**Graph algorithms - Maximum flow of minimum cost**

**Maximum flow of minimum cost problem**

As for the maximum flow, we are given a transport network (a directed graph with capacities associated to edges, plus a source and a destination vertex). Additionaly, each edge has a cost.

In addition to the maximum flow problem, a flow also has a cost. The cost is the sum, for all edges, of the flow along the edge multiplied by the cost of the edge.

In other words, the cost of an edge is the cost of transporting each unit of flow along that edge.

The goal is to find a maximum flow and, among all possibilities to achieve it, to get one that also minimizes the cost.

**Example - input graph**

A diagram of a mathematical equation

Description automatically generated

**Solution**

First step is to obtain a maximum flow regardless of the cost. Then we minimize the cost while keeping the value of the flow constant.

We repeat the following steps:

1. construct the residual graph
2. assign costs to edges: the cost of the original edge for the forward residual edges and minus the original cost for the backwards edges
3. find a negative cost cycle in the residual graph
4. increase the flow along the cycle (like for the augmenting paths in Ford-Fulkerson)
5. stop when no negative cost cycle exists any more

**Example**

Maximum flow (flow=7, cost=1\*3+2\*4+5\*3+0\*2+0\*1+4\*4+2\*3+8\*4=80):

A diagram of a mathematical equation

Description automatically generated with residual graph A diagram of a mathematical problem with Great Pyramid of Giza in the background

Description automatically generated

After using negative cost cycle 2, 5, 4, 3, 2 (capacity = 3, cost=-6):

A diagram of a mathematical equation

Description automatically generated with residual graph A diagram of a number of numbers and points

Description automatically generated with medium confidence

No negative cost cycle can be found. Final flow=7 of cost=80-18=62.