EEENG226
Signals and Systems

Chapter 1
Introduction to Singals and Systems

Classification of Signals

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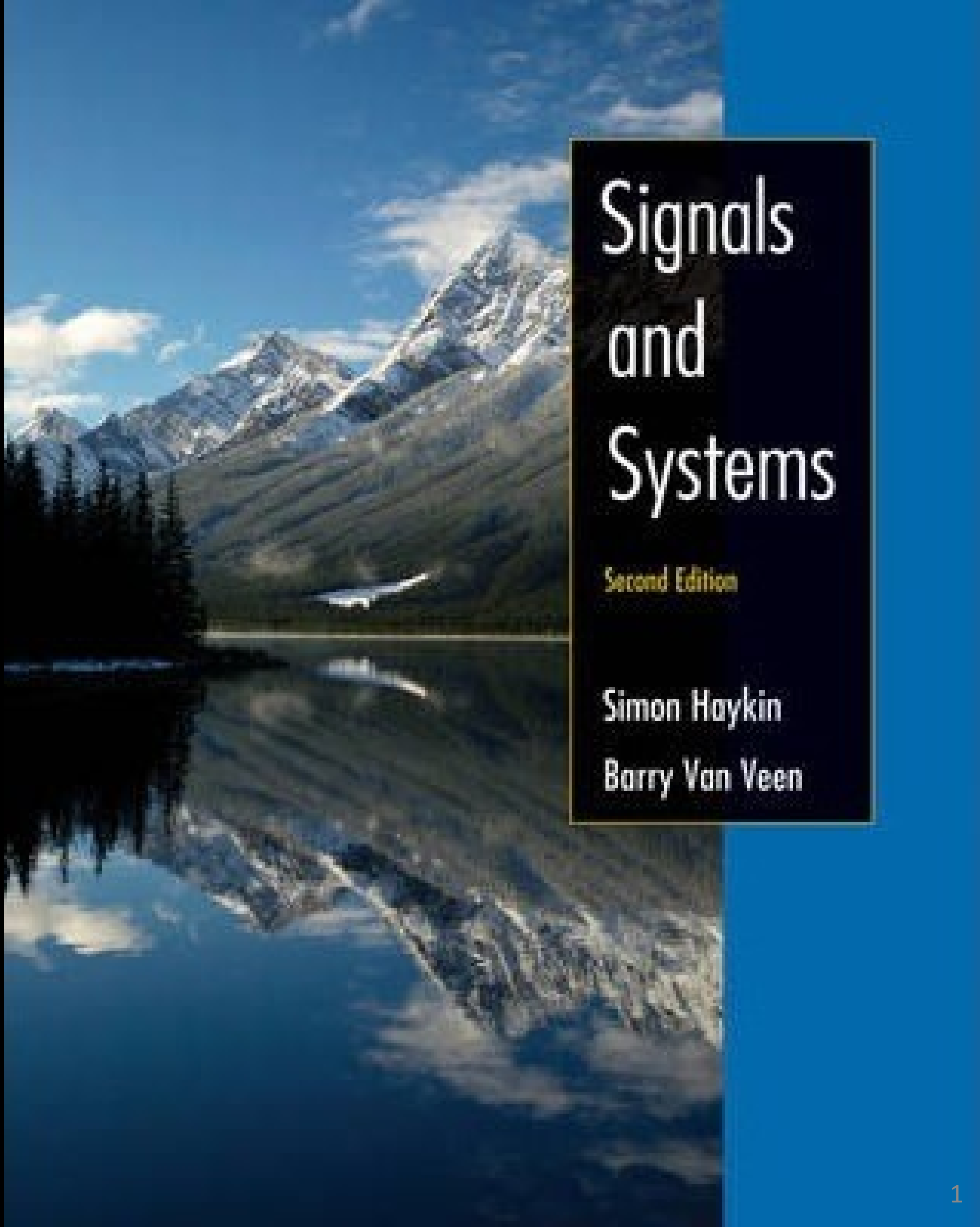
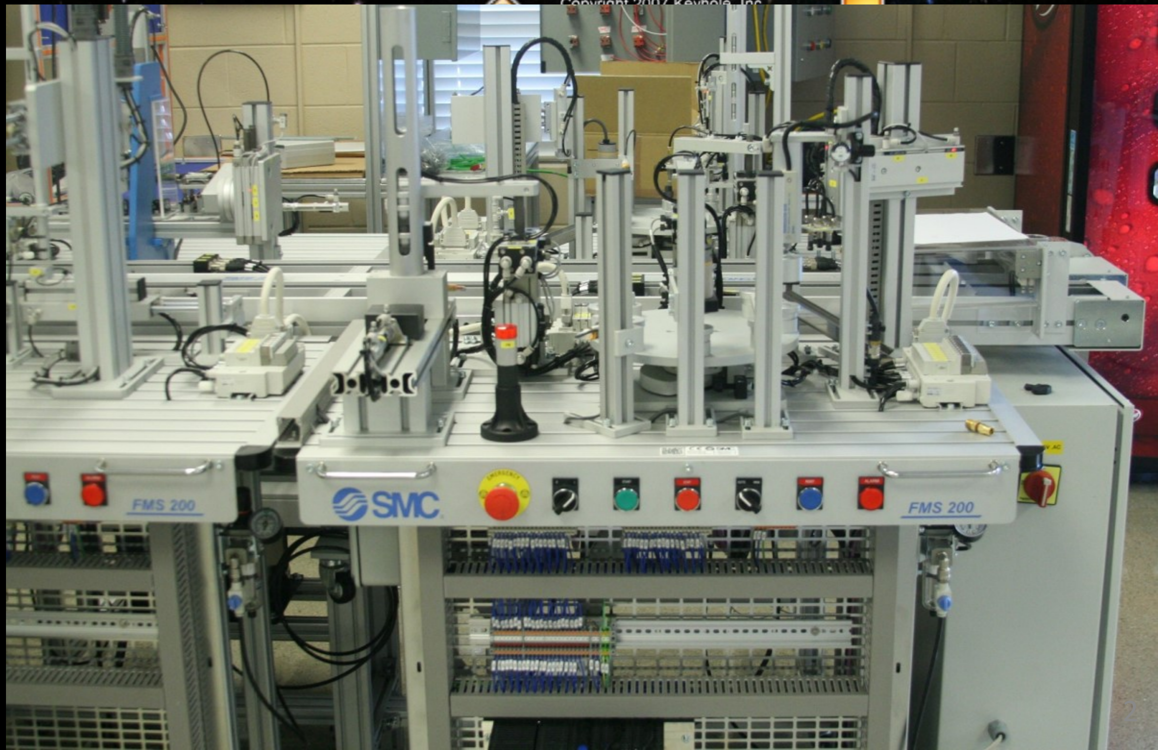


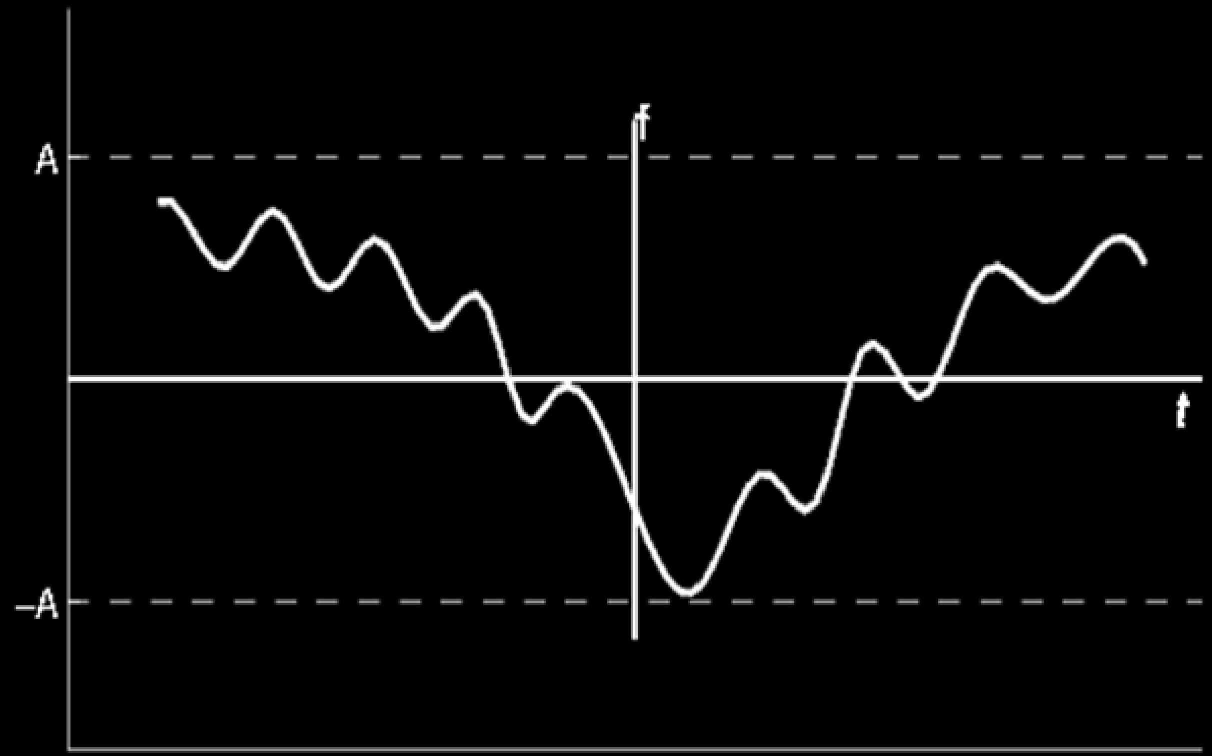
Chapter 1: Introduction

Objectives of this chapter

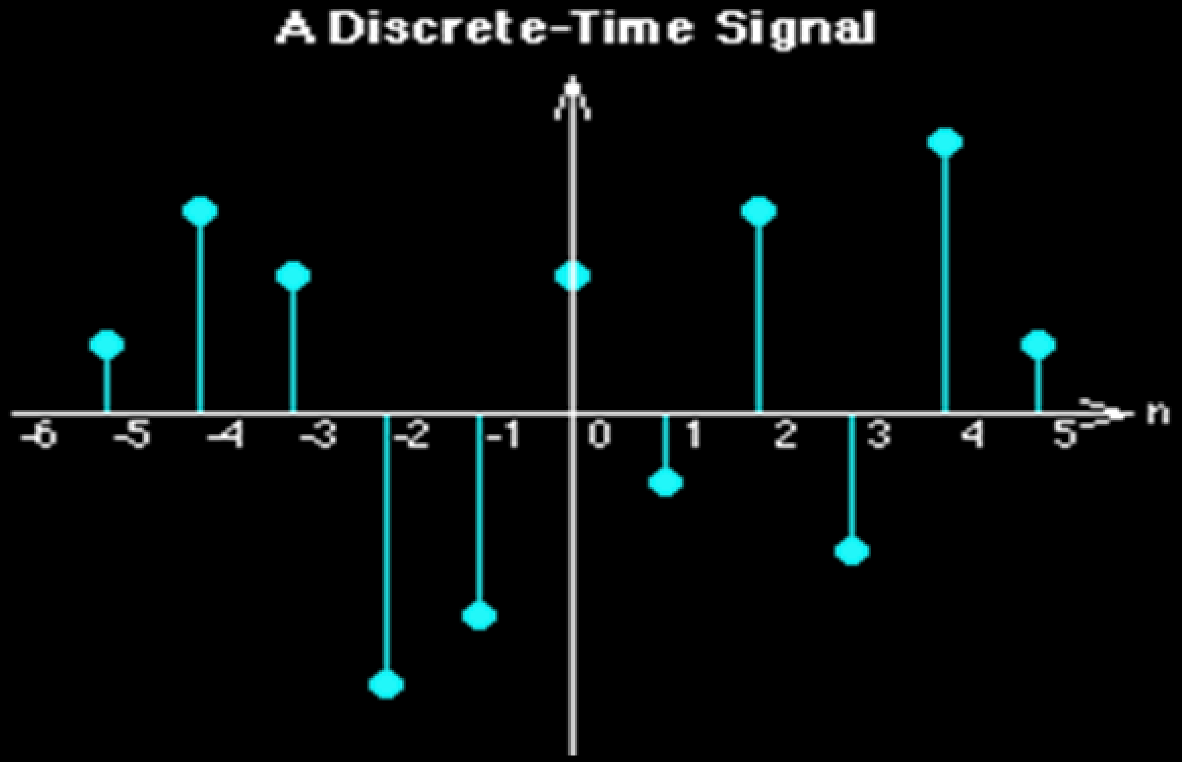


1.4 Classification of Signals

1) Continuous-time and discrete-time signals



Continuous Signal



Discrete Signal

1.4 Classification of Signals

1) Continuous-time and discrete-time signals

- A signal $x(t)$ is said to be a continuous-time signal if it is defined for all time t.
- Figure 1.11 represents an example of a continuous-time signal whose amplitude or value varies continuously with time.
- Continuous-time signals arise naturally when a physical waveform such as an acoustic wave or a light wave is converted into an electrical signal.
- Conversion is effected by a transducer; examples include the microphone, which converts variations in sound pressure into corresponding variations in voltage or current, and the photocell, which does the same for variations in light intensity.

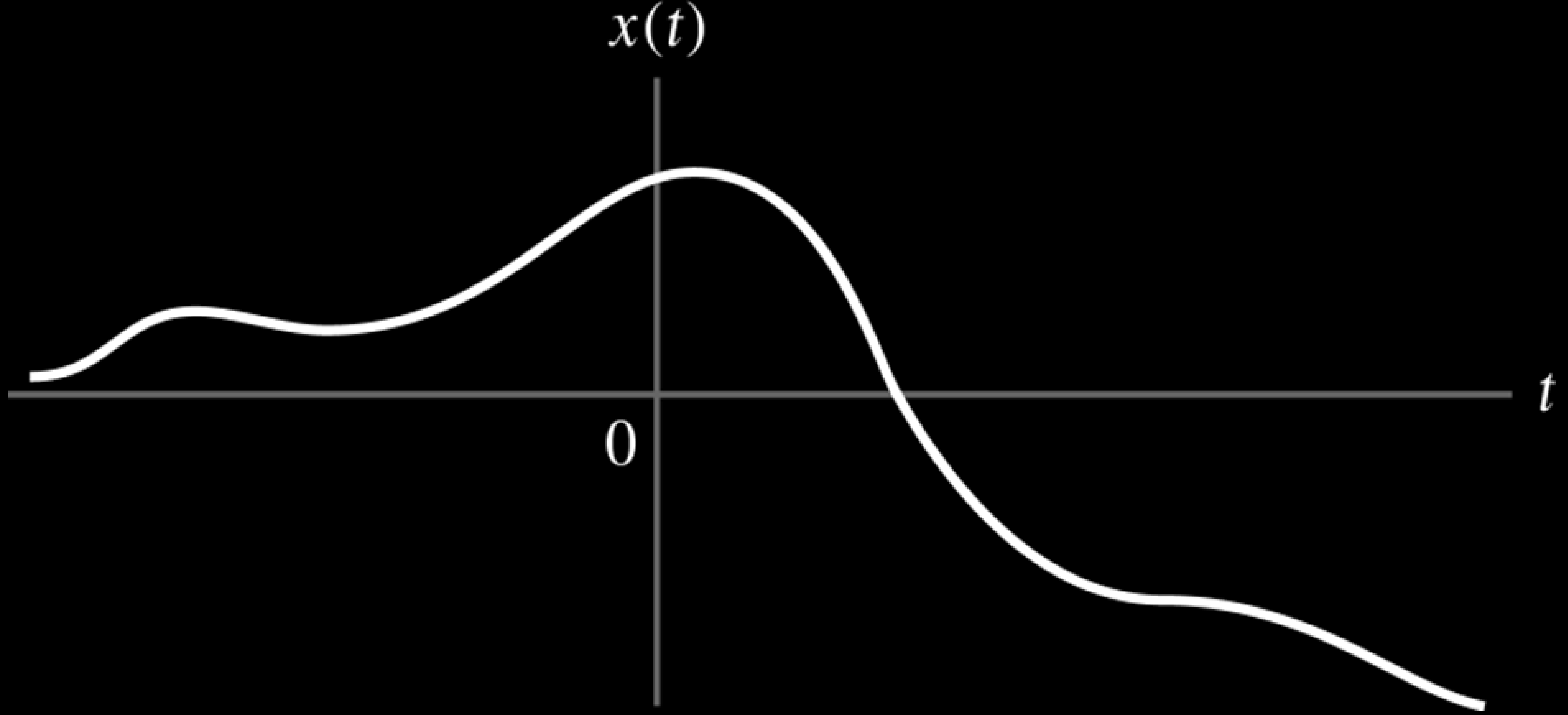


Figure 1.11 (p. 17) Continuous-time signal.

1.4 Classification of Signals

- In contrast, a discrete-time signal is defined only at discrete instants of time.
- Thus, the independent variable has discrete values only
- A discrete-time signal is often derived from a continuous-time signal by sampling it at a uniform rate. Let T_s denote the sampling period and n denote an integer that may assume positive and negative values.
- Then sampling a continuous-time signal x(t) at time $t = nT_s$ yields a sample with the value $x(nT_s)$. For convenience of presentation, we write

$$x[n] = x(nT_s),\ n = 0,\ \pm 1, \pm 2, \qquad (1.1)$$

- Consequently, a discrete-time signal is represented by the sequence of numbers . ... , x[-2], x[-1], x[0], x[1], x[2], … , which can take on a continuum of values.
- Such a sequence of numbers is referred to as a time series, written as {x[n], n = 0, ±1, ±2, … }, or simply x[n]. The latter notation is used throughout this book.

Figure 1.12 illustrates the relationship between a continuous-time signal x(t) and a discrete-time signal x[n] derived from it as described by Eq. (1.1).
Here, Symbol *t* denotes time for a continuous-time signal and symbol *n* denotes time for a discrete-time signal

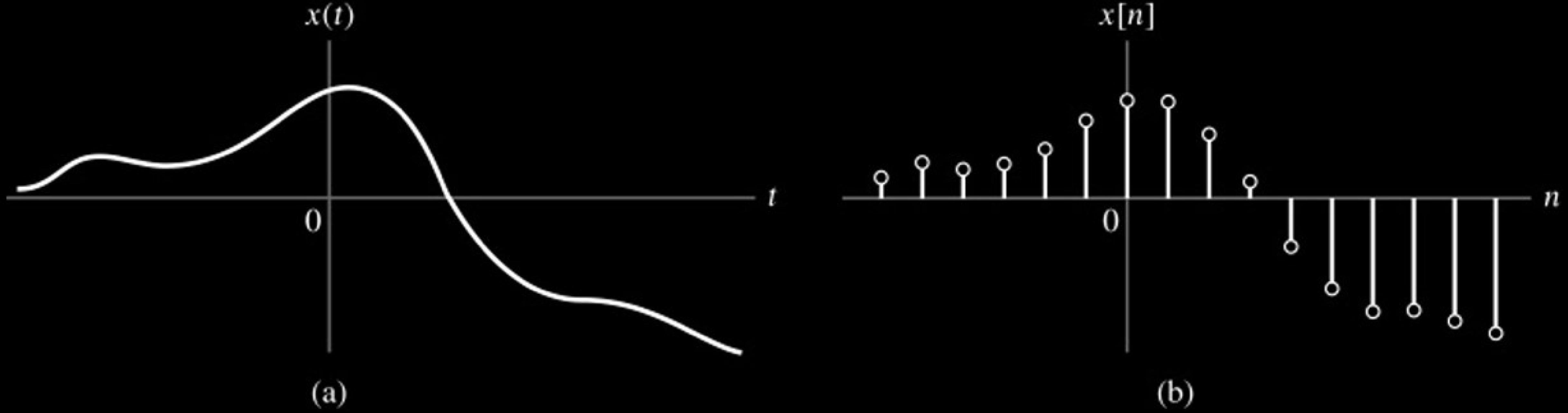
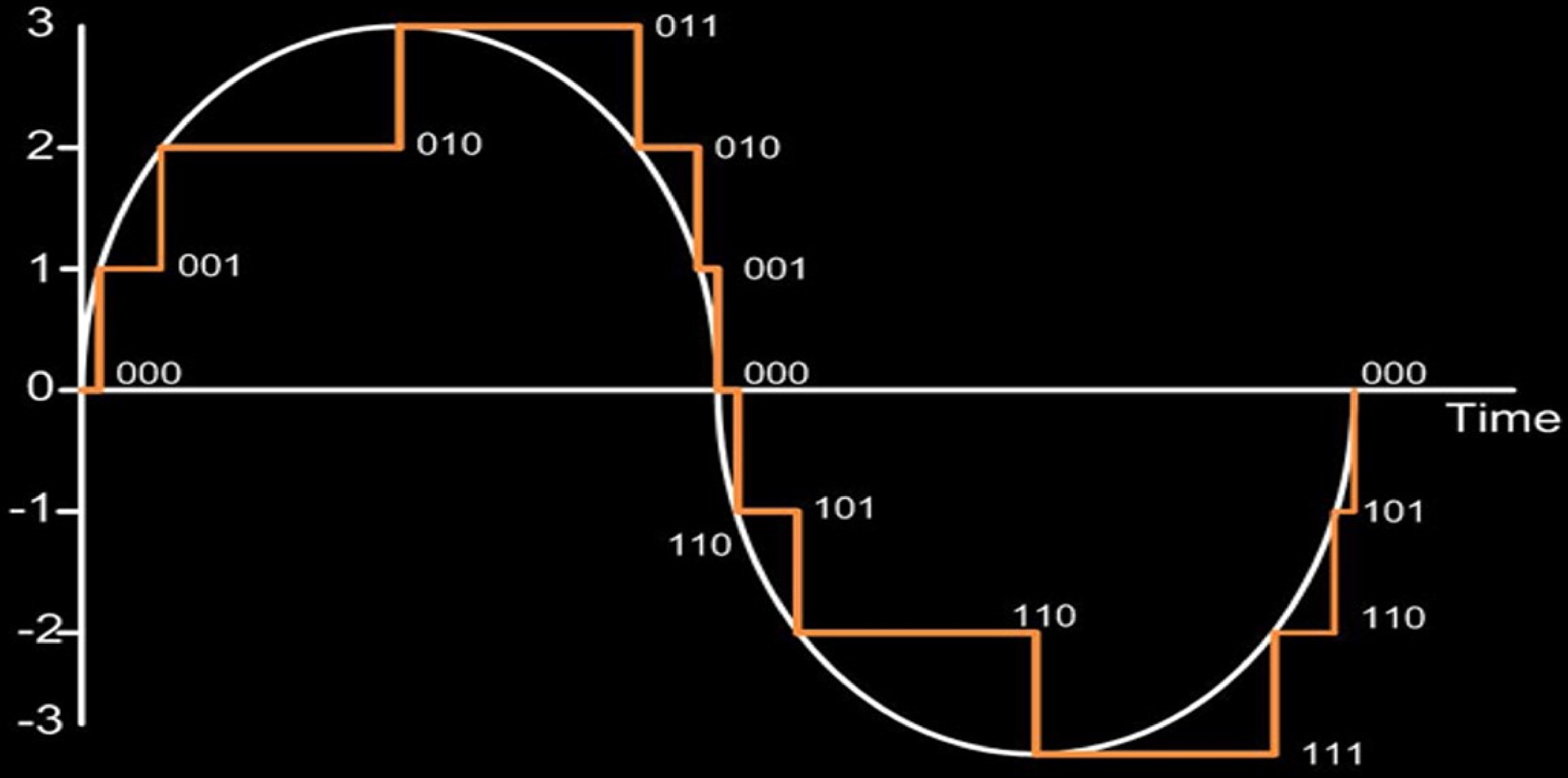


Figure 1.12 (p. 17)
a) Continuous-time signal $x(t)$. b) Representation of $x(t)$ as a discrete-time signal $x[n]$.



Converting a continuous-time signal $x(t)$ into a discrete-time signal $x[n]$

2) Even and odd signals

- Continuous-time signal x(t) is said to be an even signal if

$$x(-t) = x(t) \text{ for all } t \qquad (1\text{-}2)$$

- The signal x(t) is said to be an odd signal if

$$x(-t) = -x(t) \text{ for all t} \qquad (1.3)$$

- In other words, even signals are symmetric about the vertical axis, or time origin, whereas odd signals are antisymmetric about the time origin.

Example 1.1 EVEN AND ODD SIGNALS: Consider the signal

$$x(t) = \begin{cases} \sin\left(\frac{\pi t}{t}\right), & -T \le t \ \le T \\ 0, & \text{othervise} \end{cases}$$

Is the signal $x(t)$ an even or odd function of time t?

Solution: Replacing *t* with –*t* yields

$$x(-t) = \begin{cases} sin\left(-\frac{\pi t}{t}\right), & -T \le t \ \le T \\ 0, & othervise \end{cases} = \begin{cases} -sin\left(\frac{\pi t}{t}\right), & -T \le t \ \le T \\ 0, & othervise \end{cases} = -x(t)$$ for all *t*.

Which satisfied Eq. (1.3). Hence $x(t)$ is an odd signal.

Example 1.2 Another Example of Even and Odd Signals Find the even and odd components of the signal

$$x(t) = e^{-2t}\cos t.$$

Solution: Replacing t with $-t$ in the expression for $x(t)$ yields

$$\begin{aligned} x(-t) &= e^{2t}\cos(-t) \\ &= e^{2t}\cos t. \end{aligned}$$

Hence, applying Eqs. (1.4) and (1.5) to the problem at hand, we get

$$\begin{aligned} x_e(t) &= \frac{1}{2}(e^{-2t}\cos t + e^{2t}\cos t) \\ &= \cosh(2t)\cos t \end{aligned}$$

and

$$\begin{aligned} x_o(t) &= \frac{1}{2}(e^{-2t}\cos t - e^{2t}\cos t) \\ &= -\sinh(2t)\cos t, \end{aligned}$$

where $\cosh(2t)$ and $\sinh(2t)$ respectively denote the hyperbolic cosine and sine of time t.

► **Problem 1.1** Find the even and odd components of each of the following signals:

(a) $x(t) = \cos(t) + \sin(t) + \sin(t)\cos(t)$

(b) $x(t) = 1 + t + 3t^2 + 5t^3 + 9t^4$

(c) $x(t) = 1 + t\cos(t) + t^2\sin(t) + t^3\sin(t)\cos(t)$

(d) $x(t) = (1 + t^3)\cos^3(10t)$

Answers:

(a) Even: $\cos(t)$

Odd: $\sin(t)(1 + \cos(t))$

(b) Even: $1 + 3t^2 + 9t^4$

Odd: $t + 5t^3$

(c) Even: $1 + t^3\sin(t)\cos(t)$

Odd: $t\cos(t) + t^2\sin(t)$

(d) Even: $\cos^3(10t)$

Odd: $t^3\cos^3(10t)$

In the case of a complex-valued signal, we may speak of conjugate symmetry. A complex-
valued signal $x(t)$ is said to be conjugate symmetric if

$$x(-t) = x^*(t) \tag{1.6}$$

where the asterisk denotes complex conjugation.

Let $x(t) = a(t) + jb(t)$, where $a(t)$ is the real part of x{t), $b(t)$ is the imaginary part, and $j=\sqrt{-1}$, then the complex conjugate of $x(t)$ is $x^*(t) = a(t) - jb(t)$.

Substituting $x(t)$ and $x^*(t)$ into Eq. (1.6) yields $a(-t) + jb(-t) = a(t) – jb(t)$.

Equating the real part on the left with that on the right, and similarly for the imaginary parts, we find that $a(-t) = a(t)$ and $b(-t) = -b(t)$. It follows that a complex-valued signal $x(t)$ is conjugate symmetric if its real part is even and its imaginary part is odd. (A similar remark applies to a discrete-time signal).

3. Periodic and Nonperiodic Signals

Fundamental Period of Continuous Time Signals

To identify the period T, the frequency $f = \frac{1}{T}$ or the angular frequency $w = 2\pi f = 2\pi/T$ of a given sinusoidal or complex exponential signal, it is always helpful to write it in any of the following forms

$$\sin(wt) = \sin(2\pi f t) = \sin(2\pi t/T)$$

The fundamental frequency of a signal is the Greatest Common Divisor (GCD) of all the frequency components contained in a signal and equivalently, the fundamental period is the Least Common Multiple (LCM) of all individual periods of the components.

Example 1

Find the fundamental frequency of the following continuous signal $x(t) = cos\left(\frac{10\pi}{3}t\right) + sin\left(\frac{5\pi}{4}t\right)$

The frequencies and periods of the two terms are, respectively,

$$w_1 = \frac{10\pi}{3}, f_1 = \frac{5}{3}, T_1 = \frac{3}{5} \text{ and } w_2 = \frac{5\pi}{4}, f_2 = \frac{5}{8}, T_2 = \frac{8}{5}$$

The fundamental frequency f_0 is the GCD of $f_1 = 5/3$ and $f_2 = 5/8$

$$f_0 = GCD\left(\frac{5}{3}, \frac{5}{8}\right) = GCD\left(\frac{40}{24}, \frac{15}{24}\right) = \frac{5}{24}$$

Alternatively, the period of the fundamental T_0 is the LCM of $T_1 = \frac{3}{5}$ and $T_1 = \frac{8}{5}$

$$T_0 = LCM\left(\frac{3}{5}, \frac{8}{5}\right) = \frac{24}{5}$$

Now we get $w_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{5\pi}{12}$ and the signal can be written as

$$x(t) = cos\left(8\frac{5\pi}{12}t\right) + sin\left(3\frac{5\pi}{12}t\right) = cos(8w_0t) + sin(3w_0t)$$

i.e., the two terms are the 3rd and 8th harmonic of the fundamental frequency w_0, respectively.

Periodicity of Discrete Time Signals

Consider next the discrete-time version of a sinusoidal signal, written as

$$x[n] = A\cos(\Omega n + \phi) \tag{1.39}$$

This discrete-time signal may or may not be periodic. For it to be periodic with a period of, say, N samples, it must satisfy Eq. (1.10) for all integer n and some integer N. Substituting $n + N$ for n in Eq. (1.39) yields

$$x[n + N] = A\cos(\Omega n + \Omega \mathrm{N} + \phi)$$

To satisfy (1.10) we require that $\Omega N = 2\pi k$ radians or

$$\Omega = \frac{2\pi k}{N} \text{ radians/cycle for integer } k, N \tag{1.40}$$

The important point to note here is that, unlike continuous-time sinusoidal signals, not all discrete-time sinusoidal systems with arbitrary values of Ω are periodic. Specifically, for the discrete-time sinusoidal signal described in Eq. (1.39) to be periodic, the angular frequency Ω must be a rational multiple of 2π, as indicated in Eq. (1.40).

$$x[n] = x[n + N] \text{ for integer } n, \tag{1.10}$$

Example 4: **Discrete time sinusoidal signals**. A pair of sinusoidal signals with a common angular frequency is defined by $x_1[n] = \sin[5\pi n]$ and $x_2[n] = \sqrt{3}\cos[5\pi n]$.

a) Both $x_1[n]$ and $x_2[n]$ are periodic. Find the common fundamental period.

b) Express the composite sinusoidal signal $y[n] = x_1[n] + x_2[n]$ in the form of

$$y[n] = Acos(\Omega n + \phi)$$

and evaluate the amplitude A and phase ϕ.

Soln. 4.a) The angular frequency for both $x_1[n]$ and $x_2[n]$ is $\Omega = 5\pi$ radians/cycle. Using $N = \frac{2\pi m}{\Omega} =$

$$\frac{2\pi m}{5\pi} = \frac{2m}{5} \tag{1.40}$$

For $x_1[n]$ and $x_2[n]$ to be periodic, N must be an integer. This can be so only for

$m = 5, 10, 15, \ldots$ which results in $N = 2, 4, 6, \ldots$

b) Recalling the trigonometric identity $Acos(\Omega \mathrm{n})cos(\phi) - Asin(\Omega \mathrm{n})sin(\phi)$ and letting $\Omega = 5\pi$, the RHS of the identity will resemble $x_1[n] + x_2[n]$. We can then write

$Asin(\phi) = -1$ and $Acos(\phi) = \sqrt{3}$. Hence, $\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{amplitude\ of\ x_1[n]}{amplitude\ of\ x_2[n]} = -\frac{1}{\sqrt{3}}$. From here, we find $\phi = -\pi/3$ radians. Substituting for this inequation $Asin(\phi) = -1$ and solving for the amplitude A, we get $A = -\frac{1}{\sin\left(-\frac{\pi}{2}\right)} = 2$. Accordingly, we find $y[n] = 2cos(5\pi n - \pi/3)$.

Example 5: Is the signal $x[n] = \sin(2n)$ periodic signal?

Solution: We can rewrite the signal $x[n] = \sin(2n)$ in the form

$$x[n] = \sin\left(\frac{2\pi n}{\pi}\right) = \sin\left(\frac{2\pi n}{N}\right).$$

There is no k that satisfies $2n = 2\pi k$. Hence the signal is NOT periodic

► **Problem 1.2** The signals $x_1(t)$ and $x_2(t)$ shown in Figs. 1.13(a) and (b) constitute the real and imaginary parts, respectively, of a complex-valued signal $x(t)$. What form of symmetry does $x(t)$ have?

Answer: The signal $x(t)$ is conjugate symmetric. ◄

3. *Periodic signals and nonperiodic signals.*

A *periodic signal* $x(t)$ is a function of time that satisfies the condition

$$x(t) = x(t + T) \qquad \text{for all } t, \tag{1.7}$$

where T is a positive constant. Clearly, if this condition is satisfied for $T = T_0$, say, then it is also satisfied for $T = 2T_0, 3T_0, 4T_0, \ldots$. The smallest value of T that satisfies Eq. (1.7) is called the *fundamental period* of $x(t)$. Accordingly, the fundamental period T defines the duration of one complete cycle of $x(t)$. The reciprocal of the fundamental period T is called the *fundamental frequency* of the periodic signal $x(t)$; it describes how frequently the periodic signal $x(t)$ repeats itself. We thus formally write

$$f = \frac{1}{T}. \tag{1.8}$$

The frequency f is measured in hertz (Hz), or cycles per second. The *angular frequency*, measured in radians per second, is defined by

$$\omega = 2\pi f = \frac{2\pi}{T}, \tag{1.9}$$

since there are 2π radians in one complete cycle. To simplify terminology, ω is often referred to simply as the frequency.

Any signal $x(t)$ for which no value of T satisfies the condition of Eq. (1.7) is called an *aperiodic*, or *nonperiodic, signal*.

Figures 1.14(a) and (b) present examples of periodic and nonperiodic signals, respectively. The periodic signal represents a square wave of amplitude $A = 1$ and period $T = 0.2$ s, and the nonperiodic signal represents a single rectangular pulse of amplitude A and duration T_1.

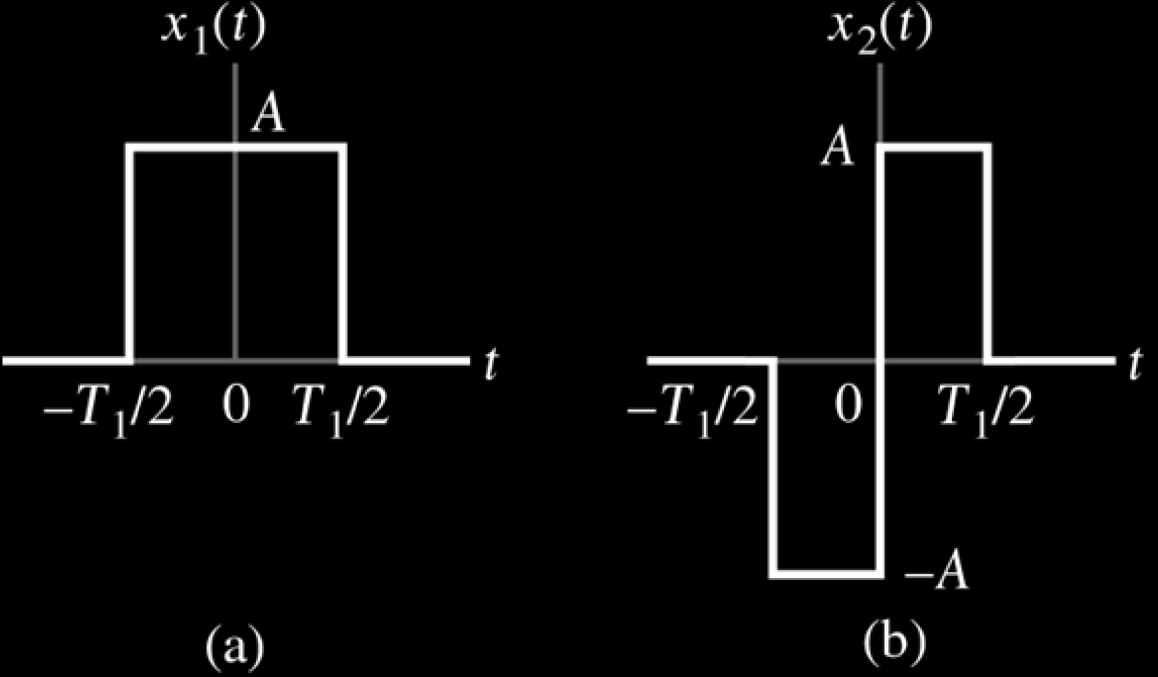


Figure 1.13 (p. 20)
(a) One example of continuous-time signal.
(b) Another example of a continuous-time signal.

► **Problem 1.3** Figure 1.15 shows a triangular wave. What is the fundamental frequency of this wave? Express the fundamental frequency in units of Hz and rad/s.

Answer: 5 Hz, or 10π rad/s. ◄

The classification of signals into periodic and nonperiodic signals presented thus far applies to continuous-time signals. We next consider the case of discrete-time signals. A discrete-time signal $x[n]$ is said to be periodic if

$$x[n] = x[n + N] \qquad \text{for integer } n, \tag{1.10}$$

where N is a positive integer. The smallest integer N for which Eq. (1.10) is satisfied is called the fundamental period of the discrete-time signal $x[n]$. The fundamental angular frequency or, simply, fundamental frequency of $x[n]$ is defined by

$$\Omega = \frac{2\pi}{N}, \tag{1.11}$$

which is measured in radians.

The differences between the defining equations (1.7) and (1.10) should be carefully noted. Equation (1.7) applies to a periodic continuous-time signal whose fundamental period T has any positive value. Equation (1.10) applies to a periodic discrete-time signal whose fundamental period N can assume only a positive integer value.

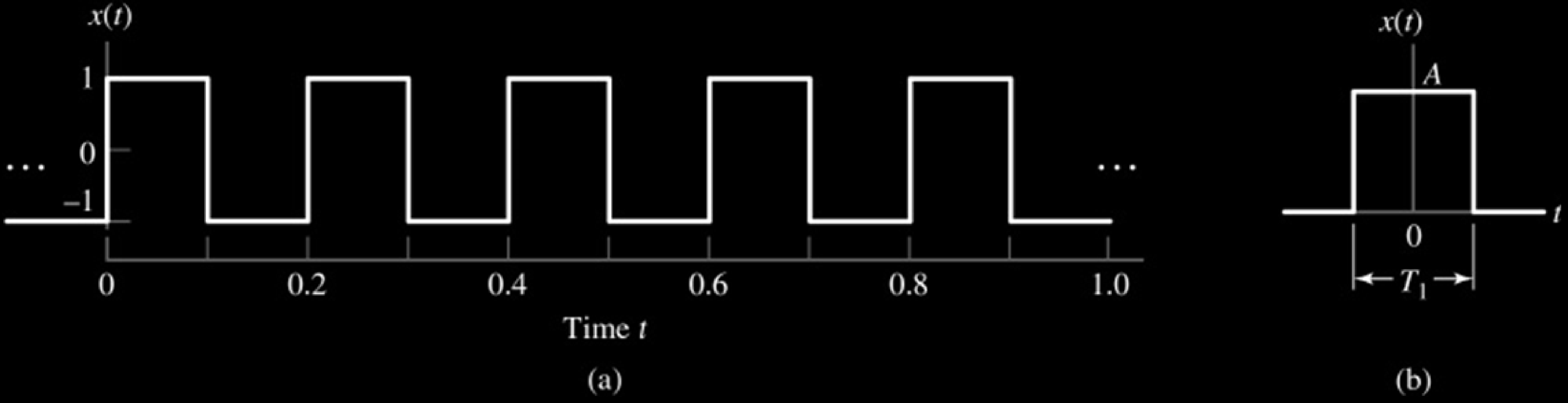


Figure 1.14 (p. 21)
(a) Square wave with amplitude A = 1 and period T = 0.2s.
(b) Rectangular pulse of amplitude A and duration T_1.

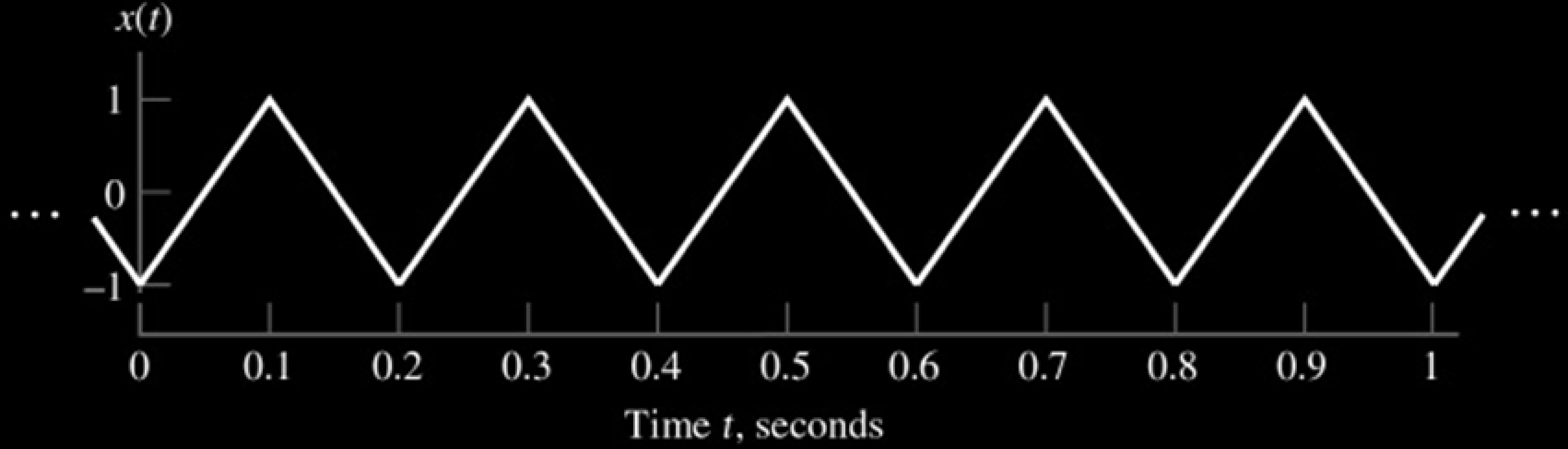


Figure 1.15 (p. 21)
Triangular wave alternative between –1 and +1 for Problem 1.3.

Two examples of discrete-time signals are shown in Figs. 1.16 and 1.17; the signal of Fig. 1.16 is periodic, whereas that of Fig. 1.17 is nonperiodic.

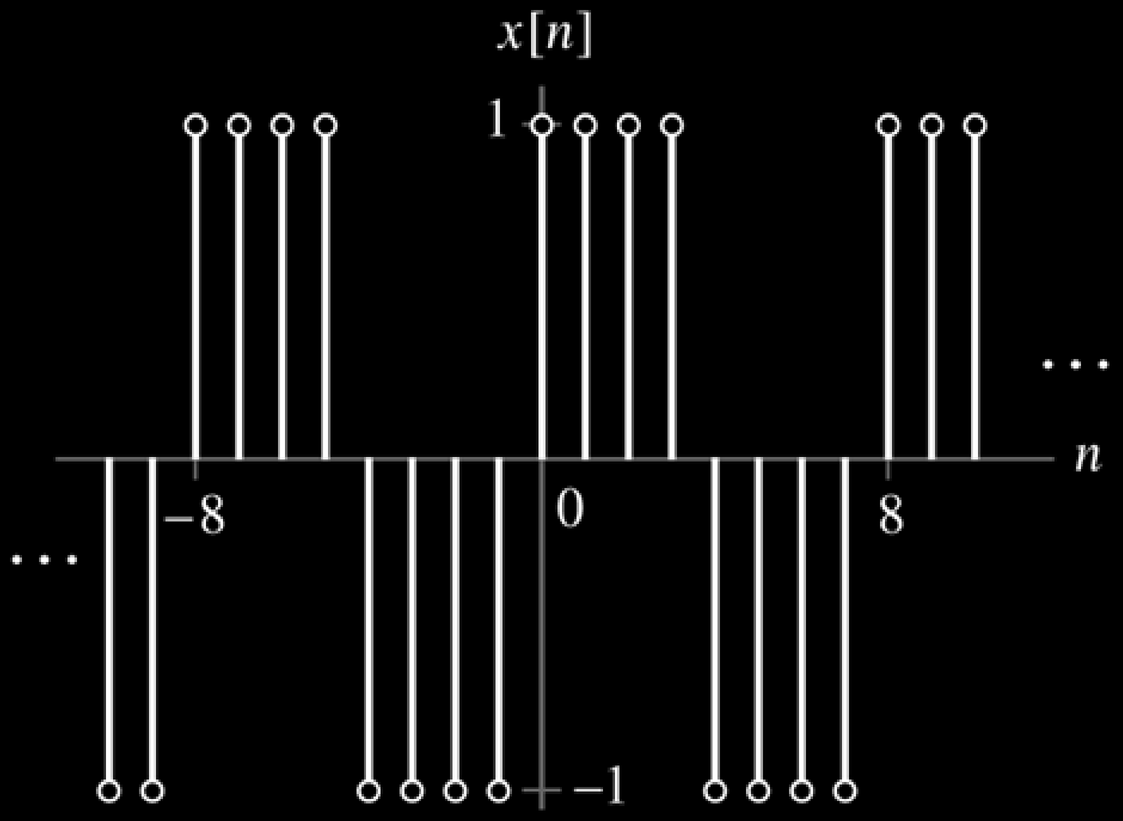


Figure 1.16 (p. 22)
Discrete-time square wave alternative between –1 and +1.

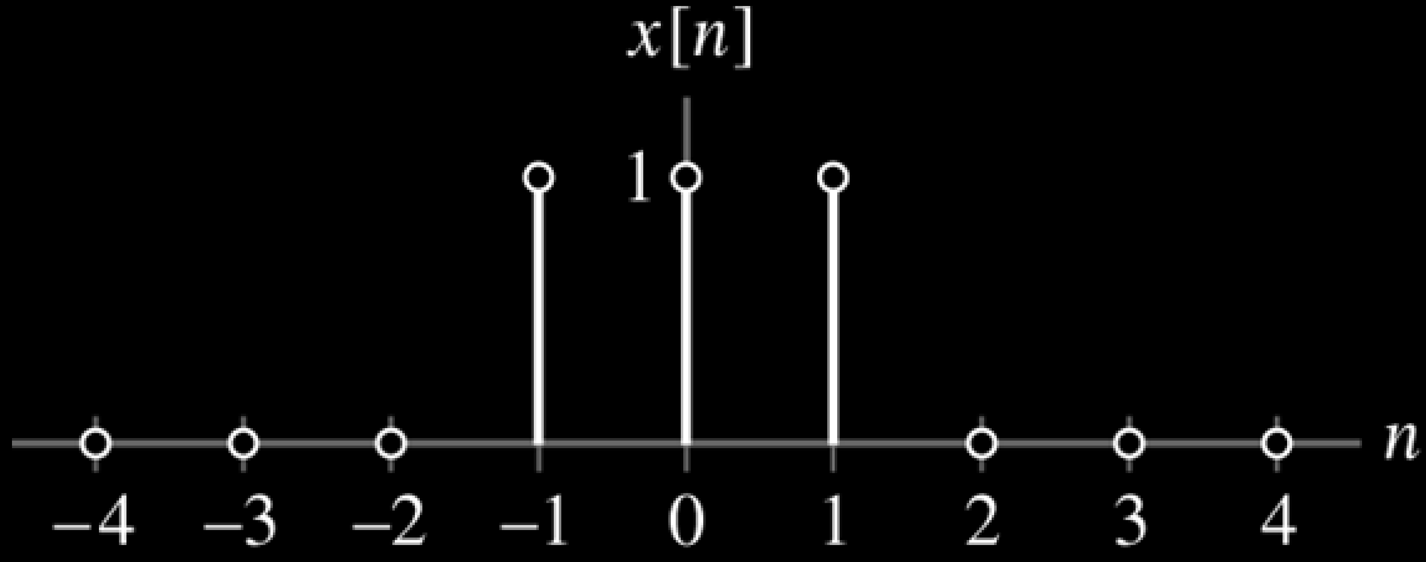


Figure 1.17 (p. 22)
Aperiodic discrete-time signal consisting of three nonzero samples.

▶ **Problem 1.4** Determine the fundamental frequency of the discrete-time square wave shown in Fig. 1.16.

Answer: $\pi/4$ radians. ◀

▶ **Problem 1.5** For each of the following signals, determine whether it is periodic, and if it is, find the fundamental period:

	Answers:
(a) $x(t) = \cos^2(2\pi t)$	(a) Periodic, with a fundamental period of 0.5 s
(b) $x(t) = \sin^3(2t)$	(b) Periodic, with a fundamental period of $(1/\pi)$ s
(c) $x(t) = e^{-2t}\cos(2\pi t)$	(c) Nonperiodic
(d) $x[n] = (-1)^n$	(d) Periodic, with a fundamental period of 2 samples
(e) $x[n] = (-1)^{n^2}$	(e) Periodic, with a fundamental period of 2 samples
(f) $x[n] = \cos(2n)$	(f) Nonperiodic
(g) $x[n] = \cos(2\pi n)$	(g) Periodic, with a fundamental period of 1 sample

4. Deterministic signals and random signals.

- A deterministic signal is a signal about which there is no uncertainty with respect to its value at any time. Deterministic signals may be modeled as completely specified functions of time
- Ex. Square wave in Fig. 1.13 and rectangular pulse in Fig. 1.14 are deterministic
- A random signal is a signal about which there is uncertainty before it occurs.
- The electrical noise generated in the amplifier of a radio or television receiver is an example of a random signal. Its amplitude fluctuates between positive and negative values in a completely random fashion
- Another example of random signal is signal received in a radio comm. system
- This signal consists of an information-bearing component, an interference component and electrical noise generated at front end of radio receiver
- Yet another example of a random signal is the EEG signal waveforms in Fig. 1.9.

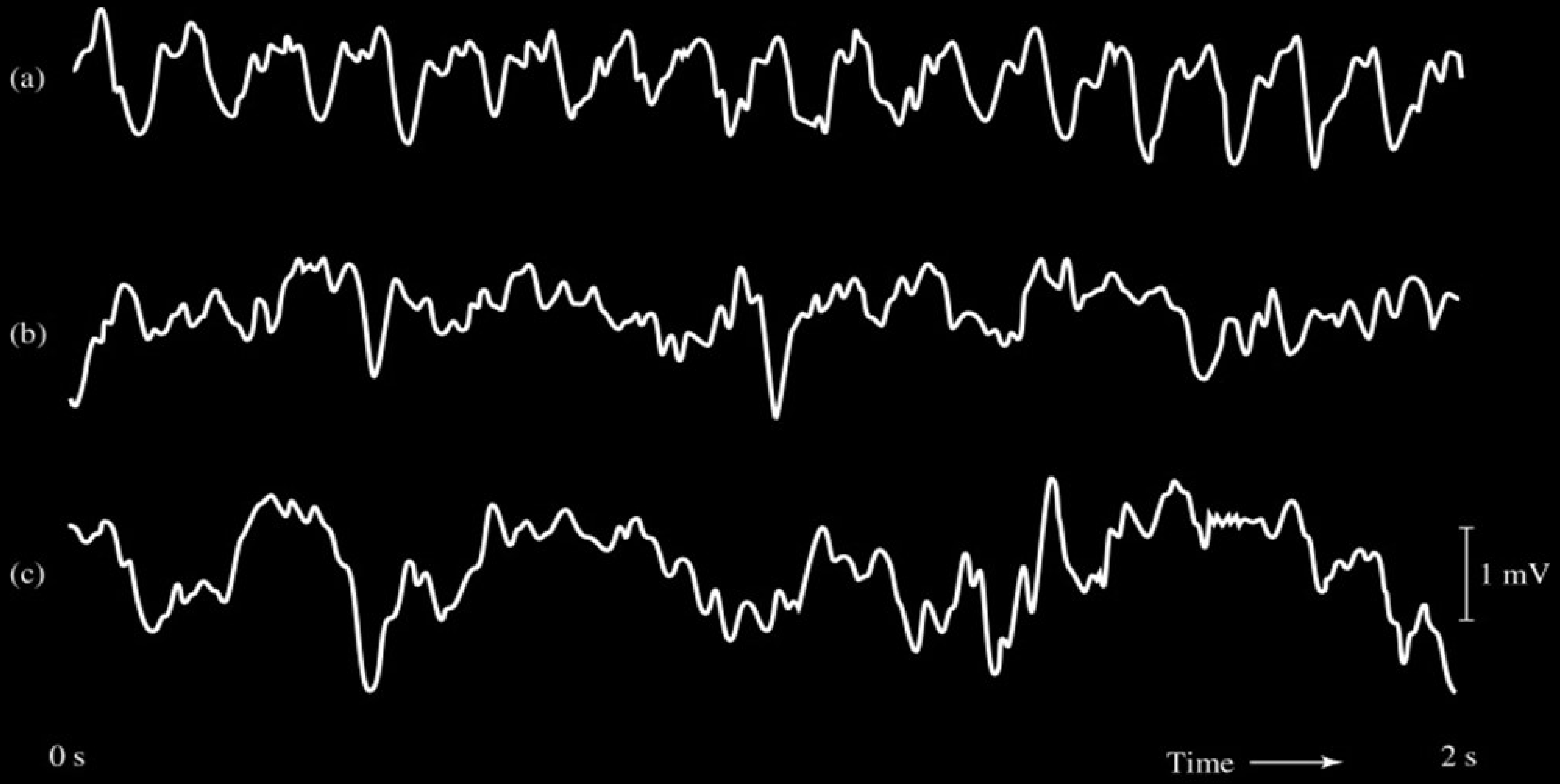


Figure 1.9 (repeated for convenience)
The traces shown in (a), (b), and (c) are three examples of EEG signals recorded from the hippocampus of a rat. Neurobiological studies suggest that the hippocampus plays a key role in certain aspects of learning and memory.

5. Energy signals and Power Signals

- In electrical systems, a signal may represent a voltage or a current.
- Consider a voltage v(t) developed across a resistor R, producing a current i(t)
- The instantaneous power dissipated in this resistor is defined by

$$p(t) = \frac{v^2(t)}{R} = Ri^2(t) \qquad (1.12, 1.13)$$

- In signal analysis, power is defined across a 1-ohm resistor, so, we may express the instantaneous power of the signal as

$$p(t) = x^2(t) \qquad (1.14)$$

- we define the total energy of the continuous-time signal x(t) as

$$\begin{aligned} E &= \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)\, dt \\ &= \int_{-\infty}^{\infty} x^2(t)\, dt \end{aligned} \qquad (1.15)$$

5. Energy signals and Power Signals

- And the average power as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt. \tag{1.16}$$

- From Eq. (1.16), we see that time-averaged power of a periodic signal x(t) of fundamental period T is given by

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt. \tag{1.17}$$

- Square root of average power P is called the roof mean-square (rms) value of the periodic signal x(t).
- In the case of a discrete-time signal x[n], integrals in Eqs. (1.15) and (1.16) are replaced by corresponding sums.

5. Energy signals and Power Signals

- Thus, the total energy of x[n] is defined by

$$E = \sum_{n=-\infty}^{\infty} x^2[n], \tag{1.18}$$

- And its average power is defined by

$$P = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} x^2[n]. \tag{1.19}$$

- From Eq. (1.19), the average power in a periodic signal x[n] with fundamental period N is given by

$$P = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]. \tag{1.20}$$

5. Energy signals and Power Signals

- A signal is referred to as an energy signal if and only if the total energy of the signal satisfies the condition

$$0 < E < \infty$$

- The signal is referred to as a power signal if and only if the average power of the signal satisfies the condition

$$0 < P < \infty$$

- The energy and power classifications of signals are mutually exclusive. In particular an energy signal has zero time-averaged power, whereas a power signal has infinite energy.
- Periodic signals and random signals are power signals, whereas deterministic and nonperiodic signals are energy signals

5. Energy signals and Power Signals

Problem 1.6:

(a) What is the total energy of the rectangular pulse shown in Fig. 1.14(b)?

(b) What is the average power of the square wave shown in Fig. 1.14(a)?

Answers; (a) A^2T_1 (b) 1.

Problem 1.7

Determine the average power of the triangular wave shown in Fig. 1.15.

Answer: 1/3

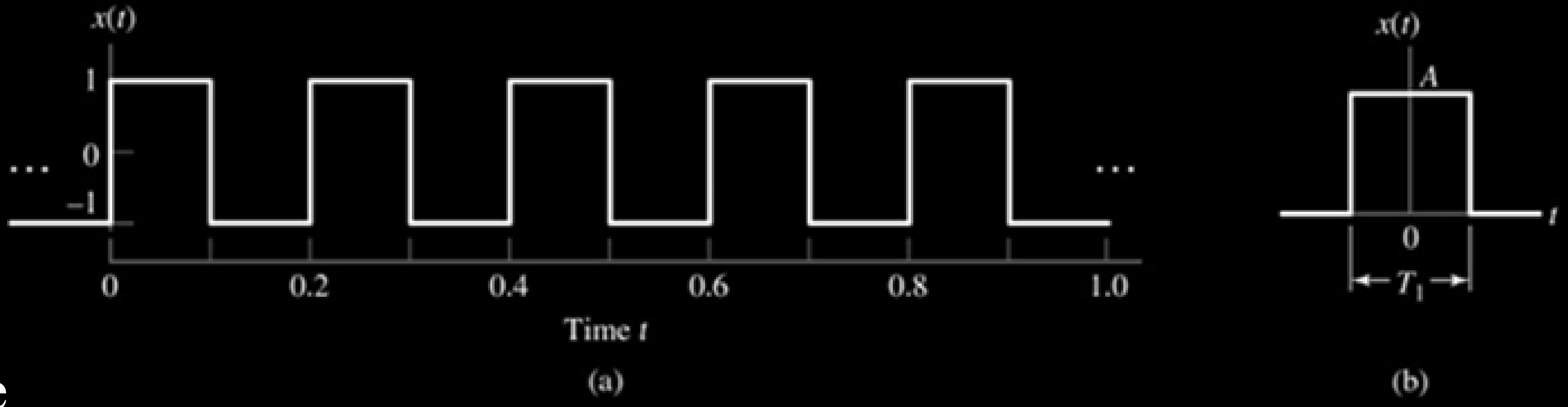


Figure 1.14 (p. 21)
(a) Square wave with amplitude A = 1 and period T = 0.2s.
(b) Rectangular pulse of amplitude A and duration T_1.

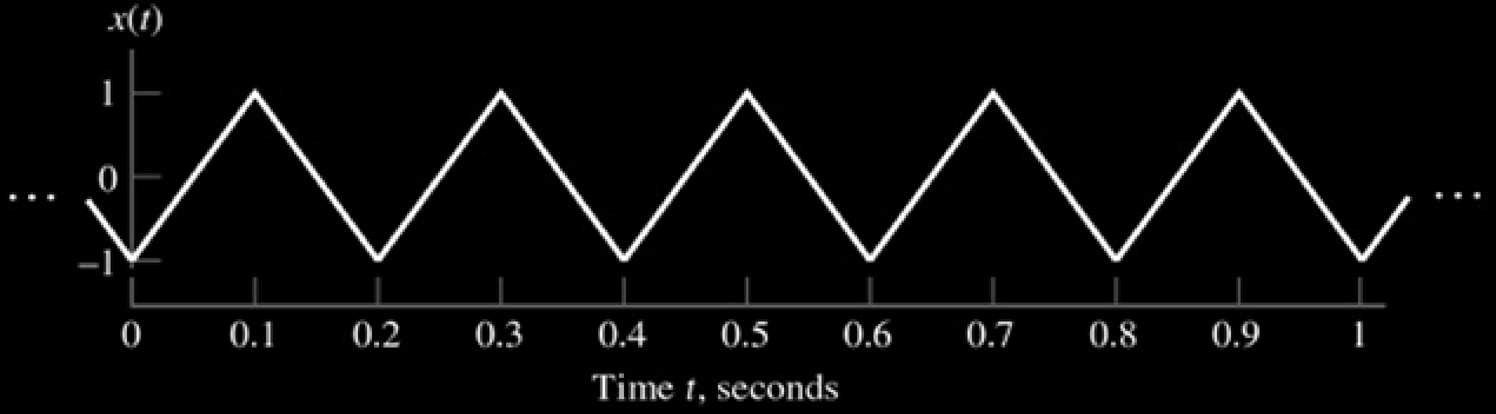


Figure 1.15 (p. 21)
Triangular wave alternative between –1 and +1 for Problem 1.3.

5. Energy signals and Power Signals

Problem 1.8
Determine the total energy of the discrete-time signal shown in Fig. 1.17.
Answer: 3 4

Problem 1.9
Categorize each of the following signals as an energy signal or a power signal, and find the energy or time-averaged power of the signal:

(a) $x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2 - t, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$

(b) $x[n] = \begin{cases} n, & 0 \le n < 5 \\ 10 - n, & 5 \le n \le 10 \\ 0, & \text{otherwise} \end{cases}$

(c) $x(t) = 5\cos(\pi t) + \sin(5\pi t), -\infty < t < \infty$

(d) $x(t) = \begin{cases} 5\cos(\pi t), & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$

(e) $x(t) = \begin{cases} 5\cos(\pi t), & -0.5 \le t \le 0.5 \\ 0, & \text{otherwise} \end{cases}$

(f) $x[n] = \begin{cases} \sin(\pi n), & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

(g) $x[n] = \begin{cases} \cos(\pi n), & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

(h) $x[n] = \begin{cases} \cos(\pi n), & n \ge 0 \\ 0, & \text{otherwise} \end{cases}$

Answers:

(a) Energy signal, energy $= \frac{2}{3}$
(b) Energy signal, energy $= 85$
(c) Power signal, power $= 13$
(d) Energy signal, energy $= 25$
(e) Energy signal, energy $= 12.5$
(f) Zero signal
(g) Energy signal, energy $= 9$
(h) Power signal, power $= \frac{1}{2}$

1.5 Basic Operations on Signals

- Two classes of operations on signals are
  - operations performed on dependent variables and
  - operations performed on independent variables

Amplitude scaling

Let x(t) denote a continuous-time signal. Then the signal y(t) resulting from amplitude scaling applied to x(t) is defined by

$$y(t) = cx(t) \tag{1.21}$$

- where c is the scaling factor.
- Device that performs amplitude scaling are electronic amplifier and resistor.
- In a manner similar to Eq. (1.21), for discrete-time signals, we write

$$y[n] = cx[n]$$

1.5 Basic Operations on Signals

Addition

- Let $x_1(t)$ and $x_2(t)$ denote a pair of continuous-time signals. Then the signal $y(t)$ obtained by the addition of $x_1(t)$ and $x_2(t)$ is defined by

$$y(t) = x_1(t) + x_2(t) \tag{1.22}$$

- Example device that adds signals is an audio mixer (combines music and voice)
- For discrete-time signals, we have

$$y[n] = x_1[n] + x_2[n]$$

Multiplication

- The multiplication of a pair of continuous-time signals $x_1(t)$ and $x_2(t)$

$$y(t) = x_1(t)x_2(t) \tag{1.23}$$

- Example of $y(t)$ is an AM radio signal
- For discrete-time signals, we write $y[n] = x_1[n]x_2[n]$

1.5 Basic Operations on Signals

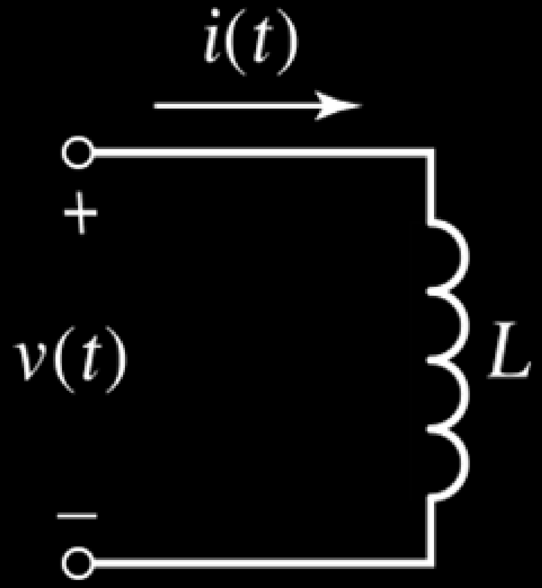


Figure 1.18 (p. 26) Inductor with current *i*(*t*), inducing voltage *v*(*t*) across its terminals.

Differentiation.

- Let $x(t)$ denote a continuous-time signal. Then the derivative of $x(t)$ with respect to time is defined by

$$y(t) = \frac{d}{dt}x(t) \qquad (1.24)$$

Example: An inductor performs differentiation.

Let $i(t)$ denote the current flowing through an inductor of inductance L, as shown in Fig. 1.18. Then the voltage $v(t)$ developed across the inductor is defined by

$$v(t) = L\frac{d}{dt}i(t) \qquad (1.25)$$

1.5 Basic Operations on Signals

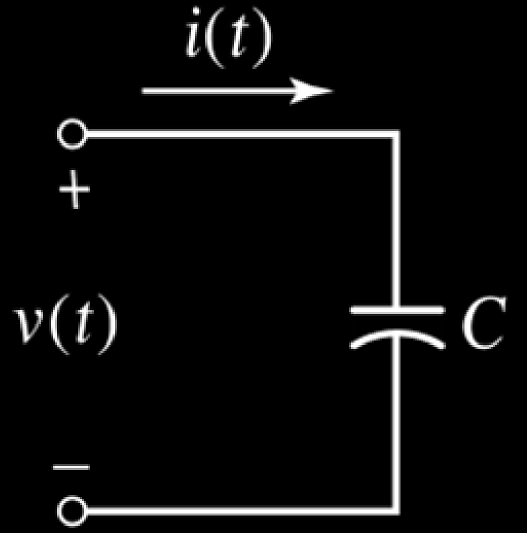


Figure 1.19 (p. 27)
Capacitor with voltage $v(t)$ across its terminals, inducing current $i(t)$.

Integration

- Let $x(t)$ denote a continuous-time signal. Then the integral of $x(t)$ with respect to time t is defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \qquad (1.26)$$

- A Capacitor performs integration.
- Let $i(t)$ denote the current flowing through a capacitor of capacitance C, as shown in Fig. 1.19. Then the voltage v(t) developed across the capacitor is defined by

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \qquad (1.27)$$

1.5.2 Operations Performed on the Independent Variable

Time scaling

- Let $x(t)$ denote a continuous-time signal. Then the signal $y(t)$ obtained by scaling the independent variable, time t, by a factor a is defined by

$$y(t) = x(at)$$

- Compressed (a > 1), expanded (0 < a < 1) version of $x(t)$ shown in Fig. 1.20
- In the discrete-time case, we write $y[n] = x[kn], \quad k > 0,$

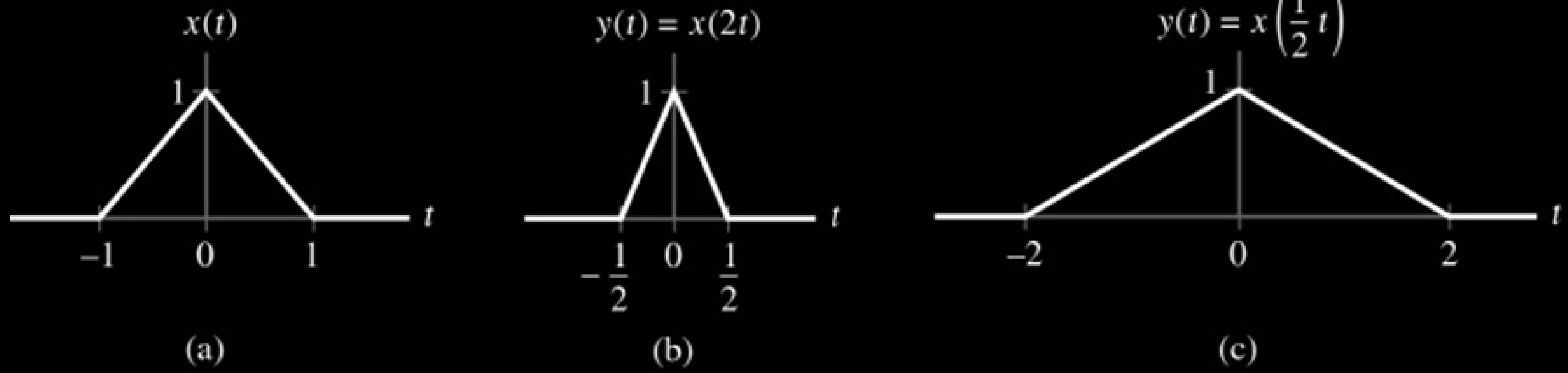


Figure 1.20 (p. 27): Time-scaling operation; (a) continuous-time signal $x(t)$, (b) version of $x(t)$ compressed by a factor of 2, and (c) version of $x(t)$ expanded by a factor of 2.

1.5.2 Operations Performed on the Independent Variable

- $y[n] = x[kn]$ is defined only for integer values of k. If $k > 1$, then some values of the discrete-time signal $y[n]$ are lost, as illustrated in Fig. 1.21 for $k = 2$
- The samples $x[n]$ for $n = \pm 1, \pm 3, \ldots$ are lost because putting $k = 2$ in $x[kn]$ causes these samples to be skipped.

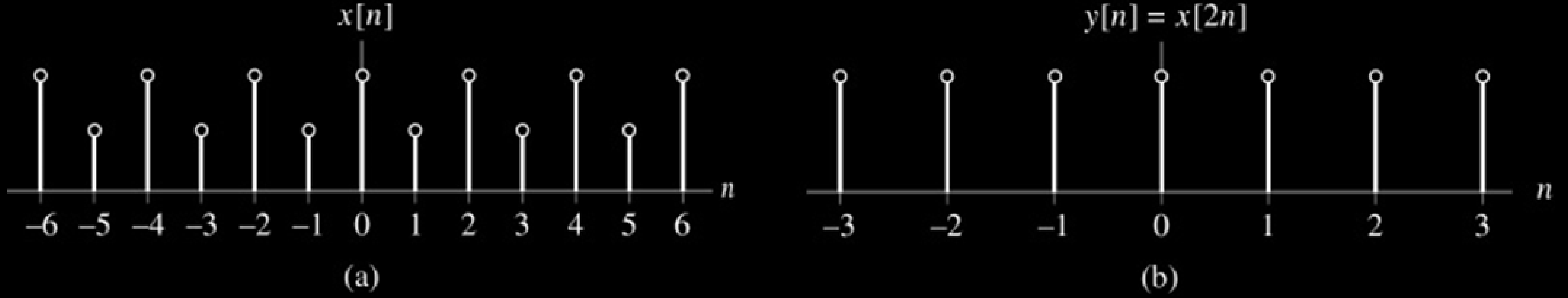


Figure 1.21 (p. 28)
Effect of time scaling on a discrete-time signal: (a) discrete-time signal $x[n]$ and (b) version of $x[n]$ compressed by a factor of 2, with some values of the original $x[n]$ lost as a result of the compression.

▶ **Problem 1.10** Let

$$x[n] = \begin{cases} n & \text{for } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Determine $y[n] = x[2n]$.

Answer: $y[n] = 0$ for all n. ◀

1.5.2 Operations Performed on the Independent Variable

Reflection

- Let $x(t)$ denote a continuous-time signal. Let $y(t)$ denote the signal obtained by replacing time t with $-t$; that is,

$$y(t) = x(-t)$$

- The signal $y(t)$ represents a reflected version of $x(t)$ about $t = 0$.
- For even signals, where $x(t) = x(t)$ for all t; signal is same as its reflected version
- For odd signals, where $x(-t) = -x(t)$ for all t; signal is negative of its reflection

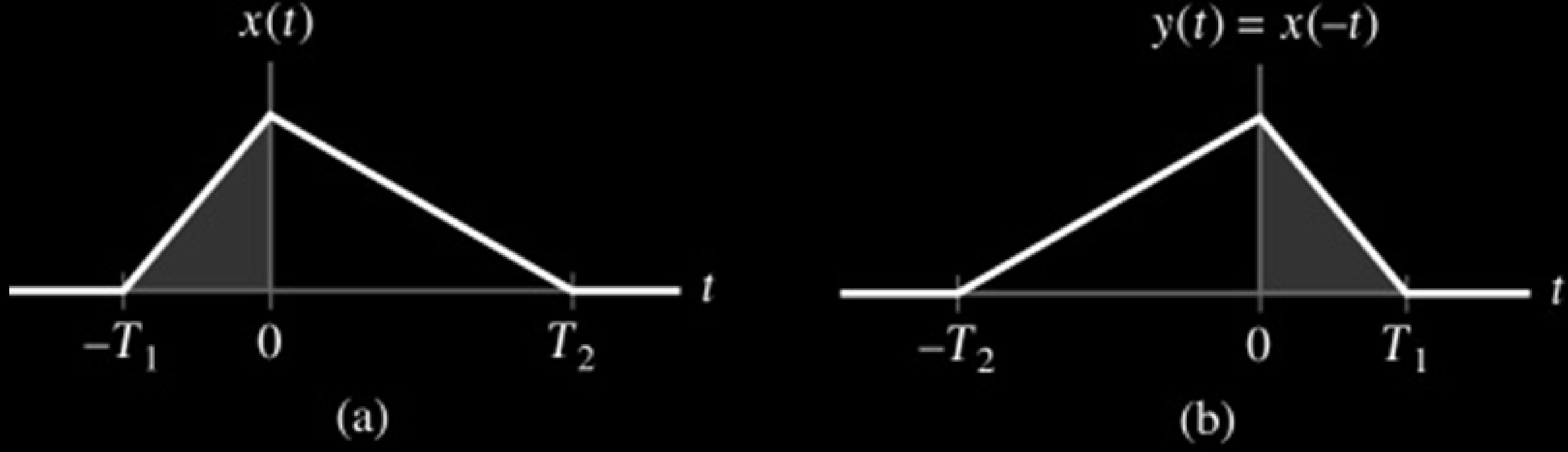


Figure 1.22 (p. 28)
Operation of reflection: (a) continuous-time signal $x(t)$ and (b) reflected version of $x(t)$ about the origin.

EXAMPLE 1.3 REFLECTION Consider the triangular pulse $x(t)$ shown in Fig. 1.22(a). Find the reflected version of $x(t)$ about the amplitude axis (i.e., the origin).

Solution: Replacing the independent variable t in $x(t)$ with $-t$, we get $y(t) = x(-t)$, as shown in the figure.

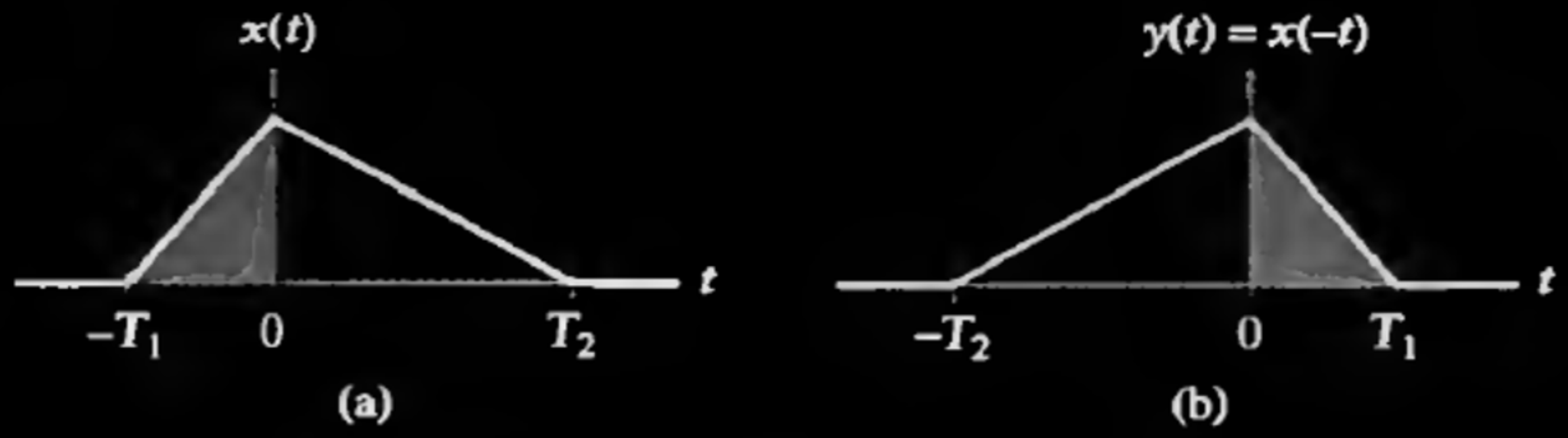


FIGURE 1.22 Operation of reflection: (a) continuous-time signal $x(t)$ and (b) reflected version of $x(t)$ about the origin.

Note that for this example, we have

$$x(t) = 0 \quad \text{for } t < -T_1 \text{ and } t > T_2.$$

Correspondingly, we find that

$$y(t) = 0 \quad \text{for } t > T_1 \text{ and } t < -T_2. \qquad \blacksquare$$

▶ **Problem 1.11** The discrete-time signal

$$x[n] = \begin{cases} 1, & n = 1 \\ -1, & n = -1 \\ 0, & n = 0 \text{ and } |n| > 1 \end{cases}.$$

Find the composite signal

$$y[n] = x[n] + x[-n].$$

Answer: $y[n] = 0$ for all integer values of n.

▶ **Problem 1.12** Repeat Problem 1.11 for

$$x[n] = \begin{cases} 1, & n = -1 \text{ and } n = 1 \\ 0, & n = 0 \text{ and } |n| > 1 \end{cases}.$$

Answer: $y[n] = \begin{cases} 2, & n = -1 \text{ and } n = 1 \\ 0, & n = 0 \text{ and } |n| > 1 \end{cases}.$

1.5.2 Operations Performed on the Independent Variable

Time shifting

- Let $x(t)$ be a continuous-time signal. Its time-shifted version is defined by

$$y(t) = x(t - t_o)$$

- where t_o is the time shift. If $t_o > 0$, the waveform of $y(t)$ is obtained by shifting $x(t)$ right, relative to the time axis. If $t_o < 0$, $x(t)$ is shifted to the left
- In the case of a discrete-time signal $x[n]$, time-shifted version is

$$y[n] = x[n - m]$$

- where the shift m must be a positive or negative integer.

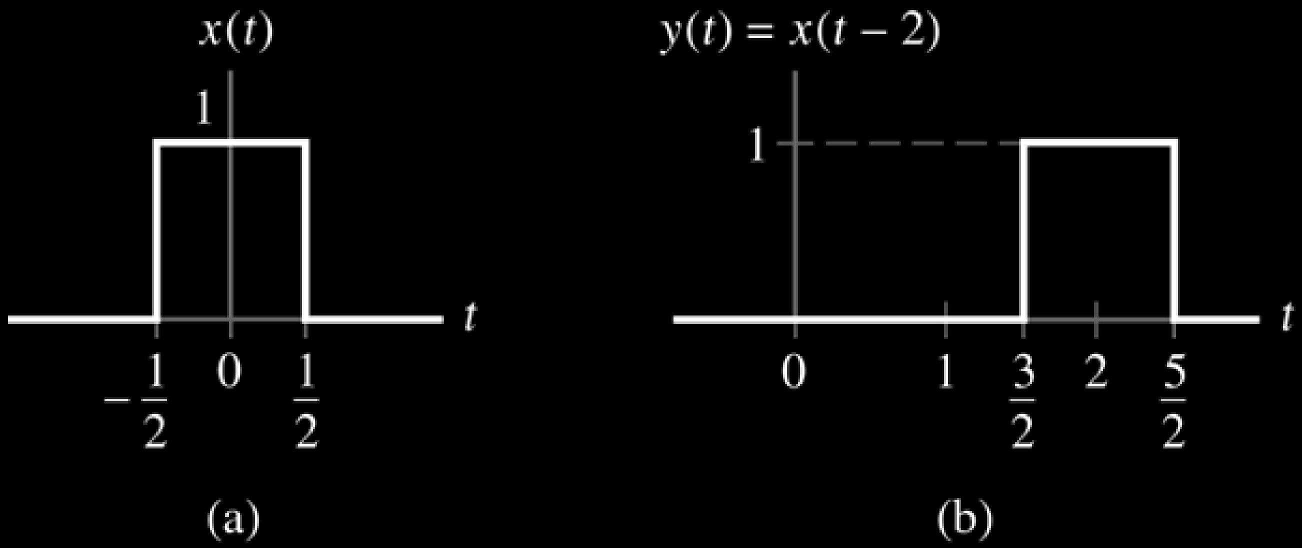


Figure 1.23 (p. 29)
Time-shifting operation: (a) continuous-time signal in the form of a rectangular pulse of amplitude 1.0 and duration 1.0, symmetric about the origin; and (b) time-shifted version of $x(t)$ by 2 time shifts.

EXAMPLE 1.4 TIME SHIFTING Figure 1.23(a) shows a rectangular pulse $x(t)$ of unit amplitude and unit duration. Find $y(t) = x(t-2)$.

Solution: In this example, the time shift t_0 equals 2 time units. Hence, by shifting $x(t)$ to the right by 2 time units, we get the rectangular pulse $y(t)$ shown in the figure. The pulse $y(t)$ has exactly the same shape as the original pulse $x(t)$; it is merely shifted along the time axis. ■

▶ **Problem 1.13** The discrete-time signal

$$x[n] = \begin{cases} 1, & n = 1, 2 \\ -1, & n = -1, -2 \\ 0, & n = 0 \text{ and } |n| > 2 \end{cases}$$

Find the time-shifted signal $y[n] = x[n+3]$.

Answer: $$y[n] = \begin{cases} 1, & n = -1, -2 \\ -1, & n = -4, -5 \\ 0, & n = -3, n < -5, \text{and } n > -1 \end{cases}.$$

1.5.3 Precedence Rule for Time Shifting and Time Scaling

- Let $y(t)$ denote a continuous-time signal derived from another continuous-time signal $x(t)$ through a combination of time shifting and time scaling;

$$y(t) = x(at - b) \tag{1.28}$$

- This relation between y(t) and x(t) satisfies the conditions and

$$y(0) = x(-b) \tag{1.29}$$

and

$$y\left(\frac{b}{a}\right) = x(0) \tag{1.30}$$

- which provide useful checks on y(t) in terms of corresponding values of x(t)

EXAMPLE 1.5 PRECEDENCE RULE FOR CONTINUOUS-TIME SIGNAL Consider the rectangular pulse $x(t)$ of unit amplitude and a duration of 2 time units, depicted in Fig. 1.24(a). Find $y(t) = x(2t + 3)$.

Solution: In this example, we have $a = 2$ and $b = -3$. Hence, shifting the given pulse $x(t)$ to the left by 3 time units relative to the amplitude axis gives the intermediate pulse $v(t)$ shown in Fig. 1.24(b). Finally, scaling the independent variable t in $v(t)$ by $a = 2$, we get the solution $y(t)$ shown in Fig. 1.24(c).

Note that the solution presented in Fig. 1.24(c) satisfies both of the conditions defined in Eqs. (1.29) and (1.30).

Suppose next that we purposely do not follow the precedence rule; that is, we first apply time scaling and then time shifting. For the given signal $x(t)$ shown in Fig. 1.25(a), the application of time scaling by factor of 2 produces the intermediate signal $v(t) = x(2t)$, which is shown in Fig. 1.25(b). Then shifting $v(t)$ to the left by 3 time units yields the signal shown in Fig. 1.25(c), which is defined by

$$y(t) = v(t + 3) = x(2(t + 3)) \neq x(2t + 3)$$

Hence, the signal $y(t)$ fails to satisfy Eq. (1.30). ■

Figure 1.24 (p. 31)
The proper order in which the operations of time scaling and time shifting should be applied in the case of the continuous-time signal of Example 1.5.
(a) Rectangular pulse $x(t)$ of amplitude 1.0 and duration 2.0, symmetric about origin
(b) Intermediate pulse $v(t)$, representing a time-shifted version of $x(t)$.
(c) Desired signal $y(t)$, resulting from the compression of $v(t)$ by a factor of 2.

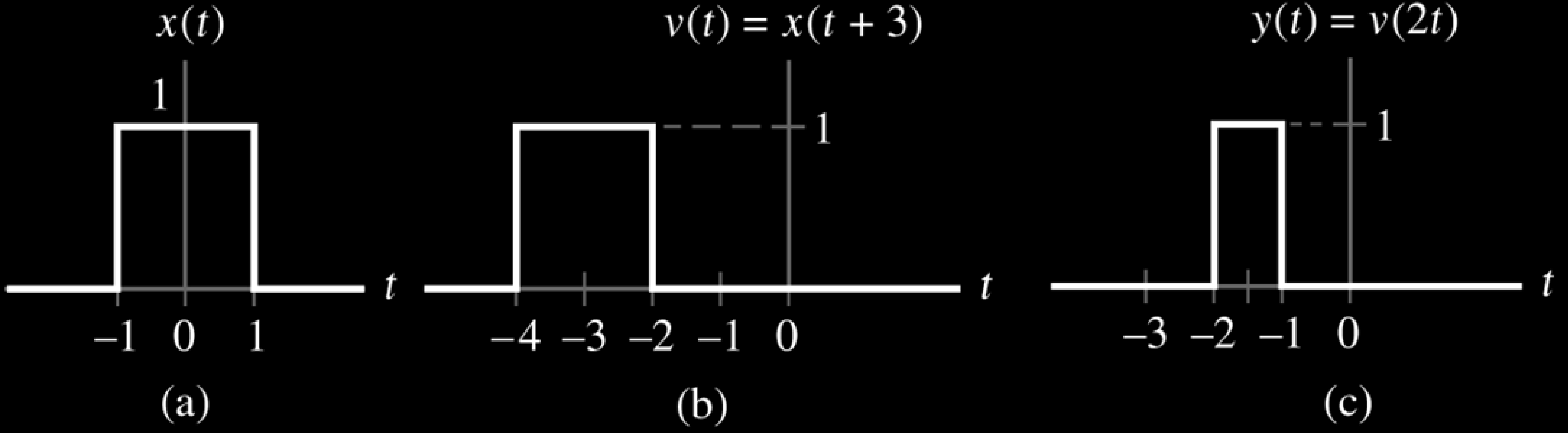
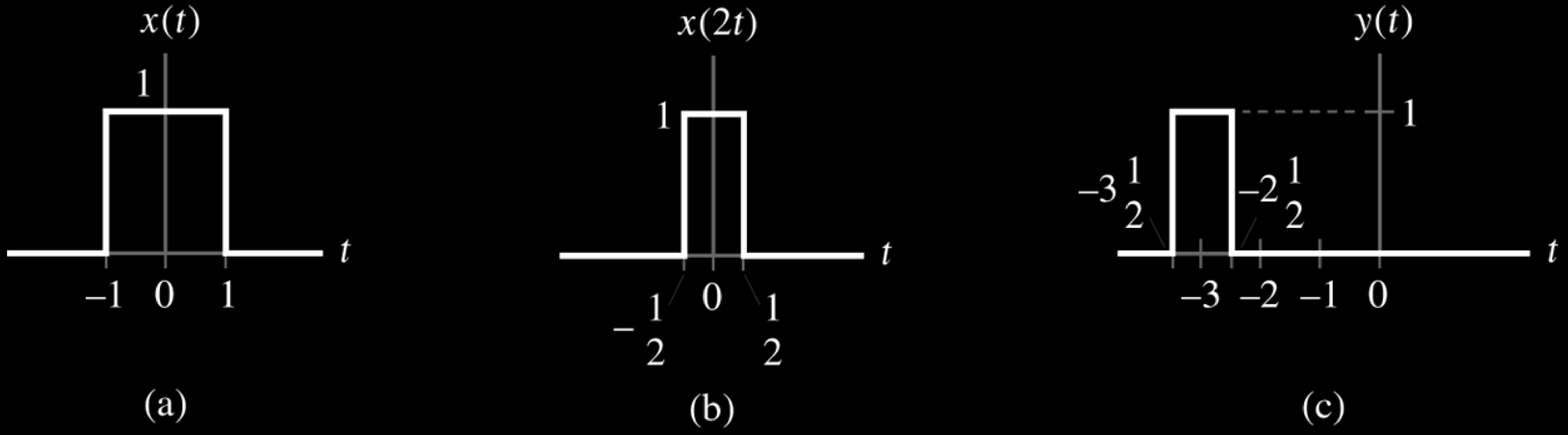


Figure 1.25 (p. 31)
The incorrect way of applying the precedence rule.
(a) Signal $x(t)$.
(b) Time-scaled signal $v(t) = x(2t)$.
(c) Signal $y(t)$ obtained by shifting $v(t) = x(2t)$ by 3 time units, which yields $y(t) = x(2(t + 3))$.



Problem 1.14

A triangular pulse signal x(t) is depicted in Fig. 1.26. Sketch each of the following signals derived from x(t):

(a) $x(3t)$	(d) $x(2(t+2))$
(b) $x(3t+2)$	(e) $x(2(t-2))$
(c) $x(-2t-1)$	(f) $x(3t)+x(3t+2)$

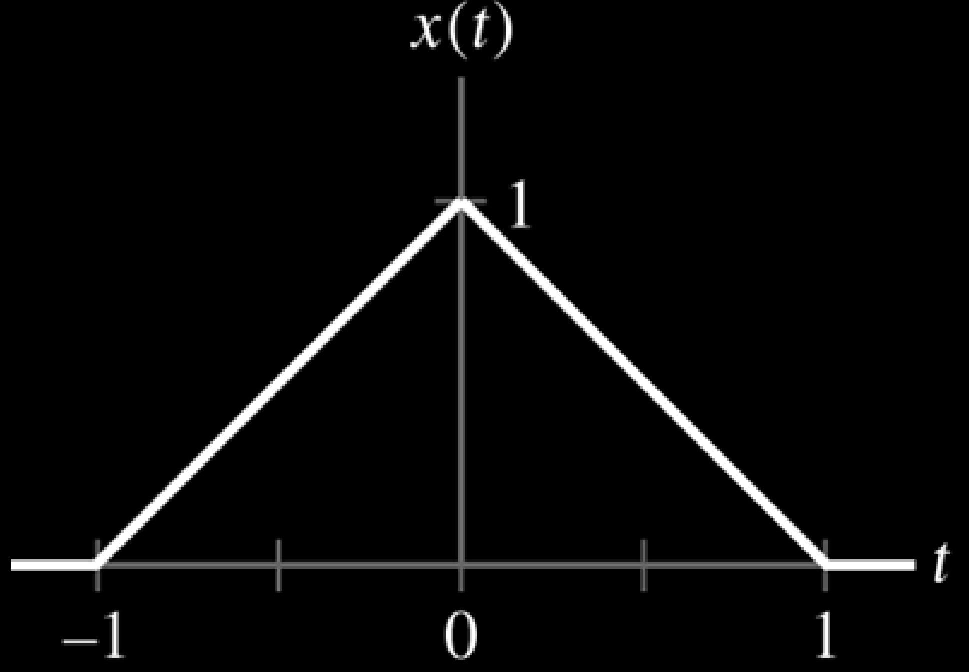
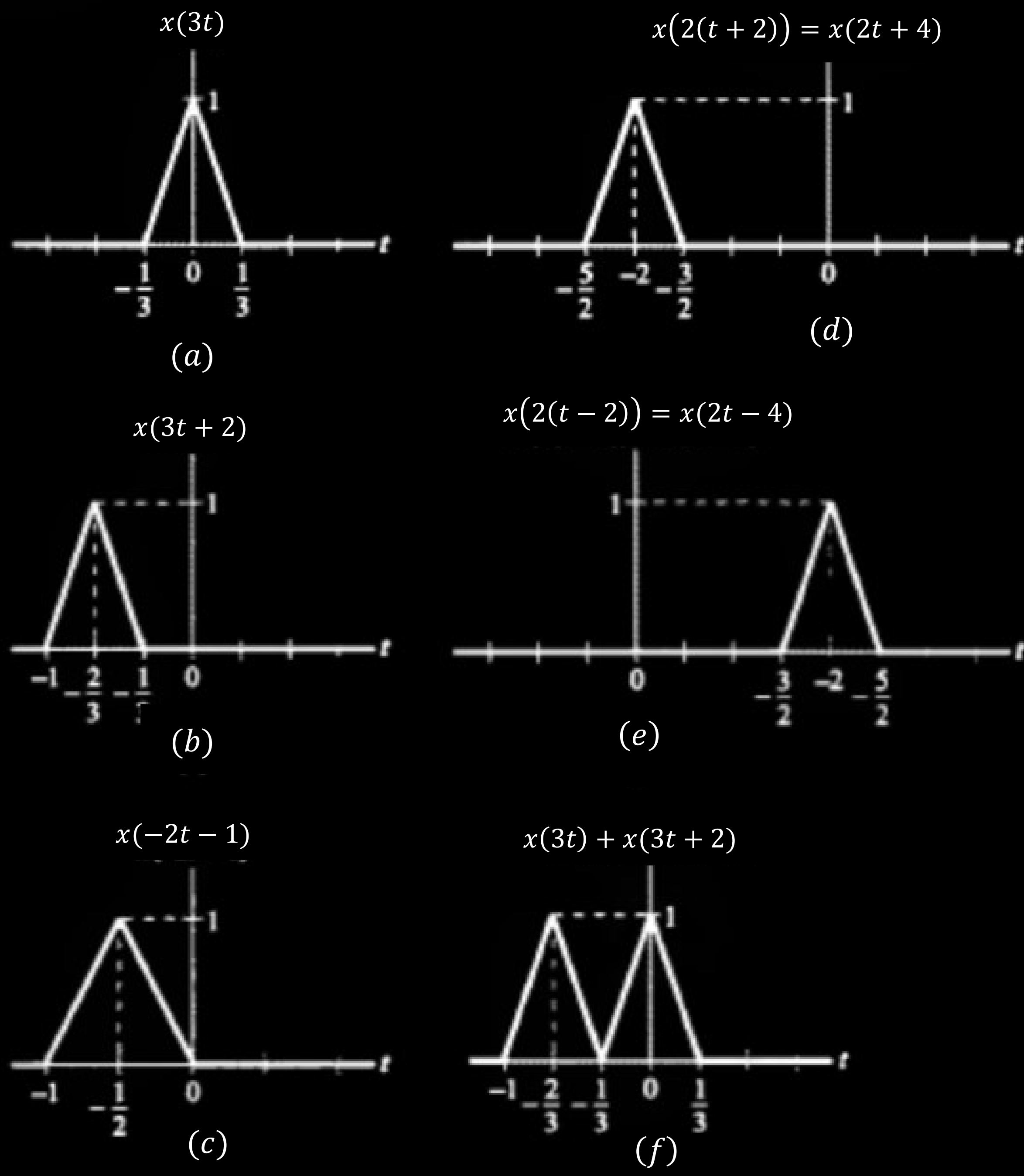


Figure 1.26 (p. 32)

Triangular pulse for Problem 1.14.

Answers:



Example 1.6: Precedence Rule for Discrete-Time Signal.

A discrete-time signal is defined by

$$x[n] = \begin{cases} 1, & n = 1, 2 \\ -1, & n = -1, -2 \\ 0, & n = 0 \text{ and } |n| > 2 \end{cases}.$$

Find $y[n] = x[2n + 3]$

Solution:

The signal $x[n]$ is displayed in Fig. 1.27(a). Time shifting $x[n]$ to the left by 3 yields the intermediate signal $v[n]$ shown in Fig. 1.27(b). Finally, scaling n in $v[n]$ by 2, we obtain the solution $y[n]$ shown in Fig. 1.27(c).

Note that as a result of the compression performed in going from $v[n]$ to $y[n] = v[2n]$, the nonzero samples of $v[n]$ at $n = -5$ and n = -1 (i.e., those contained in the original signal at $n = -2$ and $n = 2$) are lost.

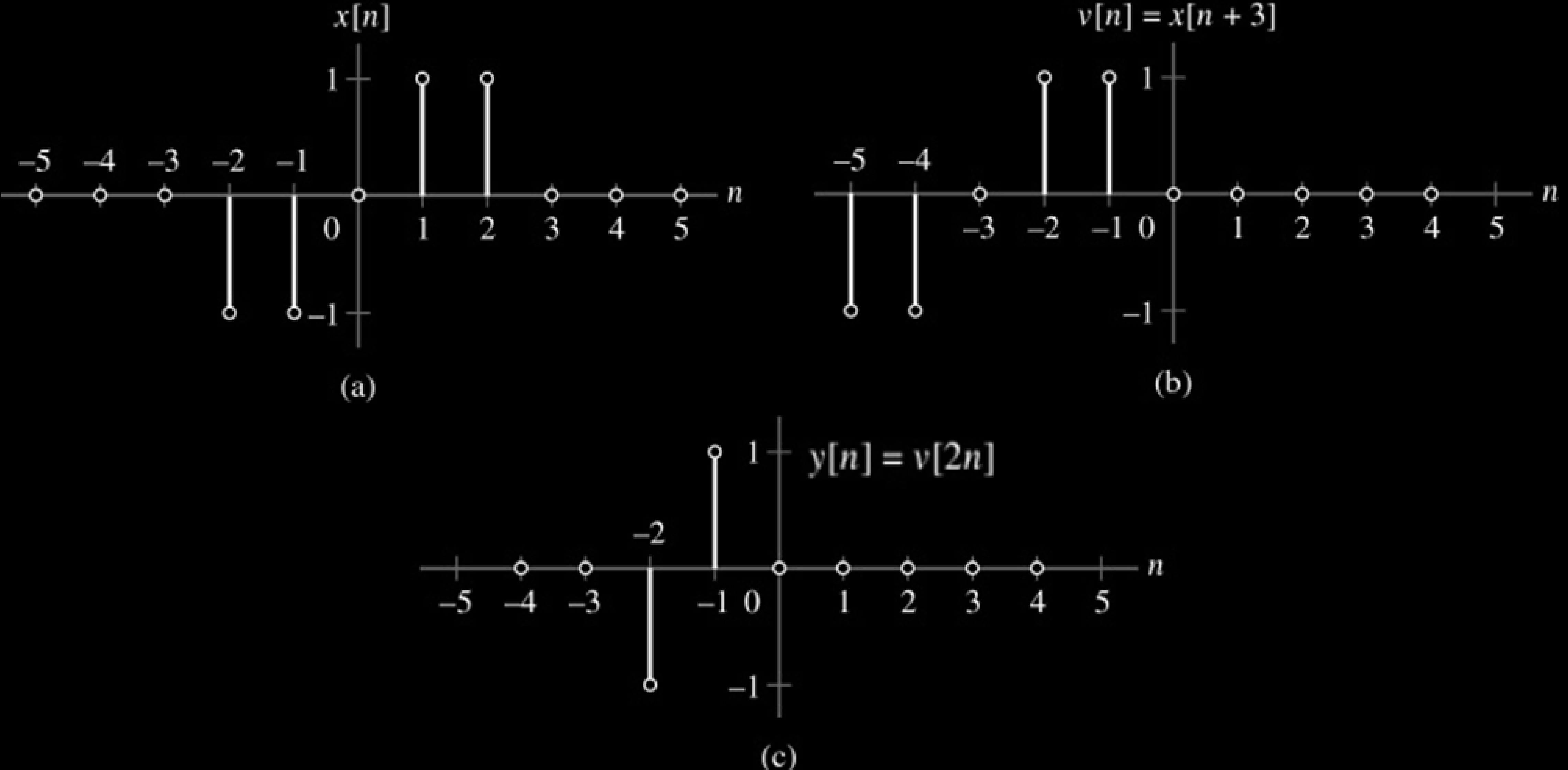
▶ **Problem 1.15** Consider a discrete-time signal

$$x[n] = \begin{cases} 1, & -2 \le n \le 2 \\ 0, & |n| > 2 \end{cases}$$

Find $y[n] = x[3n - 2]$.

Answer: $$y[n] = \begin{cases} 1, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases}.$$

Figure 1.27 (p. 33)
The proper order of applying the operations of time scaling and time shifting for the case of a discrete-time signal. (a) Discrete-time signal $x[n]$, antisymmetric about the origin. (b) Intermediate signal $v(n)$ obtained by shifting $x[n]$ to the left by 3 samples. (c) Discrete-time signal $y[n]$ resulting from the compression of $v[n]$ by a factor of 2, as a result of which two samples of the original $x[n]$, located at $n = –2, +2$, are lost.



1 .6 Elementary Signals

- These signals are exponential and sinusoidal signals, the step function, the impulse function and the ramp function, all of which serve as building blocks for the construction of more complex signals.
- They are also important in their own right, in that they may be used to model many physical signals that occur in nature.

1.6. Exponential Signals

- A real exponential signal is written as

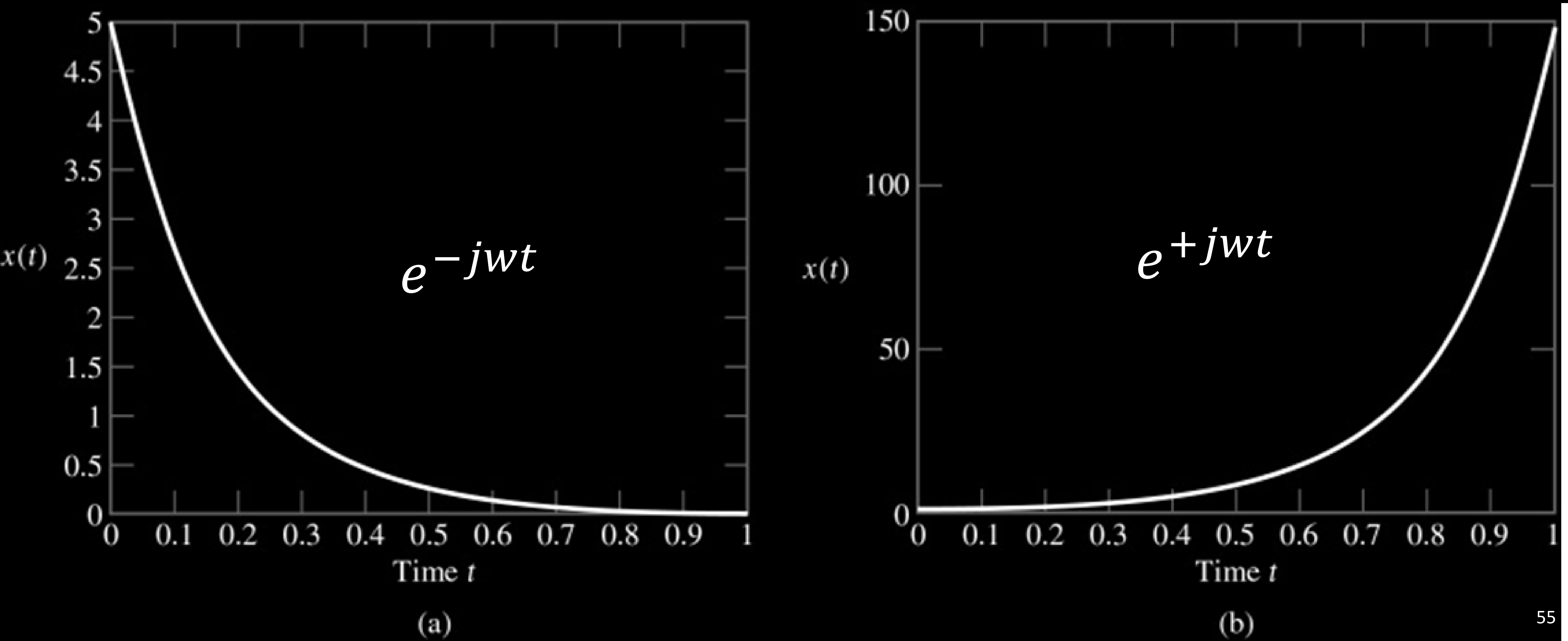
$$x(t)=Be^{at} \tag{1.31}$$

- The amplitude B and a are real parameters.

Decaying exponential, for which a < 0, *Growing exponential*, for which a > 0

- As illustrated in Fig. 1.28.

Figure 1.28 (p. 34) (a) Decaying exponential form of continuous-time signal. (b) Growing exponential form of continuous-time signal. Part (a) of the figure was generated using a = -6 and B = 5. Part (b) of figure was generated using a = 5, B = 1. If a = 0, signal $x(t)$ reduces to a dc signal of amplitude B.



From the figure, the operation of the capacitor for $t \geq 0$ is described by

$$\frac{d}{dt}v(t) + v(t) \qquad (1.32)$$

where $v(t)$ is voltage measured across capacitor at time t.

In discrete time, a real exponential signal is

$$x[n] = Br^n \qquad (1.34)$$

The exponential nature of this signal is

$$r = e^{\alpha}$$

for some α

Fig. 1.30 shows decaying and growing forms of discrete-time exponential signal corresponding to 0<r <1 and r>1,

When $r < 0$, the discrete-time exponential signal $x[n]$ assumes alternating signs for then r^n is positive for n even and negative for n odd.

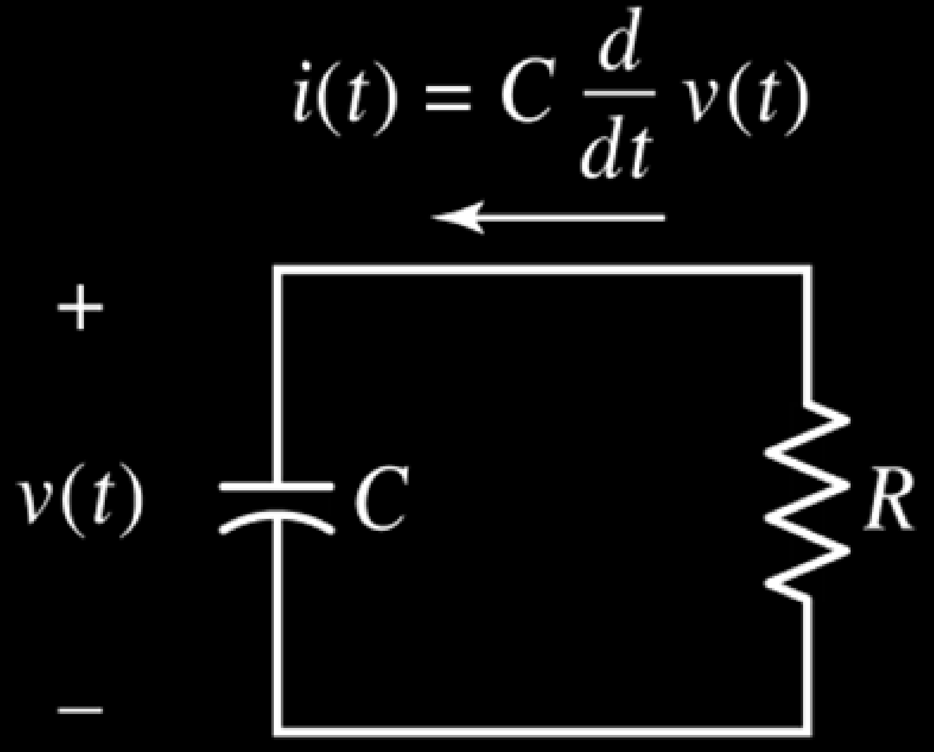
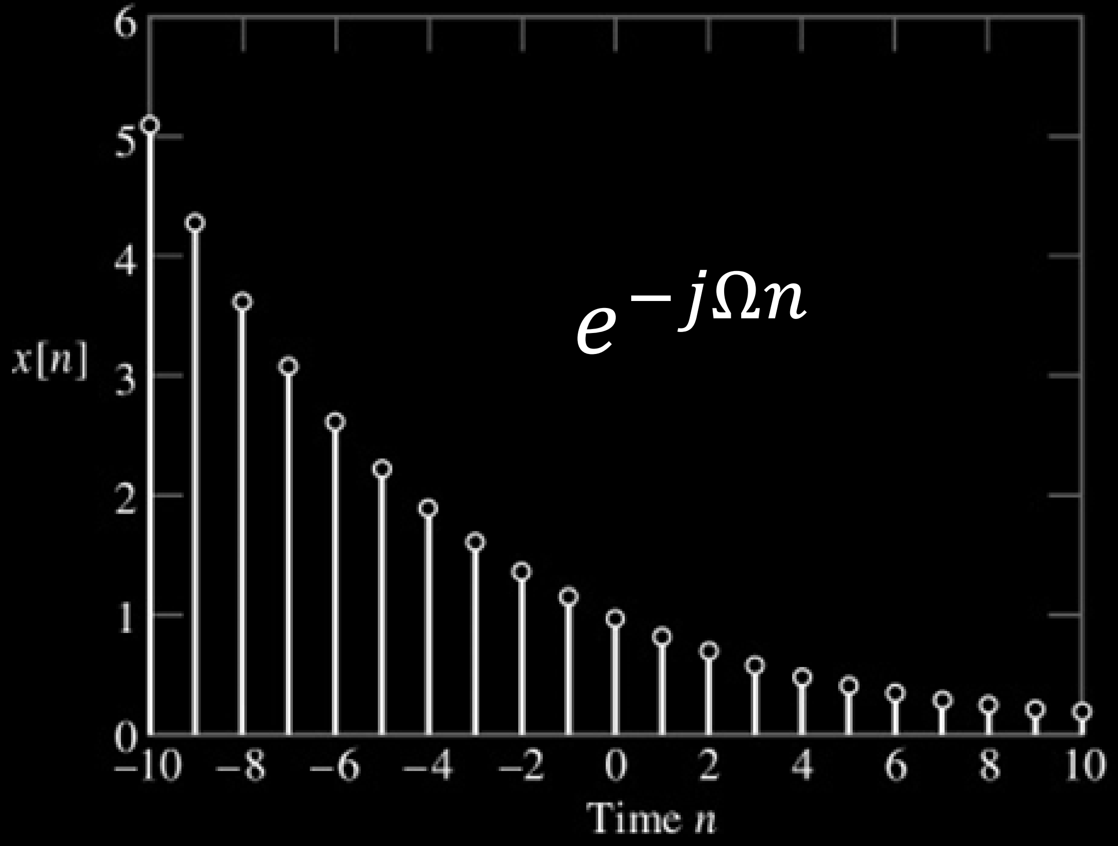


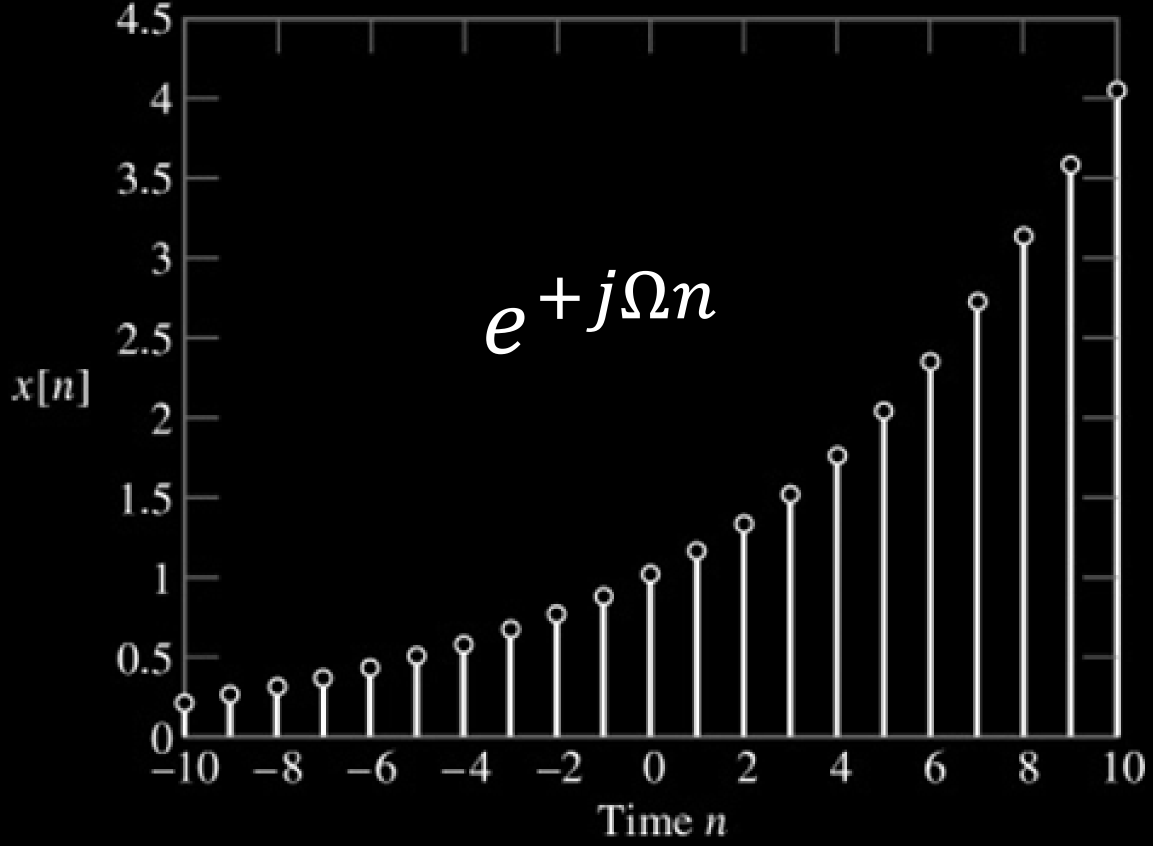
Figure 1.29 (p. 35) Lossy capacitor, with the loss represented by shunt resistance *R*.

- The exponential signals shown in Figs. 1.28 and 1.30 are all real valued.
- The mathematical forms of complex exponential signals are e^{jwt} and $e^{j\Omega n}$

Figure 1.30 (p. 35)
(a) Decaying exponential form of discrete-time signal. (b) Growing exponential form of discrete-time signal.

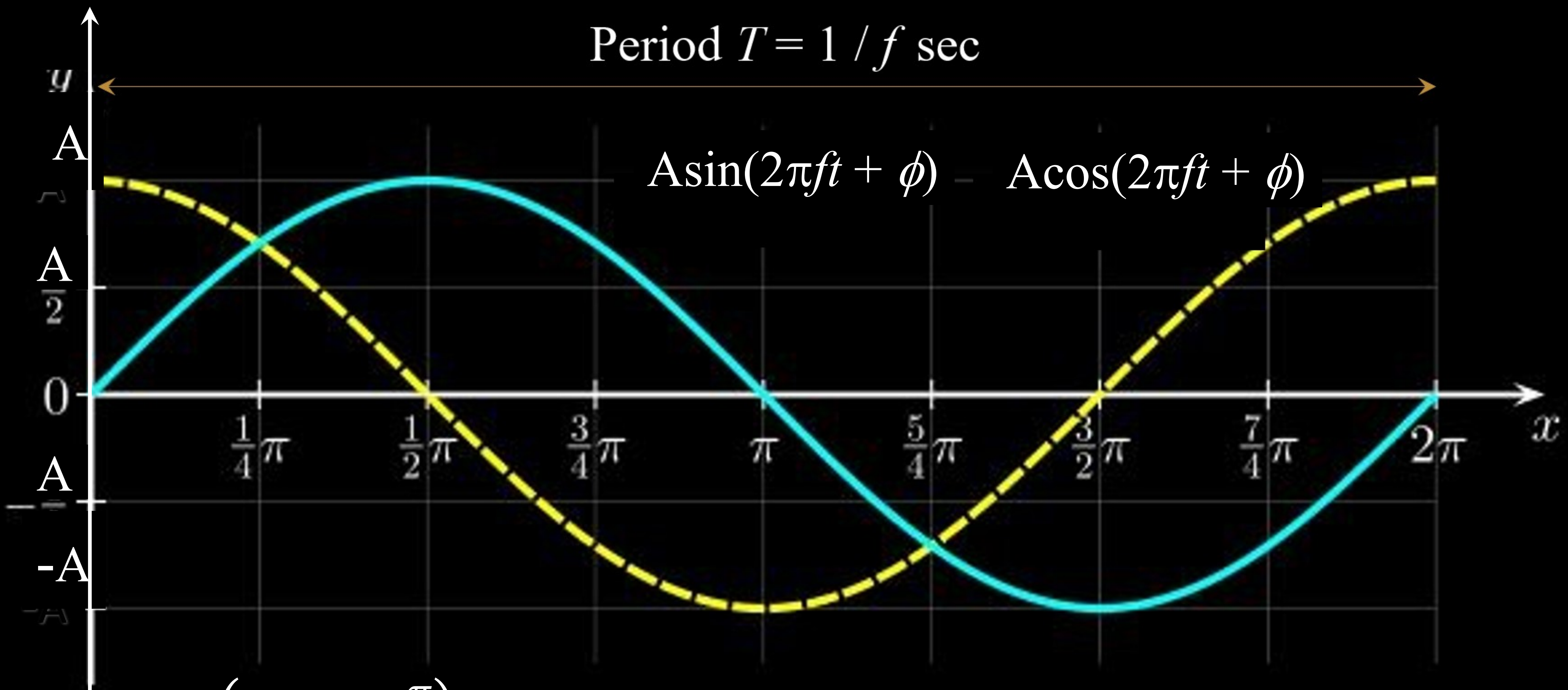


(a)



(b)

1.6.2 Sinusoidal Signals



$\cos\left(2\pi ft + \frac{\pi}{2}\right) = \sin(2\pi ft)$, Frequency $f = 1/T$ Hz, $T = 2\pi$ sec

1.6.2 Sinusoidal Signals

- The continuous-time version of a sinusoidal signal may be written as

$$x(t) = A\cos(wt + \phi), \qquad (1.35)$$

- where A is the amplitude, w is the frequency in radians per second, and ϕ is the phase angle in radians as shown in Figure 1.31
- The period of the sinusoid is $T = \frac{2\pi}{w}$ and the signal $x(t) = x(t+T)$

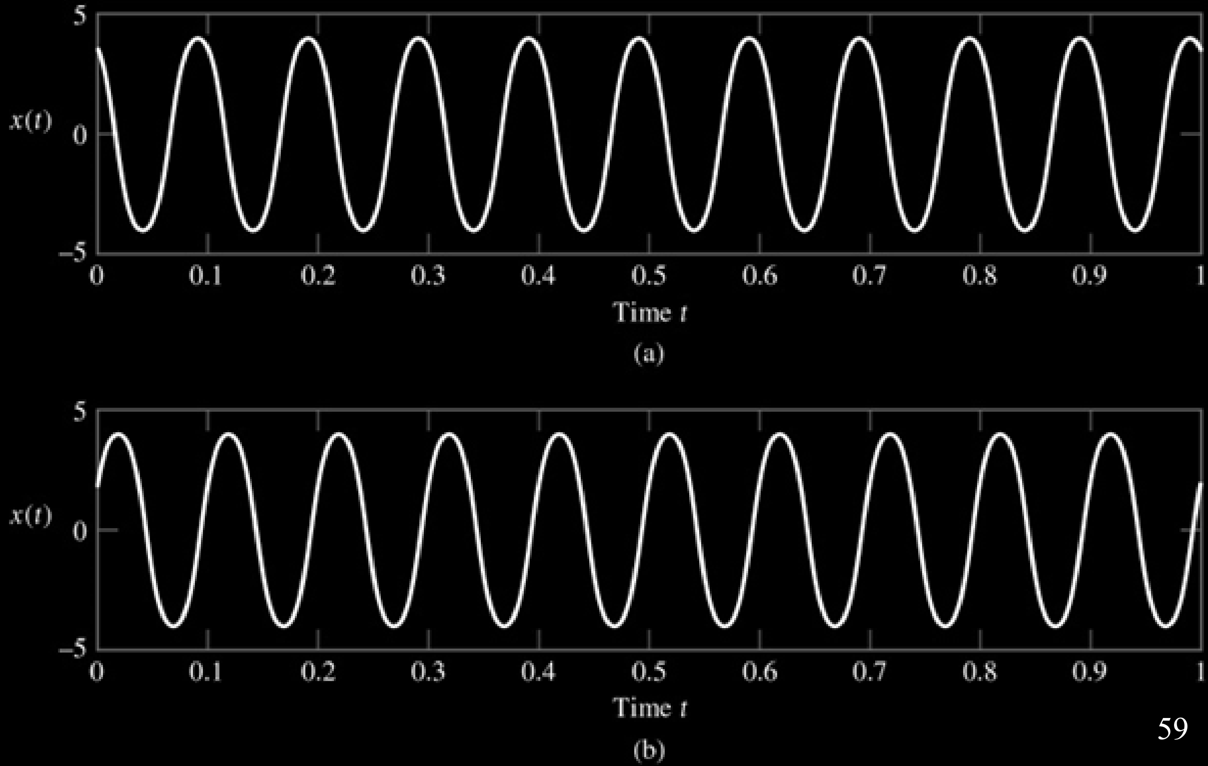


Figure 1.31 (p. 36)
(a) Sinusoidal signal $A\cos(\omega t + \Phi)$ with amplitude $A = 4$, phase $\Phi = -\pi/6$ radians
(b) Sinusoidal signal $A\sin(\omega t + \Phi)$ with amplitude $A = 4$, phase $\Phi = +\pi/6$ radians

1.6.2 Sinusoidal Signals

- In Figure 1.32, the voltage across the capacitor at time $t = 0$ is equal to V_0 . The operation of the circuit for $t \geq 0$ is described by

$$v(t) = V_0 \cos(w_0 t), \qquad t \geq 0, \tag{1.36}$$

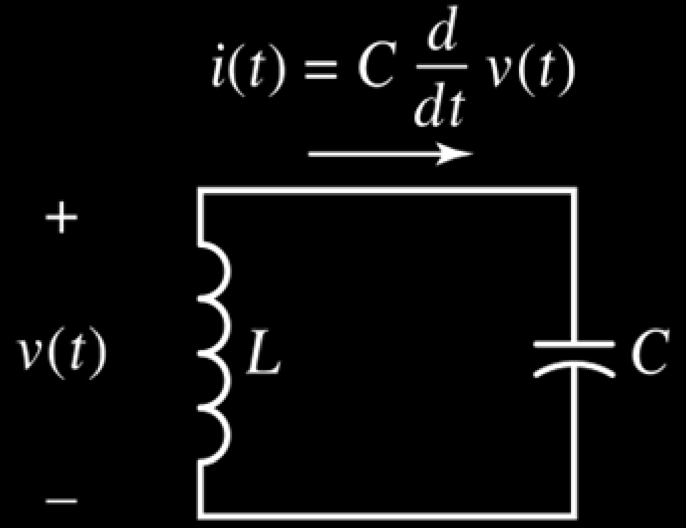
- $v(t)$ is voltage across capacitor at time t, C is capacitance and L is the inductance
- Solving (1.36) we get

$$v(t) = V_0 \cos(w_0 t), \ t \geq 0 \tag{1.37}$$

where

$$w_0 = \frac{1}{\sqrt{LC}} \tag{1.38}$$

Figure 1.32 (p. 37)
Parallel LC circuit, assuming that the inductor L and capacitor C are both ideal



1.6.2 Sinusoidal Signals

- The discrete-time version of a sinusoidal signal is

$$x[n] = A\cos(\Omega n + \phi) \tag{1.39}$$

- This discrete-time signal is periodicwith a period N samplesif it satisfies Eq. (1.10) for all integer n and some integer N given by

$$x[n+N] = A\cos(\Omega n + \Omega N + \phi)$$

- provided that $\Omega N = 2\pi m$ radians where m and n are integers, as shown in Fig.1.33

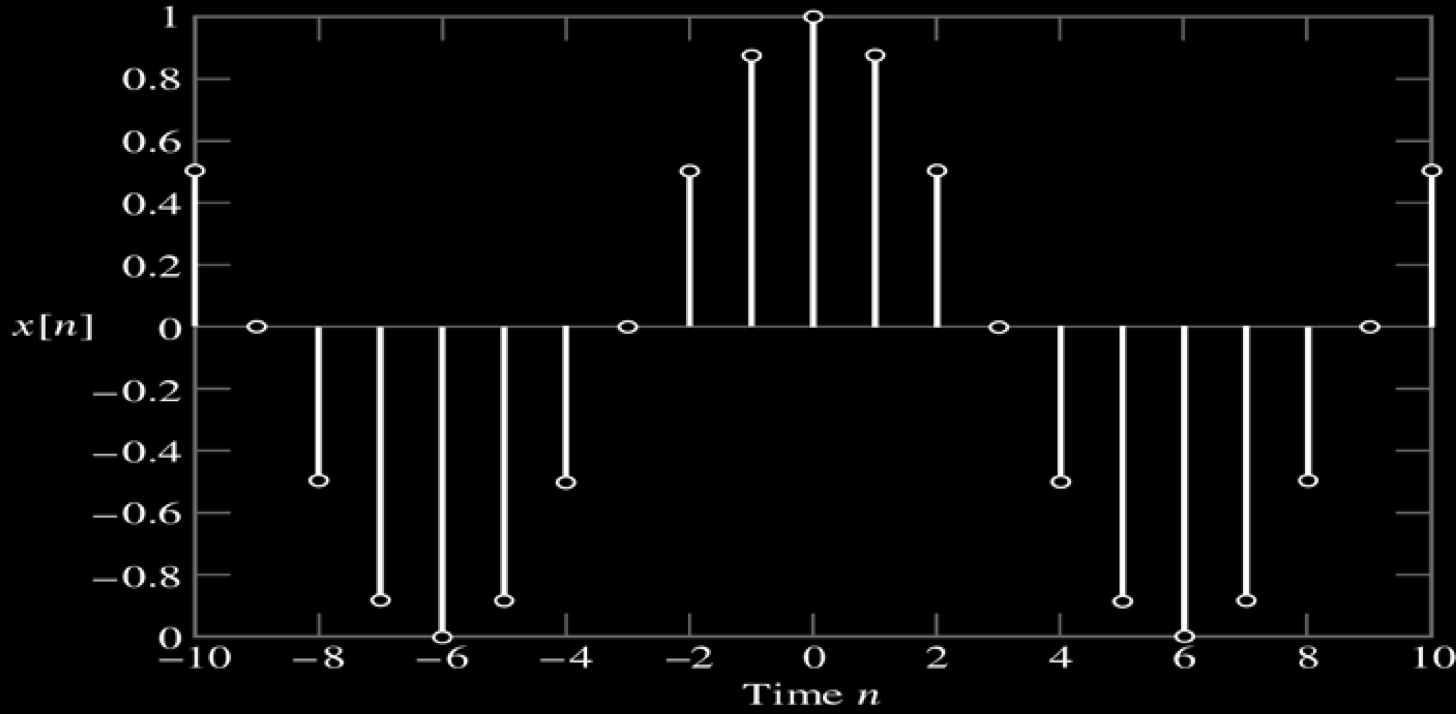


Figure 1.33 (p. 38)
Discrete-time sinusoidal signal given by (1.39) for A=1, ϕ = 0, and N = 12

EXAMPLE 1.7 DISCRETE-TIME SINUSOIDAL SIGNALS A pair of sinusoidal signals with a common angular frequency is defined by

$$x_1[n] = \sin[5\pi n]$$

and

$$x_2[n] = \sqrt{3}\cos[5\pi n].$$

(a) Both $x_1[n]$ and $x_2[n]$ are periodic. Find their common fundamental period.
(b) Express the composite sinusoidal signal

$$y[n] = x_1[n] + x_2[n]$$

in the form $y[n] = A\cos(\Omega n + \phi)$, and evaluate the amplitude A and phase ϕ.

Solution:

(a) The angular frequency of both $x_1[n]$ and $x_2[n]$ is

$$\Omega = 5\pi \text{ radians/cycle.}$$

Solving Eq. (1.40) for the period N, we get

$$N = \frac{2\pi m}{\Omega} = \frac{2\pi m}{5\pi} = \frac{2m}{5}.$$

For $x_1[n]$ and $x_2[n]$ to be periodic, N must be an integer. This can be so only for $m = 5, 10, 15, \ldots$, which results in $N = 2, 4, 6, \ldots$.

(b) Recall the trigonometric identity

$$A\cos(\Omega n + \phi) = A\cos(\Omega n)\cos(\phi) - A\sin(\Omega n)\sin(\phi).$$

Letting $\Omega = 5\pi$, we see that the right-hand side of this identity is of the same form as $x_1[n] + x_2[n]$. We may therefore write

$$A\sin(\phi) = -1 \quad \text{and} \quad A\cos(\phi) = \sqrt{3}.$$

Hence,

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{\text{amplitude of } x_1[n]}{\text{amplitude of } x_2[n]} = \frac{-1}{\sqrt{3}},$$

from which we find that $\phi = -\pi/3$ radians. Substituting this value into the equation

$$A\sin(\phi) = -1$$

and solving for the amplitude A, we get

$$A = -1/\sin\left(-\frac{\pi}{3}\right) = 2.$$

Accordingly, we may express $y[n]$ as $\quad y[n] = 2\cos\left(5\pi n - \frac{\pi}{3}\right).$

▶ **Problem 1.16** Determine the fundamental period of the sinusoidal signal

$$x[n] = 10\cos\left(\frac{4\pi}{31}n + \frac{\pi}{5}\right).$$

Answer: $N = 31$ samples **Since N should be an integer, nearest multiple of 31/2 is 31.**

▶ **Problem 1.17** Consider the following sinusoidal signals:

(a) $x[n] = 5\sin[2n]$

(b) $x[n] = 5\cos[0.2\pi n]$

(c) $x[n] = 5\cos[6\pi n]$

(d) $x[n] = 5\sin[6\pi n/35]$

Determine whether each $x(n)$ is periodic, and if it is, find its fundamental period.

Answers: (a) Nonperiodic. (b) Periodic, fundamental period = 10. (c) Periodic, fundamental period = 1. (d) Periodic, fundamental period = 35. ◀

▶ **Problem 1.18** Find the smallest angular frequencies for which discrete-time sinusoidal signals with the following fundamental periods would be periodic: (a) $N = 8$, (b) $N = 32$, (c) $N = 64$, and (d) $N = 128$.

Answers: (a) $\Omega = \pi/4$. (b) $\Omega = \pi/16$. (c) $\Omega = \pi/32$. (d) $\Omega = \pi/64$. ◀

1 .6.3 Relation Between Sinusoidal and Complex Exponential Signals

- The complex exponential $e^{j\theta}$ may be expanded as

$$e^{j\theta} = \cos\,\theta + j\,\sin\,\theta \qquad (1.41)$$

- The continuous-time sinusoidal signal of Eq. (1.35) can be expressed as real part of the complex exponential signal Be^{jwt}, where

$$B = Ae^{j\phi} \qquad (1\text{-}42)$$

- This complex quantity may be written as

$$A\cos(wt + \phi) = \mathrm{Re}\{Be^{jwt}\} \qquad (1.43)$$

- also,

$$A\sin(wt + \phi) = \mathrm{Im}\{Be^{jwt}\} \qquad (1.44)$$

1 .6.3 Relation Between Sinusoidal and Complex Exponential Signals

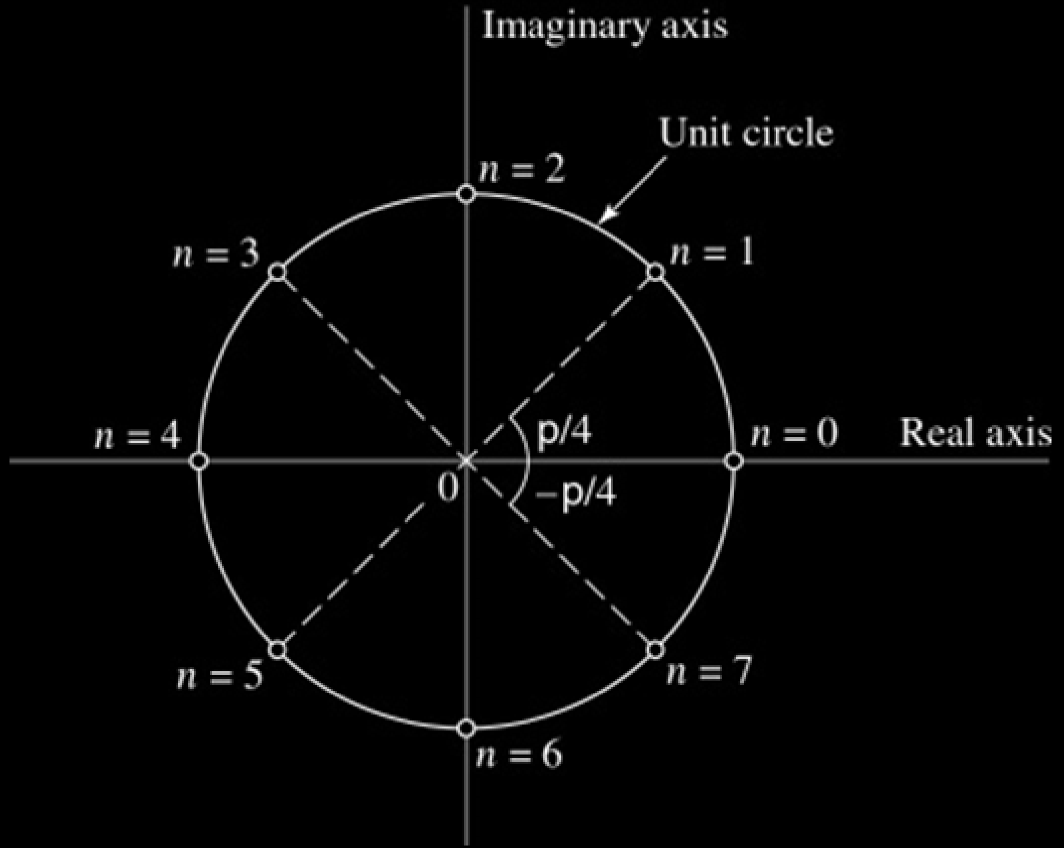
- In the discrete-time case, we may write

$$A\cos(\Omega n + \phi) = \mathrm{Re}\{Be^{j\Omega n}\} \tag{1.46}$$

and

$$A\sin(\Omega n + \phi) = \mathrm{Im}\{Be^{j\Omega n}\} \tag{1.47}$$

Figure 1.34 (p. 41)
Complex plane, showing eight points uniformly distributed on the unit circle where Ω = π / 4 and n = 0, 1, . . . , 7



1.6.4 Exponentially Damped Sinusoidal Signals

- Multiplication of sinusoidal signal by real-valued decaying exponential signal results in a new signal called exponentially damped sinusoidal signal

$$x(t) = A\, e^{j\alpha t} \sin(wt + \phi) \quad \alpha > 0 \qquad (1.48)$$

- ..

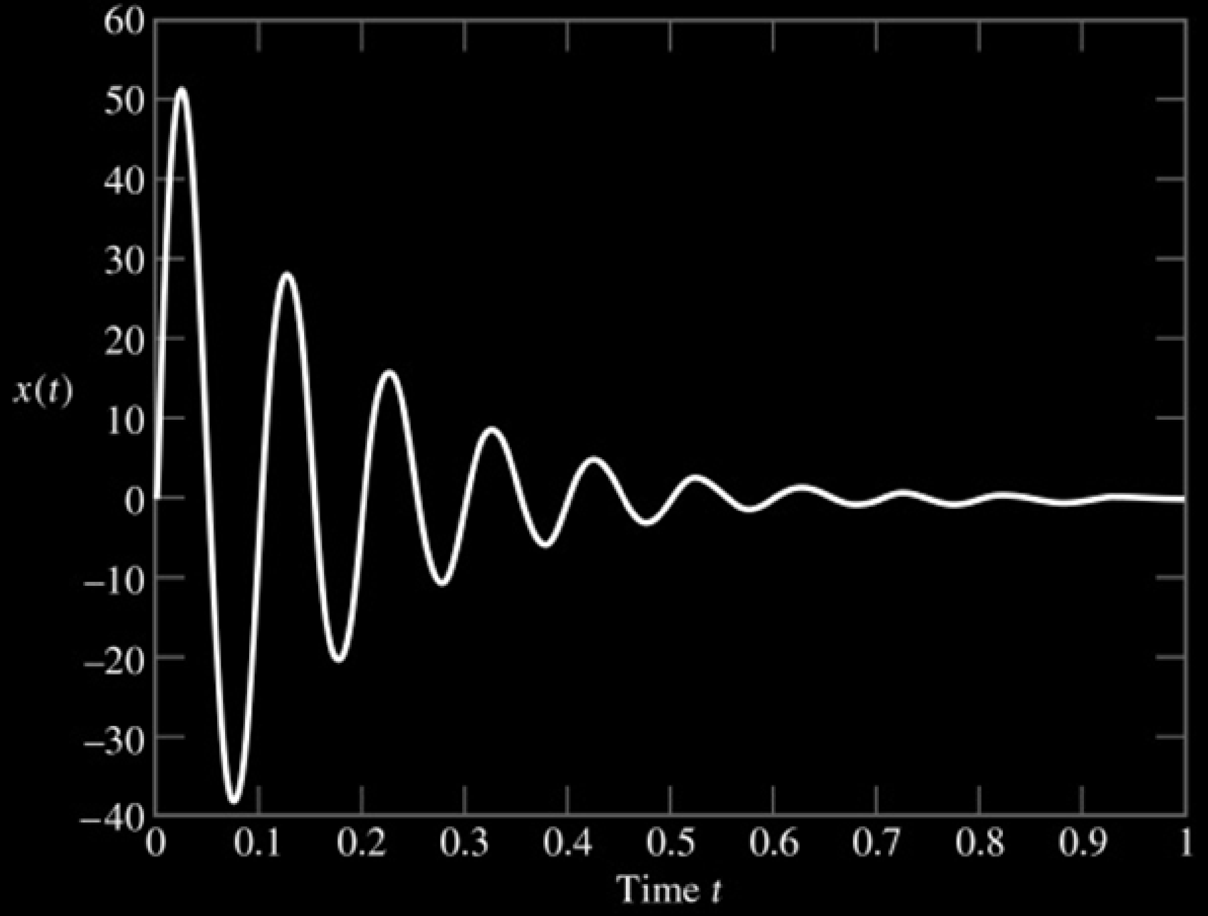


Figure 1.35 (p. 41)
Exponentially damped sinusoidal signal

$$Ae^{-at} \sin(\omega t),$$

with $A = 60$ and $\alpha = 6$

1.6.4 Exponentially Damped Sinusoidal Signals

- An exponentially damped sinusoidal signal is shown in Fig.1.36 where a capacitor of capacitance C, an inductor of inductance L, and a resistor of resistance R
- The voltage developed across the capacitor at time $t = 0$ is

$$v(t) = V_0 e^{-t/(2CR)} \cos(\omega_0 t), \qquad t \geq 0 \tag{1.50}$$

where

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{1}{4C^2R^2}}.$$

- Discrete-time version of exponentially damped sinusoidal signal of (1.48) by

$$x[n] = Br^n \sin(\Omega n + \phi), \tag{1.52}$$

$$\frac{1}{L}\int_{-\infty}^{t} v(t)dt$$

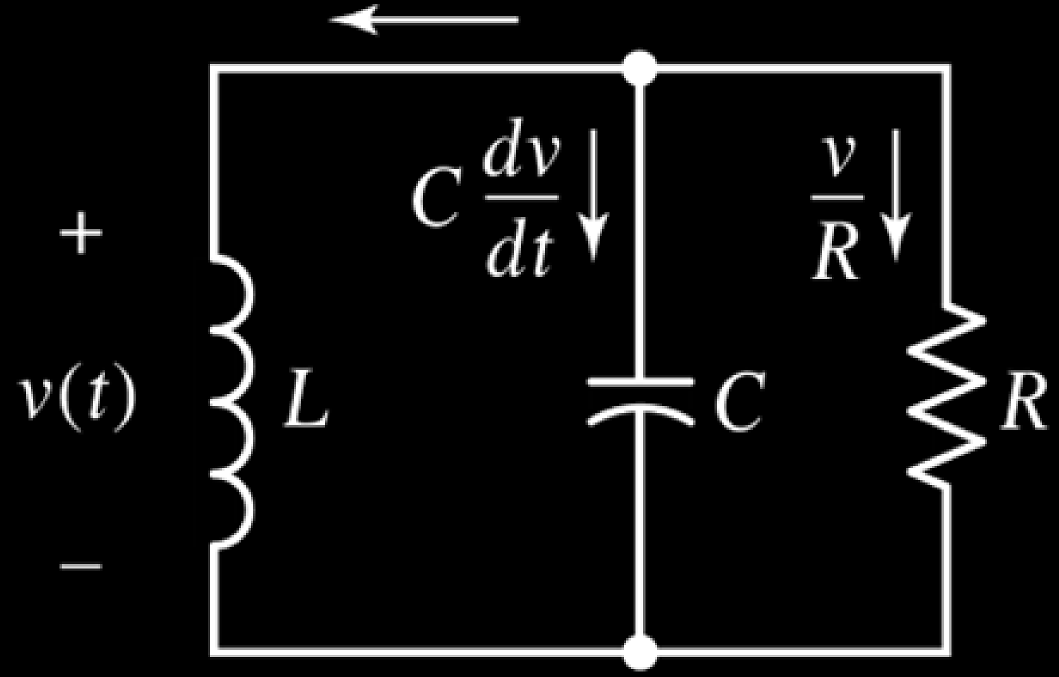


Figure 1.36 (p. 42) : Parallel *LRC,* circuit, with inductor *L*, capacitor *C*, and resistor *R* all assumed to be ideal.

▶ **Problem 1.20** Consider the complex-valued exponential signal

$$x(t) = Ae^{\alpha t + j\omega t}, \qquad a > 0.$$

Evaluate the real and imaginary components of $x(t)$ for the following cases:

(a) α real, $\alpha = \alpha_1$

(b) α imaginary, $\alpha = j\omega_1$

(c) α complex, $\alpha = \alpha_1 + j\omega_1$

Answers:

(a) $\mathrm{Re}\{x(t)\} = Ae^{\alpha_1 t}\cos(\omega t)$; $\mathrm{Im}\{x(t)\} = Ae^{\alpha_1 t}\sin(\omega t)$

(b) $\mathrm{Re}\{x(t)\} = A\cos(\omega_1 t + \omega t)$; $\mathrm{Im}\{x(t)\} = A\sin(\omega_1 t + \omega t)$

(c) $\mathrm{Re}\{x(t)\} = Ae^{\alpha_1 t}\cos(\omega_1 t + \omega t)$; $\mathrm{Im}\{x(t)\} = Ae^{\alpha_1 t}\sin(\omega_1 t + \omega t)$

▶ **Problem 1.21** Consider the pair of exponentially damped sinusoidal signals

$$x_1(t) = Ae^{\alpha t}\cos(\omega t), \qquad t \geq 0$$

and

$$x_2(t) = Ae^{\alpha t}\sin(\omega t), \qquad t \geq 0.$$

Assume that A, α, and ω are all real numbers; the exponential damping factor α is negative and the frequency of oscillation ω is positive; the amplitude A can be positive or negative.

(a) Derive the complex-valued signal $x(t)$ whose real part is $x_1(t)$ and imaginary part is $x_2(t)$.

(b) The formula

$$a(t) = \sqrt{x_1^2(t) + x_2^2(t)}$$

defines the *envelope* of the complex signal $x(t)$. Determine $a(t)$ for the $x(t)$ defined in part (a).

(c) How does the envelope $a(t)$ vary with time t?

Answers:

(a) $x(t) = Ae^{st}, \qquad t \geq 0$, where $s = \alpha + j\omega$

(b) $a(t) = |A|e^{\alpha t}, \qquad t \geq 0$

(c) At $t = 0$, $a(0) = |A|$, and then $a(t)$ decreases exponentially as time t increases; as t approaches infinity, $a(t)$ approaches zero ◀

1.6.5 Step Function

- The discrete-time version of the unit-step function in Fig. 1.37 is defined by

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{1.53}$$

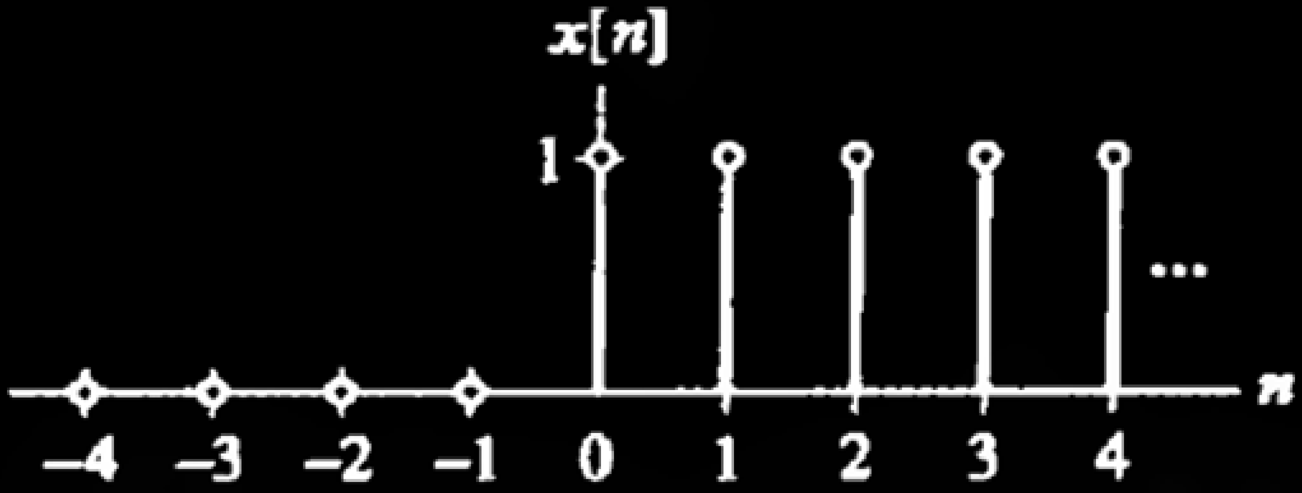


FIGURE 1.37 Discrete-time version of step function of unit amplitude.

1.6.5 Step Function

- The continous-time version of the unit-step function in Fig. 1.38 is defined by

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1.54}$$

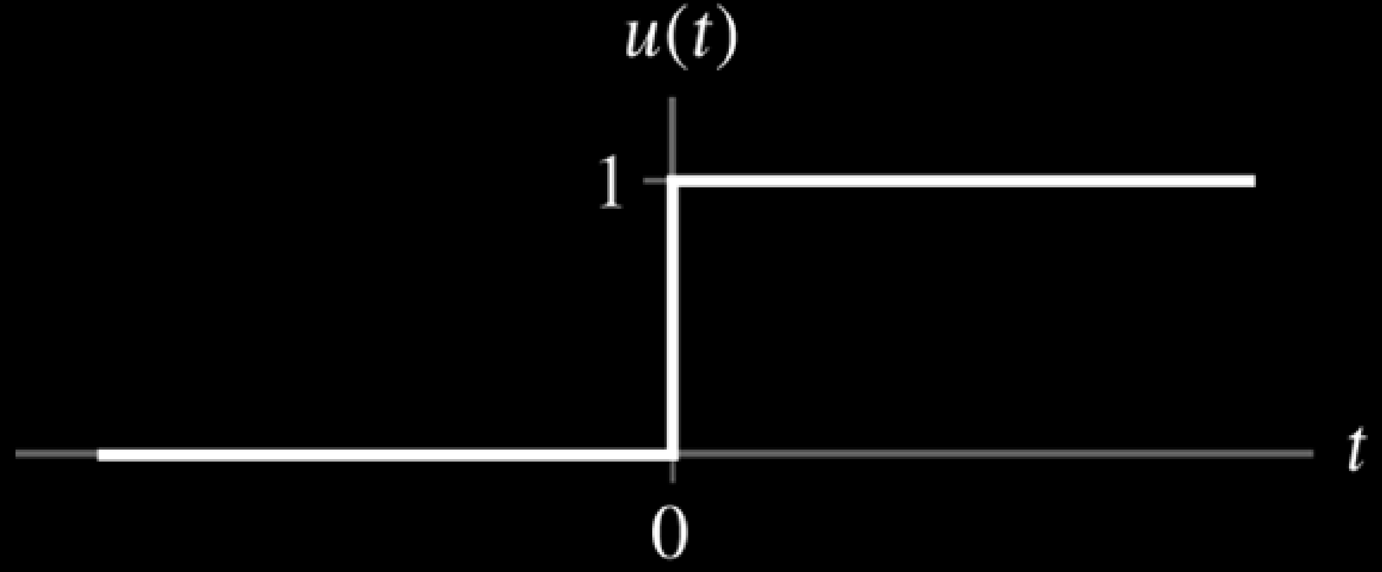


Figure 1.38 (p. 44)
Continuous-time version of the unit-step function of unit amplitude.

Example 1.8 Rectangular Pulse

- Consider the rectangular pulse x(t) shown in Fig.1.39(a). This pulse has an amplitude A and duration of 1 second. Express x(t) as a weighted sum of two step functions.
- Solution: The rectangular pulse x(t) may be written in mathematical terms as

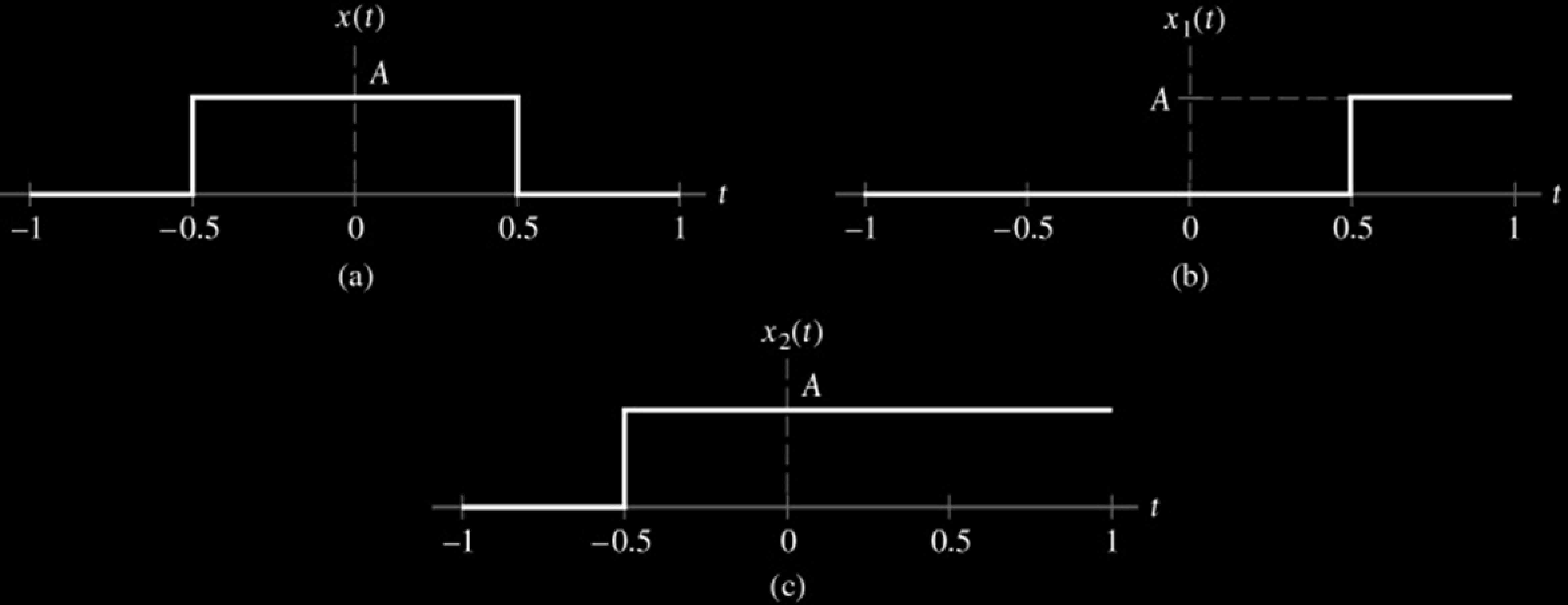
$$x(t) = \begin{cases} A & 0 \le |t| < 0.5 \\ 0 & |t| > 0.5 \end{cases} \tag{1.55}$$

- The rectangular pulse x(t) is represented as the difference of two time-shifted step functions, $x_1(t)$ and $x_2(t)$, as in Fig.1.39(b) and (c). x(t) can then be expressed as

$$x(t) = Au\left(t + \frac{1}{2}\right) - Au\left(t - \frac{1}{2}\right) \tag{1.56}$$

Figure 1.39 (p. 44)

(a) Rectangular pulse $x(t)$ of amplitude A and duration of 1 s, symmetric about the origin. (b) Representation of $x(t)$ as the difference of two step functions of amplitude A, with one step function shifted to the left by ½ and the other shifted to the right by ½; the two shifted signals are denoted by $x_1(t)$ and $x_2(t)$, respectively. Note that $x(t) = x_1(t) - x_2(t)$.



Example 1.9 RC Circuit Consider the simple RC circuit shown in Fig. 1.40(a). The capacitor C is assumed to be initially uncharged. At $t = 0$, the switch connecting the dc voltage source V_0 to the RC circuit is closed. Find the voltage $v(t)$ across the capacitor for $t \geq 0$.

Solution: The switching operation is represented by a step function $V_0 u(t)$, as shown in the equivalente circuit of Fig. 1.40(b). The capacitor cannot charge suddenly, so, with it being initially uncharged, we have

$$v(0) = 0.$$

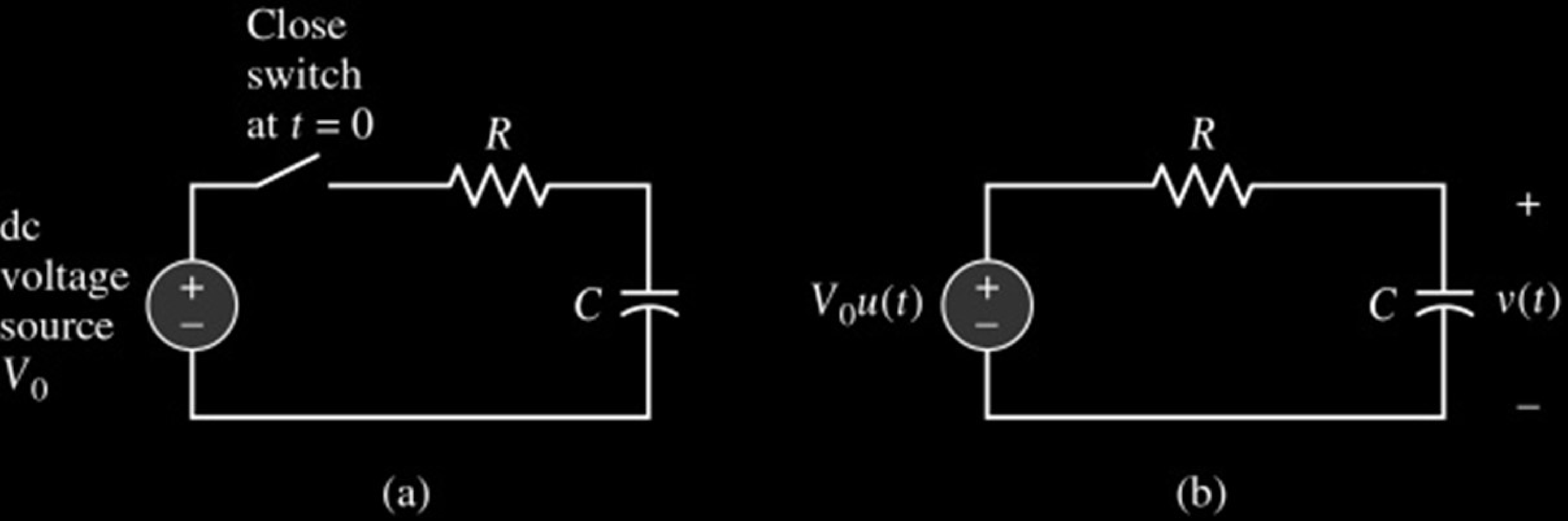
For $t = \infty$, the capacitor becomes fully charged; hence,

$$v(\infty) = V_0.$$

Recognizing that the voltage across the capacitor increases exponentially with a time constant RC, we may thus express $v(t)$ as

$$v(t) = V_0(1 - e^{-t/(RC)})u(t). \tag{1.57}$$ ■

Figure 1.40 (p. 45)
(a) Series RC circuit with a switch that is closed at time $t = 0$, thereby energizing the voltage source. (b) Equivalent circuit, using a step function to replace the action of the switch.



▶ **Problem 1.22** A discrete-time signal

$$x[n] = \begin{cases} 1, & 0 \le n \le 9 \\ 0, & \text{otherwise} \end{cases}.$$

Using $u[n]$, describe $x[n]$ as the superposition of two step functions.

Answer: $x[n] = u[n] - u[n-10]$. ◀

1.6.6 Impulse Function

- The discrete-time version of the unit impulse is defined by

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \qquad (1.58)$$

- As shown in Fig.1.41. The continous version

$$\delta(t) = 0 \quad for\ t \neq 0 \qquad (1.59)$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad (1.60)$$

The impulse $\delta(t)$ is also referred to as the *Dirac delta function*.

- See visualization in Fig.1.42

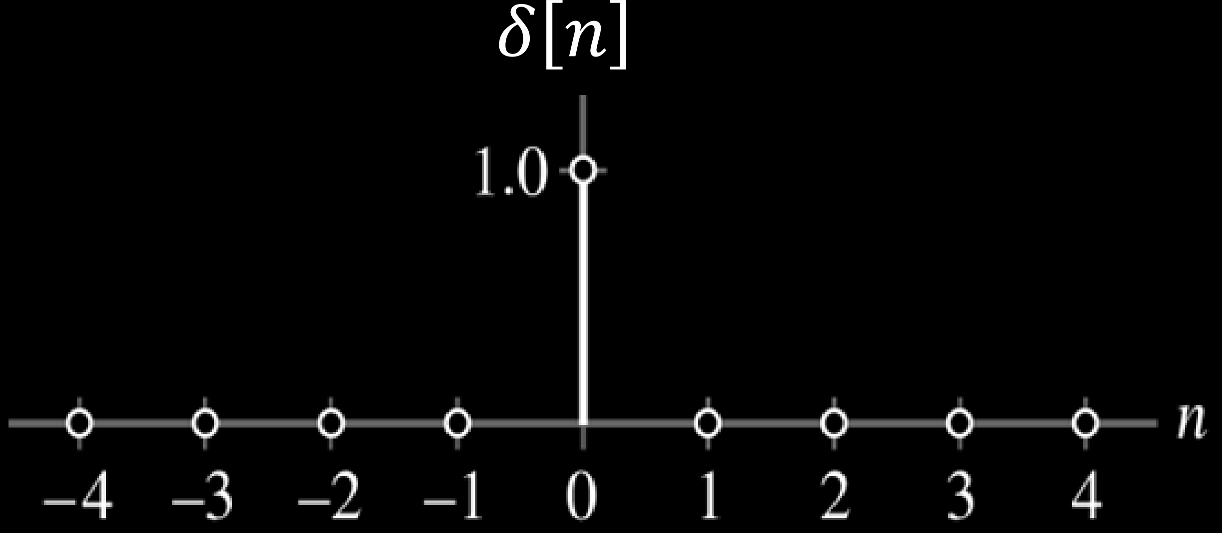


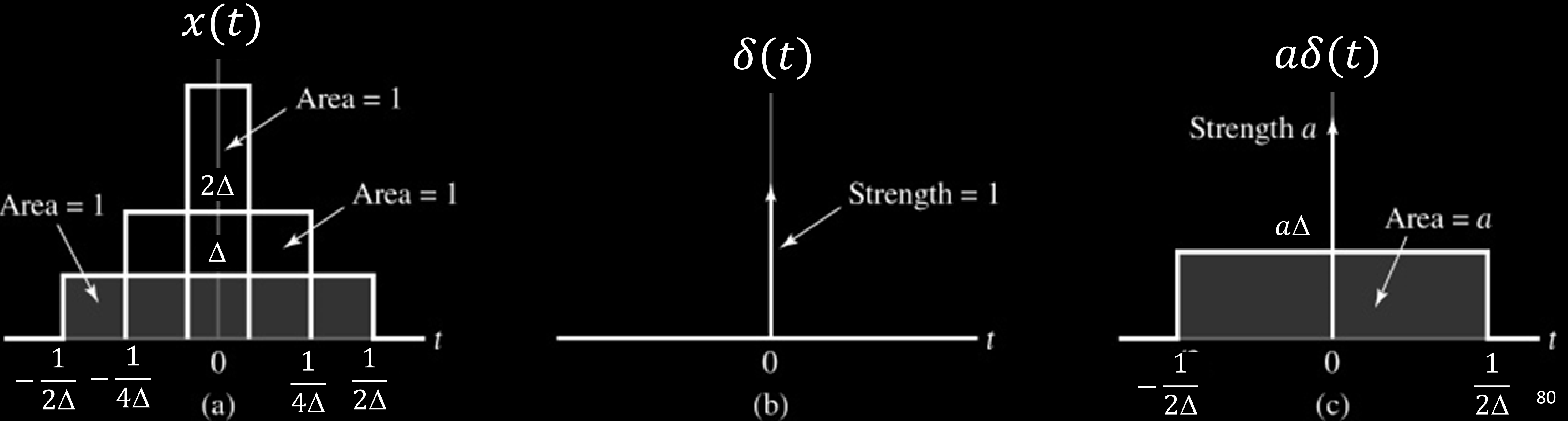
Figure 1.41 (p. 46): Discrete-time form of impulse.

Figure 1.42 (p. 46)

(a) Evolution of a rectangular pulse of unit area into an impulse of unit strength (i.e., unit impulse).

(b) Graphical symbol for unit impulse.

(c) Representation of an impulse of strength a that results from allowing the duration Δ of a rectangular pulse of area a to approach zero.



$\delta(t)$ and u(t) are related to each other by

$$\delta(t) = \frac{d}{dt} u(t) \tag{1.62}$$

and

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \tag{1.63}$$

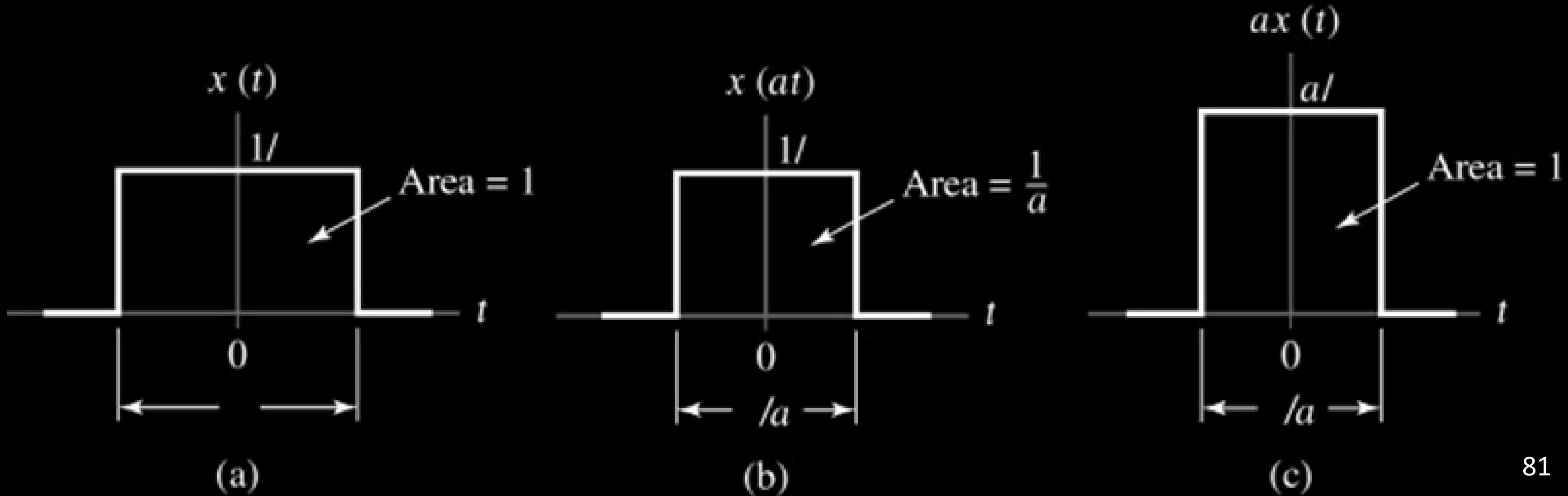
Also $\delta(-t) = \delta(t)$. The time shifting property of $\delta(t)$ function

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0) \tag{1.65}$$

And also

$$\delta(at) = \frac{1}{a}\delta(t) \quad (a > 0) \tag{1.66}$$

Figure 1.44 (p. 48)
Steps involved in proving time-scaling property of unit impulse.
(a) Rectangular pulse $x_\Delta(t)$ of amplitude $1/\Delta$ and duration Δ, symmetric about the origin.
(b) Pulse $x_\Delta(t)$ compressed by factor a.
(c) Amp. scaling of compressed pulse, restoring it to unit area.



Example 1.10 *RC* Circuit (continued) Consider the simple circuit shown in Fig. 1.43, in which the capacitor is initially uncharged and the switch connecting it to the dc voltage source V_0 is suddenly closed at time $t = 0$. (This circuit is the same as that of the *RC* circuit in Fig. 1.40, except that we now have zero resistance.) Determine the current $i(t)$ that flows through the capacitor for $t \geq 0$.

Solution: The switching operation is equivalent to connecting the voltage source $V_0 u(t)$ across the capacitor, as shown in Fig. 1.43(b). We may thus express the voltage across the capacitor as

$$v(t) = V_0 u(t).$$

By definition, the current flowing through the capacitor is

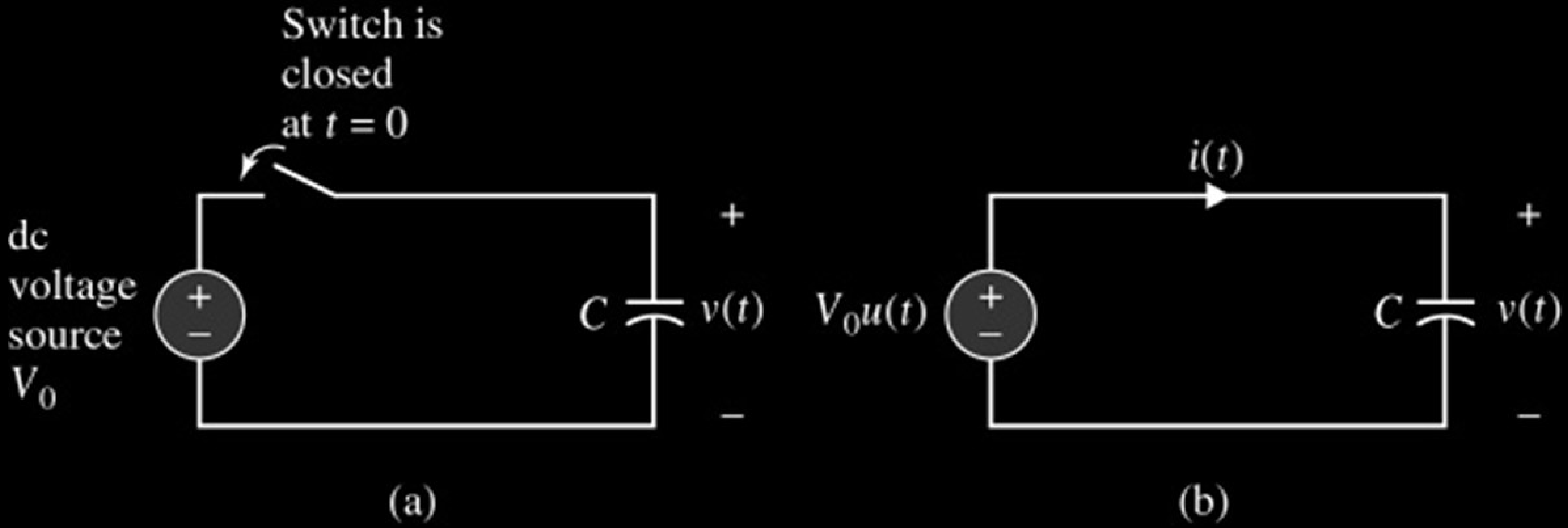
$$i(t) = C\frac{dv(t)}{dt}.$$

Hence, for the problem at hand, we have

$$i(t) = CV_0\frac{du(t)}{dt} = CV_0\delta(t),$$

where, in the second line, we have used Eq. (1.62). That is, the current that flows through the capacitor C in Fig. 1.43(b) is an impulsive current of strength CV_0. ■

Figure 1.43 (p. 47)
(a) Series circuit consisting of a capacitor, a dc voltage source, and a switch; the switch is closed at time $t = 0$. (b) Equivalent circuit, replacing the action of the switch with a step function $u(t)$.



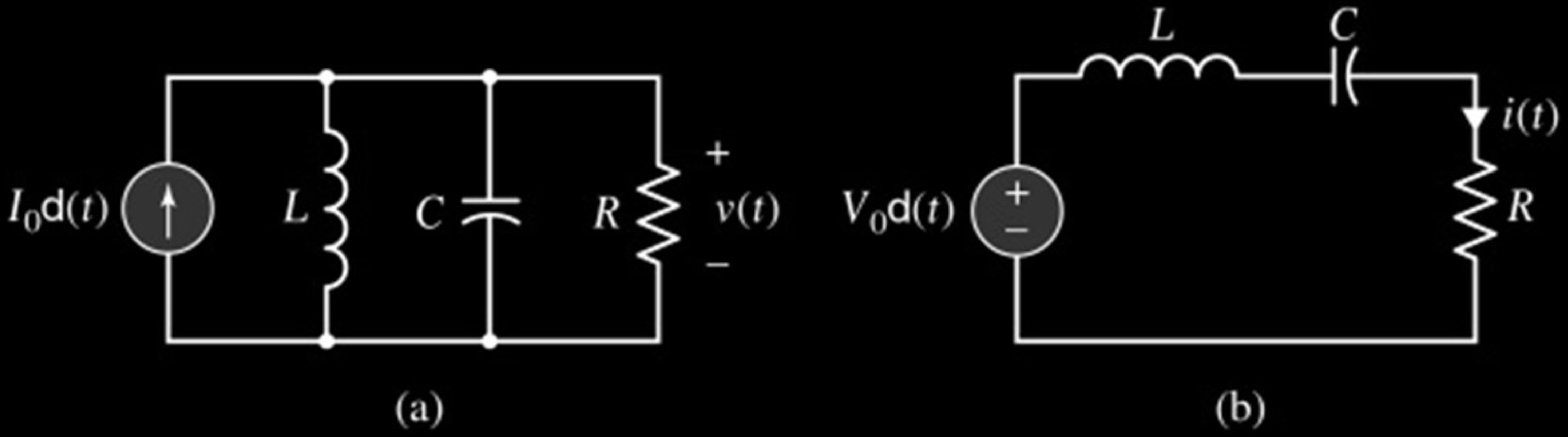
The Practical Use of a Unit Impulse

- Impulse function, is a signal of infinite amplitude at $t = 0$ and zero elsewhere.
- Practically an impulse is approximated by a very short duration pulse with very high amplitude.
- Suppose a current signal $I_0\delta(t)$ approximating an impulse function is applied across the RLC circuit in Fig. 1.45(a) at $t = 0$.
- A voltage is induced across capacitor at time t=0+ to suddenly rise to a new value

$$V_0 = \frac{1}{C}\int_{0-}^{0+} I_0\delta(t)d(t) = \frac{I_0}{C}. \qquad (1.69)$$

- Resulting value of voltage $v(t)$ across capacitor is defined by Eq. (1.50).
- The response $v(t)$ is called the transient response or the impulse response of circuit

Figure 1.45 (p. 49)
(a) Parallel *LRC* circuit driven by an impulsive current signal.
(b) Series *LRC* circuit driven by an impulsive voltage signal.



▶ **Problem 1.23** The parallel LRC circuit of Fig. 1.45(a) and the series LRC circuit of Fig. 1.45(b) constitute a pair of *dual* circuits, in that their descriptions in terms of the voltage $v(t)$ in Fig. 1.45(a) and the current $i(t)$ in Fig. 1.45(b) are mathematically identical. Given what we already know about the parallel circuit, do the following for the series LRC circuit of Fig. 1.45(b), assuming that it is initially at rest:

(a) Find the value of the current $i(t)$ at time $t = 0^+$.

(b) Write the integro-differential equation defining the evolution of $i(t)$ for $t \geq 0^+$.

Answers:

(a) $I_0 = V_0/L$

(b) $L\frac{d}{dt}i(t) + Ri(t) + \frac{1}{C}\int_{0^+}^{t} i(\tau)\, d\tau = 0$ ◀

1 .6.7 Derivatives of the Impulse

- From Fig. 1.42(a), the impulse $\delta(t)$ is the limiting form of a rectangular
- pulse of duration *A* and amplitude 1/*A*.
- Since rectangular pulse is equal to the step function (1/Δ)u(t + Δ /2) minus the step function (1/Δ)u(t - Δ /2) . From (1.62) and (1.63) we have
- the derivative of a unit-step function is a unit impulse, so differentiating the rectangular pulse with respect to time *t* yields a pair of impulses (rewritten):

$$\delta(t) = \frac{d}{dt}u(t) \qquad (1.62)$$

and

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad (1.63)$$

- One impulse of strength 1/ Δ, located at f = —Δ/2
- A second impulse of strength -1/ Δ, located at t = Δ /2

▶ Problem 1.24

(a) Evaluate the sifting property of $\delta^{(2)}(t)$.

(b) Generalize your result to describe the sifting property of the nth derivative of the unit impulse.

Answers:

(a) $$\int_{-\infty}^{\infty} f(t)\delta^{(2)}(t - t_0)\, dt = \frac{d^2}{dt^2} f(t)\Big|_{t=t_0}$$

(b) $$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t - t_0)\, dt = \frac{d^n}{dt^n} f(t)\Big|_{t=t_0}$$ ◀

- Where $\delta^1(t)$ is the 1st derivative and $\delta^2(t)$ is 2nd derivative of $\delta^2(t)$

1.6.8 Ramp Function

- The impulse function $\delta(t)$ is the derivative of the step function $u(t)$ with respect to time. Similarly, integral of step function $u(t)$ is a ramp function of unit slope shown in Fig. 1.46, defined as:

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{1.74}$$

equivalently

$$r(t) = tu(t) \tag{1.75}$$

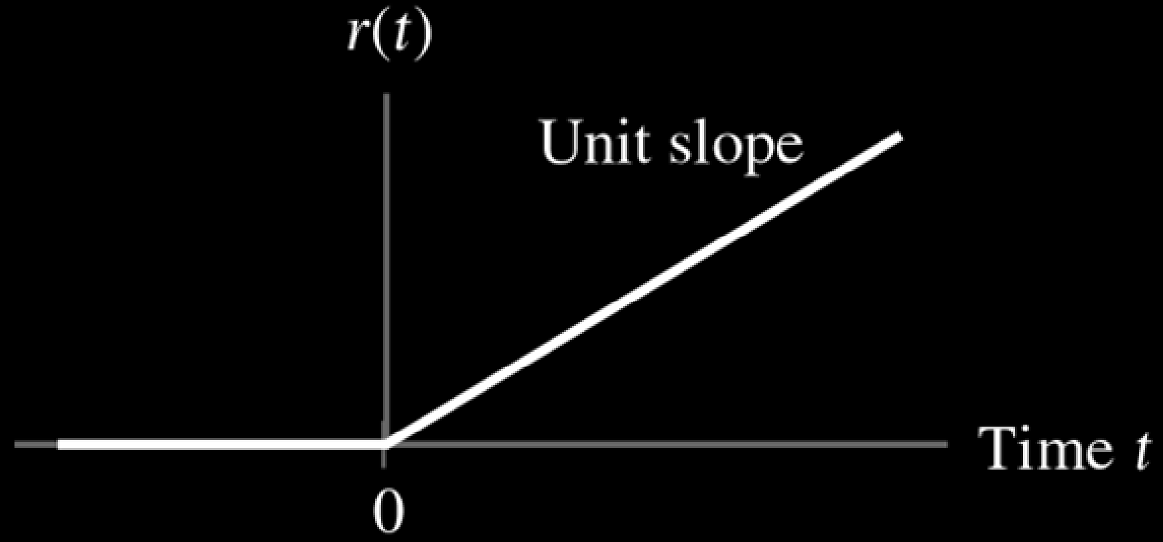


Figure 1.46 (p. 51)
Ramp function of unit slope.

• The discrete form is

$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{1.76}$$

• equivalently,

$$r[n] = nu[n] \tag{1.77}$$

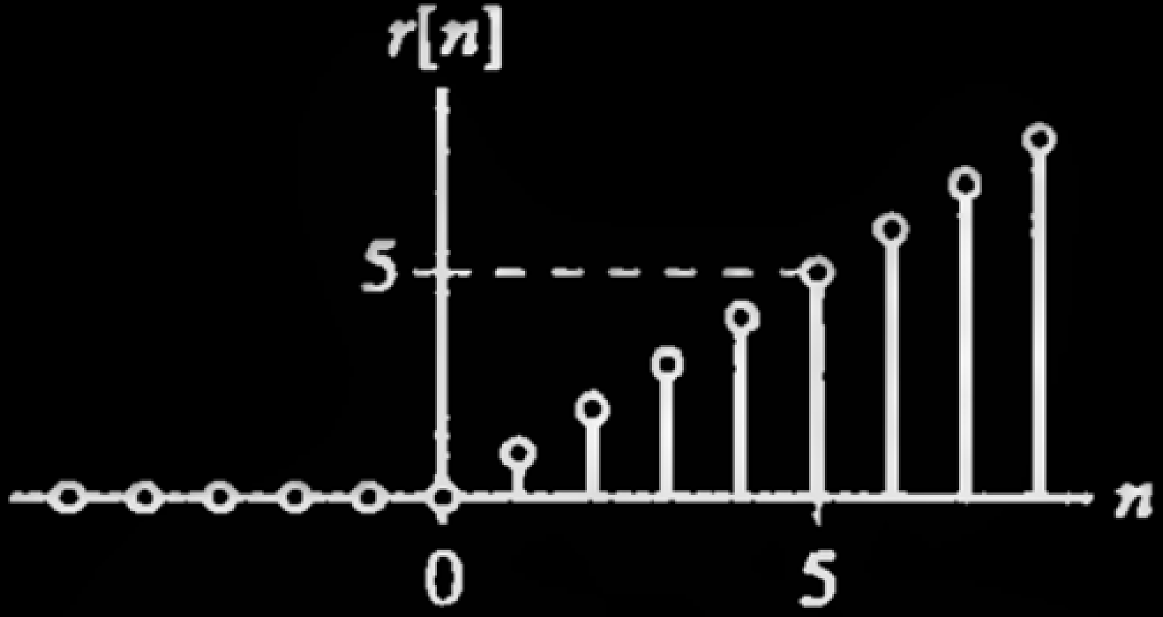


Figure 1.47 (p. 52)
Discrete-time version of the ramp function.

- **Example 1.11 Parallel Circuit:** Consider the parallel circuit of Fig. 1.48(a) involving a dc current source I_0 and an initially uncharged capacitor C. The switch across the capacitor is suddenly opened at time $t = 0$. Determine the current $i(t)$ flowing through the capacitor and the voltage $v(t)$ across it for $t \geq 0$.
- Solution: Once the switch is opened, at time $t = 0$ the current $i(t)$ jumps from zero to I_0 , and this behavior can be expressed in terms of the unit-step function as

$$i(t) = I_0 u(t)$$

- We may thus replace the circuit of Fig. 1.48(a) with the equivalent circuit shown in Fig. 1.48(b). The capacitor voltage $v(t)$ is related to the current i(t) by:

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

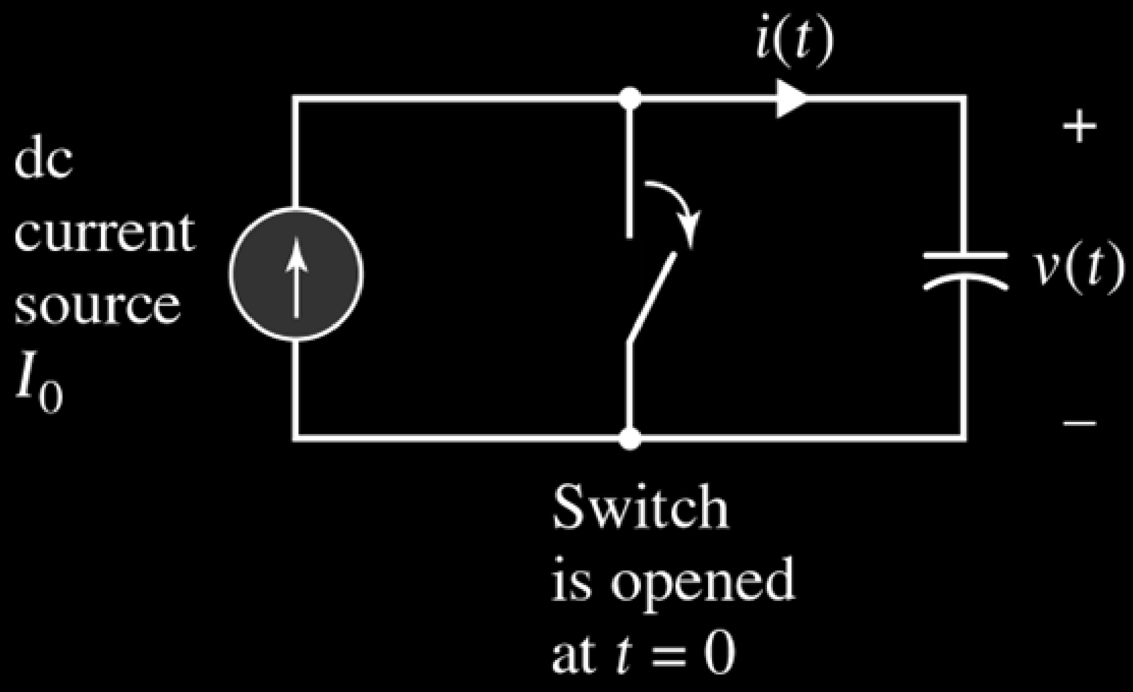
- Hence, using $i(t) = I_0 u(t)$ in this integral, we may write

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} I_0 u(\tau)\, d\tau = \begin{cases} 0 & \text{for } t < 0 \\ \frac{I_0}{C}t & \text{for } t \geq 0 \end{cases} = \frac{I_0}{C}tu(t) = \frac{I_0}{C}r(t)$$

- That is, the voltage across the capacitor is a ramp function with slope I_0/C

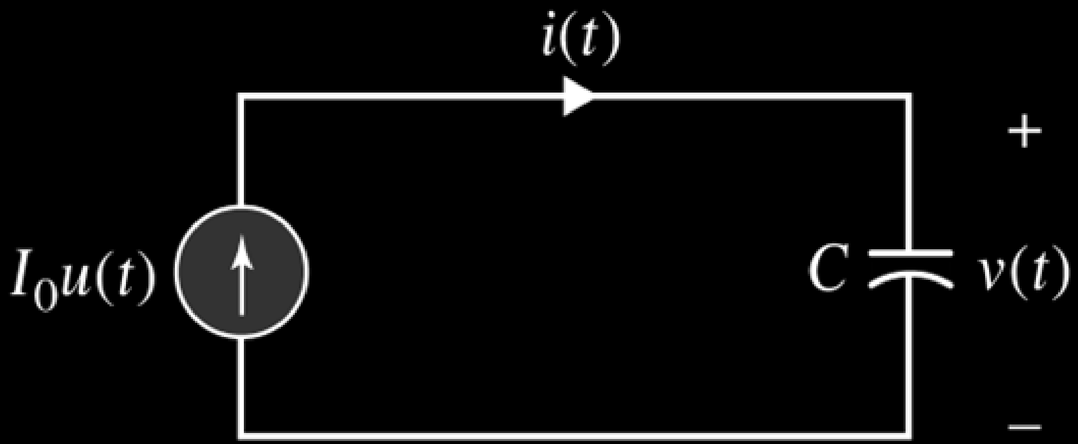
Figure 1.48 (p. 52)

(a) Parallel circuit consisting of a current source, switch, and capacitor, the capacitor is initially assumed to be uncharged, and the switch is opened at time $t = 0$.



(a)

(b) Equivalent circuit replacing the action of opening the switch with the step function $u(t)$.



(b)