Lecture 12

Discrete dynamical systems (Maps)

(1) $x_{k+1} = f(x_k)$, $k \in \mathbb{N}$, $x_0 = \eta \in \mathbb{R}^n$ given

where $f : \mathbb{R}^n \to \mathbb{R}^n$ continous.

The existence and uniqueness in the future

$\forall \eta \in \mathbb{R}^n$ the IVP $\begin{cases} x_{k+1} = f(x_k), & k \geq 0 \\ x_0 = \eta \end{cases}$ has an unique solution

$x_1 = f(\eta)$, $x_2 = f(x_1) = f(f(\eta)) = (f \circ f)(\eta) \overset{not}{=} f^2(\eta), \dots$

$x_k = f^k(\eta)$, $\forall k \geq 1$ where $f^k \overset{not}{=} \underbrace{f \circ f \circ \dots \circ f}_{k \text{ times}}$

Def: For $\eta \in \mathbb{R}^n$ we define the positive orbit of η as

$$\gamma_\eta^+ = \{\eta, f(\eta), f^2(\eta), \dots, f^k(\eta), \dots\}$$

When, in addition, f is invertible, we define the orbit of an initial state η

$$\gamma_\eta = \{\dots f^{-k}(\eta), \dots, f^{-2}(\eta), f^{-1}(\eta), \eta, f(\eta), \dots, f^k(\eta) \dots\}$$

where $f^{-k} = \underbrace{f^{-1} \circ f^{-1} \circ \dots \circ f^{-1}}_{k \text{ times}}$

Def: 1) We say that $\eta^* \in \mathbb{R}^n$ is a fixed point of (1) when $f(\eta^*) = \eta^*$, or, equivalently, $\gamma_{\eta^*}^+ = \{\eta^*\}$.

Remark: If η^* is a fixed point of f then η^* is a fixed point of f^k, $\forall k \geq 1$.

2) We say that $\eta^* \in \mathbb{R}^n$ is a <u>p-periodic point</u> of f when $p \in \mathbb{N}$, $p \geq 2$ and η^* is a fixed point of f^p but it is not a fixed point of $f, f^2, \dots, f^p$

Remark: Let η^* be a p-periodic point of f. Then

$\gamma^+_{\eta^*} = \{\eta^*, f(\eta^*), \dots, f^{p-1}(\eta^*)\}$, $\gamma^+_{\eta^*}$ is said to be a p-cycle.

- We have that any element of $\gamma^+_{\eta^*}$ is a p-periodic point.

<u>Proof</u>: $f^p(\eta^*) = \eta^* \Rightarrow f(f^p(\eta^*)) = f(\eta^*) \Rightarrow$

$\Rightarrow f^p(f(\eta^*)) = f(\eta^*) \Rightarrow f(\eta^*)$ is a fixed point of f^p

It can be proved that $f(\eta^*)$ is not a fixed point of any of the maps $f, f^2, \dots, f^{p-1}$. How? We assume by contradiction that $\exists\ k \in \{1, 2, \dots, p-1\}$ s.t. $f(\eta^*)$ is a fixed point of $f^k \Rightarrow f^k(f(\eta^*)) = f(\eta^*)$

$\Rightarrow f^{k+1}(\eta^*) = f(\eta^*)$

We know that $f^k(\eta^*) \neq \eta^*$

$$\begin{cases} x_{k+1} = f(x_k) \\ x_0 = \eta \end{cases}$$

$$\begin{cases} x_{k+1} = f(x_k) \\ x_0 = \eta \end{cases}$$

if η is a fixed point (f.p.) $\Rightarrow \eta, \eta, \dots$

if η is a p-periodic point $\Rightarrow \eta, f(\eta), \dots, f^{p-1}(\eta), \eta, f(\eta), \dots, f^{p-1}(\eta), \dots$

Property: If $\exists \lim_{k\to\infty} f^k(\eta) = \eta^*$ then η^* is a fixed point of f.

Prove $x_k = f^k(\eta)$ $\quad x_{k+1} = f(x_k)$

$\lim_{k\to\infty} x_k = \lim_{k\to\infty} x_{k+1} = \eta^*$ $\quad \Rightarrow \lim_{k\to\infty} x_{k+1} = f(\lim_{k\to\infty} x_k)$

f is continuous $\quad \Rightarrow \eta^* = f(\eta^*)$

Def: 1) Let η^* be a fixed point of f.
We say that η^* is an <u>attractor</u> of the discrete dyn. syst. (1) (or of f) when $\exists \rho > 0$ s.t. for any η s.t. $\|\eta - \eta^*\| < \rho$ we have $\lim_{k\to\infty} f^k(\eta) = \eta^*$.

For an attractor η^* we define its basin of attraction

$$A_{\eta^*} = \{ \eta \in \mathbb{R}^n : \lim_{k\to\infty} f^k(\eta) = \eta^* \}$$

2) Let η^* be a p-periodic point. We say that the p-cycle $\gamma^+_{\eta^*}$ is an attractor of the discrete dyn. syst. (1) when η^* is an attractor fixed point of f^p.

Remark: Let η^* be a p-periodic point of f.
Then $\gamma^+_{\eta^*} = \{ \eta^*, f(\eta^*), \dots, f^{p-1}(\eta^*) \}$.
Assume that $\gamma^+_{\eta^*}$ is an <u>attractor</u>.
$\Rightarrow \eta^*$ is an attractor fixed point of $f^p \Rightarrow$

$\Rightarrow \exists\, \rho > 0$ s.t. for η with $\|\eta - \eta^*\| < \rho$ we have
$\lim_{k\to\infty} (f^p)^k(\eta) = \eta^*$.

Fix such η like before ($\eta \in A_{\eta^*, f^p}$)

$$\gamma_\eta^+ : \eta, f(\eta), f^2(\eta), \ldots, f^{p-1}(\eta)$$
$$f^p(\eta), f^{p+1}(\eta), \ldots, f^{2p-1}(\eta)$$
$$f^{2p}(\eta), f^{2p+1}(\eta), \ldots$$

$$(f^p)^k = f^{kp}$$

The first column: $(f^{kp}(\eta))_{k\geq 0}$. We have
$$\lim_{k\to\infty} f^{kp}(\eta) = \eta^*.$$

The second column: $(f^{kp+1}(\eta))_{k\geq 0}$. So, we have
$$\lim_{k\to\infty} f^{kp+1}(\eta) = \lim_{k\to\infty} f(f^{kp}(\eta)) = f(\lim_{k\to\infty} f^{kp}(\eta)) = f(\eta^*)$$

Take $i \in \{1, 2, \ldots, p-1\}$. The i-th column:
$(f^{kp+i}(\eta))_{k\geq 0}$. So, $\lim_{k\to\infty} f^{kp+i}(\eta) = f^i(\eta^*)$

So, $f^k(\eta)$ is splitted in p subsequences that are convergent to an element of $\gamma_{\eta^*}^+$ (the p-cycle)

Scalar maps $(m=1)$

$f \in C^1(\mathbb{R})$ (2) $\begin{cases} x_{k+1} = f(x_k), & k \geq 0 \\ x_0 = \eta \in \mathbb{R} \text{ given} \end{cases}$

Example: For $f(x) = 1-2x^2$. Find its fixed points and its 2-periodic points.

- fixed points: $f(x) = x \iff 1-2x^2 = x \iff 2x^2+x-1=0$

  $\cdot\ \Delta = 9 \Rightarrow x_{1,2} = \dfrac{-1 \pm 3}{4} \nearrow -1,\ \searrow \dfrac{1}{2}$

  So, two fixed points $\eta_1^* = -1$ and $\eta_2^* = \dfrac{1}{2}$

  This means that the solution of the IVP $\begin{cases} x_{k+1} = 1-2x_k^2 \\ x_0 = -1 \text{ or } \frac{1}{2} \end{cases}$

  is $x_k = -1, \forall k \geq 0$ or, resp. $x_k = \frac{1}{2}, \forall k \geq 0$

- 2-periodic points: $f^2(x) = x$, where $f^2 = f \circ f$

  Compute $f^2(x) = f(f(x)) = 1-2[f(x)]^2 = 1-2(1-2x^2)^2 =$

  $= 1-2(1-4x^2+4x^4) = -8x^4+8x^2-1$

  So, $f^2(x) = x \iff -8x^4+8x^2-x-1 = 0 \iff$

  $8x^4-8x^2+x+1=0$

  (*) -1 and $\frac{1}{2}$ are fixed points of $f \Rightarrow -1$ and $\frac{1}{2}$ are fixed points of $f^2 \Rightarrow$ eq. (3) has -1 and $\frac{1}{2}$ as roots.

  $\Rightarrow (2x^2+x-1) \mid (8x^4-8x^2+x+1)$

  $8x^4-8x^2+x+1 = (x+1)\underbrace{(8x^3-8x^2+1)}_{}$

  $= 8x^3-4x^2-4x^2+1 = 4x^2(2x-1)-(2x+1)(2x-1)$

$4x^2-2x-1=0$

$\Delta' = 4+16 = 20$

$x_{1,2} = \frac{1\pm\sqrt{5}}{4}$

So, f^2 has the f.p $\underbrace{-1, \frac{1}{2}}_{\text{the f.p of } f}, \frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}$

Thus, the 2-periodic points of f are $\frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}$.

Hence, $\left\{\frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}\right\}$ the unique 2-cycle of f.

This means that the sol. of the IVP $\begin{cases} x_{k+1} = 1-2x_k^2 \\ x_0 = \frac{1-\sqrt{5}}{4} \end{cases}$ is

$x_{2k+1} = \frac{1+\sqrt{5}}{4}, \quad x_{2k} = \frac{1-\sqrt{5}}{4}, \forall k \geq 0$

Theorem (The linearization method)

I. Let $\eta^* \in \mathbb{R}$ be a fixed point of $f \in C^1(\mathbb{R})$.
If $|f'(\eta^*)| < 1$ then η^* is an attractor.

II. Let $\eta_1^* \in \mathbb{R}$ be a 2-periodic point of $f \in C^1(\mathbb{R})$.
Denote by $\eta_2^* = f(\eta_1^*)$ s.t. $\{\eta_1^*, \eta_2^*\}$ is the corresp. 2-cycle.
If $|f'(\eta_1^*) \cdot f'(\eta_2^*)| < 1$ then the 2-cycle $\{\eta_1^*, \eta_2^*\}$ is an attractor.

Comment on I

$\dot{x} = ax \qquad \eta^* = 0$ equilibrium point $\qquad x(t) = \eta e^{at} \xrightarrow[t\to\infty]{} 0$ when $a<0$

$x_{k+1} = a x_k \qquad \eta^* = 0$ fixed point $\qquad x_k = \eta \cdot a^k \xrightarrow[k\to\infty]{} 0$ when $|a|<1$

Comment on II :

$$f^2(\eta_1^*) = \eta_1^*$$

$$(f^2)'(\eta_1^*) \overset{?}{=} f'(\eta_1^*) \cdot f'(\eta_2^*)$$

$$f^2(x) = f(f(x)) \Rightarrow (f^2)'(x) = f'(f(x)) = f'(f(x)) \cdot f'(x) \Rightarrow$$

$$\Rightarrow (f^2)'(\eta_1^*) = f'(\eta_1^*) \cdot f'(\eta_2^*)$$

Example: $f(x) = 1 - 2x^2$. Study the stability of the fixed points and of the 2-cycle.

$$f'(x) = -4x$$

$f'(-1) = 4 \Rightarrow |f'(-1)| > 1 \Rightarrow -1$ a fixed point, but not attractor

$f'(\frac{1}{2}) = -2 \Rightarrow |f'(\frac{1}{2})| > 1 \Rightarrow \frac{1}{2}$ is not an attractor

$\left\{ \frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4} \right\}$ is a 2-cycle.

$$f'\left(\frac{1-\sqrt{5}}{4}\right) \cdot f'\left(\frac{1+\sqrt{5}}{4}\right) = -4 \cdot \frac{1-\sqrt{5}}{4} \cdot (-4) \cdot \frac{1+\sqrt{5}}{4} = (1-\sqrt{5})(1+\sqrt{5}) = -4$$

$\Rightarrow$ the 2-cycle is not an attractor