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Symmetric Matrices

Consider an $n \times n$ (real) symmetric matrix A.

1. All eigenvalues of A are real numbers.
2. All eigenvectors of A are orthogonal to each other (except for colinear ones belong to the same eigenvalue).
3. A is diagonalizable.
4. So we have a decomposition $A = X\Lambda X^T = X^{-1}\Lambda X$, where X is orthogonal and Λ is symmetric.

Positive Definite Matrices

Still consider an $n \times n$ (real) symmetric matrix A.

1. We call A *positive definite* if for any vector $0 \neq v \in \mathbb{R}^n$, we have $x^T Ax > 0$.
2. We call A *positive semidefinite* if for any vector $v \in \mathbb{R}^n$, we have $x^T Ax \geq 0$.
3. We call A *negative definite* if for any vector $0 \neq v \in \mathbb{R}^n$, we have $x^T Ax < 0$.
4. A is positive definite if and only if all the eigenvalues are positive.
5. A simple example: For any invertible matrix B, $B^T B$ is always positive definite. (Why?)

Problems

1. Is the set of positive definite $n \times n$ matrices a vector space? What about positive semidefinite?
2. Let A be an $n \times n$ anti-symmetric matrix ($A^T = -A$). Show that
   (a) $x^T Ax = 0$ for any vector $x \in \mathbb{R}^n$.
   (b) The eigenvalues of A are pure imaginary.
   (c) The determinant of A is non-negative.

3. True or false? Explain why or why not.

   (a) Every positive definite matrix is invertible.

   (b) The only positive definite projection matrix is I.

   (c) A diagonal symmetric matrix with positive diagonal entries is positve definite.

   (d) A symmertric matrix with positive determinant might not be positve

   (e) If C is positive definite and A has linearly independent columns, then A^TCA is positive definite.