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Lecture 4

1. Scalar first order LDE's
2. Linear homogeneous systems with constant coefficients

1. Scalar first order LDE's

(1) $x' + a(t)x = f(t)$; $t \in I$, where $a, f \in C(I)$

For $t_0 \in I$ fixed and consider $A(t) = -\int_{t_0}^{t} a(s)ds$. Thus

Thus $A'(t) = -a(t)$
$A(t_0) = 0$

I method : The integrating factor method to find the general sol. of (1).

Theorem 1: The function $\mu : I \to \mathbb{R}$, $\mu(t) = e^{-A(t)}$ is an integrating factor for (1).

The general sol. for (1) is

$$x = c \cdot e^{A(t)} + e^{A(t)} \cdot \int_{t_0}^{t} f(s) e^{-A(s)} ds \quad , c \in \mathbb{R} \text{ arbitrary}$$

Proof: $x' + a(t)x = f(t) \;\Big|\cdot e^{-A(t)} \iff \underbrace{x' e^{-A(t)} + x \cdot a(t) \cdot e^{-A(t)}}_{(x \cdot e^{-A(t)})'} = f(t) \cdot e^{-A(t)}$

$\iff (x \cdot e^{-A(t)})' = f(t) \cdot e^{-A(t)}$, $\forall t \in I$

integrate

$x \cdot e^{-A(t)} = \int_{t_0}^{t} f(s) \cdot e^{-A(s)} ds + c \;\Big|\; \forall t \in I \;\Big|\cdot e^{A(t)} \iff$

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$\Leftrightarrow \quad x = c\cdot e^{A(t)} + e^{A(t)}\cdot\int_{t_0}^{t} f(s)\cdot e^{-A(s)}\,ds \quad , \forall t\in I$, where $c\in\mathbb{R}$ arbitr.

<u>II method</u>: The <u>separation of variables</u> method to find the general sol. of $x' + a(t)\cdot x = 0$

<u>Def</u>: A DE of the form $x' = b(t)\cdot g(x)$ is said to be a <u>separable eq.</u>

$$\left(x' \longrightarrow \frac{dx}{dt} \qquad \frac{dx}{dt} = b(t)\cdot g(x) \qquad \frac{dx}{g(x)} = b(t)\,dt \right)$$

we separated the variables

Notice that $x' + a(t)x = 0$ can be written as $x' = -a(t)x$, thus is a separable equation.

<u>Remark</u>. We already know that $x = c\cdot e^{A(t)}$, $c\in\mathbb{R}$ is the general sol. From here we deduce that $x=0$ is a (constant) solution and any other sol. satisfies $x(t)\neq 0$, $\forall t\in I$.

The separation of var. method: $x'(t) = -a(t)x(t)$

$x=0$ is a sol.

Now we look for sol. $x\neq 0$ $\quad \dfrac{x'(t)}{x(t)} = -a(t) \Leftrightarrow$

$$\left(\ln|x(t)|\right)' = -a(t)$$

integrate

$\Leftrightarrow \quad \ln|x(t)| = A(t) + c$, $c\in\mathbb{R}$ $\Leftrightarrow$ $|x(t)| = e^{A(t)+c}$ $\Leftrightarrow$

$x(t) = \pm e^{c}\cdot e^{A(t)}$, $c\in\mathbb{R}$ arbitrary

Recall $x=0$ sol.

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Note that $\{e^c, -e^c, 0 : c \in \mathbb{R}\} = \mathbb{R}$

$\Leftrightarrow \quad x(t) = c_1 \cdot e^{A(t)}, \quad c_1 \in \mathbb{R}$ arbitrary

<u>Remark</u>: we can use the sep. of var. method to find just a solution $x_1 \neq 0$. How? when we integrate we take $c=0$, that is $\ln|x(t)| = A(t) \Rightarrow x_1(t) = e^{A(t)}$

this is a sol. a first order LHDE

we apply the fundamental th. for LHDE we obtain that the general sol. is $x = c \cdot e^{A(t)}, c \in \mathbb{R}$.

<u>III method</u>: The Lagrange method (the variation of constant method) to find a particular solution of

(1) $x' + a(t)x = f(t)$ (linear nonhom.)

The idea of Lagrange was to look at the general sol. of the LHDE $x' + a(t)x = 0$, which is $x_h = c \cdot e^{A(t)}$, $c \in \mathbb{R}$ arbitrary and, then, to look for a partic. sol. of (1) of the form $x_p = \varphi(t) \cdot e^{A(t)}$, $\varphi = ?$

x_p is a sol. of (1) $\Leftrightarrow \varphi'(t) \cdot e^{A(t)} + \cancel{\varphi(t) \cdot e^{A(t)} \cdot (-a(t))} + \cancel{a(t) \cdot \varphi(t) \cdot e^{A(t)}} = f(t)$

$\Leftrightarrow \varphi'(t) = f(t) \cdot e^{-A(t)}$

$\Leftrightarrow \varphi(t) = \int_{t_0}^{t} f(s) \cdot e^{-A(s)} ds, \forall t \in I$

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So, $x_p = e^{A(t)} \cdot \int_{t_0}^{t} f(s) e^{-A(s)} ds$

<u>Remark</u>: Thus, the general sol. of (1) is $x = x_h + x_p$, that is we found again (2).

<u>Exercises</u> Find the general sol. of a) $x' - 2tx = t$, $t \in \mathbb{R}$

b) $x' - 2tx = \frac{2}{\sqrt{\pi}}$, $t \in \mathbb{R}$

(HW) c) $x' + \frac{1}{t} x = \frac{1}{t} e^{-2t+1}$

$t \in (0, \infty)$

a) $x' - 2tx = t$ first order L nonhom. DE

<u>Step 1</u>: we write the LHDE associated $x' - 2tx = 0$

apply the sep. of var. method $x' = 2tx$ $\frac{dx}{dt} = 2tx$

$x = 0$ is a sol.

$$\int \frac{dx}{x} = \int 2t \, dt$$

$$\ln|x| = t^2 + c$$

$$|x| = e^{t^2 + c}$$

$$\left.\begin{array}{l} x = \pm e^c \cdot e^{t^2} \\ x = 0 \end{array}\right\} \Rightarrow x_h = c_1 \cdot e^{t^2}, \; c_1 \in \mathbb{R}$$

<u>Step 2</u>: $x_p = ?$ Note that $x_p = -\frac{1}{2}$ is a solution

<u>Step 3</u>: the gen. sol. $x = c_1 \cdot e^{t^2} - \frac{1}{2}$, $c_1 \in \mathbb{R}$

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b) $x' - 2tx = \frac{2}{\sqrt{\pi}}$

<u>Step 1</u>: it has the same hom. part as a). Thus

$$x_h = c_1 \cdot e^{t^2}, \; c_1 \in \mathbb{R}$$

<u>Step 2</u>: $x_p = ?$ we apply the Lagrange method

$$x_p = \varphi(t) \cdot e^{t^2}$$

x_p is a sol $\Leftrightarrow$ $\varphi'(t) \cdot e^{t^2} + \cancel{\varphi(t) \cdot e^{t^2} \cdot 2t - 2t \cdot \varphi(t) e^{t^2}} = \frac{2}{\sqrt{\pi}} \Leftrightarrow$

$$\Leftrightarrow \varphi'(t) = \frac{2}{\sqrt{\pi}} e^{-t^2} \Leftrightarrow \varphi(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} \, ds$$

it is known that the primitive of e^{-t^2} is not a finite combination of elementary functions, thus we are not able to write its expression

The Gauss error function

$$\operatorname{erf}(t) \overset{not}{=} \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} \, ds, \; \forall t \in \mathbb{R}$$

So, $\varphi(t) = \operatorname{erf}(t) \Rightarrow x_p = \operatorname{erf}(t) \cdot e^{t^2}$

<u>Step 3</u>: $x = c \cdot e^{t^2} + \operatorname{erf}(t) \cdot e^{t^2}$, $c \in \mathbb{R}$ arbitrary

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Find an integrating factor for a) and b). It should be the same since both eq. have the same homogeneous part.

$$\underbrace{x' - 2tx}_{} = t \quad \Big| \cdot e^{-t^2}$$

$$\Leftrightarrow x' \cdot e^{-t^2} - 2tx \cdot e^{-t^2} = t \cdot e^{-t^2}$$

$a(t) = -2t$

$A(t) = -\int_0^t a(s) \, ds = 2\int_0^t s \, ds$

$= s^2 \Big|_0^t = t^2$

$\mu(t) = e^{-A(t)} = e^{-t^2}$ an integrating factor

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$$\Leftrightarrow \left(x \cdot e^{-t^2}\right)' = t\, e^{-t^2} \Leftrightarrow x \cdot e^{-t^2} = -\frac{1}{2} e^{-t^2} + c \;\Big|\cdot e^{t^2}, \; c \in \mathbb{R} \Leftrightarrow$$

$$\left(\int t e^{-t^2}\, dt = -\frac{1}{2} \int \left(e^{-t^2}\right)' dt = \frac{1}{2} e^{-t^2} \right)$$

$$\Leftrightarrow \boxed{x = c \cdot e^{t^2} - \frac{1}{2}, \; c \in \mathbb{R}}$$

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