Seminar 6 - W11

1) $\dot{X} = AX \qquad A = \begin{pmatrix} -10 & 4 \\ m & 10 \end{pmatrix} \qquad m \in \mathbb{R} \setminus \{-25\}$

$\det A \neq 0$

i) Study the type and stability of the equilibrium point $(0,0)$

$$\det(A - \lambda I_2) = \begin{vmatrix} -10-\lambda & 4 \\ m & 10-\lambda \end{vmatrix} = -(10+\lambda)(10-\lambda) - 4m$$
$$= -10^2 + \lambda^2 - 4m$$

$$\det(A - \lambda I_2) = 0 \iff \lambda^2 - 4m - 100 = 0$$
$$\lambda^2 = 4m + 100$$
$$\lambda_{1,2} = \pm\sqrt{4(m+25)}$$

I. $4m + 100 > 0$

$m > -25$

$\Rightarrow \lambda_{1,2} = \pm\sqrt{4(m+25)} = \pm 2\sqrt{m+25}$

$\Rightarrow (0,0)$ saddle unstable

II. $m < -25$

$\lambda_{1,2} = \pm i\sqrt{-4m-100} \Rightarrow$ center stable

1) $\pm 2i$

$-1 \pm 2i$

ii) Find $a=?$ such that $H: \mathbb{R}^2 \to \mathbb{R}$ $H(x,y) = ax^2 - 2y^2 + 10xy$ is a global first integral of the system.

$$\frac{dy}{dx}$$

$$\begin{cases} \dot{x} = \underbrace{-10x + 4y}_{f_1} \\ \dot{y} = \underbrace{mx + 10y}_{f_2} \end{cases}$$

$$f_1 \cdot \frac{\partial H}{\partial x}(x,y) + f_2 \frac{\partial H}{\partial y}(x,y) = 0$$

$$(-10x + 4y)(2ax + 10y) + (mx + 10y)(-4y + 10x) = 0$$

$$(10x - 4y)\left[(-2ax - 10y) + mx + 10y\right] = 0 \quad , \forall (x,y) \in \mathbb{R}^2, A^2$$

$$(10x - 4y)(mx - 2ax) = 0 \quad , \forall (x,y) \in \mathbb{R}^2$$

$$mx - 2ax = 0$$

$$(m - 2a)x = 0 \; ; x, \forall x \in \mathbb{R}$$

$$m - 2a = 0$$

$$a = \frac{m}{2}$$

$$H(x,y) = \frac{m}{2}x^2 - 2y^2 + 10xy$$

$$U \subseteq \mathbb{R}^2$$

iii) Find $m = ?$ such that $\varphi(t,1,3) = (\underbrace{\underbrace{\cos 2t + \sin 2t}_{x(t)}}_{\text{flow}}, \underbrace{3\cos 2t + 2\sin 2t}_{y(t)}), \forall t \in \mathbb{R}$

$m \in \mathbb{R}$

$$\begin{cases} \dot{x} = -10x + 4y \\ \dot{y} = mx + 10y \\ x(0) = 1 \\ y(0) = 3 \end{cases}$$

$\dot{x} = x'(t)$

$$\begin{cases} x(t) = \cos 2t + \sin 2t \\ y(t) = 3\cos 2t + 2\sin 2t \end{cases}$$

$$\begin{cases} \dot{x} = -2\sin 2t + 2\cos 2t \\ \dot{y} = -6\sin 2t + 4\sin t \end{cases}$$

$$\dot{x} = -10(\cos 2t + \sin 2t) + 4(3\cos 2t + 2\sin 2t)$$

$$= -2\sin 2t + 2\cos 2t$$

$$\dot{y} = m(\cos 2t + \sin 2t) + 10(3\cos 2t + 2\sin 2t)$$

$$= \underbrace{(m+20)}_{-6}\sin 2t + \underbrace{(m+30)}_{4}\cos 2t$$

$$\begin{cases} m+20 = -6 \\ m+30 = 4 \end{cases} \rightarrow \begin{matrix} m = -26 \\ m = -26 \end{matrix}$$