Proof of Divergence Harmonic Series

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Analysis I

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Question [Divergence of Harmonic Series]. Prove that the harmonic series diverges.

Proof. (Proof by contradiction) Suppose that the harmonic series converges. Let S be the sum of the harmonic series, i.e.

$$S = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

We may group the terms of the harmonic series into groups of 3 as follows:

$$S = 1 + (\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7}) + \cdots$$

For all $x \in \mathbb{R}, x > 1$, we have that $\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} > \frac{3}{x}$. This can be shown by expanding the left hand side and simplifying as follows:

$$\begin{aligned}
\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} &= \frac{(x^2+x) + (x^2-1) + (x^2-x)}{x(x-1)(x+1)} \\
&= \frac{3x^2-1}{x(x^2-1)} \\
&= \frac{2x^2 + (x^2-1)}{x(x^2-1)} \\
&= \frac{2x}{x^2-1} + \frac{1}{x} \\
&> \frac{2x}{x^2} + \frac{1}{x} \\
&= \frac{3}{x}.
\end{aligned}$$

Therefore, choosing $x = 3, 6, 9, \cdots$ we have that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} > \frac{3}{3} = \frac{1}{1}$$
$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} > \frac{3}{6} = \frac{1}{2}$$
$$\frac{1}{8} + \frac{1}{9} + \frac{1}{10} > \frac{3}{9} = \frac{1}{3}$$
$$\vdots$$

Proceeding in this manner, we have that

$$\begin{aligned} S &= 1 + (\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7}) + \cdots \\ &> 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots \\ &= 1 + S. \end{aligned}$$

$$\therefore S > 1 + S.$$

Which is clearly impossible for any finite S. Therefore, the harmonic series diverges. $\square$