Planes in the Euclidean space $\mathbb{E}^3$.

1. Determine parametric equations for the plane π in the following cases:

1. π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1(3,-1)$ and $\mathbf{a}_2(0,3,1)$,
2. π contains the point $A(1,2,1)$ and is parallel to $\mathbf{i}$ and $\mathbf{j}$,
3. π contains the point $M(1,7,1)$ and is parallel coordinate plane Oyz,
4. π contains the points $M_1(5,3,4)$ and $M_2(1,0,1)$, and is parallel to the vector $\mathbf{a}(1,3,-3)$,
5. π contains the point $A(1,5,7)$ and the coordinate axis Ox.

2. Determine Cartesian equations for the plane π in the following cases:

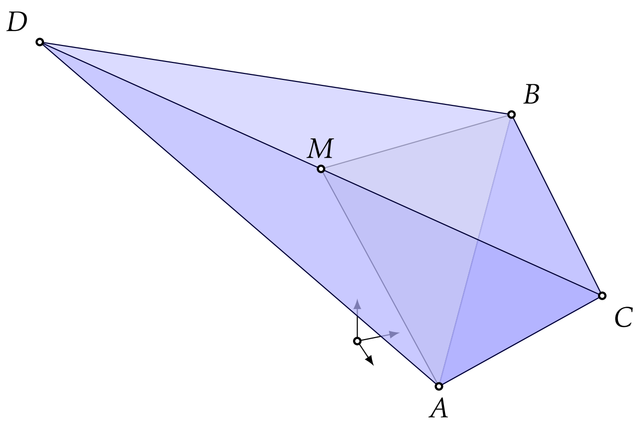
1. $\pi : x = 2+3u-4v,\ y = 4-v,\ z = 2+3u$;
2. $\pi : x = u+v,\ y = u-v,\ z = 5+6u-4v$.

3. Determine parametric equations for the plane π in the following cases:

1. $3x-6y+z=0$;
2. $2x-y-z-3=0$;

4. Determine an equation for each plane passing through $A(3,5,-7)$ and intersecting the coordinate axes in congruent segments.

5. Let $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$ be vertices of a tetrahedron. Determine a Cartesian equation of the plane containing $[AB]$ and the midpoint of $[CD]$.



6. Show that a parallelepiped with faces in the planes $2x+y-2z+6=0$, $2x-2y+z-8=0$ and $x+2y+2z+1=0$ is rectangular.

7. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.

8. Determine the distance between the planes $x-2y-2z+7=0$ and $2x-4y-4z+17=0$.

Lines in the Euclidean space $\mathbb{E}^3$.

9. Determine parametric equations for the line ℓ in the following cases:

1. ℓ contains the point $M_0(2,0,3)$ and is parallel to the vector $\mathbf{a}(3,-2,-2)$,
2. ℓ contains the point $A(1,2,3)$ and is parallel to the Oz-axis,
3. ℓ contains the points $M_1(1,2,3)$ and $M_2(4,4,4)$.

10. Give Cartesian equations for the lines ℓ in the previous exercise.

11. Determine parametric equations for the line contained in the planes $x+y+2z-3=0$ and $x-y+z-1=0$.

12. Determine the relative positions of the lines $x=-3t, y=2+3t, z=1$ and $x=1+5s, y=1+13s, z=1+10s$.

13. Let $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,-5)$ be vertices of a triangle. Determine the equation of the internal angle bisector of $\measuredangle A$.

14. Determine the parameter m for which the line $x=-1+3t, y=2+mt, z=-3-2t$ doesn't intersect the plane $x+3y+3z-2=0$.

15. Determine the values a and d for which the line $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-3}{-2}$ is contained in the plane $ax+y-2z+d=0$.

16. Determine the values a and c for which the line $3x-2y+z+3=0 \cap 4x-3y+4z+1=0$ is perpendicular to the plane $ax+8y+cz+2=0$.

17. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x+4y-3z+7=0$.

18. Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $2x-3y+z-4=0$.

19. Determine the orthogonal projection of the point $A(1,3,5)$ on the line $2x+y+z-1=0 \cap 3x+y+2z-3=0$.

1. Determine parametric equations for the plane π in the following cases:

1. π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1\overset{(3,-1,1)}{\cancel{(3,-1)}}$ and $\mathbf{a}_2(0,3,1)$,
2. π contains the point $A(1,2,1)$ and is parallel to $\mathbf{i}$ and $\mathbf{j}$,
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4. π contains the points $M_1(5,3,4)$ and $M_2(1,0,1)$, and is parallel to the vector $\mathbf{a}(1,3,-3)$,
5. π contains the point $A(1,5,7)$ and the coordinate axis Ox.

1.) $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad t,s \in \mathbb{R}$

2.) $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \lambda,\mu \in \mathbb{R}$

3) $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix} + u\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad u,v \in \mathbb{R}$

4.) $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} + t\overset{\overrightarrow{M_2M_1}}{\begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}} + s\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \quad t,s \in \mathbb{R}$

5.) $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} + \mu\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \lambda,\mu \in \mathbb{R}$

2. Determine Cartesian equations for the plane π in the following cases:

1. $\pi : x = 2 + 3u - 4v,\ y = 4 - v,\ z = 2 + 3u$;
2. $\pi : x = u + v,\ y = u - v,\ z = 5 + 6u - 4v$.

1. $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} + u\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} + v\begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix} \iff \begin{array}{l} x = 2 + 3u - 4v \\ y = 4 - v \\ z = 2 + 3u \end{array}$

$$\pi: \begin{vmatrix} x-2 & y-4 & z-2 \\ 3 & 0 & 3 \\ -4 & -1 & 0 \end{vmatrix} = 0 \iff \pi: \cancel{3}(x-2) - \overset{4}{\cancel{12}}(y-4) - \cancel{3}(z-2) = 0$$

$$x - 2 - 4y + 16 - z + 2 = 0$$

$$\iff \pi: x - 4y - z + 16 = 0$$

2. $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} + u \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} + v \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$

$$\pi: \begin{vmatrix} x & y & z-5 \\ 1 & 1 & 6 \\ 1 & -1 & -4 \end{vmatrix} = 0 \iff \pi: x(-4+6) - y(-4-6) + (z-5)(-2) = 0$$

$$\iff \pi: 2x + 10y - 2z + 10 = 0$$

3. Determine parametric equations for the plane π in the following cases:

1. $3x - 6y + z = 0$;
2. $2x - y - z - 3 = 0$;

1. $3x - 6y + z = 0 \iff \begin{cases} z = -3x + 6y \\ x = x \\ y = y \end{cases} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$ $x, y \in \mathbb{R}$

2. $2x - y - z - 3 = 0 \iff \begin{cases} y = 2x - z - 3 \\ x = x \\ z = z \end{cases} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$x, z \in \mathbb{R}$

4. Determine an equation for each plane passing through $P(3, 5, -7)$ and intersecting the coordinate axes in congruent segments.

$$\pi: \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \pi \cap Ox = (a, 0, 0) = A$$

$$\pi \cap Oy = (0, b, 0) = B$$

$$\pi \cap Oz = (0, 0, c) = C$$

"congruent segments" $\iff |OA| = |OB| = |OC| \iff |a| = |b| = |c| = m.$

$\iff a = \pm m, \quad b = \pm m, \quad c = \pm m$

I $\quad a = m, \quad b = m, \quad c = m$

$$\pi: \frac{x}{m} + \frac{y}{m} + \frac{z}{m} = 1 \iff \pi: x + y + z = m$$

$\Pi \ni P(3,5,-7) \iff 3+5-7=m \quad$ So $\quad m=1$

so $\quad \Pi : x+y+z=1$

$\underline{\text{II}}$ $\quad a=m, \quad b=m, \quad c=-m$

$\Pi : x+y-z=m \ni P(3,5-7) \implies 3+5+7=m$

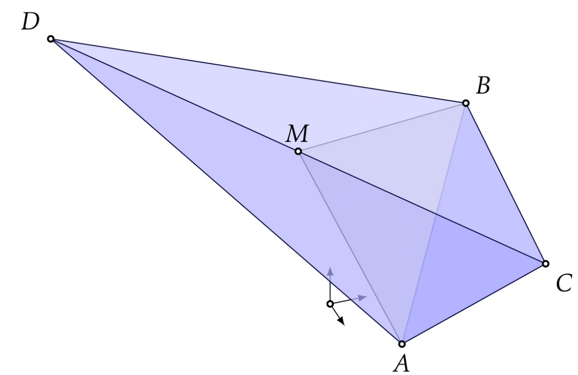
so $\quad \Pi : x+y-z=15$

$\underline{\text{II}}$ $\quad a=m, \quad b=-m, \quad c=m$

...

all the other cases are treated similarly

5. Let $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$ be vertices of a tetrahedron. Determine a Cartesian equation of the plane containing $[AB]$ and the midpoint of $[CD]$.



$M = \left(\frac{6+0}{2}, \frac{3-7}{2}, \frac{4+8}{2}\right) = (3,-2,6) \qquad \overrightarrow{AB} = (-1,2,5) \quad \overrightarrow{AM} = (1,-3,-6)$

$$ABM : \begin{vmatrix} x-2 & y-1 & z \\ -1 & 2 & 5 \\ 1 & -3 & -6 \end{vmatrix} = 0 \iff ABM : (x-2)\overset{3}{(-12+15)} - (y-1)\overset{1}{(6-5)} + z\overset{1}{(3-2)} = 0$$

$$\iff ABM : 3x - y + z - 5 = 0$$

6. Show that a parallelepiped with faces in the planes $\underbrace{2x + y - 2z + 6 = 0}_{\pi_1}$, $\underbrace{2x - 2y + z - 8 = 0}_{\pi_2}$ and $\underbrace{x + 2y + 2z + 1 = 0}_{\pi_3}$ is rectangular.

$\Downarrow$

if the angles between these planes are right angles

$n_1(2,1,-2)$ is a normal vector for π_1

$n_2(2,-2,1)$ —— $\parallel$ —— π_2

$n_3(1,2,2)$ —— $\parallel$ —— π_3

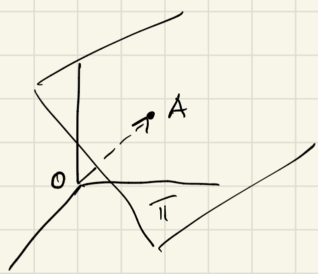
$\pi_1 \perp \pi_2 \iff n_1 \perp n_2 \iff n_1 \cdot n_2 = 0$, and $n_1 \cdot n_2 = 4 - 2 - 2 = 0$ so $\pi_1 \perp \pi_2$

$n_1 \cdot n_3 = 2 + 2 - 4 = 0 \Rightarrow \pi_1 \perp \pi_3$

$n_2 \cdot n_3 = 2 - 4 + 2 = 0 \Rightarrow \pi_2 \perp \pi_3$

7. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.

A = orthogonal projection of O on π



$\Rightarrow OA \perp \pi$

$\Rightarrow \vec{OA} \perp \pi$

$\Rightarrow \vec{OA}$ is a normal vector for π

$\Rightarrow \pi: 1\cdot(x-1) + (-1)(y-(-1)) + 3(z-3) = 0$ (since $A \in \pi$)

so $\pi: x - y + 3z - 11 = 0$

8. Determine the distance between the planes $\overbrace{x-2y-2z+7=0}^{\pi_1}$ and $\overbrace{2x-4y-4z+17=0}^{\pi_2}$.

$$\downarrow \qquad\qquad \downarrow$$

$n_1(1,-2,-2)$ is a normal vector for π_1

$n_2(2,-4,-4)$ is a normal vector of π_2

$2n_1 = n_2$, they are proportional

$\Rightarrow$ π_1 and π_2 have the same normal vectors $\Rightarrow \pi_1 \parallel \pi_2$

(Another way to see this is by noticing that $\operatorname{rank}\begin{bmatrix} 1 & -2 & -2 \\ 2 & -4 & -4 \end{bmatrix} = 1$)

So $\pi_1 \parallel \pi_2$

$$\Rightarrow d(\pi_1, \pi_2) = d(\pi_1, P) \quad \forall\, P \in \pi_2$$

choose $P = \left(-\frac{17}{2}, 0, 0\right) \in \pi_2$

then $d(\pi_1, P) = \dfrac{\left|-\frac{17}{2}+7\right|}{\sqrt{1+4+4}} = \dfrac{\frac{3}{2}}{\sqrt{9}} = \dfrac{1}{2}$

9. Determine parametric equations for the line ℓ in the following cases:

1. ℓ contains the point $M_0(2,0,3)$ and is parallel to the vector $\mathbf{a}(3,-2,-2)$,
2. ℓ contains the point $A(1,2,3)$ and is parallel to the Oz-axis,
3. ℓ contains the points $M_1(1,2,3)$ and $M_2(4,4,4)$.

1. $\ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \quad t \in \mathbb{R}$

2. $\ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad s \in \mathbb{R}$

3. $\ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \underbrace{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}_{\overrightarrow{M_1M_2}} \quad \lambda \in \mathbb{R}$

10. Give Cartesian equations for the lines ℓ in the previous exercise.

1. $\ell: \dfrac{x-2}{3} = \dfrac{y}{-2} = \dfrac{z-3}{-2} \iff \ell: \begin{cases} -2x+4 = 3y \\ y = z-3 \end{cases} \iff \ell: \begin{cases} 2x+3y-4=0 \\ y-z+3=0 \end{cases}$

2. $\ell: \begin{cases} x = 1 \\ y = 2 \\ z = 3+s \end{cases} \iff \ell: \begin{cases} x = 1 \\ y = 2 \end{cases}$

3. $\ell: \left.\begin{cases} x = 1+3\lambda \\ y = 2+2\lambda \\ z = 3+\lambda \implies \lambda = z-3 \end{cases}\right\} \implies \ell \begin{cases} x = 1+3z-9 \\ y = 2+2z-6 \end{cases} \iff \ell: \begin{cases} x-3z+8=0 \\ y-2z+4=0 \end{cases}$

11. Determine parametric equations for the line contained in the planes $x+y+2z-3=0$ and $x-y+z-1=0$.

method I

$$l: \begin{cases} x+y+2z-3=0 \\ x-y+z-1=0 \end{cases} \quad \begin{pmatrix} 1 & 1 & 2 & -3 \\ 1 & -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & -2 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 3 & -4 \\ 0 & -2 & -1 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2x = -3z+4 \\ -2y = z-2 \end{cases} \Rightarrow \begin{cases} x = -\frac{3}{2}z+2 \\ y = -\frac{1}{2}z+1 \\ z = z \end{cases} \Rightarrow l: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$z \in \mathbb{R}$

method II

$n_1 (1,1,2)$ is a normal vector for $\Pi_1 : x+y+2z-3=0$

$n_2 (1,-1,1)$ —— 11 —— $\Pi_2 : x-y+z-1=0$

$\Rightarrow$ $n_1 \times n_2$ is a direction vector for l $\quad n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 3i + j - 2z$

choose a $\overset{P}{\text{point}}$ in $l \in \Pi_1 \cap \Pi_1$, i.e a point P such that the coordinates of P satisfy the equations of Π_1 and Π_2

for example $P = (2,1,0)$

then

$$l: \begin{bmatrix} x \\ y \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{P} + \lambda \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \qquad \lambda \in \mathbb{R}$$

12. Determine the relative positions of the lines $x = -3t, y = 2 + 3t, z = 1$ and $x = 1 + 5s, y = 1 + 13s, z = 1 + 10s$.

$$l_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{= P_1} + t \underbrace{\begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}}_{= v_1 \text{ a direction vector for } l_1} \quad t \in \mathbb{R} \qquad l_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{= P_2} + s \underbrace{\begin{bmatrix} 5 \\ 13 \\ 10 \end{bmatrix}}_{= v_2 \text{ a direction vector for } l_2} \quad s \in \mathbb{R}$$

v_1 and v_2 are linearly independent (they are not proportional)

$\Rightarrow$ l_1 is not parallel to l_2

$\Rightarrow$ either l_1 and l_2 intersect or they are skew

$l_1 \cap l_2 \neq \emptyset \iff$ the two lines are coplanar

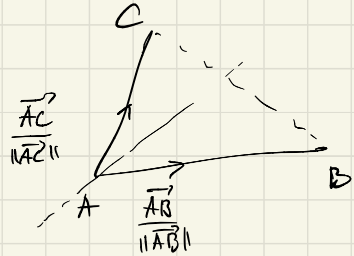
$\iff$ $\overrightarrow{P_1P_2}, v_1, v_3$ are linearly dependent

$$\iff \underbrace{\begin{vmatrix} 1 & -1 & 0 \\ -3 & 3 & 0 \\ 5 & 13 & 10 \end{vmatrix}}_{= 10 \begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} = 0} = 0$$

so, yes the two lines intersect.

13. Let $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,-5)$ be vertices of a triangle. Determine the equation of the internal angle bisector of $\measuredangle A$.

a direction vector for the angle bisector ℓ is $\dfrac{\vec{AB}}{\|\vec{AB}\|} + \dfrac{\vec{AC}}{\|\vec{AC}\|}$



$$\vec{AB} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \frac{\vec{AB}}{\|\vec{AB}\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{AC} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \frac{\vec{AC}}{\|\vec{AB}\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} + \lambda \begin{bmatrix} 1 + 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

14. Determine the parameter m for which the line $x = -1 + 3t, y = 2 + mt, z = -3 - 2t$ doesn't intersect the plane $x + 3y + 3z - 2 = 0$.

$$\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + t \underbrace{\begin{bmatrix} 3 \\ m \\ -2 \end{bmatrix}}_{=\, v \text{ a dir. vector for } \ell}$$

$\pi : x + 3y + 3z - 2 = 0$

$\hookrightarrow n(1,3,3)$ is a normal vector for π

$$\ell \cap \pi = \emptyset \iff \ell \parallel \pi \iff v \perp n \iff v \cdot n = 0$$

$$v \cdot n = 3 + 3m - 6 = 0 \iff m = 1$$

15. Determine the values a and d for which the line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$ is contained in the plane $ax + y - 2z + d = 0$.

$$l: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}_{P} + t \underbrace{\begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}}_{v}$$

$\Pi: ax + y + 2z + d = 0$

$\hookrightarrow n(a, 1, 2)$ is a normal vector for Π

$l \subseteq \Pi \iff l \parallel \Pi$ and any point in l, for example $P = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ belongs to Π

$l \parallel \Pi \iff n \cdot v = 0 \iff 3a + 2 - 4 = 0 \iff a = \frac{2}{3}$

$P \in \Pi \iff \frac{2}{3} \cdot 2 - 1 + 6 + d = 0 \iff d = \frac{-19}{3}$

16. Determine the values a and c for which the line $3x - 2y + z + 3 = 0 \cap 4x - 3y + 4z + 1 = 0$ is perpendicular to the plane $ax + 8y + cz + 2 = 0$.

$$l: \begin{cases} 3x - 2y + z + 3 = 0 & \leftarrow \text{eq. of plane } \Pi_1 \text{ with normal. vect } n_1(3, -2, 1) \\ 4x - 3y + 4z + 1 = 0 & \leftarrow \text{—"— } \Pi_2 \text{ —"— } n_2(4, -3, 4) \end{cases}$$

$\Rightarrow$ a direction vector for l is $n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 4 & -3 & 4 \end{vmatrix} = -5i - 8j - k$

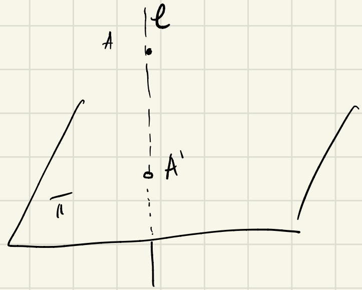
$\Pi: ax + 8y + cz + 2 = 0 \Rightarrow n(a, 8, c)$ is a normal vector for Π

$l \perp \Pi \iff v \parallel n \iff \frac{a}{-5} = \frac{8}{-8} = \frac{c}{-1} \Rightarrow a = 5$ and $c = 1$

$\Rightarrow \Pi: 5x + 8y + z + 2 = 0$

17. Determine the $\underbrace{\text{orthogonal projection}}$ of the point $A(2,11,-5)$ on the plane $\underbrace{x+4y-3z+7=0}_{\pi}$.

is the intersection of the line ℓ, passing through A and orthogonal to π, with the plane π



$$Pr^{\perp}_{\pi}(A) = A'$$

$\ell \perp \pi \Rightarrow$ the normal vectors of π are direction vectors of ℓ

from the eq. of π we see that $n(1,4,-3)$ is a normal vector for ℓ

$$\Rightarrow \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix}}_{A} + t \underbrace{\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}}_{n} \quad t \in \mathbb{R} \quad \Leftrightarrow \quad \begin{cases} x = 2+t \\ y = 11+4t \\ z = -5-3t \end{cases}$$

$$\ell \cap \pi : \quad (2+t) + 4(11+4t) - 3(-5-3t) + 7 = 0$$

$$2+t+44+16t+15+9t+7=0$$

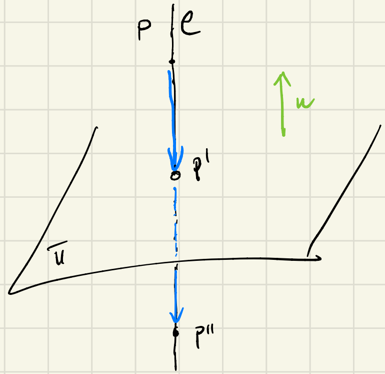
$$26t = -68 \Leftrightarrow t = -\frac{34}{13}$$

$$\Rightarrow \ell \cap \pi = \begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix} - \frac{34}{13} \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -8 \\ 7 \\ 37 \end{bmatrix}$$

18. Determine the $\underbrace{\text{orthogonal reflection}}$ of the point $P(6,-5,5)$ in the plane $\underbrace{2x-3y+z-4=0}_{\pi}$.

of the point P is the point P'' such that

$Pr^{\perp}_{\pi}(P) = P'$ is the midpoint of the segment $[PP'']$



we calculate P' first.

$P' = \ell \cap \pi$ where $\ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

↳ normal vector of π is direction vector for ℓ

so $\ell: \begin{cases} x = 6 + 2t \\ y = -5 - 3t \\ z = 5 + t \end{cases}$

$\Rightarrow \ell \cap \pi: \; 2(6+2t) - 3(-5-3t) + (5+t) - 4 = 0$

$12 + 4t + 15 + 9t + 5 + t - 4 = 0$

$14t = -28 \Rightarrow t = -2 \quad \Rightarrow \quad \ell \cap \pi = \begin{bmatrix} 6 \\ -5 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Now we can obtain $P'' = Ref^{\perp}_{\pi}(P)$ as $P'' = P + 2\overrightarrow{PP'}$

or by using the fact that P' is the midpoint of PP''

$\Rightarrow P'' = (x'', y'', z'') \quad \begin{cases} 2 = \dfrac{x'' + 6}{2} \\ 1 = \dfrac{y'' - 5}{2} \\ 3 = \dfrac{z'' + 5}{2} \end{cases} \Rightarrow P'' = \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \ldots$

19. Determine the orthogonal projection of the point $A(1,3,5)$ on the line $2x+y+z-1=0 \cap 3x+y+2z-3=0$.

denote this line by ℓ

Let A' be the orthogonal proj of A on ℓ

$A' = \ell \cap \Pi$ where Π is the plane containing A and orthogonal to ℓ