

Lecture 14

30.05.2022

Exam 2021

2. (1p) Using the integrating factor method find (an integral representation of) the solution of the IVP $\begin{cases} x' + tx = -1 \\ x(0) = 0 \end{cases}$

3. (2.75p) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 -function s.t. $f(-5) < 0$, $f(0) = 1$, $f(5) < 0$

a) (0.5p) Justify that the dynamical system $\dot{x} = f(x)$ does not have a global repeller equilibrium point.

b) (0.25p) Give a simple example of such function f .

c) (1.25p) Consider f from b). Represent the phase portrait in $\mathbb{R}^2 \setminus \{(0,0)\}$ of the system given in polar coordinates by $\dot{r} = f(r)$, $\dot{\theta} = 3$

d) Transform in cartesian coordinates the system from c).

4. (2p) Find the values of $h > 0$ such that the attractor equilibrium point of $\dot{x} = -3x^2 + 6x + 24$, is also an attractor fixed point of the discrete dynamical system associated to the Euler's numerical formula with stepsize $h > 0$ for the given differential equation.

4.

$$\dot{x} = -3x^2 + 6x + 24$$

$$f(x) = -3x^2 + 6x + 24$$

$$-3x^2 + 6x + 24 = 0 \quad | : (-3)$$

$$x^2 - 2x - 8 = 0$$

$$x_1 = -2$$

$$x_2 = 4$$

$$f'(x) = -6x + 6$$

$$f'(-2) = 12 + 6 = 18 > 0 \Rightarrow -2 \text{ is a repeller}$$

$$f'(4) = -24 + 6 = -18 < 0 \Rightarrow 4 \text{ is an attractor}$$

Euler's formula

$$y'(x) = f(x, y(x)), \quad x \in [x_0, x^*]$$

$$\left[\begin{array}{c} | \quad | \\ x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m = x^* \end{array} \right]$$

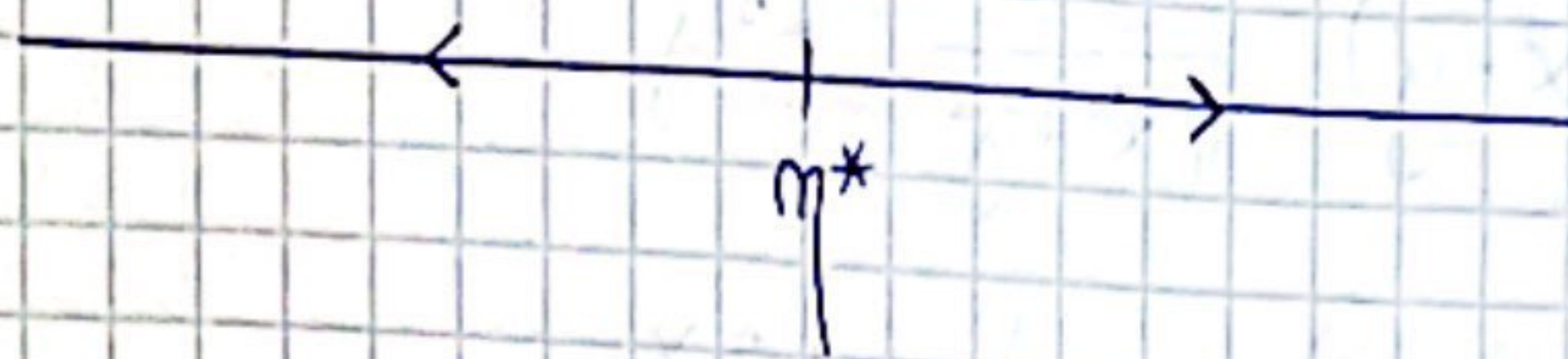
$$x_{k+1} = x_k + h$$

$$x_k = x_0 + kh, \quad k = \overline{0, m}$$

$$y_{k+1} = y_k + h f(x_k, y_k)$$

$$x_{k+1} = x_k + h(-3x_k^2 + 6x_k + 24)$$

3. a)



$\exists \eta_1^*$ and η_2^* equilibrium points

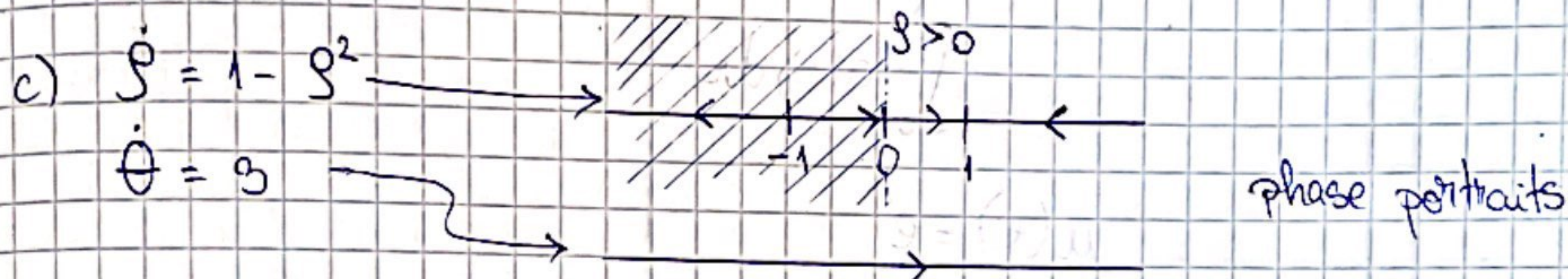
$$\eta_1^* \in (-5; 0)$$

$$\eta_2^* \in (0, 5)$$

$$\lim_{t \rightarrow -\infty} \varphi(t, \eta) = \eta_1^*, \quad \forall \eta \in \mathbb{R}$$

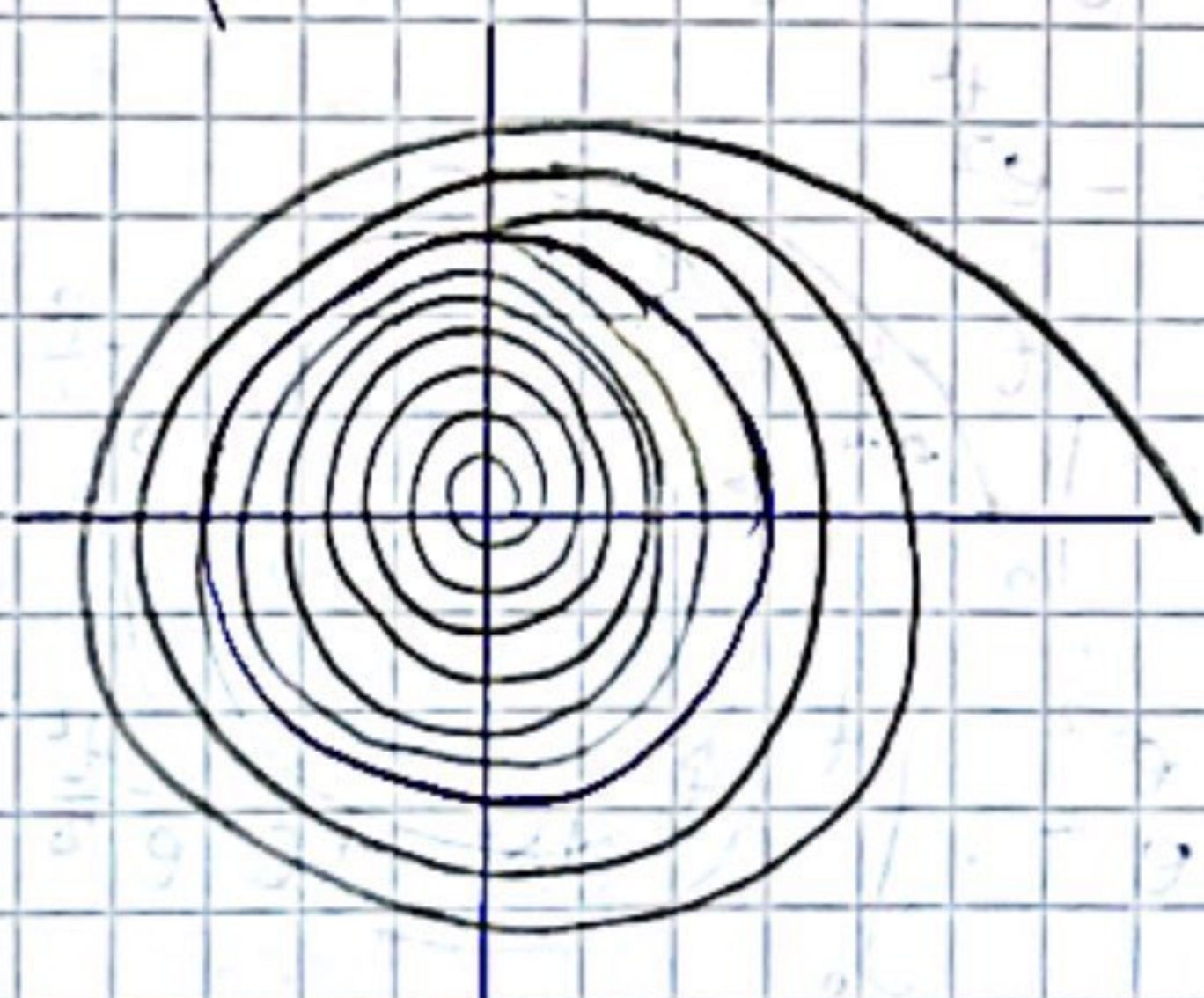
$$\lim_{t \rightarrow -\infty} \varphi(t, \eta_2^*) = \eta_1^* \Rightarrow \eta_2^* = \eta_1^* \text{ - contradiction}$$

b) $f(x) = 1 - x^2$



$\dot{\theta} > 0 \Rightarrow \forall$ orbit rotates anticlockwise around the origin

$\rho = 1 \Rightarrow$ correspond to an unit circle (radius = 1; center in origin)



$$d) \begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{\rho} \cos \theta - \rho \sin \theta \cdot \dot{\theta} \\ \dot{y} = \dot{\rho} \sin \theta + \rho \cos \theta \cdot \dot{\theta} \end{cases} ; \rho = x^2 + y^2$$

$$\Rightarrow \begin{cases} \dot{x} = (1 - \rho^2) \cos \theta - 3\rho \sin \theta \\ \dot{y} = (1 - \rho^2) \sin \theta + 3\rho \cos \theta \end{cases}$$

$$\begin{cases} \dot{x} = (1-x^2-y^2) \cdot \frac{x}{\sqrt{x^2+y^2}} - 3y \\ \dot{y} = (1-x^2-y^2) \cdot \frac{y}{\sqrt{x^2+y^2}} + 3x \end{cases}$$

2. $[x(t) \cdot \mu(t)]' = x'(t) \cdot \mu(t) + x(t) \cdot \mu'(t)$

$$x' \cdot e^{t^2} + x \cdot t \cdot e^{t^2} = (x \cdot e^{t^2})' \quad \text{False}$$

$$x' \cdot e^{\frac{t^2}{2}} + x \cdot t \cdot e^{\frac{t^2}{2}} = (x \cdot e^{\frac{t^2}{2}})' \quad \text{True}$$

$\mu(t) = e^{\frac{t^2}{2}}$ is an integrating factor

Theory

$$x' + a(t)x$$

$$= \int_0^s a(s) ds$$

$$\mu(t) = e^{\int a(t) dt}$$

$$x' + tx = -1 \quad | \cdot e^{\frac{t^2}{2}}$$

first order lin. nonhom. d.e. with nonconstant coeff.

$$x' \cdot e^{\frac{t^2}{2}} + t \cdot x \cdot e^{\frac{t^2}{2}} = -e^{\frac{t^2}{2}}$$

$$(x \cdot e^{\frac{t^2}{2}})' = -e^{\frac{t^2}{2}}$$

$$x \cdot e^{\frac{t^2}{2}} = -\int_0^t e^{\frac{s^2}{2}} ds + c \quad | \cdot e^{-\frac{t^2}{2}}, c \in \mathbb{R}$$

$$x = -e^{-\frac{t^2}{2}} \cdot \int_0^t e^{\frac{s^2}{2}} ds + c \cdot e^{-\frac{t^2}{2}}$$

$$x(0) = c$$

$$x(0) = 0$$

$$\Rightarrow c = 0$$

$$\varphi(t) = -e^{-\frac{t^2}{2}} \cdot \int_0^t e^{\frac{s^2}{2}} ds$$

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

a) Find the principal matrix sol. of $\dot{X} = AX$.

b) Compute e^{At}

c) Find $a, b \in \mathbb{R}$ s.t. $H(x, y) = x^2 + ay^2 + bxy$ is a global first integral of $\dot{X} = AX$.

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{X} = AX \Leftrightarrow \begin{cases} \dot{x} = 2x - 5y \\ \dot{y} = x - 2y \end{cases}$$

solutions

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

matrix sol.

$$(ivp_1) \begin{cases} \dot{X} = AX \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

$$(ivp_2) \begin{cases} \dot{X} = AX \\ x(0) = 0 \\ y(0) = 1 \end{cases}$$

$H: \mathbb{R}^2 \rightarrow \mathbb{R}$, C^1 -function

H is f.i. of $\dot{X} = AX$ iff $\frac{\partial H}{\partial x} \cdot (2x - 5y) + \frac{\partial H}{\partial y} \cdot (x - 2y) = 0$,
 $\forall (x, y) \in \mathbb{R}^2$

$$(2x + by)(2x - 5y) + (2ay + bx)(x - 2y) = 0$$

$$4x^2 + \underline{2bxy} - \underline{10xy} - 5by^2 + \underline{2axy} + \underline{bx^2} - 4ay^2 - \underline{2bxy} = 0$$

$$\begin{cases} 4 + b = 0 \end{cases}$$

$$\begin{cases} 2b - 10 + 2a - 2b = 0 \end{cases}$$

$$\begin{cases} -5b - 4a = 0 \end{cases}$$

$$b = -4, \quad a = 5$$