Lecture 6

Linear homogeneous differential systems with constant coefficients

$A \in \mathcal{M}_n(\mathbb{R})$ $\quad X' = AX$, where the unknown $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$

e^{tA}

Recall $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$, $\forall t \in \mathbb{R}$

Example: 1) For $A = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ we have $e^{tA} = \operatorname{diag}(e^{t\lambda_1}, e^{t\lambda_2}, \dots, e^{t\lambda_n})$

2) We found, using the initial definition, $e^{t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ $\forall t \in \mathbb{R}$

Second method: Using the equivalent def (1)

$$\cos t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots$$

$$\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$$

$A \overset{\text{not}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$

$A^3 = -A$

$A^4 = -A \cdot A = -A^2 = I_2$

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$$\Rightarrow A^k = \begin{cases} (-1)^p I_2, & k = 2p \\ (-1)^p A, & k = 2p+1 \end{cases}, \ p \in \mathbb{N}$$

$$e^{tA} = \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p}}{(2p)!} I_2 + \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p+1}}{(2p+1)!} A$$

$$= (\cos t) I_2 + (\sin t) A$$

$A \in \mathcal{M}_n(\mathbb{R})$ $\quad X' = AX$

Theorem 1 For any $\eta \in \mathbb{R}^n$ the unique sol. of the IVP $\begin{cases} X' = AX \\ X(0) = \eta \end{cases}$ is $\varphi : \mathbb{R} \to \mathbb{R}^n$, $\varphi(t) = e^{tA}\eta$!

Remark: 1) The general sol. of $X' = AX$ can be written as $X = e^{At} \cdot C$, $C \in \mathbb{R}^n$ arbitrary

2) Let $P \in \mathcal{M}_n(\mathbb{R})$ be invertible. Then the gen. sol. of $X' = AX$ can also be written as $X = e^{At} \cdot P \cdot C$, $C \in \mathbb{R}^n$ arbitrary

$$(C' = PC \Leftrightarrow C = P^{-1} \cdot C')$$

Similar matrices

Def 1: Let $A, B \in \mathcal{M}_n(\mathbb{C})$. We say that A is similar to B when $\exists P \in \mathcal{M}_n(\mathbb{C})$, $\exists P^{-1}$ such that $\boxed{A = PBP^{-1}}$

Propr 1: If $\lambda_1 \in \mathbb{C}$ is an eigenvalue of A and $u_1 \in \mathbb{C}^n$ is an eigenvector of A then λ_1 is an eigenvalue of B and $P^{-1}u_1$ is the corresp. eigenvector of B, when A and B are similar

Proof $Au_1 = \lambda_1 u_1$, $u_1 \neq 0$

$$PBP^{-1}\underbrace{u_1} = \lambda_1 u_1 \;\Big|\cdot P^{-1} \Rightarrow B\cdot \underbrace{P^{-1}u_1}_{\neq 0} = \lambda_1 \underbrace{P^{-1}u_1}$$

Property 2. Assume that A and B are similar

1) $A^k = PB^kP^{-1}$, $\forall k \in \mathbb{N}$

2) $e^{tA} = Pe^{tB}P^{-1}$, $\forall t \in \mathbb{R}$

Proof 1) Prove by induction

$k=0$ $\quad J_n = P\cdot J_n \cdot P^{-1}$ ✓

$k=1$ $\quad A = PBP^{-1}$

$$A^{k+1} = A^k \cdot A = (PB^k\underbrace{P^{-1})\cdot P}_{J_n}BP^{-1} = PB^{k+1}P^{-1}$$

Def 2 We say that $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable over $\mathbb{R}$ when $\exists$ a diagonal matrix $B \in \mathcal{M}_n(\mathbb{R})$ s.t. A and B are similar.

- We say $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable over $\mathbb{C}$ when $\exists$ a diagonal matrix $B \in \mathcal{M}_n(\mathbb{C})$ s.t. A and B are similar

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Property 3 $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable over $\mathbb{C}$ if and only if there exist n linearly independent eigenvectors of A in $\mathbb{C}^n$.

$A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable over $\mathbb{R}$ if and only if any eigenvalue of A is in $\mathbb{R}$ and there exist n lin. indep. eigenvectors of A in $\mathbb{R}^n$.

Example: a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$A \in \mathcal{M}_2(\mathbb{R})$ $\quad \lambda_1, \lambda_2 \in \mathbb{C}$ eigenvalues

b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$ $\qquad$ d) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\det(A - \lambda I_2) = 0$

a) $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda_{1,2} = \pm i$

Property 4 If the eigenvalues of A are not repeated, (there are n distinct eigenvalues) then $\exists$ n lin. indep. eigenvectors.

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b) $\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 = 0$

$1-\lambda = 0$

$\lambda_1 = \lambda_2 = 1$

d) $\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 = 0$

$\lambda_1 = \lambda_2 = 1$

c) $u = \begin{pmatrix} a \\ b \end{pmatrix}$ $\quad Au = \lambda u$ $\quad$ u is an eigenvector corresp. to the eigenvalue λ.

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{cases} a+b=a \\ b=b \end{cases}$ $\quad \begin{pmatrix} a \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ there 2 lin. indep. eignvectors

A - is not diagonalizable

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d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$a = a$
$b = b$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are 2 lin. indep. eigenvectors of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\hookrightarrow$ A - diagonalizable over $\mathbb{R}$ and $\mathbb{C}$

The general sol. of a system $X' = AX$ in the case that A is diagonalizable matrix over $\mathbb{C}$ / over $\mathbb{R}$.

<u>Step 1</u>: $A \in \mathcal{M}_n(\mathbb{R})$ compute its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ (they can be repeated) and the corresp. eigenvectors $u_1, u_2, \dots, u_n \in \mathbb{C}^n$. Then decide whether it is diagonalizable or not. If it is, continue.

<u>Step 2</u>: Write $D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $P = (u_1, u_2, \dots, u_n)$ (its columns are the eigenvectors where the order is important)

$$A = PDP^{-1}$$

<u>Step 3</u>: $e^{tA} = Pe^{tD}P^{-1} = P \operatorname{diag}(e^{t\lambda_1}\ e^{t\lambda_2} \dots e^{t\lambda_n})P^{-1} \in \mathcal{M}_n(\mathbb{R})$

The gen. sol. $X = e^{tA} \cdot C$, $C \in \mathbb{R}^n$ arbitrary

Assume that A is diagonalizable over $\mathbb{R}$, then we already know that the gen. sol. of $X' = AX$

$$X = e^{tA} \cdot P \cdot C = P \cdot \operatorname{diag}(e^{t\lambda_1}\ e^{t\lambda_2} \dots e^{t\lambda_n}) \cdot C$$
$$= (e^{t\lambda_1}u_1\ e^{t\lambda_2}u_2 \dots e^{t\lambda_n}u_n)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$= c_1 e^{t\lambda_1}u_1 + c_2 e^{t\lambda_2}u_2 + \dots + c_n e^{t\lambda_n}u_n, \quad c_1, c_2, \dots, c_n \in \mathbb{R}$$

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HW: Let $u \in \mathbb{R}^n$ be an eigenvector of A corresp. to the eigenvalue $\lambda \in \mathbb{R}$.

~~Prove~~ Check that $\varphi : \mathbb{R} \to \mathbb{R}^n$, $\varphi(t) = e^{t\lambda} u$ is a sol. of $X'=AX$

Example: $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ is diagonalizable over $\mathbb{R}$

Find the general sol. of $X' = AX$ in two ways.

$$\det(A - \lambda I_2) = 0 \iff \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0 \iff$$

$$\iff -(1-\lambda)(1+\lambda) - 3 = 0 \iff -1 + \lambda^2 - 3 = 0 \iff \lambda^2 - 4 = 0 \iff$$

$$\lambda_1 = -2 \text{ and } \lambda_2 = 2$$

So A has 2 real eigenvalues $\lambda_1 \neq \lambda_2 \Rightarrow A$ is diagonalizable over $\mathbb{R}$

$u_1 \neq 0 \quad Au_1 = -2u_1 \quad u_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -2\begin{pmatrix} a \\ b \end{pmatrix} \iff \begin{cases} a + 3b = -2a \\ a - b = -2b \end{cases} \iff \begin{cases} 3a + 3b = 0 \\ a + b = 0 \end{cases}$$

$$\iff \begin{cases} a + b = 0 \\ a + b = 0 \end{cases} \iff a + b = 0 \iff b = -a \Rightarrow u_1 = a\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad a = 1$$

$$u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$u_2 \neq 0 \quad Au_2 = 2u_2 \quad u_2 = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix} \iff \begin{cases} a + 3b = 2a \\ a - b = 2b \end{cases} \iff \begin{cases} -a + 3b = 0 \\ a - 3b = 0 \end{cases} \iff$$

$$\iff a = 3b \iff u_2 = b\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad b = 1 \quad u_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

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I method to write the gen. sol

$$D = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix} \qquad P^{-1} = ?$$

$$e^{tA} = P \cdot \begin{pmatrix} e^{-2} & 0 \\ 0 & e^{2t} \end{pmatrix} P^{-1} \qquad X = e^{tA} \cdot C, \ C \in \mathbb{R}^2 \text{ arbitrary}$$

II method to write the gen. sol.

$$X = c_1 \cdot e^{-2t} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \cdot e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} + 3c_2 e^{2t} \\ -c_1 e^{-2t} + c_2 e^{2t} \end{pmatrix}, \ c_1, c_2 \in \mathbb{R}$$

III method

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