18.06 Recitation April 21

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Eigenvalues

Consider an $n \times n$ matrix A.

1. We call λ an eigenvalue of A if there exists some vector $x \in \mathbb{C}^n$, such that $Ax = \lambda x$. Also we call x is the eigenvector.

2. A number λ is an eigenvalue of A if and only if $det(A - \lambda \mathrm{Id}_{n\times n}) = 0$

3. We call A diagonalizable, if we have $Q^{-1}AQ = \Sigma$ for some $n \times n$ matrix Q, where $\Sigma = diag(x_1, x_2, \ldots, x_n)$.

4. The determinant is the product of all the eigenvalues, the trace is the sum of all the eigenvalues.

Differential Equations

1. For a sequence of linear differential equatioins, we can express them using a matrix equation.

$$\frac{d}{dt}\mathbf{u}(t) = A(t)\mathbf{u}(t)$$

   where $\mathbf{u}(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ and $A(t) = (A_{ij}(t))$.

2. We have the exponential function

$$e^{At} = \mathrm{Id} + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \ldots$$

   Take the derivative of e^{At}, we have

$$\frac{d}{dt}e^{At} = A + A^2t + \cdots + \frac{1}{(n-1)!}A^nt^{n-1} + \cdots = Ae^{At}$$

3. Pay attention that $e^{A+B} \neq e^Ae^B$ in general. But if $AB = BA$, then this is true.

Problems

1. Fibonacci sequence. Let $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2} (n \geq 2)$.

   (a) Find a matrix A such that $\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$

   (b) Compute the eigenvalues and eigenvectors of A.

   (c) Find a formula for F_n.

2. Let A be an $n \times n$ diagonalizable matrix. Suppose all the eigenvalues λ_i satisfy $|\lambda_i| < 1$. Show that $A^n \to 0$ when $n \to \infty$.

3. Compute e^{At} for the following A.

   (a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

   (d) $A = Q^{-1} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} Q$, where $\det Q \neq 0$

4. (a) Suppose $A^T = -A$, show that e^{At} is an orthogonal matrix for all t. Deduce that any solution to
   $$\frac{d}{dt}\mathbf{u}(t) = e^{At}\mathbf{u}(t)$$
   satisfies $||\mathbf{u}(t)|| = ||\mathbf{u}(0)||$.

   (b) (challenging) If $\text{trace}(A) = 0$, show that $\det(e^A) = 1$.