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Individual Homework
Predicate Resolution

Exercise 4.2

Using a refinement of predicate resolution prove:

- the semi-distributivity of '∃' over '∧':

$U_1 = \vdash (\exists x)(P(x) \wedge Q(x)) \rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

$U_2 = \nvdash (\exists x)P(x) \wedge (\exists x)Q(x) \rightarrow (\exists x)(P(x) \wedge Q(x))$

1. $\neg U_1 = \neg \cancel{(\exists x)}((\exists x)(P(x) \wedge Q(x)) \rightarrow (\exists x)P(x) \wedge (\exists x)Q(x))$

$\equiv \neg(\neg(\exists x)(P(x) \wedge Q(x)) \vee (\exists x)P(x) \wedge (\exists x)Q(x))$

$\equiv (\exists x)(P(x) \wedge Q(x)) \wedge \neg((\exists x)P(x) \wedge (\exists x)Q(x))$

$\equiv (\exists x)(P(x) \wedge Q(x)) \wedge (\neg(\exists x)P(x) \vee \neg(\exists x)Q(x))$

$\equiv (\exists x)(P(x) \wedge Q(x)) \wedge ((\forall x)\neg P(x) \vee (\forall x)\neg Q(x))$

$\equiv (\exists x)(P(x) \wedge Q(x)) \wedge ((\forall y)\neg P(y) \vee (\forall z)\neg Q(z))$

$\equiv (\exists x)(\forall y)(\forall z)(P(x) \wedge Q(x) \wedge (\neg P(y) \vee \neg Q(z)))$

$(\neg U_1)^P = (\exists x)(\forall y)(\forall z)(P(x) \wedge Q(x) \wedge (\neg P(y) \vee \neg Q(z)))$

└ the prenex form

$[x \leftarrow a]$, a - Skolem constant

$(\neg U_1)^S = (\forall y)(\forall z)(P(a) \wedge Q(a) \wedge (\neg P(y) \vee \neg Q(z)))$

└ the Skolem form

$(\neg U_1)^C = P(a) \wedge Q(a) \wedge (\neg P(y) \vee \neg Q(z))$ – the clausal normal form

$S_1 = \{ C_1 = P(a),\ C_2 = Q(a),\ C_3 = \neg P(y) \vee \neg Q(z) \}$

$C_4 = Res^{Pr}_{[y \leftarrow a]}(C_1, C_3) = \neg Q(z)$

$C_5 = Res^{Pr}_{[z \leftarrow a]}(C_2, C_4) = \square$

$(\neg U_1)^C \vdash^{Pr}_{Res} \square \Rightarrow \vdash U_1$

2. $\neg U_2 = \neg((\exists x)P(x) \wedge (\exists x)Q(x) \rightarrow (\exists x)(P(x) \wedge Q(x)))$

$\equiv \neg(\neg((\exists x)P(x) \wedge (\exists x)Q(x)) \vee (\exists x)(P(x) \wedge Q(x)))$

$\equiv (\exists x)P(x) \wedge (\exists x)Q(x) \wedge \neg(\exists x)(P(x) \wedge Q(x))$

$\equiv (\exists x)P(x) \wedge (\exists x)Q(x) \wedge (\forall x)\neg(P(x) \wedge Q(x))$

$\equiv (\exists x)P(x) \wedge (\exists x)Q(x) \wedge (\forall x)(\neg P(x) \vee \neg Q(x))$

$\equiv (\exists x)P(x) \wedge (\exists y)Q(y) \wedge (\forall z)(\neg P(z) \vee \neg Q(z))$

$\equiv (\exists x)(\exists y)(\forall z)(P(x) \wedge Q(y) \wedge (\neg P(z) \vee \neg Q(z)))$

$(\neg U_2)^P = (\exists x)(\exists y)(\forall z)(P(x) \wedge Q(y) \wedge (\neg P(z) \vee \neg Q(z)))$

└ the prenex form

$[x \leftarrow a], [y \leftarrow b]$, a, b - Skolem constants

$(\neg U_2)^S = (\forall z)(P(a) \wedge Q(b) \wedge (\neg P(z) \vee \neg Q(z)))$ – the Skolem form

$(\neg U_2)^C = P(a) \wedge Q(b) \wedge (\neg P(z) \vee \neg Q(z))$ – the clausal normal form

$S_2 = \{ C_1' = P(a),\ C_2' = Q(b),\ C_3' = \neg P(z) \vee \neg Q(z) \}$

$C_4' = Res^{Pr}_{[z \leftarrow a]}(C_1', C_3') = \neg Q(a)$

$C_5' = Res^{Pr}_{[z \leftarrow b]}(C_2', C_3') = \neg P(b)$

$Q(a)$ and $Q(b)$ are not unifiable because a and b are constants, so the clauses C_2' and C_4' do not resolve. Likewise for $P(a)$ and $P(b)$ (C_1' and C_5')

$(\neg U_2)^C \nvdash^{Pr}_{Res} \square \Rightarrow \nvdash U_2$

$\vdash U_1$ and $\nvdash U_2 \Rightarrow$ '$\exists$' is only semi-distributive over '$\wedge$'