1. Markov matrices

Definition. We say a matrix or vector is *positive* if its entries are positive.

Theorem. (Perron–Frobenius) If A is a *positive* square matrix, then it has an eigenvalue $\lambda_{\max}$ with the following properties:

(i) $\lambda_{\max}$ is a positive real number.

(ii) The algebraic multiplicity of $\lambda_{\max}$ is 1, and all other eigenvalues have absolute value $< \lambda_{\max}$.

(iii) There is a *positive* vector x with eigenvalue $\lambda_{\max}$.

(iv) x and its multiples are the only *positive* eigenvectors of A.

Definition. A square matrix A is *Markov* if it is positive and each column has entries summing to 1.

Let $\mathbf{1}$ be the all-ones column vector. The condition that each column of A has entries summing to 1 can be expressed as $\mathbf{1}^\top A = \mathbf{1}^\top$.

Lemma. If A is Markov, and v is any vector, then the sum of entries of v equals that of Av.

Proof. Since A is Markov, $\mathbf{1}^\top A = \mathbf{1}^\top$. Therefore $\mathbf{1}^\top (Av) = \mathbf{1}^\top v$, as desired. □

Proposition. If A is Markov, then $\lambda_{\max} = 1$.

Proof. Let x be the corresponding eigenvector, so $Ax = \lambda_{\max} x$. The previous lemma implies that

$$\mathbf{1}^\top x = \mathbf{1}^\top Ax = \lambda_{\max} \mathbf{1}^\top x.$$

Since x is positive, $\mathbf{1}^\top x$ is nonzero, and dividing gives $1 = \lambda_{\max}$. □

In recitation, we will explain how Markov matrices can be interpreted as transition probabilities in diagrams called *Markov chains*. We will also discuss the example

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

from Problem 3 in Section 10.3 on page 480 of Strang's book.

2. Differential equations

A path in $\mathbb{R}^n$ can be thought of as a rule which assigns, to every time t, an n-vector $\boldsymbol{u}(t)$. A path may be the solution to a differential equation

$$\frac{d}{dt}\boldsymbol{u}(t) = A\,\boldsymbol{u}(t).$$

This says that, at time t, the velocity vector of the path equals the matrix A times the position $\mathbf{u}(t)$ of the path at that time. It tells you the direction the point must travel, given its current location.

Definition. If A is a square matrix, then

$$e^A := \mathrm{Id} + A + \frac{1}{2}A^2 + \cdots + \frac{1}{n!}A^n + \cdots,$$

just like the usual series for the exponential function. For any A, it converges absolutely.

If we plug in the matrix At, we find

$$e^{At} = \mathrm{Id} + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \cdots.$$

This is an equality of square matrices whose entries depend on t. Differentiating, we find

$$\begin{aligned}\frac{d}{dt}\,e^{At} &= A + A^2 t + \frac{1}{2}A^3\,t^2 + \cdots + \frac{1}{(n-1)!}A^n t^{n-1} + \cdots \\ &= A\left(\mathrm{Id} + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{(n-1)!}A^{n-1}t^{n-1} + \cdots\right) \\ &= A\,e^{At}.\end{aligned}$$

If $AB = BA$, then one can show $e^A e^B = e^{A+B}$ with the usual power series manipulations. This implies, for example, that $e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$ and $e^{\mathrm{Id}}e^A = e^{\mathrm{Id}+A}$. Note that $e^{\mathrm{Id}} = e\,\mathrm{Id}$.

Problem. What is the inverse of e^A?

Problem. Compute e^{At} when $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (See Example 5 in Section 6.3 on page 328 of Strang's book.)

Problem. Compute e^{At} when $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. (Cf. Example 4 in Section 6.3 on page 327 of Strang's book.)

The matrix exponential allows us to solve the differential equation from before. If we take

$$\boldsymbol{u}(t) = e^{At}v$$

for any fixed $v \in \mathbb{R}^n$, then

$$\begin{aligned}\frac{d}{dt}\,\boldsymbol{u}(t) &= \frac{d}{dt}\,e^{At}v \\ &= A\,e^{At}v \\ &= A\,\boldsymbol{u}(t),\end{aligned}$$

as desired. Note that $\boldsymbol{u}(0) = e^{A0}v = v$, so v is the 'starting point' of our solution path.

Problem. Convert the one-variable second-order linear differential equation

$$\frac{d^2}{dt^2}\,y(t) + y(t) = 0$$

to a two-variable system of first order linear differential equations, and solve it.

(See Example 3 in Section 6.3 on page 323 of Strang's book.)

Problem. If v is an eigenvector of A with eigenvalue λ, show that v is also an eigenvector of e^{At} with eigenvalue $e^{\lambda t}$. Describe the behavior of $e^{\lambda t}v$ as $t \to \infty$, based on where λ lies in the complex plane.

Problem. If A satisfies $A^\top = -A$, show that e^{At} is orthogonal, and deduce that any solution to

$$\frac{d}{dt}\,\boldsymbol{u}(t) = A\,\boldsymbol{u}(t).$$

must satisfy $\|\boldsymbol{u}(t)\| = \|\boldsymbol{u}(0)\|$ for all t.

(See page 328 of Strang's book.)