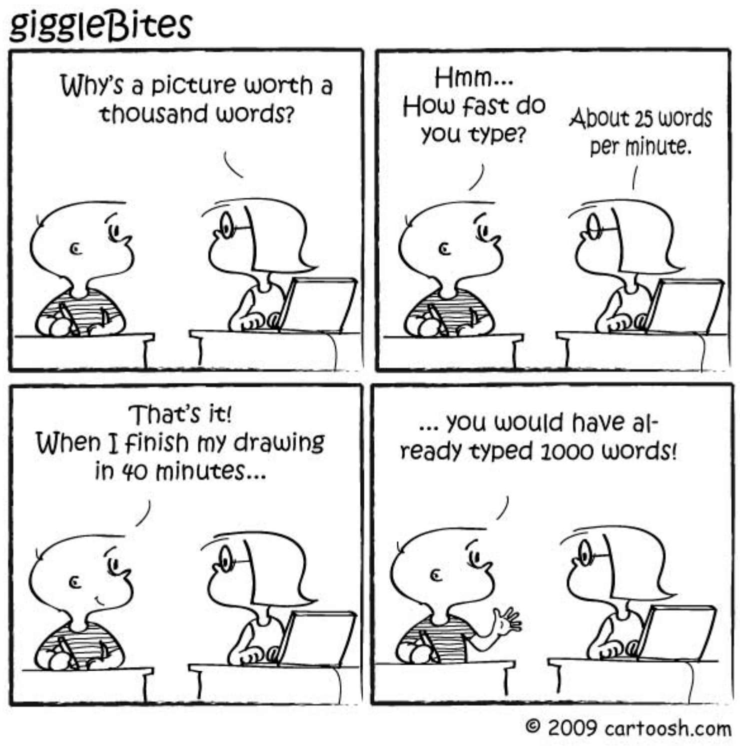
Mathematical & Graphical Review

A picture is worth a thousands words.

Chinese Proverb



Functions

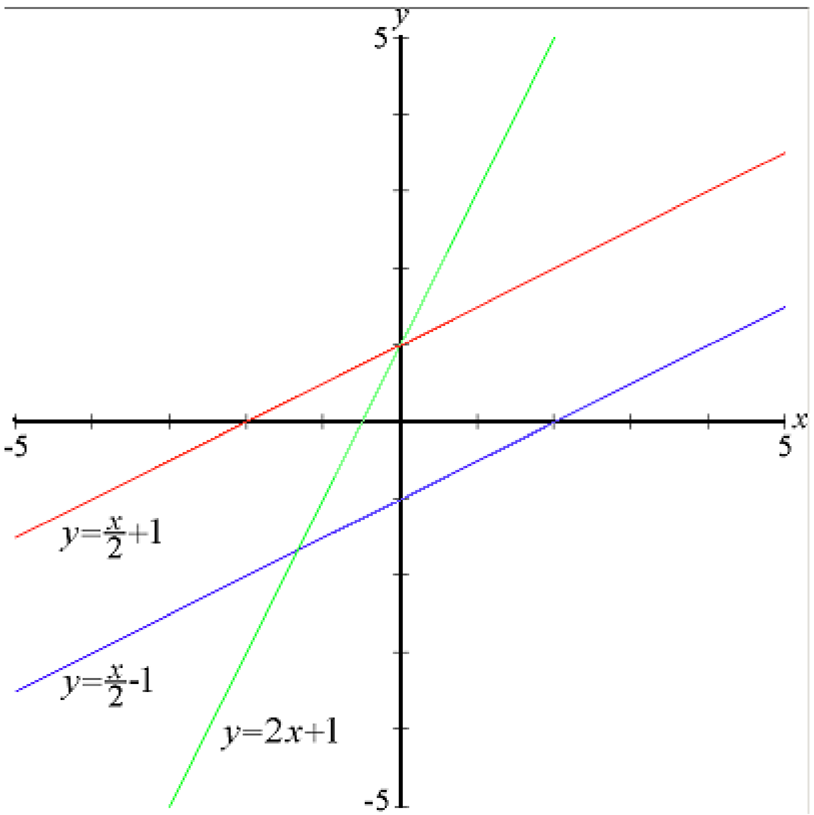
• A **function** is a relation between a given set of elements (the domain) and another set of elements (the codomain), which associates each element in the domain with exactly one element in the codomain.

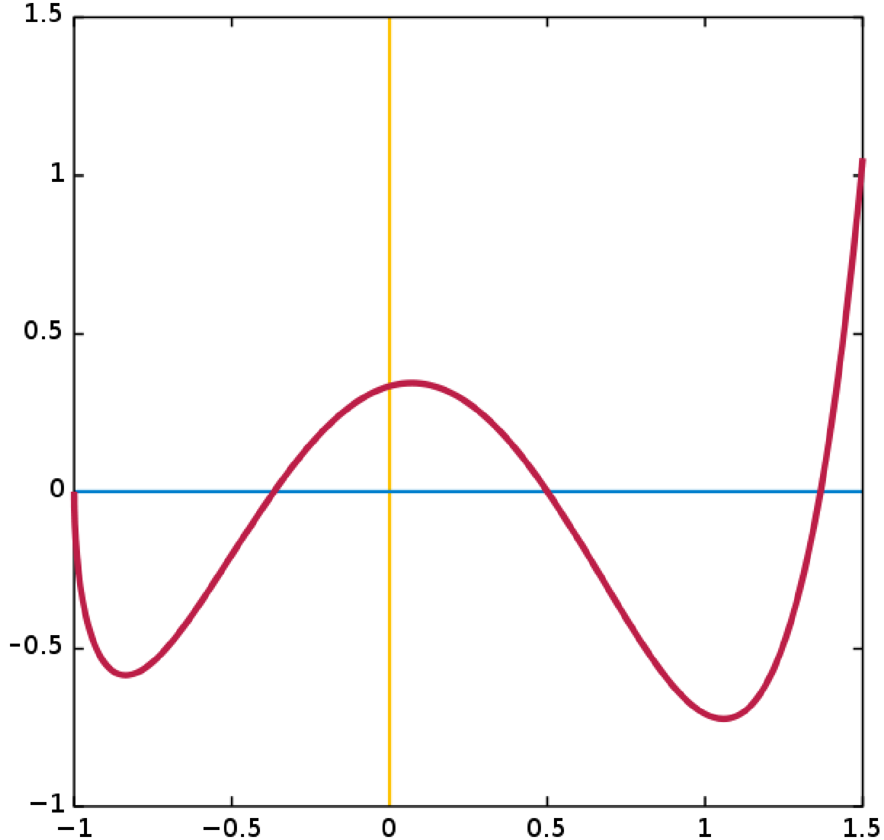
• The symbol for the input to a function is often called the **independent variable** or **argument** and is often represented by the letter x. The symbol for the output is called the **dependent variable** or **value** and is often represented by the letter y. The function itself is most often called f, and thus the notation $\boldsymbol{y = f(x)}$ indicates that a function named f has an input named x and an output named y.

• Linear functions can be written as $\boldsymbol{f(x) = bx + a}$ *(algebraic expression)*, where b and a are real constants and x is a real variable. The constant $\boldsymbol{b}$ is often called the **slope**, while a is the **y-intercept**.

• A function that cannot be expressed as $\boldsymbol{f(x) = bx + a}$ is called **a non-linear function**.

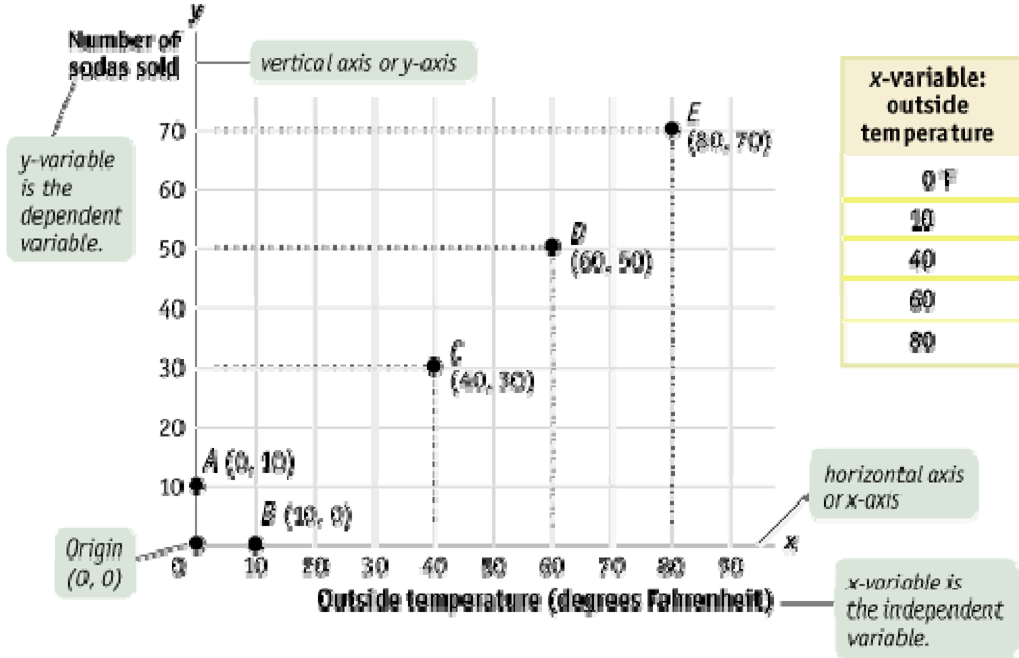
Linear & non-linear Functions





How Graphs Work

Two-Variable Graphs



x-variable: outside temperature	y-variable: number of sodas sold	Point
0°F	10	A
10	0	B
40	30	C
60	50	D
80	70	E

How Graphs Work

Two-Variable Graphs

• A **curve** is a line on a graph that depicts a relationship between two variables. It may be either a straight line or a curved line. If the curve is a straight line, the variables have a **linear relationship**. If the curve is not a straight line, the variables have a **nonlinear relationship**.

• Two variables have a **positive (negative) relationship** when an increase in the value of one variable is associated with an increase (decrease) in the value of the other variable. It is illustrated by a curve that slopes upward (downward) from left to right.

• The **horizontal intercept** of a curve is the point at which it hits the horizontal axis; it indicates the value of the x-variable when the value of the y-variable is zero.

• The **vertical intercept** of a curve is the point at which it hits the vertical axis; it shows the value of the y-variable when the value of the x-variable is zero.

A Key Concept: The Slope of a Curve

- The **slope** of a line or curve is a measure of how steep it is. The slope of a line is measured by "rise over run"—the change in the *y*-variable between two points on the line divided by the change in the *x*-variable between those same two points.

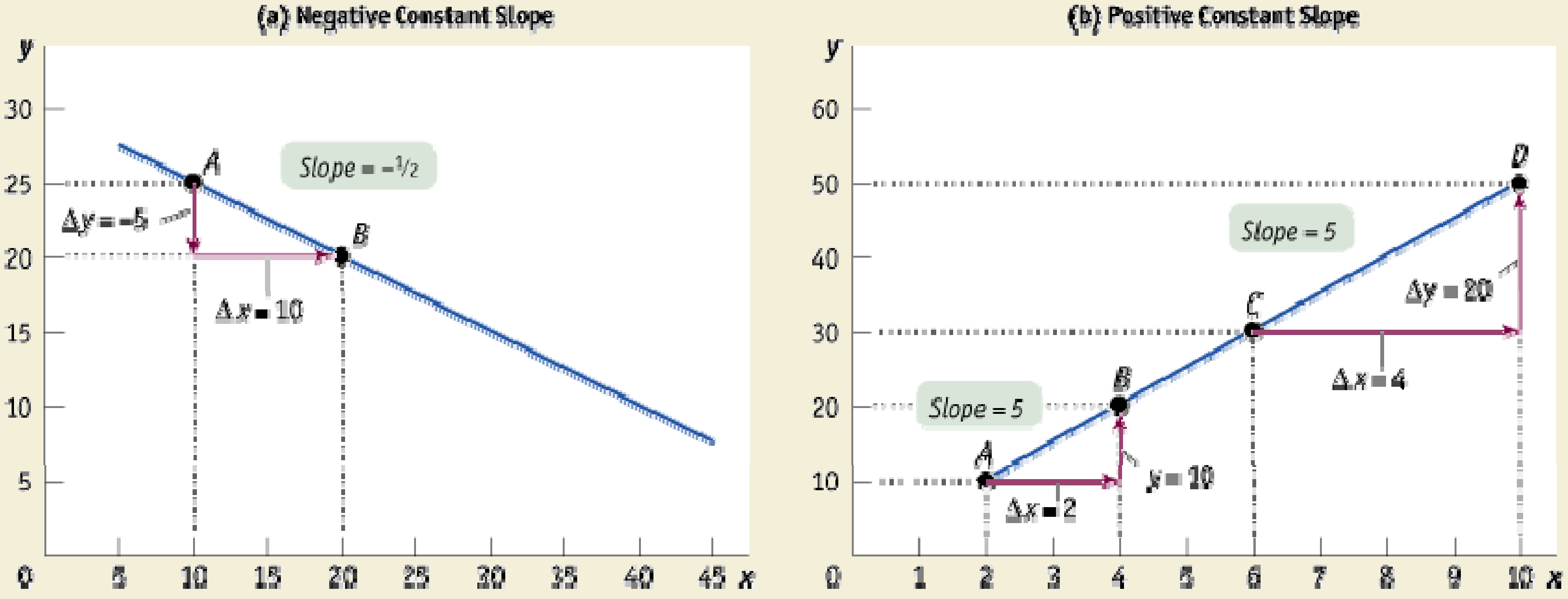
The Slope of a Linear Curve

$$\frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \text{Slope}$$

A Key Concept: The Slope of a Curve

The Slope of a Linear Curve

Figure 2A-3 Calculating the Slope



Panels (a) and (b) show two linear curves. Between points *A* and *B* on the curve in panel (a), the change in *y* (the rise) is –5 and the change in *x* (the run) is 10. So the slope from *A* to *B* is $\Delta y/\Delta x = -5/10 = -1/2 = -0.5$, where the negative sign indicates that the curve is downward sloping. In panel (b), the curve has a slope from *A* to *B* of $\Delta y/\Delta x = 10/2 = 5$. The slope from *C* to *D* is $\Delta y/\Delta x = 20/4 = 5$. The slope is positive, indicating that the curve is upward sloping. Furthermore, the slope between *A* and *B* is the same as the slope between *C* and *D*, making this a linear curve. The slope of a linear curve is constant: it is the same regardless of where it is calculated along the curve.

A Key Concept: The Slope of a Curve

Horizontal and Vertical Curves and Their Slopes

• When a curve is horizontal, the value of **y** along that curve never changes—it is **constant**. The slope of a horizontal curve is always zero.

• If a curve is vertical, the value of **x** along the curve never changes—it is **constant**. The slope of a vertical line is equal to infinity.

• A vertical or a horizontal curve has a special implication: it means that the ***x*-variable and the *y*-variable are unrelated**.

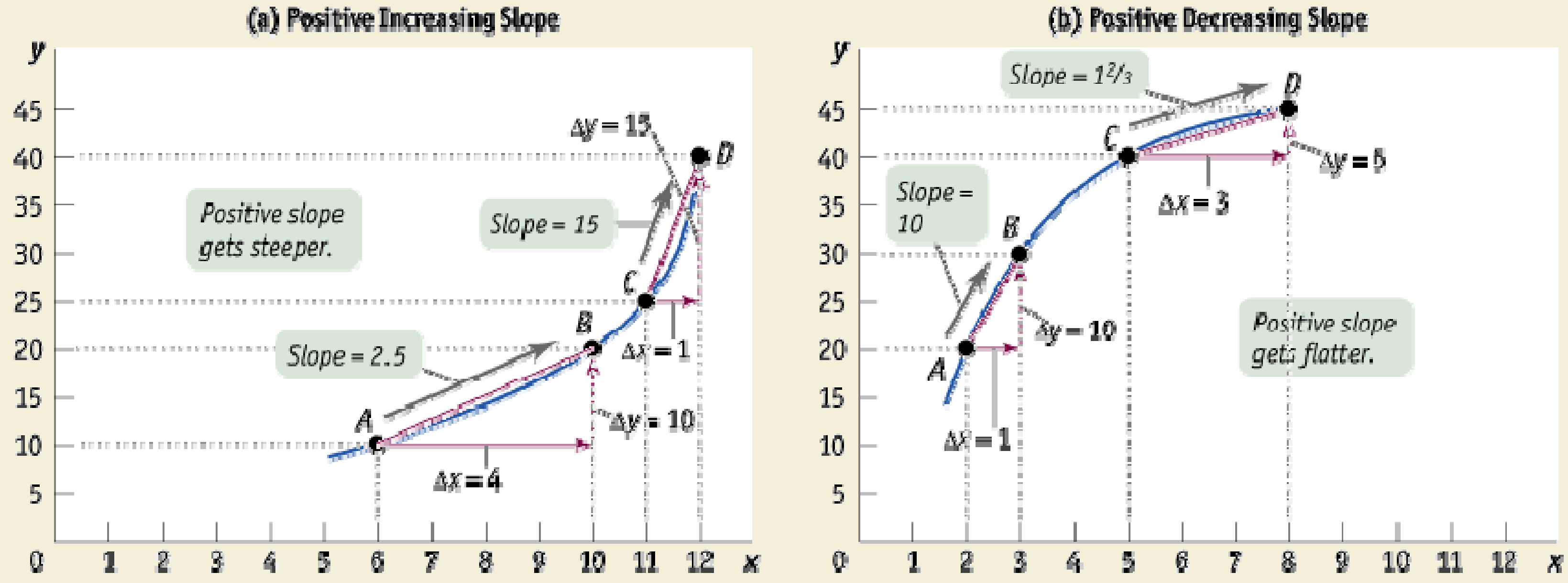
• The slope of a **nonlinear curve** is not the same between every pair of points.

• The **absolute value** of a negative number is the value of the negative number without the minus sign.

A Key Concept: The Slope of a Curve

The Slope of a Nonlinear Curve

Figure 2A-4 **Nonlinear Curves**



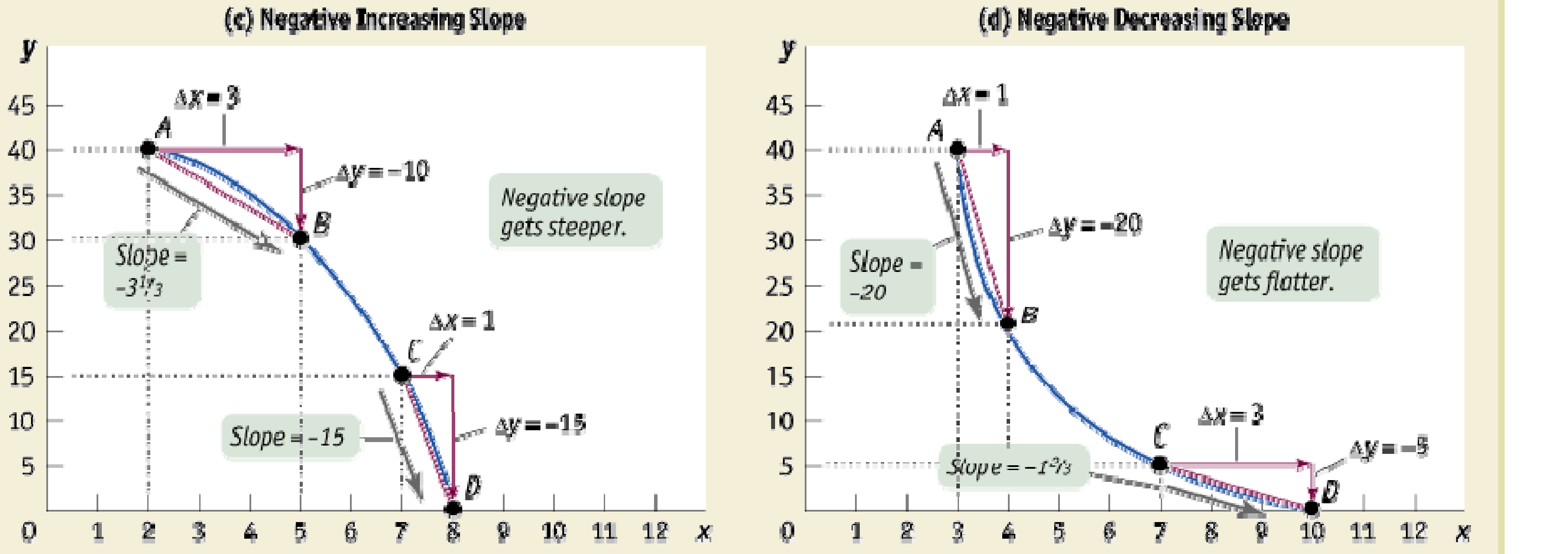
In panel (a) the slope of the curve from A to B is $\Delta y/\Delta x = 10/4 = 2.5$, and from C to D it is $\Delta y/\Delta x = 15/1 = 15$. The slope is positive and increasing; it gets steeper as you move to the right.

In panel (b) the slope of the curve slope from A to B is $\Delta y/\Delta x = 10/1 = 10$, and from C to D it is $\Delta y/\Delta x = 5/3 = 1\frac{2}{3}$. The slope is positive and decreasing; it gets flatter as you move to the right.

A Key Concept: The Slope of a Curve

The Slope of a Nonlinear Curve

Figure 2A-4 Nonlinear Curves, cont'd.



In panel (c) the slope from A to B is $\Delta y/\Delta x = -10/3 = -3\frac{1}{3}$, and from C to D it is $\Delta y/\Delta x = -15/1 = -15$. The slope is negative and increasing; it gets steeper as you move to the right.

And in panel (d) the slope from A to B is $\Delta y/\Delta x = -20/1 = -20$, and from C to D it is $\Delta y/\Delta x = -5/3 = -1\frac{2}{3}$. The slope is negative and decreasing; it gets flatter as you move to the right.

The slope in each case has been calculated by using the arc method—that is, by drawing a straight line connecting two points along a curve. The average slope between those two points is equal to the slope of the straight line between those two points.

A Key Concept: The Slope of a Curve

• To calculate the slope along a nonlinear curve, you draw a straight line between two points of the curve.

• The slope of that straight line is a measure of the average slope of the curve between those two end-points

• We use the derivative, which is the slope of a straight line that is tangent to the curve at one point, to have information of the slope of a curved line (at this point):

$$\lim_{x\to 0}\frac{\Delta y}{\Delta x}\Big|_{a_0}=\frac{dy}{dx}\Big|_{a_0}=\frac{df(x)}{dx}\Big|_{a_0}=\text{Slope}\,\Big|_{a_0}$$

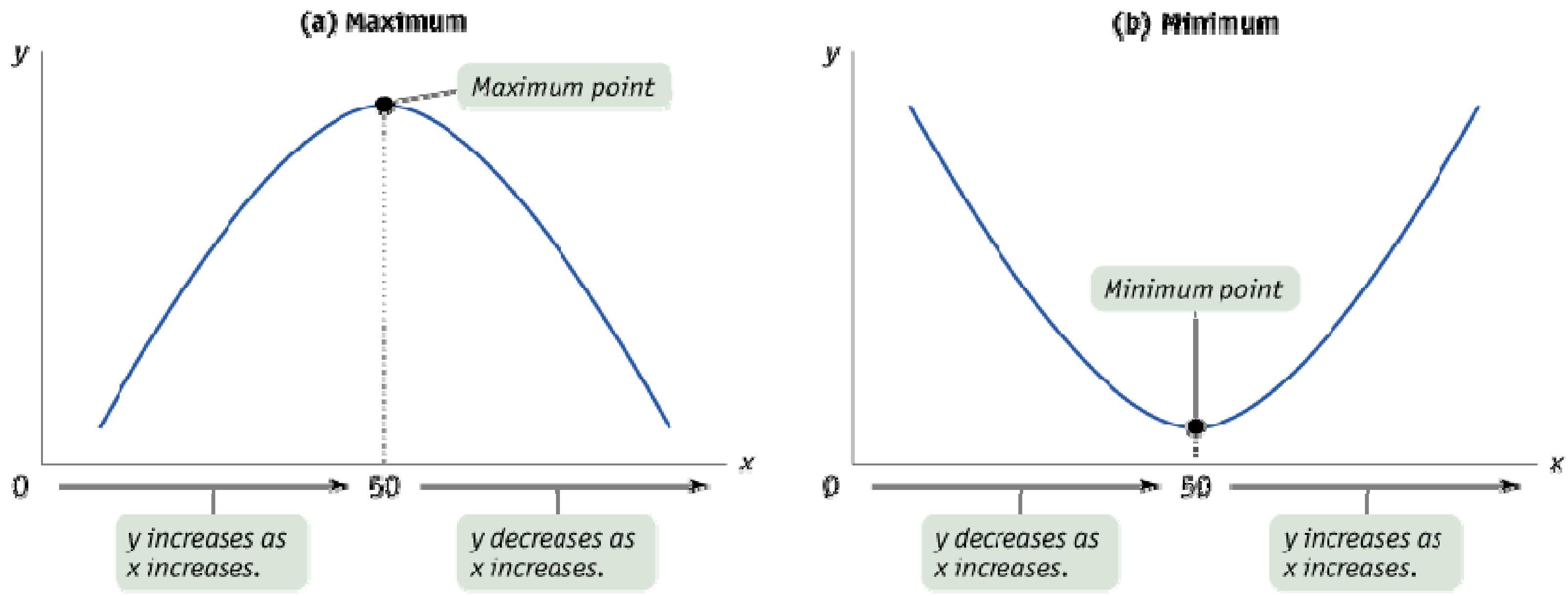
See: http://en.wikipedia.org/wiki/File:Graph_of_sliding_derivative_line.gif

A Key Concept: The Slope of a Curve

Maximum and Minimum Points

- A nonlinear curve may have a **maximum** point, the highest point along the curve. At the maximum, the slope of the curve changes from positive to negative.

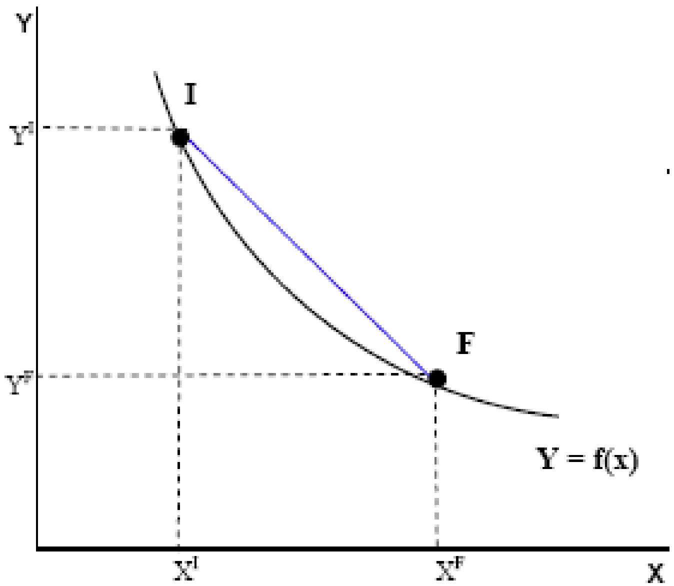
- A nonlinear curve may have a **minimum** point, the lowest point along the curve. At the minimum, the slope of the curve changes from negative to positive.

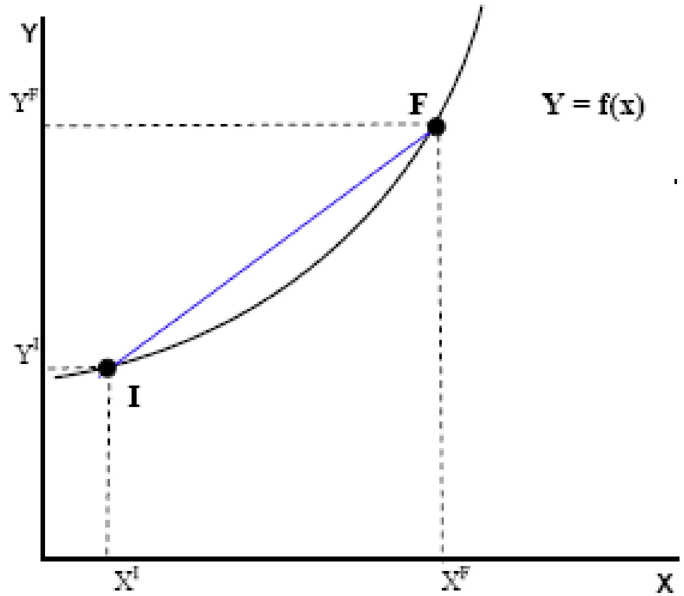


Related Concepts: Concavity and Convexity

Convexity

- A non-linear function (*f*) is called **convex** (in an interval) if **it lies below the straight line segment connecting two points**, for any two points (in the interval).



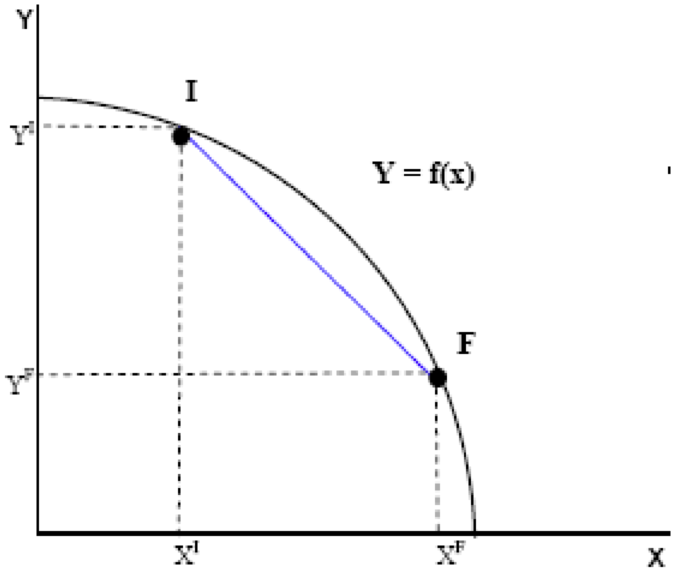


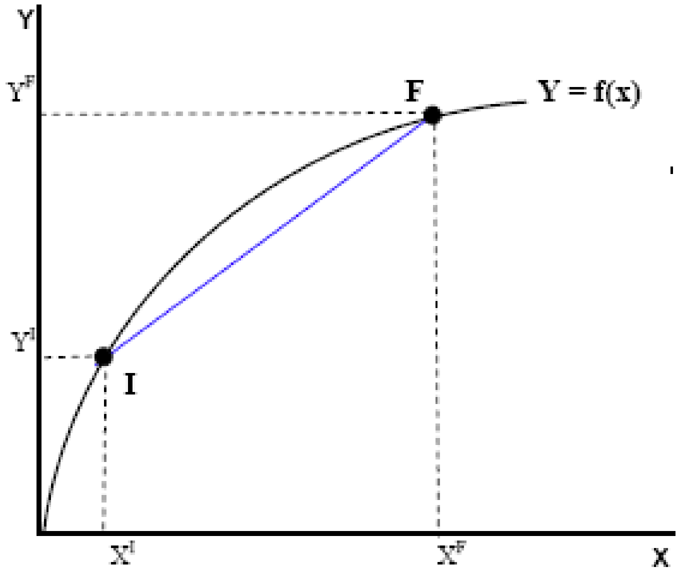
$$f \text{ is convex} \Leftrightarrow f'' = \frac{d^2 f(x)}{x^2} \geq 0$$

Related Concepts: Concavity and Convexity

Concavity

• A non-linear function (*f*) is called **concave** (in an interval) if **it lies above the straight line segment connecting two points**, for any two points (in the interval).





$$f \text{ is concave} \Leftrightarrow f'' = \frac{d^2 f(x)}{x^2} \leq 0$$

Related Concepts: Functions of two variables

• Consider the function *Y* = *f(X,Z)*. In this case, the dependent variable can vary because of *X*, because of *Z* or because of *X* and *Z* simultaneously.

• When we are interested in analysing the effect on *Y* of shifts in just one variable (for instance, *X*), we will use the clause "*Ceteris Paribus*" which means "*all other things being equal*".

• Mathematically, we will calculate de partial derivative, which is the derivative of Y with respect to one of its variables with the other held constant :

$$\left.\frac{\delta Y}{\delta X}\right|_Z = \left.\frac{\delta f(X)}{\delta X}\right|_Z = \lim_{\Delta X \to 0} \left.\frac{\Delta Y}{\Delta X}\right|_Z$$

Related Concepts: Useful derivatives

• During the course we will use linear functions like: ***f*(*x*) = *bx* + *a*** whose general derivative is ***f'*(*x*) = *b***.

• In addition, we will also use some non-linear functions (very often, but not always!) of the form: $y = f(x) = c\sqrt[j]{x} = cx^{1/j}$.

The derivative of the above function is: $f'(x) = \frac{c}{j} x^{\frac{1}{j}-1}$.

• Some examples:

$$y = 10\sqrt{x} \Rightarrow f'(x) = \frac{5}{\sqrt{x}}$$

$$y = \frac{3}{2}x + \frac{2}{3} \Rightarrow f'(x) = \frac{3}{2}$$

$$y = 12\sqrt[3]{x} \Rightarrow f'(x) = \frac{4}{x^{2/3}}$$

KEY TERMS

Function (linear, non-linear)

Dependent, Independent variable

Curve

Graph

Slope

Horizontal intercept

Vertical intercept

Maximum

Minimum

Concavity

Convexity

Derivative

Absolute value