Method II HW

24.03.20

Seminar 3 - W5

$\lambda > 0$

$x'' - 4x = e^{\lambda t}$

i) If $\lambda \neq 2 \Rightarrow$ find a solution of the form $x_p(t) = a \cdot e^{\lambda t}$ $(a = ?)$

$x_p'(t) = a \cdot \lambda \cdot e^{\lambda t}$

$x_p''(t) = a \cdot \lambda^2 \cdot e^{\lambda t}$

$a \cdot \lambda^2 \cdot e^{\lambda t} - 4 \cdot a \cdot e^{\lambda t} = e^{\lambda t}$

$(a \cdot \lambda^2 - 4a) e^{\lambda t} = e^{\lambda t} \mid : e^{\lambda t} \ (e^{\lambda t} \neq 0)$

$a(\lambda^2 - 4) = 1$

$a = \frac{1}{\lambda^2 - 4}$

$x_p(t) = \frac{1}{\lambda^2 - 4} \cdot e^{\lambda t}$

ii) If $\lambda = 2$ Find a solution of the form

$x_p(t) = a \cdot t \cdot e^{2t}$

$x_p'(t) = a \cdot e^{2t} + 2a \cdot t e^{2t}$

$x_p''(t) = 2a e^{2t} + 2a \cdot e^{2t} + 4a \cdot t e^{2t}$

$x_p''(t) = 4a(e^{2t} + t \cdot e^{2t})$

$x_p''(t) - 4x_p(t) = 4ae^{2t} + \cancel{4a \cdot t \cdot e^{2t}} - \cancel{4a \cdot t \cdot e^{2t}} = 4ae^{2t} = e^{2t}$

$a = \frac{1}{4} \Rightarrow x_p = \frac{1}{4} \cdot t \cdot e^{2t}$

$$x'' - 4x = 0$$

$$r^2 - 4 = 0 \Rightarrow r = \pm 2 \to e^{-2t}, e^{2t}$$

$$x_h(t) = c_1 \cdot e^{-2t} + c_2 \cdot e^{2t}, c_i \in \mathbb{R}$$

$\lambda \neq 2$

$$x(t) = x_h(t) + x_p(t) = c_1 \cdot e^{-2t} + c_2 \cdot e^{2t} + \frac{1}{\lambda^2 - 4} \cdot e^{\lambda t}$$

$\lambda = 2$

$$x(t) = c_1 \cdot e^{-2t} + c_2 e^{2t} + \frac{t}{4} \cdot e^{2t}, c_i \in \mathbb{R}$$

$r_{1,2} = 2$ multiplicity 2

$$c_1 \cdot e^{2t} + c_2 \cdot t \cdot e^{2t}$$

2) $x'' + x = \cos(\lambda t)$

$$r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i$$

$$\downarrow$$

$\sin t, \cos t$ $\quad\quad$ $x_h(t) = c_1 \cdot \sin t + c_2 \cdot \cos t$

$\lambda \neq 1$

$$x_p(t) = a \cdot \cos \lambda t + b \cdot \sin \lambda t, a, b \in \mathbb{R}$$

$\lambda = 1$

$$x_p(t) = t(a \cdot \cos t + b \cdot \sin t)$$

$r = i$ mult. 2

$t \cdot \sin t, t \cos t$

$x'' + x = 2$

$x_p = 2$

$x'' + x = t^3$

$x_p = a \cdot t^3 + b \cdot t^2 + ct + d + et^4$

$x' + a(t) \cdot x = f(t) \quad , \quad t \in I$

$x' + \frac{1}{t} \cdot x = \frac{1}{t} \cdot e^{-2t+1} \quad , \quad t \in (0, \infty)$

$A(t) = -\int_{t_0}^{t} a(s) ds$

$A'(t) = a(t)$

$A'(t_0) = 0$

$x_p = \varphi(t) \cdot e^{A(t)}$

$x(t) = x_h(t) + x_p(t)$

$x_h : \quad x' + \frac{1}{t} x = 0$

$\frac{dx}{dt} + \frac{x}{t} = 0 \Rightarrow \frac{dx}{dt} = -\frac{x}{t} \Rightarrow \frac{dx}{x} = -\frac{dt}{t} \; \Big| \int$

$\int \frac{dx}{x} = -\int \frac{dt}{t} \Rightarrow \boxed{\ln|x| = -\ln t} + \mathscr{C}$

$\Rightarrow |x| = e^{-\ln t + \mathscr{C}} = e^{\mathscr{C}} \cdot \frac{1}{t}$

$\Rightarrow x_h = \begin{cases} \pm e^{c} \cdot \frac{1}{t} \\ 0 \end{cases} \Rightarrow x_h(t) = a \cdot \frac{1}{t}$

Guess one non-zero solution

$$\ln|x| = \underbrace{-\ln t}_{= \ln\frac{1}{t}} + C \Rightarrow x(t) = \frac{1}{t}$$

$\frac{1}{t}$ is a solution $\Rightarrow x_h(t) = c_1 \cdot \frac{1}{t}$

$x_p = ?$ Lagrange method $\quad x_p = \varphi(t) \cdot e^{A(t)}$

$$A(t) = -\ln t \Rightarrow x_p = \varphi(t) \cdot \frac{1}{t} \Rightarrow x_p' + \frac{1}{t} x_p =$$

$$x_p' + \frac{1}{t} x_p = \varphi'(t) \cancel{\frac{1}{t}} + \cancel{\varphi(t)\left(-\frac{1}{t^2}\right)} + \cancel{\frac{1}{t} \cdot \varphi(t) \cdot \frac{1}{t}} = \cancel{\frac{1}{t}} \cdot e^{-2t+1}$$

$$\Rightarrow \varphi'(t) = e^{-2t+1} \Rightarrow \varphi(t) = -\frac{1}{2} e^{-2t+1}$$

$$\Rightarrow x_\varphi(t) = -\frac{1}{2} e^{-2t+1} \cdot \frac{1}{t}$$

$$x(t) = x_p(t) + x_h(t) = -\frac{1}{2} e^{-2t+1} \cdot \frac{1}{t} + c_1 \cdot \frac{1}{t}$$