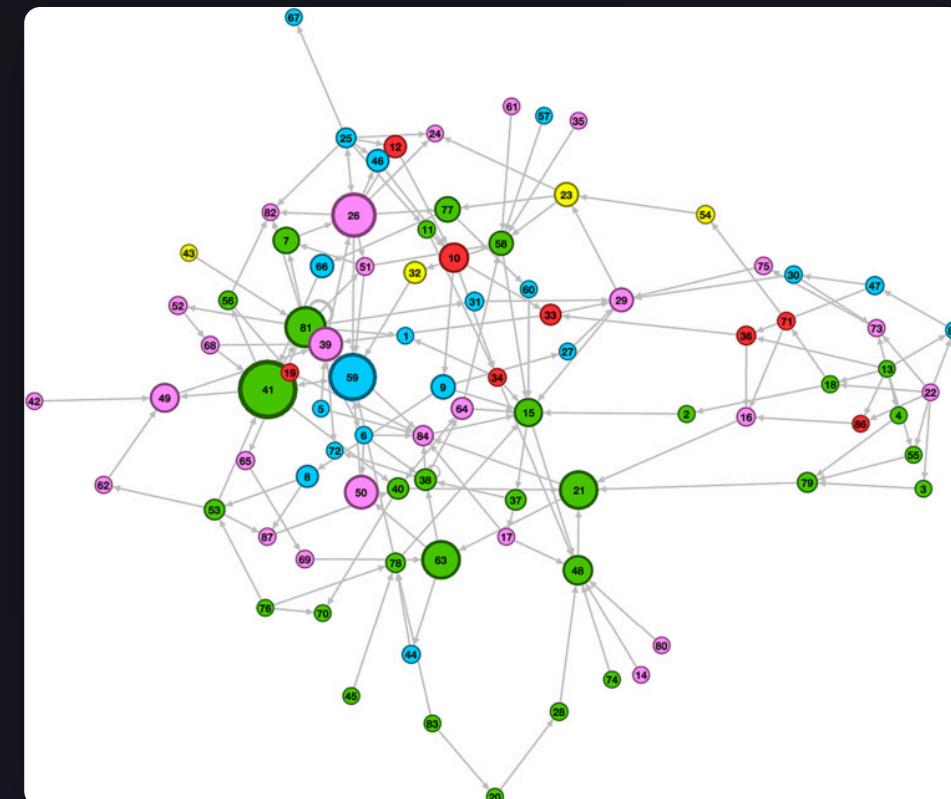
- ▶ Feature extraction for graph neural networks
 - ▶ Distinguishing **graph families** by complexity profile
 - ▶ Dimensionality reduction for graph embeddings

Pattern Recognition

- ▶ Identifying **anomalous structures** in complex data
 - ▶ Detecting **community transitions** in social networks
 - ▶ **Phase transitions** in dynamic systems

Scientific Applications

- ▶ Brain connectivity analysis
 - ▶ Molecular structure complexity quantification
 - ▶ Ecological networks stability assessment



Real-World Use Cases

Social Media Analysis

Protein Interaction Networks

Power Grid Optimization

Transportation Networks

Citation Networks

Financial Risk Assessment

Disease Spread Modeling

Quantum Information Systems

Conclusion

$$\Omega = \sum \lambda_i^2$$

Universal measure of spectral complexity for graphs

Key Takeaways

Energy Interpretation

Ω quantifies **total vibrational energy** across all eigenmodes

Complexity Indicator

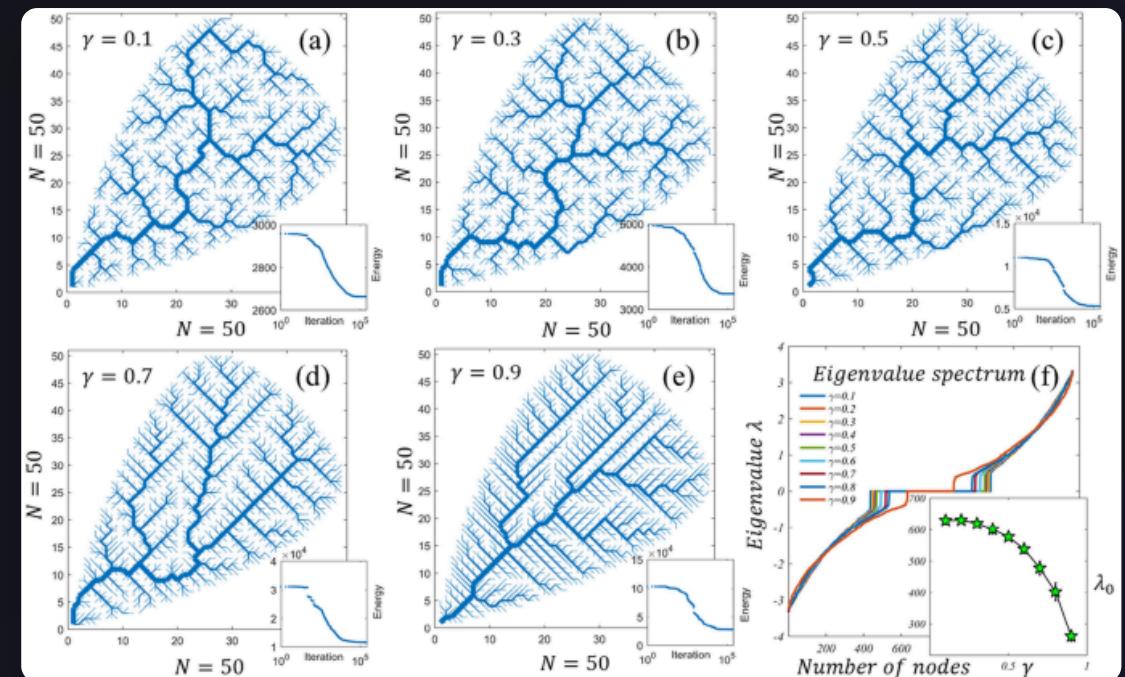
Low Ω : simple • High Ω : **complex dynamics**

Graph Comparison

Enables **objective comparison** of graph structures

Network Analysis

Reveals **intrinsic properties** of complex networks



Future Directions

Integration with **graph neural networks** for enhanced learning

Applications in **quantum information** and molecular structures

Dynamic **temporal analysis** of evolving networks

Development of **approximation algorithms** for large-scale graphs