Lecture 3

Linear homogeneous differential equations with constant coefficients (continue)

Given the coefficients $a_1, a_2, \dots, a_n \in \mathbb{R}$, with $n \in \mathbb{N}^*$, we consider the CHDE with CC

(1) $x^{(n)} + a_1 \cdot x^{(n-1)} + \dots + a_{n-1} \cdot x' + a_n \cdot x = 0$

Looking for sol. $x = e^{rt}$, $\boxed{r \in \mathbb{R}}$! we found that

(2) $r^n + a_1 r^{n-1} + \dots + a_{n-1} \cdot r + a_n = 0$ (called the characteristic eq. of (1))

it is evident that for $r \in \mathbb{R}$ a root of (2) we have that e^{rt} is a solution of (1).

Q1 we want to justify now why for the roots of (2) $r_{1,2} = \alpha \pm i\beta$ (with $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$) we have that $e^{\alpha t} \cos \beta t$ and $e^{\alpha t} \sin \beta t$ are solutions of (1)

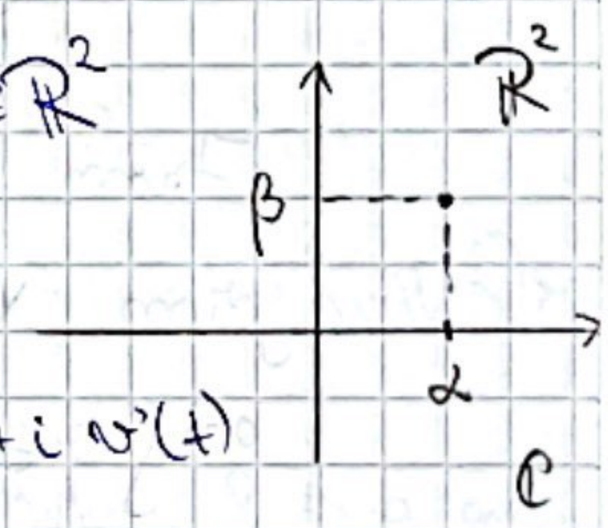
A1 The complex exponential: $e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i \sin\beta)$, $\forall \alpha, \beta \in \mathbb{R}$ (Euler's formula)

A complex function of real variable: $\gamma : I \to \mathbb{C}$, where $I \subset \mathbb{R}$ interval

$\gamma(t) = u(t) + i v(t)$, $u(t) \overset{nd}{=} \mathrm{Re}(\gamma(t))$, $v(t) \overset{not}{=} \mathrm{Im}(\gamma(t))$

we have $u, v : I \to \mathbb{R}$ (real functions)

$\mathbb{C} \approx \mathbb{R}^2$ $\quad \alpha + i\beta \in \mathbb{C} \leftrightarrow (\alpha, \beta) \in \mathbb{R}^2$



When $u, v \in C^1(I)$ we have that $\gamma'(t) = u'(t) + i v'(t)$

$\gamma' : I \to \mathbb{C}$

Proposition 1 Let $u, v \in C^n(I)$. If $\gamma : I \to \mathbb{C}$, $\gamma = u + iv$, satisfies eq. (1) then u and v are solutions of eq. (1) (this prop. is valid also for LHDE with varing coefficients)

Scanned with CamScanner

Proof $\mathcal{L}(x)(t) = x^{(n)}(t) + a_1 \cdot x^{(n-1)}(t) + \ldots + a_n \cdot x(t), \forall t \in \mathbb{R}$

$$\left.\begin{array}{l} \mathcal{L}(\gamma) = \mathcal{L}(u+iv) \underset{\mathcal{L} \text{ is linear}}{=} \underbrace{\mathcal{L}(u)}_{\text{real}} + i\underbrace{\mathcal{L}(v)}_{\text{real}} \Rightarrow \begin{array}{l} \mathcal{L}(u) = Re(\mathcal{L}(\gamma)) \\ \mathcal{L}(v) = Im(\mathcal{L}(\gamma)) \end{array} \\ \text{ip is that } \gamma \text{ satisfies eq (1)} \Rightarrow \mathcal{L}(\gamma) = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} \mathcal{L}(u) = 0 \\ \mathcal{L}(v) = 0 \end{array}\right\} \Rightarrow u \text{ and } v \text{ are solutions of (1)}$$

(recall that u and v are real functions and $u, v \in C^n(\mathbb{R}, \mathbb{R})$)

Proposition 2 a) $e^{rt} = e^{\alpha t}\cos\beta t + ie^{\alpha t} \cdot \sin\beta t, \forall t \in \mathbb{R}$

b) $(e^{rt})' = re^{rt}, \forall t \in \mathbb{R}$

Proof a) Use Euler Formula. $u(t) \overset{not}{=} e^{\alpha t}\cos\beta t$, $v(t) = e^{\alpha t} \cdot \sin\beta t$

Note that $u, v \in C^\infty(\mathbb{R})$.

b) $$\left.\begin{array}{l} e^{rt} = u(t) + iv(t), \forall t \in \mathbb{R} \Rightarrow (e^{rt})' = u'(t) + iv'(t) \\ u'(t) = \alpha e^{\alpha t}\cos\beta t - \beta e^{\alpha t} \cdot \sin\beta t \\ v'(t) = \alpha \cdot e^{\alpha t}\sin\beta t + \beta e^{\alpha t}\cos\beta t \end{array}\right\} \Rightarrow$$

$$\Rightarrow (e^{rt})' = \alpha \cdot e^{\alpha t} + i\beta \cdot e^{rt} = (\alpha + i\beta)e^{rt} = re^{rt} \quad \square$$

From here we have: If we look for a solution of (1) of the form $x = e^{rt}$, with $r \in \mathbb{C}$, then r must satisfy eq. (2). Hence any $r \in \mathbb{C}$ root of (2) gives e^{rt}, a complex function, that satisfies eq. (1)

Now we apply Prop 1 and Prop 2 a) and obtain that:

for $r = \alpha + i\beta$, $\beta \neq 0$ root of (2) we get $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$ solutions of (1).

Scanned with CamScanner

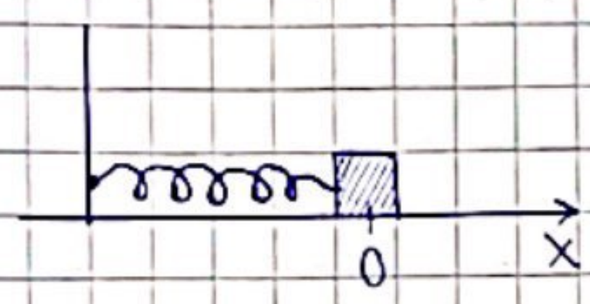
Note that eq. (2) has real coefficients, thus if $r_1 = \alpha + i\beta$, $\beta \neq 0$ is a root of (2) then $r_2 = \alpha - i\beta$ is also a root of (2).

$$e^{(\alpha - i\beta)t} = e^{\alpha t} \cdot \cos(-\beta t) + i e^{\alpha t} \sin(-\beta t) = \underbrace{e^{\alpha t} \cos \beta t}_{Re} + i \cdot [\underbrace{-e^{\alpha t} \sin \beta t}_{Im}]$$

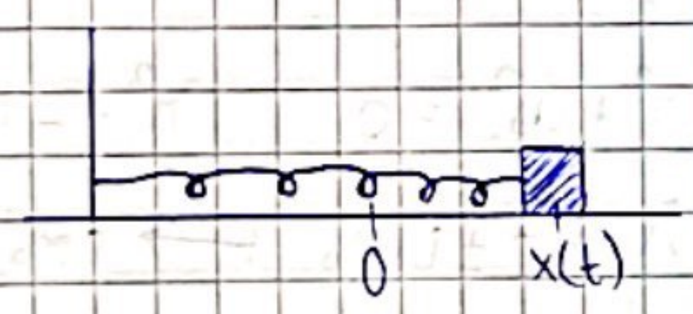
$\Rightarrow$ $e^{\alpha t} \cos \beta t$ and $-e^{\alpha t} \sin \beta t$ are solutions

Nothing new!

An application : <u>the spring-mass system</u>



this is the situation when the system is at an equilibrium position (there is no movement)



- $x(t)$ measures the displacement of the mass from the equilibrium position $x = 0$
- $x'(t)$ is the velocity, while $x''(t)$ is the acceleration

Newton's second law : $F = m \cdot a$ $\quad m > 0 \quad a = x''(t)$

The forces : - the elastic (or restoring) force : $F_R = -k x(t)$ with $k > 0$ (Hooke's law)

- the friction (damping) force : $F_d = -\nu \cdot x'(t)$ with $\nu > 0$
- an external force : $F_{ext} = m f(t)$ where f is a given function

Then $F = F_R + F_d + F_{ext} = -kx - \nu x' + m f(t)$
$F = ma$
$\Rightarrow -kx - \nu x' + m f(t) = m x''$

$$\Rightarrow \boxed{x'' + \frac{\nu}{m} x' + \frac{k}{m} x = f(t)}$$

second order (linear) Non-HDE with constant coefficients $\frac{\nu}{m} > 0$, $\frac{k}{m} > 0$

Scanned with CamScanner

we consider the following cases:

Case 1: Motion without damping, without external force

$$x'' + \frac{k}{m} x = 0$$

Case 2: Motion with damping, without external force

$$x' + \frac{D}{m} x' + \frac{k}{m} x = 0$$

Case 3: Motion without damping, with external force

$$x'' + \frac{k}{m} x = A \cos \omega t$$, where $A > 0$ and $\omega > 0$ parameters

---

<u>Case 1</u>: $\omega_0^2 = \frac{k}{m}$ $\quad \omega_0 > 0$

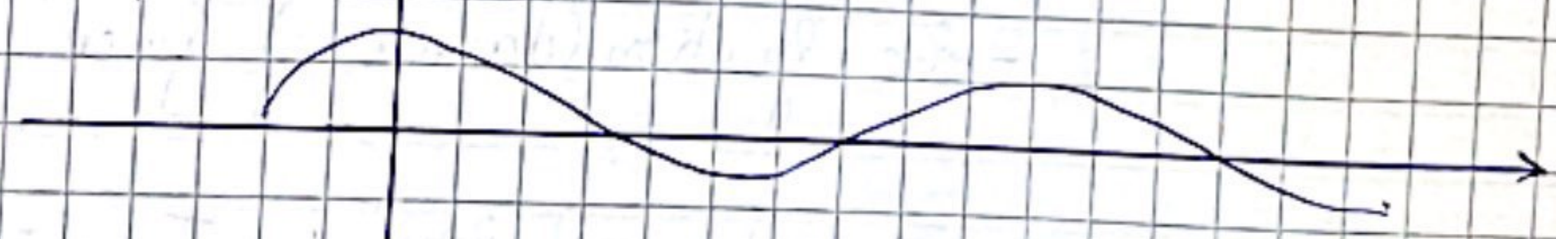
$$x'' + \omega_0^2 x = 0 \qquad r^2 + \omega_0^2 = 0 \qquad r^2 = -\omega_0^2$$

$$r_{1,2} = \pm i \omega_0 \longmapsto \cos \omega_0 t, \sin \omega_0 t$$

$$x = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \quad c_1, c_2 \in \mathbb{R} \text{ arbitrary.}$$

Any function like this is periodic with minimal period $\frac{2\pi}{\omega_0}$, it ~~as~~ oscillates around $x = 0$ with constant amplitude is bounded.

perpetuum mobile
(no realistic)



Scanned with CamScanner

Case 2: $x'' + \frac{\vartheta}{m} x' + \frac{k}{m} x = 0$

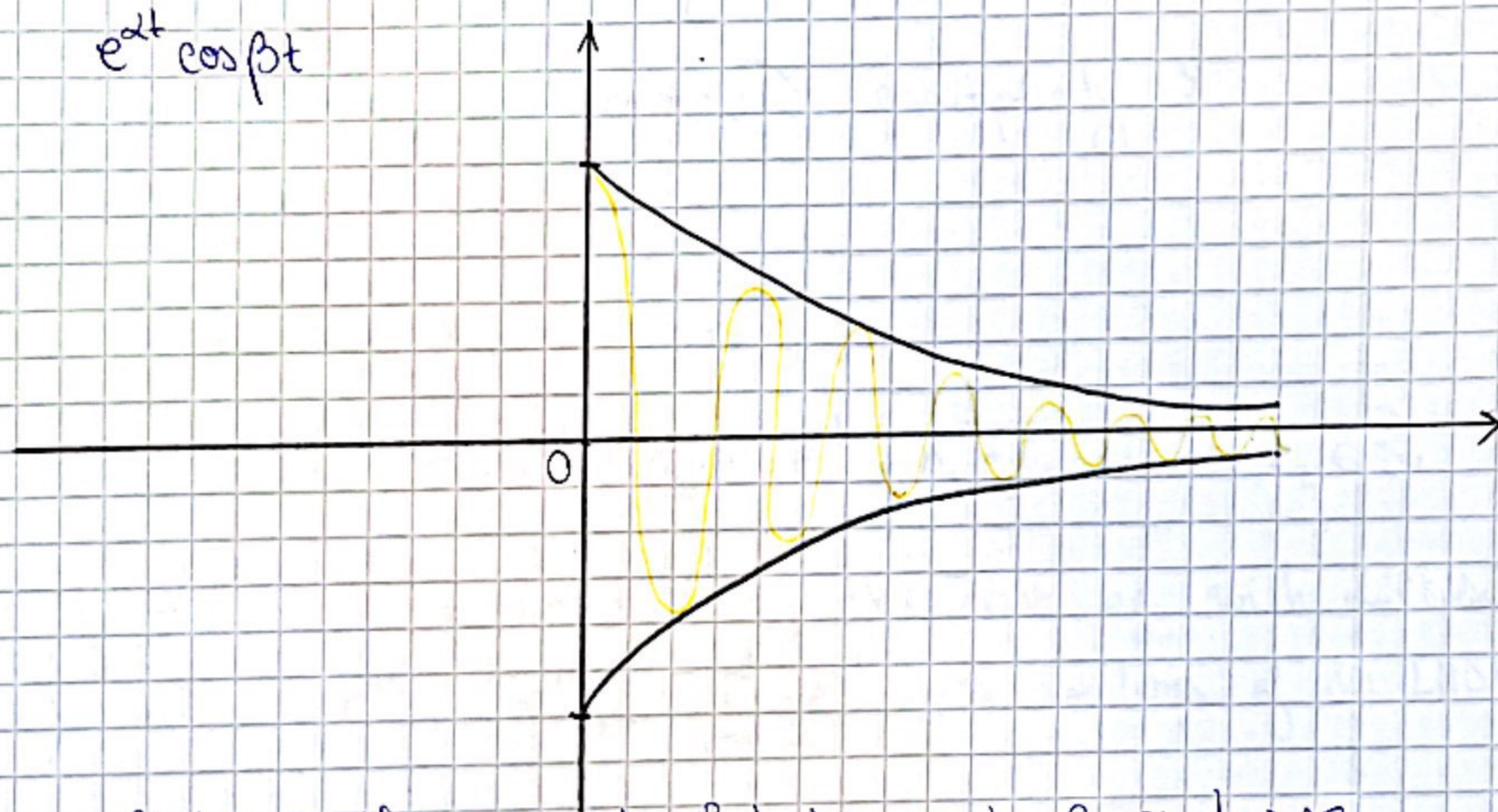
$$r^2 + \frac{\vartheta}{m} r + \frac{k}{m} = 0 \qquad \Delta = \frac{\vartheta^2}{m^2} - 4\frac{k}{m} = \frac{\vartheta^2 - 4km}{m^2}$$

Case 2.1. $\vartheta^2 < 4km$ (the damping coeff. is smaller than a critical value $\sqrt{4km}$)

$$r_{1,2} = \frac{-\frac{\vartheta}{m} \pm i\frac{\sqrt{4km-\vartheta^2}}{m}}{2} = \underbrace{-\frac{\vartheta}{2m}}_{\alpha} \pm i \underbrace{\frac{\sqrt{4km-\vartheta^2}}{2m}}_{\beta} \longmapsto e^{\alpha t}\cos\beta t,\ e^{\alpha t}\cdot\sin\beta t$$

$$x = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t, \quad c_1, c_2 \in \mathbb{R}$$

$e^{\alpha t}\cos\beta t$



oscillates with amplitude that decreases to 0 as $t \to \infty$

(the damping coeff is bigger than a critical value)

Case 2.2. $\vartheta^2 > 4km$ $\quad r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2 \longmapsto e^{r_1 t}, e^{r_2 t}$

$$r_1 + r_2 = -\frac{\vartheta}{m} < 0 \qquad r_1 \cdot r_2 = \frac{k}{m} > 0 \implies r_1 < 0 \text{ and } r_2 < 0$$

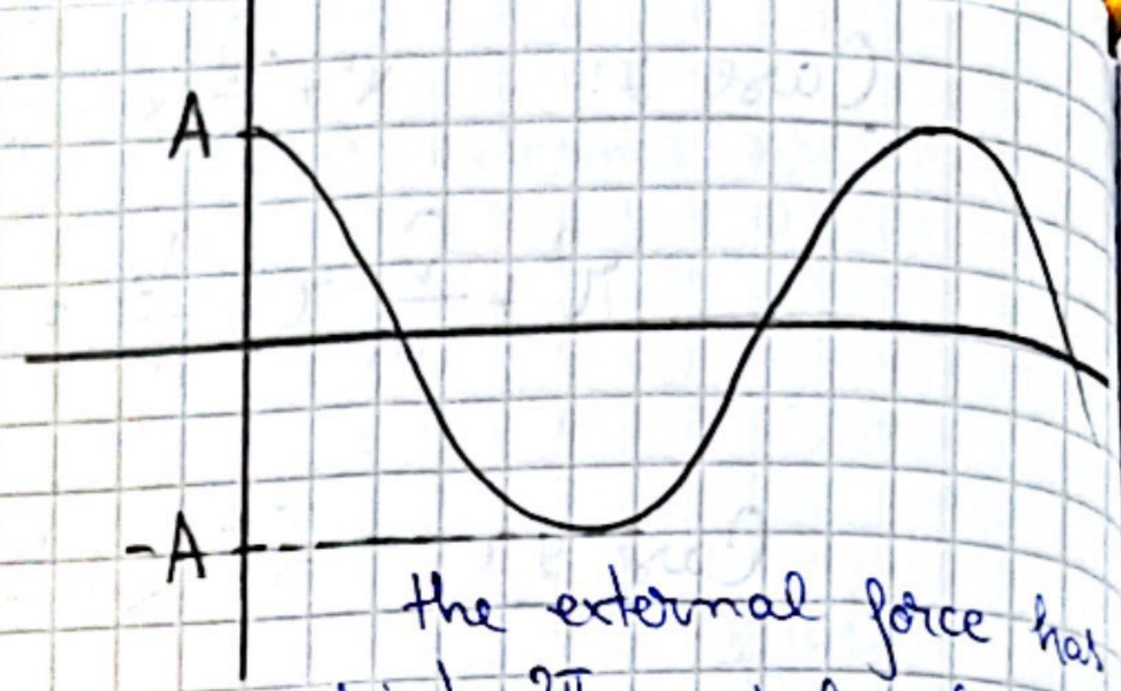
$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad c_1, c_2 \in \mathbb{R}$$

any such function decreases exponentially to 0 and will not oscillate around 0.

Scanned with CamScanner

Short resumtation

Case 3: $x'' + \omega_0^2 x = A\cos\omega t$



the external force has period $\frac{2\pi}{\omega}$ and has frequency ω

$$x'' + \omega_0^2 x = 0$$

$$x = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$$

$$x_p = ?$$

- $\omega \neq \omega_0$

  $x_p = a\cos\omega t + b\sin\omega t$, $a, b \in \mathbb{R}$ $a = ?$, $b = ?$

- $\omega = \omega_0$ $x_p = t(a\cos\omega t + b\sin\omega t)$

Questions: $\omega \neq \omega_0$ is any solution periodic?

$\omega = \omega_0$ <u>any solution is unbounded</u> <u>resonance</u>

Tacoma bridge disaster

Scanned with CamScanner