# Seminar 1

**1.** Which ones of the usual symbols of addition, subtraction, multiplication and division define an operation (composition law) on the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$?

**2.** Let $A = \{a_1, a_2, a_3\}$. Determine the number of:
$(i)$ operations on $A$;
$(ii)$ commutative operations on $A$;
$(iii)$ operations on $A$ with identity element.
Generalization for a set $A$ with $n$ elements ($n \in \mathbb{N}^*$).

**3.** Decide which ones of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are groups together with the usual addition or multiplication.

**4.** Let "$*$" be the operation defined on $\mathbb{R}$ by $x * y = x + y + xy$. Prove that:
$(i)$ $(\mathbb{R}, *)$ is a commutative monoid.
$(ii)$ The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

**5.** Let "$*$" be the operation defined on $\mathbb{N}$ by $x * y = \text{g.c.d.}(x, y)$.
$(i)$ Prove that $(\mathbb{N}, *)$ is a commutative monoid.
$(ii)$ Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ ($n \in \mathbb{N}^*$) is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid.
$(iii)$ Fill in the table of the operation "$*$" on $D_6$.

**6.** Determine the finite stable subsets of $(\mathbb{Z}, \cdot)$.

**7.** Let $(G, \cdot)$ be a group. Show that:
$(i)$ $G$ is abelian $\Longleftrightarrow \forall x, y \in G$, $(xy)^2 = x^2 y^2$.
$(ii)$ If $x^2 = 1$ for every $x \in G$, then $G$ is abelian.

**8.** Let "$\cdot$" be an operation on a set $A$ and let $X, Y \subseteq A$. Define an operation "$*$" on the power set $\mathcal{P}(A)$ by

$$X * Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

Prove that:
$(i)$ If $(A, \cdot)$ is a monoid, then $(\mathcal{P}(A), *)$ is a monoid.
$(ii)$ If $(A, \cdot)$ is a group, then in general $(\mathcal{P}(A), *)$ is not a group.

## Seminar 2

**1.** Let $r, s, t, v$ be the homogeneous relations defined on the set $M = \{2,3,4,5,6\}$ by

$$x\, r\, y \Longleftrightarrow x < y$$

$$x\, s\, y \Longleftrightarrow x | y$$

$$x\, t\, y \Longleftrightarrow g.c.d.(x, y) = 1$$

$$x\, v\, y \Longleftrightarrow x \equiv y \pmod 3 .$$

Write the graphs $R, S, T, V$ of the given relations.

**2.** Let $A$ and $B$ be sets with $n$ and $m$ elements respectively ($m, n \in \mathbb{N}^*$). Determine the number of:
$(i)$ relations having the domain $A$ and the codomain $B$;
$(ii)$ homogeneous relations on $A$.

**3.** Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

**4.** Which ones of the properties of reflexivity, transitivity and symmetry hold for the following homogeneous relations: the strict inequality relations on $\mathbb{R}$, the divisibility relation on $\mathbb{N}$ and on $\mathbb{Z}$, the perpendicularity relation of lines in space, the parallelism relation of lines in space, the congruence of triangles in a plane, the similarity of triangles in a plane?

**5.** Let $M = \{1,2,3,4\}$, let $r_1$, $r_2$ be homogeneous relations on $M$ and let $\pi_1$, $\pi_2$, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.
$(i)$ Are $r_1, r_2$ equivalences on $M$? If yes, write the corresponding partition.
$(ii)$ Are $\pi_1, \pi_2$ partitions on $M$? If yes, write the corresponding equivalence relation.

**6.** Define on $\mathbb{C}$ the relations $r$ and $s$ by:

$$z_1\, r\, z_2 \Longleftrightarrow |z_1| = |z_2| ; \qquad z_1\, s\, z_2 \Longleftrightarrow arg\, z_1 = arg\, z_2 \text{ or } z_1 = z_2 = 0 .$$

Prove that $r$ and $s$ are equivalence relations on $\mathbb{C}$ and determine the quotient sets (partitions) $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometric interpretation).

**7.** Let $n \in \mathbb{N}$. Consider the relation $\rho_n$ on $\mathbb{Z}$, called the *congruence modulo* $n$, defined by:

$$x\, \rho_n\, y \Longleftrightarrow n | (x - y) .$$

Prove that $\rho_n$ is an equivalence relation on $\mathbb{Z}$ and determine the quotient set (partition) $\mathbb{Z}/\rho_n$. Discuss the cases $n = 0$ and $n = 1$.

**8.** Determine all equivalence relations and all partitions on the set $M = \{1,2,3\}$.

**9.** Let $M = \{0,1,2,3\}$ and let $h = (\mathbb{Z}, M, H)$ be a relation, where

$$H = \{(x, y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : x = 4z + y\} .$$

Is $h$ a function?

**10.** Consider the following homogeneous relations on $\mathbb{N}$, defined by:

$$m\, r\, n \Longleftrightarrow \exists a \in \mathbb{N} : m = 2^a n ,$$

$$m\, s\, n \Longleftrightarrow (m = n \text{ or } m = n^2 \text{ or } n = m^2) .$$

Are $r$ and $s$ equivalence relations?

# Seminar 3

**1.** Let $M$ be a non-empty set and let $S_M = \{f : M \to M \mid f \text{ is bijective}\}$. Show that $(S_M, \circ)$ is a group, called the *symmetric group* of $M$.

**2.** Let $M$ be a non-empty set and let $(R, +, \cdot)$ be a ring. Define on $R^M = \{f \mid f : M \to R\}$ two operations by: $\forall f, g \in R^M$,

$$f + g : M \to R\,, \quad (f+g)(x) = f(x) + g(x)\,, \quad \forall x \in M\,,$$

$$f \cdot g : M \to R\,, \quad (f \cdot g)(x) = f(x) \cdot g(x)\,, \quad \forall x \in M\,.$$

Show that $(R^M, +, \cdot)$ is a ring. If $R$ is commutative or has identity, does $R^M$ have the same property?

**3.** Prove that $H = \{z \in \mathbb{C} \mid |z| = 1\}$ is a subgroup of $(\mathbb{C}^*, \cdot)$, but not of $(\mathbb{C}, +)$.

**4.** Let $U_n = \{z \in \mathbb{C} \mid z^n = 1\} \quad (n \in \mathbb{N}^*)$ be the *set of $n$-th roots of unity*. Prove that $U_n$ is a subgroup of $(\mathbb{C}^*, \cdot)$.

**5.** Let $n \in \mathbb{N}$, $n \geq 2$. Prove that:
($i$) $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) \neq 0\}$ is a stable subset of the monoid $(M_n(\mathbb{C}), \cdot)$;
($ii$) $(GL_n(\mathbb{C}), \cdot)$ is a group, called the *general linear group of rank $n$*;
($iii$) $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) = 1\}$ is a subgroup of the group $(GL_n(\mathbb{C}), \cdot)$.

**6.** Show that the following sets are subrings of the corresponding rings:
($i$) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ in $(\mathbb{C}, +, \cdot)$.
($ii$) $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$ in $(M_2(\mathbb{R}), +, \cdot)$.

**7.** ($i$) Let $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = |z|$. Show that $f$ is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

($ii$) Let $g : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $g(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that $g$ is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}), \cdot)$.

**8.** Let $n \in \mathbb{N}$, $n \geq 2$. Prove that the groups $(\mathbb{Z}_n, +)$ of residue classes modulo $n$ and $(U_n, \cdot)$ of $n$-th roots of unity are isomorphic.

**9.** Let $n \in \mathbb{N}$, $n \geq 2$. Consider the ring $(\mathbb{Z}_n, +, \cdot)$ and let $\widehat{a} \in \mathbb{Z}_n^*$.
($i$) Prove that $\widehat{a}$ is invertible $\iff (a, n) = 1$.
($ii$) Deduce that $(\mathbb{Z}_n, +, \cdot)$ is a field $\iff n$ is prime.

**10.** Let $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$. Show that $(\mathcal{M}, +, \cdot)$ is a field isomorphic to $(\mathbb{C}, +, \cdot)$.

## Seminar 4

**1.** Let $K$ be a field. Show that $K[X]$ is a $K$-vector space, where the addition is the usual addition of polynomials and the scalar multiplication is defined as follows: $\forall\, k \in K$, $\forall f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$,

$$k \cdot f = (ka_0) + (ka_1)X + \cdots + (ka_n)X^n .$$

**2.** Let $K$ be a field and $m, n \in \mathbb{N}$, $m, n \geq 2$. Show that $M_{m,n}(K)$ is a $K$-vector space, with the usual addition and scalar multiplication of matrices.

**3.** Let $K$ be a field, $A \neq \emptyset$ and denote $K^A = \{f \mid f : A \to K\}$. Show that $K^A$ is a $K$-vector space, where the addition and the scalar multiplication are defined as follows: $\forall f, g \in K^A$, $\forall k \in K$, $f + g \in K^A$, $kf \in K^A$,

$$(f+g)(x) = f(x) + g(x)\,, \quad (k \cdot f)(x) = k \cdot f(x)\,, \forall x \in A\,.$$

**4.** Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define the operations: $x \perp y = xy$ and $k \top x = x^k$, $\forall k \in \mathbb{R}$ and $\forall x, y \in V$. Prove that $V$ is a vector space over $\mathbb{R}$.

**5.** Let $K$ be a field and let $V = K \times K$. Decide whether $V$ is a $K$-vector space with respect to the following addition and scalar multiplication:

$(i)$ $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + 2y_2)$ and $k \cdot (x_1, y_1) = (kx_1, ky_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

$(ii)$ $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k \cdot (x_1, y_1) = (kx_1, y_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

**6.** Let $p$ be a prime number and let $V$ be a vector space over the field $\mathbb{Z}_p$.

$(i)$ Prove that $\underbrace{x + \cdots + x}_{p \text{ times}} = 0$, $\forall x \in V$.

$(ii)$ Is there a scalar multiplication endowing $(\mathbb{Z}, +)$ with a structure of a vector space over $\mathbb{Z}_p$?

**7.** Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:

$(i)$ $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
$(ii)$ $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;
$(iii)$ $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;
$(iv)$ $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
$(v)$ $E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;
$(vi)$ $F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$?

**8.** Which ones of the following sets are subspaces:

$(i)$ $[-1, 1]$ of the real vector space $\mathbb{R}$;
$(ii)$ $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ of the real vector space $\mathbb{R}^2$;
$(iii)$ $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Q} \right\}$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;
$(iv)$ $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

**9.** Which ones of the following sets are subspaces of the $K$-vector space $K[X]$:

$(i)$ $K_n[X] = \{f \in K[X] \mid \text{degree}(f) \leq n\}$ $(n \in \mathbb{N})$;
$(ii)$ $K'_n[X] = \{f \in K[X] \mid \text{degree}(f) = n\}$ $(n \in \mathbb{N})$.

**10.** Show that the set of all solutions of a homogeneous system of two equations and two unknowns with real coefficients is a subspace of the real vector space $\mathbb{R}^2$.

# Seminar 5

**1.** Determine the following generated subspaces:
$(i)$ $< 1, X, X^2 >$ in the real vector space $\mathbb{R}[X]$.
$(ii)$ $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in the real vector space $M_2(\mathbb{R})$.

**2.** Consider the following subspaces of the real vector space $\mathbb{R}^3$:
$(i)$ $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
$(ii)$ $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
$(iii)$ $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write $A, B, C$ as generated subspaces with a minimal number of generators.

**3.** Consider the following vectors in the real vector space $\mathbb{R}^3$:

$$a = (-2, 1, 3), b = (3, -2, -1), c = (1, -1, 2), d = (-5, 3, 4), e = (-9, 5, 10).$$

Show that $\langle a, b\rangle = \langle c, d, e\rangle$.

**4.** Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that $S$ and $T$ are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

**5.** Let $S$ and $T$ be the set of all even functions and of all odd functions in $\mathbb{R}^{\mathbb{R}}$ respectively. Prove that $S$ and $T$ are subspaces of the real vector space $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{R}} = S \oplus T$.

**6.** Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x + y, x - y),$$

$$g(x, y) = (2x - y, 4x - 2y),$$

$$h(x, y, z) = (x - y, y - z, z - x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

**7.** Which ones of the following functions are endomorphisms of the real vector space $\mathbb{R}^2$:
$(i)$ $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (ax + by, cx + dy)$, where $a, b, c, d \in \mathbb{R}$;
$(ii)$ $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (a + x, b + y)$, where $a, b \in \mathbb{R}$?

**8.** Let $a \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x \cos a - y \sin a, x \sin a + y \cos a)\,.$$

Prove that $f \in End_{\mathbb{R}}(\mathbb{R}^2)$.

**9.** Determine the kernel and the image of the endomorphisms from Exercise **6.**

**10.** Let $V$ be a vector space over $K$ and $f \in End_K(V)$. Show that the set

$$S = \{x \in V \mid f(x) = x\}$$

of fixed points of $f$ is a subspace of $V$.

# Seminar 6

**1.** Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:
$(i)$ $v_1$, $v_2$, $v_3$ are linearly dependent and determine a dependence relationship.
$(ii)$ $v_1$, $v_2$ are linearly independent.

**2.** Prove that the following vectors are linearly independent:
$(i)$ $v_1 = (1, 0, 2)$, $v_2 = (-1, 2, 1)$, $v_3 = (3, 1, 1)$ in ${}_{\mathbb{R}}\mathbb{R}^3$.
$(ii)$ $v_1 = (1, 2, 3, 4)$, $v_2 = (2, 3, 4, 1)$, $v_3 = (3, 4, 1, 2)$, $v_4 = (4, 1, 2, 3)$ in ${}_{\mathbb{R}}\mathbb{R}^4$.

**3.** Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$. Determine $a \in \mathbb{R}$ such that the vectors $v_1$, $v_2$, $v_3$ are linearly independent.

**4.** Let $v_1 = (1, -2, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (0, a, 1, 2)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^4$. Determine $a \in \mathbb{R}$ such that the vectors $v_1$, $v_2$, $v_3$ are linearly dependent.

**5.** Let $v_1 = (1, 1, 0)$, $v_2 = (-1, 0, 2)$, $v_3 = (1, 1, 1)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$.
$(i)$ Show that the list $(v_1, v_2, v_3)$ is a basis of the real vector space $\mathbb{R}^3$.
$(ii)$ Express the vectors of the canonical basis $(e_1, e_2, e_3)$ of $\mathbb{R}^3$ as a linear combination of the vectors $v_1$, $v_2$ and $v_3$.
$(iii)$ Determine the coordinates of $u = (1, -1, 2)$ in each of the two bases.

**6.** Let $n \in \mathbb{N}^*$. Show that the vectors

$$v_1 = (1, \dots, 1, 1), v_2 = (1, \dots, 1, 2), v_3 = (1, \dots, 1, 2, 3), \dots, v_n = (1, 2, \dots, n-1, n)$$

form a basis of the real vector space $\mathbb{R}^n$ and write the coordinates of a vector $(x_1, \dots, x_n)$ in this basis.

**7.** Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the lists $(E_1, E_2, E_3, E_4)$ and $(A_1, A_2, A_3, A_4)$ are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in each of the two bases.

**8.** Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid \mathit{degree}(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X - a, (X - a)^2)$ $(a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial $f = a_0 + a_1 X + a_2 X^2 \in \mathbb{R}_2[X]$ in each basis.

**9.** Determine the number of bases of the vector space $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$.

**10.** Determine the number of elements of the general linear group $(GL_3(\mathbb{Z}_2), \cdot)$ of invertible $3 \times 3$-matrices over $\mathbb{Z}_2$.

# Seminar 7

**1.** Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

**2.** Let $K$ be a field and $S = \{(x_1, \dots, x_n) \in K^n \mid x_1 + \cdots + x_n = 0\}$.
$(i)$ Prove that $S$ is a subspace of the canonical vector space $K^n$ over $K$.
$(ii)$ Determine a basis and the dimension of $S$.

**3.** Determine a basis and the dimensions of the vector spaces $\mathbb{C}$ over $\mathbb{C}$ and $\mathbb{C}$ over $\mathbb{R}$. Prove that the set $\{1, i\}$ is linearly dependent in the vector space $\mathbb{C}$ over $\mathbb{C}$ and linearly independent in the vector space $\mathbb{C}$ over $\mathbb{R}$.

**4.** Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (y, -x)$. Prove that $f$ is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

**5.** Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x, y, z) = (-y + 5z, x, y - 5z)$. Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

**6.** Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

**7.** Determine a complement for the following subspaces:
$(i)$ $A = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ in the real vector space $\mathbb{R}^3$;
$(ii)$ $B = \{aX + bX^3 \mid a, b \in \mathbb{R}\}$ in the real vector space $\mathbb{R}_3[X]$.

**8.** Let $V$ be a vector space over $K$ and let $S, T$ and $U$ be subspaces of $V$ such that $dim(S \cap U) = dim(T \cap U)$ and $dim(S + U) = dim(T + U)$. Prove that if $S \subseteq T$, then $S = T$.

**9.** Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\},$$

$$T =< (0, 1, 1), (1, 1, 0) >$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

**10.** Determine the dimensions of the subspaces $S$, $T$, $S + T$ and $S \cap T$ of the real vector space $M_2(\mathbb{R})$, where

$$S = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle, \qquad T = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle.$$

# Seminar week 9

$$r = (\overset{\text{domain}}{A}, \overset{\text{codomain}}{B}, \overset{\text{graph}}{R}) \qquad \text{relation}$$

$A, B$ sets, $R \subseteq A \times B$

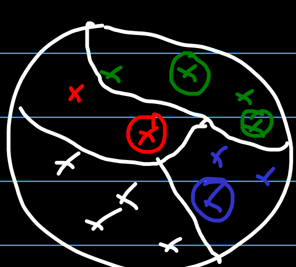
homogeneous relation : $r = (A, A, R)$

<u>Equivalence relation</u> : (binary homogeneous) relation $r = (A, A, R)$ that satisfies the following properties

- reflexivity : $\forall x \in A$: $x\, r\, x$ ($\Leftrightarrow (x, x) \in R$)
- symmetry : $\forall x, y \in A$: if $x\, r\, y$, then $y\, r\, x$
- transitivity : $\forall x, y, z \in A$ if $x\, r\, y$ and $y\, r\, z$, then $x\, r\, z$

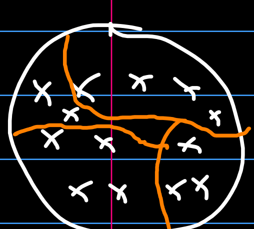
<u>Prop</u> : If $A$ is a set, then we have a bijection :

$$\{\text{equivalence relations on } A\} \xleftrightarrow{\sim} \{\text{partition of } A\}$$



$$r \longmapsto A/r \quad \text{quotient set of } r$$

$$r_P \longleftarrow\!\!\!\mid \; P \subseteq \mathcal{P}(A)$$



$$A/r = \{ \underbrace{r\langle x \rangle}_{=:\,\hat{x}} \mid x \in A \} = \{\text{green stuff, red stuff, blue stuff, white stuff}\}$$

$$r\langle x \rangle = \{ y \in A \mid x\, r\, y \}$$

$\mathcal{P}$ partition of $A$ $\Rightarrow$ $\mathcal{P} = \{ A_i \mid i \in I \}$

s.t. $\forall i,j \in I : A_i \cap A_j = \emptyset$

$$\bigcup_{i \in I} A_i = A$$

$x \, r_{\mathcal{P}} \, y \iff \exists \, S \in \mathcal{P}$ s.t. $x, y \in S$

2.1.

**1.** Let $r, s, t, v$ be the homogeneous relations defined on the set $M = \{2, 3, 4, 5, 6\}$ by

$$x \, r \, y \iff x < y$$

$$x \, s \, y \iff x | y \iff y \, \vdots \, x$$

$$x \, t \, y \iff g.c.d.(x, y) = 1$$

$$x \, v \, y \iff x \equiv y \pmod 3 . \iff 3 \, | \, (x - y) \iff$$

Write the graphs $R, S, T, V$ of the given relations. $\iff x \bmod 3 = y \bmod 3$

Sol.:

$R = \{ (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6) \}$

$S = \{ (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6) \}$

$T = \{ (2,3), (2,5), (3,4), (3,5), (4,5), (5,6), (3,2), (5,2), (4,3), (5,3), (5,4), (6,5) \}$

$V = \{ (5,2), (6,3), (2,2), (2,5), (3,3), (3,6), (4,4), (5,5), (6,6) \}$

**3.** Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

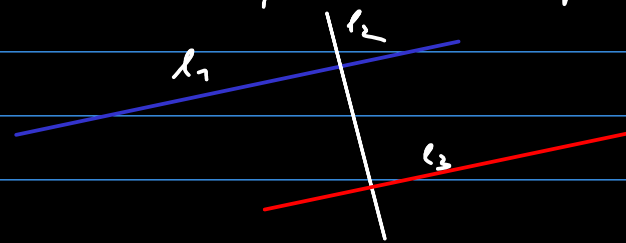
Sol: $(r)$, $\neg(s)$, $\neg(t)$: $M = \{1,2,3\}$

Graph: $\{(1,1), (1,2), (2,2), (2,3), (3,3)\}$ ✓

$\neg(r)$, $(s)$, $\neg(t)$: $M$ = set of lines in the plane

$M = \{1,2,3\}$ $\quad l_1 \sim l_2 \iff l_1 \perp l_2$

Graph: $\{(1,2), (2,1), (2,3), (3,2)\}$




$\neg(r)$, $\neg(s)$, $(t)$:

$M = \mathbb{R}$

$x\, r\, y \iff x < y$ ✓

$M = \{1,2\}$

Graph: $\{(1,2)\}$ ✓

**5.** Let $M = \{1,2,3,4\}$, let $r_1$, $r_2$ be homogeneous relations on $M$ and let $\pi_1$, $\pi_2$, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.

$(i)$ Are $r_1, r_2$ equivalences on $M$? If yes, write the corresponding partition.

$(ii)$ Are $\pi_1, \pi_2$ partitions on $M$? If yes, write the corresponding equivalence relation.

$\Delta_M = \{(1,1), (2,2), (3,3), (4,4)\}$

(i) $r_1$ refl., because $\Delta_M \subseteq R_1$

$r_1$ symmetrical

$r_1$ transitive

$\Rightarrow M/r_1 = \{\{1,2,3\},\{4\}\}$

$(1,2) \in R_2$, but $(2,1) \notin R_2 \Rightarrow R_2$ not

symmetric $\Rightarrow R_2$ not an equivalence

Ex: $A = \{1, 2\}$. $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

$P = \{\{(1,1), (2,2)\}, \{(1,2), (2,1)\}\}$

$r = (A, A, R)$, $R = A \times A$

$$\Rightarrow A/r = \{A\}$$

$$R = \Delta_A$$

$$\Rightarrow A/r = \{\{x\} \mid x \in A\}$$

(ii)

**5.** Let $M = \{1,2,3,4\}$, let $r_1$, $r_2$ be homogeneous relations on $M$ and let $\pi_1$, $\pi_2$, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.
($i$) Are $r_1, r_2$ equivalences on $M$? If yes, write the corresponding partition.
($ii$) Are $\pi_1, \pi_2$ partitions on $M$? If yes, write the corresponding equivalence relation.

$\pi_1$ is a partition, because $\forall x \in M$ $\exists! A \in \pi_1$

so that $x \in A$

$$R_{\pi_1} = \{(1,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$$

$\pi_2$ is not a partition, because $\{1\} \cap \{1,2\} \neq \emptyset$

**8.** Determine all equivalence relations and all partitions on the set $M = \{1, 2, 3\}$.

Sol.: Partitions:

$$\underbrace{\{\{1,2,3\}\}}_{1\text{ element}} \qquad \underbrace{\begin{matrix}\{\{1,2\},\{3\}\} \\ \{\{1\},\{2,3\}\} \\ \{\{2\},\{1,3\}\}\end{matrix}}_{2\text{ elements}} \qquad \underbrace{\{\{1\},\{2\},\{3\}\}}_{3\text{ elements}}$$

$\Rightarrow$ 5 partitions $\Rightarrow$ 5 equivalences

Ex. 6,7

12.10.2021

# Seminar WZ - 911

- $(R, +, \cdot)$ ring

  $R$ → set, $+, \cdot$ → operations

  → $(R, +)$ abelian group

  → $(R, \cdot)$ semigroup

  (if monoid ⇒ unital ring or ring with unity)

  → distributivity:

  $\forall x, y, z \in R: x \cdot (y+z) = x \cdot y + x \cdot z$

  $(y+z) \cdot x = y \cdot x + z \cdot x$

  if commutativity of $\cdot$ ⇒ commutative ring

- $(K, +, \cdot)$ field

  → $(K, +, \cdot)$ unital ring

  → $\forall x \in K \setminus \{0\}\ \exists x' : xx' = x'x = 1$

  → $\cdot$ commutative (if not, then division ring)

<u>Def</u>: $(G, \cdot)$ group, $S \subseteq G$, then:

$S \leq G \iff$ (i) $S \neq \emptyset$

(S is a subgroup of G) (ii) $\forall x, y \in S: xy^{-1} \in S$

<u>Def</u>: $(R,+,\cdot)$ ring, $S \subseteq R$, then:

$S \le R$ $\iff$ (i) $S \neq \emptyset$
(ii) $\forall x,y \in S: x-y \in S$
(iii) $\forall x,y \in S: x \cdot y \in S$

($S$ is a subring of $R$)

3.5. **5.** Let $n \in \mathbb{N}$, $n \geq 2$. Prove that:
$(i)$ $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) \neq 0\}$ is a stable subset of the monoid $(M_n(\mathbb{C}), \cdot)$;
$(ii)$ $(GL_n(\mathbb{C}), \cdot)$ is a group, called the *general linear group of rank* $n$;
$(iii)$ $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) = 1\}$ is a subgroup of the group $(GL_n(\mathbb{C}), \cdot)$.

(You don't have to show that $\det(AB) = \det(A)\det(B)$, it can be assumed)

<u>Sol.</u>: (i) We need to show that $\forall A, B \in GL_n(\mathbb{C}) : AB \in GL_n(\mathbb{C})$

$A, B \in GL_n(\mathbb{C}) \Rightarrow \det A, \det B \neq 0 \Rightarrow \det(A)\det(B) \neq 0 \Rightarrow$

$\Rightarrow \det(AB) \neq 0 \Rightarrow AB \in GL_n(\mathbb{C})$

(ii) $GL_n(\mathbb{C}) \subseteq M_n(\mathbb{C}) \Rightarrow \cdot$ is associative

$\det(I_n) = 1 \Rightarrow I_n \in GL_n(\mathbb{C}) \Rightarrow I_n$ is the neutral element for $\cdot$ in $GL_n(\mathbb{C})$

$A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$ $\quad B = (b_{kl})_{\substack{k=\overline{1,n} \\ l=\overline{1,p}}}$

$\Rightarrow$ the element $(\alpha, \beta)$ in $AB$ is:

$$\sum_{\gamma=1}^{n} a_{\alpha\gamma} \cdot b_{\gamma\beta}$$

$$\forall A \in GL_n(\mathbb{C}) \quad \exists\, A^{-1} = \frac{1}{\det(A)} \cdot A^{*} \in GL_n(\mathbb{C})$$

$$AA^{-1} = A^{-1}A = I_n$$

(ii) $SL_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) \mid \det A = 1\}$

$\det(I_n) = 1 \Rightarrow I_n \in SL_n(\mathbb{C}) \Rightarrow SL_n(\mathbb{C}) \neq \emptyset$

$\forall A, B \in SL_n(\mathbb{C}) \overset{?}{\Rightarrow} AB^{-1} \in SL_n(\mathbb{C})$

$A, B \in SL_n(\mathbb{C}) \Rightarrow \det A = \det B = 1$

$\det(AB^{-1}) = \det A \cdot \underbrace{\det(B^{-1})}$

$\det(B) \cdot \det(B^{-1}) = \det(BB^{-1}) = \det(I_n) = 1$

$$\det(AB^{-1}) = \det A \cdot \underbrace{\det(B^{-1})}_{\frac{1}{\det B}} = \frac{\det A}{\det B} = 1$$

$\Rightarrow AB^{-1} \in SL_n(\mathbb{C})$

**6.** Show that the following sets are subrings of the corresponding rings:
$(i)$ $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ in $(\mathbb{C}, +, \cdot)$.
$(ii)$ $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$ in $(M_2(\mathbb{R}), +, \cdot)$.

Sol: (i) $\mathbb{Z}[i] \neq \emptyset$, because $1 \in \mathbb{Z}[i]$

$\forall z_1, z_2 \in \mathbb{Z}[i] \overset{?}{:} z_1 - z_2 \in \mathbb{Z}[i]$

Let $z_1 = a + bi$, $z_2 = c + di$, $a, b, c, d \in \mathbb{Z}$

$$z_1 - z_2 = \underbrace{(a-c)}_{\in \mathbb{Z}} + \underbrace{(b-d)}_{\in \mathbb{Z}} \cdot i \in \mathbb{Z}[i]$$

$$\forall z_1, z_2 \in \mathbb{Z}[i] : z_1 z_2 \overset{?}{\in} \mathbb{Z}[i]$$

$$z_1 = a + bi, \quad z_2 = c + di$$

$$z_1 z_2 = ac + adi + cbi + bdi^2 =$$

$$= \underbrace{(ac - bd)}_{\in \mathbb{Z}} + \underbrace{(ad + bc)}_{\in \mathbb{Z}} \cdot i \in \mathbb{Z}[i]$$

$$\Rightarrow \mathbb{Z}[i] \leq \mathbb{C}$$

__Remark:__ $\mathbb{C}$ is a field, but $\mathbb{Z}[i]$ isn't a field

$$1+i \in \mathbb{C} \quad (1+i)^{-1} = \frac{1}{1+i} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2} \cdot i \notin \mathbb{Z}[i]$$

**7.** (*i*) Let $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = |z|$. Show that $f$ is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

(*ii*) Let $g : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $g(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that $g$ is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}), \cdot)$.

__Def:__ $(G_1, *)$, $(G_2, \square)$ groups

$f : G_1 \to G_2$ is a group homomorphism if:

$$\forall x, y \in G_1 : f(x * y) = f(x) \square f(y)$$

$$\left( \Rightarrow f(x^{-1}) = f(x)^{-1}, \quad e_{G_2} = f(e_{G_1}) \right)$$

**7.** $(i)$ Let $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = |z|$. Show that $f$ is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

$(ii)$ Let $g : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $g(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that $g$ is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}), \cdot)$.

Sol.: 7.(ii)

Let $x = a+bi$, $y = c+di$

We will show that $g(xy) = g(x) \cdot g(y)$

$$g(xy) = g((a+bi)(c+di)) = g(ac-bd+(ad+bc)\cdot i) =$$

$$= \begin{pmatrix} ac-bd & bc+ad \\ -bc-ad & ac-bd \end{pmatrix}$$

$$g(x)\cdot g(y) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac-bd & ad+bc \\ -bc-ad & -bd+ac \end{pmatrix}$$

$\Rightarrow g(xy) = g(x)\cdot g(y) \Rightarrow g$ group morphism

Terminology: homomorphism = morphism $f: A_1 \to A_2$
endomorphism = morphism $f: A \to A$
isomorphism = morphism $f: A_1 \to A_2$ + bijectivity
automorphism = endo + iso = morphism $f: A \to A$ + bijectivity

**10.** Let $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \Big| a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$. Show that $(\mathcal{M}, +, \cdot)$ is a field isomorphic to $(\mathbb{C}, +, \cdot)$.

<u>Sol.</u>: $(\mathcal{M}, +, \cdot)$ is a ring ?

$\rightarrow$ $+$ is an operation on $\mathcal{M}$;

$$A_1, A_2 \in \mathcal{M}, \quad A_1 = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}$$

$$A_1 + A_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ -b_1 - b_2 & a_1 + a_2 \end{pmatrix} \in \mathcal{M}$$

$\rightarrow$ assoc. of $+$ is inherited

$\rightarrow$ the neutral element, $O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}$

$\rightarrow$ invertibility of $+$:

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \Rightarrow -A = \begin{pmatrix} -a & -b \\ b & -a \end{pmatrix} \in \mathcal{M}$$

$$A + (-A) = O_2$$

$\rightarrow$ commutativity of $+$ is inherited

$\rightarrow$ $\cdot$ is an operation on $\mathcal{M}$:

$$A_1 \cdot A_2 = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} =$$

$$= \begin{pmatrix} a_1 a_2 - b_1 b_2 & a_1 b_2 + b_1 a_2 \\ -a_2 b_1 - b_2 a_1 & -b_1 b_2 + a_1 a_2 \end{pmatrix} \in \mathcal{M}$$

$\rightarrow$ associativity of $\cdot$ is inherited

$\rightarrow$ the neutral element of $\cdot$ is $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}$

$\rightarrow$ distributivity is inherited

$$\Rightarrow \mathcal{M} \underset{\text{(subring)}}{\leq} \mathcal{M}_2(\mathbb{C})$$

We will now show that $\forall A \in \mathcal{M} \setminus \{0\}\ \exists A'$:

$$AA' = A'A = I_2$$

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Let $A' = \frac{1}{a^2+b^2} \cdot A^* = \frac{1}{a^2+b^2} \cdot \begin{pmatrix} a & -b \\ b & a \end{pmatrix} =$

$$= \begin{pmatrix} \frac{a}{a^2+b^2} & \frac{-b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix} \in \mathcal{M}$$

$$\Rightarrow \mathcal{M} \text{ field}$$

We use the function:

$$g : \mathcal{M} \to \mathbb{C}$$
$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \longmapsto a+ib$$

Show that $g$ is a field isomorphism

19.10.2021

# Seminar W4 – 911

**Def.**: $(V,+)$ abelian group, $(K,+,\cdot)$ field

$$\cdot : K \times V \to V \quad \text{external operation}$$
$$(k, v) \mapsto k\cdot v$$

$\to$ $V$ <u>$K$-vector space</u> if:

- $\forall \alpha,\beta \in K, \forall v \in V: \quad (\alpha+\beta)\cdot v = \alpha v + \beta v$
- $\forall \alpha \in K, \forall v_1, v_2 \in V: \quad \alpha\cdot(v_1+v_2) = \alpha v_1 + \alpha v_2$
- $\forall \alpha,\beta \in K, \forall v \in V: \quad (\alpha\beta)v = \alpha\cdot(\beta v)$
- $\forall v \in V: \quad 1\cdot v = v$

**4.** Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define the operations: $x \perp y = xy$ and $k \top x = x^k$, $\forall k \in \mathbb{R}$ and $\forall x, y \in V$. Prove that $V$ is a vector space over $\mathbb{R}$.

**Sol.**: $(V, \perp)$ abelian group, external operation: $\top$

- $\perp$ is an op: $\forall v_1, v_2 \in V \overset{?}{\Rightarrow} v_1 \perp v_2 \in V$

$$v_1 \perp v_2 = \underbrace{v_1}_{>0} \cdot \underbrace{v_2}_{>0} > 0 \Rightarrow v_1 \perp v_2 \in V$$

- assoc. of $\perp$: $\forall v_1, v_2, v_3 \in V \overset{?}{\Rightarrow} (v_1 \perp v_2) \perp v_3 = v_1 \perp (v_2 \perp v_3)$

$$(v_1 \perp v_2) \perp v_3 = (v_1 v_2) \perp v_3 = v_1 v_2 v_3$$
$$v_1 \perp (v_2 \perp v_3) = v_1 \perp (v_2 v_3) = v_1 v_2 v_3 \quad \checkmark$$

$\Rightarrow$ assoc.

- commutativity of $\perp$ : $\forall v_1, v_2 \in V$ :

$$v_1 \perp v_2 = v_1 v_2 = v_2 v_1 = v_2 \perp v_1$$

- neutral element of $\perp$ : $1 \in V$, so $\forall v \in V$: $1 \perp v = v = v \perp 1$
- invertibility of $\perp$ : $\forall v \in V \Rightarrow v > 0 \Rightarrow \frac{1}{v} \in V \Rightarrow$

$\Rightarrow \forall v$ has an inverse

$\Rightarrow (V, \perp)$ abelian group

We will now prove the axioms :

$\underline{\forall \alpha, \beta \in \mathbb{R}, \forall v \in V : \quad (\alpha + \beta) T v \stackrel{?}{=} (\alpha T v) \perp (\beta T v)}$

$$(\alpha + \beta) T v = v^{\alpha + \beta}$$

$$(\alpha T v) \perp (\beta T v) = v^{\alpha} \perp v^{\beta} = v^{\alpha} \cdot v^{\beta} = v^{\alpha + \beta}$$

$\underline{\forall \alpha \in \mathbb{R}, \forall v_1, v_2 \in V : \quad \alpha T (v_1 \perp v_2) = (\alpha T v_1) \perp (\alpha T v_2)}$

$$\left.\begin{aligned} LHS &= \alpha T (v_1 v_2) = (v_1 v_2)^{\alpha} \\ RHS &= v_1^{\alpha} \perp v_2^{\alpha} = v_1^{\alpha} \cdot v_2^{\alpha} = (v_1 v_2)^{\alpha} \end{aligned}\right\} \Rightarrow \checkmark$$

$\underline{\forall \alpha, \beta \in \mathbb{R}, \forall v \in V : \quad (\alpha \beta) T v \stackrel{?}{=} \alpha T (\beta T v)}$

$LHS = v^{\alpha \beta} \qquad RHS = \alpha T (v^{\beta}) = v^{\beta \alpha} = v^{\alpha \beta} \Rightarrow \checkmark$

$$\underline{\forall v \in V : \quad 1 \top v \overset{?}{=} v}$$

$$1 \top v = v^1 = v$$

$\Rightarrow$ $V$ is an $\mathbb{R}$-vector space

**5.** Let $K$ be a field and let $V = K \times K$. Decide whether $V$ is a $K$-vector space with respect to the following addition and scalar multiplication:

(*i*) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + 2y_2)$ and $k \cdot (x_1, y_1) = (kx_1, ky_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

(*ii*) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k \cdot (x_1, y_1) = (kx_1, y_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

$\underline{\text{Sol.}}$ : (i) If we assume that $V$ is a vector space

$$(1+1) \cdot (1,1) = (2,2)$$

$2 := 1+1$ , $3 := 1+1+1$

$$(1+1) \cdot (1,1) = 1 \cdot (1,1) + 1 \cdot (1,1) = (1+1, 1+2 \cdot 1) = (2,3)$$

$\Rightarrow$ $2 = 3$ $\Rightarrow$ $1 = 0$ $\Rightarrow K$ is not a field $\Rightarrow$ contradiction $\Rightarrow$

$\Rightarrow$ $V$ is not a vector space.

Another approach: $\forall \alpha, \beta \in K$, $\forall v \in V$ : , $v = (v_1, v_2)$

$$(\alpha + \beta) \cdot v = \alpha v + \beta v$$

$$\Rightarrow ((\alpha+\beta) v_1, (\alpha+\beta) v_2) = (\alpha v_1, \alpha v_2) + (\beta v_1, \beta v_2)$$

$$\Rightarrow \beta v_2 = 2\beta v_2 \quad \Rightarrow \quad \beta v_2 = 0$$

$$\forall \beta, v_2 \in K$$

(ii) We assume that $V$ is a vector space.

$$\forall \alpha, \beta \in K, \forall v = (x,y) \in V \Rightarrow (\alpha+\beta)\cdot(x,y) = \alpha(x,y) + \beta\cdot(x,y)$$

$$(\alpha+\beta)\cdot(x,y) = ((\alpha+\beta)x, y)$$

$$\alpha(x,y) + \beta(x,y) = (\alpha x, y) + (\beta x, y) = ((\alpha+\beta)x, 2y)$$

$$\Rightarrow 2y = y, \ \forall y \in K \Rightarrow y = 0, \ \forall y \in K \quad \text{contradiction}$$

<u>Def</u>: $V$ $K$-vector space, $S \subseteq V$

$S \leq_K V$ ($K$-subspace) $\Leftrightarrow$

(i) $S \neq \emptyset$

(ii) $(S,+) \leq (V,+)$ (subgroup): $\forall x, y \in S$: $x - y \in S$

(iii) $S$ is compatible with scalar multiplication: $\forall x \in S, \forall k \in K$: $kx \in S$

$\Big\}$ $\forall \alpha, \beta \in K, \forall x, y \in S$: $\alpha x + \beta y \in S$

**7.** Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;
(iii) $C = \{(x,y,z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;
(iv) $D = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(v) $E = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;
(vi) $F = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}$?

<u>Sol</u>: (i) $A \neq \emptyset$, because $(0,0,0) \in A$

Let $v_1 = (\overset{0}{\overset{\shortparallel}{x_1}}, y_1, z_1)$, $v_2 = (\overset{0}{\overset{\shortparallel}{x_2}}, y_2, z_2)$

$$v_1 - v_2 = (\overbrace{x_1 - x_2}^{=0}, y_1 - y_2, z_1 - z_2) \in A$$

Let $\alpha \in \mathbb{R}$ : $\alpha v_1 = \alpha \cdot (x_1, y_1, z_1) = (\underbrace{\alpha x_1}_{=0}, \alpha y_1, \alpha z_1) \in \mathbb{R}^3 \Rightarrow$

$$\Rightarrow \alpha v_1 \in A \Rightarrow A \leq_{\mathbb{R}} \mathbb{R}^3$$

(ii) $\underset{\substack{\Uparrow \\ B}}{(0,1,1)} + \underset{\substack{\Uparrow \\ B}}{(1,1,0)} = (1,2,1) \notin B \Rightarrow B \not\leq \mathbb{R}^3$

(iii) $\underbrace{\frac{1}{2}}_{\in \mathbb{R}} \cdot \underset{\substack{\Uparrow \\ C}}{(1,2,3)} = (\frac{1}{2}, 1, \frac{3}{2}) \notin C \Rightarrow C \not\leq \mathbb{R}^3$

(iv) $D \neq \emptyset$, because $(0,0,0) \in D$

Let $v = (v_1, v_2, v_3)$ , $u = (u_1, u_2, u_3) \in D$

$$\Rightarrow v_1 + v_2 + v_3 = u_1 + u_2 + u_3 = 0$$

$$v - u = (v_1 - u_1, v_2 - u_2, v_3 - u_3)$$

$$(v_1 - u_1) + (v_2 - u_2) + (v_3 - u_3) = (v_1 + v_2 + v_3) - (u_1 + u_2 + u_3) = 0$$

$$\Rightarrow v - u \in D$$

Let $v = (v_1, v_2, v_3)$ , $a \in \mathbb{R}$

$$av = (av_1, av_2, av_3)$$

$$av_1 + av_2 + av_3 = a(v_1 + v_2 + v_3) = 0$$

$$\Rightarrow av \in D \Rightarrow D \leq_{\mathbb{R}} \mathbb{R}^3$$

26.10.2021

# Seminar W5 – 911

**Prop:** $V$ $K$-vector space, $X \subseteq V$

The subspace generated by $X$ is:

$$\langle X \rangle = \bigcap_{\substack{S \leq_K V \\ S \supseteq X}} S = \left\{ \sum_{i=1}^{n} k_i \cdot v_i \;\middle|\; n \in \mathbb{N}, k_i \in K, v_i \in X \right\} =$$

= "the set of all finite linear combinations of elements from $X$"

**2.** Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
(iv) $D = \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x+2y=0 \\ y+t=0 \end{cases} \right\}$
(v) $E = \{ (x,y,z) \in \mathbb{R}^3 \mid y+2z = 0 \}$
Write $A, B, C$ as generated subspaces with a minimal number of generators.

**Sol:** (i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\} = \{(0,y,z) \mid y,z \in \mathbb{R}\} =$

$= \{(0,y,0) + (0,0,z) \mid y,z \in \mathbb{R}\} = \{ y \cdot (0,1,0) + z \cdot (0,0,1) \mid y,z \in \mathbb{R}\}$

$= \langle \{(0,1,0), (0,0,1)\} \rangle = \langle (0,1,0), (0,0,1) \rangle$

(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x = -y-z\} = \{(-y-z, y, z) \mid y,z \in \mathbb{R}\} =$

$= \{(-y,y,0) + (-z,0,z) \mid y,z \in \mathbb{R}\} = \{ y \cdot (-1,1,0) + z \cdot (-1,0,1) \mid y,z \in \mathbb{R}\}$

$= \langle (-1,1,0), (-1,0,1) \rangle$

It is the minimal number of generators, since $(-1,0,1) \notin \langle (-1,1,0) \rangle$

(iii) $C = \{(x,y,z) \mid x=y=z\} = \{(x,x,x) \mid x \in \mathbb{R}\} =$

$= \{x \cdot (1,1,1) \mid x \in \mathbb{R}\} = \langle (1,1,1) \rangle$

(iv) $D = \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x+2y=0 \\ y+t=0 \end{cases} \right\} =$

$= \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x=-2y \\ t=-y \end{cases} \right\} =$

$= \{(-2y, y, z, -y) \mid y,z \in \mathbb{R}\} =$

$= \{(-2y, y, 0, -y) + (0,0,z,0) \mid y,z \in \mathbb{R}\} =$

$= \{y(-2,1,0,-1) + z \cdot (0,0,1,0) \mid y,z \in \mathbb{R}\} =$

$= \langle (-2,1,0,-1), (0,0,1,0) \rangle$

This the minimal number of generators, because $(0,0,1,0) \notin \langle (-2,1,0,-1) \rangle$

Proof: if $(0,0,1,0) \in \langle (-2,1,0,-1) \rangle$, then $\exists \alpha \in \mathbb{R}$:

$$(0,0,1,0) = \alpha \cdot (-2,1,0,-1)$$

$\Rightarrow \begin{cases} 0=0 \\ \alpha=0 \\ 0=1 \\ -\alpha=0 \end{cases}$, which has no solutions

$\Rightarrow (0,0,1,0) \notin \langle (-2,1,0,-1) \rangle$

(v) $E = \{(x,y,z) \in \mathbb{R}^3 \mid y+2z=0\}$

$= \{(x,y,z) \in \mathbb{R}^3 \mid y=-2z\} = \{(x,-2z,z) \mid x,z \in \mathbb{R}\}$

$= \{ (x,0,0) + (0,-2z,z) \mid x,z \in \mathbb{R} \} = \{ x\cdot(1,0,0) + z\cdot(0,-2,1) \mid x,z \in \mathbb{R} \}$

$= \langle (1,0,0), (0,-2,1) \rangle$

This is the minimal number of generators, because none of them can be obtained as a linear combination of the others

<u>Def</u>: $V, W$ $K$-vector spaces, $f: V \to W$ is a

<u>(homo)morphism of vector spaces</u> (<u>linear map</u>) if:

$\cdot \ \forall v_1, v_2 : \ f(v_1+v_2) = f(v_1) + f(v_2)$

$\cdot \ \forall k \in K, \forall v \in V: \ f(kv) = k f(v)$

$\forall k_1, k_2 \in K, \forall v_1, v_2 \in V: \ f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2)$

**6.** Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x,y) = (x+y, x-y),$$

$$g(x,y) = (2x-y, 4x-2y),$$

$$h(x,y,z) = (x-y, y-z, z-x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

<u>Sol</u>: Let $v_1, v_2 \in \mathbb{R}^2$, $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$

Let $k_1, k_2 \in \mathbb{R}$:

$f(k_1 v_1 + k_2 v_2) = f(k_1 (x_1, y_1) + k_2 (x_2, y_2)) =$

$$= f\left( (k_1x_1, k_1y_1) + (k_2x_2, k_2y_2) \right) = f(k_1x_1 + k_2x_2, k_1y_1 + k_2y_2) =$$

$$= (k_1x_1 + k_2x_2 + k_1y_1 + k_2y_2, k_1x_1 + k_2x_2 - k_1y_1 - k_2y_2) =$$

$$= (k_1x_1 + k_1y_1, k_1x_1 - k_1y_1) + (k_2x_2 + k_2y_2, k_2x_2 - k_2y_2) =$$

$$= k_1(x_1 + y_1, x_1 - y_1) + k_2(x_2 + y_2, x_2 - y_2) =$$

$$= k_1 \cdot f(x_1, y_1) + k_2 \cdot f(x_2, y_2) = k_1 f(u_1) + k_2 f(u_2)$$

$$g(x, y) = (2x - y, 4x - 2y)$$

$$\forall k_1, k_2 \in \mathbb{R}, \quad \forall u_1 = (x_1, y_1), \quad u_2 = (x_2, y_2) \in \mathbb{R}^2$$

$$k_1 g(u_1) + k_2 g(u_2) = k_1 \cdot (2x_1 - y_1, 4x_1 - 2y_1) +$$

$$+ k_2 \cdot (2x_2 - y_2, 4x_2 - 2y_2) = (2k_1x_1 - k_1y_1, 4k_1x_1 - 2k_1y_1) +$$

$$+ (2k_2x_2 - k_2y_2, 4k_2x_2 - 2k_2y_2) = (2k_1x_1 - k_1y_1 + 2k_2x_2 - k_2y_2,$$

$$4k_1x_1 - 2k_1y_1 + 4k_2x_2 - 2k_2y_2) = (\underbrace{2k_1x_1 + 2k_2x_2}_{2(k_1x_1 + k_2x_2)} - (k_1y_1 + k_2y_2),$$

$$4(k_1x_1 + k_2x_2) - 2(k_1y_1 + k_2y_2)) =$$

$$= g(k_1x_1 + k_2x_2, k_1y_1 + k_2y_2) = g(k_1u_1 + k_2u_2)$$

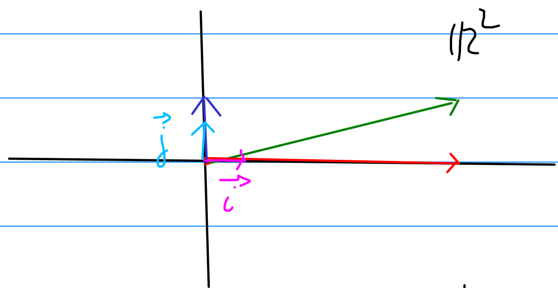
$$\Rightarrow g \in \operatorname{End}_{\mathbb{R}}(\mathbb{R}^2)$$

Def: ${}_kV$, $S,T \leq_k V$

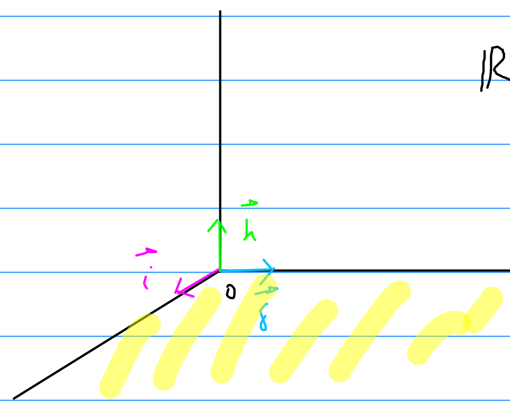
$$V = S+T \iff \forall v \in V\ \exists s \in S,\ \exists t \in T: \quad v = s+t$$

$$V = S \oplus T \iff \begin{matrix} V = S+T \\ S \cap T = \{0\} \end{matrix} \iff \forall v \in V\ \exists!\, s \in S, t \in T: \quad v = s+t$$

("direct sum")




$$\mathbb{R}^2 = \underbrace{\mathbb{R}\vec{i}}_{=\langle \vec{i} \rangle} \oplus \underbrace{\mathbb{R}\vec{j}}_{=\langle \vec{j} \rangle}$$




$$\mathbb{R}^3 = \mathbb{R}\vec{i} \oplus \mathbb{R}\vec{j} \oplus \mathbb{R}\vec{k} = \langle \vec{k} \rangle \oplus \underbrace{\langle \vec{i}, \vec{j} \rangle}_{\simeq \mathbb{R}^2}$$

**4.** Let

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0\},$$
$$T = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that $S$ and $T$ are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

Sol.: $S = \{(-y-z, y, z) \mid y,z \in \mathbb{R}\} = \{(-y,y,0) + (-z,0,z) \mid y,z \in \mathbb{R}\}$

$= \{y \cdot (-1,1,0) + z \cdot (-1,0,1) \mid y,z \in \mathbb{R}\}$

Let $(x,y,z) \in S \cap T \Rightarrow \begin{cases} x+y+z = 0 \\ x = y = z \end{cases} \Rightarrow x = y = z = 0$

$\Rightarrow S \cap T \subseteq 0 \Rightarrow S \cap T = 0 = \{0\}$

$\mathbb{R}^3 = S + T$

We have to show that $\forall v = (x,y,z) \in \mathbb{R}^3 \ \exists s \in S, t \in T$:

$$v = s + t$$

Assume we have found such a decomposition.

Say $t = (a,a,a) \Rightarrow s = (x-a, y-a, z-a) \in S$

$$\Rightarrow x - a + y - a + z - a = 0 \Rightarrow a = \frac{x+y+z}{3}$$

Now, for any $v = (x,y,z)$, we can decompose:

$$(x,y,z) = \underbrace{\left(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}\right)}_{=:t} +$$

$$+ \underbrace{\left(x - \frac{x+y+z}{3}, y - \frac{x+y+z}{3}, z - \frac{x+y+z}{3}\right)}_{=:s}$$

We can now clearly see that $s \in S$ and $t \in T$

$\Rightarrow$ we have, thus, shown, that $\forall v \in \mathbb{R}^3$: $\exists s \in S, \exists t \in T$:

$$v = s + t$$

Hence we have shown that $\mathbb{R}^3 = S + T$

Because $S \cap T = 0 \Rightarrow \mathbb{R}^3 = S \oplus T$

02.11.2021

# Seminar W6 – 911

Def: $V$ $K$ – vector space.

We say that $v_1, v_2, \dots, v_n \in V$ are linearly independent if:

$\forall \alpha_1, \alpha_2, \dots, \alpha_n \in K$, if $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$, then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

Conversely, $v_1, \dots, v_n \in V$ are linearly dependent if:

$\exists \alpha_1, \dots, \alpha_n \in K$, not all zero, so that $\underbrace{\alpha_1 v_1 + \dots + \alpha_n v_n = 0}_{\text{"dependence relationship"}}$

**1.** Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:
(*i*) $v_1$, $v_2$, $v_3$ are linearly dependent and determine a dependence relationship.
(*ii*) $v_1$, $v_2$ are linearly independent.

Sol: We want to find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ so that:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\alpha_1 \cdot (1, -1, 0) + \alpha_2 \cdot (2, 1, 1) + \alpha_3 \cdot (1, 5, 2) = (0, 0, 0)$$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_2 + 5\alpha_3 = 0 \\ \alpha_2 + 2\alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = -2\alpha_3 \\ \alpha_1 - 4\alpha_3 + \alpha_3 = 0 \\ -\alpha_1 - 2\alpha_3 + 5\alpha_3 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_2 = -2\alpha_3 \\ \alpha_1 = 3\alpha_3 \\ \alpha_1 = 3\alpha_3 \end{cases}$$

$\Rightarrow \forall \alpha_1, \alpha_2, \alpha_3$ that satisfy these conditions, we can get a dependence relationship:

$\Rightarrow 3v_1 - 2v_2 + v_3 = 0$

(ii) Let $\alpha_1, \alpha_2 \in \mathbb{R}$ so that

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

$$\alpha_1 \cdot (1,-1,0) + \alpha_2 \cdot (2,1,1) = 0$$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ -\alpha_1 + \alpha_2 = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0$$

$\Rightarrow v_1, v_2$ are linearly independent

**3.** Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$. Determine $a \in \mathbb{R}$ such that the vectors $v_1$, $v_2$, $v_3$ are linearly independent.

Sol: Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ so that:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow \begin{cases} \alpha_1 + a\alpha_2 + \alpha_3 = 0 \\ a\alpha_1 + \alpha_2 = 0 \\ \alpha_2 + a\alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} a(\alpha_1 - \alpha_3) = 0 \\ \alpha_1 + a\alpha_2 + \alpha_3 = 0 \\ a\alpha_1 + \alpha_2 = 0 \end{cases}$$

If $a \neq 0 \Rightarrow \alpha_1 = \alpha_3$: $\begin{cases} 2\alpha_1 + a\alpha_2 = 0 \\ a\alpha_1 + \alpha_2 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_2 = -a\alpha_1 \\ 2\alpha_1 - a^2\alpha_1 = 0 \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} \alpha_2 = -a\alpha_1 \\ \alpha_1(2-a^2) = 0 \\ \alpha_3 = \alpha_1 \end{cases}$$

If $a \neq \pm\sqrt{2} \Rightarrow \alpha_1 = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$ so $v_1, v_2, v_3$ lin. indep.

If $a = \pm\sqrt{2} \Rightarrow \begin{cases} \alpha_1 = \alpha_1 \\ \alpha_2 = -a\alpha_1 \\ \alpha_3 = \alpha_1 \end{cases}$ in this case $v_1, v_2, v_3$ lin. dependent

If $a=0$: $\begin{cases} \alpha_1+\alpha_3=0 \\ \alpha_2=0 \end{cases}$ This system is compatible indetermined, therefore $v_1, v_2, v_3$ are linearly dependent.

<u>Def</u>: $V$ $K$-vector space. $X \subseteq V$ <u>basis</u> for $V$ if:

- $X$ linearly independent
- $X$ system of generators for $V$ $\left( V = \langle X \rangle \right)$

$\left( \dim V = \# \text{ of elements in every basis} \right)$

**7.** Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the lists $(E_1, E_2, E_3, E_4)$ and $(A_1, A_2, A_3, A_4)$ are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in each of the two bases.

<u>Sol</u>: We prove first that $(E_1, E_2, E_3, E_4)$ is lin. indep.

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ so that $\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 = 0$

$$\Rightarrow \alpha_1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha_4 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

So $E_1, E_2, E_3, E_4$ lin. indep.

We prove that $M_2(\mathbb{R}) = \langle E_1, E_2, E_3, E_4 \rangle$

Let $M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. We have:

$$M = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4$$

$$\Rightarrow \mathcal{M}_2(\mathbb{R}) = \langle E_1, E_2, E_3, E_4 \rangle$$

$$\Rightarrow (E_1, E_2, E_3, E_4) \text{ basis for } {}_{\mathbb{R}}\mathcal{M}_2(\mathbb{R})$$

$$\dim \mathcal{M}_2(\mathbb{R}) = 4$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ so that

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \alpha_1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \alpha_4 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \\ \alpha_2 + \alpha_3 + \alpha_4 = 0 \\ \alpha_3 + \alpha_4 = 0 \\ \alpha_1 + \alpha_4 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -\alpha_4 \\ \alpha_3 = -\alpha_4 \\ \alpha_2 - \alpha_4 + \alpha_4 = 0 \\ -\alpha_4 + \alpha_2 - \alpha_4 + \alpha_4 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -\alpha_4 \\ \alpha_3 = -\alpha_4 \\ \alpha_2 = 0 \\ \alpha_4 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

$$\Rightarrow A_1, A_2, A_3, A_4 \text{ are linearly independent}$$

We will show now that $\mathcal{M}_2(\mathbb{R}) = \langle A_1, A_2, A_3, A_4 \rangle$

Let $M = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$.

$$M = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4$$

$$\Rightarrow \begin{cases} x = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ y = \alpha_2 + \alpha_3 + \alpha_4 \\ z = \alpha_3 + \alpha_4 \\ t = \alpha_1 + \alpha_4 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = t - \alpha_4 \\ \alpha_3 = z - \alpha_4 \\ x = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ y = \alpha_2 + \alpha_3 + \alpha_4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_1 = t - \alpha_4 \\ \alpha_3 = z - \alpha_4 \\ x = t - \alpha_4 + \alpha_2 + z - \alpha_4 + \alpha_4 \\ y = \alpha_2 + z - \alpha_4 + \alpha_4 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = t - \alpha_4 \\ \alpha_3 = z - \alpha_4 \\ x = z + t + \alpha_2 - \alpha_4 \\ y = \alpha_2 + z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_1 = t - \alpha_4 \\ \alpha_3 = z - \alpha_4 \\ \alpha_2 = y - z \\ x = z + t + y - z - \alpha_4 \end{cases} \Leftrightarrow \begin{cases} \alpha_4 = y + t - x \\ \alpha_2 = y - z \\ \alpha_3 = z + x - y - t \\ \alpha_1 = x - y \end{cases}$$

$$\Rightarrow \quad M_2(\mathbb{R}) = \langle A_1, A_2, A_3, A_4 \rangle$$

Not.: If $V$ $K$-v.s., $B = (v_1, v_2, \dots, v_n)$ basis for $V$

$\forall v \in V$: the coordinates of $v$ in the basis $B$ are:

$\alpha_1, \alpha_2, \dots, \alpha_n$ (denoted $[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$)

if $\boxed{v = \alpha_1 v_1 + \dots + \alpha_n v_n}$

We have to determine $\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{(E_1, E_2, E_3, E_4)}$, $\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{(A_1, A_2, A_3, A_4)}$

$$\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{(E_1, E_2, E_3 E_4)} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \text{ because}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = 2 \cdot E_1 + 1 \cdot E_2 + 1 \cdot E_3 + 0 \cdot E_4$$

$$\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{(A_1, A_2, A_3, A_4)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

so that $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4$

$$\begin{cases} \alpha_4 = y + t - x \\ \alpha_2 = y - z \\ \alpha_3 = z + x - y - t \\ \alpha_1 = x - y \end{cases} \Rightarrow \begin{cases} \alpha_4 = 1 + 0 - 2 = -1 \\ \alpha_2 = 1 - 1 = 0 \\ \alpha_3 = 1 + 2 - 1 - 0 = 2 \\ \alpha_1 = 2 - 1 = 1 \end{cases}$$

$$\Rightarrow \left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{(A_1, A_2, A_3, A_4)} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$

09. 11. 2021

# Seminar W7 – 911

**Th.:** $V$ $K$-vector space, $\dim_K V = n$. $B = (e_1, \ldots, e_n)$ a list of vectors

Then:

$B$ basis $\iff$ $B$ linearly independent $\iff$ $B$ system of generators for $V$

**1*.** Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + z = 0\}$$

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid \cancel{x = y = z}\}.$$

$$\begin{cases} x + 2y = 0 \\ z + y = 0 \end{cases}$$

**Sol:** $B = \{(x,y,z) \in \mathbb{R}^3 \mid z = -x - 2y\} = \{(x, y, -x-2y) \mid x, y \in \mathbb{R}\} =$

$= \{x \cdot (1, 0, -1) + y \cdot (0, 1, -2) \mid x, y \in \mathbb{R}\} =$

$= \langle (1,0,-1), (0,1,-2) \rangle$

$b = ((1,0,-1), (0,1,-2))$ basis, because $b$ linearly independent

$\dim B = 2$

$$\overbrace{f \in \mathrm{Hom}_k(V, W)}^{\text{"} f \text{ is a linear map between } V \text{ and } W \text{"}}$$

<u>Def</u>: $V, W$ $k$-v.s., $f \in \mathrm{Hom}_k(V, W)$, Then:

$\mathrm{Ker}\, f$ ("kernel") $= \{ u \in V \mid f(u) = 0_W \}$

$\mathrm{Im}\, f = \{ f(u) \mid u \in V \} = \{ w \in W \mid \exists u \in V : f(u) = w \}$

<u>Example</u>: $f : \mathbb{R}^3 \to \mathbb{R}^3$, $(x,y,z) \mapsto (x+2y, y+x, z)$. Find a basis for $\mathrm{Ker}\, f$ and $\mathrm{Im}\, f$.

$$\mathrm{Ker}\, f = \{ (x,y,z) \in \mathbb{R}^3 \mid (x+2y, y+x, z) = (0,0,0) \} =$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} x+2y = 0 \\ y+x = 0 \\ z = 0 \end{cases} \right\} =$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} z = 0 \\ y = -x \\ y = -\frac{x}{2} \end{cases} \right\} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} z = 0 \\ y = 0 \\ x = 0 \end{cases} \right\} =$$

$$= \{ (0,0,0) \} \Rightarrow \mathrm{Ker}\, f = 0$$

$$\mathrm{Im}\, f = \{ (x+2y, y+x, z) \mid x,y,z \in \mathbb{R} \} =$$

$$= \{ x \cdot (1,1,0) + y \cdot (2,1,0) + z \cdot (0,0,1) \mid x,y,z \in \mathbb{R} \} =$$

$$= \langle (1,1,0), (2,1,0), (0,0,1) \rangle$$

Sneak peek: $u_1, u_2, \ldots, u_n \in_k V$, $\mathrm{rank}(u_1, \ldots, u_n) :=$ max # of lin. indep. vectors among $u_1, u_2, \ldots, u_n$

If $V = k^n$, then $\mathrm{rank}(u_1, \ldots, u_n) = \mathrm{rank}\begin{pmatrix} u_1 \\ \hline u_2 \\ \hline \vdots \\ \hline u_n \end{pmatrix} = \mathrm{rank}(u_1 | u_2 | \cdots | u_n)$

To decide if $(1,1,0)$, $(2,1,0)$, $(0,0,1)$ are lin. indep. we can either use the definition of linear independence, or just calculate the rank of the matrix formed by the vectors.

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|M| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \neq 0 \Rightarrow \text{rank } M = 3 \Rightarrow$$

$\Rightarrow (1,1,0), (2,1,0), (0,0,1)$ are linearly independent $\Rightarrow$ they form a basis of $Im f$

**4.** Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (y, -x)$. Prove that $f$ is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

<u>Sol.</u>: $v_1, v_2 \in \mathbb{R}^3$, $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2)$

$$\underline{f(v_1 + v_2) \overset{?}{=} f(v_1) + f(v_2)}$$

$$f(v_1 + v_2) = f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (y_1 + y_2, -(x_1 + x_2))$$

$$f(v_1) + f(v_2) = f(x_1, y_1, z_1) + f(x_2, y_2, z_2) = (y_1, -x_1) + (y_2, -x_2)$$

$$\Rightarrow f(v_1 + v_2) = f(v_1) + f(v_2)$$

$$\underline{f(k v_1) \overset{?}{=} k f(v_1)}$$

$$f(ku_1) = f(k(x_1, y_1, z_1)) = f(kx_1, ky_1, kz_1) = (ky_1, -kx_1) =$$

$$= k \cdot (y_1, -x_1) = k f(v_1)$$

$$\Rightarrow f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$$

$$\operatorname{Ker} f = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = (0, 0)\} =$$

$$= \{(x, y, z) \in \mathbb{R}^3 \mid (y, -x) = (0, 0)\} =$$

$$= \{(x, y, z) \in \mathbb{R}^3 \mid y = x = 0\} =$$

$$= \{(0, 0, z) \mid z \in \mathbb{R}\} = \{z \cdot (0, 0, 1) \mid z \in \mathbb{R}\}$$

$$= \langle (0, 0, 1) \rangle, \quad B = ((0, 0, 1)) \Rightarrow \dim \operatorname{Ker} f = 1$$

$$\operatorname{Im} f = \{(y, -x) \mid x, y \in \mathbb{R}\} = \{y \cdot (1, 0) + x \cdot (0, -1) \mid x, y \in \mathbb{R}\} =$$

$$= \langle (1, 0), (0, -1) \rangle$$

$$\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 \neq 0 \Rightarrow (1, 0), (0, -1) \text{ are linearly independent}$$

$$\Rightarrow B = ((1, 0), (0, 1)) \text{ basis for } \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 2$$

**Th (Steinitz)**: $V$ $K$-vector space, $\dim V = n$
(rephrased)

If $u_1, u_2, \dots, u_m \in V$, $m \leq n$ are linearly independent,

then $\exists\, w_{m+1}, w_{m+2}, \dots, w_n \in V$ so that

$B = (u_1, u_2, \dots, u_m, w_{m+1}, \dots, w_n)$ basis for $V$

In order to complete a linearly independent family $u_1, \dots, u_m$ to a basis:

- We choose $u_{m+1} \in V \setminus \langle u_1, \dots, u_m \rangle$
- Do we have enough vectors ($n$)?

Yes | No

Yes, basis! | We choose $u_{m+2} \in V \setminus \langle u_1, \dots, u_m, u_{m+1} \rangle$

Yes | No

**6.** Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Sol: $A = \langle (1,0,0), (0,1,0) \rangle$

$(0,0,1) \notin A \Rightarrow (1,0,0), (0,1,0), (0,0,1)$ lin. indep $\Rightarrow$

$\Rightarrow \big((1,0,0),\ (0,1,0),\ (0,0,1)\big)$ basis for $\mathbb{R}^3$

$B = \{ (x,y,z) \in \mathbb{R}^3 \mid x+y+z=0 \}$

$= \langle (1,0,-1),\ (0,1,-1) \rangle$

We choose to add $(0,0,1)$, because $\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$, hence

$(1,0,-1), (0,1,-1), (0,0,1)$ lin. indep $\overset{\dim \mathbb{R}^3 = 3}{\Longrightarrow}$ this is a basis for $\mathbb{R}^3$

$C = \{ (x,y,z) \in \mathbb{R}^3 \mid x=y=z \} =$

$= \langle (1,1,1) \rangle$

We first add $(0,1,0) \in \mathbb{R}^3 \setminus C$. So $(1,1,1)$ and $(0,1,0)$

are linearly independent. To complete this to a basis of $\mathbb{R}^3$,

we need another vector, which must not belong to $\langle (1,1,1), (0,1,0) \rangle$

$\langle (1,1,1), (0,1,0) \rangle = \{ a(1,1,1) + b\cdot(0,1,0) \mid a,b \in \mathbb{R} \} =$

$= \{ (a, a+b, a) \mid a,b \in \mathbb{R} \} =$ we could just stop here and choose a vector not of the form $(a, a+b, a)$

$= \left\{ (x,y,z) \;\middle|\; \begin{cases} x = a \\ y = a+b \\ z = a \end{cases},\ a,b \in \mathbb{R} \right\} =$

$= \left\{ (x,y,z) \;\middle|\; \begin{matrix} a = x \\ b = y-a \\ z = x \end{matrix} \right\} =$

$= \{ (x,y,z) \mid z = x \}$

We can just choose, for instance

$(1,0,2) \notin \mathbb{R}^3 \setminus \langle (1,1,1), (0,1,0) \rangle$

$\Rightarrow (1,0,2), (1,1,1), (0,1,0)$ lin. indep $\Rightarrow$ they form a basis of $\mathbb{R}^3$

<u>Th. ($1^{st}$ dim. theorem)</u>: $f: V \to W$ $K$-linear map, then:

$$\dim V = \dim(\ker f) + \dim(\operatorname{Im} f)$$

<u>Th. ($2^{nd}$ dim theorem)</u> $V$ $K$-v.s., $S, T \leq_K V$:

$$\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$$

**9.** Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\},$$

$$T = < (0,1,1), (1,1,0) >$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

<u>Sol.</u>: $T = \{ a \cdot (0,1,1) + b \cdot (1,1,0) \mid a, b \in \mathbb{R} \} =$

$= \{ (b, a+b, a) \mid a, b \in \mathbb{R} \} =$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} x = b \\ y = a+b \\ z = a \end{cases} \right\} =$

$= \{ (x,y,z) \in \mathbb{R}^3 \mid y = x + z \}$

$S \cap T = \{ (x, y, z) \in \mathbb{R}^3 \mid y = x+z \text{ and } x = 0 \} =$

$= \{ (0, y, y) \in \mathbb{R}^3 \mid y \in \mathbb{R} \} =$

$= \langle (0,1,1) \rangle \Rightarrow \dim(S \cap T) = 1$

In order to show that $S+T = \mathbb{R}^3$, since $S+T \leq \mathbb{R}^3$,

it suffices to show that $\dim(S+T) = \dim(\mathbb{R}^3) = 3$

$S = \{ (x, y, z) \in \mathbb{R}^3 \mid x = 0 \}$

because $(0,1,1)$ and $(1,1,0)$ are lin. indep.

$T = \langle (0,1,1), (1,1,0) \rangle \Rightarrow \dim T = 2$

because $(0,1,0)$ and $(0,0,1)$ are lin. indep.

$S = \langle (0,1,0), (0,0,1) \rangle \Rightarrow \dim S = 2$

$\dim(S+T) = \dim S + \dim T - \dim(S \cap T) = 2+2-1 = 3$

$\Rightarrow S+T = \mathbb{R}^3$

16.11.2021

# Seminar W8-911

**Prop:** $S$ linear system. $M$ its matrix, $\overline{M}$ its extended matrix

$$S \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \overline{M} = \left( M \,\middle|\, \begin{matrix} b_1 \\ \vdots \\ b_m \end{matrix} \right)$$

Kronecker-Capelli thm: $(S)$ is compatible $\Leftrightarrow$ $\operatorname{rank}(M) = \operatorname{rank} \overline{M}$

Rouché's theorem: We find a principal minor in $M$ $\Delta$

[minor = determinant formed by selecting $k$ rows and $k$ columns from $M$; principal minor = minor of maximal size (minor of size $\operatorname{rank} M$)]

We form all the characteristic minors

$$\begin{vmatrix} \text{principal minor} & \text{elements from column of free terms} \\ \text{row from } M & \end{vmatrix}$$

$(S)$ is compatible $\Leftrightarrow$ all the char. minors are zero

**Cramer's rule:** If we have a square system $S$ ($n$ unknowns, $n$ equations)

$\to$ solution is unique

$S$ compatible determined $\Leftrightarrow \Delta = \det(M) \neq 0$

If $S$ compatible $\Rightarrow$ $x_1 = \frac{\Delta_{x_1}}{\Delta}, \dots, x_n = \frac{\Delta_{x_n}}{\Delta}$

**2.** Using the Kronecker-Capelli theorem, decide if the following linear systems are compatible and then solve the compatible ones:

$$(i)\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases} \qquad (ii)\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

$$(iii)\begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$$

**3.** Using the Rouché theorem, decide if the systems from **2.** are compatible and then solve the compatible ones.

Sol.: 3(ii)

$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

$$M = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & -1 \\ 1 & -2 & 1 & 5 \end{pmatrix} \Rightarrow \operatorname{rank} M \leq 3$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \neq 0 \Rightarrow \operatorname{rank} M \geq 2$$

Any 3 columns we choose, two of them will be proportional, so all minors of order 3 are 0.

$\Rightarrow \operatorname{rank} M = 2$

Principal unknowns: $x_1, x_4$

Principal equations: first, third

Now we apply Rouché's theorem to see if the system is compatible

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \qquad M = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & -1 \\ 1 & -2 & 1 & 5 \end{pmatrix}$$

$$\overline{M} = \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 & 5 \end{array}\right)$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 5 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 4 & 4 \end{vmatrix} = -8 - (-8) = 0 \overset{\text{Rouché}}{\Longrightarrow} \text{the system is compatible}$$

We take the secondary unknowns as parameters:

$$x_2 = \alpha \qquad x_3 = \beta.$$

So our system is:

$$\begin{cases} x_1 + x_4 = 1 + 2\alpha - \beta \\ x_1 + 5x_4 = 5 + 2\alpha - \beta \end{cases}$$

$$\Rightarrow \begin{cases} x_1 + x_4 = 1 + 2\alpha - \beta \\ 4x_4 = 4 \end{cases} \Rightarrow \begin{cases} x_4 = 1 \\ x_1 = 2\alpha - \beta \\ x_2 = \alpha \\ x_3 = \beta \end{cases}$$

Solve the following linear systems by the Gauss and Gauss-Jordan methods:

**5.** $(i)\begin{cases}2x+2y+3z=3\\ x-y\quad=1\\ -x+2y+z=2\end{cases}$ $(ii)\begin{cases}2x+5y+z=7\\ x+2y-z=3\\ x+y-4z=2\end{cases}$ $(iii)\begin{cases}x+y+z=3\\ x-y+z=1\\ 2x-y+2z=3\\ x+z=4\end{cases}$

**6.** $\begin{cases}2x_1+x_2+x_3+x_4=1\\ x_1+2x_2-x_3+4x_4=2\\ x_1+5x_2-4x_3+11x_4=\lambda\end{cases}$ $(\lambda\in\mathbb{R})$

**7.** $\begin{cases}ax+y+z=1\\ x+ay+z=a\\ x+y+az=a^2\end{cases}$ $(a\in\mathbb{R})$

Sol. : (i) $\begin{cases}2x+2y+3z=3\\ x-y=1\\ -x+2y+z=2\end{cases}$

row echelon form

$$\left(\begin{array}{ccc|c}2&2&3&3\\1&-1&0&1\\-1&2&1&2\end{array}\right)\underset{L_1\leftrightarrow L_2}{\sim}\left(\begin{array}{ccc|c}1&-1&0&1\\2&2&3&3\\-1&2&1&2\end{array}\right)\sim$$

$$\underset{\substack{L_2\leftarrow L_2-2L_1\\ L_3\leftarrow L_3+L_1}}{\sim}\left(\begin{array}{ccc|c}1&-1&0&1\\0&4&3&1\\0&1&1&3\end{array}\right)\underset{L_2\leftrightarrow L_3}{\sim}\left(\begin{array}{ccc|c}1&-1&0&1\\0&1&1&3\\0&4&3&1\end{array}\right)\sim$$

$$\underset{L_3\leftarrow L_3-4L_2}{\sim}\left(\begin{array}{ccc|c}1&-1&0&1\\0&1&1&3\\0&0&-1&-11\end{array}\right)\rightarrow$$ we have obtained a row echelon form

Gauss: from the row echelon form we rewrite the system and solve it.

Gauss-Jordan: keep eliminating zeros above the diagonal and you get solutions

**Gauss:**

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right)$$

$$\begin{cases} x - y = 1 \\ y + z = 3 \\ -z = -11 \end{cases} \Rightarrow \begin{cases} z = 11 \\ y = 3 - 11 = -8 \\ x = -8 + 1 = -7 \end{cases}$$

**Gauss-Jordan**

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right) \underset{\sim}{L_2 \leftarrow L_2 + L_3}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & -1 & -11 \end{array}\right) \begin{array}{c} L_1 \leftarrow L_1 + L_2 \\ \sim \\ L_3 \leftarrow -L_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 11 \end{array}\right)$$

$$\Rightarrow \begin{cases} x = -7 \\ y = -8 \\ z = 11 \end{cases}$$

When we're using Gaussian elimination to solve a system, there are some remarkable situations:

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 \cdots & 0 & a \end{array}\right), a \neq 0 \Rightarrow \text{incompatible system}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 \cdots & 0 & 0 \cdots & 0 & 0 \end{array}\right) \Rightarrow \text{redundancy (redundant equation)}$$

**5.** $(i)\begin{cases}2x+2y+3z=3\\ x-y\quad=1\\ -x+2y+z=2\end{cases}$ $(ii)\begin{cases}2x+5y+z=7\\ x+2y-z=3\\ x+y-4z=2\end{cases}$ $(iii)\begin{cases}x+y+z=3\\ x-y+z=1\\ 2x-y+2z=3\\ x+z=4\end{cases}$

**6.** $\begin{cases}2x_1+x_2+x_3+x_4=1\\ x_1+2x_2-x_3+4x_4=2\\ x_1+5x_2-4x_3+11x_4=\lambda\end{cases}$ $(\lambda\in\mathbb{R})$

**7.** $\begin{cases}ax+y+z=1\\ x+ay+z=a\\ x+y+az=a^2\end{cases}$ $(a\in\mathbb{R})$

Sol :

$$\left(\begin{array}{ccc|c}1&1&1&3\\1&-1&1&1\\2&-1&2&3\\1&0&1&4\end{array}\right)\underset{L_2\leftarrow L_2-L_1}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&-2&0&-2\\2&-1&2&3\\1&0&1&4\end{array}\right)\underset{L_3\leftarrow L_3-2L_1}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&-2&0&-2\\0&-3&0&-3\\1&0&1&4\end{array}\right)\sim$$

$$\underset{L_4\leftarrow L_4-L_1}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&-2&0&-2\\0&-3&0&-3\\0&-1&0&-1\end{array}\right)\underset{\substack{L_2\leftarrow \frac{L_2}{-2}\\ L_3\leftarrow\frac{L_3}{-3}\\ L_4\leftarrow\frac{L_4}{-1}}}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&1&0&1\\0&1&0&1\\0&1&0&1\end{array}\right)\sim$$

$$\underset{L_3\leftarrow L_3-L_2}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&1&0&1\\0&0&0&0\\0&1&0&1\end{array}\right)\underset{L_4\leftarrow L_4-L_2}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&1&0&1\\0&0&0&0\\0&0&0&0\end{array}\right)$$

$$\Rightarrow\begin{cases}x+y+z=3\\ y=1\end{cases}\Rightarrow\begin{cases}x=2-z\\ y=1\end{cases}$$

(ii)

$$\left(\begin{array}{ccc|c}2&5&1&7\\1&2&-1&3\\1&1&-4&2\end{array}\right)\underset{\substack{L_2\leftarrow 2L_2-L_1\\ L_3=2L_3-L_1}}{\sim}\left(\begin{array}{ccc|c}2&5&1&7\\0&-1&-3&-1\\0&-3&-9&-3\end{array}\right)\sim$$

Gauss-Jordan

$\Longrightarrow$

$$L_3 \leftarrow L_3 - 3L_2 \sim \left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad L_1 \leftarrow L_1 + 5L_2 \sim \left(\begin{array}{ccc|c} 2 & 0 & -14 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} 2x - 14z = 2 \\ -y - 3z = -1 \end{cases} \Longrightarrow \begin{cases} x = 7z + 1 \\ y = -3z + 1 \end{cases}$$

23.11.2021

# Seminar W 9 - 911

Compute by applying elementary operations the ranks of the matrices:

**1.** $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Sol: 1.

$$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_4 \leftarrow L_4 - L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ rank $M$ = # of nonzero rows in the row echelon form $= 3$

Compute by applying elementary operations the ranks of the matrices:

**1.** $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Sol: 3.

$$\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \sim$$

$$\begin{array}{c} L_2 - \beta L_1 \\ \sim \\ L_3 - 2L_1 \end{array} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix}$$

If $\alpha = 0$ : $\begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3-3\beta & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix}$ row echelon form

$\Rightarrow$ rank $M = 3$

If $\alpha \neq 0$ : $\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix} \sim$

$$\begin{array}{c} L_3 \leftarrow L_3 - \frac{1-\beta\alpha}{\alpha} \cdot L_2 \\ \sim \end{array} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 0 & 3-3\beta+2\cdot\frac{1-\beta\alpha}{\alpha} & 4-3\beta-\frac{1-\beta\alpha}{\alpha} \end{pmatrix}$$

$\Rightarrow$ rank $M \in \{2, 3\}$

rank $M = 2 \iff \begin{cases} 3 - 3\beta + \frac{2\cdot(1-\beta\alpha)}{\alpha} = 0 \\ 4 - 3\beta - \frac{1-\beta\alpha}{\alpha} = 0 \end{cases} \iff$

$\iff \begin{cases} 3\alpha - 5\beta\alpha + 1 = 0 \\ 4\alpha - 2\beta\alpha - 1 = 0 \end{cases} \Rightarrow \begin{cases} 7\alpha - 7\beta\alpha = 0 \\ 3\alpha - 5\beta\alpha + 1 = 0 \end{cases} \iff$

$\iff \begin{cases} 7\alpha(1-\beta) = 0 \\ 3\alpha - 5\beta\alpha + 1 = 0 \end{cases} \overset{\alpha \neq 0}{\iff} \begin{cases} 1 - \beta = 0 \\ 3\alpha - 5\beta\alpha + 1 = 0 \end{cases} \Rightarrow \begin{cases} \beta = 1 \\ -2\alpha + 1 = 0 \end{cases}$

$\Rightarrow$ rank $M = 2 \iff \alpha = \frac{1}{2}$ $\beta = 1$

rank $M = 3 \iff \alpha \neq \frac{1}{2}$ or $\beta \neq 1$

Compute by applying elementary operations the inverses of the matrices:

**4.** $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

$$\left(A \mid I_n\right) \sim \overset{\text{Gauss-Jordan}}{\cdots} \sim \left(I_n \mid A^{-1}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow L_2 \cdot \frac{1}{-3}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 6L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow \frac{1}{9} \cdot L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \underset{L_1 \leftarrow L_1 - 2L_3}{\overset{L_2 \leftarrow L_2 - 2L_3}{=}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{5}{9} & \frac{4}{9} & -\frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$\overset{L_1 \leftarrow L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$\Rightarrow \quad A^{-1} = \begin{pmatrix} 1/9 & 2/9 & 2/9 \\ 2/9 & 1/9 & -2/9 \\ 2/9 & -2/9 & 1/9 \end{pmatrix}$$

Compute by applying elementary operations the inverses of the matrices:

**4.** $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.  **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

Sol.:

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{L_2 \leftarrow L_2 - 2L_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\underset{L_3 \leftarrow L_3 - 3L_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \underset{L_2 \leftarrow \frac{1}{-5} L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \sim$$

$$\underset{L_3 \leftarrow L_3 + 12L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 1 \end{array}\right) \underset{L_3 \leftarrow 5L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & \frac{-1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \sim$$

$$\underset{L_2 \leftarrow L_2 - \frac{3}{5} L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \underset{L_1 \leftarrow L_1 - 2L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

$$\underset{L_1 \leftarrow L_1 - 4L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \quad = A^{-1}$$

Ex.: $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \\ 1 & -1 & 2 \end{pmatrix}$ is not invertible

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 5 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & -3 & -1 & -1 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow L_3 - L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right)$$

$\Rightarrow$ the matrix is not invertible, because we have a zero row on the left.

**7.** In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1, 0, 4)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 5, -36)$ and $v_4 = (2, 10, -72)$. Determine $dim < X >$ and a basis of $< X >$.

Sol.: To find a basis of $\langle X \rangle$, we just need to bring the matrix whose rows are the elements of $X$ to a row echelon form.

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 5 & -36 \\ 2 & 10 & -72 \end{pmatrix} \underset{\substack{L_4 \leftarrow L_4 - 2L_1 \\ L_3 \leftarrow L_3 - L_1}}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 5 & -40 \\ 0 & 10 & -80 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 10L_2}{\overset{L_3 \leftarrow L_3 - 5L_2}{\sim}}$$

$$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \dim \langle X \rangle = 2$$

A basis $\langle X \rangle$ is:

$$\big( (1,0,4),\ (0,1,-8) \big)$$

**9.** Determine the dimension of the subspaces $S$, $T$, $S+T$ and $S\cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S = < (1,0,4), (2,1,0), (1,1,-4) > ,$$

$$T = < (-3,-2,4), (5,2,4), (-2,0,-8) > .$$

Sol.:

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ basis for $S$: $\big((1,0,4), (0,1,-8)\big)$ $\Rightarrow$ $\dim S = 2$

$$\begin{pmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} -2 & 0 & -8 \\ 5 & 2 & 4 \\ -3 & -2 & 4 \end{pmatrix} \overset{L_1 \leftarrow L_1 - L_3}{\sim}$$

$$\sim \begin{pmatrix} 1 & 2 & -12 \\ 5 & 2 & 4 \\ -3 & -2 & 4 \end{pmatrix} \underset{L_3 \leftarrow L_3 + 3L_1}{\overset{L_2 \leftarrow L_2 - 5L_1}{\sim}} \begin{pmatrix} 1 & 2 & -12 \\ 0 & -8 & 64 \\ 0 & 4 & -32 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 + \frac{1}{2}L_2}{\sim} \begin{pmatrix} 1 & 2 & -12 \\ 0 & -8 & 64 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ basis for $T$: $\big((1,2,-12), (0,-8,64)\big)$ $\Rightarrow$ $\dim T = 2$

$S+T = < S \cup T >$

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \\ -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \begin{matrix} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - L_1 \\ \sim \\ L_4 \leftarrow L_4 + 3L_1 \\ L_5 \leftarrow L_5 - 5L_1 \\ L_6 \leftarrow L_6 + 2L_1 \end{matrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \\ 0 & -2 & 16 \\ 0 & 2 & -16 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} L_3 \leftarrow L_3 - L_2 \\ \sim \\ L_4 \leftarrow L_4 + 2L_2 \\ L_5 \leftarrow L_5 - 2L_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ basis for $S+T$:

$\big((1,0,4), (0,1,-8)\big) \Rightarrow \dim(S+T) = 2$

$$\dim(S \cap T) = \dim S + \dim T - \dim(S+T) =$$

$$= 2 + 2 - 2 = 2$$

**6.** Let $K$ be a field, let $B = (e_1, e_2, e_3, e_4)$ be a basis and let $X = (v_1, v_2, v_3)$ be a list in the canonical $K$-vector space $K^4$, where

$$v_1 = 3e_1 + 2e_2 - 5e_3 + 4e_4\,,$$

$$v_2 = 3e_1 - e_2 + 3e_3 - 3e_4\,,$$

$$v_3 = 3e_1 + 5e_2 - 13e_3 + 11e_4\,.$$

Write the matrix of the list $X$ in the basis $B$, determine an echelon form for it and deduce that $X$ is linearly dependent.

Sol: $X = (v_1, v_2, \dots, v_m)$ list of vectors

$B = (b_1, b_2, \dots, b_n)$ basis

$v_1 = a_{11} b_1 + a_{12} b_2 + \dots + a_{1n} b_n$

$\vdots$

$v_m = a_{m1} b_1 + a_{m2} b_2 + \dots + a_{mn} b_n$

$$[X]_B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$[x]_B = \begin{pmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \end{pmatrix} \sim$$

$$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - L_1}{\sim}} \begin{pmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 3 & -8 & 7 \end{pmatrix} \overset{L_3 \leftarrow L_3 + L_2}{\sim} \begin{pmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ the initial vectors were linearly dependent

07.12.2021

# Seminar W10 - 911

Def: $V, V'$ $K$-vector spaces, $B = (v_1, \dots, v_m)$ basis of $V$ ("source basis")

$B' = (v_1', \dots, v_n')$ basis of $V'$ ("target basis")

$f : V \to V'$ $K$-linear map.

$$[f]_{B,B'} = \left( [f(v_1)]_{B'} \quad [f(v_2)]_{B'} \quad \cdots \quad [f(v_m)]_{B'} \right) \in M_{n,m}(K)$$

Prop: $V, V'$ $K$-v.s., $B, B'$ bases of $V, V'$, $f: V \to V'$ linear map,

$$\forall v \in V : \quad [f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B$$

**2.** Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1,1,0), (0,1,1), (1,0,1))$ of $\mathbb{R}^3$, $B' = (v_1', v_2') = ((1,1), (1,-2))$ of $\mathbb{R}^2$ and let $E' = (e_1', e_2')$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

$e_1' = (1,0)$, $e_2' = (0,1)$

Sol. Ex: $[f]_{E,B'}$, $E = (e_1, e_2, e_3)$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$$[f]_{E,B'} = \left( [f(e_1)]_{B'} \quad [f(e_2)]_{B'} \quad [f(e_3)]_{B'} \right)$$

$$f(e_1) = f(1,0,0) = (0,-1) = \alpha_1 \cdot v_1' + \beta_1 \cdot v_2'$$

$$\Rightarrow (0,-1) = \alpha_1 \cdot (1,1) + \beta_1 \cdot (1,-2) \Rightarrow \begin{cases} \alpha_1 + \beta_1 = 0 \\ \alpha_1 - 2\beta_1 = -1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 2\beta_1 - 1 \\ 2\beta_1 - 1 + \beta_1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \beta_1 = \frac{1}{3} \\ \alpha_1 = -\frac{1}{3} \end{cases} \Rightarrow [f(e_1)]_{B'} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix};$$

$$f(e_2) = f(0,1,0) = (1,0) = \alpha_2 \cdot e_1' + \beta_2 \cdot e_2'$$

$$\Rightarrow (1,0) = \alpha_2 \cdot (1,1) + \beta_2 \cdot (1,-2) \Rightarrow \begin{cases} \alpha_2 + \beta_2 = 1 \\ \alpha_2 - 2\beta_2 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = \frac{2}{3} \\ \beta_2 = \frac{1}{3} \end{cases}$$

$$\Rightarrow [f(e_2)]_{B'} = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$f(e_3) = f(0,0,1) = (0,0) = \alpha_3 \cdot (1,1) + \beta_3 \cdot (1,-2)$$

$$\Rightarrow \begin{cases} 0 = \alpha_3 + \beta_3 \\ 0 = \alpha_3 - 2\beta_2 \end{cases} \Rightarrow \begin{cases} \alpha_3 = 0 \\ \beta_3 = 0 \end{cases}$$

$$\Rightarrow [f(e_3)]_{B'} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} [f(e_1)]_{B'} &= \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix} \\ [f(e_2)]_{B'} &= \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \\ [f(e_3)]_{B'} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}\right\} \Rightarrow [f]_{E,B'} = \begin{pmatrix} -1/3 & 2/3 & 0 \\ 1/3 & 1/3 & 0 \end{pmatrix}$$

$[f]_{B,B'} = ?$

$f(u_1) = f(1,1,0) = (1,-1) = \alpha_1 \cdot u_1' + \beta_1 \cdot u_2'$

$\Rightarrow (1,-1) = \alpha_1 \cdot (1,1) + \beta_1 \cdot (1,-2)$

$\Rightarrow \begin{cases} 1 = \alpha_1 + \beta_1 \\ -1 = \alpha_1 - 2\beta_1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 1 - \beta_1 \\ -1 = 1 - 3\beta_1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = \frac{1}{3} \\ \beta_1 = \frac{2}{3} \end{cases}$

$\Rightarrow [f(u_1)]_{B'} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$

$f(u_2) = f(0,1,1) = (1,0) = \alpha_2 \cdot u_1' + \beta_2 \cdot u_2' = \alpha_2 (1,1) + \beta_2 \cdot (1,-2)$

$\Rightarrow \begin{cases} 1 = \alpha_2 + \beta_2 \\ 0 = \alpha_2 - 2\beta_2 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = 1 - \beta_2 \\ 0 = 1 - 3\beta_2 \end{cases} \Leftrightarrow \begin{cases} \beta_2 = \frac{1}{3} \\ \alpha_2 = \frac{2}{3} \end{cases}$

$\Rightarrow [f(u_2)]_{B'} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

$f(u_3) = f(1,0,1) = (0,-1) = \alpha_3 \cdot (1,1) + \beta_3 \cdot (1,-2)$

$\Rightarrow \begin{cases} 0 = \alpha_3 + \beta_3 \\ -1 = \alpha_3 - 2\beta_3 \end{cases} \Leftrightarrow \begin{cases} \alpha_3 = -\beta_3 \\ -1 = -3\beta_3 \end{cases} \Leftrightarrow \begin{cases} \beta_3 = \frac{1}{3} \\ \alpha_3 = -\frac{1}{3} \end{cases}$

$\Rightarrow [f(u_3)]_{B'} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$

$\Rightarrow [f]_{B,B'} = \begin{pmatrix} 1/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & 1/3 \end{pmatrix}$

$$f(v_1) = (1,-1) = \alpha_1 \cdot (1,0) + \beta_1 \cdot (0,1)$$

$$\Rightarrow \alpha_1 = 1 \ , \ \beta_1 = -1$$

$$\Rightarrow [f(v_1)]_E = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$[f(v_2)]_E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$[f(v_3)]_E = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow [f]_E = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

**4.** Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis $E$ of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

$(i)$ Show that $v = (1,4,1,-1) \in Ker\, f$ and $v' = (2,-2,4,2) \in Im\, f$.
$(ii)$ Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
$(iii)$ Define $f$.

Prop: $V, V'$ K-v.s., $B, B'$ bases of $V, V'$, $f: V \to V'$ linear map,

$$\forall v \in V: \quad [f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B$$

(i) $$[f(v)]_E = [f]_E \cdot [v]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow f(v) = 0 \Rightarrow v \in Ker f$.

We have to find $v'' = (x,y,z,t)$ so that $f(v'') = v'$

$$[f(v'')] = [f]_E \cdot [v'']_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

We have to show that $\exists\, x, y, z, t$ so that

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\overset{L_2 \leftrightarrow L_4}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 2 & -2 & 6 & 0 \end{array}\right) \begin{array}{l} L_3 \leftarrow L_3 + L_2 \\ \sim \\ L_4 \leftarrow L_4 - 2L_2 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ the system is compatible $\Rightarrow \exists x, y, z, t \Rightarrow v' \in \operatorname{Im} f$.

(ii) $\operatorname{Ker} f = \{ v = (x, y, z, t) \mid f(v) = 0 \} =$

$$= \left\{ v = (x,y,z,t) \;\middle|\; [f]_E \cdot [v]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ v = (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 2 & 1 & -5 & 1 & 0 \\ 1 & 2 & -4 & 5 & 0 \end{array}\right) \begin{array}{l} L_2 \leftarrow L_2 + L_1 \\ \sim \\ L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1 \end{array} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\begin{array}{l} L_3 \leftarrow L_3 + \frac{1}{2} L_2 \\ \sim \\ L_4 \leftarrow L_4 - \frac{1}{2} L_2 \end{array} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \overset{L_2 \leftarrow \frac{1}{2} L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x + y - 3z + 2t = 0 \\ y - z + 3t = 0 \end{cases}$$

$$\operatorname{Ker} f = \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x + y - 3z + 2t = 0 \\ y - z + 3t = 0 \end{cases} \right\} =$$

$$= \left\{ (x,y,z,t) \;\middle|\; \begin{cases} y = z - 3t \\ x = 3z - 2t - z + 3t = 2z + t \end{cases} \right\} =$$

$$= \left\{ (2z+t,\ z-3t,\ z,\ t) \;\middle|\; z,t \in \mathbb{R} \right\} =$$

$$= \left\{ (2z,\ z,\ z,\ 0) + (t,\ -3t,\ 0,\ t) \;\middle|\; z,t \in \mathbb{R} \right\} =$$

$$= \left\{ z \cdot (2,1,1,0) + t \cdot (1,-3,0,1) \;\middle|\; z,t \in \mathbb{R} \right\} =$$

$$= \langle (2,1,1,0),\ (1,-3,0,1) \rangle$$

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & -3 & 0 & 1 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & -3 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix} \overset{L_2 \leftarrow L_2 - 2L_1}{\sim} \begin{pmatrix} 1 & -3 & 0 & 1 \\ 0 & 7 & 1 & -2 \end{pmatrix}$$

$\Rightarrow$ basis for $\operatorname{Ker} f$ : $((1,-3,0,1),\ (0,7,1,-2))$

$\Rightarrow \dim(\operatorname{Ker} f) = 2$

$$\operatorname{Im} f = \left\{ f(u) \;\middle|\; u \in \mathbb{R}^4 \right\} = \left\{ f(x,y,z,t) \;\middle|\; x,y,z,t \in \mathbb{R} \right\} =$$

$$= \left\{ w = (a,b,c,d) \;\middle|\; \exists\, u = (x,y,z,t) : f(u) = w \right\} =$$

$$= \left\{ w = (a,b,c,d) \;\middle|\; \exists\, u = (x,y,z,t) : [f]_E \cdot [u]_E = [w]_E \right\}$$

$$= \left\{ w = (a,b,c,d) \;\middle|\; \exists\, u = (x,y,z,t) : \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right\}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 2 & -2 & 6 & a+b \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 1 & -1 & 3 & d-a \end{array}\right) \sim$$

$$\overset{L_2 \leftrightarrow L_4}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 2 & -2 & 6 & a+b \end{array}\right) \underset{L_4 \leftarrow L_4 - 2L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}}$$

$$= \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & 0 & 0 & 0 & c+d-3a \\ 0 & 0 & 0 & 0 & a+b-2d+2a \end{array}\right)$$

The system is compatible $\Leftrightarrow \begin{cases} c+d-3a=0 \\ 3a+b-2d=0 \end{cases}$

$$\operatorname{Im} f = \left\{ (a,b,c,d) \in \mathbb{R}^4 \;\middle|\; \begin{cases} c+d-3a=0 \\ 3a+b-2d=0 \end{cases} \right\} =$$

$$= \left\{ (a,b,c,d) \;\middle|\; \begin{cases} c = 3a-d \\ b = 2d-3a \end{cases} \right\} =$$

$$= \{ (a, 2d-3a, 3a-d, d) \mid a,d \in \mathbb{R} \}$$

$$= \langle (1,-3,3,1), (0,2,-1,1) \rangle$$

$\Rightarrow$ basis for $\operatorname{Im} f$ : $\left( (1,-3,3,1), (0,2,-1,1) \right)$

$\Rightarrow \dim \operatorname{Im} f = 2$

Verification: $\underbrace{\dim \ker f}_{2} + \underbrace{\dim \operatorname{Im} f}_{2} = \underbrace{\dim(\mathbb{R}^4)}_{4}$ ✓

(iii)

$f(x,y,z,t) = ?$

$$[f(x,y,z,t)]_E = [f]_E \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+y-3z+2t \\ -x+y+z+4t \\ 2x+y-5z+t \\ x+2y-4z+5t \end{pmatrix}$$

$$\Rightarrow f(x,y,z,t) = (x+y-3z+2t,\ -x+y+z+4t,\ 2x+y-5z+t,\ x+2y-4z+5t)$$

14.12.2021

# Seminar W11 – 911

**Def**: $V, V'$ K-vector space, $\mathcal{B} = (v_1, v_2, \dots, v_n)$ basis for $V$

$\mathcal{B}' = (v_1', v_2', \dots, v_n')$ basis for $V'$

$f : V \to V'$ K-linear map

$$\boxed{[f]_{\mathcal{B},\mathcal{B}'} = \left( [f(v_1)]_{\mathcal{B}'} \;\; [f(v_2)]_{\mathcal{B}'} \;\; \cdots \;\; [f(v_n)]_{\mathcal{B}'} \right)}$$

**Prop**: $\forall v \in V$: $\boxed{[f(v)]_{\mathcal{B}'} = [f]_{\mathcal{B},\mathcal{B}'} \cdot [v]_{\mathcal{B}}}$

**Def**: $[id_V]_{\mathcal{B},\mathcal{B}'} =: T_{\mathcal{B}',\mathcal{B}}$ = "the base-change matrix from $\mathcal{B}'$ to $\mathcal{B}$"

**Corollary**: $f = id_V \Rightarrow [v]_{\mathcal{B}'} = [id]_{\mathcal{B},\mathcal{B}'} \cdot [v]_{\mathcal{B}}$

$$\left( [v]_{\mathcal{B}'} = T_{\mathcal{B}',\mathcal{B}} \cdot [v]_{\mathcal{B}} \right)$$

**Prop**: $\underline{[id]^{-1}_{\mathcal{B},\mathcal{B}'} = [id]_{\mathcal{B}',\mathcal{B}}}$ → This doesn't work for any linear map $f$, just for id! $[f]^{-1}_{\mathcal{B},\mathcal{B}'} \neq [f]_{\mathcal{B}',\mathcal{B}}$

$$\left( T^{-1}_{\mathcal{B}',\mathcal{B}} = T_{\mathcal{B},\mathcal{B}'} \right)$$

Prop: $V, V', V''$ $K$-v.s., $B, B', B''$ bases

$f: V' \to V''$, $g: V \to V'$ $K$-linear maps

$f \circ g: V \to V''$

$$\boxed{[f \circ g]_{B,B''} = [f]_{B',B''} \cdot [g]_{B,B'}}$$

Corollary: $V, V'$ $K$-v.s., $B_1, B_2$ bases for $V$, $B_1', B_2'$ bases for $V'$, $f: V \to V'$ linear map

$$[f]_{B_2, B_2'} = [id_{V'}]_{B_1', B_2'} \cdot [f]_{B_1, B_1'} \cdot [id_V]_{B_2, B_1}$$

$$= T_{B_2', B_1'} \cdot [f]_{B_1, B_1'} \cdot T_{B_1, B_2}$$

**2.** In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

Sol.: $[2f]_B = 2 \cdot [f]_B = 2 \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$

$[f+g]_B = [f]_B + [g]_B$

$[g]_B = [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'}$

In order to find $[id]_{B',B}$.

1st approach (tedious, but simple one):

$$[id]_{B',B} = \left([e_1']_B \quad [e_2']_B\right)$$

$$[e_1']_B = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow e_1' = \alpha e_1 + \beta e_2$$

$$\Rightarrow (1,0) = \alpha \cdot (1,2) + \beta \cdot (1,3)$$

$$\Rightarrow \begin{cases} 1 = \alpha + \beta \\ 0 = 2\alpha + 3\beta \end{cases} \Rightarrow \begin{cases} \beta = 1 - \alpha \\ 0 = 2\alpha + 3 - 3\alpha \end{cases} \Rightarrow \begin{cases} \beta = 1 - \alpha \\ 0 = -\alpha + 3 \end{cases} \Rightarrow \begin{cases} \alpha = 3 \\ \beta = -2 \end{cases}$$

$$\Rightarrow [e_1']_B = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$[e_2']_B = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \Rightarrow e_2' = \alpha_2 e_1 + \beta_2 e_2$$

$$\Rightarrow (2,1) = \alpha_2 \cdot (1,2) + \beta_2 \cdot (1,3)$$

$$\Rightarrow \begin{cases} \alpha_2 + \beta_2 = 2 \\ 2\alpha_2 + 3\beta_2 = 1 \end{cases} \Rightarrow \begin{cases} \alpha_2 = 5 \\ \beta_2 = -3 \end{cases} \Rightarrow [e_2']_B = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

$$\Rightarrow [id]_{B',B} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

2nd approach (the fancier one):

$$[id]_{B',B} = [id]_{E,B} \cdot [id]_{B',E} = [id]_{B,E}^{-1} \cdot [id]_{B',E}$$

$$B = ((1,2),(1,3)) \quad , \quad B' = ((1,0),(2,1))$$

$$[id]_{B,E} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \qquad [id]_{B',E} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow [id]_{B',B} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$[g]_B = [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'}$$

$$[id]_{B,B'} = [id]_{B',B}^{-1} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$\Rightarrow [g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$\Rightarrow [f+g]_B = [f]_B + [g]_B$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'}$$

$$[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$$

$$[f]_{B'} = [id]_{B,B'} \cdot [f]_{B} \cdot [id]_{B',B} =$$

$$= \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$$

$$\Rightarrow [f \circ g]_{B'} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

# Eigenvalues and eigenvectors

**Def**: $V$ $K$-v.s., $f: V \to V$ linear map.

$v \in V \setminus \{0\}$ eigenvector for $f$ if $\exists \lambda \in K$ (called an eigenvalue):

$$f(v) = \lambda \cdot v$$

$V(\lambda) = \{ v \in V \mid f(v) = \lambda v \} = \{\text{the set of eigenvectors corresponding to } \lambda\} \cup \{0\}$

$\hookrightarrow$ the eigenspace of $f$ corresponding to $\lambda$.

**Prop**: $\lambda$ eigenvalue of $f$ $\Leftrightarrow$ $\lambda$ is a root of the characteristic polynomial $p_f(X) = \det([f]_B - X I_n)$

($B$ basis of $V$, $n = \dim V$)

**Rem.**: eigenvalues & eigenvectors for $f \in \mathrm{End}_K(V)$ $\Leftrightarrow$ eigenvalues and eigenvectors for $A \in M_n(K)$, $A = [f]_B$

Compute the eigenvalues and the eigenvectors of the (endomorphisms having) matrices:

**7.** $\begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{pmatrix}$ $(a, b \in \mathbb{R}^*)$.

**Sol**: $p_A(X) = \det(A - X I_3) = \begin{vmatrix} a-X & 0 & b \\ 0 & a-X & 0 \\ b & 0 & a-X \end{vmatrix} = (a-X)^3 - b^2(a-X)$

$$= (a-x)\left((a-x)^2 - b^2\right) = (a-x)(a-x-b)(a-x+b)$$

$\Rightarrow$ the eigenvalues are:

$$\lambda_1 = a, \quad \lambda_2 = a-b, \quad \lambda_3 = a+b$$

To find $V(\lambda)$, we just need to solve the equation:

$$A \cdot [v]_E = \lambda \cdot [v]_E$$

$$(A - \lambda \cdot I_3) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$V(\lambda_1) = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} bz = 0 \\ bx = 0 \end{cases} \right\} =$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} x = 0 \\ z = 0 \end{cases} \right\} =$$

$$= \{ (0,y,0) \mid y \in \mathbb{R} \} = \langle (0,1,0) \rangle$$

$$V(\lambda_2) = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} b & 0 & b \\ 0 & b & 0 \\ b & 0 & b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ (x,y,z) \;\middle|\; \begin{cases} bx + bz = 0 \\ by = 0 \end{cases} \right\} =$$

$$\overset{b \neq 0}{=} \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} y = 0 \\ x = -z \end{cases} \right\} =$$

$$= \{ (-z, 0, z) \mid z \in \mathbb{R} \} = \langle (-1, 0, 1) \rangle$$

$$V(\lambda_3) = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -b & 0 & b \\ 0 & -b & 0 \\ b & 0 & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} -b(x-z) = 0 \\ -by = 0 \end{cases} \right\} =$$

$$= \{ (z, 0, z) \mid z \in \mathbb{R} \} = \langle (1,0,1) \rangle$$

$\dim V(\lambda_1) = 1 = \dim V(\lambda_2) = \dim V(\lambda_3)$

$\lambda_1, \lambda_2, \lambda_3$ have multiplicity 1.

$\Rightarrow$ we can make a basis of eigenvectors:

$$B = \big( (0,1,1), (-1,0,1), (1,0,1) \big)$$

Let $f$ be a linear map so that $[f]_E = A = \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{pmatrix}$

$$[f]_B = [id]_{E,B} \cdot [f]_E \cdot [id]_{B,E}$$

$$[id]_{B,E} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

We will get $[1]_B = \begin{pmatrix} a & 0 & 0 \\ 0 & a-b & 0 \\ 0 & 0 & a+b \end{pmatrix}$

Which is a diagonal matrix.