9.05.2022.

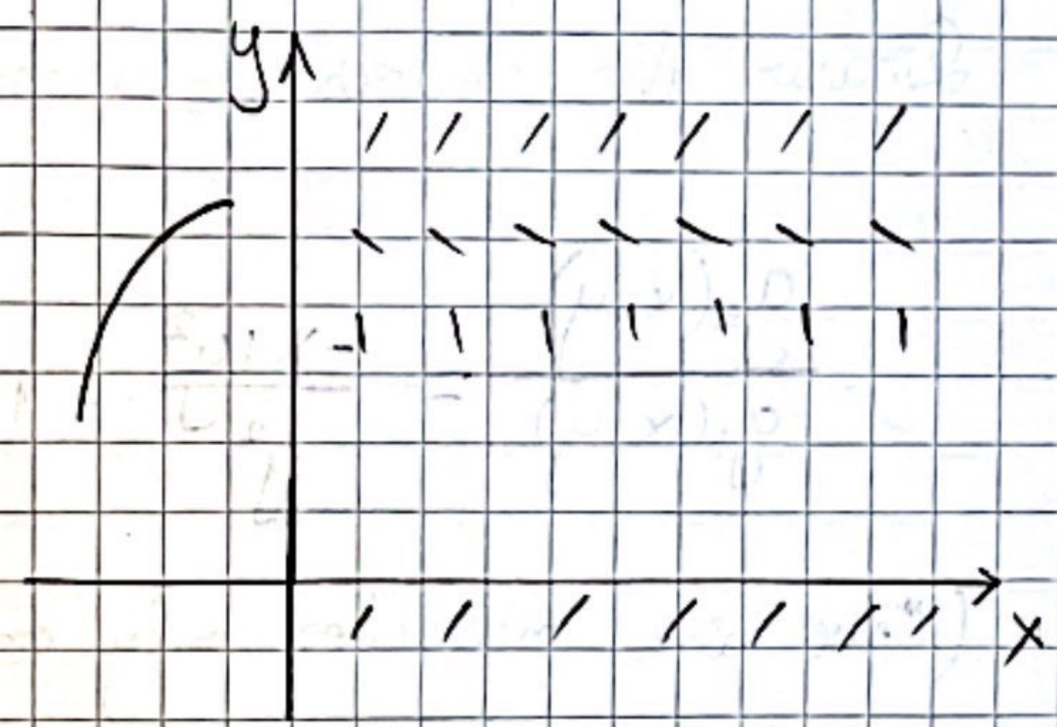
Lecture 11

I. The direction field associated to a differential eq.

II. Short introduction to numerical methods

I. (1) $y' = f(x, y(x))$

(2) $\begin{cases} \dot{x}(t) = g_1(x(t), y(t)) \\ \dot{y}(t) = g_2(x(t), y(t)) \end{cases}$



Assumptions: $f \in C^1(\mathbb{R}^2)$, $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$

Recall: We have existence and uniqueness for any IVP.
The graph of a sol. of (1) is a plane curve (in $\mathbb{R}^2$)
an orbit of (2) is a plane curve.

Def 1: • The direction field associated to eq. (1) is a collection of vectors.

To a point $(x_0, y_0) \in \mathbb{R}^2$ we associate a vector of slope $m = f(x_0, y_0)$

• The direction field associated to system (2) is a collection of vectors.
To a point $(x_0, y_0) \in \mathbb{R}^2$ we associate a vector of slope $m = \dfrac{g_2(x_0, y_0)}{g_1(x_0, y_0)}$

Scanned with CamScanner

Example: (3) $y' = 1 - \frac{x}{y^2}$

$$(4) \begin{cases} \dot{x} = y^2 \\ \dot{y} = -x + y^2 \end{cases}$$

Note that (3) and (4) have the same direction field.
Draw the vectors corresponding to the points $(1,1)$, $(1,0)$, $(0,1)$

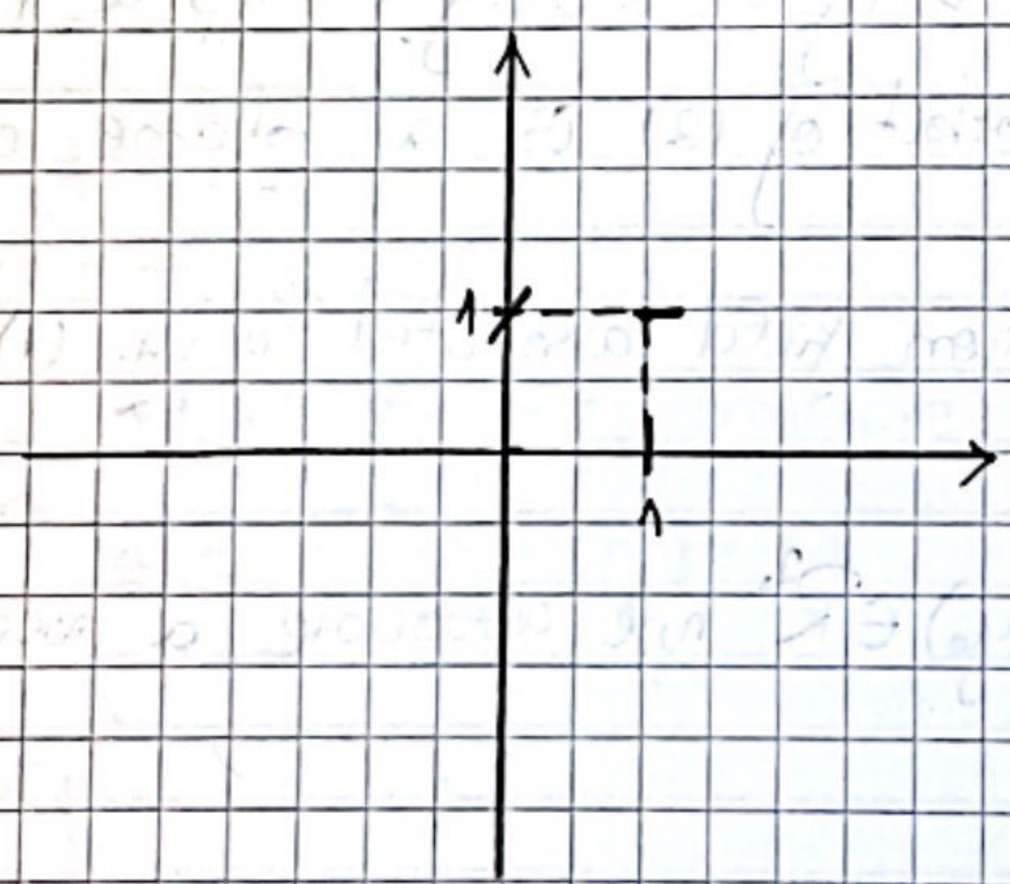
$$\frac{g_2(x,y)}{g_1(x,y)} = \frac{-x+y^2}{y^2} = 1 - \frac{x}{y^2} = f(x,y) \Rightarrow$$ (3) and (4) have the same direction field

(*we are interested only on their slope*)

$(1,1) \rightarrow m_1 = f(1,1) = 0$

$(1,0) \rightarrow m_2 = f(1,0) = \infty$

$(0,1) \rightarrow m_3 = f(0,1) = 1$



$g(x,y) = (g_1(x,y), g_2(x,y))$ this vector has the slope $\frac{g_2(x,y)}{g_1(x,y)}$

Scanned with CamScanner

Property: Let $(x_0, y_0) \in \mathbb{R}^2$:

I. Denote by $\varphi : I \to \mathbb{R}$ the solution of the IVP $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$.

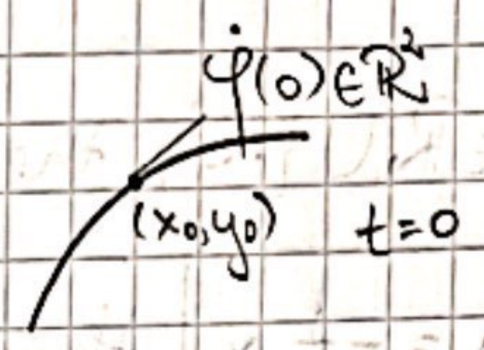
Then $\varphi'(x_0) = f(x_0, y_0)$, i.e. the vector of the direction field associated to (x_0, y_0) is tangent to the solution curve that passes through this point.

II. Denote by $\varphi : I \to \mathbb{R}^2$ the sol. of the IVP $\begin{cases} \dot{x} = g_1(x,y) \\ \dot{y} = g_2(x,y) \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$

Then the tangent vector to the orbit of φ in (x_0, y_0) is

$$\dot{\varphi}(0) = \begin{pmatrix} g_1(x_0,y_0) \\ g_2(x_0,y_0) \end{pmatrix} \text{ whose slope is } \frac{g_2(x_0,y_0)}{g_1(x_0,y_0)}$$

orbit $\begin{cases} x = \varphi_1(t) \\ y = \varphi_2(t) \end{cases}$



- In other words the direction field is tangent to the orbit system too.

Def: Let $m \in \mathbb{R} \cup \{\infty\}$. The m-izocline of (1) in $\{(x,y) \in \mathbb{R}^2 : m = f(x,y)\}$.

The m-izocline of (2) is $\{(x,y) \in \mathbb{R}^2 : m = \frac{g_2(x,y)}{g_1(x,y)}\}$

Scanned with CamScanner

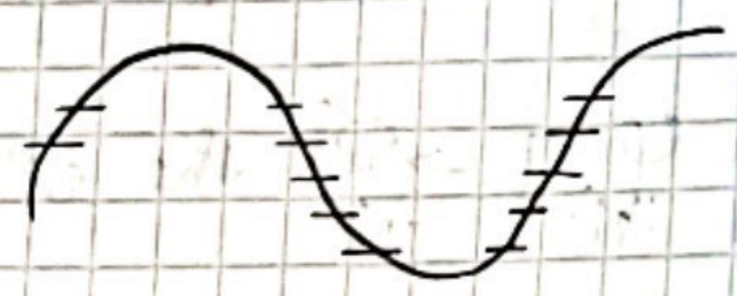
Example: (5) $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ (6) $y' = -\frac{x}{y}$

a) Note that they have the same direction field.

b) Find and represent the isoclines.

c) Find the slope of the orbits of (5)

0-isocline



a) We have $f(x,y) = -\frac{x}{y}$, $\begin{matrix} g_1(x,y) = -y \\ g_2(x,y) = x \end{matrix}$ $\Rightarrow \frac{g_2(x,y)}{g_1(x,y)} = f(x,y) \Rightarrow$

$\Rightarrow$ the direction field is the same

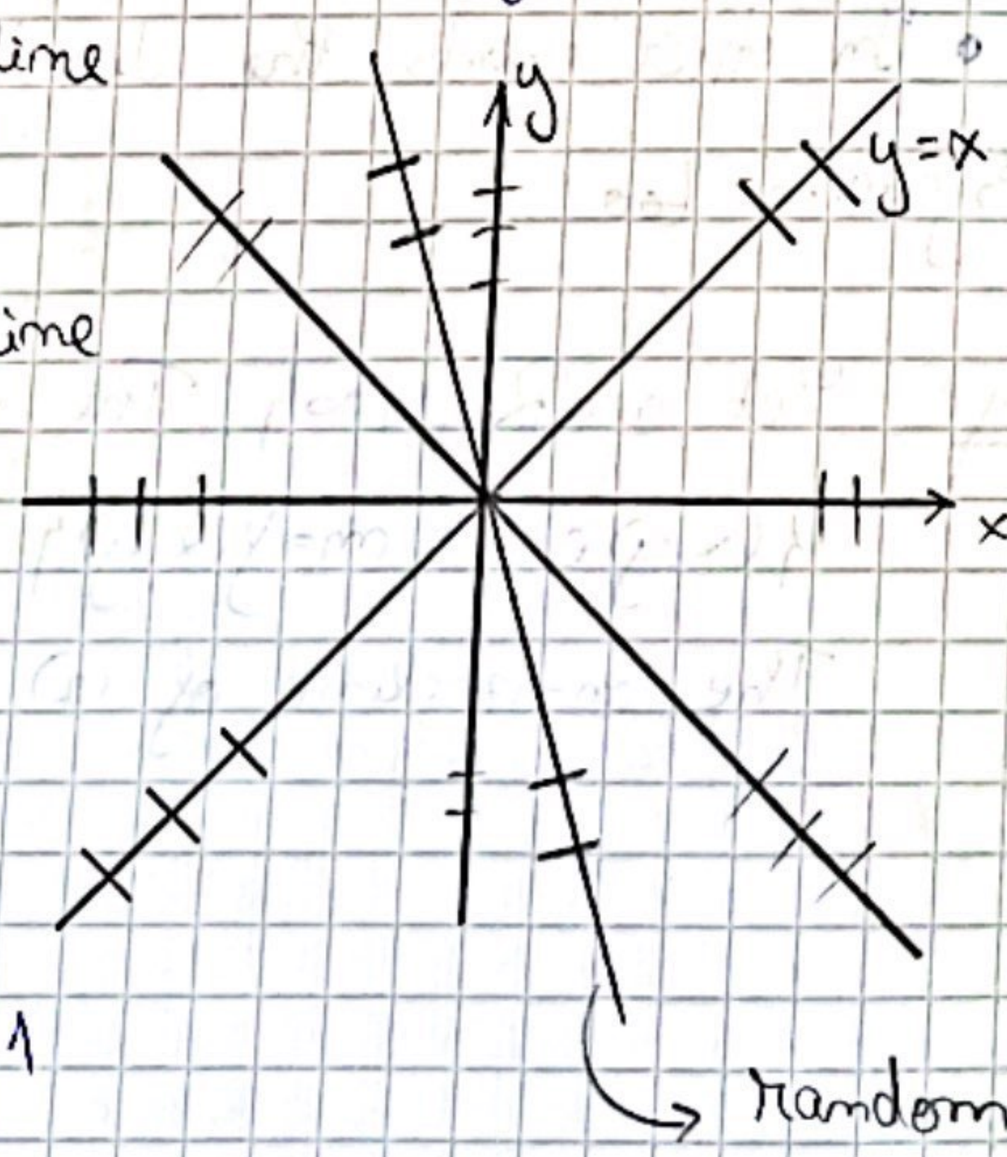
b) $m \in \mathbb{R} \cup \{\infty\}$ the m-isocline has the equation $m = -\frac{x}{y} \Leftrightarrow$

$\boxed{y = -\frac{1}{m}x}$ this is a line through the origin

this is the m-isocline

$m = -1 \Rightarrow y = x$

this is the -1-isocline



! Note that $m \cdot \left(-\frac{1}{m}\right) = -1$

random-isocline

- The direction field is orthogonal to any isocline of this example, thus, to any line that passes through the origin.

So, any orbit is orthogonal to any line that passes through the origin

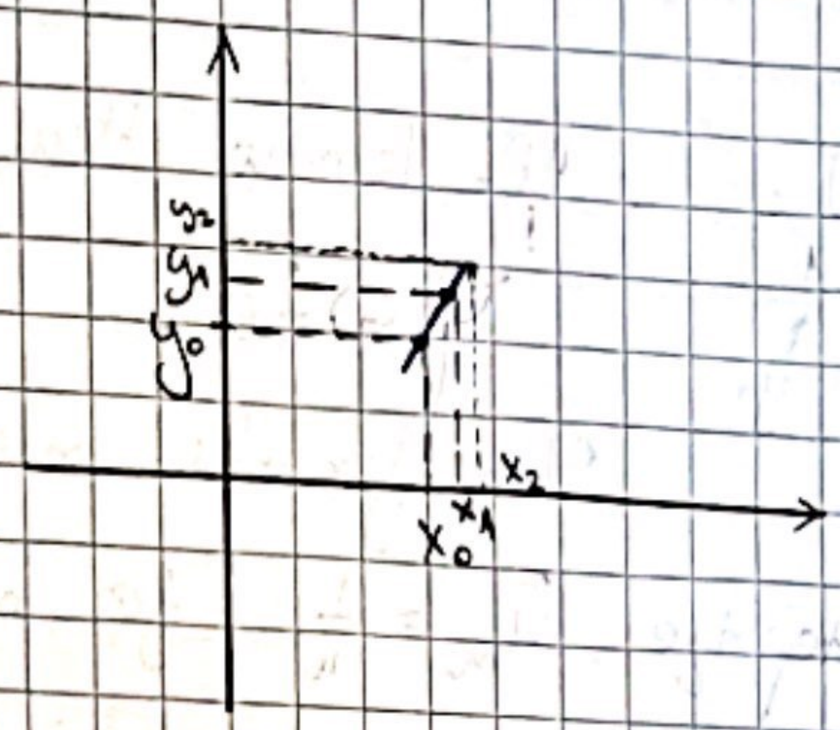
Thus any orbit in a circle centered in the origin.

Scanned with CamScanner

II. Short introduction to numerical methods

$$(1)\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \qquad f \in C^1(\mathbb{R}^2)$$

has a unique solution $\varphi : I \to \mathbb{R}$.



$m_0 = f(x_0, y_0)$

$m_1 = f(x_1, y_0)$

The Euler's numerical method

$[x_0, x^*] \subset I$

take a partition of $[x_0, x^*]$

$$x_0 < x_1 < x_2 < \dots < x_m = x^*$$

$\boxed{m}$ $m \in \mathbb{N}, m \geq 1$ is the number of steps

$$y - y_0 = m_0(x - x_0)$$

$$y_1 = y_0 + (x_1 - x_0) \cdot f(x_0, y_0)$$

$$\dots$$

$$y_{k+1} = y_k + (x_{k+1} - x_k) f(x_k, y_k), \quad k = \overline{0, m-1}$$

y_i is an "approximation" of the exact value $\varphi(x_i)$.

the result of a numerical method

Scanned with CamScanner

Usually, $x_{k+1} - x_k = h$, $\forall k = \overline{0, n-1}$ $\quad h > 0$ is called the <u>stepsize</u>

$$\Rightarrow \begin{cases} x_{k+1} = x_k + h \\ y_{k+1} = y_k + h \cdot f(x_k, y_k) \end{cases}, \quad k = \overline{0, n-1}$$

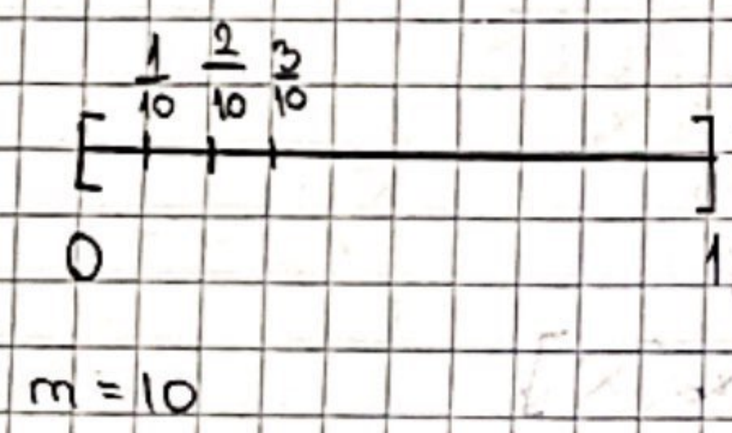
<u>Exercise:</u> $\begin{cases} y' = y \\ y(\underset{x_0}{0}) = \underset{y_0}{1} \end{cases}$. We know that the unique sol. is $\varphi(x) = e^x$.

We fix the interval $[0, 1]$

We take the stepsize $h_n = \frac{1}{n}$ for $n \geq 1$ fixed

We compute y_n, an approximation of $\varphi(1) = e$

Prove that $\lim\limits_{n \to \infty} y_n = e$



Note that, $h_n = \frac{1}{n}$. the number of steps to cover the interval $[0, 1]$ is n. $\quad f(x, y) = y$

Write the Euler's numerical formula $\begin{cases} x_{k+1} = x_k + \frac{1}{n} \\ y_{k+1} = y_k + \frac{1}{n} y_k \end{cases}, \quad k = \overline{0, n}$

$x_0 = 0$, $y_0 = 1$

$$\Rightarrow \begin{cases} x_k = \frac{k}{n} \\ y_{k+1} = \left(1 + \frac{1}{n}\right) y_k \end{cases}, \quad k = \overline{0, n-1} \qquad x_0 = 0, \quad y_0 = 1$$

$$\begin{cases} x_k = \frac{k}{n} \\ y_k = \left(1 + \frac{1}{n}\right)^k \end{cases}, \quad k = \overline{0, n-1}$$

Scanned with CamScanner

$$y_n = \left(1 + \frac{1}{n}\right)^n \approx e \,, \quad \forall n \in \mathbb{N}^*$$

$$\Rightarrow \lim_{n \to \infty} y_n = e$$

Scanned with CamScanner