Seminar 4 - W7

7.04

$$x' + \frac{1}{t}x = \frac{1}{t}\cdot e^{-2t+1} \quad \Big| \cdot \mu(t) \qquad \varphi(t)\cdot e^{A(t)}$$

$$a(t) = \frac{1}{t} \qquad A(t) = -\int_{t_0}^{t} a(s)\,ds = -\ln t$$

$$\mu(t) = e^{-A(t)} = e^{\ln t} = t$$

$$\Rightarrow x't + x = e^{-2t+1} \Rightarrow (x\cdot t)' = e^{-2t+1}$$

$$\Rightarrow x\cdot t = -\frac{1}{2}e^{-2t+1} + c$$

$$\Rightarrow x(t) = \underbrace{-\frac{1}{2t}\cdot e^{-2t+1}}_{x_p} + \underbrace{\frac{c}{t}}_{x_h},\ c\in\mathbb{R}$$

1) Let $\eta \in \mathbb{R}$ be a fixed parameter

a) Find $\varphi_\eta : I_\eta \to \mathbb{R}$, the unique solution of the IVP.

open interval

$$\begin{cases} x' = -x \\ x(0) = \eta \end{cases} \qquad x' + x = 0$$

b) Study the properties of φ_η, find its image denoted by γ_η (orbit of η), represent the graph of φ_η.

$x(t)$

$$x'(t) = \frac{d}{dt}x(t)$$

$$x' + x = 0$$

$$\frac{dx}{dt} + x = 0$$

$$\frac{dx}{dt} = -x$$

$$\frac{dx}{-x} = dt \quad \Big| \int$$

$$-\int \frac{1}{x}dx = \int dt$$

$$-\ln|x| = t + c$$

$$\ln|x| = -t + c$$

$$x = \pm e^{-t+c} = \underbrace{\pm e^{c}}_{e^c,\ -e^c} \cdot e^{-t}$$

$$\Rightarrow x(t) = c \cdot e^{-t},\ c \in \mathbb{R}$$

$$t = 0 \Rightarrow x(0) = c \iff \eta = c$$

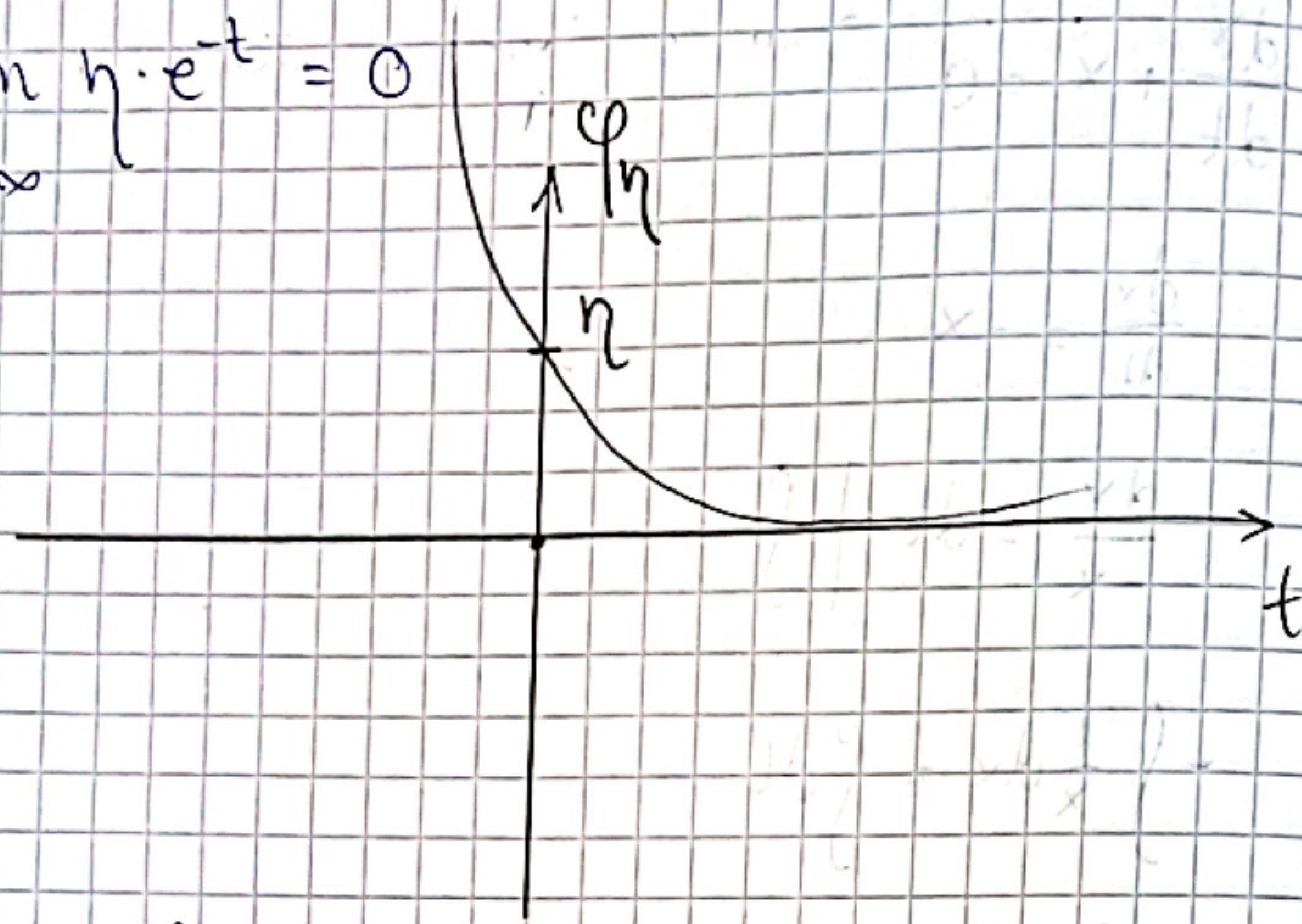
$$\varphi_\eta(t) = \eta \cdot e^{-t}$$

Scanned with CamScanner

b) $\varphi_\eta(t) = \eta \cdot e^{-t}$

i) $\eta > 0 \Rightarrow \eta \cdot e^{-t}$ decreasing

$\lim_{t \to \infty} \eta \cdot e^{-t} = 0$



$\lim_{t \to -\infty} \eta \cdot e^{-t} = +\infty$

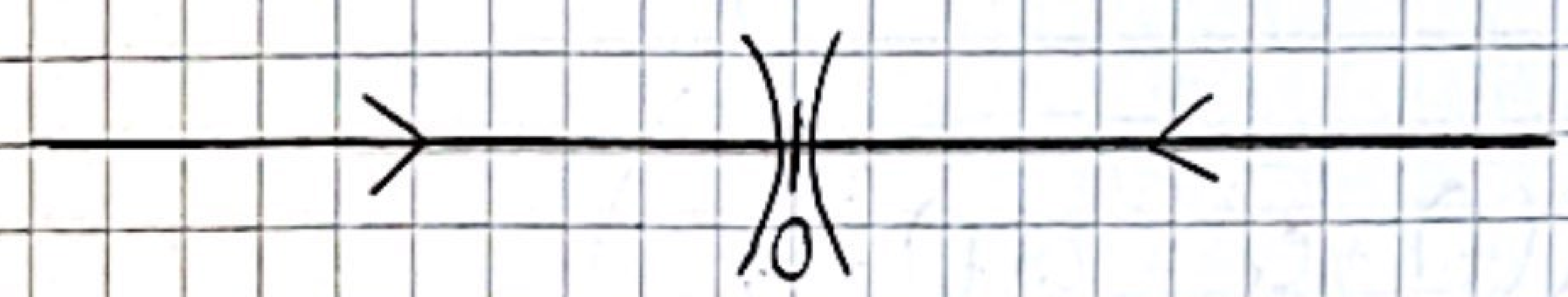
$\Rightarrow \gamma_\eta = (0, +\infty)$

ii) $\eta < 0 \Rightarrow \eta \cdot e^{-t}$ increasing

$\gamma_\eta = (-\infty, 0)$

iii) $\eta = 0 \Rightarrow \varphi_0(t) = 0 \Rightarrow \gamma_0 = \{0\}$

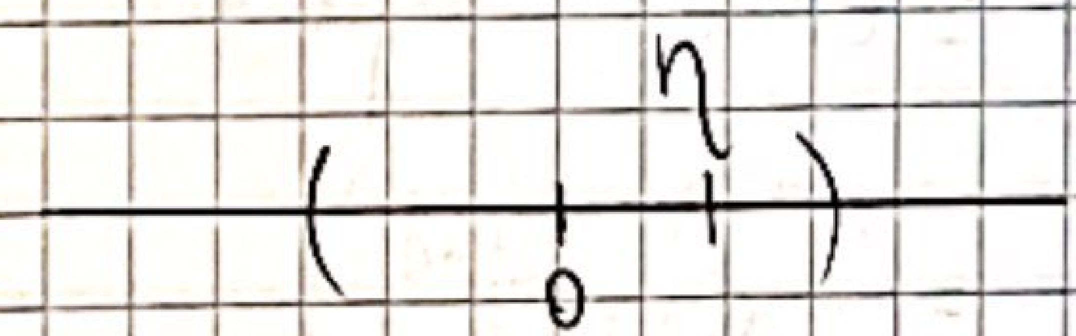
c) Represent γ_η and insert an arrow to indicate the future.



If $\gamma_{\eta^*} = \{\eta^*\} \Leftrightarrow \varphi_{\eta^*}(t) = \eta^* \Leftrightarrow \eta^*$ is an equilibrium

η^* attractor $\Leftrightarrow \exists\, V \in \mathcal{V}(\eta^*) : \forall \eta \in V$

$$\lim_{t \to \infty} \varphi_\eta(t) = \eta^*$$



η^* repeller $\Leftrightarrow \lim_{t \to -\infty} \varphi_\eta(t) = \eta^*, \forall \eta \in V$

2) $x' = x$ HW

3) $x' = 1 - x^2$

$\frac{dx}{dt} = 1 - x^2$ $\qquad \frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1} \;\Big|\cdot\left(-\frac{1}{2}\right)$

$\frac{dx}{1-x^2} = dt \;\Big|\int$

$\int \frac{dx}{1-x^2} = \int dt$

$$-\frac{1}{2}\int \frac{1}{x-1}-\frac{1}{1+x}\,dx=\int dt$$

$$-\frac{1}{2}\left(\ln|x-1|-\ln|x+1|\right)=t+c \quad \Big| \cdot(-2)$$

$$\left(f(t)+c_1=g(t)+c_2\right)$$

$$\ln\left|\frac{x-1}{x+1}\right|=-2t+c$$

$$\frac{x-1}{x+1}=\pm e^{-2t+c}=\pm e^{c}\cdot e^{-2t}$$

$$\frac{x+1-2}{x+1}=\pm e^{c}\cdot e^{-2t+c} \quad \Big| ()^{-1}$$

$$\left(1-\frac{2}{x+1}=\pm e^{c}\, e^{-2t+c}\right) \qquad e^{-2t}=\frac{1}{e^{2t}}$$

$$c_1\in\mathbb{R}^*$$

$$\left(-\frac{2}{x+1}=\pm e^{c}\cdot e^{-2t+c}-1\right) \qquad \frac{1}{c_1\cdot e^{-2t}}$$

$$\frac{x+1}{x-1}=c_2\cdot e^{2t}$$

$$\frac{x-1}{x-1}+\frac{2}{x-1}=1+\frac{2}{x-1}=c_2\cdot e^{2t}$$

$$\frac{2}{x-1}=c_2\cdot e^{2t}-1$$

$$x-1=\frac{2}{c_2\cdot e^{2t}-1}$$

$$x=\frac{2}{c_2\cdot e^{2t}-1}+1$$

$$x=\frac{c_2\cdot e^{2t}+1}{c_2\cdot e^{2t}-1}$$

$$x(t) = \frac{c_2 \cdot e^{2t} + 1}{c_2 \cdot e^{2t} - 1}$$

$$x(0) = \frac{c_2 + 1}{c_2 - 1} = \eta$$

$$\frac{c_2 - 1 + 2}{c_2 - 1} = \eta$$

$$1 + \frac{2}{c_2 - 1} = \eta$$

$$\frac{2}{c_2 - 1} = \eta - 1$$

$$c_2 = \frac{2}{\eta - 1} + 1$$

$$c_2 = \frac{2 + \eta - 1}{\eta - 1}$$

$$c_2 = \frac{\eta + 1}{\eta - 1}$$

$$\varphi_\eta(t) = \frac{\frac{\eta+1}{\eta-1} \cdot e^{2t} + \frac{\eta-1}{1}}{\frac{\eta+1}{\eta-1} \cdot e^{2t} - \frac{\eta-1}{1}} = \frac{(\eta+1) \cdot e^{2t} + (\eta - 1)}{(\eta+1) \cdot e^{2t} - (\eta - 1)}$$

Choose $\eta \in \{-2, -1, 0, 1, 2\}$

$\varphi_{-1}(t) = -1 \Rightarrow \gamma_{-1} = \{-1\} \Rightarrow -1$ equilibrium

$\varphi_1(t) = 1 \Rightarrow \gamma_1 = \{1\} \Rightarrow 1$ equilibrium

$$\varphi_0(t) = \frac{e^{2t}-1}{e^{2t}+1}$$

Hint: $-1 < \varphi_0 < 1$ and $\varphi_0' > 0$

$\Rightarrow \varphi_0$ increasing $\Big\} \Rightarrow \gamma_0 = (-1, 1)$

$$\lim_{t\to\infty} \varphi_0(t) = 1$$

$$\lim_{t\to-\infty} \varphi_0(t) = -1$$

$$\varphi_{-2}(t) = \frac{-e^{2t}-3}{-e^{2}+3} = \frac{e^{2t}+3}{e^{2t}-3}$$

Hint: $\boxed{\varphi_{-2} < -1}$ and $\varphi_{-2}' < 0 \quad \forall t \in \underbrace{(-\infty, \ln\sqrt{3})}_{I_{-2}}$

$$\frac{e^{2t}+3}{e^{2t}-3} < -1 \Leftrightarrow \frac{e^{2t}+\cancel{3}+e^{2t}-\cancel{3}}{\underbrace{e^{2t}-3}} < 0$$

$$e^{2t}-3<0 \Rightarrow e^{2t}<3$$

$$2t < \ln 3$$

$$t < \ln\sqrt{3}$$

$x' = 1 - x^2$

$\varphi_{-2}' = 1 - \underbrace{\varphi_{-2}^2}_{>1} < 0 \Rightarrow \varphi_{-2}$ is strictly decreasing

$$\lim_{t\to-\infty} \varphi_{-2}(t) = \lim_{t\to-\infty} \frac{e^{2t}+3}{e^{2t}-3} = -1$$

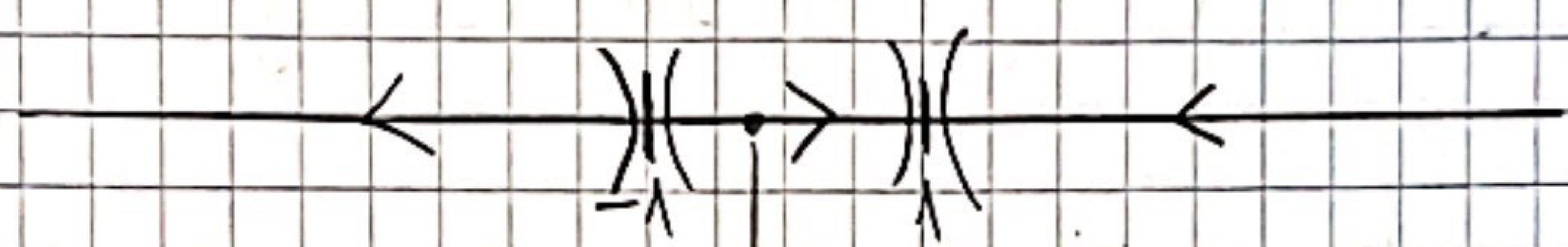
$$\lim_{t\to\ln\sqrt{3}} \varphi_{-2}(t) = \lim_{t\to\ln\sqrt{3}} \frac{\overset{\to 3}{e^{2t}+3}}{\underbrace{e^{2t}-3}_{\to 0^-}} = -\infty$$

$\Bigg\} \Rightarrow \gamma_{-2} = (-\infty, -1)$

$$\varphi_2(t) = \frac{3e^{2t}+1}{3e^{2t}-1} = \frac{e^{2t}+\frac{1}{3}}{e^{2t}-\frac{1}{3}}$$

$$\varphi_2 > 1,\ \varphi_2' < 0,\ \forall t \in (-\ln\sqrt{3}, +\infty) = I_2$$

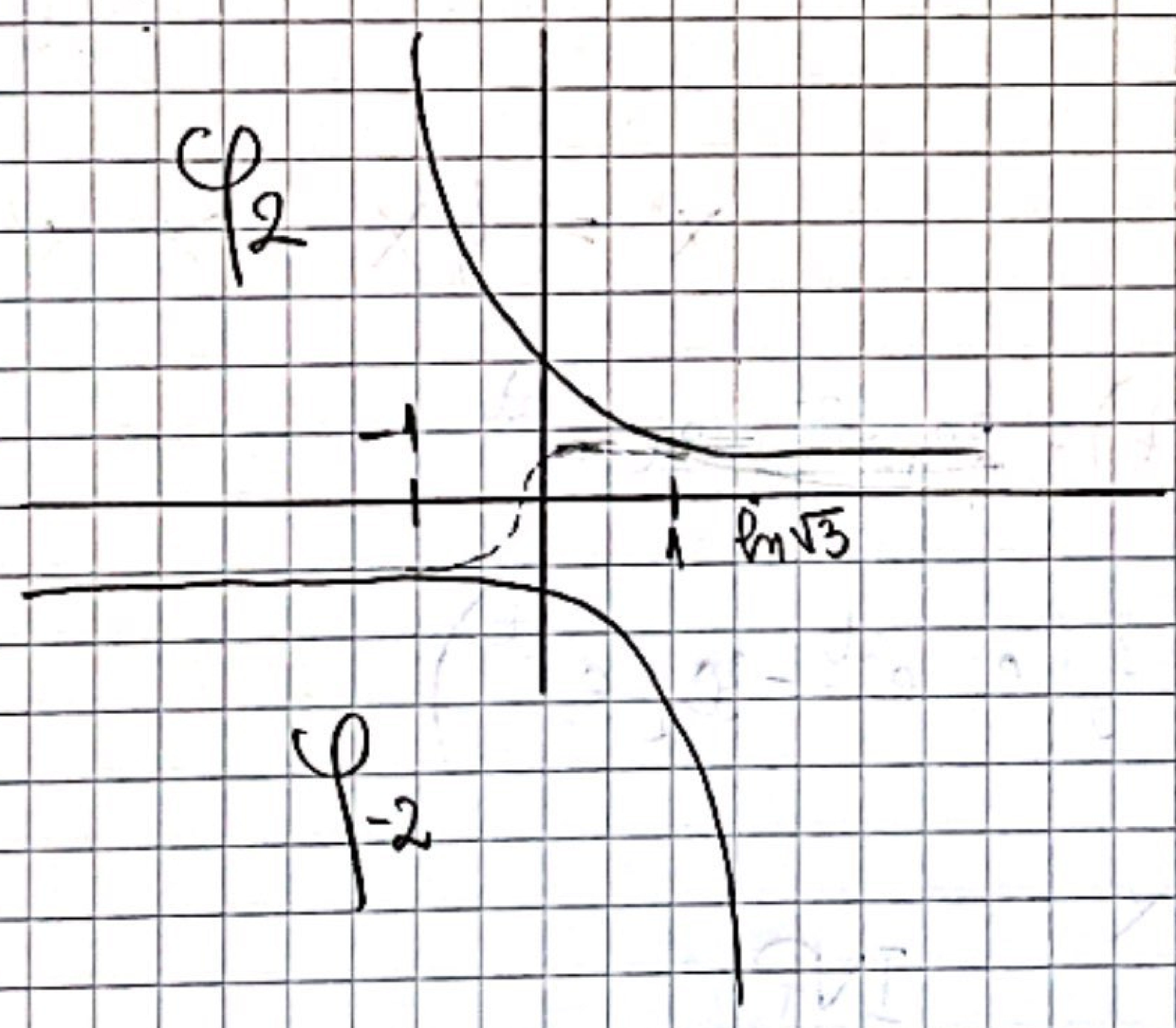
$$\Rightarrow \gamma_2 = (1, +\infty)$$



$\lim_{t\to\infty} \varphi_\eta = \varphi_1 = 1$

-1 repeller

1 attractor



$x' = 1 - x^2$ $\qquad$ $\dot{x} = 1 - x^2$

x			-1			1	
$1-x^2$	$-$	$-$	-0	$+$	$+$	0	$-$

4) Using the reduction method find the general solution of $X' = AX$

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad X' = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} x' = x + 3y \,\Big|' \rightarrow \\ y' = x - y \end{cases}$$

$$\rightarrow y = \frac{1}{3}(x' - x)$$

$$\left.\begin{aligned} x'' &= x' + 3y' \\ y' &= x - y \end{aligned}\right\} \Rightarrow x'' = x' + 3(x - y)$$

$$x'' = \cancel{x'} + 3x - \cancel{3} \cdot \frac{1}{\cancel{3}}(\cancel{x'} - x)$$

$$x'' = 4x \Rightarrow x(t) = c_1 \cdot e^{2t} + c_2 \cdot e^{-2t}$$

$$x'(t) = 2c_1 \cdot e^{2t} - 2c_2 \cdot e^{-2t}$$

$$y = \frac{1}{3}\left(c_1 \cdot e^{2t} - 3c_2 \cdot e^{-2t}\right)$$

$$\begin{cases} X' = AX \\ X(0) = \eta \end{cases} \quad \text{IVP}$$

$$X(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ \frac{1}{3}c_1 - c_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 = \dfrac{3(\eta_1 + \eta_2)}{4} \\ c_2 = \dfrac{1}{4}\eta_1 - \dfrac{3}{4}\eta_2 \end{cases}$$