18.06 Recitation May 5

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Similar Matrices

Consider an $n \times n$ matrix A.

1. B is similar to A if $B = Q^{-1}AQ$ for some $n \times n$ matrix Q.

2. Similar matrices have the same eigenvalues, determinant, trace.

3. (Jordan form) Any $n \times n$ matrix is similar to a matrix called its Jordan form.

Statistics

Suppose we have a series of data $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

1. The average $\bar{x} = (x_1 + x_2 + \cdots + x_n)/n$.

2. The sample variance
$$s = \frac{1}{n-1} \sum_i (x_i - \bar{x})^2$$
.

3. Suppose we have a random variable X, $P(X = x_i) = p_i$, $p_1 + p_2 + \cdots + p_N = 1$. Then the variance
$$\text{Var}(X) = \sum_{i=1}^{N} p_i (x_i - \mu)^2$$
where μ is the expected value
$$\mu = E[X] = \sum_{i=1}^{N} p_i x_i$$
If all p_i are the same, then
$$\text{Var}(X) = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2$$

Problems

1. Let A, B be two positive definite matrices. Prove that the real eigenvalues of AB is positive.

2. True or false? Explain why or why not.

   (a) A symmetric matrix cannot be similar to a nonsymmetric one.

   (b) An invertible matrix cannot be similar to a singular one.

   (c) A cannot be similar to $-A$ unless $A = 0$.

   (d) A cannot be similar to $A + I$.

   (e) If A is similar to A^{-1}, then all the eigenvalues of A must be 1 or -1.

3. Suppose all the eigenvalues of $A_{n\times n}$ are 0. Show that $A^n = 0$.

4. Suppose we have a set of data $x_1, \ldots, x_N$. N is very large, so we can only take a sample $Y_1, \ldots, Y_n$ to analyze ($n < N$). Directly taking the variance of the sample data gives

   $$\sigma_Y^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

   where $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Since Y_i are selected randomly, both $\bar{Y}$ and σ_Y^2 are random variables. We can compute the expected values of them. Show that

   $$E[\sigma_Y^2] = \frac{n-1}{n}\sigma^2$$

   where

   $$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

   (Therefore

   $$s^2 = \frac{n}{n-1}\sigma_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

   is called the *unbiased sample variance*)