$X'' + X = 0$

- linear combination

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Lecture 5

Linear differential systems

Let $n \in \mathbb{N}^*$, $A \in C(I, \mathcal{M}_n(\mathbb{R}))$ where $I \subset \mathbb{R}$ open interval, $F \in C(I, \mathbb{R}^n)$.

We consider the system

(1) $X' = A(t)X + F(t)$

this is a linear non-hom (when $F \not\equiv 0$) differential system in $\mathbb{R}^n$.

Here $A(t)$ is the coefficients matrix, $F(t)$ is the non-hom. part of (1), while $X' = A(t)X$ is

Now we write the system by components

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \cdots & & & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \quad F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \cdots \\ f_n(t) \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

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$$(1) \iff \begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + f_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + f_2(t) \\ \dots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + f_n(t) \end{cases}$$

<u>Def 2</u>. A <u>matrix solution</u> of $X' = A(t)X$ is a function $U : I \to \mathcal{M}_n(\mathbb{R})$ of class C^1 s.t. $U'(t) = A(t)U(t)$, $\forall t \in I$

<u>Property</u>: A matrix function $U : I \to \mathcal{M}_n(\mathbb{R})$, of class C^1, is a matrix solution of $X' = A(t)X$ if and only if each column of $U(t)$ is a solution of $X' = A(t)X$.

<u>Theorem 1</u> (The existence and uniqueness theorem for the IVP).
Let $t_0 \in I$ and $\eta \in \mathbb{R}^n$. The IVP $\begin{cases} X' = A(t)X + F(t) \\ X(t_0) = \eta \end{cases}$ has a unique solution

<u>Definition</u> The matrix function $E : \mathbb{R} \to \mathcal{M}_n(\mathbb{R})$ which is the unique matrix function such that $\begin{cases} E'(t) = A\,E(t) \\ E(0) = I_n \end{cases}$ is said to be the exponential of the matrix (tA) and is denoted by $E(t) = e^{tA}$ or $E(t) = \exp(tA)$.
Here $A \in \mathcal{M}_n(\mathbb{R})$ and $I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{pmatrix}$ is the identity matrix.

<u>Property</u> Let $A \in \mathcal{M}_n(\mathbb{R})$. The series

$$I_n + A + \frac{1}{2!}A^2 + \dots + \frac{1}{k!}A^k + \dots$$ is convergent to e^A.

Thus, $\boxed{e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k}$ (an equivalent definition of the matrix

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Examples: Compute $e^{t\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}}$ where $\lambda_1, \lambda_2 \in \mathbb{R}$.

in two ways

Sol $E(t) = e^{tA}$, where $A\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$. $X = \begin{pmatrix}x_1\\ x_2\end{pmatrix}$

First we use the definition of e^{tA}.

Step 1: we write the LHS with matrix A

$$X' = AX \iff \begin{cases} x_1' = \lambda_1 x_1 \\ x_2' = \lambda_2 x_2 \end{cases}$$

this is an uncoupled system, so it is easy to solve it.

Step 2: we formulate 2 IVP's

$$(2)\begin{cases} x_1' = \lambda_1 x_1 \\ x_2' = \lambda_2 x_2 \\ x_1(0) = 1 \\ x_2(0) = 0 \end{cases}$$

has the unique solution $\varphi(t) = \begin{pmatrix} e^{\lambda_1 t} \\ 0 \end{pmatrix}$

$$(3)\begin{cases} x_1' = \lambda_1 x_1 \\ x_2' = \lambda_2 x_2 \\ x_1(0) = 0 \\ x_2(0) = 1 \end{cases}$$

has the unique solution $\psi(t) = \begin{pmatrix} 0 \\ e^{\lambda_2 t} \end{pmatrix}$

Step 3: $E(t) = \begin{pmatrix} \text{Sol. of } (2) & \text{Sol. of } (3) \end{pmatrix} = \begin{pmatrix} \varphi(t) & \psi(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$

$$x_1' = \lambda_1 x_1 \iff x_1' - \lambda_1 x_1 = 0$$

$$r - \lambda_1 = 0$$

$$r = \lambda_1 \longmapsto e^{\lambda_1 t} \Rightarrow \begin{cases} x_1 = c_1 e^{\lambda_1 t}, & c_1 \in \mathbb{R} \\ x_2 = c_2 e^{\lambda_2 t}, & c_2 \in \mathbb{R} \end{cases}$$

Conclusion: $e^{t\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}, \forall t \in \mathbb{R}$

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Now we use the equivalent definition with Taylor series.

$$e^{tA} = I_2 + tA + \frac{1}{2!}(tA)^2 + \dots + \frac{1}{k!}(tA)^k + \dots =$$

$$= I_2 + tA + \frac{t^2}{2!}A^2 + \dots + \frac{t^k}{k!}A^k + \dots$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad A^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

By induction one can prove that $A^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}, \forall k \in \mathbb{N}$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda_1 t)^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda_2 t)^k \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

<u>Linear homogeneous systems of dimension 2 with constant coefficients solved using the reduction method.</u>

$$(4)\begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})$$

we assume that (4) is a coupled system, i.e. either $a_{12} \neq 0$ or $a_{21} \neq 0$.

To fix, consider that $a_{12} \neq 0$. Our aim is to find a second order LDE for x, i.e. a relation between x, x' and x''.

(Comment: if $a_{12} = 0$ but $a_{21} \neq 0$ we find a relation between y, y', y'')

$$x' = a_{11}x + a_{12}y \Rightarrow y = \frac{1}{a_{12}}(x' - a_{11}x) \quad (5)$$

$$x' = a_{11}x + a_{12}y \Rightarrow x'' = a_{11}x' + a_{12}y' \Rightarrow x'' = a_{11}x' + a_{12}(a_{21}x + a_{22}y)$$

$y' = a_{21}x + a_{22}y$

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$$\Rightarrow x'' = a_{11}x' + a_{12}a_{21}x + a_{22}(x' - a_{11}x)$$
(5)

$$\Rightarrow x'' - (a_{11} + a_{22})x' + (a_{11}a_{22} - a_{12}a_{21})x = 0 \quad (6)$$ this is a second order LHDE with constant coefficients

that can be solved using the characteristic equation method

<u>Remarks</u>: The characteristic polynomial of (6) is

$$\ell(r) = r^2 - \underbrace{(a_{11} + a_{22})}_{\operatorname{tr} A} r + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{\det A}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Note that this is the characteristic polynomial of A.

Recall that the roots of $\ell(r)$ are the eigenvalues of A.

- Recall the definition of an eigenvalue and an eigenvector of a matrix $A \in \mathcal{M}_n(\mathbb{R})$:

we say that $\lambda \in \mathbb{C}$ is an eigenvalue of the matrix A when $\exists\, u \in \mathbb{R}^n$, $u \neq 0$ such that $Au = \lambda u$. This u is said to be an eigenvector of A corresp. to the eigenvalue λ.

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Example: Compute in two ways $e^{t\begin{pmatrix}0&1\\-1&0\end{pmatrix}}$, $t\in\mathbb{R}$

Method I. $A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ the LHDS having A as the coeff. matrix is:

Step 1:

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

this is a coupled system. We use the reduction method to find the general sol.

$$x'' = y' \Rightarrow x'' = -x \Rightarrow x''+x=0$$

$$r^2+1=0$$

$$r^2=-1$$

$$r_{1,2}=\pm i \longmapsto \cos t,\ \sin t$$

$$(\alpha\pm i\beta \longmapsto e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t)$$

$$\begin{cases} x = c_1\cdot\cos t + c_2\sin t, & c_1,c_2\in\mathbb{R} \\ y = x' \Rightarrow y = -c_1\sin t + c_2\cos t, & c_1,c_2\in\mathbb{R} \end{cases}$$

Step 2

$$\begin{cases} x' = y \\ y' = -x \\ x(0)=1 \\ y(0)=0 \end{cases}$$

$x = c_1\cos t + c_2\sin t$ $\quad x(0)=c_1$

$y = -c_1\sin t + c_2\cos t$ $\quad x(0)=c_2$

$\Rightarrow c_1=1$ and $c_2=0$

the unique solution of this IVP is

$$\varphi(t)=\begin{pmatrix}\cos t\\-\sin t\end{pmatrix}$$

$$\begin{cases} x' = y \\ y' = -x \\ x(0)=0 \\ y(0)=1 \end{cases}$$

$c_1=0$, $c_2=1$

the unique sol. of this IVP is

$$\psi(t)=\begin{pmatrix}\sin t\\\cos t\end{pmatrix}$$

Step 3: $e^{t\begin{pmatrix}0&1\\-1&0\end{pmatrix}} = \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$, $\forall t\in\mathbb{R}$

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