Recitation 5/5

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Vector space of functions

Consider a vector space $\mathbb{R}^n$ with some orthonormal basis $b = \{b_1, b_2, \ldots, b_n\}$. Given $x \in \mathbb{R}^n$, how can we express x in terms of linear combination of b? How do we compute the coefficients?

As we learned in previous lectures, functions on $\mathbb{R}$ form a vector space. (Why?)

We will now think about the vector space of nice functions, (nice in a sense where the integral is nicely defined on these functions - let's not go too deep with this) and the inner product of two functions f, g is defined as

$$< f, g >= \int_{[a,b]} n(x) f(x) g(x) dx$$

with an appropriate domain $[a, b]$ and some normalizing function $n(x)$.

Important example - the vector space of all polynomials with degree less than 3.
- The basis (functions) can be chosen as $\{1, x, x^2\}$. Let's set $a = -1, b = 1$. Now, compute the coefficients of $3x^2+2x+4$. Why can't we use the previous inner product method for obtaining the linear combination coefficients?

- Now we take another basis, $\{1, x, \frac{1}{3}(x^2 - 1)\}$. It is orthonormal, indeed. Now let's compute the coefficient of linear combination for $3x^2 + 2x + 4$ with the inner product.

- For a given domain $[-1, 1]$, we can expand this basis to any degree to make a orthonormal basis of the whole polynomial functions.(Or with any domain $[a, b]$) This is called the **Orthogonal Polynomial**. The example here is a famous Legendre Polynomial.

- Abstractions and comparison between infinite dimensional vector space and vector space $\mathbb{R}^n$

Fourier Series

(Unfortunately) Mathematicians found many more basis for the whole function vector space. The one that is most frequently used is the Fourier basis.

The basis is $\{\sin x, \sin 2x, \ldots\} \cup \{\frac{1}{\sqrt{2}}, \cos x, \cos 2x, \ldots\}$ and the inner product is given as,

$$< f, g >= \frac{1}{\pi} \int_{-[\pi,\pi]} f(x) g(x) dx$$

Fourier basis is important since
- Any function(nice function, of course) can be expressed as a sum of **periodic** functions
- It is proven that for usual functions, the linear combination coefficients of $\sin nx, \cos nx$ with large n is very small. In other words, we only need to compute first few coefficients to nicely approximate the original function.

Problems

1. Prove that Fourier basis is orthonormal.

2. Function $f(x)$ is 0 on $\dots, [-3\pi, -2\pi], [-\pi, 0], [\pi, 2\pi], \dots$ and 1 on $\dots, [-2\pi, -\pi], [0, \pi], [2\pi, 3\pi], \dots$. Find Fourier series of $f(x)$.

3. Consider function $f(x) = x^2$. Denote Fourier expansion of f as

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{j=1}^{\infty} a_j \cos jx + \sum_{j=1}^{\infty} b_j \sin jx$$

Compute first few b_i coefficients. What do we get? Explain why.

4. Consider function $g(x) = x^3$. Without any computation, what can we deduce about Fourier coefficients of g?(Hint : similar to problem 3)

5. Similarly explain why some coefficients in problem 2 can be easily deduced without hard computation(i.e. without integration).

ANSWERS

1. For $i \neq j$,

$$\int_{-\pi}^{\pi} \sin ix \sin jx dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos(i+j)x - \cos(i-j)x) dx = 0$$

similarly product of cosines are zeros. Product of sine and cosine is also zero since it is an odd function.

2. If we let $f(x) = \frac{a_0}{\sqrt{2}} + \sum_{j=1}^{\infty} a_j \cos jx + \sum_{j=1}^{\infty} b_j \sin jx$, $a_0 = \frac{1}{\pi} \int_0^{\pi} \frac{1}{\sqrt{2}} dx = \frac{1}{\sqrt{2}}$. For cosine functions,

$$a_n = \frac{1}{\pi} \int_0^{\pi} \cos nx dx = \frac{1}{n\pi} \sin nx \bigg|_{x=0}^{\pi} = 0$$

and for sine functions,

$$b_n = \frac{1}{\pi} \int_0^{\pi} \sin nx dx = -\frac{1}{n\pi} \cos nx \bigg|_{x=0}^{\pi} = \frac{1 - \cos n\pi}{n\pi} = \frac{1 - (-1)^n}{n\pi}$$

So the Fourier series is,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n\pi} \sin nx$$

3. Computing first few coefficients, we obtain all b_j zeros. This is because the function x^3 is an odd function, so it is only expressed as sum of odd basis functions(cosines).

4. Similarly, since x^2 is an even function, the coefficients of cosine basis a_j all become zero.

5. For problem 2, observe that if we subtract $1/2$ from the original function, it returns an odd function. So $a_0 = \frac{1}{\sqrt{2}}$ and $a_j = 0$ is easily deduced without any hard computation.