Recitation 4/21

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Eigenvalues, Continued

- Eigenvalues are values λ such that shifted matrix $A - \lambda I$ have nonempty nullspace.

$$\exists x \neq 0 \text{ such that } (A - \lambda I)x = 0$$

- Determinant of $A - \lambda I$ is a degree n polynomial of λ. n solutions of the polynomial are exactly the eigenvalues. (Why?)
- We have exactly n eigenvalues, counting multiplicities (i.e. for 3 by 3 matrix we can have three eigenvalues, 1, 1, 2)

Diagonalization

- Let $A \in \mathbb{R}^{n \times n}$ have eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $x_1, \ldots, x_n$. n equations $Ax_i = \lambda x_i$ can be simultaneously represented as,

$$AX = X\Lambda$$

where X is a matrix with i^{th} column x_i, Λ is a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$.

- **If X is invertible**(X has linearly independent eigenvectors), We can now express A as

$$A = X\Lambda X^{-1}$$

and this is called a **Diagonalization or Eigendecomposition** of A.

- Why Diagonalization is so powerful and important?

Problems

1. Compute eigenvalues of $A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 3 & 1 \\ 2 & 4 & -1 \end{pmatrix}$ using the polynomial $\det(A - \lambda I)$.

2. (a) Let $f_0, f_1, \ldots$ be Fibonacci sequence with $f_0, f_1 = 0, 1$. Find 2 by 2 matrix A such that $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = A\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$. Then, express $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$ in terms of A.

(b) Find eigenvectors, eigenvalues, and eigendecomposition of A.

(c) (Challenging) Express the eigendecomposition of A^{100}. With a small assumption(Regard a very small number as 0), prove that the ratio f_{101}/f_{100} is same as the largest eigenvalue.

(d) Find formula for f_n.

3. Think about another sequence $g_0, g_1, \ldots$ with relationship $g_{i+1} = 2g_i + g_{i-1}$ and $g_0, g_1 = 0, 1$. Find formula for g_n.

4. True or false. Prove or give counterexample.

(a) Diagonalizable matrices are invertible.

(b) Invertible matrices are diagonalizable.

(c) Non-diagonalizable matrices can be invertible.

(d) Non-invertible matirice can be diagonalizable.

(e) If A is diagonalizable then A^5 is diagonalizable.

(f) Squared singular values of A are eigenvalues of A^2

(g) Squared singular values of A are eigenvalues of A^TA

(h) Squared singular values of A are eigenvalues of AA^T

ANSWERS

1. $\det(A-\lambda I) = (\lambda-5)(\lambda-1)(\lambda+3)$ so eigenvalues are 5, 1, -3.

2.(a) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = A \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \cdots = A^n \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(b) $\det(A-\lambda I) = \lambda^2 - \lambda - 1$ so $\lambda = \frac{1\pm\sqrt{5}}{2}$. To find eigenvectors, $A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 \end{pmatrix} = \frac{1+\sqrt{5}}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ we have $x_1 = \frac{1+\sqrt{5}}{2} x_2$, and from the second eigenvalue similarly we obtain $x_1 = \frac{1-\sqrt{5}}{2} x_2$. So two eigenvectors are $v_1, v_2 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} x \\ x \end{pmatrix}, \begin{pmatrix} \frac{1-\sqrt{5}}{2} x \\ x \end{pmatrix}$. We can take $x = 1$ for simplicity. Eigendecomposition becomes,

$A = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1}$

(c) From Eigendecomposition, $A^{100} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}^{100} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1}$ and assuming that $\left(\frac{1-\sqrt{5}}{2}\right)^{100} \sim 0$(power of number less than 1 approaches 0), we can say

$$A^{100} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{100} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1}$$

and since $\begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix}$ we finally deduce

$$\begin{pmatrix} f_{101} \\ f_{100} \end{pmatrix} = A^{100} \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{100} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{101} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{100} \end{pmatrix}$$

and the ratio between two numbers is $\frac{1+\sqrt{5}}{2}$.

(d) Generalizing problem (c) but without assumption,

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = A^n \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and computing the matrix multiplication we have

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right)$$

3. Set up another A, in this case it is $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$. The eigendecomposition is given as,

$$A = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & -1+\sqrt{2} \\ -1 & 1+\sqrt{2} \end{pmatrix}$$

and we can deduce from

$$\begin{pmatrix} g_{n+1} \\ g_n \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (1+\sqrt{2})^n & 0 \\ 0 & (1-\sqrt{2})^n \end{pmatrix} \begin{pmatrix} 1 & -1+\sqrt{2} \\ -1 & 1+\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

that $g_n = \frac{1}{2\sqrt{2}}((1+\sqrt{2})^n - (1-\sqrt{2})^n)$

4. (a) Not if they have zero eigenvalues. For example zero matrix

(b) False. They can have all nonzero eigenvalues but the eigenvectors can be linearly independent. For example, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ has eigenvalues 1, 1 and they share same eigenvectors so they form a linearly dependent eigenvectors thus not diagonalizable.

(c) True. Exactly same example as above.

(d) True. the ones with zero eigenvalues.

(e) True. $A = X\Lambda X^{-1}$ implies $A^5 = X\Lambda^5 X^{-1}$, a diagonalization of A^5.

(f) False. $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$. (g) True. $A = U\Sigma V^T$ implies $A^T A = V\Sigma^2 V^T$ which is a eigendecomposition of $A^T A$. (h) True. $A = U\Sigma V^T$ implies $AA^T = U\Sigma^2 U^T$ which is a eigendecomposition of $A^T A$.