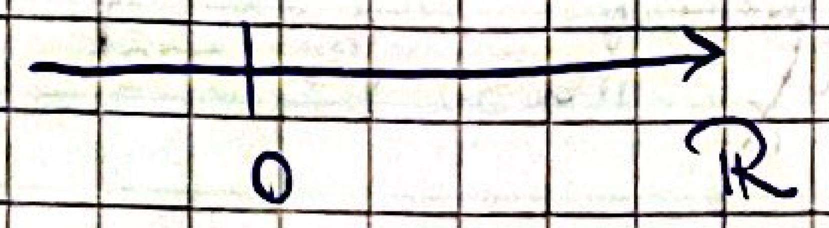
# Dynamical Systems

Many laws of nature can be stated as dynamical systems, more precisely using either differential equations or difference equations.

I. $x : \mathbb{R} \to \mathbb{R}^n$ $\quad t \longmapsto x(t)$ $\quad t$ is the time, a continuous time

$x(t)$ describes the state of a phenomenon that changes in time




• A differential equation is a relation between $x(t)$ and its derivatives $x'(t)$, $x''(t)$, ... up to some order

In mechanics $x(t)$ is the position of a material point, $x'(t)$ is its velocity, $x''(t)$ is the acceleration.

In chemistry, $x(t)$ can be the quantity or the concentration of a chemical substance, $x'(t)$ is the rate of change.

In population dynamics $x(t)$ is the number or the density of individuals in a population (people, animals, fish...) and $x'(t)$ is the rate of growth.

II. $x : \mathbb{N} \to \mathbb{R}$ $\quad k \longmapsto x(k) = x_k$ $\quad (x_k)_{k \geq 0}$ is a sequence

$k$ is the time, a discrete time

• A difference equation is a relation between $x_k, x_{k+1}, x_{k+2}, \ldots$

There are two main parts of dynamical systems theory
I. Continuous dynamical systems
II. Discrete dynamical systems

The aim of this theory is to make „predictions" for the evolution of a phenomenon when one knows the initial state and some "laws" of "nature".
Mathematically this means that either we intend to find the expression of $x(t)$ or $(x_k)_{k\geq 0}$ or, if this is not possible, to be able to establish its properties.

A general form of a differential equation is ~~$F(t, x(t),$~~
$F(t, x(t), x'(t), \dots, x^{(n)}(t)) = 0$

1. Linear differential equations of order $n$, $n \in \mathbb{N}^*$

(1) $x^{(n)} + a_1(t)\, x^{(n-1)} + \dots + a_{n-1}(t)\, x' + a_n(t)\, x = f(t)$

where $a_1, \dots, a_n \in C(I)$
$a_1, \dots, a_n : I \to \mathbb{R}$ continuous called the coefficients are given, known
$I \subset \mathbb{R}$ nonempty open interval
$f : I \to \mathbb{R}$ continuous (called the force or the nonhomogeneous part of (1)). is given

The unknown is $x : I \to \mathbb{R}$ a scalar function

Definition: A solution of (1) is a function $\varphi : I \to \mathbb{R}$ which is $n$ times differentiable and $\varphi^{(n)} : I \to \mathbb{R}$ is continuous such that
$$\varphi^{(n)}(t) + a_1(t)\varphi^{(n-1)}(t) + \dots + a_n(t)\varphi(t) = f(t), \ \forall t \in I$$

Notations: Let $I \subset \mathbb{R}$ interval

$$C(I) = \{ f : I \to \mathbb{R} \text{ continuous} \}$$

$$C^n(I) = \{ f : I \to \mathbb{R} \text{ s.t. } \exists f, f', f'', \dots, f^{(n)} \text{ continuous on } I \}$$

So $a_1, a_2, \dots, a_n, f \in C(I)$ and a solution $\varphi \in C^n(I)$.

We consider the addition between two functions

Let $f_1, f_2 \in C(I)$, $f_1 + f_2 \in C(I)$ $\quad (f_1+f_2)(t) = f_1(t) + f_2(t)$ $\forall t \in I$

$\alpha \in \mathbb{R}$ (scalar) $\quad \alpha \cdot f \in C(I) \quad (\alpha f)(t) = \alpha \cdot f(t), \forall t \in I$

<u>$C(I)$ is a vector space</u>

$C^n(I)$ is a vector space.

why (1) is called linear?

For each $x \in C^n(I)$ we define the function $\mathcal{L}x : I \to \mathbb{R}$,

$$\mathcal{L}x(t) \overset{\text{def}}{=} x^{(n)}(t) + a_1(t) x^{(n-1)}(t) + \dots + a_n(t) x(t), \ \forall t \in I$$

Moreover, we define now $\mathcal{L} : C^n(I) \to C(I)$

$$x \longmapsto \mathcal{L}x$$

So, $\mathcal{L}$ is a map between two vector spaces.

known

<u>Proposition</u> $\mathcal{L}$ is a linear map, i.e.

$$\cancel{\mathcal{L}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathcal{L}(f_1) + \alpha_2 \mathcal{L}(f_2)}$$

$$\mathcal{L}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \mathcal{L}(x_1) + \alpha_2 \mathcal{L}(x_2), \ \forall x_1, x_2 \in C^n(I)$$
$$\forall \alpha_1, \alpha_2 \in \mathbb{R}$$

<u>Proof</u>: Let $t \in I$, ~~let~~

$$\mathcal{L}(\alpha_1 x_1 + \alpha_2 x_2)(t) = (\alpha_1 x_1 + \alpha_2 x_2)^{n}(t) + a_1(t)(\alpha_1 x_1 + \alpha_2 x_2)^{(n-1)} + \dots + a_n(t)(\alpha_1 x_1 + \alpha_2 x_2)(t)$$

$$= \alpha_1 x_1^{(n)}(t) + \alpha_2 (x_2^{(n)}(t) + a_1(t)\alpha_1 \cdot x_1^{(n-1)}(t) + a_1(t) \cdot \alpha_2 \cdot x_2^{(n-1)} + \dots + a_n(t)\alpha_1 x_1(t)$$

$x_2(t)$

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$= \alpha_1 \cdot \mathcal{L} x_1(t) + \alpha_2 \cdot \mathcal{L} x_2(t)$.

with this notation, eq (1) can be written as

$$(2) \quad \mathcal{L} x = f \quad \text{(equation with } x \text{ as unknown)}$$

A particular case of (2) is when $f = 0$, i.e.

$$(3) \quad \mathcal{L} x = 0$$ This is called linear homogeneous differential equation

The set of solutions of (3) is (called, from linear algebra course) the kernel of $\mathcal{L}$, denoted $\operatorname{Ker} \mathcal{L}$.

when $f \neq 0$ eq (2) is called linear nonhomogeneous.

Examples. 1) $x + 2x' = 0$ first order differential eq.

$x' + \frac{1}{2} x = 0$ linear homogeneous, with constant coeff.

$x = 0$ is a solution (in fact $x = 0$ is a solution of any linear hom. dif. eq.)

$x = -2$ "$(-2)' + \frac{1}{2} \cdot (-2) = 0$" $\Leftrightarrow$ "$-1 = 0$" F

Thus $x = -2$ is not a solution

$x = a$ $(a \in \mathbb{R})$ check "$\frac{a}{2} = 0$" True $\Leftrightarrow a = 0$

So, $x = 0$ is the only constant solution of $x' + \frac{1}{2} x = 0$

Try $x = at + b$ $\quad x' = a$ check "$a + \frac{1}{2}(at + b) = 0 \mid \cdot 2 \ \forall t \in \mathbb{R}$"

$\Leftrightarrow$ "$at + (2a + b) = 0 \ \forall t \in \mathbb{R}$"

$$\Leftrightarrow \begin{cases} a = 0 \\ 2a + b = 0 \end{cases} \Leftrightarrow a = b = 0$$

Try $x = e^{-\frac{t}{2}}$

$x' = -\frac{1}{2}e^{-\frac{t}{2}}$

check " $-\frac{1}{2}e^{-\frac{t}{2}} + \frac{1}{2}e^{-\frac{t}{2}} = 0, \forall t \in \mathbb{R}$ " TRUE

So, $x = e^{-\frac{t}{2}}$ is another solution. In addition, $x = ce^{-\frac{t}{2}}$, $c \in \mathbb{R}$ arbitrary is also a solution

2) $x'' + t x' = 3$ second order linear nonhomogeneous diff. eq.

coeff is not constant

nonhom. part

3) $x \cdot x' = 0$ first order nonlinear differential eq.

## Consequences of the linearity of $\mathcal{L}$ (The linearity principle or The superposition principle)

C1) $\mathcal{L}(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k) = \alpha_1 \mathcal{L}(x_1) + \alpha_2 \mathcal{L}(x_2) + \ldots + \alpha_k \mathcal{L}(x_k)$

$\forall x_1, \ldots, x_k \in C^n(I), \forall \alpha_1, \ldots, \alpha_k \in \mathbb{R}, k \geq 1, k \in \mathbb{N}$

Proof by induction

C2) If $x_1, \ldots, x_k$ are solutions of the LHDE $\mathcal{L}(x) = 0$ then $\forall \alpha_1, \ldots, \alpha_k \in \mathbb{R}$ we have that $\alpha_1 x_1 + \ldots + \alpha_k x_k$ is another solution of $\mathcal{L}(x) = 0$.

C3) Let $f_1, \ldots, f_k \in C(I)$, let $x_i$ be a sol. of $\mathcal{L}(x) = f_i$, $i = \overline{1,k}$.

Let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Then for the eq. $\mathcal{L}(x) = \alpha_1 f_1 + \ldots + \alpha_k f_k$ we have the solution $\alpha_1 x_1 + \ldots + \alpha_k x_k$.

For the proof of (2) and (3) we apply (1).

$x' + \frac{1}{2}x = 0 \Big| \cdot e^{\frac{t}{2}}$ we know $x = c \cdot e^{-\frac{t}{2}}$ is a solution $\forall c$

this is an integrating factor for this equation

$$x' \cdot e^{\frac{t}{2}} + x \cdot \underbrace{\frac{1}{2} e^{\frac{t}{2}}}_{(e^{\frac{t}{2}})'} = 0 \quad \forall t \in \mathbb{R} \iff \left(x \cdot e^{\frac{t}{2}}\right)' = 0 \iff$$

$$x \cdot e^{\frac{t}{2}} = c \Big|, c \in \mathbb{R} \; \Big| \cdot e^{-\frac{t}{2}} \iff$$

$$\boxed{x = c \cdot e^{-\frac{t}{2}}, c \in \mathbb{R}}$$

Hence

There are no other solutions.

$x = c \cdot e^{-\frac{t}{2}}, c \in \mathbb{R}$ is the general solution of ord. d. e

S3, S6 - tests

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