24.02.2022.

Seminar Nr 1

Differential equations = equation that involves derivatives of a function and the unknown and the variables are all functions

$x(t)$ - unknown

General form of an n^{th} order differential equation:

$$x^{(n)}(t) + a_{n-1}(t)\cdot x^{(n-1)}(t) + \ldots + a_1(t)\cdot x'(t) + a_0(t)\cdot x(t) = f(t)$$

$f(t)$ - non-homogeneous part

a) $x' = 0$

$x'(t) = 0$

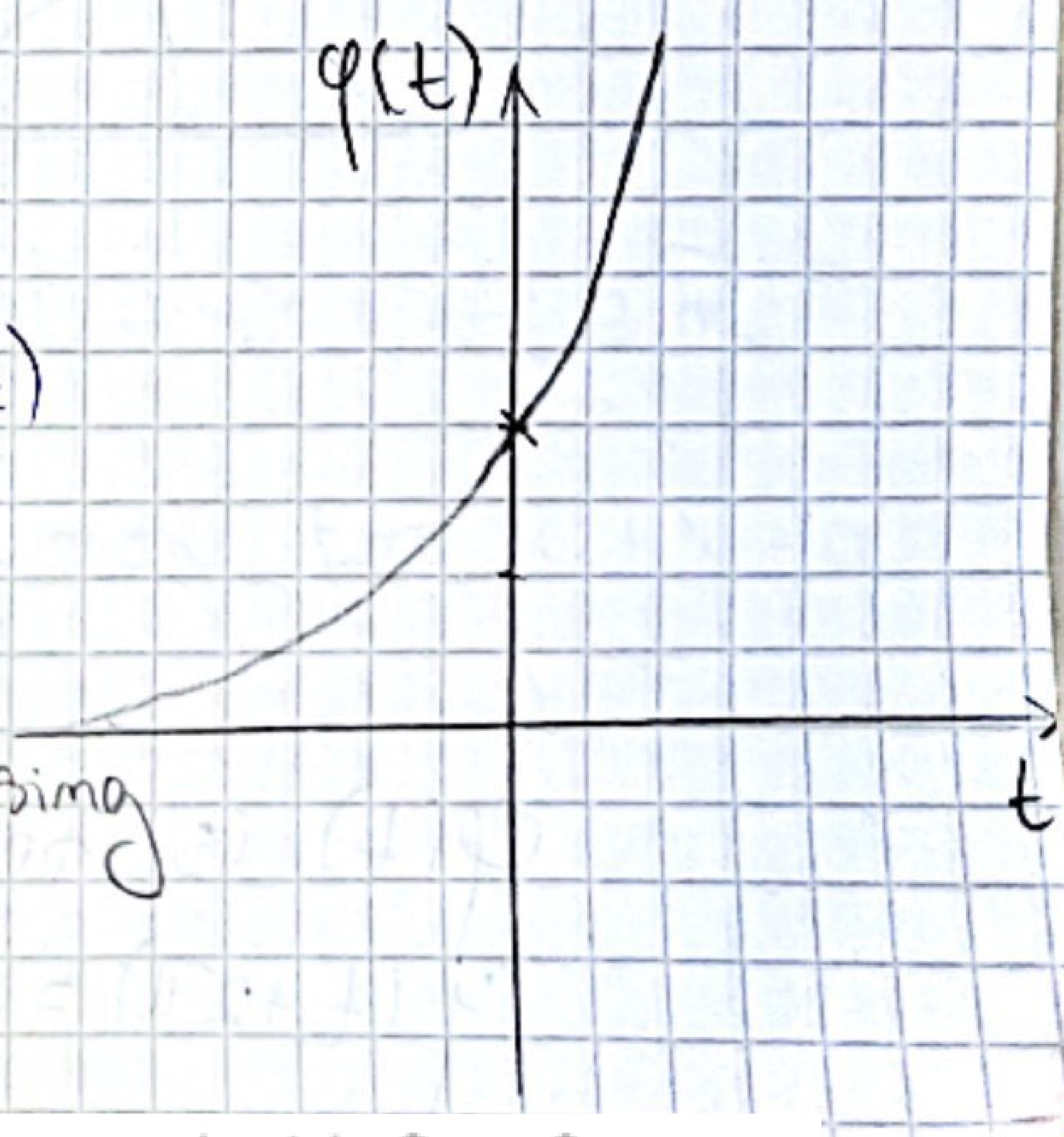
$$\int_{t_0}^{t} x'(s)\,ds = \int_{t_0}^{t} 0\,ds = 0 \Rightarrow x(s)\Big|_{t_0}^{t} = x(t) - \underbrace{\overbrace{x(t_0)}^{c}}_{const \in \mathbb{R}} = 0$$

t_0 fixed const.

$x(t) = c,\ c \in \mathbb{R}$

1) Show that the function $\varphi: \mathbb{R} \to \mathbb{R}$, $\varphi(t) = 2e^{3t}$, $\forall t \in \mathbb{R}$ is a solution of the Initial Value Problem (IVP)

$x' = 3x \quad x(0) = 2$



$\varphi'(t) = 6e^{3t} = 3\cdot 2e^{3t} = 3\varphi(t)$

$\varphi(0) = 2\cdot e^0 = 2$

$\varphi'(t) = 6e^{3t} > 0 \Rightarrow \varphi$ is increasing

$\lim\limits_{t\to -\infty} \varphi(t) = 0$

$\lim \varphi(t) = \infty$

$\varphi(t)$ is not periodic

$\varphi(t)$ is bounded from below by 0 but unbounded from above

2. Let $\eta \in \mathbb{R}^*$ be fixed. Show that the function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \eta \cdot \sin t$ is a solution of the IVP: $x'' + x = 0$, $x(0) = 0$, $x'(0) = \eta$

$\varphi'(t) = \eta \cdot \cos t$

$\varphi''(t) = -\eta \cdot \sin t$

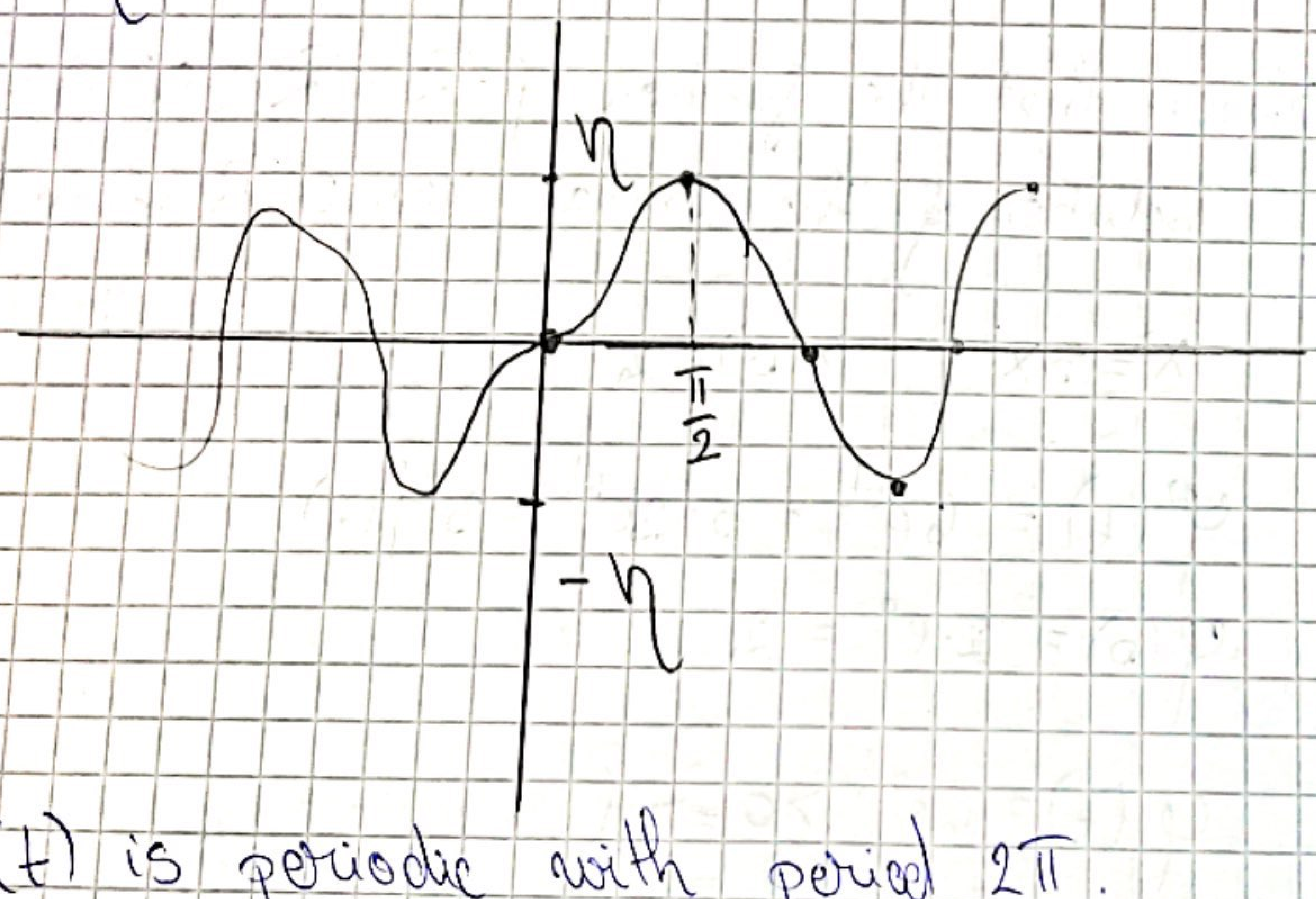
$\varphi''(t) + \varphi(t) = -\cancel{\eta \cdot \sin t} + \cancel{\eta \sin t} = 0$

$\varphi(0) = \eta \cdot \sin 0 = 0$

$\varphi'(0) = \eta \cdot \cos 0 = \eta$

Represent the integral curve

$\eta > 0$



$\varphi(t)$ is periodic with period 2π.

$\varphi(t+2\pi) = \varphi(t)$

$\varphi(t)$ is bounded by η and $-\eta$

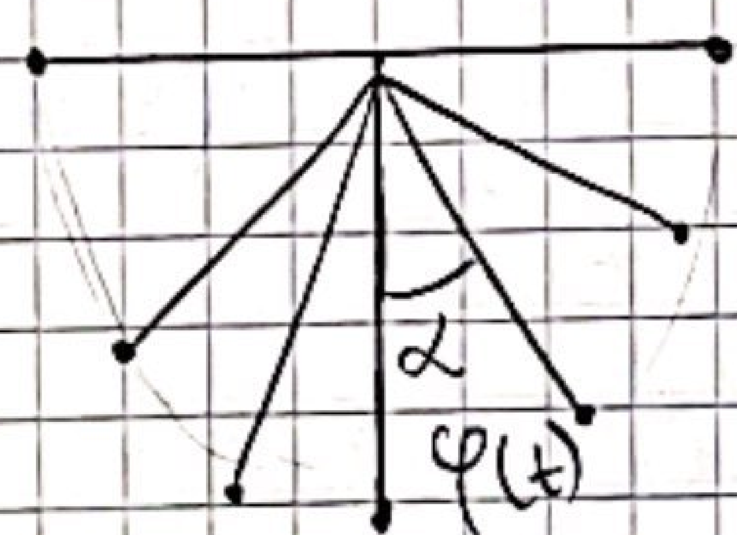
$\lim\limits_{t \to \pm\infty} \varphi(t)$ ~~doesn't~~ don't exist

$\eta : \sin t$

$x_n = (\eta, \eta, \eta, \ldots)$

$t = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi, \ldots \to \infty$

$\eta = \frac{\pi}{2}$



4) Decide whether $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \cos t$, $\forall t \in \mathbb{R}$ is a solution of $x' + x = 0$

$\varphi'(t) = -\sin t$

$\varphi'(t) + \varphi(t) = -\sin t + \cos t \neq 0$, $\forall t \in \mathbb{R}$

$\Rightarrow \varphi(t) = \cos t$ is not a solution for $x' + x = 0$

$x' = -x$

$(e^{-t})' = -e^{-t} = -(e^{-t})$

$x' + x = 0 \quad | \cdot e^t$

$x(t) = c \cdot e^{-t}$

$x' \cdot e^t + x \cdot e^t = 0$

$(x \cdot e^t)' = 0 \quad \Rightarrow \quad x \cdot e^t = c, \ c \in \mathbb{R}$

$x' = 0 \qquad x = c \cdot e^{-t}, \ c \in \mathbb{R}$

$x'' - x = 0$

$\cancel{\varphi'(t) = -\sin t}$

$$\left.\begin{array}{l} x_1(t) = c_1 \cdot e^t \\ x_2(t) = c_2 \cdot e^{-t} \end{array}\right\} \Rightarrow x(t) = c_1 \cdot e^t + c_2 \cdot e^{-t} \text{ is the solution}$$

$(\sin t)' = \cos t$

$(\sin t)' = -\sin t$ linearly independent

e^t

derivata de 2 ori $\Rightarrow$ 2 sol.

5) Find the constant solutions of the following eq:

a) $x' = x - x^3$

$x(t) = c \Rightarrow x' = 0$ $\Rightarrow x - x^3 = 0$

$\cancel{x = x^3}$

$x(1 - x^2) = 0$

I. $x_1 = 0$

II. $x^2 = 1 \Rightarrow x_2 = -1$

$x_3 = 1$

$x(t) \in \{-1, 0, 1\}$

b) $x' = \sin x$

$x(t) = c \Rightarrow x'(t) = 0 \Rightarrow \sin x = 0 \Rightarrow x \in \{0, \pi, \dots\}$

$x \in \{k\pi \mid k \in \mathbb{Z}\}$

7) Prove that $\cos t$, $\sin t$ and e^t are linearly independent in $C(\mathbb{R})$

$f(t), g(t)$

$(f+g)(t) = f(t) + g(t)$

$\cos t, \sin t, e^t$ are lin. indep. if:

$a \cdot \cos t + b \cdot \sin t + c \cdot e^t = 0 \Rightarrow a = b = c = 0$

if $t = 0 \Rightarrow a \cdot \cos 0 + b \cdot 0 + c \cdot e^0 = a + c = 0$

if $t = \pi \Rightarrow a \cdot \cos \pi + b \cdot 0 + c \cdot e^{\pi} = c \cdot e^{\pi} - a = 0$ | +

$c(1 + e^{\pi}) = 0 \Rightarrow c = 0$

$\left.\begin{matrix} a + c = 0 \\ c = 0 \end{matrix}\right\} \Rightarrow a = 0$

$b \cdot \sin t = 0, \forall t \in \mathbb{R} \Rightarrow b = 0$

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$\Rightarrow \cos t, \sin t, e^t$ are linearly independent

ii) Find $a, b, c \in \mathbb{R}$: $x(t) = a \cdot \sin t + b \cos t + c \cdot e^t$ is a solution of : $x' + x = -3 \sin t + 2e^t$

$$x'(t) = a \cdot \cos t - b \sin t + c \cdot e^t$$

$$x'(t) + x(t) = (a-b) \sin t + (a+b) \cos t + 2c \cdot e^t = -3 \sin t + 2e^t$$

$$\left.\begin{aligned} a-b &= -3 \\ a+b &= 0 \\ c &= 1 \end{aligned}\right\} \Rightarrow 2a = -3 \Rightarrow a = -\frac{3}{2} \Rightarrow b = \frac{3}{2}$$

8) Find $r \in \mathbb{R}$: $x(t) = e^{rt}$ is a solution of

$$x'' - 5x' + 6x = 0$$

$$x'(t) = r e^{rt}$$

$$x''(t) = r^2 e^{rt}$$

$$r^2 e^{rt} - 5r e^{rt} + 6e^{rt} = 0$$

$$e^{rt}(r^2 - 5r + 6) = 0$$

$$e^{rt} \neq 0 \Rightarrow r^2 - 5r + 6 = 0$$

$$\Rightarrow r \in \{+2, +3\}$$

$$\begin{cases} x_1(t) = e^{+2t} \\ x_2(t) = e^{+3t} \end{cases}$$

$$\Downarrow$$

$$x(t) = c_1 \cdot e^{2t} + c_2 \cdot e^{3t}$$