10.03.2022

Seminar 2 - W3

Find the general solution of the following differential equations:

e^{rt} $\qquad e^{rt}(\underline{\qquad}) = 0$

1) $x' + 6x = 0 \rightarrow \underbrace{r + 6 = 0}$ characteristic equation

$r = -6 \in \mathbb{R} \rightarrow \underbrace{e^{-6t}}$ is a solution

the general solution $x(t) = c \cdot e^{-6t}, \ c \in \mathbb{R}$

2) $x'' = 0 \rightarrow r^2 = 0$ charact. eq.

$\Rightarrow r = 0$ is a solution with multiplicity 2

$r_{1,2} = 0$

$\Rightarrow$ the general solution:

$$x(t) = c_1 \cdot e^{0t} + c_2 \cdot t \cdot e^{0t} = c_1 + c_2 \cdot t, \ c_1, c_2 \in \mathbb{R}$$

If $r \in \mathbb{R}$ is a solution of the char. eq. with multiplicity α then $e^{rt}, t \cdot e^{rt}, t^2 \cdot e^{rt}, \ldots, t^{\alpha-1} \cdot e^{rt}$ are solutions of the LHDE

3) $x'' - 2x' - 15x = 0 \rightarrow r^2 - 2r - 15 = 0$

$\Delta = 4 + 60 = 64 \Rightarrow \sqrt{\Delta} = 8$

$r_1 = \frac{2+8}{2} = 5 \qquad\qquad (r-5)(r+3) = 0$

$r_2 = \frac{2-8}{2} = -3$

$\Rightarrow$ we have 2 different solutions of multiplicity 1

$\Rightarrow$ the general solution: $x(t) = c_1 \cdot e^{5t} + c_2 \cdot e^{-3t}, c_1, c_2 \in \mathbb{R}$

$\cdot \mathbb{R}$

4) $x^{(4)} - x = 0 \rightarrow r^4 - 1 = 0 \Rightarrow (r^2+1)(r^2-1) = 0$

$(r^2+1)(r-1)(r+1) = 0$

$\Rightarrow r_{1,2} \in \{1, -1\} \rightarrow e^t, e^{-t}$ solution

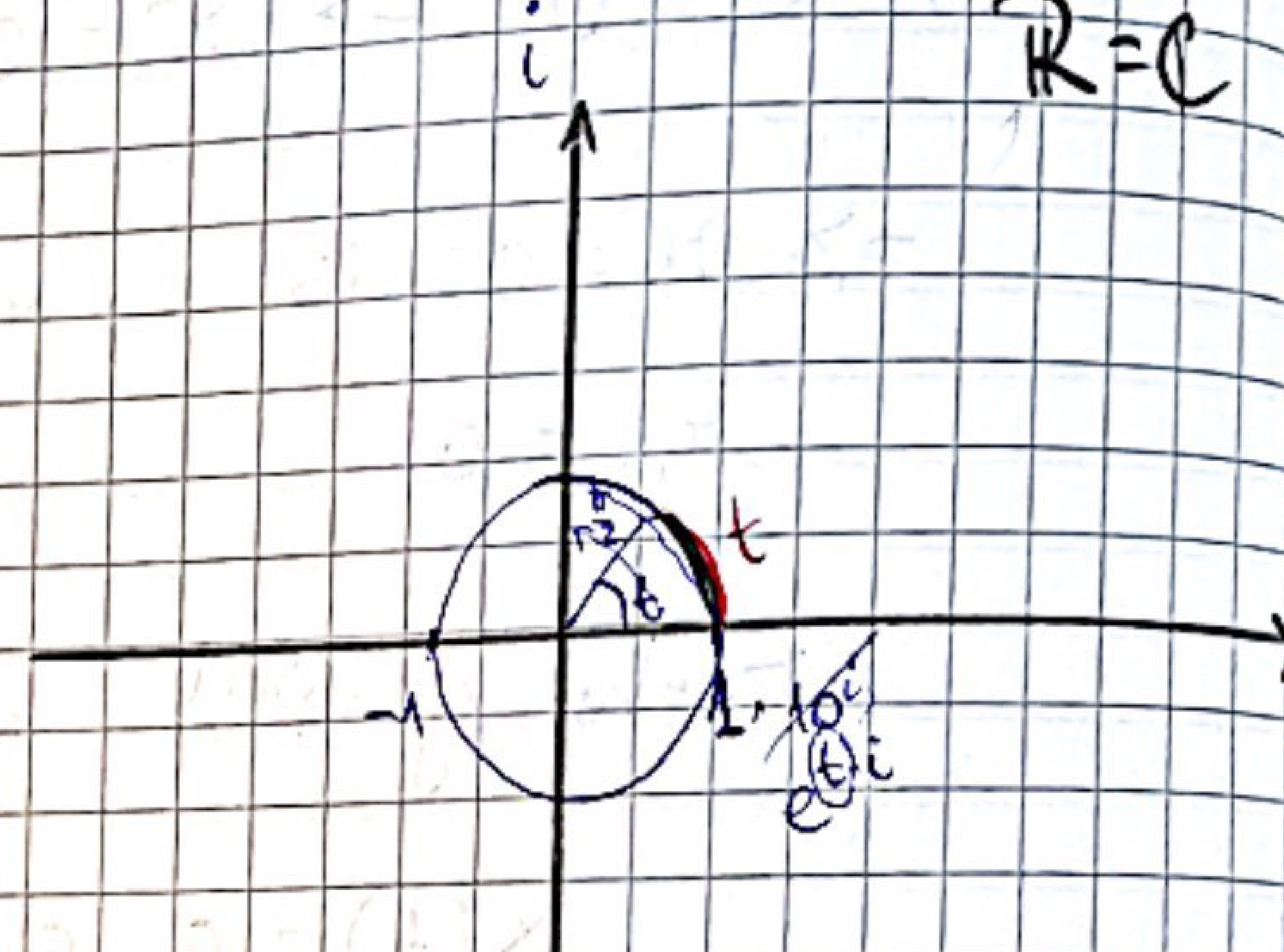
$r_{3,4} = \pm i$

$e^{a+bi} = e^a \cdot e^{bi}$ $\qquad$ $e^{t \cdot i} = \cos t + i \sin t$

$\left| \begin{array}{l} 10^i = a + bi, \ i^2 = -1 \\ 10^{-i} = a - bi, \ (\bar{i})^2 = i^2 = -1 \end{array} \right.$

$1 = \underset{\sin}{a^2} + \underset{\cos}{b^2}$

$\mathbb{R}^2 = \mathbb{C}$



$e^{t \cdot i} = \cos t + i \sin t$

$e^{\pi \cdot i} = -1$

$e^{\frac{\pi}{2} \cdot i} = i$

$10^i = (e^{\ln 10})^{i \cdot \pi} \neq e^i$

$r_{3,4} = \pm i$

$\left. \begin{array}{l} \cos t + i \sin t \\ \cos t - i \sin t \end{array} \right| \begin{array}{l} = e^{it} \\ = e^{-it} \end{array} \longrightarrow \sin t, \cos t$ are solutions

$\Rightarrow$ the general solution $x(t) = c_1 \cdot e^t + c_2 \cdot e^{-t} + c_3 \cdot \cos t + c_4 \cdot \sin t, \ c_i \in \mathbb{R}$

$\boxed{r = a + bi \rightarrow e^{at} \cdot \cos bt, \ e^{at} \cdot \sin bt}$

$x'' + \underbrace{a}_{t^2} x' + \underbrace{b}_{} x$

5) Find the LHDE with constant coefficient and of minimal order having the following solutions.

a) $5t \cdot e^{-3t}$ and $-3e^{5t}$

$\hookrightarrow$ $\boxed{r_{1,2} = -3 \text{ with multiplicity } 2}$ $\hookrightarrow$ $\boxed{r_3 = 5 \text{ with multiplicity } 1}$

! r sol. with mult. m

$e^{rt}, t \cdot e^{rt}, t^2 \cdot e^{rt}, \dots, t^{m-1} \cdot e^{rt}$

$(r+3)^2(r-5) = 0$ the charac. eq

the general solution: $x(t) = c_1 \cdot e^{-3t} + c_2 \cdot t \cdot e^{-3t} + c_3 \cdot e^{5t}$, $c_1, c_2, c_3 \in \mathbb{R}$

$(r^2 + 6r + 9)(r-5) = 0$

$r^3 + 6r^2 + 9r - 5r^2 - 30r - 45 = 0$

$r^3 + r^2 - 21r - 45 = 0$

$\Rightarrow x^{(3)} + x'' - 21x' - 45x = 0$ LHDE

b) $e^{5t} \cdot \sin 3t$ $\rightarrow$ $r_1 = 5 + 3i$ with multiplicity 1

$r_2 = 5 - 3i$ with multiplicity 1

$bi \rightsquigarrow \sin bt$

$(r - r_1)(r - r_2) = 0$

$(r - 5 - 3i)(r - 5 + 3i) = 0$

$(r-5)^2 - 9(-1) = 0$

$r^2 - 10r + 25 + 9 = 0$

$r^2 - 10r + 34 = 0$

$\boxed{x'' - 10x' + 34x = 0}$ LHDE

the general sol: $x(t) = c_1 \cdot e^{5t} \cdot \cos 3t + c_2 \cdot e^{5t} \cdot \sin 3t$, $c_1, c_2 \in \mathbb{R}$

, $c_i \in \mathbb{R}$

6) Find all solutions for each of the following boundary value problem (BVP)

$x'' + x = 0 \qquad x(0) = x(\pi) = 0$

$r^2 + 1 = 0$

$r^2 = -1$

$r_{1,2} = \pm i$ with multiplicity 1

$\hookrightarrow e^{0 \cdot t} \cdot \sin t \, , \, e^{0 \cdot t} \cdot \cos t$ - the solutions

the gen. sol. $x(t) = c_1 \cdot \sin t + c_2 \cdot \cos t \, , \, c_1, c_2 \in \mathbb{R}$

$x(0) = c_1 \cdot \sin 0 + c_2 \cdot \cos 0 = c_2 \, , \, x(0) = 0 \Rightarrow c_2 = 0$

$x(\pi) = c_1 \cdot \sin \pi + c_2 \cdot \cos \pi = -c_2 \, , \, x(\pi) = 0 \Rightarrow c_2 = 0$

$\Rightarrow c_1 \in \mathbb{R}$

$\Rightarrow x(t) = c_1 \cdot \sin t \, , \, c_1 \in \mathbb{R}$

7) Find $\lambda \in \mathbb{R}$ with the property that there exist nonnull 2π periodic solutions of $x'' + \lambda x = 0$
$x(t) \neq 0$

$r^2 + \lambda = 0$

$r^2 = -\lambda$

$\cancel{r = \pm i\sqrt{\lambda}}$

CI. $\lambda > 0$

$r_{1,2} = \pm i\sqrt{\lambda}$

$\hookrightarrow e^{0t} \cdot \sin \sqrt{\lambda} t + e^{0t} \cdot \cos \sqrt{\lambda} \cdot t$

$x(t) = c_1 \cdot \sin \sqrt{\lambda} \cdot t + c_2 \cdot \cos \sqrt{\lambda} \cdot t \, , \, c_1, c_2 \in \mathbb{R}$

$f(t + \varepsilon) = f(t)$

$\sin(\sqrt{\lambda}\cdot t)$

$$\sin\left(\sqrt{\lambda}\left(t+\frac{2\pi}{\sqrt{\lambda}}\right)\right) = \sin(\sqrt{\lambda}t + 2\pi) =$$

$$= \sin(\sqrt{\lambda}\cdot t)$$

the period is $\frac{2\pi}{\sqrt{\lambda}}$

$$\frac{2\pi}{\sqrt{\lambda}} = 2\pi \Rightarrow \sqrt{\lambda} = 1$$
$$\lambda = 1$$

C_{II} : $\lambda = 0$

$r^2 = 0$

$\Rightarrow r = 0$ with multiplicity 2

$x(t) = c_1 + t\cdot c_2 \rightarrow$ it's not periodic

C_{III} : $\lambda < 0$

$r^2 = -\lambda \Rightarrow r_{1,2} = \pm\sqrt{\lambda}$

$x(t) = c_1\cdot e^{\sqrt{\lambda}\cdot t} + c_2\cdot e^{-\sqrt{\lambda}\cdot t} \rightarrow$ it's not periodic