

Lecture 13

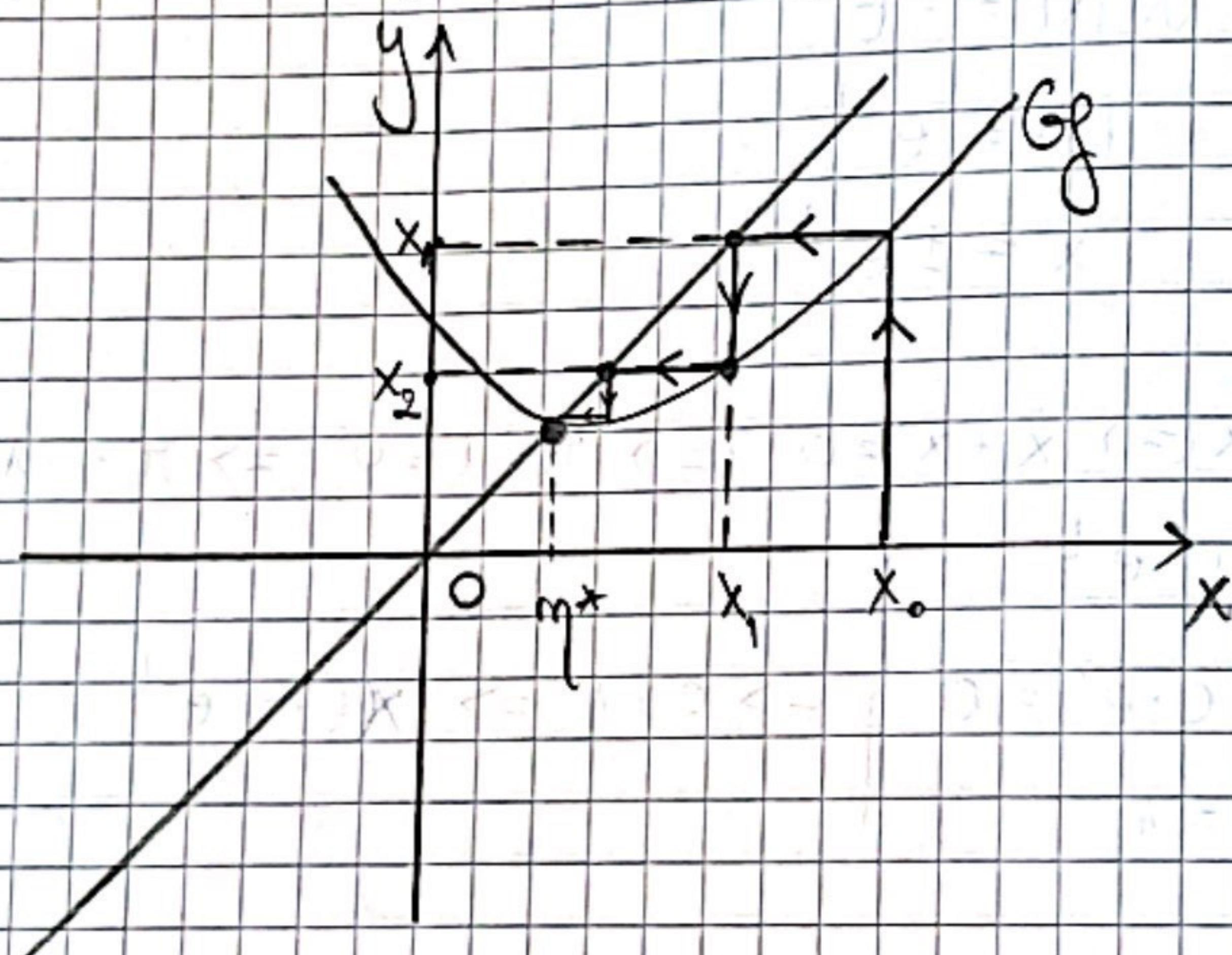
Scalar discrete dynamical systems

(1) $x_{k+1} = f(x_k)$, $k \geq 0$, $k \in \mathbb{N}$, where $f \in C^1(\mathbb{R}, \mathbb{R})$.

Theorem (L12) Let m^* be a fixed point of f .
If $|f'(m^*)| < 1$ then m^* is an attractor.

The cobweb or stair-step diagram to "construct" geometrically a sequence $(x_k)_{k \geq 0}$ that satisfies (1).

- We represent the graph of f and the line $y = x$



- $Gf \cap y = x$ gives the fixed points of f

Def: If $f(m^*) = m^*$ then we say that m^* is a fixed point of f

$x_0 \in \mathbb{R}$ fix

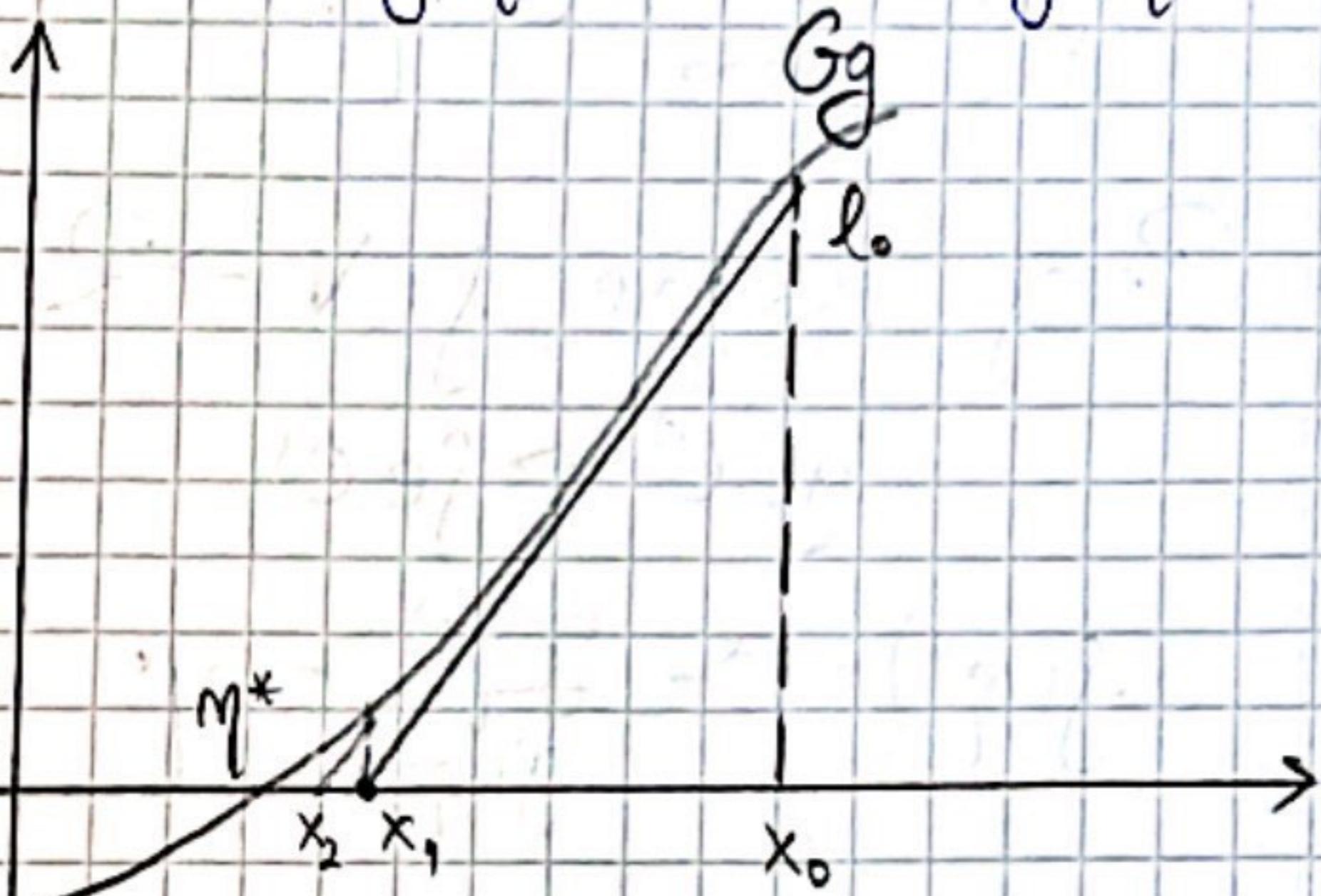
$$x_1 = f(x_0)$$

$$x_2 = f(x_1)$$

The Newton-Raphson method to find approximations for a zero of a function

Take $g: I \rightarrow \mathbb{R}$, is C^2 on the open, nonempty interval I .

We know that $\exists m^* \in I$ s.t. $g(m^*) = 0$ and $g'(m^*) \neq 0$



$x_0 \in I$ fixed.

$(x_0, g(x_0)) \in Gg$;

the line l_0 is tangent to
 Gg in $(x_0, g(x_0))$

Our aim is to find good approximations of m^* , or, more precisely, to find $(x_k)_{k \geq 0}$ such that $\lim_{k \rightarrow \infty} x_k = m^*$

$$l_0: y - g(x_0) = g'(x_0)(x - x_0)$$

$l_0 \cap x: y = 0$ in the eq. of l_0 .

$$\Rightarrow -g(x_0) = g'(x_0)(x_1 - x_0)$$

$$\Rightarrow x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}$$

In general

$$\left\{ \begin{array}{l} x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}, \quad k \geq 0 \\ x_0 \text{ given} \end{array} \right.$$

Theorem 2: Let $V \subset I$ open, nonempty interval be such that $\eta^* \in V$ and $g'(\eta) \neq 0$, $\forall \eta \in V$.

Then $\forall x_0 \in V$ given the sequence $(x_k)_{k \geq 0}$ given by (3) satisfies $\lim_{k \rightarrow \infty} x_k = \eta^*$.

Proof: Define $f: V \rightarrow \mathbb{R}$, $f(x) = x - \frac{g(x)}{g'(x)}$, $\forall x \in V$
 $g \in C^2 \Rightarrow f \in C^1$

$$f(\eta^*) = \eta^* - \frac{g(\eta^*)}{g'(\eta^*)} = \eta^* \Rightarrow \eta^* \text{ is a fixed point of } f$$

$$f'(x) = 1 - \frac{g'(x) \cdot g'(x) - g(x) \cdot g''(x)}{[g'(x)]^2}$$

$$\Rightarrow f'(\eta^*) = 1 - 1 = 0 \Rightarrow |f'(\eta^*)| = 0 < 1$$

Lim method \implies the f.p. of f , η^* is an attractor

Then $\exists \delta > 0$ s.t. $|x_0 - \eta^*| < \delta$, the seq. $(x_k)_{k \geq 0}$ given by (3) satisfies $\lim_{k \rightarrow \infty} x_k = \eta^*$.

Example: $g(x) = x^2 - 3$, $g: (0, \infty) \rightarrow \mathbb{R}$

Try to estimate the basin of attraction of the attractor as in the proof of Th. 2 for this g , using the stair-step diagram.

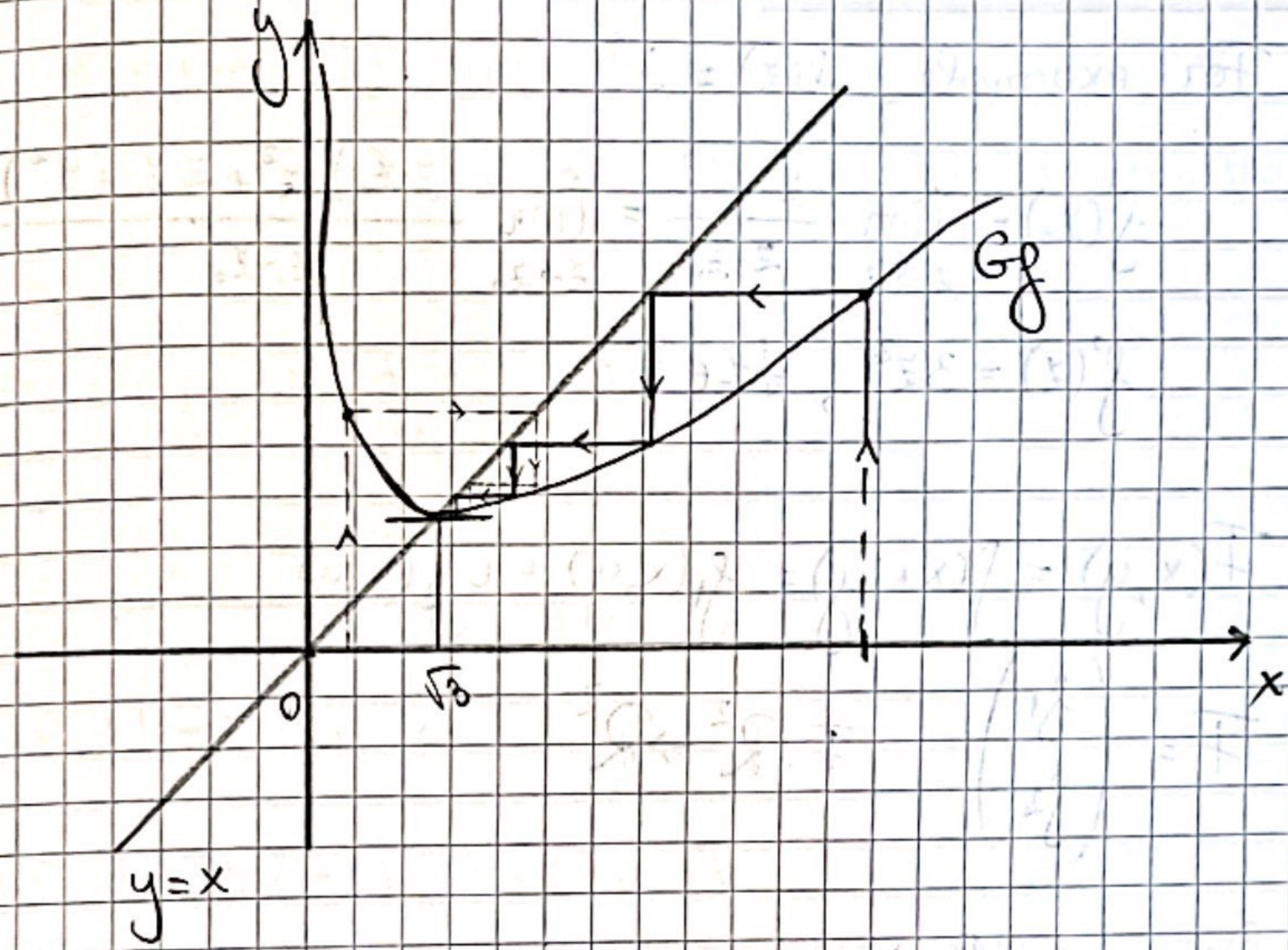
$$\eta^* = \sqrt{3}$$

$$f(x) = x - \frac{g(x)}{g'(x)} = x - \frac{x^2 - 3}{2x} = \frac{2x^2 - x^2 + 3}{2x} = \frac{x^2 + 3}{2x} = \frac{1}{2}x + \frac{3}{2} \cdot \frac{1}{x}$$

$$\left\{ \begin{array}{l} x_{k+1} = \frac{1}{2}x_k + \frac{3}{2} \cdot \frac{1}{x_k} \\ x_0 \text{ given} \end{array} \right.$$

From the proof of Th2 we know that $\sqrt{3}$ is an attracting fixed point of f , thus (as we wrote in the conclusion of Th2) $\exists \delta > 0$ s.t. for x_0 with $|x_0 - \sqrt{3}| < \delta$ we have $\lim_{k \rightarrow \infty} x_k = \sqrt{3}$

So, we want to estimate the basin of attraction of $\sqrt{3}$, denoted $A_{\sqrt{3}}$.



$$f: (0, \infty) \rightarrow \mathbb{R}$$

$$f'(x) = \frac{1}{2} - \frac{3}{2} \cdot \frac{1}{x^2} = \frac{x^2 - 3}{2x^2}$$

x	0	$\sqrt{3}$	$+\infty$
f'	- - -	0	+
f	$+\infty$	$\sqrt{3}$	$+\infty$

$f'(m^*) = 0$

It seems that $A_{\sqrt{3}} = (0, \infty)$ $|f'| < 1$

Complex maps

$$f: \mathbb{C} \rightarrow \mathbb{C}$$

Def: The derivative in $z_0 \in \mathbb{C}$ is (if exists) the limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

For example, $f(z) = z^3$

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{z^3 - z_0^3}{z - z_0} = \lim_{z \rightarrow z_0} \frac{(z-z_0)(z^2 + z \cdot z_0 + z_0^2)}{z - z_0} = 3z_0^2$$

$$f'(z) = 3z^2, \forall z \in \mathbb{C}$$

$$F(x, y) = f(x+iy) = f_1(x, y) + i f_2(x, y)$$

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

Def: $m^* \in \mathbb{C}$ is a fixed point of $f: \mathbb{C} \rightarrow \mathbb{C}$ when $f(m^*) = m^*$

Define $(z_k)_{k \geq 0}$ a sequence in \mathbb{C} by $z_k = f^k(z_0)$, or

$$\left\{ \begin{array}{l} z_{k+1} = f(z_k) \\ z_0 \in \mathbb{C} \text{ given} \end{array} \right.$$

We define a similar notion of attractor.

Th: If $m^* \in \mathbb{C}$ is a fixed point of $f: \mathbb{C} \rightarrow \mathbb{C}$ (of class C^1) such that $|f'(m^*)| < 1$ then m^* is an attractor.

Example: $g: \mathbb{C} \rightarrow \mathbb{C}$, $g(z) = z^4 - 1$

Take $g(z) = z - \frac{g(z)}{g'(z)}$

$$g(z) = 0 \Leftrightarrow z^4 - 1 = 0 \Leftrightarrow (z^2 - 1)(z^2 + 1) = 0 \Leftrightarrow z_1 = -1, z_2 = 1 \\ z_3 = i, z_4 = -i$$

each of them is an attractor of g

A_{-1}, A_1, A_i, A_{-i} the corresponding basins of attraction, as subsets of \mathbb{C}

Their shape is of a fractal

The tent map

$$T: [0, 1] \rightarrow \mathbb{R}$$

$$T(x) = 1 - |2x - 1|, \forall x \in [0, 1]$$

$$T^k(x_0), x_0 \in [0, 1]$$

→ these sequences can have very complicated behaviour. "the tent map is chaotic"

- represent the graph of T

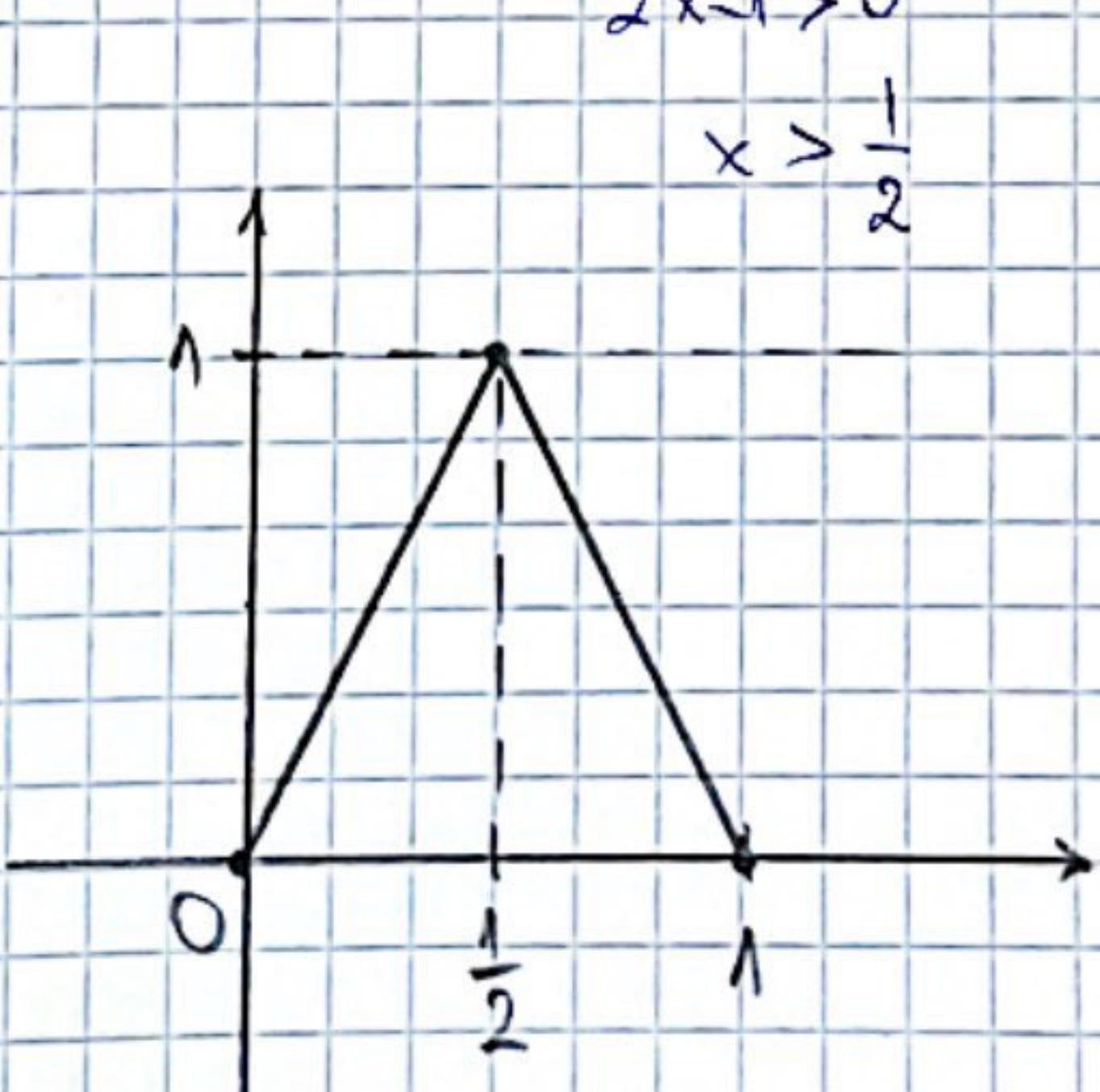
- note a similarity with $f(x) = 4x(1-x)$

- $T^k(m)$ for $m = \frac{3}{2^n}, n \geq 2, m \in \mathbb{N}$

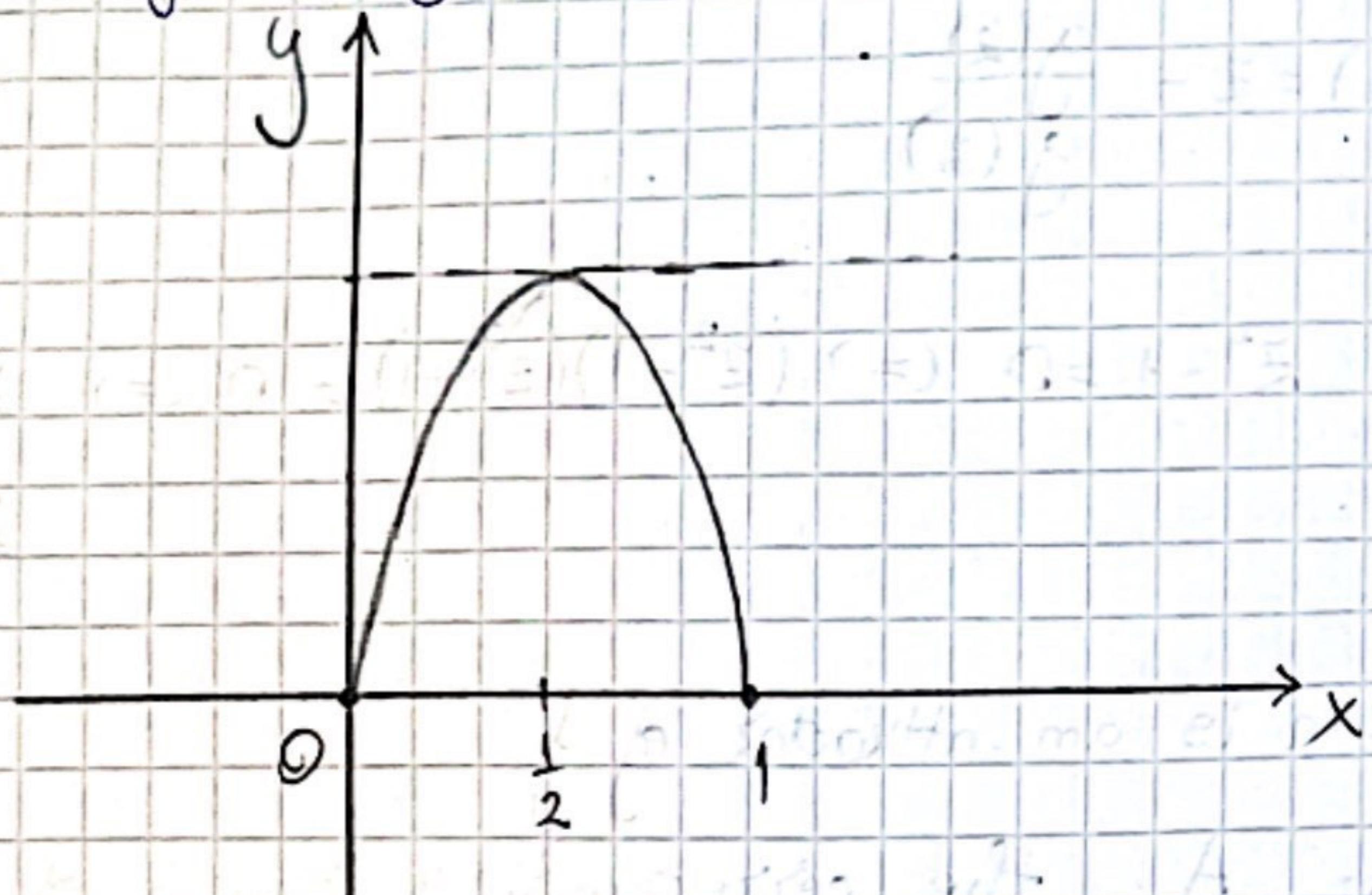
- graphs of T^2 and T^3 (+w)

$$T(x) = \begin{cases} 2(1-x), & x \in [\frac{1}{2}, 1] \\ 2x, & x \in [0, \frac{1}{2}] \end{cases}$$

$$T(0) = T(1) = 0, T(\frac{1}{2}) = 1$$



The graph of $f(x) = 4x(1-x)$, $x \in [0, 1]$



$$f\left(\frac{1}{2}\right) = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$$

$$x_{k+1} = 4 \times x_k \times (1 - x_k)$$

$$x_0 = 0.67$$

$$y_{k+1} = 4 \times y_k - 4 \times y_k \times y_k$$

$$y_0 = 0.67$$

$$x_k \in [0, 1]$$

$$x_{30} = 0.01\dots$$

$$y_k \in [0, 1]$$

$$y_{30} = 0.91\dots$$