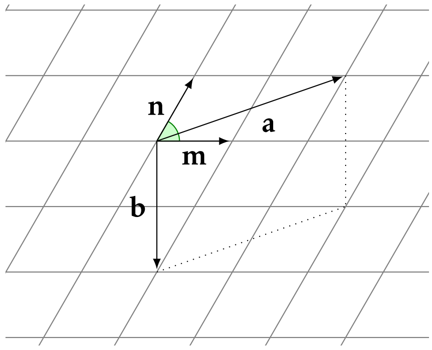
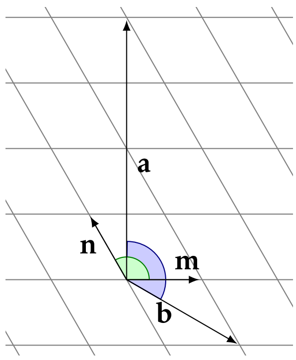
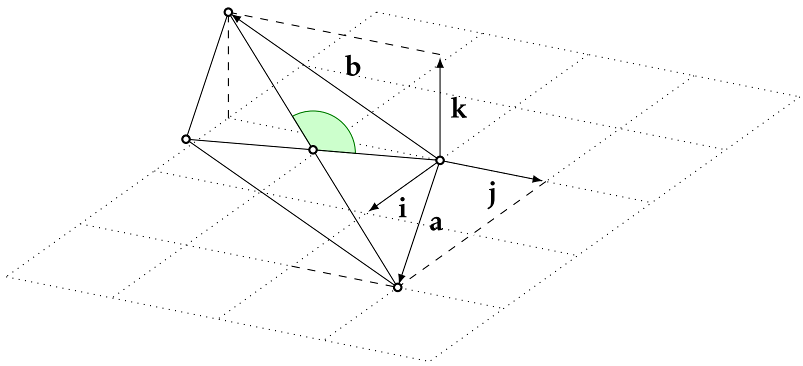
1. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 60°$. Determine the length of the diagonals in the parallelogram spanned by the vectors $\mathbf{a} = 2\mathbf{m}+\mathbf{n}$ and $\mathbf{b} = \mathbf{m}-2\mathbf{n}$.



2. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 120°$. Determine the angle between the vectors $\mathbf{a} = 2\mathbf{m}+4\mathbf{n}$ and $\mathbf{b} = \mathbf{m}-\mathbf{n}$.



3. You are given two vectors $\mathbf{a}(2,1,0)$ and $\mathbf{b}(0,-2,1)$ with respect to an orthonormal basis. Determine the angles between the diagonals of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.



4. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis. Consider the vectors $\mathbf{q} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{p} = \mathbf{i}+2\mathbf{j}+\lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p},\mathbf{q})$ is $5/12$.

5. Let ABC be a triangle. Show that

$$\overrightarrow{AB}^2 + \overrightarrow{AC}^2 - \overrightarrow{BC}^2 = 2\overrightarrow{AB}\cdot\overrightarrow{AC}$$

and deduce the law of cosines in a triangle.

6. Let $ABCD$ be a rectangle. Show that for any point O

$$\overrightarrow{OA}\cdot\overrightarrow{OC} = \overrightarrow{OB}\cdot\overrightarrow{OD} \quad \text{and} \quad \overrightarrow{OA}^2 + \overrightarrow{OC}^2 = \overrightarrow{OB}^2 + \overrightarrow{OD}^2.$$

7. Show that the Gram-Schmidt orthogonalization process yields an orthonormal basis.

8. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

9. Show that the orthogonal reflection of a vector $\mathbf{b}$ parallel to $\mathbf{a}$ is

$$\mathrm{Ref}^{\|}_{\mathbf{a}}(\mathbf{b}) = \mathbf{b} - 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = \mathbf{b} - 2\,\mathrm{Pr}^{\perp}_{\mathbf{a}}(\mathbf{b}).$$

Show that the orthogonal reflection of a vector $\mathbf{b}$ in the vector $\mathbf{a}$ is

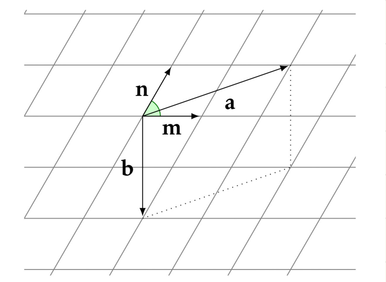
$$\mathrm{Ref}^{\perp}_{\mathbf{a}}(\mathbf{b}) = -\mathbf{b} + 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = -\mathbf{b} + 2\,\mathrm{Pr}^{\perp}_{\mathbf{a}}(\mathbf{b}) = -\mathrm{Ref}^{\|}_{\mathbf{a}}(\mathbf{b}).$$

10. Let $\mathbf{v} \in \mathbb{V}^n$ be a vector. Show that

1. The set $\mathbf{v}^{\perp}$ is a vector subspace of $\mathbb{V}^n$.
2. There is a basis $\mathbf{v},\mathbf{v}_2,\mathbf{v}_3,\ldots,\mathbf{v}_n$ of $\mathbb{V}^n$ with $\mathbf{v}_2,\ldots,\mathbf{v}_{n-1}$ a basis of $\mathbf{v}^{\perp}$.

11. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\phi : \mathbb{V}^3 \to \mathbb{R}$ be the map $\phi(\mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to an orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w}\cdot\mathbf{v}$?

1. Let **m** and **n** be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 60°$. Determine the length of the diagonals in the parallelogram spanned by the vectors $\mathbf{a} = 2\mathbf{m} + \mathbf{n}$ and $\mathbf{b} = \mathbf{m} - 2\mathbf{n}$.



- The length of the two diagonals are

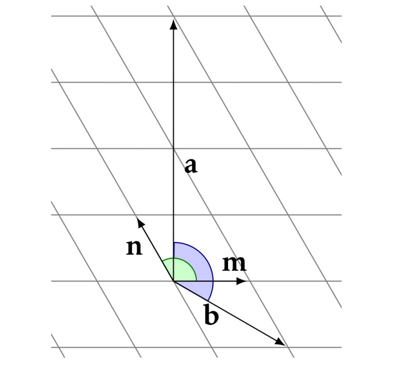
$\|a+b\|$ and $\|a-b\|$

- $\|a+b\| = \|2m+n+m-2n\|$

$$\|a+b\|^2 = \|3m-n\|^2 = (3m-n)^2 = \underbrace{9m^2}_{m^2=1} - 6\underbrace{mn}_{\|m\|\cdot\|n\|\cdot\cos\measuredangle(m,n) \,=\, 1\cdot 1\cdot\cos 60°} + \underbrace{n^2}_{=1} = 10 - 6\cdot\frac{1}{2} = 7 \Rightarrow \|a+b\| = \sqrt{7}$$

- $\|a-b\|^2 = \|2m+n-m+2n\| = (m+3n)^2 = m^2+6mn+9n^2 = 10+6\cdot\frac{1}{2} = 13 \Rightarrow \|a-b\| = \sqrt{13}$

2. Let **m** and **n** be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 120°$. Determine the angle between the vectors $\mathbf{a} = 2\mathbf{m} + 4\mathbf{n}$ and $\mathbf{b} = \mathbf{m} - \mathbf{n}$.



$$\cos\measuredangle(a,b) = \frac{a\cdot b}{\|a\|\cdot\|b\|}$$

$$a\cdot b = (2m+4n)\cdot(m-n)$$

$$= 2m^2 - 2mn + 4nm - 4n^2$$

$$= 2m^2 + 2nm - 4n^2$$

$$= 2\|m\|^2 + 2\|n\|\cdot\|m\|\cdot\cos\measuredangle(n,m) - 4\|n\|^2$$

$$= -2 + 2\cdot\frac{-1}{2} = -3$$

$$\|a\|^2 = (2m+4n)^2 = 4m^2 + 16mn + 16n^2 = 20 + 16\cdot\frac{-1}{2} = 12 \quad \Rightarrow \|a\| = \sqrt{12}$$

$$\|b\|^2 = (m-n)^2 = m^2 - 2mn + n^2 = 2 - 2\frac{-1}{2} = 3 \quad \Rightarrow \|b\| = \sqrt{3}$$

$$\Rightarrow \cos\measuredangle(a,b) = \frac{-3}{2\sqrt{3}\cdot\sqrt{3}} = -\frac{1}{2} \Rightarrow \measuredangle(a,b) = 120°$$

3. You are given two vectors $\mathbf{a}(2,1,0)$ and $\mathbf{b}(0,-2,1)$ with respect to an orthonormal basis. Determine the angles between the diagonals of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

$$\cos \measuredangle(a+b, a-b) = \frac{(2,-1,1)\cdot(2,3,-1)}{\|(2,-1,1)\|\cdot\|(2,3,-1)\|} = \frac{4-3-1}{\sqrt{4+1+1}\sqrt{4+9+1}} = 0 \Rightarrow$$ the two diagonals are perpendicular to eachother.

4. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis. Consider the vectors $\mathbf{q} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{p} = \mathbf{i}+2\mathbf{j}+\lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p},\mathbf{q})$ is $5/12$.

$$\cos \measuredangle(p,q) = \frac{p\cdot q}{\|p\|\cdot\|q\|} = \frac{3+2}{\sqrt{9+1}\sqrt{1+4+\lambda}} = \frac{5}{12} \iff \sqrt{10}\sqrt{5+\lambda} = 12 \iff 10(5+\lambda^2) = 144$$

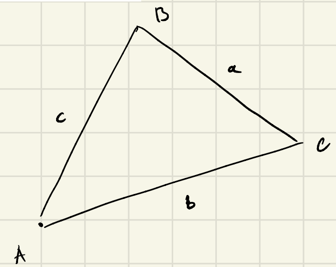
So $\lambda^2 = 9.4 \Rightarrow \lambda = \sqrt{9.4}$

$47/5$

5. Let ABC be a triangle. Show that

$$\overrightarrow{AB}^2 + \overrightarrow{AC}^2 - \overrightarrow{BC}^2 = 2\overrightarrow{AB}\cdot\overrightarrow{AC} \quad (*)$$

and deduce the law of cosines in a triangle.



$$(*) \iff \overrightarrow{AB}^2 - 2\overrightarrow{AB}\cdot\overrightarrow{AC} + \overrightarrow{AC}^2 = \overrightarrow{BC}^2$$

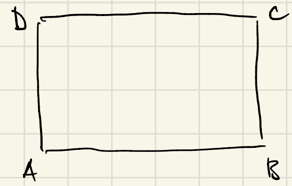
$$\iff \underbrace{(\overrightarrow{AB} - \overrightarrow{AC})^2}_{=\overrightarrow{BC}^2} = \overrightarrow{BC}^2$$

on the other hand $$(*) \iff c^2 + b^2 - a^2 = 2\underbrace{\|\overrightarrow{AB}\|}_{c}\cdot\underbrace{\|\overrightarrow{AC}\|}_{b}\underbrace{\cos\measuredangle(AB,AC)}_{\cos\hat{A}}$$

so $a^2 = c^2 + b^2 - 2bc\cos\hat{A}$

6. Let $ABCD$ be a rectangle. Show that for any point O

$$\overrightarrow{OA}\cdot\overrightarrow{OC} = \overrightarrow{OB}\cdot\overrightarrow{OD} \quad \text{und} \quad \overrightarrow{OA}^2 + \overrightarrow{OC}^2 = \overrightarrow{OB}^2 + \overrightarrow{OD}^2.$$



1) $\underbrace{\overrightarrow{OA}\cdot\overrightarrow{OC}} = \overrightarrow{OB}\cdot\overrightarrow{OD}$

$$(\overrightarrow{OB}+\overrightarrow{BA})(\overrightarrow{OD}+\overrightarrow{DC}) = \overrightarrow{OB}\,\overrightarrow{OD} + \overrightarrow{BA}\cdot\overrightarrow{OD} + \overrightarrow{OB}\cdot\overrightarrow{DC} + \overrightarrow{BA}\cdot\overrightarrow{DC}$$

$$= \overrightarrow{OB}\cdot\overrightarrow{OD} + \overrightarrow{BA}\cdot\overrightarrow{OC} + \overrightarrow{OB}\,\overrightarrow{DC}$$

$$- \overrightarrow{OB}\cdot\overrightarrow{BA} \qquad (\text{since } \overrightarrow{BA} = -\overrightarrow{DC})$$

$$= \overrightarrow{OB}\cdot\overrightarrow{OD} + \overrightarrow{BA}\,(\overrightarrow{OC} - \overrightarrow{OD})$$

$$\overrightarrow{CB}$$

$$\underbrace{\qquad}_{=0} \quad (\text{since } \overrightarrow{BA}\perp\overrightarrow{CB}$$

2) $$\frac{\overrightarrow{OA}+\overrightarrow{OC}}{2} = \frac{\overrightarrow{OB}+\overrightarrow{OD}}{2}$$

$$\Rightarrow (\overrightarrow{OA}+\overrightarrow{OC})^2 = (\overrightarrow{OB}+\overrightarrow{OD})^2$$

$$\Rightarrow \overrightarrow{OA}^2 + 2\overrightarrow{OA}\,\overrightarrow{OC} + \overrightarrow{OC}^2 = \overrightarrow{OB}^2 + \underbrace{2\overrightarrow{OB}\cdot\overrightarrow{OD}}_{2\overrightarrow{OA}\,\overrightarrow{OC}\ (\text{by part } 2)} + \overrightarrow{OD}^2$$

$$\Rightarrow \overrightarrow{OA}^2 + \overrightarrow{OC}^2 = \overrightarrow{OB}^2 + \overrightarrow{OD}^2$$

7. Show that the Gram-Schmidt orthogonalization process yields an orthonormal basis.

Let $\{e_1, \dots, e_n\}$ be a basis of V^n

Consider

$$v_1 = e_1 = v$$

$$v_2 = e_2 - \frac{v_1 \cdot e_2}{v_1 \cdot v_1} e_1$$

$$v_3 = e_3 - \frac{v_1 e_3}{v_1 \cdot v_1} v_1 - \frac{v_2 e_3}{v_2 \cdot v_2} v_2$$

$$\vdots$$

$$v_j = e_j - \sum_{i=1}^{j-1} \frac{v_i e_j}{v_i v_i} v_i$$

$$\vdots$$

Claim $\{v_1, \dots, v_n\}$ is a basis of V^n and $\{v_2, \dots v_n\}$ is a basis of $v^\perp$

- $\forall j$ $\{v_1, \dots, v_j\}$ is linearly independent

else v_j = linear combination of $\{v_1 \dots v_{j-1}\}$

$\Leftrightarrow$ v_j = linear combination of $\{e_1, \dots, e_{j-1}\}$

but $v_j = e_j$ + linear combination of $\{e_1, \dots, e_{j-1}\}$

contradiction

$\Rightarrow$ $\{v_1, \dots, v_j\}$ is lin. indep. $\forall j \in \{1, \dots, n\}$

$\Rightarrow$ $\{v_1, \dots, v_n\}$ is a basis of V^n

Claim $\{v_1, \dots, v_j\}$ is a set of mutually orthogonal vectors

i.e. $\forall i < j$ $v_j \perp v_i$ we show this by induction

$$v_1 \cdot v_2 = v_1 \cdot \left(e_2 - \frac{v_1 e_2}{v_1 v_1} v_1 \right) = v_1 \cdot e_2 - \frac{v_1 e_2}{v_1 v_1} \cdot v_1 v_1 = 0$$

assume statement holds for $\{v_1, \dots, v_{j-1}\}$

then, $\forall k<j$ we have

$$v_k \cdot v_j = v_k \cdot \left(e_j - \sum_{i=1}^{j-1} \frac{v_i \cdot e_j}{v_i \cdot v_i} v_i \right)$$

$$= v_k \cdot e_j - \frac{v_k e_j}{\cancel{v_k v_k}} \cancel{v_k v_k} - \sum_{\substack{i=1 \\ i \neq k}}^{j-1} \frac{v_i e_j}{v_i v_i} \cdot \underbrace{v_k \cdot v_i}_{\overset{\shortparallel}{0} \text{ by induction}}$$

$$= 0$$

$\Rightarrow$ the basis $\{v_1, \dots, v_n\}$ is orthogonal

$\Rightarrow$ the basis $\left\{ \frac{v_1}{\|v_1\|}, \dots, \frac{v_n}{\|v_n\|} \right\}$ is orthonormal. (why?)

8. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

$w_1 = v_1$

$$w_2 = v_2 - \frac{w_1 v_2}{w_1 w_1} w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}{1^2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$w_3 = v_3 - \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$$

So $w_1(0,1,0)$, $w_2(2,0,0)$ and $w_3(0,0,1/2)$ form an orthogonal basis

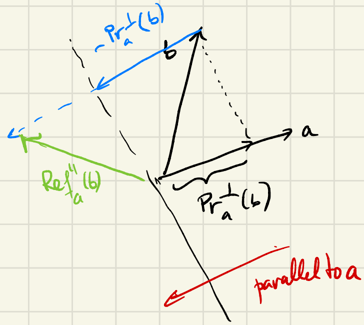
the corresponding orthonormal basis is $\frac{w_1}{\|w_1\|} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\frac{w_2}{\|w_2\|} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\frac{w_3}{\|w_3\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

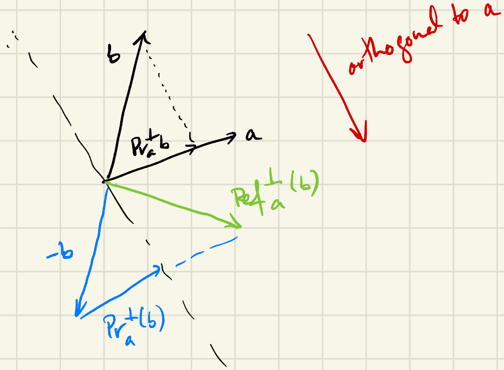
9. Show that the orthogonal reflection of a vector **b** parallel to **a** is

$$\mathrm{Ref}^{\parallel}_{\mathbf{a}}(\mathbf{b}) = \mathbf{b} - 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = \mathbf{b} - 2\,\mathrm{Pr}^{\perp}_{\mathbf{a}}(\mathbf{b}).$$

Show that the orthogonal reflection of a vector **b** in the vector **a** is

$$\mathrm{Ref}^{\perp}_{\mathbf{a}}(\mathbf{b}) = -\mathbf{b} + 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = -\mathbf{b} + 2\,\mathrm{Pr}^{\perp}_{\mathbf{a}}(\mathbf{b}) = -\,\mathrm{Ref}^{\parallel}_{\mathbf{a}}(\mathbf{b}).$$





10. Let $\mathbf{v} \in \mathbb{V}^n$ be a vector. Show that

1. The set $\mathbf{v}^{\perp}$ is a vector subspace of $\mathbb{V}^n$.
2. There is a basis $\mathbf{v}, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ of $\mathbb{V}^n$ with $\mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$ a basis of $\mathbf{v}^{\perp}$.

1.) $v^{\perp} = \{ w \in \mathbb{V}^n : w \cdot v = 0 \}$

$\forall\, w, u \in v^{\perp}\ \forall\, \alpha, \beta \in \mathbb{R} \quad (\alpha w + \beta u)\cdot v = \alpha \overset{0}{\widetilde{w\cdot u}} + \beta \overset{0}{\widetilde{u \cdot v}} = 0$

$\Rightarrow \alpha w + \beta u \in v^{\perp}$ so $v^{\perp}$ is a vector subspace

2.) Follow the Gram-Schmidt process

Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{V}^n$

By Steiniz Theorem (Algebra, Lecture 6) we may assume that $e_1 = v$

Now consider

$$v_1 = e_1 = v$$

$$v_2 = e_2 - \frac{v_1 \cdot e_2}{v_1 \cdot v_1} e_1$$

$$v_3 = e_3 - \frac{v_1 e_3}{v_1 \cdot v_1} v_1 - \frac{v_2 e_3}{v_2 \cdot v_2} v_2$$

$$\vdots$$

$$v_j = e_j - \sum_{i=1}^{j-1} \frac{v_i e_j}{v_i v_i} v_i$$

by Exercise 7 $\{\overset{v}{\overset{\parallel}{v_1}}, \dots, v_n\}$ is a basis and $\overset{v}{\overset{\parallel}{v_1}} \perp v_k \;\; \forall k \in \{2, \dots, n\}$

So, since $v^\perp$ is a proper subspace of V^n it has dimension at most $n-1$

$\Rightarrow \{v_2, \dots, v_n\}$ is a basis of $v^\perp$

11. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\phi : \mathbb{V}^3 \to \mathbb{R}$ be the map $\phi(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to an orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w} \cdot \mathbf{v}$?

- That the map is linear follows from the properties of the scalar product v is fixed and

$$\phi(\alpha w + \beta u) = v \cdot (\alpha w + \beta u) = \alpha\, v \cdot w + \beta\, v \cdot u = \alpha\, \phi(w) + \beta \phi(u)$$

for any two vectors w, u and scalars $\alpha, \beta \in \mathbb{R}$

- if $v = v(v_1, \dots, v_n)$ with respect to the orthonormal basis $e_1, \dots, e_n$ then

$$\phi(e_1) = (v_1 e_1 + \dots + v_n e_n) \cdot e_1 = v_1 \cdot e_1^2 = v_1 \cdot \|e_1\|^2 = v_1$$

$$\vdots$$

$$\phi(e_j) = v_j \quad \Rightarrow \quad [\phi]_{e_i} = [v_1 \; v_2 \; \dots \; v_n]$$

- Nothing changes if we permute v and w since $w \cdot v = v \cdot w$.

(12.) $v = v(v_1, v_2, v_3) = v_1 i + v_2 j + v_3 k$

$v \perp a \iff v \cdot a = 0 \iff 4v_1 - 2v_2 - 3v_3 = 0$

$v \perp b \iff v \cdot b = 0 \iff \quad v_2 + 3v_3 = 0 \quad \Rightarrow v_2 = -3v_3$

$\Big\} \Rightarrow 4v_1 + 3v_3 = 0$

$4v_1 = -3v_3 = v_2$

$\Downarrow$

$\left(\frac{1}{4}v_2, v_2, -\frac{1}{3}v_2\right)$

the vectors orthogonal to both a and b.

$\|v\| = 26 \iff v_1^2 + v_2^2 + v_3^2 = 676$

$\|v\| = 26 \Rightarrow \frac{1}{16}v_2^2 + v_2^2 + \frac{1}{9}v_2^2 = 26^2$

$\frac{9 + 144 + 16}{4^2 \cdot 3^2} v_2^2 = 26^2 \iff \frac{169}{4^2 \cdot 3^2} v_2^2 = 26^2$

$\iff v_2 = \pm \frac{12 \cdot 26}{13} = \pm 24$

so $v = (6, 24, -8)$ or $(-6, -24, 8)$

the angle with Ox is $\sphericalangle(v, i)$ It is acute if $\cos\sphericalangle(v,i) > 0$

$\Rightarrow v = (6, 24, -8)$