18.06 Recitation May 12

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Linear Transformations

1. A linear transformation T is a linear map between two vector spaces $T : V \to W$. We say T is linear if $T(av + bw) = aT(v) + bT(w)$ for any $v, w \in V$ and $a, b \in \mathbb{R}$.

2. We can use a matrix to denote the linear transofmration. Choose a basis $v = (v_1, \ldots, v_n)^T$ of V and a basis $w = (w_1, \ldots, w_m)^T$ of W, then we will have $T(v_i) = a_{i1}w_1 + \cdots + a_{i_m}w_m$. The matrix $A = (a_{ij})$ satisfies $Tv = Aw$.

3. When $V = W$, it is called an endormorphism. In this case $Tv = Av$, and A is an n $\times$ n matrix.

4. If we change the basis, say $v' = (v'_1 \ldots, v'_n)$ is another basis for V. We will have $Tv' = A'v'$. if we denote $v' = Qv$ where Q is an $n \times n$ invertible matrix. Then $A = Q^{-1}A'Q$. So the matrix of T under different base are similar. (Prove it!)

Summary on similar matrices

Consider an $n \times n$ matrix A.

1. B is similar to A if $B = Q^{-1}AQ$ for some $n \times n$ matrix Q.

2. Similar matrices have the same eigenvalues, determinant, trace.

3. (Jordan form) Any $n \times n$ matrix is similar to a matrix called its Jordan form.

4. Similar matrices stand for the same linear transformation under different base.

Problems

1. Why does every linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ takes squares to paralleograms? Rectangles also go to paralleograms (squashed if T is not invertible)?

2. If we change the basis, say $v' = (v'_1 \ldots, v'_n)$ is another basis for V. We will have $Tv' = A'v'$. if we denote $v' = Qv$ where Q is an $n \times n$ invertible matrix. Show that $A = Q^{-1}A'Q$.

3. (a) On the xy-plane, let S be reflection across the 45-degree line, and T be reflection across the y axis. Show that $ST \neq TS$.

   (b) Let T' be rotating 180 degrees (centered at the origin). Does T' commute with S?

(c) What about in the 3-dimensional case? i.e. S is relection across the $xy-$plane, and T be mapping $x \to -x$. Does $ST = TS$?

4. True or false? Explain why or why not.

   (a) If we know $T(v)$ for n different nonzero vectors $v \in \mathbb{R}^n$, then we know $T(v)$ for any v.

   (b) A cannot be similar to $-A$ unless $A = 0$.

   (c) If A is similar to A^{-1}, then all the eigenvalues of A must be 1 or -1.

   (d) If A and B are similar, then $A - B$ has trace 0.

5. For a positive definite symmetric matrix A. Show that we can always find a symetric matrix B such that $B^2 = BB^T = A$.