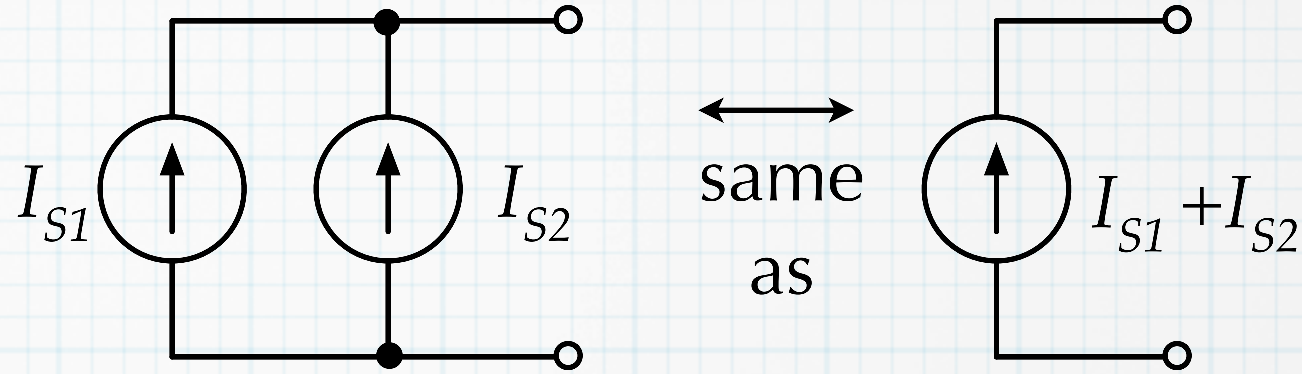
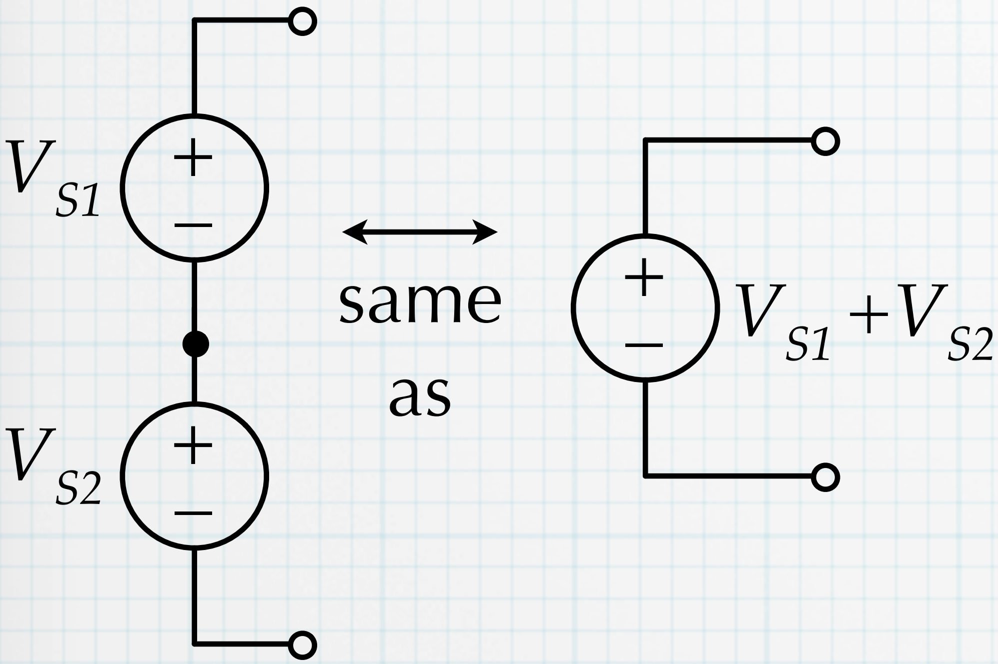
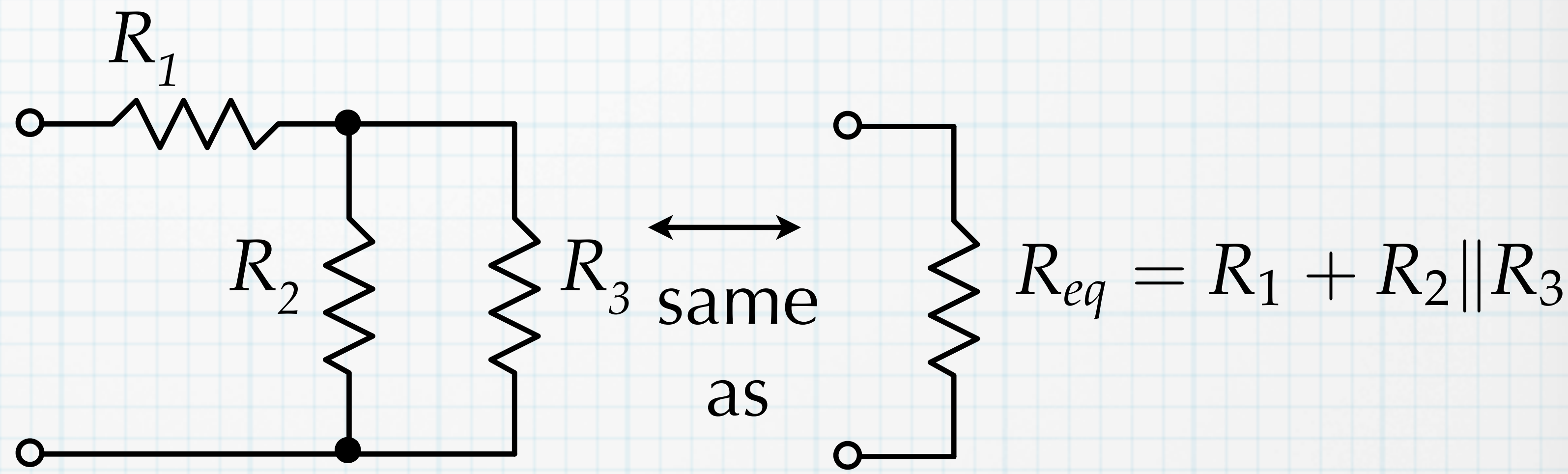
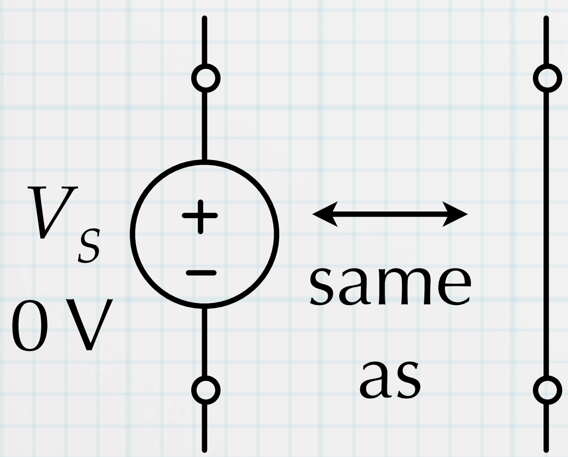
Thevenin equivalent circuits

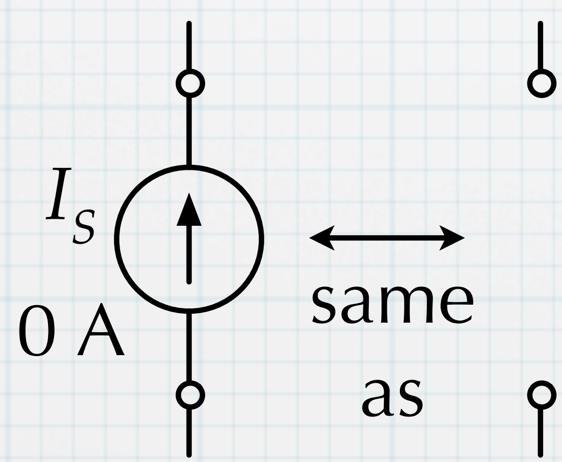
We have seen the idea of equivalency used in several instances already.

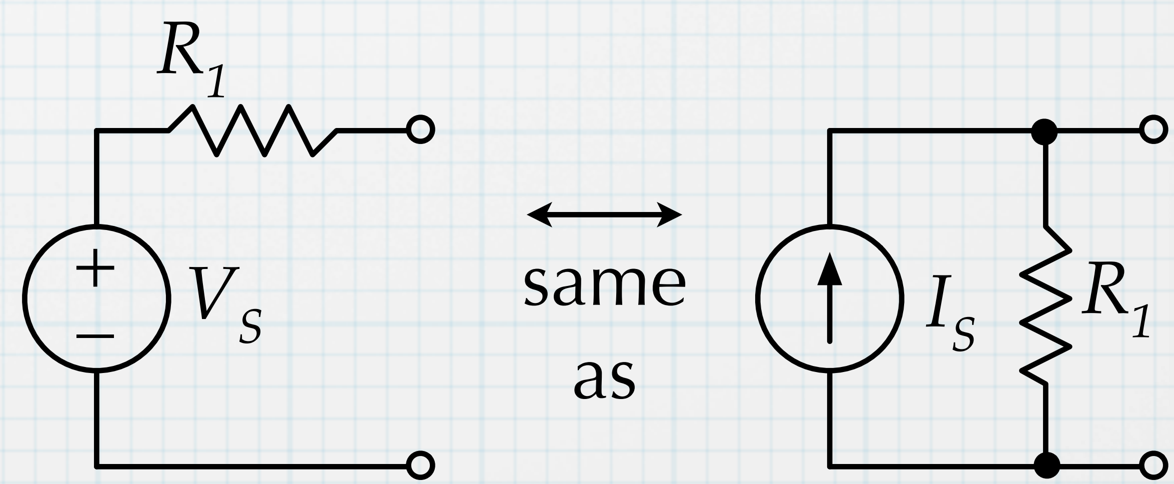






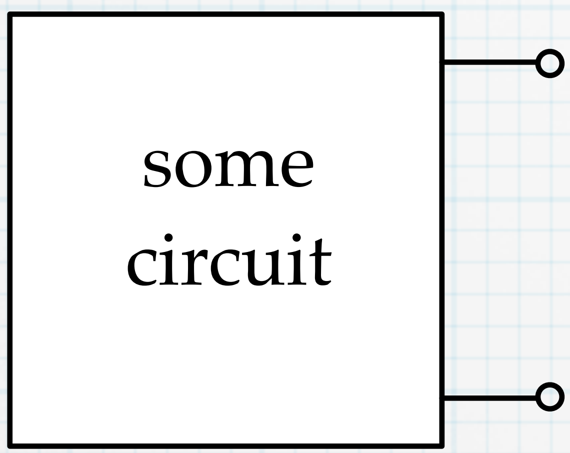




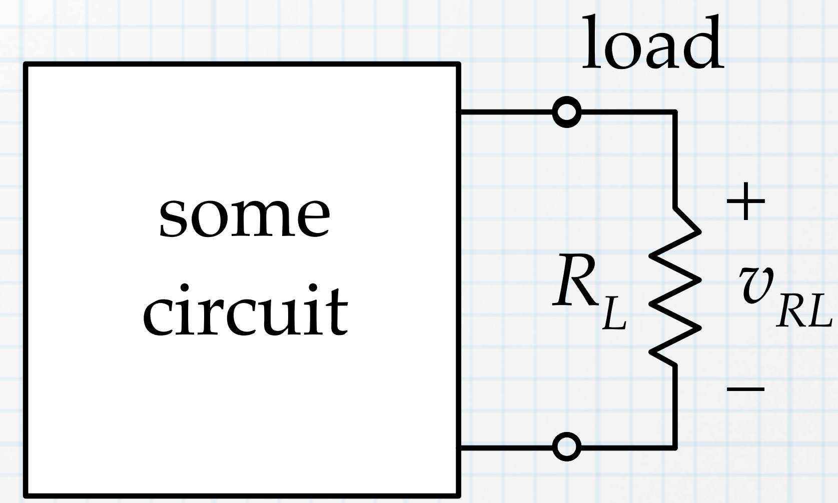


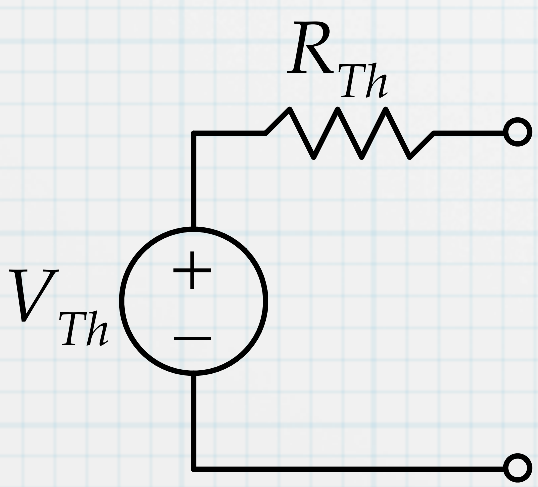
$$V_S = I_S R_1$$

The behavior of any circuit, with respect to a pair of terminals (port) can be represented with a *Thevenin equivalent*, which consists of a voltage source in series with a resistor.

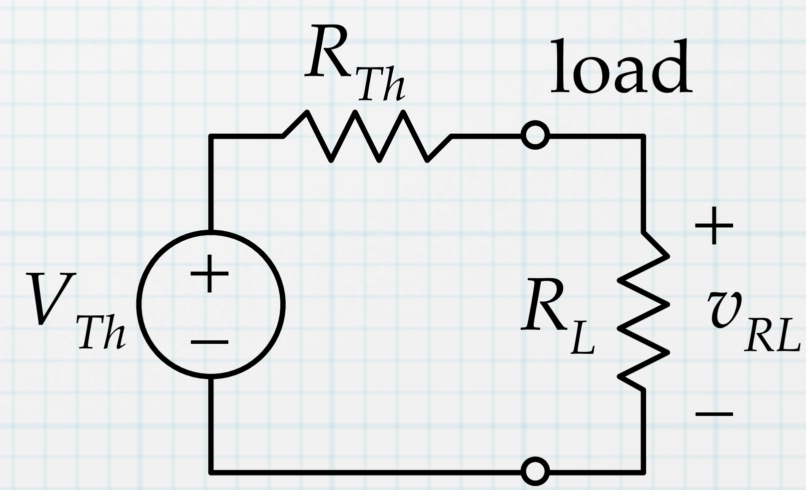


two terminals
(two nodes) =
port



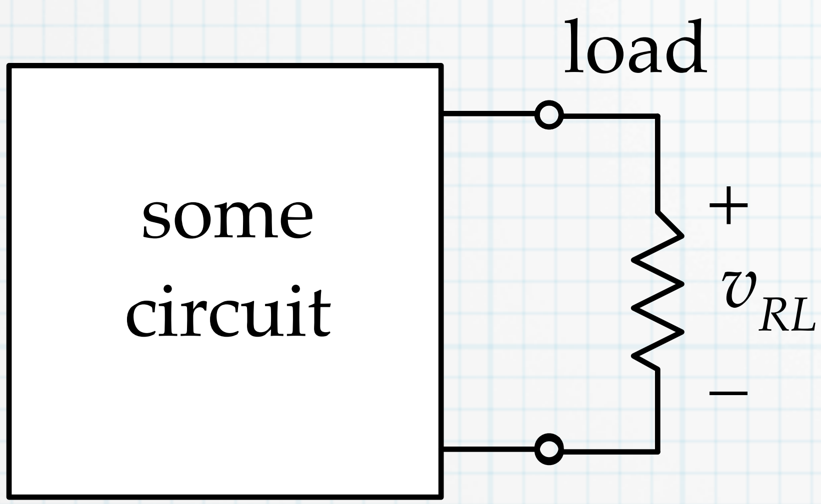


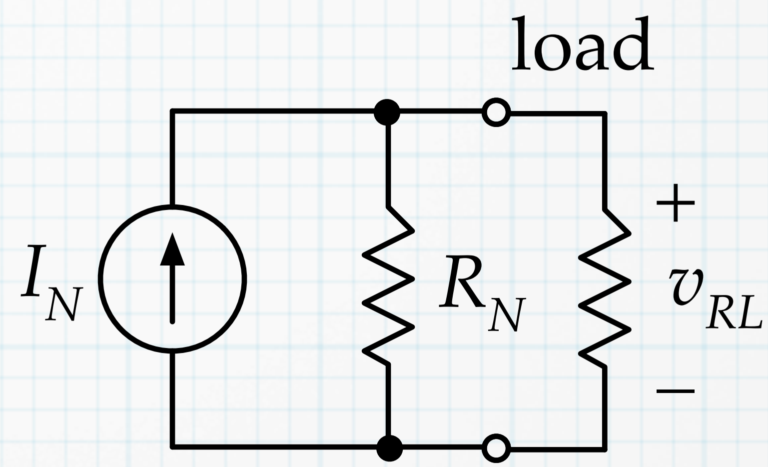
Thevenin
equivalent



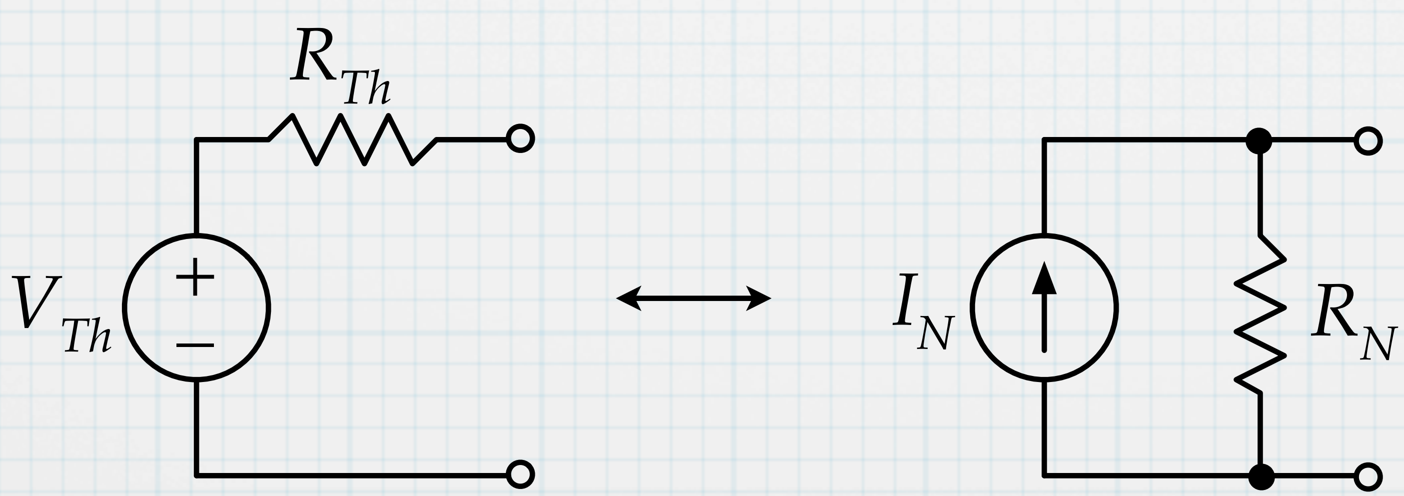
Need to determine V_{Th} and R_{Th} so that the model behaves just like the original.

Norton equivalent





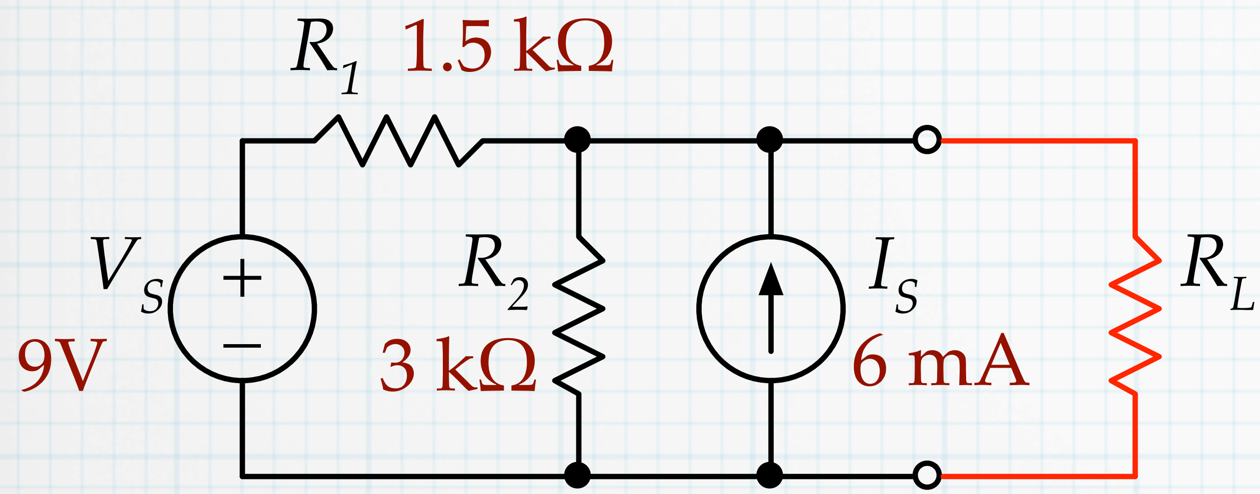
Ideas developed independently (Thevenin in 1880's and Norton in 1920's). But we recognize the two forms as identical because they are source transformations of each other. In EE 201, we won't make a distinction between the methods for finding Thevenin and Norton. Find one and we have the other.

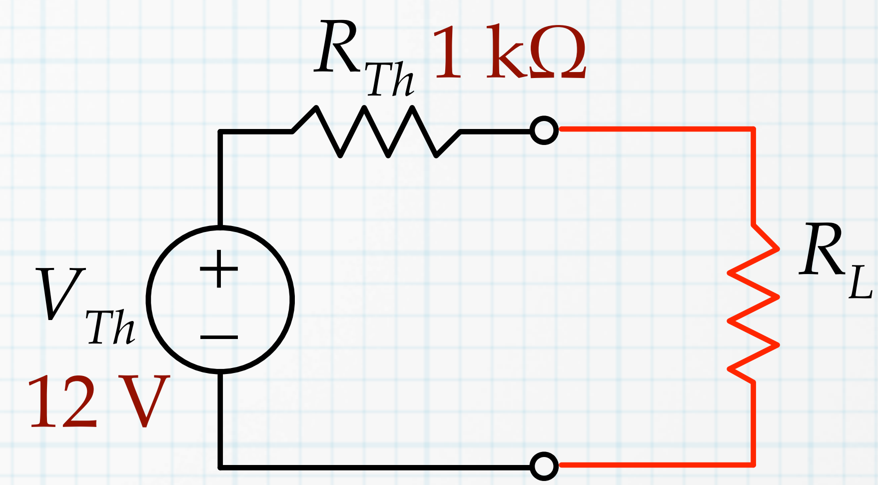


$$R_{Th} = R_N$$

$$V_{Th} = I_N R_{Th}$$

Example





Attach various *load* resistors to the original circuit. Do the same for the equivalent circuit. For each load resistance, calculate the load voltage (and current and power) for each of the circuits. The results are *identical*. In terms of the load that is attached at the port, the two circuits are indistinguishable.

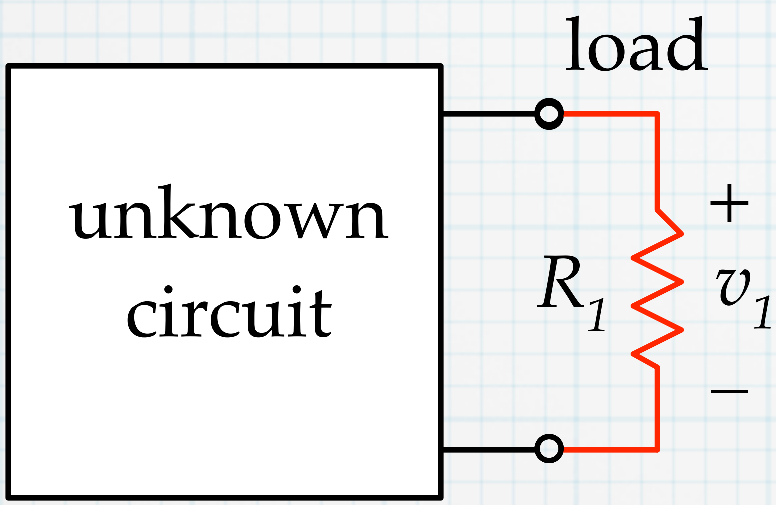
Check it yourself.

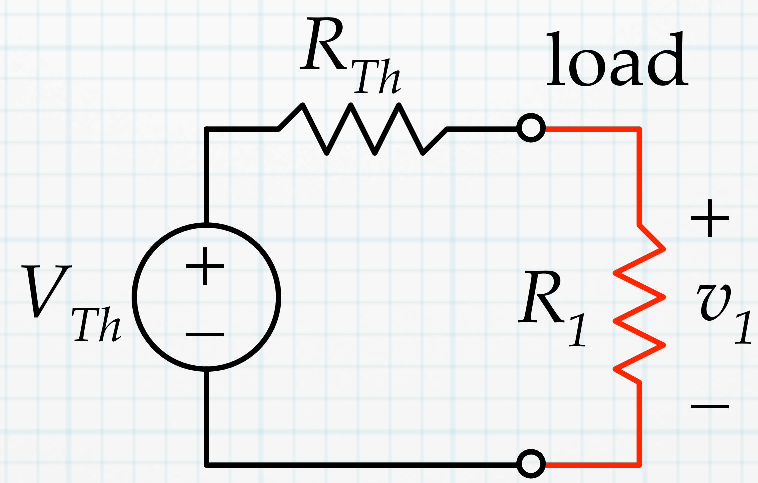
R	v	i	P
1 Ω	11.99 mV	11.99 mA	0.144 mW
10 Ω	118.8 mV	11.88 mA	1.41 mW
100 Ω	1.091 V	10.91 mA	11.90 mW
1 kΩ	6.0 V	6.0 mA	36 mW
10 kΩ	10.91 V	1.09 mA	11.9 mW
100 kΩ	11.88 V	0.119 mA	1.41 mW

Determining the Thevenin (or Norton) components

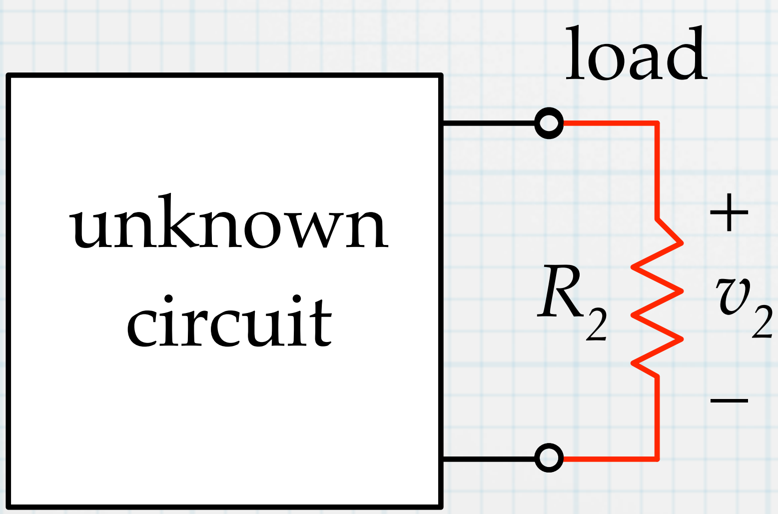
How to find V_{Th} and R_{Th}?

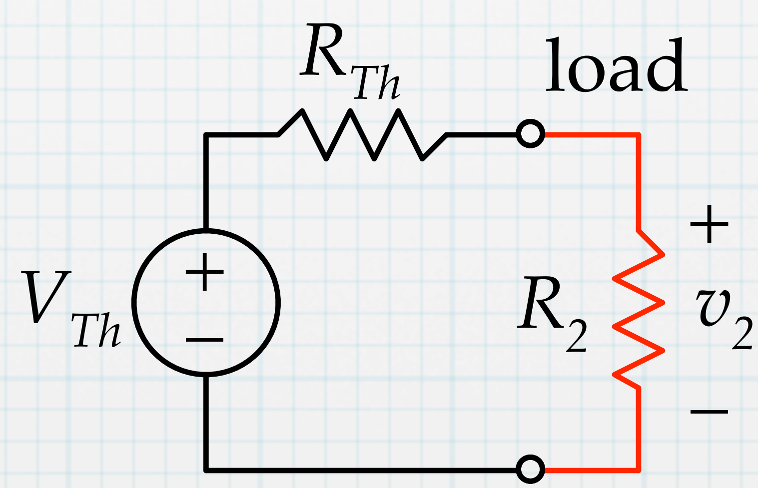
Need two components, so two measurements or calculations should suffice. Use two different load resistors.





$$v_1 = \frac{R_1}{R_{Th} + R_1} V_{Th}$$



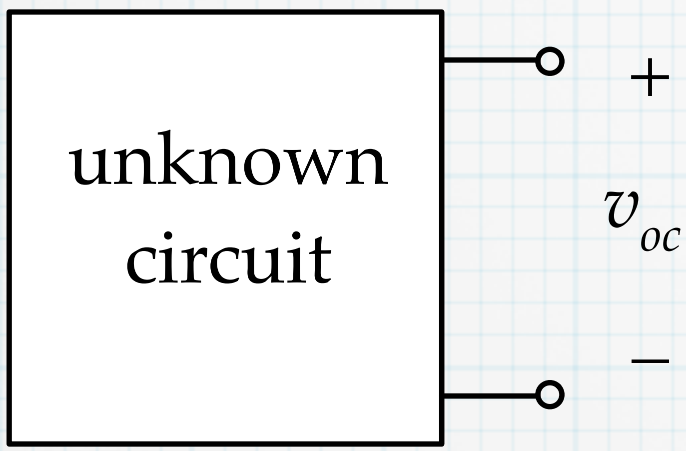


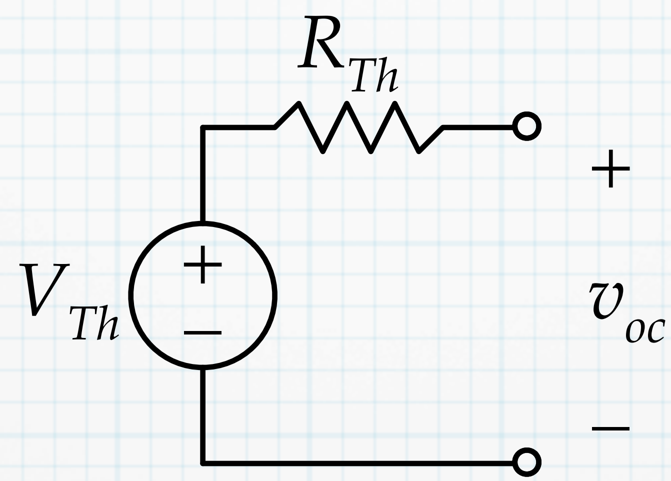
$$v_2 = \frac{R_2}{R_{Th} + R_2} V_{Th}$$

2 equations, 2 unknowns: $V_{Th} = \dfrac{v_1 v_2 (R_1 - R_2)}{R_1 v_2 - R_2 v_1}$ $\quad R_{Th} = \dfrac{R_1 R_2 (v_1 - v_2)}{R_1 v_2 - R_2 v_1}$

More directly: open-circuit voltage, short-circuit current

1. Leave port *open-circuited*. ($R_L \rightarrow \infty$, $i_L = 0$) Measure open-circuit voltage.

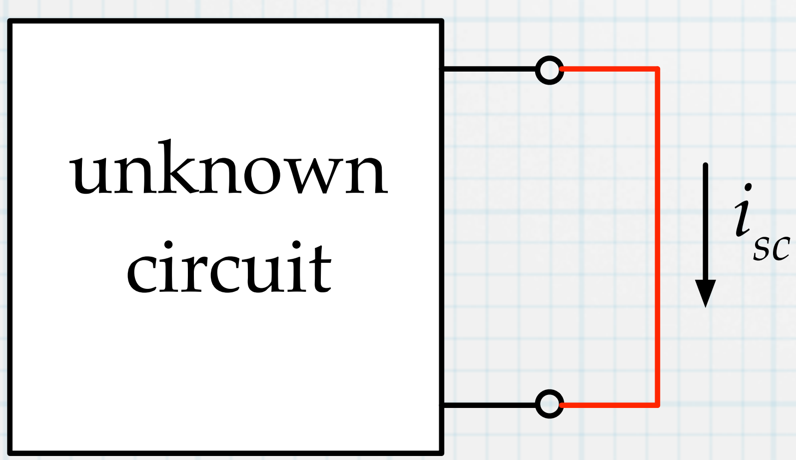


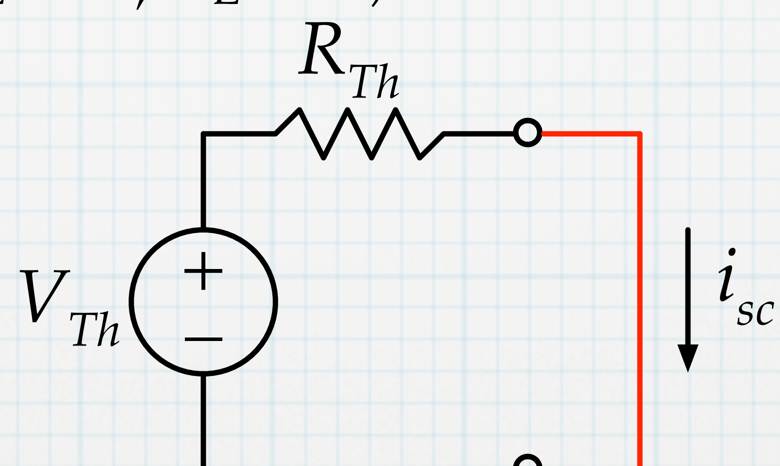


$$v_{oc} = V_{Th}$$

open-circuit voltage is a direct measure of V_{Th}.

2. *Short* the output port. ($R_L = 0$, $v_L = 0$) Measure short-circuit current.





$$i_{sc} = \frac{V_{Th}}{R_{Th}}$$

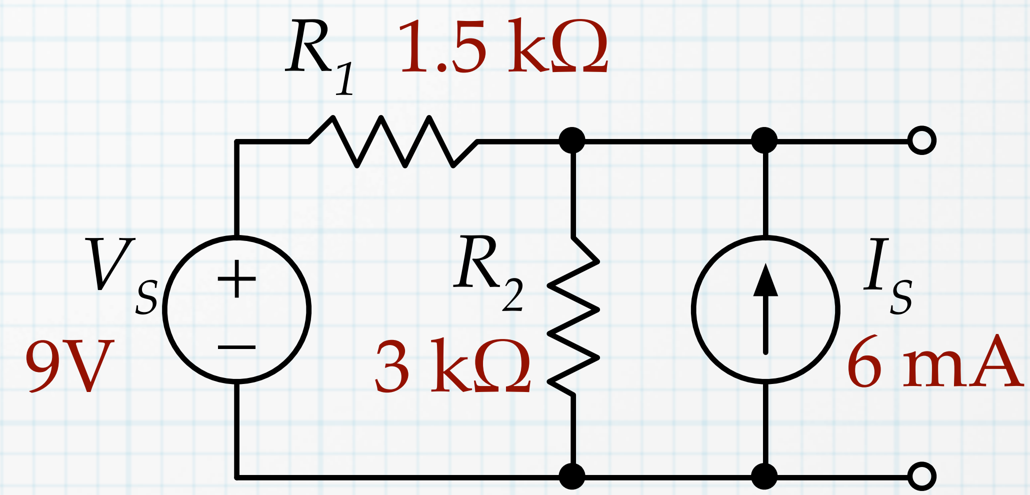
$$R_{Th} = \frac{V_{Th}}{i_{sc}} = \frac{v_{oc}}{i_{sc}}$$

Note, that i_{sc} can also be interpreted as a direct measurement of I_N: $i_{sc} = I_N$.

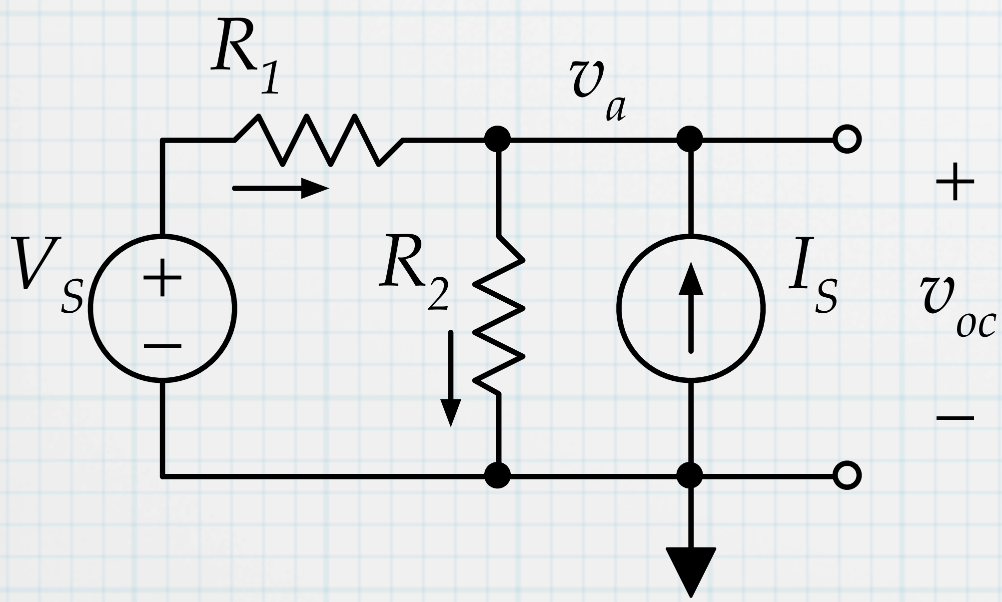
Calculating Thevenin equivalent

The open-circuit voltage / short-circuit current approach can be used to calculate the Thevenin equivalent for a known circuit.

Consider the circuit from slide 4:



Open-circuit voltage – Use whatever method you prefer. We'll use node voltage in this case.



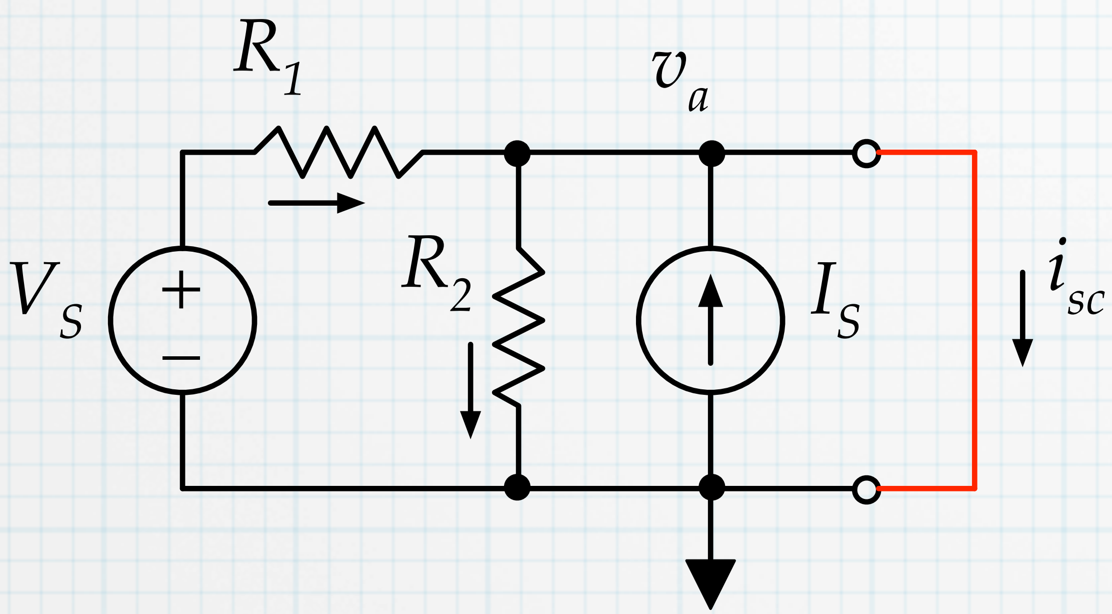
$$v_{oc} = v_a - 0$$

$$\frac{V_S - v_a}{R_1} + I_S = \frac{v_a}{R_2}$$

$$v_a = \frac{V_S + R_1 I_S}{1 + \frac{R_1}{R_2}} = \frac{9\text{V} + (1.5\text{k}\Omega)(6\text{mA})}{1 + \frac{1.5\text{k}\Omega}{3.0\text{k}\Omega}} = 12\text{ V}$$

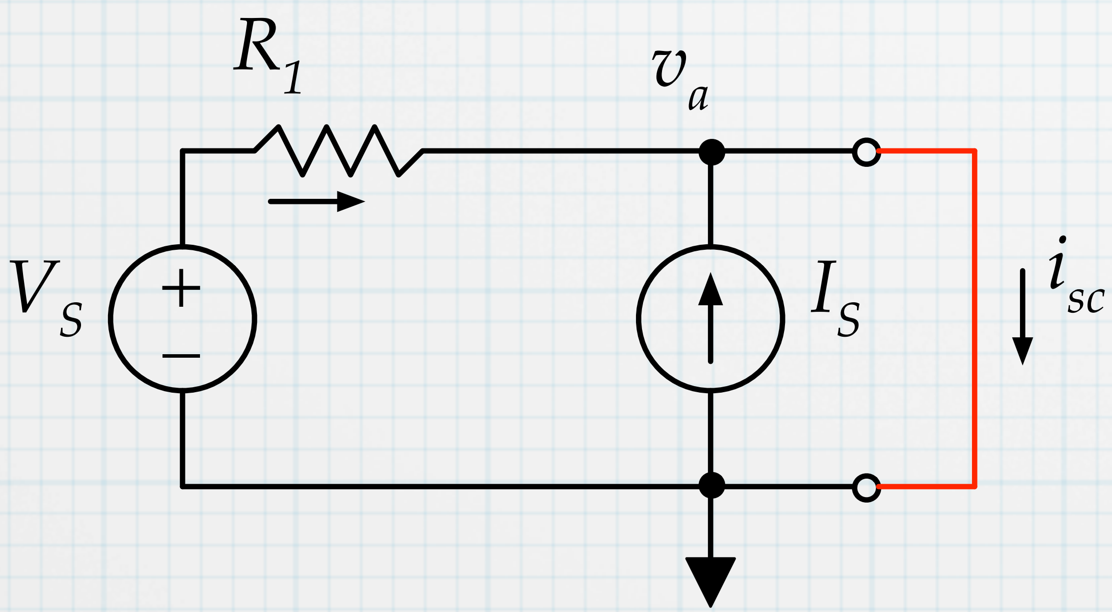
$$V_{Th} = v_{oc} = 12\text{ V}.$$

Short-circuit current – Use whatever method you prefer. We'll use node voltage in this case. But proceed carefully – the short circuit introduces some unusual wrinkles into the circuit analysis.



a: Because of the short circuit, $v_a = 0$!

b: Because of the short, $v_{R2} = 0$ and $i_{R2} = 0$. So R_2 plays no role and can be removed.



$i_{sc} = i_{R1} + I_S$

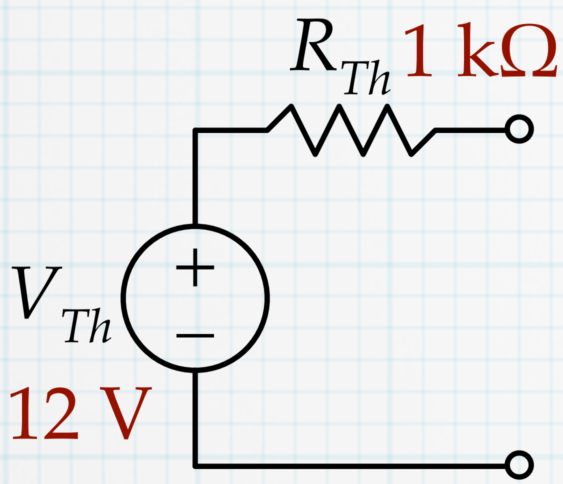
$$= \frac{V_S - v_a}{R_1} + I_S$$

$$= \frac{V_S}{R_1} + I_S$$

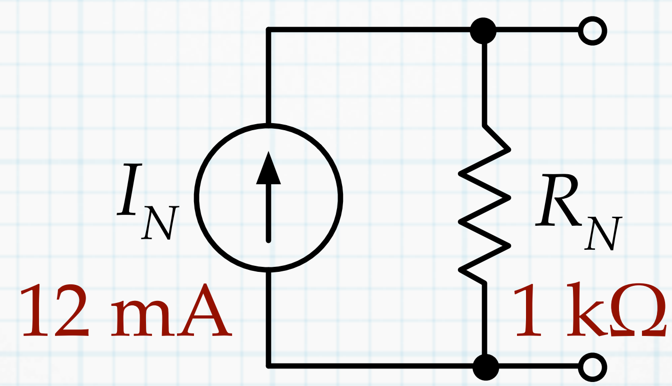
$$= \frac{9\text{V}}{1.5\text{k}\Omega} + 6\text{mA} = 12\text{mA}$$

$$R_{Th} = \frac{V_{Th}}{i_{sc}} = \frac{12\text{V}}{12\text{mA}} = 1\text{k}\Omega$$

Thevenin



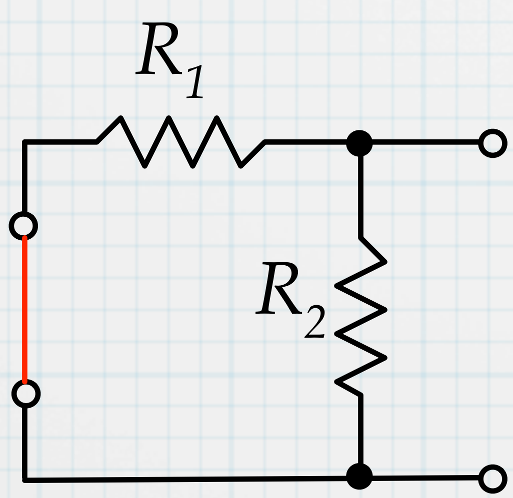
Norton



Alternate method for R_{Th}

If the circuit consists of independent sources and resistors *only*, then the Thevenin resistance can also be found by de-activating the independent sources and finding the equivalent resistance as seen from the port.

De-activate the sources



$$\leftarrow R_{Th} = R_1 \| R_2 = (1.5\text{k}\Omega) \,\|\, (3\text{k}\Omega) = 1\text{k}\Omega$$

Summary

To *measure* V_{Th} and R_{Th}

1. Use a voltmeter to measure the *open-circuit* voltage at the port of the circuit: $v_{oc} = V_{Th}$.

2. Connect a short circuit across the output and use an ammeter to measure the short-circuit current: $i_{sc} = I_N$.

3. Calculate $R_{Th} = V_{Th} / I_N$.

Note that shorting the output may not always be practical. For example, some devices may have over-current protection circuitry that prevents large short-circuit currents from flowing. Or the device might not be able to handle the large current that might flow when the output is shorted without being damaged. In those cases:

1. Use a voltmeter to measure the *open-circuit* at the port of the circuit: $v_{oc} = V_{Th}$.

2. Attach a load resistance, R_L that is small enough so that an appreciable current is flowing. Measure the resulting load voltage, v_L.

3. Calculate $R_{Th} = R_L \left(\frac{v_{oc}}{v_L} - 1 \right)$

Summary

To *calculate* V_{Th} and R_{Th}

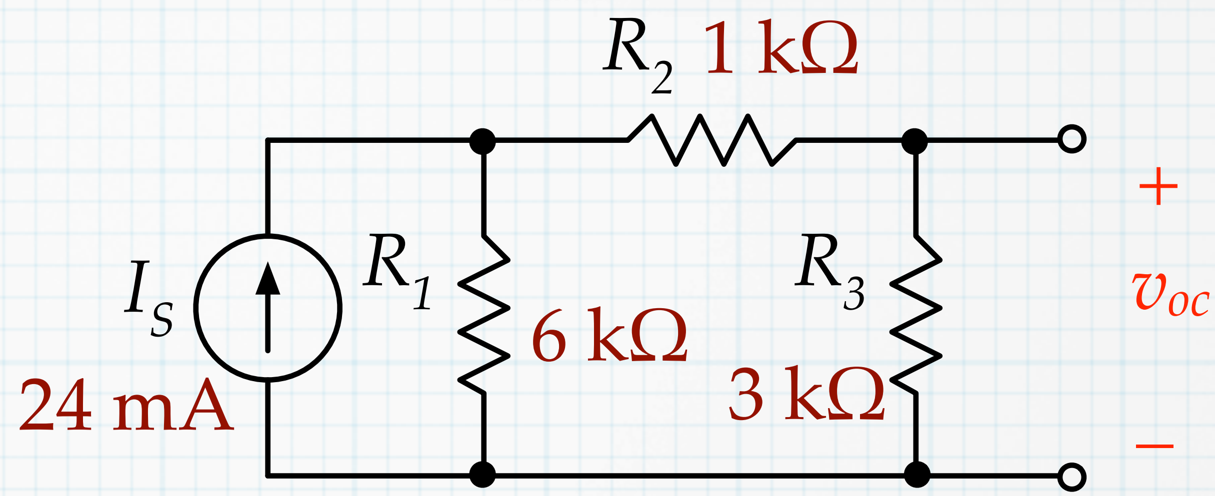
1. Using whatever techniques are appropriate, calculate the *open-circuit* voltage at the port of the circuit: $v_{oc} = V_{Th}$.
2. Connect a short circuit across the output. Using whatever techniques are appropriate, calculate the short-circuit current: $i_{sc} = I_N$.
3. Calculate $R_{Th} = V_{Th} / I_N$.

Alternate method (for circuits that consist only of independent sources and resistors).

1. Using whatever techniques are appropriate, calculate the *open-circuit* voltage at the port of the circuit: $v_{oc} = V_{Th}$.
2. De-activate all independent sources. Calculate the equivalent resistance as seen from the port. (If dependent sources are present in the circuit, the *test generator* method can be used to find equivalent resistance. See the equivalent resistance notes to review the test generator technique.)

Example 1

Find the Thevenin and Norton equivalents of the circuit at right, with the port as shown.



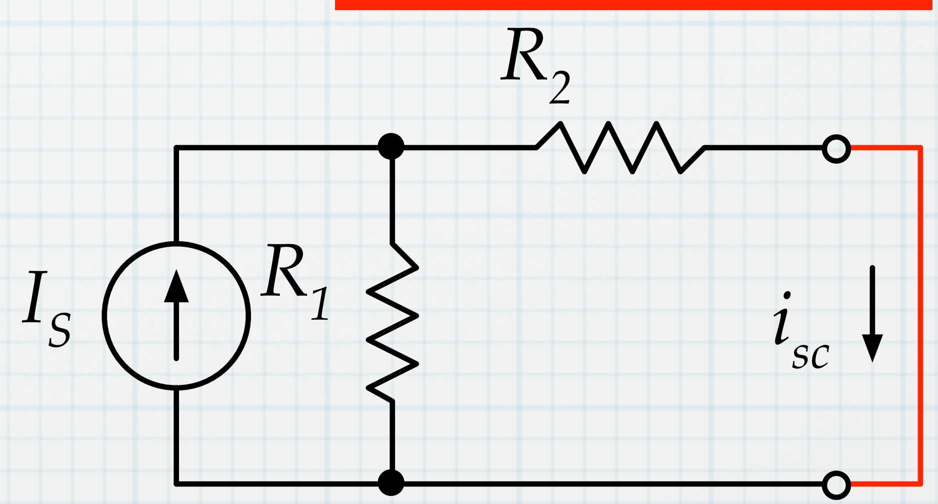
Find v_{oc}. Start with a current divider.

$$i_{R3} = \frac{\frac{1}{R_2+R_3}}{\frac{1}{R_1}+\frac{1}{R_2+R_3}} I_S = \frac{\frac{1}{1+3}}{\frac{1}{6}+\frac{1}{1+3}}(24\text{mA}) = 14.4\text{mA}$$

$$v_{oc} = i_{R3}R_3 = (14.4\text{mA})(3\text{k}\Omega) = 43.2\text{V}$$

$$\boxed{V_{Th} = v_{oc} = 43.2\ \text{V}}$$

Find i_{sc}. Note that R_3 is shorted out. Use a current divider again.

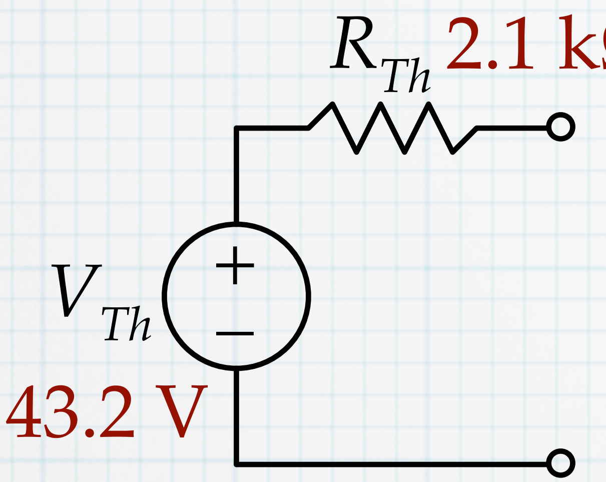


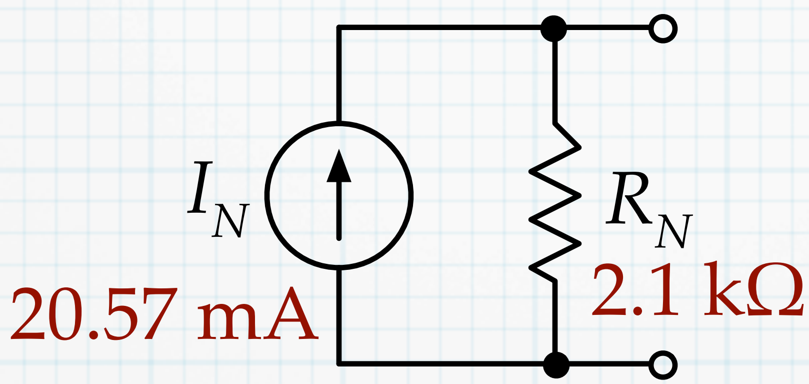
$$i_{sc} = \frac{\frac{1}{R_2}}{\frac{1}{R_2}+\frac{1}{R_2}} I_S$$

$$= \frac{\frac{1}{1}}{\frac{1}{6}+\frac{1}{1}}(24\text{mA}) = 20.57\text{mA}$$

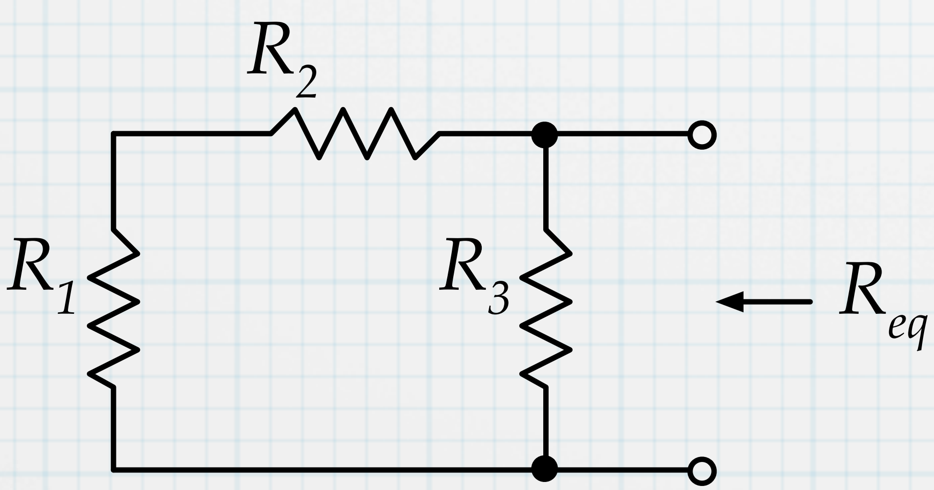
$$\boxed{I_N = i_{sc} = 20.57\ \text{mA}}$$

$$\boxed{R_{Th} = \frac{V_{Th}}{i_{sc}} = \frac{43.2\text{V}}{20.57\text{mA}} = 2.1\text{k}\Omega}$$



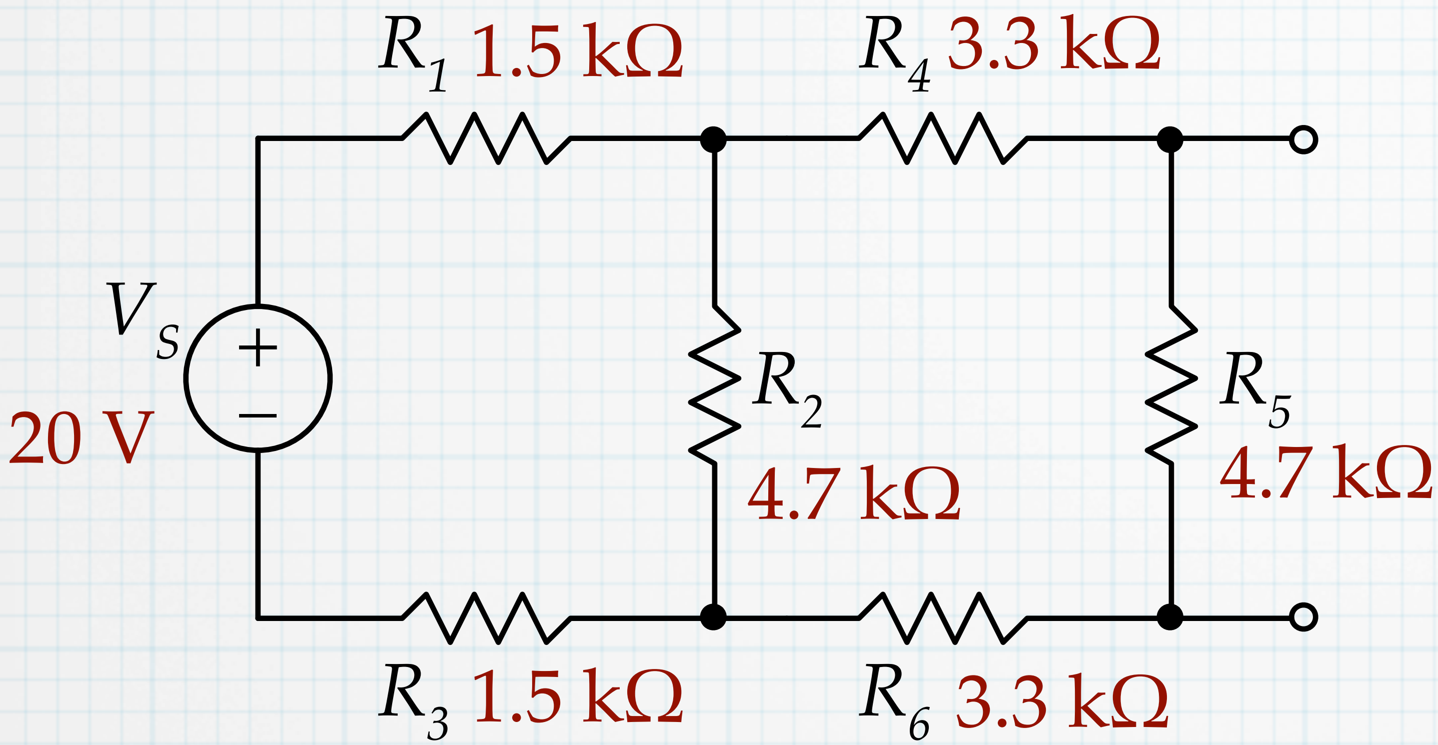


Alternatively, we could use the short-cut method to find R_{Th}.
De-activating the current source:

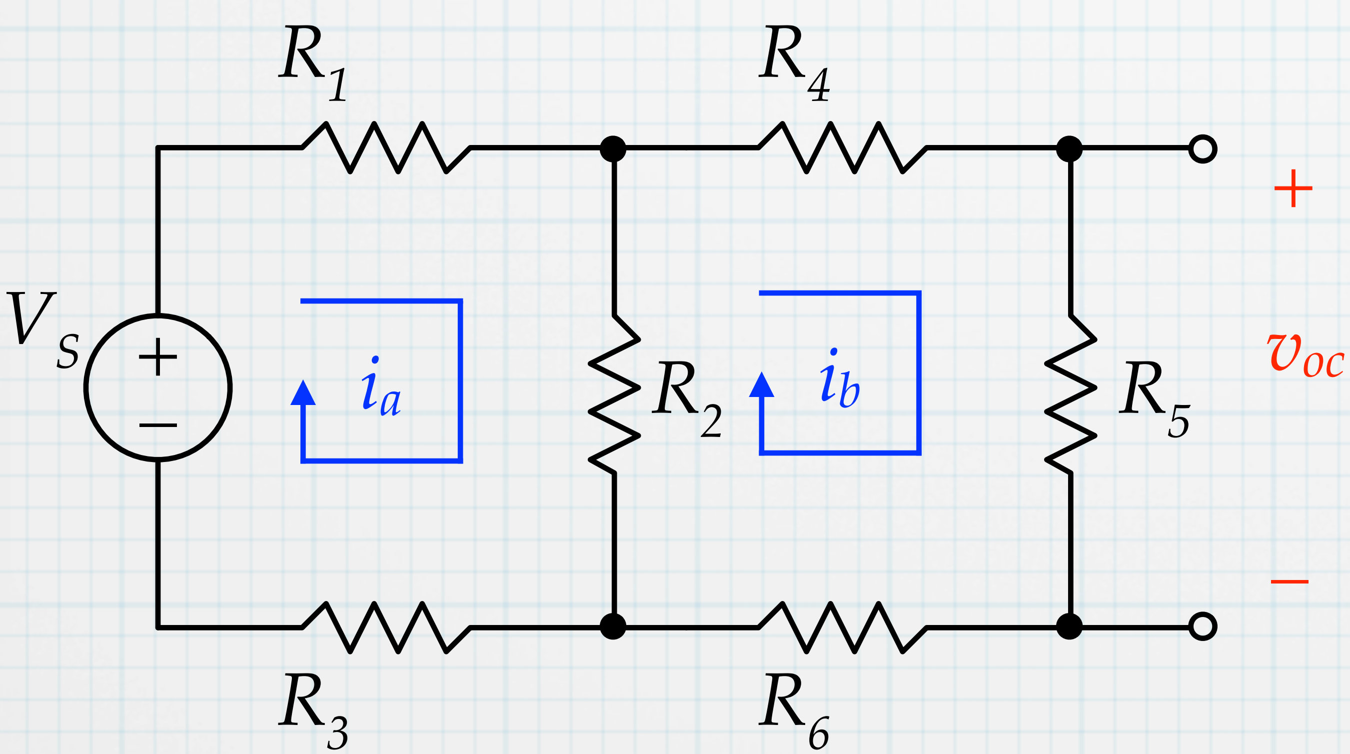


$$R_{eq} = R_3 \| (R_1 + R_2)$$
$$= (3\text{k}\Omega) \| (6\text{k}\Omega + 1\text{k}\Omega) = 2.1\text{k}\Omega$$

Example 2



Find the Thevenin and Norton equivalents of the circuit at left, with the port as shown.



Find v_{oc}. Use mesh current method.

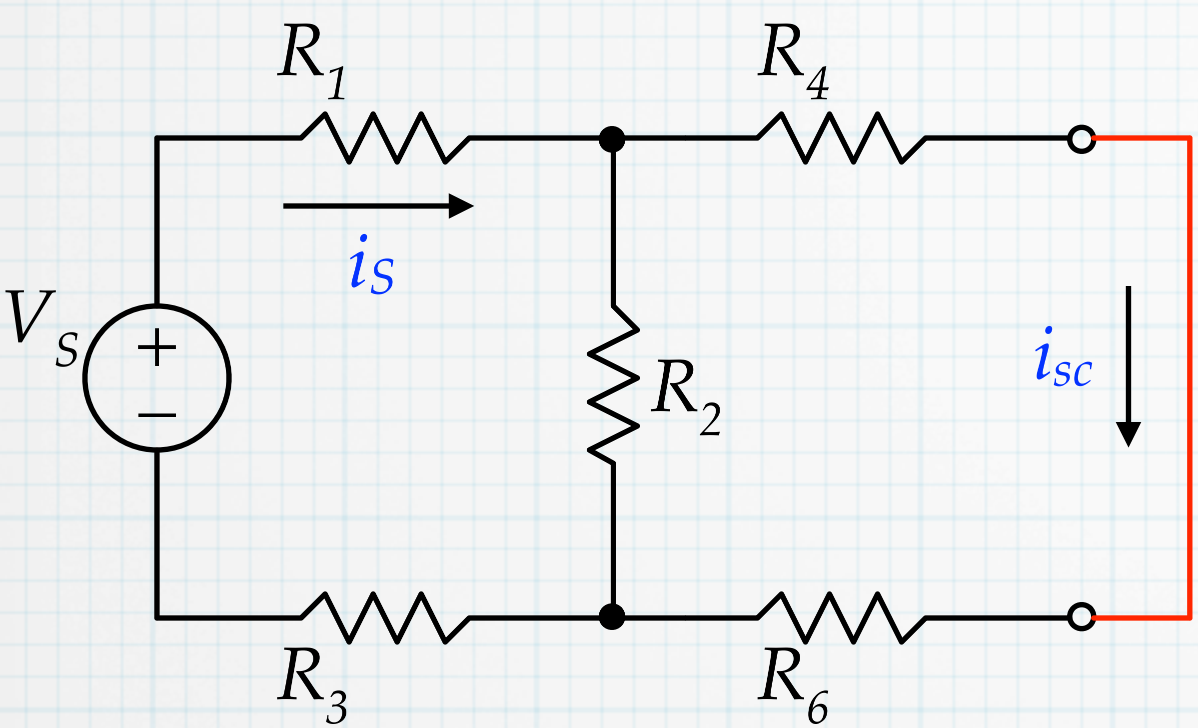
$$V_S - v_{R1} - v_{R2} - v_{R3} = 0$$
$$v_{R2} - v_{R4} - v_{R5} - v_{R6} = 0$$

$$V_S - R_1 i_a - R_2\left(i_a - i_b\right) - R_3 i_a = 0$$
$$R_2\left(i_a - i_b\right) - R_4 i_b - R_5 i_b - R_6 i_b = 0$$

$$\left(R_1 + R_2 + R_3\right) i_a - R_2 i_b = V_S$$
$$R_2 i_a - \left(R_2 + R_4 + R_5 + R_6\right) i_b = 0$$

Insert values and solve: $i_b = 0.930$ mA.

$$\boxed{v_{oc} = i_b\, R_5 = 4.37\text{ V.}}$$



Find i_{sc}. Note that R_5 is shorted out by the short circuit.

Use equivalent resistance to find i_S.

$$R_S = R_1 + R_3 + R_2 \| (R_4 + R_6) = 5.75\text{k}\Omega$$

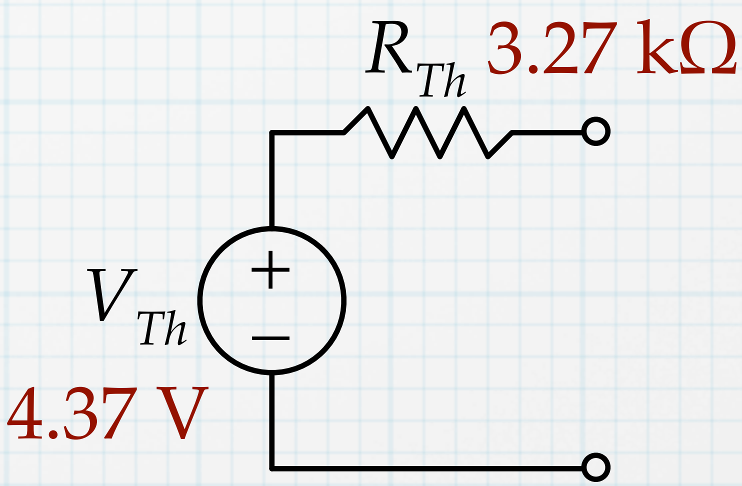
$$i_S = \frac{V_S}{R_S} = 3.48\text{mA}$$

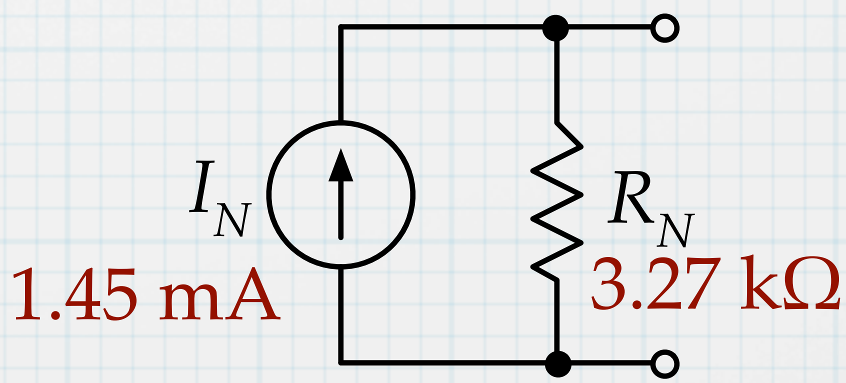
Use current divider to find i_{sc}.

$$i_{sc} = \frac{\frac{1}{R_4+R_6}}{\frac{1}{R_4+R_6} + \frac{1}{R_2}} i_S$$

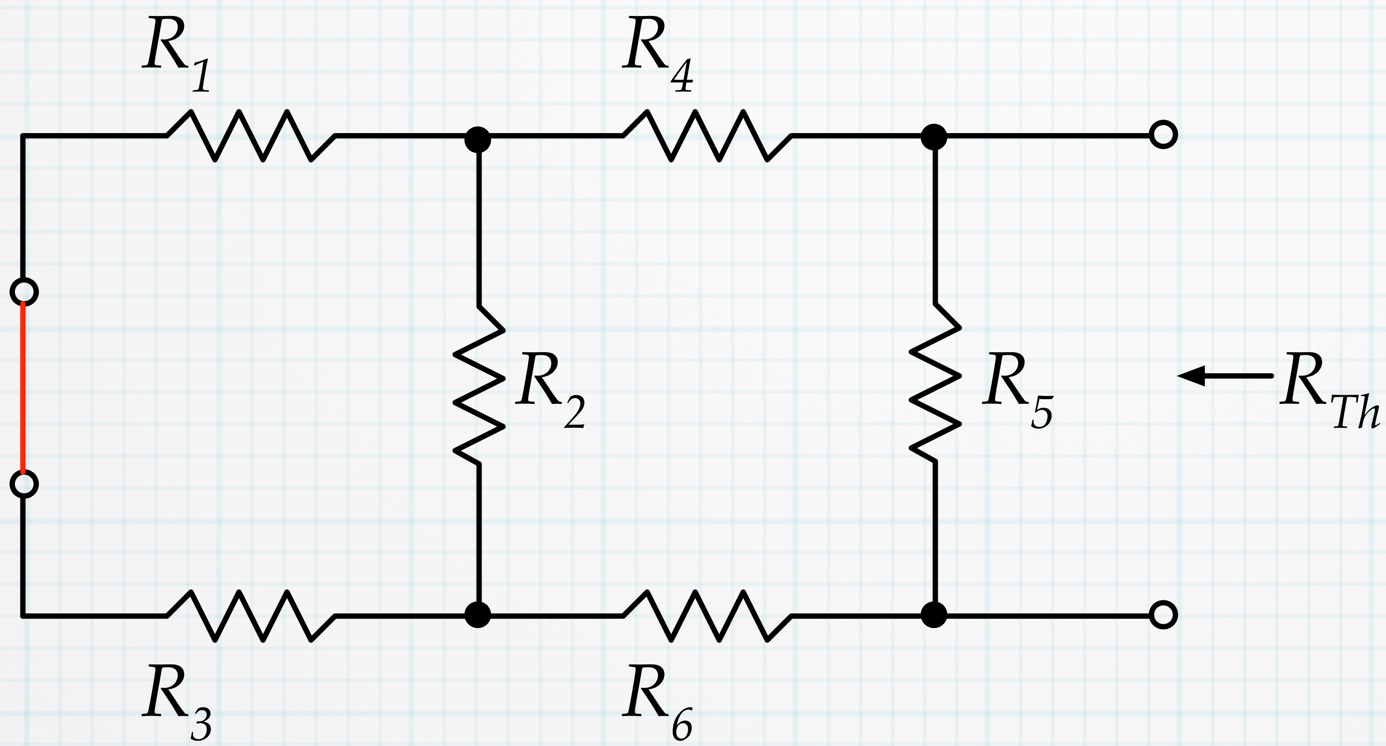
$$\boxed{i_{sc} = 1.45\text{ mA}}$$

$$\boxed{R_{Th} = \frac{V_{Th}}{i_{sc}} = \frac{4.37\text{V}}{1.45\text{mA}} = 3.02\text{k}\Omega}$$





Alternatively, we can use the short-cut method to find R_{Th}.

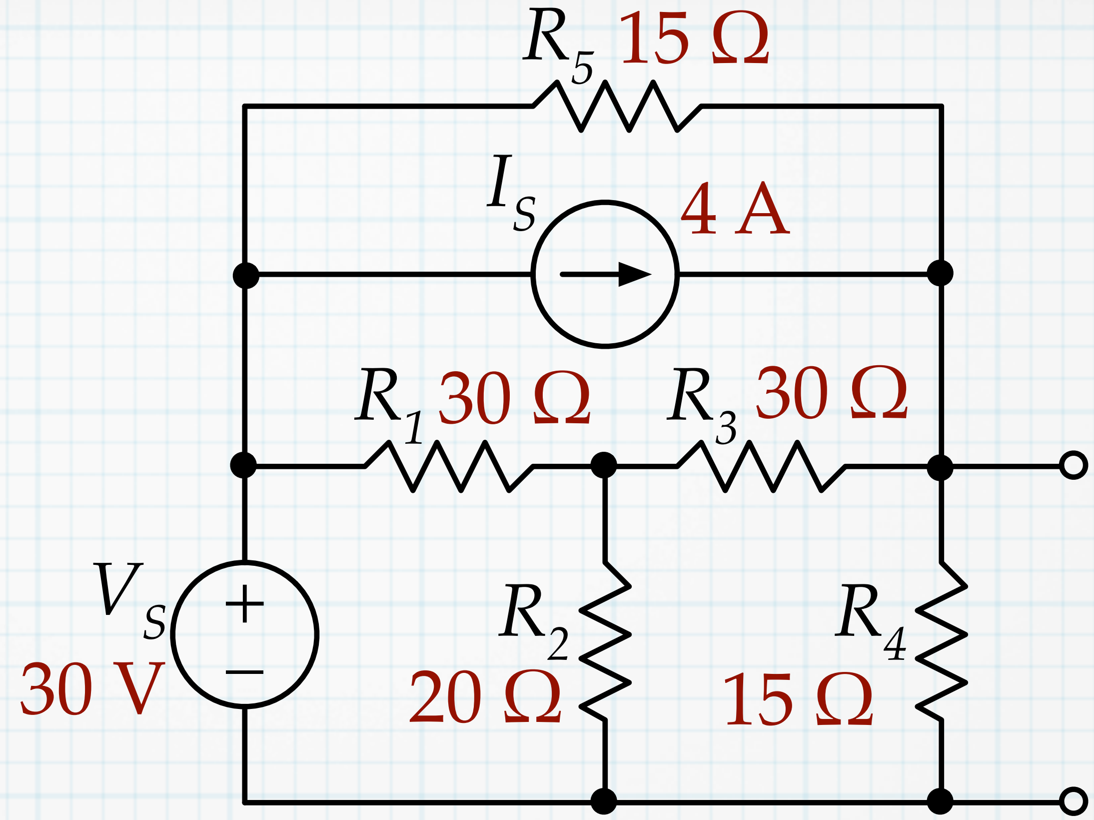


$$R_{Th} = R_5 \| \left[R_4 + R_6 + R_2 \| \left(R_1 + R_3 \right) \right]$$

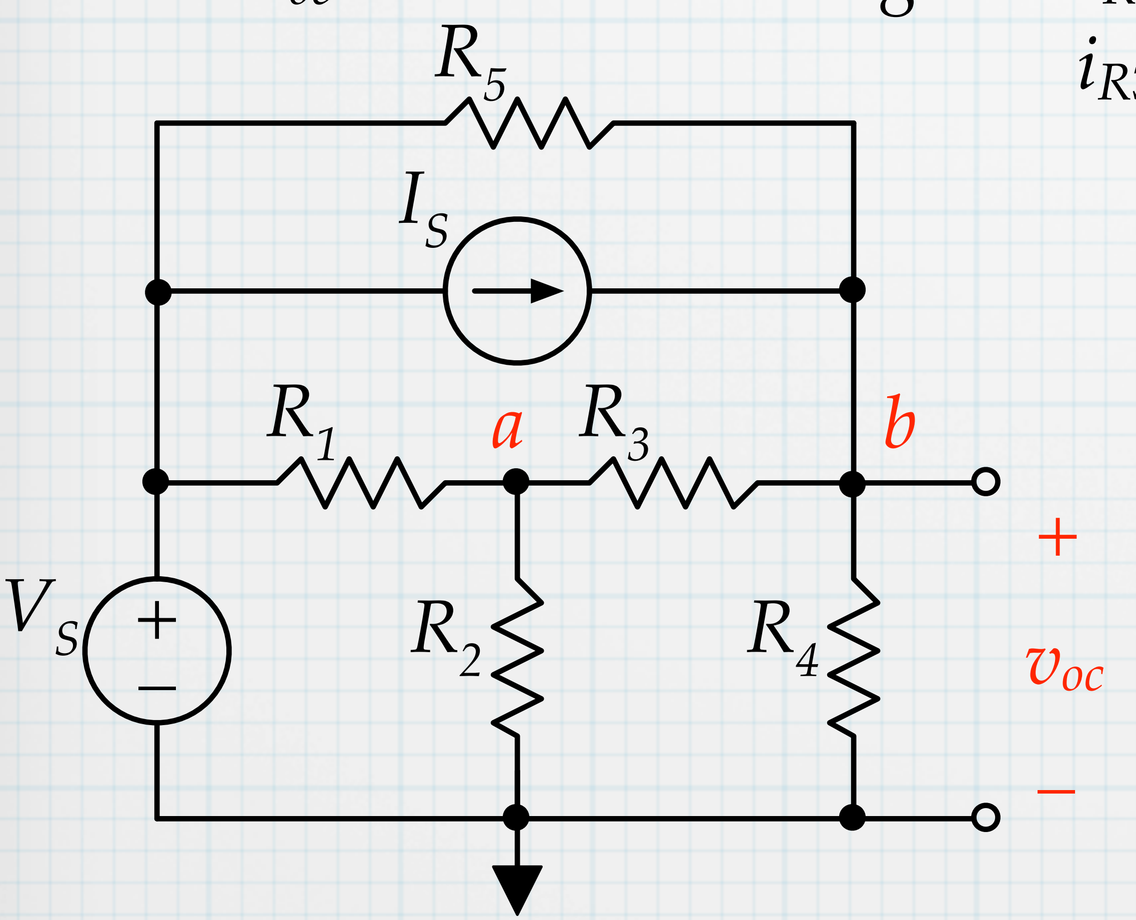
$$= 3.02\ \text{k}\Omega$$

Example 3

Find the Thevenin and Norton equivalents of the circuit at right, with the port as shown.



Find v_{oc}. Use node voltage.



$$i_{R1} = i_{R2} + i_{R3}$$

$$i_{R3} + i_{R5} + I_S = i_{R4}$$

$$\frac{V_S - v_a}{R_1} = \frac{v_a}{R_2} + \frac{v_a - v_b}{R_3}$$

$$\frac{v_a - v_b}{R_3} + \frac{V_S - v_b}{R_5} + I_S = \frac{v_b}{R_4}$$

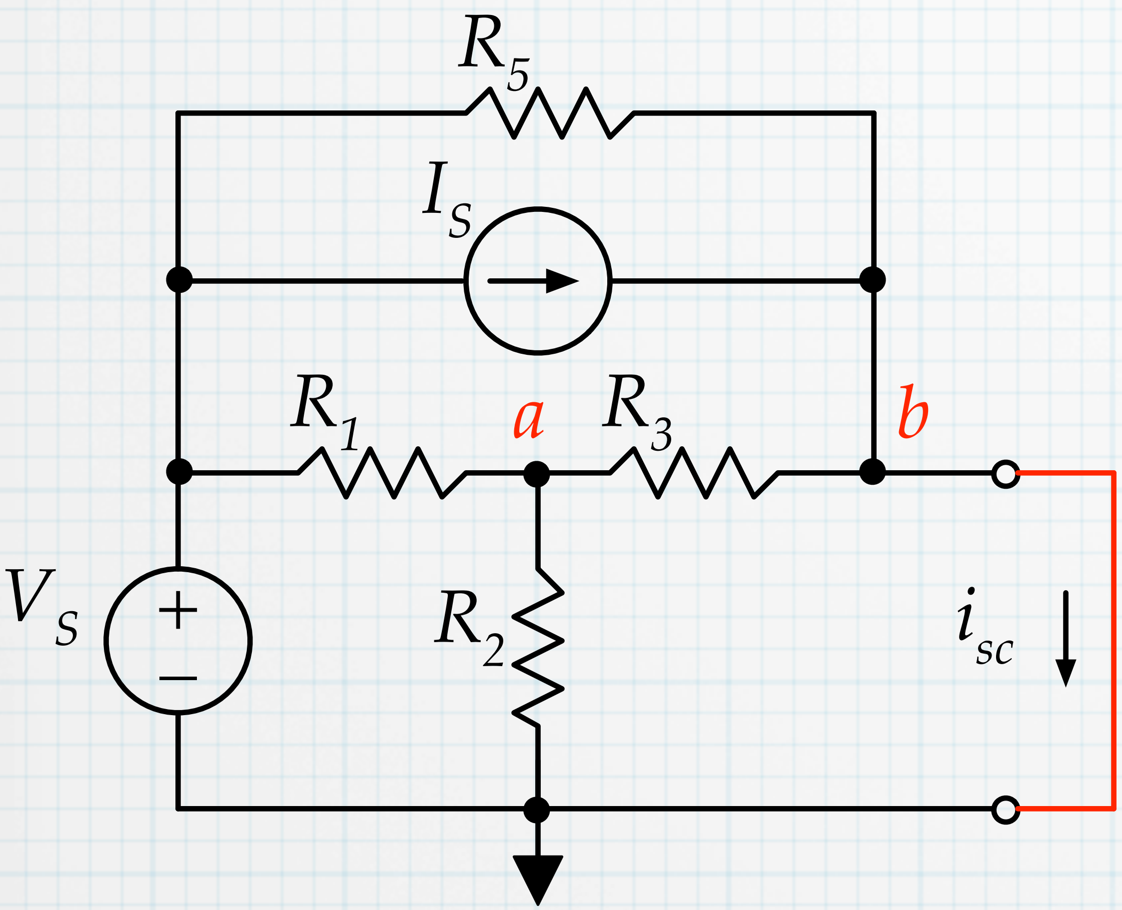
$$\left(1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}\right) v_a - \frac{R_1}{R_3} v_b = V_S$$

$$-\frac{R_5}{R_3} v_a + \left(1 + \frac{R_5}{R_3} + \frac{R_5}{R_4}\right) v_b = V_S + R_5 I_S$$

solve: $v_a = 20$ V, $v_b = 40$ V.

$v_{oc} = v_b = 40$ V

Find i_{sc}. Use node voltage, again. Note that R_4 is shorted out (so ignore it) and node b is shorted to ground, $v_b = 0$.



$$i_{sc} = i_{R3} + i_{R5} + I_S$$

$$= \frac{v_a - v_b}{R_3} + \frac{V_S - v_b}{R_5} + I_S$$

$$= \frac{v_a}{R_3} + \frac{V_S}{R_5} + I_S$$

$$i_{sc} = \frac{8.57\text{V}}{30\Omega} + \frac{30\text{V}}{15\Omega} + 4\text{A} = 6.28\text{A}$$

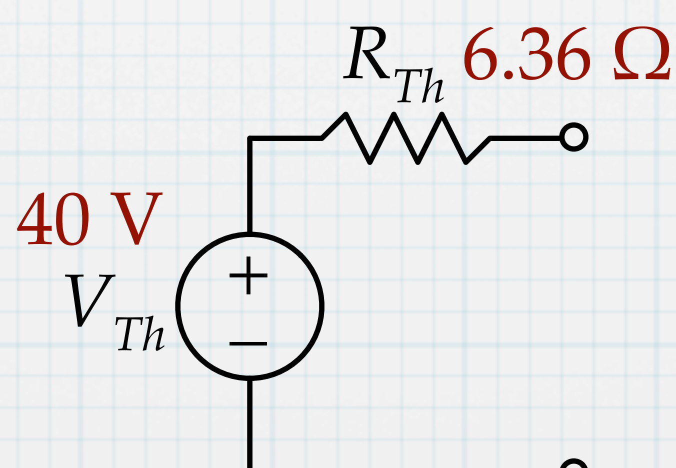
$$R_{Th} = \frac{40\text{V}}{6.28\text{A}} = 6.36\Omega$$

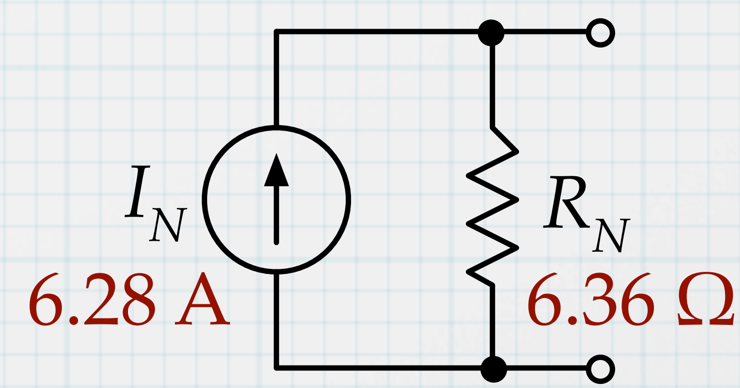
Need v_a.

$$i_{R1} = i_{R2} + i_{R3}$$

$$\frac{V_S - v_a}{R_1} = \frac{v_a}{R_2} + \frac{v_a}{R_3}$$

$$v_a = \frac{V_S}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}} = 8.57\text{ V}$$

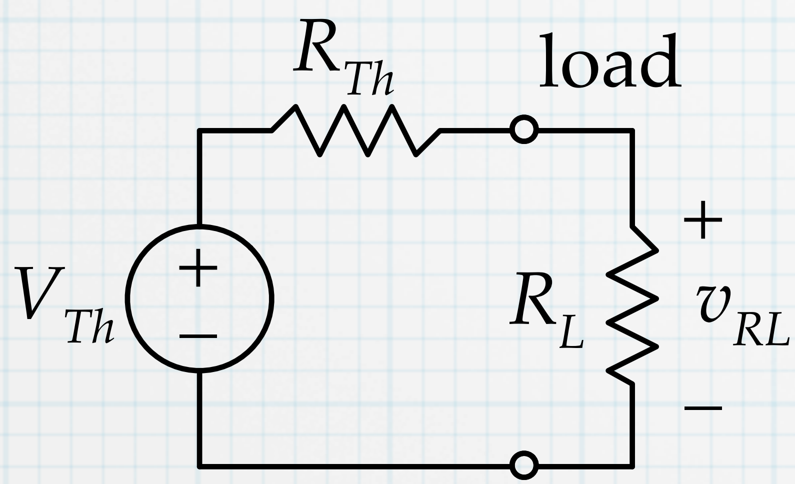




Maximum power transfer

Now that we have the ability to model any circuit using a simple Thevenin (or Norton) equivalent, we can answer another important question: How much power can a given circuit supply to an attached load?

Start with the Thevenin equivalent and determine the load resistance that would lead to the maximum amount of power being dissipated in the load.



$$P_L = \frac{v_{RL}^2}{R_L} = \frac{R_L V_{Th}^2}{(R_L + R_{Th})^2}$$

In the usual way, find the max by setting the derivative to zero and solving.

$$\frac{dP_L}{dR_L} = 0 \qquad \frac{V_{Th}^2}{(R_L + R_{Th})^2} - 2\frac{R_L V_{Th}^2}{(R_L + R_{Th})^3} = 0$$

$$R_L + R_{Th} - 2R_L = 0 \longrightarrow R_L = R_{Th}$$

For maximum power to the load

See slide 4 for an example – look at power column in the table.