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CS103
April 28, 2017

Problem Set Three

Checkpoint:
i)
Theorem: A binary relation R over a set A is a strict order if and only if the relation R is irreflexive and transitive.
Proof:
Lemma one:
A binary relation R over a set A is irreflexive and transitive if the relation R is a strict order. The definition of a strict order is a relation that is irreflexive, transitive, and asymmetric. Therefore, if relation R is a strict order, it is irreflexive and transitive.

Lemma two:
A binary relation R over a set A is a strict order if the relation R is irreflexive and transitive. Let R be an arbitrary irreflexive and transitive binary relation over a set A. We will prove that R is asymmetric.

By contradiction; suppose R is not asymmetric. That means that for some $a \in A$ and $b \in A$, if aRb holds then bRa holds.

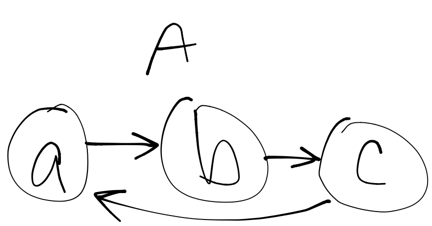
Since R is transitive, if aRb and bRa hold, then aRa holds. However, we know that R is irreflexive, so $a \not R a$ is true. Thus, aRa cannot hold.

A contradiction is found, so our original assumption was wrong. R is a binary relation over a set A that is asymmetric.

Since we have proved that R is a binary relation over a set A that is asymmetric, irreflexive, and transitive, by definition, R is a strict order.

Conclusion:
Since we have proved both that **a binary relation R over a set A is irreflexive and transitive if the relation R is a strict order**, and that **a binary relation R over a set A is a strict order if the relation R is irreflexive and transitive**, we can conclude that **a binary relation R is a strict order if and only if the relation R is irreflexive and transitive.**

ii)



We could not leave transitive out because there is no way to prove transitivity with only the other two. You need transitivity because Transitive combined with irreflexive implies asymmetry and transitive combined with asymmetry implies irreflexivity. As seen above, cRa can hold under the rules of irreflexivity and asymmetry. However, in a strict order, a would need to go before c, since a goes before b, so this is not a strict order. Thus, a binary relation that is irreflexive and asymmetric can be not a strict order.

Problem One: Odd Rational Numbers

i) *Theorem*: 3/2 is not an odd rational number.
Proof: By definition, an odd rational number is a number x that can be expressed as x = p/q, where q is odd. By contradiction; assume 3/2 is an odd rational number. 3/2 can be expressed as p/q where p = 3 and q = 2. However, q is 2, which is even. This contradicts our definition, since q must be odd, so our original assumption was incorrect. Therefore, 3/2 is not an odd rational number.

ii) Definition: An equivalence relation is a binary relation over a set A that is transitive, reflexive, and symmetric.

Theorem: *x ~ y if y - x is odd rational number* is an equivalence relation.
Proof:
Lemma 1: *x ~ y if y - x is odd rational number* is reflexive.
Proof: Consider arbitrary rational numbers *x* and *y*. Prove that if *x = y*, *y - x* is an odd rational number. If *x = y*, the expression can be re-written as: *x - x* is an odd rational number. *x - x = 0*, which can be expressed as *p/q* where *q* is odd, namely *p* = 0 and *q* = 1, 0/1.

Lemma 2: *x ~ y if y - x is odd rational number* is symmetric.
Proof: Consider arbitrary rational numbers *x* and *y*. Prove that if *y - x* is an odd rational number, *x - y* is an odd rational number. If *y - x* is an odd rational number, it can be expressed as *y - x = p/q* where *p* and *q* are integers and *q* is odd. This can be rewritten as *x - y = -pq*. Thus, *x - y* can be expressed as *p1/q* where *p1 = -p*. Since *p* is an integer, *-p* is also an integer, so *p1/q* satisfies our definition of an odd rational number. Thus, *x ~ y* is symmetric because if *y - x* is an odd rational number, *x - y* is also an odd rational number.

Lemma 3: *x ~ y if y - x is odd rational number* is transitive.
Proof: Consider arbitrary rational numbers *x*, *y*, and *z*. Prove that if *y - x* is an odd rational number and *z - y* is an odd rational number, then *z - x* is odd rational number. Assume *y - x* is an odd rational number. It can then be expressed as *p/q* where *p* and *q* are integers and *q* is odd. *y* can then be expressed as *x + p/q*. Thus, *z - y* can be expressed as *z - (x + p/q)*. Assume *z - (x + p/q)* is an odd rational number, expressed as *j/k* where *j* and *k* are integers and *k* is odd. So the expression is *z - x - p/q = j/k* where *p*, *q*, *j*, and *k* are integers and *q* and *k* are odd. This expression can be rewritten as *z - x = p/q + j/k*. This can be rewritten as *z - x = (pk + jq) / qk*. Since *q* and *k* are odd integers, *qk* is also an odd integer (lemma 4), and *p*, *k*, *j*, and *q* are integers, *pk + jq* is an integer. Thus, we can rewrite the expression as *z - x = m/n*, where *m = pk + jq* and *n = qk*. Now, we have *z - x = m/n*, where *m* is an integer and *n* is an odd integer, which satisfies our definition of an odd rational number.

Lemma 4: The product of two odd numbers is odd.
Proof: Let *a* and *b* be arbitrary odd numbers that can be expressed as *2k + 1* and *2j + 1* respectively, where *k* and *j* are integers.

$$(2k+1)(2j+1) = 4jk + 2k + 2j + 1 = 2(2jk + k + j) + 1$$

There exists an integer *m*, where *m = 2jk + k + j*, so that the above expression can be expressed as *2m + 1,* which is odd.

Conclusion: Since we have proved that *x ~ y if y - x is an odd rational number* is reflexive, symmetric, and transitive, then ~ is an equivalence relation by definition.

iii) [0]~ = { $y \in \mathbb{R}$ | y is an odd rational number }

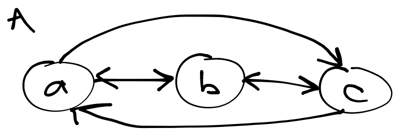
Theorem: For all real numbers *y* where *y* is odd, *y - 0* will always be an odd rational number.

Proof: Consider an arbitrary real number *y* that is an odd rational number. *y - 0 = y*. Thus, if *y* is an odd rational number, then *y - 0* is the same number and thus is also an odd rational number.

By contradiction; Assume *y* is not an odd rational number. Then, *y - 0* cannot be an odd rational number, since *y - 0 = y* and *y* is not an odd rational number. So any number that does not follow the rule of the given equivalent set does not belong there. Thus, the provided equivalence set rule covers all elements that belong in the set and excludes inequivalent elements.

Problem Two : Redefining Equivalence Relations?

i)

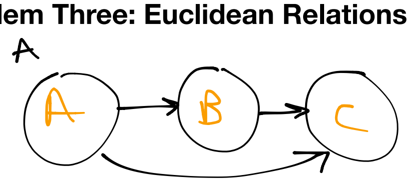


None of the elements point back to themselves, so the relation is not reflexive. However, for any relation that is true for one direction, it is true in the other as well, i.e if *aRb* then *bRa*. Also, if *aRb* and *bRc*, then *aRc* is true as well.

ii) The proof is inaccurate in the case of set A being one element. Thus, elements *x* and *y* in A would be equivalent. Proceeding with the assumption that *xRy* is true, would be the same as assuming *xRx* is true, since *x* = *y*. This would be assuming reflexivity, which is what you're trying to prove. So this is an invalid proof.

Problem Three: Euclidean Relations

i)



aRb and *aRc* --> *bRc*. Where *aRb* holds and *aRc* holds, *bRc* holds. There is no symmetry, no arrow pointing out has one coming back in.

ii) *Theorem*: R is an equivalence relation if and only if it is reflexive and Euclidean.
Proof part 1: Prove that R is transitive, symmetric, and reflexive if it is reflexive and Euclidean. Assume R is Euclidean and reflexive, prove that it is transitive and symmetric.
Lemma 1: R is symmetric
Proof: If R is Euclidean, then for R and arbitrary elements of A *x*, *y*, and *z*, if *xRy* holds and *xRz* holds, then *yRz* holds. Now, let *z* = *x*. Now we can say that if *xRy* holds and *xRx* holds, then *yRx* holds. We know *xRx* holds because R is reflexive. So if *xRy* holds, then *yRx* holds, proving symmetry.
Lemma 2: R is transitive.
Proof: Let *x*, *y* and *z* be arbitrary elements in A. Prove that if *xRy* and *yRz* hold, *xRz* holds. Assume *xRy* and yRz hold. Since R is symmetric, we know that yRx and zRy hold. Since yRx and yRz hold, then, since R is Euclidean, xRz. Thus, R is transitive.

Proof part 2: Prove that an equivalence relation is reflexive and Euclidean.
Equivalence relations are reflexive, symmetric, and transitive.
Prove that equivalence relations are Euclidean.
Assume R is an equivalence relation, making R reflexive, symmetric, and transitive. Show that for all elements x, y, and z in A, if xRy and xRz, then yRz. Let x, y, and z be arbitrary elements in A. Assume xRy and xRz hold. Since R is symmetric, we know that yRx holds. Since R is transitive, and yRx and xRz holds, then yRz holds.

Conclusion:
Since we proved both that a binary relation R that is an equivalence relation is reflexive and Euclidean, and that a binary relation R that is Euclidean and reflexive is an equivalence relation, then a binary relation R over set A is an equivalence relation if and only if R is Euclidean and reflexive.

Problem Four: The Less-Than Relation

Theorem: The less than relation is a strict order.

Proof: A binary relation R is a strict order if R is transitive and irreflexive, as proved in the checkpoint. We will prove that < is irreflexive and transitive.

Lemma 1: < is irreflexive

Proof: By definition, x < y if there exists a natural number k where k is not 0 and y = x + k. By contradiction; assume < is reflexive. For arbitrary integers x and y, x = y, to test if reflexivity, x < x holds. The only time x = x + k, where k is a natural number is when k = 0. However, this is impossible by definition. A contradiction is found, so our original assumption is false. Therefore, < is irreflexive.

Lemma 2: < is transitive

Proof: Let x and y and z be arbitrary integers where x < y and y < z. Prove that x < z. By definition, since x < y, y = x + k, where k is a natural number that is not 0. Similarly, since y < z, z = y + j, where j is a natural number that is not 0. Since y = x + k, z = x + k + j. This can be rewritten as z = x + m where m is k + j. Since k and j are nonzero natural numbers, so is m. So x < z because there exists a nonzero natural number m where z = x + m. Therefore, < is transitive.

Conclusion: Since < is irreflexive and transitive, it is a strict order.

Problem Five: Hasse Diagrams and Covering Relations

i) For $\mathbb{N}$, $x \lessdot y$ if y - x = 1.

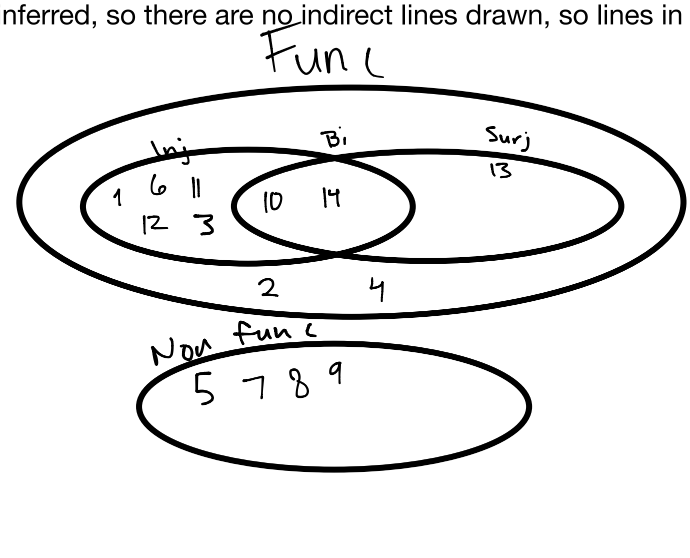
ii) *Theorem*: $\lessdot$ is not a strict order.

Proof: Strict orders must be transitive. Prove that $\lessdot$ is not transitive by finding a set A where for arbitrary elements x, y, and z, $x \lessdot y$ and $y \lessdot z$ hold, but $x \lessdot z$. In part i, we defined the set $\mathbb{N}$ and for the set, $x \lessdot y$ if y - x = 1. Let x, y, z be natural numbers where x = 1, y = 2, and z = 3. Since 2 - 1 = 1 and 3 - 2 = 1, we can say that $x \lessdot y, y \lessdot z$. But 3 - 1 ≠ 1 so $x \not\lessdot z$. Therefore, $\lessdot$ is not transitive, and thus is not a strict order.

iii) For $\wp(\{1, 2, 3, 4\})$, $x \subsetneq y$ if all elements in x are in y, and $|x|=|y|-1$. If all the elements in x are in y, then it is a subset of y. However, x cannot equal y, so they must not have the same cardinality. Finally, x must have exactly one less element than y to be covered because otherwise, there would exist an element z where $x \subseteq z$ and $z \subseteq y$. E.g x = {1}, y = {1, 2, 3} and z = {1, 2}.

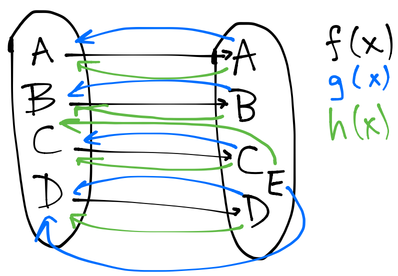
iv) The Hasse diagram shows the relationship between elements in a partially ordered set by drawing lines between them and positioning them on the diagram according to upward direction. The line segments between elements denote covering, since they show a direct relation where no other element is in between. Transitivity is inferred, so there are no indirect lines drawn, so lines in Hasse diagrams always imply covering.

Problem Six: Properties of Functions



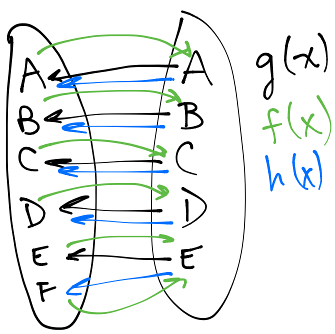
Problem Seven: Left, Right, and True Inverses

i)



ii)
Theorem: Let f: A --> B be a function. If f has a left inverse, then f is injective.
Proof: An injective function has one input for every output, so if the outputs are the same, the inputs are the same. Let x and y be elements of A. Assume f(x) = f(y). Plug in these values to the left inverse function g: B --> A. Since f(x) = f(y), g(f(x)) = g(f(y)) by since functions are deterministic. Since g is a left inverse of f, for any element a in A, g(f(a)) = a. Therefore, g(f(x)) = x and g(f(y)) = y. Thus, x = y. We have proved that if the outputs are the same, the inputs are the same, which makes a function injective by the definition of injectivity.

iii)



iv) *Theorem*: Let function f: A --> B be a function. If f has a right inverse, then f is surjective.
Proof: By contradiction; assume f has a right inverse and is NOT surjective. If f is NOT surjective, then for every possible output, there is NOT a possible input that produces it. That means that there is an element b in f's codomain B, where there is no possible input a from A that produces it. But this is impossible because f has a right inverse, so f(g(b)) = b where g: B --> A is a function and b is an element of B. This means for all elements b in B, g(b) is an element a in A where a is g(b) and f(a) is b. Thus, all elements b in B have a corresponding element a in A where f(a) produces b, so there cannot be any elements in b that are impossible to produce from an element a in A. Since we have reached a contradiction, our initial assumption was wrong. Therefore, if f has a right inverse, it must be surjective.

v) Theorem: If f: A --> B is a function, and both g1: B --> A and g2: B --> A are inverses of f, then g1(b) = g2(b) for all b$\in$B.
Proof: If g1 and g2 are inverses of f, then g1 and g2 are simultaneously left and right inverses of f. Since we proved that if f has a left inverse it is injective, and if f has a right inverse it is surjective, f is both injective and surjective. Therefore, it is a bijection. Since a bijection uniquely associates each element in its domain A with a unique element in the codomain B, there is only one mapping that associates each unique element in domain B with a unique element in domain A. Therefore, g1 and g2 have the same mapping from B to A, which means that g1(b) = g2(b) for all b$\in$B.

vi) My proof in part v would not work if g1 and g2 were just left inverses, because it would no longer be implied that f is a bijection (since it is not injective) which my proof relies on to prove unique mapping.

vii) My proof in part v would not work if g1 and g2 are just right inverses of f, because it would no longer be implied that f is a bijection (since it is not surjective), which my proof relies on to prove unique mapping.

Problem Eight: Function Composition
i) Theorem: If f is surjective and g is not a bijection, then g∘f is not a bijection.
Proof: Let f: A --> B be a function and g: B --> C be a function. If f is surjective, then there exists an output b in codomain B for which there is no possible input a in A that maps to b. If g is not a bijection, then g has the quality of being not surjective or not injective.
Case 1 - g is not surjective:
If g is not surjective, then there are elements c in codomain C that have no possible mapping from an element b in B. Since g∘f's codomain is that of g's, then g∘ f is not surjective, since there are unmapped elements in its codomain. Since it is not surjective, it is not a bijection.

Case 2 - g is not injective:
If g is not injective, then there are elements b in domain B that map to the same output element c in C. Since f is surjective, every element in f's codomain B can be produced by an input in f's domain A. Then, g's domain covers all of domain B, but g is not injective, so multiple inputs from B point to a single element in codomain C. Since g∘f's domain is that of f's, and g∘ f's codomain is that of g's, then g∘f is not injective, since multiple inputs from f's domain A can map to a single output in g's codomain C.

ii) Theorem: If f is not a bijection and g is injective, then g∘f is not a bijection.
Proof: Let f: A --> B be and function and g: B --> C be a function. If g is injective, then every element in g's domain B maps to exactly one unique element in g's domain C. If f is not a bijection, it has the quality of being either not surjective or not injective.

Case 1 - f is not injective
If f is not injective, then there are elements in f's domain A that map to the same output in f's codomain B. Since g is injective, all elements in g's domain B are mapped one to one with g's codomain C, which is also g ∘ f's codomain. Since multiple inputs from A are mapped to a single output in B, which is mapped to a single output in C, there exists multiple elements in A that output the same element c in C. Thus, g ∘ f is not injective, and therefore not a bijection.

Case 2 - f is not surjective
If f is not surjective, then there are elements in f's codomain B where no input is mapped to. Since g is injective, each output of f is mapped to one element in C. However, the elements that g could map in C from B are lost because the output of f did not cover the entire domain of B. Since there are elements c in C that have no possible inputs, g ∘ f is not surjective, and therefore not a bijection.

iii)