Let A be an $n \times n$ matrix.

The eigenvalues of A are the roots of the characteristic polynomial $\det(A - \lambda \operatorname{Id}_{n\times n})$. If an eigenvalue λ_0 corresponds to a root with multiplicity r in this polynomial, meaning that $(\lambda - \lambda_0)^r$ is a factor of the polynomial, then we say that the eigenvalue λ_0 has *algebraic multiplicity* r. Since the characteristic polynomial is degree n, there are n eigenvalues, counted according to algebraic multiplicity.

If λ_0 is an eigenvalue, then $\operatorname{null}(A - \lambda_0 \operatorname{Id}_{n\times n}) > 0$, i.e. there exists a nonzero vector v such that $Av = \lambda_0 v$. We say that v is an *eigenvector* for the eigenvalue λ_0, and $\operatorname{null}(A - \lambda_0 \operatorname{Id}_{n\times n})$ is the *eigenspace* for eigenvalue λ_0. The dimension of $\operatorname{null}(A - \lambda_0 \operatorname{Id}_{n\times n})$ is called the *geometric multiplicity*, and we have

$$1 \leq (\text{geometric multiplicity of } \lambda_0) \leq (\text{algebraic multiplicity of } \lambda_0).$$

The matrix A is *diagonalizable* if we can write $A = VDV^{-1}$ where V is invertible and D is diagonal. If $d_1, \ldots, d_n$ are the diagonal entries of D, and $v_1, \ldots, v_n$ are the columns of V, this equation is equivalent to asserting that $Av_i = d_i v_i$ for all $i \in \{1, \ldots, n\}$. In other words, this equation says that the $v_1, \ldots, v_n$ are eigenvectors, and v_i has eigenvector d_i. A matrix is diagonalizable if and only if, for each eigenvalue λ_0, its geometric multiplicity equals its algebraic multiplicity. In particular, a matrix is diagonalizable if it has n distinct eigenvalues, because then the algebraic multiplicities are all equal to 1.

Problems

(1) Find the eigenvalues, their geometric and algebraic multiplicities, and eigenvectors for the matrix $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(2) If the eigenvalues, their geometric and algebraic multiplicities, and eigenvectors for A are known, what are the corresponding data for $A + t \operatorname{Id}_{n\times n}$, where t is a given scalar?

(3) Find the eigenvalues, their geometric and algebraic multiplicities, and eigenvectors for the matrix $A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

(4) Show that the eigenvalues of A^2 are the squares of the eigenvalues of A, and that this correspondence respects algebraic multiplicity. Does it always respect geometric multiplicity? What about for higher powers of A?

(5) Diagonalize the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Write down a closed-form expression for A^n. Does there exist B such that $B^2 = A$ and B has real eigenvalues?

(6) Suppose all eigenvalues of A are equal to r, and A is diagonalizable. Show that $A = r \operatorname{Id}_{n\times n}$.

(7) Does there exist a 2×2 matrix A such that $(A^n)_{12} = n$ for all $n \geq 1$, and A is diagonalizable?

(8) Let v_1, v_2 be linearly independent eigenvectors of A. If $v_1 + v_2$ is an eigenvector, what can you conclude about the eigenvalues of v_1, v_2, and $v_1 + v_2$?

Solutions

(1) There is only the eigenvalue $\lambda = 0$, with algebraic multiplicity 4 and geometric multiplicity 1. The corresponding eigenspace is $\left\{ \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for arbitrary } x \right\}$.

(2) The eigenvalues are modified by adding t, while the other data (multiplicities and eigenspaces) stay the same. This is essentially because

$$(A + t\,\mathrm{Id}_{n\times n}) - \lambda\,\mathrm{Id}_{n\times n} = A - (\lambda - t)\,\mathrm{Id}_{n\times n}\,.$$

(3) By (1) and (2), there is only the eigenvalue $\lambda = 3$, with algebraic multiplicity 4 and geometric multiplicity 1. The corresponding eigenspace is $\left\{ \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for arbitrary } x \right\}$.

(4) Let $P(\lambda)$ be the characteristic polynomial of A. Note that

$$\begin{aligned}\det(A^2 - \lambda\,\mathrm{Id}_{2\times 2}) &= \det((A - \sqrt{\lambda}\,\mathrm{Id}_{2\times 2})(A + \sqrt{\lambda}\,\mathrm{Id}_{2\times 2})) \\ &= \det(A - \sqrt{\lambda}\,\mathrm{Id}_{2\times 2})\ \det(A + \sqrt{\lambda}\,\mathrm{Id}_{2\times 2}) \\ &= P(\sqrt{\lambda})\ P(-\sqrt{\lambda}).\end{aligned}$$

Suppose for notational simplicity that A is 2×2. Then we can write $P(\lambda) = (\lambda - \lambda_0)(\lambda - \lambda_1)$, and the preceding expression equals

$$\begin{aligned}&(\sqrt{\lambda} - \lambda_0)(\sqrt{\lambda} - \lambda_1)(-\sqrt{\lambda} - \lambda_0)(-\sqrt{\lambda} - \lambda_1) \\ &= (\lambda - \lambda_0^2)(\lambda - \lambda_1^2).\end{aligned}$$

Therefore, the eigenvalues of A^2 are the squares of the eigenvalues of A, and algebraic multiplicity is preserved. The general $n \times n$ case is identical.

Squaring the matrix doesn't always preserve geometric multiplicity. Look at the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The geometric multiplicity of $\lambda = 0$ is 1, but the geometric multiplicity of $\lambda = 0$ for A^2 is 2, because A^2 is the zero matrix.

Similar conclusions apply for A^3, A^4, and so on. Instead of using a 'difference of squares' factorization, one uses the factorizations

$$a^n - b^n = (a - b)(a - \zeta b) \cdots (a - \zeta^{n-1})$$

where $\zeta = e^{\frac{2\pi i}{n}}$ is a primitive n-th root of unity.

(5) We have $A = VDV^{-1}$ where

$$\begin{aligned}V &= \begin{pmatrix} 2 & 2 \\ \frac{3+\sqrt{33}}{2} & \frac{3-\sqrt{33}}{2} \end{pmatrix} \\ D &= \begin{pmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{pmatrix}.\end{aligned}$$

Using that

$$V^{-1} = -\frac{1}{2\sqrt{33}} \begin{pmatrix} \frac{3-\sqrt{33}}{2} & -2 \\ \frac{-3-\sqrt{33}}{2} & 2 \end{pmatrix},$$

and the fact that $A^n = VD^nV^{-1}$, we find that

$$A^n = -\frac{1}{2\sqrt{33}} \begin{pmatrix} (3-\sqrt{33})(\frac{5+\sqrt{33}}{2})^n + (-3-\sqrt{33})(\frac{5-\sqrt{33}}{2})^n & -4(\frac{5+\sqrt{33}}{2})^n + 4(\frac{5-\sqrt{33}}{2})^n \\ -3(5+\sqrt{33})^n + 3(5-\sqrt{33})^n & (-3-\sqrt{33})(\frac{5+\sqrt{33}}{2})^n + (3-\sqrt{33})(\frac{5-\sqrt{33}}{2})^n \end{pmatrix}.$$

Note that one of the eigenvalues of A is negative. Therefore, (4) implies that the desired matrix B doesn't exist.

(6) We can write $A = VDV^{-1}$ where $D = r\,\mathrm{Id}_{n\times n}$. This implies that $A = r\,\mathrm{Id}_{n\times n}$.

(7) No. Suppose such an A exists, and write $A = VDV^{-1}$. Then $A^n = VD^nV^{-1}$ for all $n \geq 1$. If $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, then each entry of A^n can be expressed as $c_1d_1^n + c_2d_2^n$ for some scalars c_1, c_2. (The solution to (5) gives an example of this.) There do not exist scalars c_1, c_2, d_1, d_2 such that $c_1d_1^n + c_2d_2^n = n$ for all $n \geq 1$. This gives the contradiction.

(8) Suppose v_1, v_2, v_1+v_2 have eigenvalues $\lambda_1, \lambda_2, \lambda_3$, respectively. Note that

$$\begin{aligned} \lambda_1 v_1 + \lambda_2 v_2 &= Av_1 + Av_2 \\ &= A(v_1+v_2) \\ &= \lambda_3 v_1 + \lambda_3 v_2. \end{aligned}$$

Since v_1, v_2 are linearly independent, we conclude that $\lambda_1 = \lambda_2 = \lambda_3$.