18.06 Recitation April 14

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Volume, Determinant, Trace

Consider an $n \times n$ matrix A.

1. A region in $\mathbb{R}^n$ is transformed into another region in $\mathbb{R}^n$ under $v \mapsto Av$.
2. The determinant is the scaling factor of volumes berween the two regions.
3. The trace of A is the sum of its diagonal elements. We have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Eigenvalues

Still consider an $n \times n$ matrix A.

1. We call λ an eigenvalue of A if there exists some vector $x \in \mathbb{C}^n$, such that $Ax = \lambda x$. Also we call x is the eigenvector.
2. Geometric meaning of eigenvalues and eigenvectors
3. If A is diagonalizable, i.e. we have $Q^{-1}AQ = \Sigma$ for some $n \times n$ matrix Q, where $\Sigma = diag(x_1, x_2, \ldots, x_n)$. What are the eigenvalues of A?
4. A number λ is an eigenvalue of A if and only if $det(A - \lambda \mathrm{Id}_{n\times n}) = 0$
5. Suppose A is diagonal, what is the relationship between the determinant/trace of A and the eigenvalues of A? What if A is not diagonal but diagnalizable?

Problems

1. What is the area of the triangle in $\mathbb{R}^2$ whose vertices are (1,1), (2,4), (4,2)?
2. Compute the eigenvalues and corresponding eigenvectors of the following matrices.

   (a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

   (b) $B = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, where $a, b, c \neq 0$

   (c) $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

3. If $AB = BA$, and v is an eigenvector of B. Show that Av is also an eigenvector of B. What about $A^n v$?

4. (a) Is the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalizable?

   (b) What is the relationship between the eigenvalues of an $n \times n$ matrix B and eigenvalues of B^2? (hiint: one side is easy)

   (c) If B is diagonalizable, is B^2 also diagonalizable?