Rec 8 solu

Monday, April 13, 2020 4:41 PM

Volumes, Matrix calculus

1. Geometric meaning of the determinate of a linear transformation A.

   Volume of parallelpiped generated by columns of A.

2. Matrix calculation: Given a function $f(x, y, z)$, write df in a matrix multiplication form.

   $$df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

3. Given a $n \times n$ matrix A, trace $tr(A) =$ $\sum_{i=1}^{n} a_{ii}$

Eigenvalues

1. Suppose A is a $n \times n$ matrix, λ is an **eigenvalue** of A if $\exists\, u \in \mathbb{R}^n \quad Au = \lambda u$,

   u is an **eigenvector** of A.

2. If A is diagonalizable, then

   $$P^{-1}AP = \Sigma,$$

   where $\Sigma = diag(\lambda_1, \cdots, \lambda_n)$, $P = (u_1, \cdots, u_n)$, and u_i is the eigenvector for the eigenvalue λ_i.

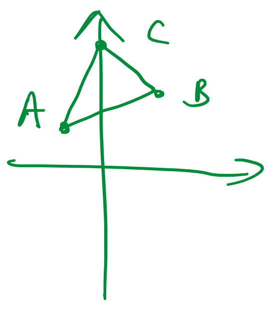
Problems

1. What is area of the triangle whose vertices are $(-1, 1), (1, 2), (0, 3)$?

   $\vec{AB} = (2, 1)$

   $\vec{AC} = (1, 2)$

   $\text{Area} = \frac{1}{2} \cdot \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{3}{2}$



2. Suppose A is a 2×2 matrix with eigenvalues being $2, 1$ and the corresponding eigenvectors being u and v.

   (a) What is $(2A + I)u$?

   $Au = 2u$. $\quad (2A + I)u = 2Au + u = 2 \cdot 2u + u = 5u$.

   (b) Is $2A + I$ diagonalizable? If yes, what are the eigenvalues of $2A + I$?

   Yes, $\quad (2A + I)v = 2Av + v = 2v + v = 3v$

   So the eigenvalues of $2A + I$ are 5 and 3.

3. Suppose A, B are $n \times n$ matrices.

   (a) ~~Is $tr(AB) = tr(A)tr(B)$?~~ Is $tr(A^T) = tr(A)$? Yes

   No. Example $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B$. $\quad tr(A) = 0 = tr(B)$ but $tr(AB) = tr\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$.

   (b) If A is diagonalizable, is A^2 also diagonalizable?

   Yes $\quad P^{-1}AP = \Sigma \iff A = P\Sigma P^{-1}$

   $A^2 = P\Sigma P^{-1} P \Sigma P^{-1} = P\Sigma^2 P^{-1}$

   So $P^{-1}A^2 P = \Sigma^2$ is diagonalizable.

   ~~(c) Are eigenvectors of A always linearly independent?~~

   (c) If the eigenvector matrix of A is the identity matrix, what can you say about A?

   $A \cdot I = I\Sigma$

   So $A = \Sigma$ is diagonalizable.