Boolean functions ex 2.2

Author: Cojocaru Catalin-Victor

Teoretical Results

- Monom – conjunction of variables
- Minterm – monom that contains all n variables, it also takes value 1 for only one argument

Definition 1: Let $f:(B_2)^n \to B_2$ be a Boolean function of n variables.

The set $S_f = \{(x_1, x_2, \ldots, x_n) \mid f(x_1, x_2, \ldots, x_n) = 1\}$, containing all the groups $(x_1, x_2, \ldots, x_n) \in B_2^n$ for which f takes the value 1, is called the support of f.

Definition 2: The monom m is smaller or equal than the monom m' if and only if the support of m is included or equal to the support of m': $m \le m' \Leftrightarrow S_m \subseteq S_{m'}$

For a Boolean function of n variables let us consider the monoms:

$$m = x_{k_1}^{\alpha_{k_1}} \wedge \dots \wedge x_{k_j}^{\alpha_{k_j}} \wedge x_{k_i} \wedge x_{k_l}^{\alpha_{k_l}} \wedge \dots \wedge x_{k_s}^{\alpha_{k_s}} \text{ and}$$

$$m' = x_{k_1}^{\alpha_{k_1}} \wedge \dots \wedge x_{k_j}^{\alpha_{k_j}} \wedge \overline{x_{k_i}} \wedge x_{k_l}^{\alpha_{k_l}} \wedge \dots \wedge x_{k_s}^{\alpha_{k_s}} \text{, where:}$$

$k_1, \dots, k_j, k_i, k_l, \dots, k_s \in \{1,2,\dots,n\}$ and $k_1 < \dots < k_j < k_i < k_l < \dots < k_s$,

1. m and m' are called ***adjacent (neighbor) monoms*** because they differ only by the power of the variable with the index "k_i".
   Adjacency (neighborhood) relation is defined by a single variable change.
2. *the factorization of the monoms m and m'* is:
   $m \vee m' = x_{k_1}^{\alpha_{k_1}} \wedge \dots \wedge x_{k_j}^{\alpha_{k_j}} \wedge x_{k_l}^{\alpha_{k_l}} \wedge \dots \wedge x_{k_s}^{\alpha_{k_s}}$ obtained by eliminating the variable "k_i".

Definition 5: The set M(*f*) is called the *set of maximal monoms* for the function $f:(B_2)^n \to B_2$ if:

1. $\forall m \in M(f), m \in FB(n), m \leq f$ and
2. $\forall m \in M(f), \nexists m' \in FB(n)$ such that $m < m' \leq f$.

The maximal monoms are minterms or monoms obtained by using factorization.

Definition 6: The set C(*f*) is called the *set of central monoms* for the function $f:(B_2)^n \to B_2$ if:

1. $\forall m \in C(f), m \in M(f)$ and
2. $\forall m \in C(f), m \not\leq \vee_{m' \in M(f)-\{m\}} m'$)

Teoretical Results

Simplification Algorithm

<u>Input data</u>: *f – a Boolean function in disjunctive canonical form*

<u>Output data</u>: *$f'_1, f'_2, \ldots, f'_k$ all simplified forms of f*

<u>Step 1</u>: Compute $M(f)$ and $C(f)$

<u>Step 2</u>: If $M(f) = C(f)$ then $f' = \bigvee_{m \in M(f)} m$, <u>STOP1</u> // case1--- one solution

else

If $C(f) \neq \varnothing$ then // there exist central monoms

$g = \bigvee_{m \in C(f)} m$

$f_i' = g \vee h_i$, $i = \overline{1,k}$, h_i is a disjunction of a minimum number of maximal monoms such that $S_{h_i} = S_f - S_g$

<u>STOP2</u> // case2 --- k solutions

else // there are no central monoms

$f_i' = h_i$, $i = \overline{1,k}$, h_i is a disjunction of a minumum number of maximal monoms such that $S_{h_i} = S_f$

<u>STOP3</u> // case3 --- k solutions

End_if

End_if

End_algorithm

The <u>simplification process</u> is formalized by the steps below.

The steps 2, 3, 4 are specific to the applied simplification method.

1. **The initial function f is transformed into DCF(f).**
2. **Factorization process => the set of maximal monoms $M(f)$.**
3. **From the set of maximal monoms the central monoms are selected => $C(f)$**
4. **The case of the simplification algorithm (presented below) is identified and all simplified forms are obtained.**

Teoretical Results

DCF form

$$f_2 = \overset{m_{15}}{x_1 x_2 x_3 x_4} \vee \overset{m_{14}}{x_1 x_2 x_3 \bar{x}_4} \vee \overset{m_6}{\bar{x}_1 x_2 x_3 \bar{x}_4} \vee \overset{m_8}{x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4} \vee$$
$$\vee \overset{m_0}{\bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4} \vee \overset{m_2}{\bar{x}_1 \bar{x}_2 x_3 \bar{x}_4} \vee \overset{m_9}{x_1 \bar{x}_2 \bar{x}_3 x_4} \vee \overset{m_1}{\bar{x}_1 \bar{x}_2 \bar{x}_3 x_4} \vee \overset{m_3}{\bar{x}_1 \bar{x}_2 x_3 x_4}$$

$$= m_{15} \vee m_{14} \vee m_6 \vee m_8 \vee m_0 \vee m_2 \vee m_9 \vee m_1 \vee m_3$$

$x_1 x_2 \setminus x_3 x_4$	00	01	11	10
00	m_0	m_1	m_3	m_2
01				m_6
11			m_{15}	m_{14}
10	m_8	m_9		

$m_0 \vee m_1 \vee m_3 \vee m_2 = \bar{x}_1 \bar{x}_2 = max_1$

$m_0 \vee m_1 \vee m_8 \vee m_9 = \bar{x}_2 \bar{x}_3 = max_2$

$m_2 \vee m_6 = \bar{x}_1 x_3 \bar{x}_4 = max_3$

$m_6 \vee m_{14} = \cancel{m}\, x_2 x_3 \bar{x}_4 = max_4$

$m_{15} \vee m_{14} = \cancel{x}\, x_1 x_2 x_3 = max_5$

$M(f) = \{ max_1, max_2, max_3, max_4, max_5 \}$

$C(f) = \{ max_1, max_2, max_5 \}$ - Central monoms

Let $g = max_1 \vee max_2 \vee max_5 = \overline{x_1 x_2} \vee \overline{x_2 x_3} \vee x_1 x_2 x_3$

m_6 - not covered minterm, can be covered using max_3 or max_4

So the simplest forms are:

$$f_1^s(x_1, x_2, x_3, x_4) = g \vee max_3 = \overline{x_1 x_2} \vee \overline{x_2 x_3} \vee x_1 x_2 x_3 \vee \bar{x}_1 x_3 \bar{x}_4$$

$$f_2^s(x_1, x_2, x_3, x_4) = g \vee max_4 = \overline{x_1 x_2} \vee \overline{x_2 x_3} \vee x_1 x_2 x_3 \vee x_2 x_3 \bar{x}_4$$

DCF form

$$f_2 = \overset{m_{15}}{x_1 x_2 x_3 x_4} \vee \overset{m_{14}}{x_1 x_2 x_3 \bar{x}_4} \vee \overset{m_6}{\bar{x}_1 x_2 x_3 \bar{x}_4} \vee \overset{m_8}{x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4} \vee$$
$$\vee \overset{m_0}{\bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4} \vee \overset{m_2}{\bar{x}_1 \bar{x}_2 x_3 \bar{x}_4} \vee \overset{m_9}{x_1 \bar{x}_2 \bar{x}_3 x_4} \vee \overset{m_1}{\bar{x}_1 \bar{x}_2 \bar{x}_3 x_4} \vee \overset{m_3}{\bar{x}_1 \bar{x}_2 x_3 x_4}$$

$$= m_{15} \vee m_{14} \vee m_6 \vee m_8 \vee m_0 \vee m_2 \vee m_9 \vee m_1 \vee m_3$$

$x_1x_2 \backslash x_3x_4$	00	01	11	10
00	m_0	m_1	m_3	m_2
01				m_6
11			m_{15}	m_{14}
10	m_8	m_9		

$$m_0 \vee m_1 \vee m_3 \vee m_2 = \bar{x}_1 \bar{x}_2 = max_1$$

$$m_0 \vee m_1 \vee m_8 \vee m_9 = \bar{x}_2 \bar{x}_3 = max_2$$

$$m_2 \vee m_6 = \bar{x}_1 x_3 \bar{x}_4 = max_3$$

$$m_6 \vee m_{14} = x_2 x_3 \bar{x}_4 = max_4$$

$$m_{15} \vee m_{14} = x_1 x_2 x_3 = max_5$$

$$M(f) = \{ max_1, max_2, max_3, max_4, max_5 \}$$

$C(f) = \{ max_1, max_2, max_5 \}$ - Central monoms

Let $g = max_1 \vee max_2 \vee max_5 = \bar{x}_1\bar{x}_2 \vee \bar{x}_2\bar{x}_3 \vee x_1x_2x_3$

m_6 - not covered minterm, can be covered using max_3 or max_4

So the simplest forms are:

$$f_1^s(x_1, x_2, x_3, x_4) = g \vee max_3 = \bar{x}_1\bar{x}_2 \vee \bar{x}_2\bar{x}_3 \vee x_1x_2x_3 \vee \bar{x}_1 x_3 \bar{x}_4$$

$$f_2^s(x_1, x_2, x_3, x_4) = g \vee max_4 = \bar{x}_1\bar{x}_2 \vee \bar{x}_2\bar{x}_3 \vee x_1x_2x_3 \vee x_2x_3\bar{x}_4$$