GRAMMARS

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1. Given the grammar $G = (N,\ \Sigma,\ P,\ S)$

$N = \{S, C\}$

$\Sigma = \{a, b\}$

$P:\ S \ \rightarrow ab \mid aCSb$

$\quad C \ \rightarrow S \mid bSb$

$\quad CS \ \rightarrow b$,

prove that $w \ = \ ab(ab^2)^2 \ \in L(G)$.

$Obs.:\ (ab)^2 = abab \ \neq a^2b^2 = aabb$

Sol.: [redacted]

S => aCSb => abSbSb $\overset{2}{\Rightarrow}$ ababbabb
(2) (4) (1)

=> $S \overset{4}{\Rightarrow}$ ababbabb = w => $w \ \in L(G)$

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2. Given the grammar $G = (N,\ \Sigma,\ P,\ S)$

$N = \{S\}$

$\Sigma = \{a, b, c\}$

$P:\ S \ \rightarrow a^2S \mid bc$,

find $L(G)$.

Sol.: [redacted]

Let $L \ = \{a^{2n}bc \mid n \ \in \ \mathbb{N}\}$

? $L \ = \ L(G)$

(1) ? $L \subseteq L(G)$ (all sequences of that shape are generated by G)

? $\forall\, n \in \mathbb{N},\ a^{2n}bc \in L(G)$

Take P(n): $a^{2n}bc \in L(G)$ and prove P(n) true, $\forall\, n \in \mathbb{N}$
We'll prove by mathematical induction

(a) Verification step: ? P(0): $a^0bc \in L(G)$ is true

$$S \underset{(2)}{\Rightarrow} bc = a^0bc \Rightarrow \text{P(0) true}$$

(b) Proof step: We suppose P(k) is true and then prove that P(k+1) is also true, where $k \in \mathbb{N}$

$$\text{P(k) true} \Rightarrow a^{2k}bc \in L(G) \Rightarrow S \overset{*}{\Rightarrow} a^{2k}bc \text{ (induction hypothesis)}$$

$$S \underset{(1)}{\Rightarrow} a^2S \underset{\text{(ind. hypo.)}}{\overset{*}{\Rightarrow}} a^2a^{2k}bc = a^{2(k+1)}bc$$

$$\Rightarrow S \overset{*}{\Rightarrow} a^{2(k+1)}bc \Rightarrow \text{P(k+1) is true}$$

(a) + (b) => (1)

(2) ? $L \supseteq L(G)$ (G generates only sequences of that shape)

$$\begin{aligned} S &\Rightarrow bc = a^0bc \\ &\Rightarrow a^2S \Rightarrow a^2bc \\ &\qquad \Rightarrow a^4S \Rightarrow a^4bc \\ &\qquad\qquad \Rightarrow a^6S \Rightarrow \ldots \end{aligned}$$

We notice that starting from S and using all grammar productions in all possible combinations, we only get, as sequences of terminals, sequences of the shape $a^{2n}bc$ where $n \in \mathbb{N}$. It follows that the grammar doesn't generate anything else.

Obs.: This inclusion may also be discharged by induction.

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3. Find a grammar that generates $L = \{0^n 1^n 2^m \mid n, m \in \mathbb{N}^*\}$

Sol.: [redacted]

$G = (N,\ \Sigma,\ P,\ S)$
$N = \{S,\ V,\ C\}$
$\Sigma = \{0,\ 1,\ 2\}$
P : S -> VC
V -> 0V1 | 01
C -> 2 | 2C

(1) ? $L \subseteq L(G)$
? $\forall\ n,\ m \in N^*,\ 0^n 1^n 2^m \in L(G)$

Let $n,\ m \in N^*$

$$S \underset{(1)}{\Rightarrow} VC \underset{(a)}{\overset{n}{\Rightarrow}} 0^n 1^n C \underset{(b)}{\overset{m}{\Rightarrow}} 0^n 1^n 2^m \quad \Rightarrow \quad S \overset{*}{\Rightarrow} 0^n 1^n 2^m \quad \Rightarrow \quad 0^n 1^n 2^m \in L(G)$$

(a) $V \overset{n}{\Rightarrow} 0^n 1^n,\ \forall\ n \in N^*$

(b) $C \overset{m}{\Rightarrow} 2^m,\ \forall\ m \in N^*$

HW: Prove (a) and (b) above by induction
Justify the reverse inclusion