Rec 10-solu

Sunday, April 26, 2020 11:14 AM

Rec 10

1. Let A be a $n \times n$ symmetric matrix.

   (a) A is diagonalizable and the eigenvalues of A are Real.

   (b) A can be decomposed as $Q \Sigma Q^T$.

   Q: orthogonal square matrix

   Σ: diagonal matrix

   $Q = (u_1 \cdots u_n)$

   $$(u_1 \cdots u_n)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\begin{pmatrix} u_1^T \\ \vdots \\ u_n^T \end{pmatrix}$$

   $$A = \lambda_1 u_1 u_1^T + \cdots + \lambda_n u_n u_n^T$$

2. Equivalent conditions for positive definite (semi-positive definite.)

   - All the eigenvalues are >0 (≥ 0)
   - All the pivots are >0 (≥ 0)
   - All upper-left determinats >0 (≥ 0)
   - Any vector x, $x^TAx > 0$ $(\geq 0$

   $$x^TAx = y^TQ^TAQy = y^TQ^TQ\Sigma Q^TQy$$
   $(y = Q^Tx)$
   $$= y^T\Sigma y = \sum \lambda_i (y_i)^2 > 0$$

Problems

1. Is the set of ~~positive~~ semi-positive definite $n \times n$ matrices a vector space?

   No $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ Not semi-positive.

2. Let A be a 2×2 symmetric matrix with two different eigenvalues λ_1 and λ_2. The corresponding eigenvectors are u_1 and u_2.

   (a) Prove that u_1 and u_2 are perpendicular to each other.

   $$u_1^TAu_2 = u_1^T\lambda_2 u_2 = \lambda_2 u_1^Tu_2$$
   $$u_1^TAu_2 = u_1^TA^Tu_2 = (Au_1)^Tu_2 = \lambda_1 u_1^Tu_2$$
   So $\lambda_1(u_1^Tu_2) = \lambda_2 u_1^Tu_2$

   Since $\lambda_1 \neq \lambda_2$, $u_1^Tu_2 = 0$

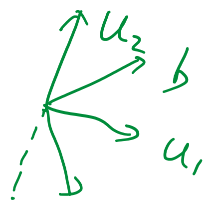
   (b) If $\lambda_1 = 0$, $\lambda_2 = 1$, interpret Ab using projection of a vector b.

   $$A = (u_1 u_2)\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}\begin{pmatrix} u_1^T \\ u_2^T \end{pmatrix}$$
   $$= u_2u_2^T$$
   So Ab is the projection of b to u_2

   (c) If $\Lambda_1 = 1$, $\lambda_2 = -1$, interpret Ab geometrically.

   $Ab = u_1u_1^Tb - u_2u_2^Tb$,

   Ab is the reflection of b along u_1.



3. Given an invertible matrix A, can $A^TA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$? How about $A^TA = \begin{pmatrix} 0&0&0&0&1 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \end{pmatrix}$?

   ① No $\det(A^TA) = \det(A^T)\det A$
   $= (\det A)^2 \neq -1$

   or $tr(A^TA) = \sum (a_{ij})^2 \neq 0$.

   ② No $(A^TA)_{11} = \sum_{j=1}^{5} (a_{j1})^2 \neq 0$ If $=0$, then $A = \begin{pmatrix} 0\,0\,0\,0\,0 \\ * \end{pmatrix}$ is not invertible.