Seminar 7

26.05.2022.

Linear difference equations with constant coefficients

$$x_{k+1} - a x_k = f_k \ (1), \quad a \in \mathbb{R}^*, \ (f_k) \subseteq \mathbb{R}$$

Example: $x_{k+1} - a \cdot x_k = 0 \ (2) \qquad x_0 = c, c \in \mathbb{R}$

$$x_{k+1} = a \cdot x_k$$

$$x_1 = a \cdot c$$

$$x_2 = a^2 \cdot c$$

$$\dots$$

$$x_k = a^k \cdot c, \ c \in \mathbb{R}$$

<u>Theorem</u>: Let x^h be the general solution of (2) and x^p is a particular solution of (1).

Then the general solution of (1) is $\underline{x = x^h + x^p}$

<u>Proof</u>: $x^h_{k+1} - a \cdot x^h_k = 0, \ \forall k \geq 0$

$$x^p_{k+1} - a x^p_k = f_k, \ \forall k \geq 0$$

$$(1) \Rightarrow \underbrace{(x^h_{k+1} + x^p_{k+1})}_{x_{k+1}} - a \underbrace{(x^h_k + x^p_k)}_{x_k} =$$

$$= \underbrace{(x^h_{k+1} - a \cdot x^h_k)}_{0} + \underbrace{(x^p_{k+1} - a \cdot x^p_k)}_{f_k} = f_k$$

<u>Theorem</u>: For any solution x of (1) we have that $\underbrace{x - x^p}_{x^h}$ is a solution of (2)

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1) Find a particular solution of the form $x_k^p = a\cdot(-5)^k$ $(a=?)$

of $x_{k+1} + 2x_k = (-5)^{k+1}$ $(*)$

Find the general solution of $(*)$ and of the IVP:

$$\begin{cases} x_{k+1} + 2x_k = (-5)^{k+1} \\ x_0 = 0 \end{cases}$$

$$a\cdot(-5)^{k+1} + 2a\cdot(-5)^k = (-5)^{k+1} \quad | \ (-5)^k$$

$$a\cdot(-5) + 2a = -5$$

$$-3a = -5$$

$$a = \frac{5}{3} \Rightarrow x_k^p = \frac{5}{3}(-5)^k$$

$$x_{k+1} + 2x_k = 0$$

$$x_0 = c \Rightarrow x_1 = -2x_0$$

$$x_2 = 4x_0$$

$$x_k^h = (-2)^k\cdot c, \ c\in\mathbb{R}$$

$$\Rightarrow x_k = (-2)^k\cdot c + \frac{5}{3}(-5)^k \quad \text{gen. sol.}$$

$$k=0 \Rightarrow x_0 = c + \frac{5}{3} = 0 \Rightarrow c = -\frac{5}{3}$$

unique sol. is $(-2)^k\cdot\left(-\frac{5}{3}\right) + \frac{5}{3}\cdot(-5)^k$

char. equation method

$$x_{k+2} + a\cdot x_{k+1} + b\cdot x_k = 0, \ k\geq 0 \quad a,b\in\mathbb{R}, \ b\neq 0$$

Let $r\in\mathbb{R}^*$ and look for solutions of the form r^k.

$$\Rightarrow r^{k+2} + a\cdot r^{k+1} + b\cdot r^k = 0 \quad | : r^k \neq 0$$

$$r^2 + a\cdot r + b = 0$$

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Step 1: Write the char. eq. $r^2 + ar + b = 0$
Find the solutions $r_{1,2} \in \mathbb{C}$.

Step 2: We have two sequences

a) $r_{1,2} \in \mathbb{R}$ $\quad r_1 \neq r_2 \longmapsto r_1^k$ and r_2^k

b) $r_{1,2} \in \mathbb{R}$ $\quad r_1 = r_2 \longmapsto r_1^k$ and $k \cdot r_1^k$

c) $r_{1,2} \in \mathbb{C}$ $\quad r_{1,2} = \alpha \pm i\beta$, $\beta \neq 0 \to Re((\alpha+i\beta)^k)$
$\to Im((\alpha+i\beta)^k)$

Theorem: The solutions are lin. independent.

Theorem: If x^1 and x^2 are lin. indep. solutions then the general solution is $c_1 \cdot x^1 + c_2 \cdot x^2$

1) Find the general solution of $x_{k+2} - 2x_{k+1} + x_k = 0$, $k \geq 0$

char. eq.: $r^2 - 2r + 1 = 0 \iff (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1$

$\to \underbrace{1^k}_{1}$ and $\underbrace{k \cdot 1^k}_{k}$ sol. $\Rightarrow$ gen sol. $x_k = c_1 + c_2 \cdot k$
$c_1, c_2 \in \mathbb{R}$

2) Find the solution of the IVP.

$$\begin{cases} x_{k+2} - x_{k+1} - x_k = 0 & \text{(Fibonacci sequence)} \\ x_0 = 0 \\ x_1 = 1 \end{cases}$$

$r^2 - r - 1 = 0$

$\Delta = 1 + 4 = 5 \to r_1 = \frac{1+\sqrt{5}}{2} \in \mathbb{R}$

$r_2 = \frac{1-\sqrt{5}}{2} \in \mathbb{R}$

$\Rightarrow$ the sol. are $\left(\frac{1+\sqrt{5}}{2}\right)^k$ and $\left(\frac{1-\sqrt{5}}{2}\right)^k$

$\Rightarrow$ The gen. sol. x_k is $c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^k + c_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^k$

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$x_0 = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$x_1 = c_1 \cdot \frac{1+\sqrt{5}}{2} + c_2 \cdot \frac{1-\sqrt{5}}{2} = 1 \Rightarrow -c_2 \cdot \frac{1+\sqrt{5}}{2} + c_2 \cdot \frac{1-\sqrt{5}}{2} = 1$

$-\sqrt{5} c_2 = 1$

$c_2 = -\frac{1}{\sqrt{5}} \Rightarrow c_1 = \frac{1}{\sqrt{5}}$

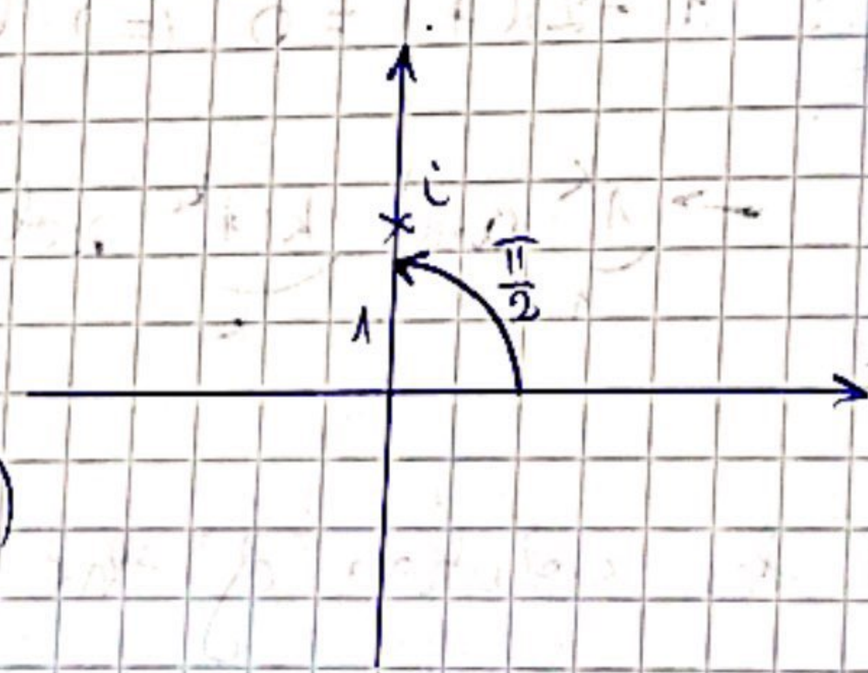
$$x_k = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^k$$

3) Find the general solution of $x_{k+2} + x_k = 0$, $k \geq 0$

$r^2 + 1 = 0$

$r^2 = -1 \Rightarrow r_{1,2} = \pm i \longrightarrow Re(i^k)$

$Jm(i^k)$



$i = r(\cos\varphi + i \sin\varphi)$

$i = \cos\frac{\pi}{2} + i \cdot \sin\frac{\pi}{2}$

$i^k = \cos k\frac{\pi}{2} + i \sin k \cdot \frac{\pi}{2}$

$Re(i^k) = \cos k \cdot \frac{\pi}{2}$

$Jm(i^k) = \sin k \cdot \frac{\pi}{2}$

The gen. sol.: $x_k = c_1 \cdot \cos k \cdot \frac{\pi}{2} + c_2 \cdot \sin k \cdot \frac{\pi}{2}$, $c_1, c_2 \in \mathbb{R}$

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5) Find the LHDifference E with CC (constant coefficients) of minimal order with solution x_k.

a) $x_k = 7 \cdot \underbrace{\frac{1}{2^k}}_{\downarrow} + 3 \cdot \underbrace{\frac{1}{3^k}}_{\downarrow}$, $k \geq 0$ $\qquad r^k$

$r_1 = \frac{1}{2} \qquad r_2 = \frac{1}{3}$

char. eq.: $\left(r - \frac{1}{2}\right)\left(r - \frac{1}{3}\right) = 0$

$$r^2 - \frac{5}{6} r + \frac{1}{6} = 0$$

$$x_{k+2} - \frac{5}{6} x_{k+1} + \frac{1}{6} x_k = 0$$

$f'(\underbrace{\sqrt{2}}_{\uparrow}) = 5 > 0 \Rightarrow \sqrt{2}$ repeller

b) $Re(i^k) \quad Im(i^k)$

$r_{1,2} = \pm i \Rightarrow$ char. eq.: $r^2 + 1 = 0$

$$x_{k+2} + x_k = 0$$

c) $x_0 = 1$, $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = -1$, $x_5 = 5$

Can it be of degree 1?

$$x_{k+1} - a x_k = 0$$

$\underbrace{x_1}_{0} - a \cdot \underbrace{x_0}_{1} = 0 \Rightarrow a = 0 \Rightarrow x_{k+1} = 0, \forall k \geq 0$

$x_3 = 1 \neq 0$

$\Downarrow$

not of degree 1

$x_2 - x_1 =$

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Degree 2

$$x_{k+2} + ax_{k+1} + bx_k = 0$$

$$x_2 = -a\cancel{x_1} - bx_0 = -bx_0 = 0 \Rightarrow b = 0$$

$$x_3 = -a\underbrace{x_2}_{0} = 0 \text{ but } x_3 = 1 \Rightarrow \text{not of degree } 2$$

Degree 3

$$x_{k+3} + ax_{k+2} + bx_{k+1} + cx_k = 0$$

$$x_3 + ax_2 + bx_1 + cx_0 = 0$$

$$1 + c = 0$$

$$c = -1$$

$$x_4 + ax_3 + \cancel{bx_2} - \cancel{x_1} = 0$$

$$-1 + a = 0$$

$$a = 1$$

$$x_5 + x_4 + bx_3 - x_2 = 0$$

$$5 - 1 + b = 0$$

$$b = -4$$

$$\Rightarrow \text{DE}: \quad x_{k+3} + x_{k+2} - 4x_{k+1} - x_k = 0$$

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