Robotics – 34753

Robot Kinematics II & Inverse Kinematics

Konstantinos Poulios
Associate Professor
Department of Civil and Mechanical Engineering
DTU Lyngby, building 404 / room 124

Robot Kinematics II & Inverse Kinematics – Lecture Overview

1. **Repetition**

2. **Kinematic Chains**

   - **Denavit-Hartenberg Convention (D-H)**
   - Coordinate Frame Assignment
   - End Effector Frame
   - Examples

3. **Inverse Kinematics**

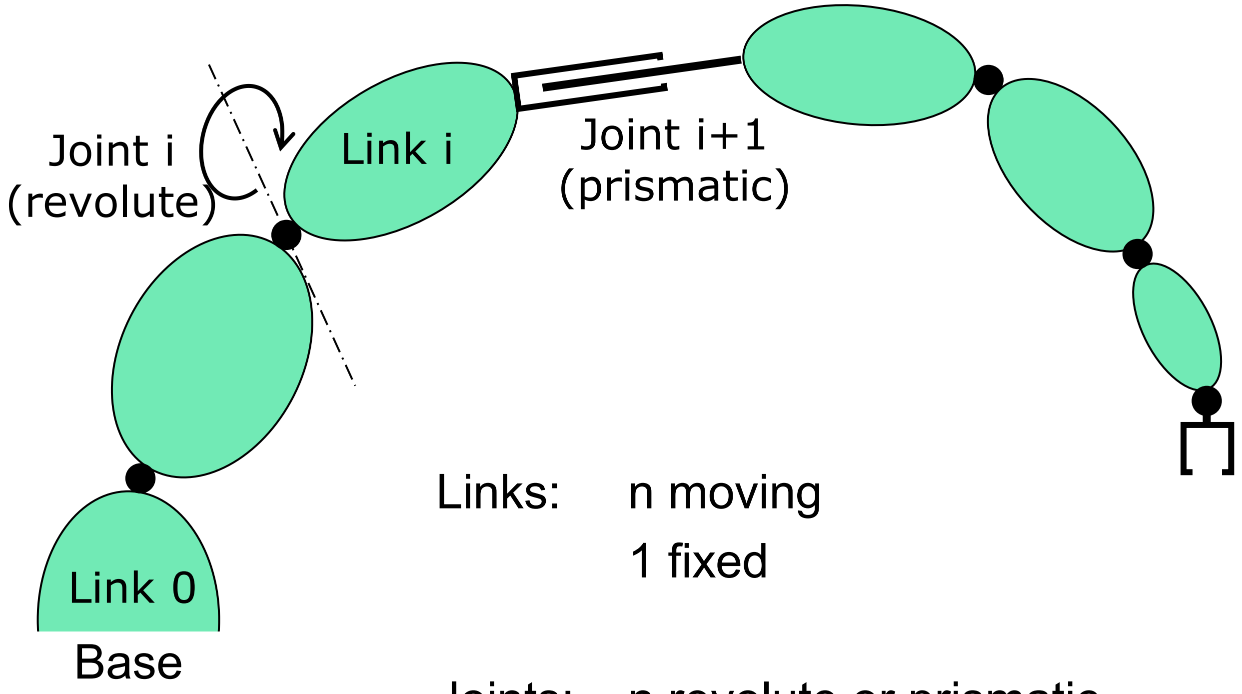
   - Inverse Position
   - Inverse Orientation

Repetition

Repetition

- Serial link manipulator (a.k.a. robot arm, industrial robot)
  - An open chain of rigid bodies (links) connected by joints (revolute or prismatic)

- Manipulator specification
  - Degrees of freedom: n
  - Joint space
  - Work space
  - Redundancy: n > m



Links: n moving
1 fixed

Joints: n revolute or prismatic
1 dof each

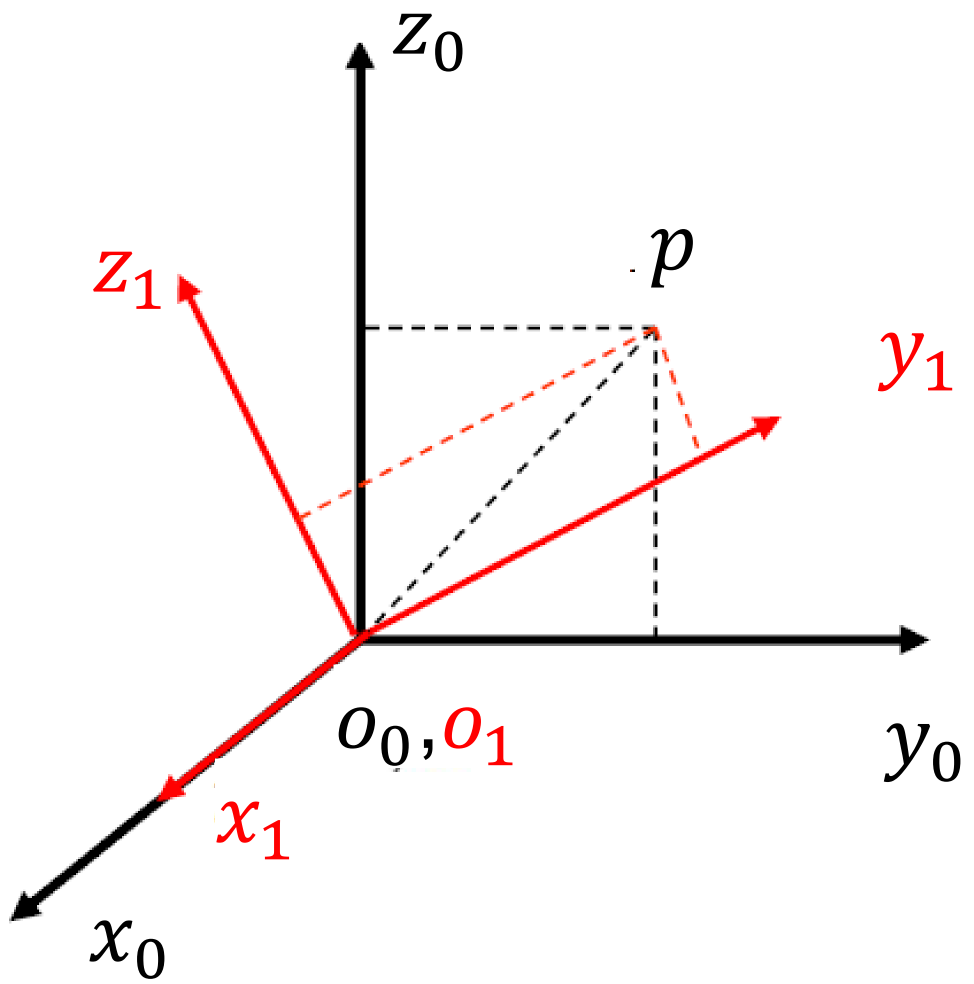
Repetition

- Basic rotation matrix between frame $o_0 x_0 y_0 z_0$ and frame $o_1 x_1 y_1 z_1$

$$p^0 = \begin{bmatrix} x_0 \cdot x_1 & x_0 \cdot y_1 & x_0 \cdot z_1 \\ y_0 \cdot x_1 & y \cdot y_1 & y_0 \cdot z_1 \\ z_0 \cdot x_1 & z_0 \cdot y_1 & z_0 \cdot z_1 \end{bmatrix} p^1$$

$$p^0 = R_1^0 \, p^1$$

$$p^1 = (R_1^0)^{-1} \, p^0$$



Repetition

- Basic rotation matrices

  – About x-axis with θ

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}$$

  – About y-axis with θ

$$R_{y,\theta} = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}$$

  – About z-axis with θ

$$R_{z,\theta} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Repetition

- Coordinate transformation from {1} to {0}

$$p^0 = R_1^0\, p^1 + o_1^0$$

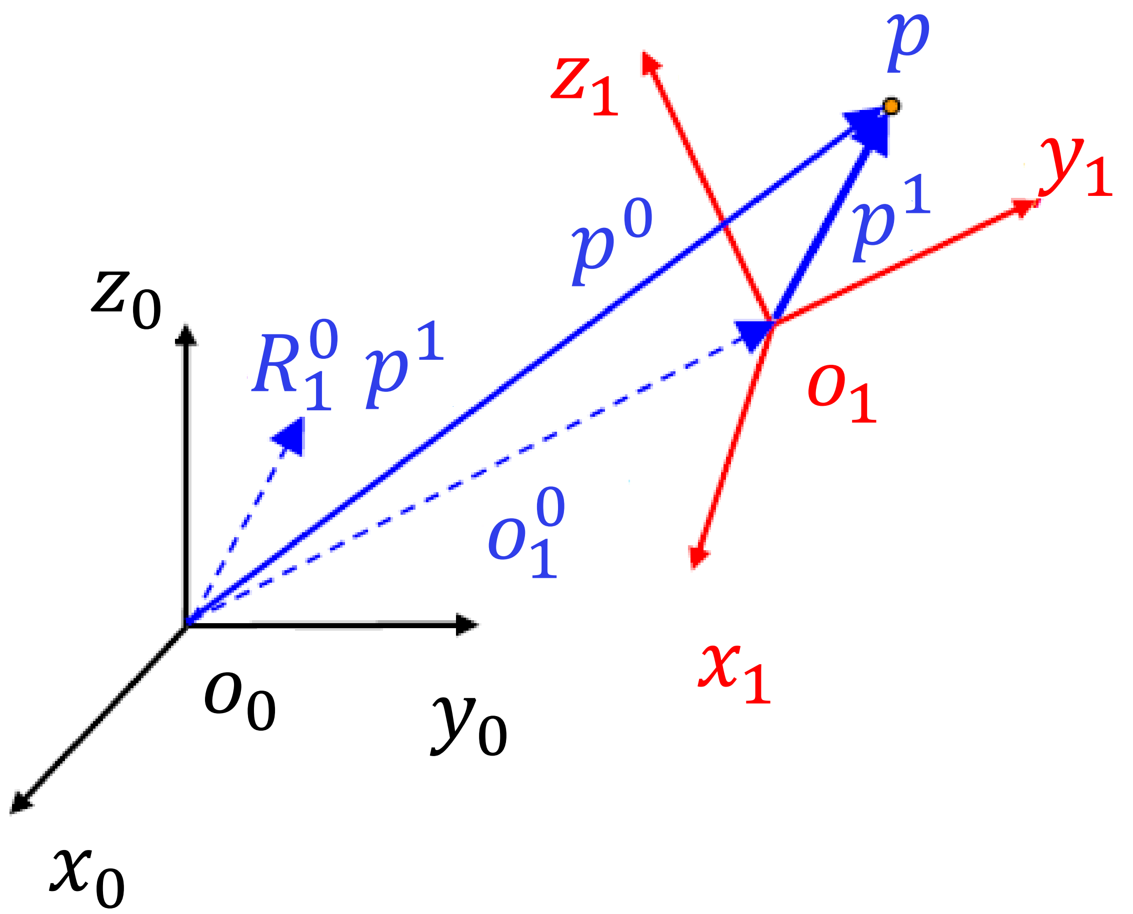
$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & o_1^0 \\ 0_{1x3} & 1 \end{bmatrix} \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

- Homogeneous transformation matrix

rotation matrix

position vector

$$T_1^0 = \begin{bmatrix} R_1^0 & o_1^0 \\ 0_{1x3} & 1 \end{bmatrix}$$



Repetition

- Special cases

  – Translation

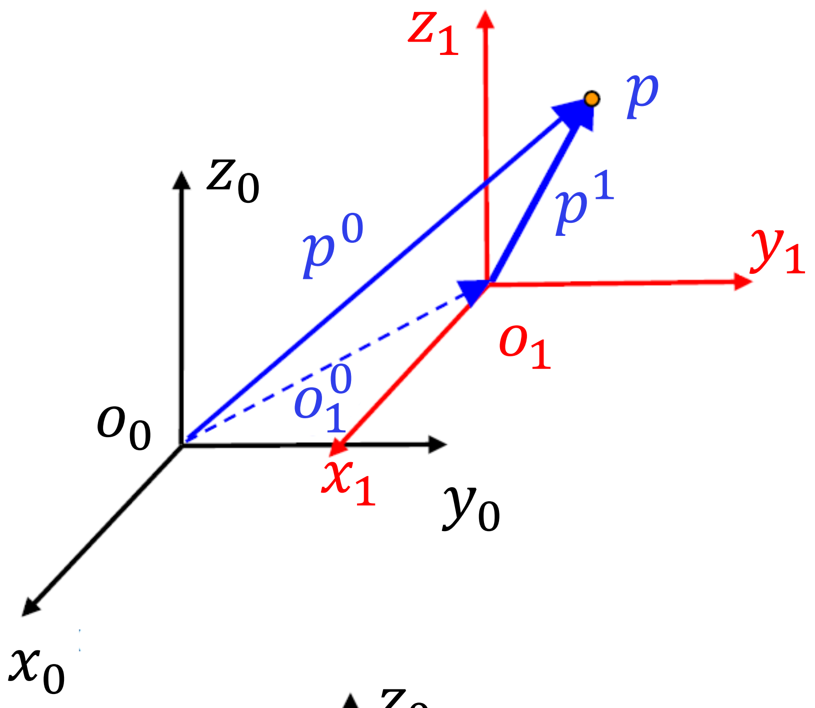
$$H_1^0 = \begin{bmatrix} I_{3\times3} & o_1^0 \\ 0_{1\times3} & 1 \end{bmatrix}$$

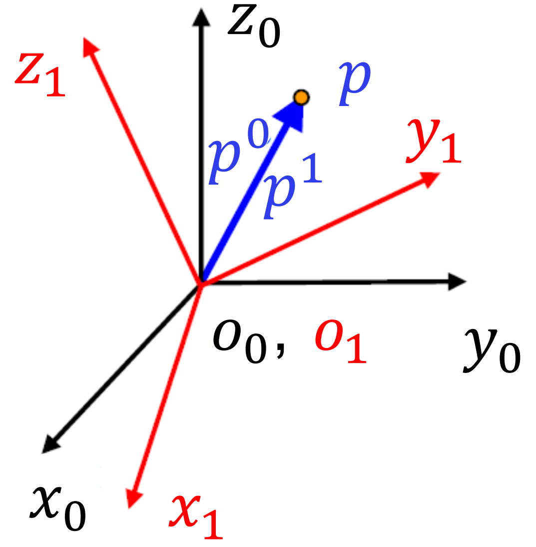
  – Rotation

$$H_1^0 = \begin{bmatrix} R_1^0 & 0_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix}$$

- General case

$$H_1^0 = \begin{bmatrix} x_1^0 & y_1^0 & z_1^0 & o_1^0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Repetition

- Basic homogeneous transformation matrices

$$\mathrm{Trans}_{\mathrm{x},a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{\mathrm{x},\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{\mathrm{y},b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{\mathrm{y},\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{\mathrm{z},c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{\mathrm{z},\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Repetition

- Composition of Transformations

  - Rotation/Translation wrt. original/fixed/global/world frame → **Pre-Multiplication**

  - Rotation/Translation wrt. current frame → **Post-Multiplication**

- Example with rotations

  1. θ-rotation about **current** x
  2. ϕ-rotation about **current** z
  3. α-rotation about **fixed** z
  4. β-rotation about **current** y
  5. δ-rotation about **fixed** x

$$R_5^0 = \underbrace{R_{x,\delta}}_{5}\underbrace{R_{z,\alpha}}_{3}\underbrace{R_{x,\theta}}_{1}\underbrace{R_{z,\phi}}_{2}\underbrace{R_{y,\beta}}_{4}$$

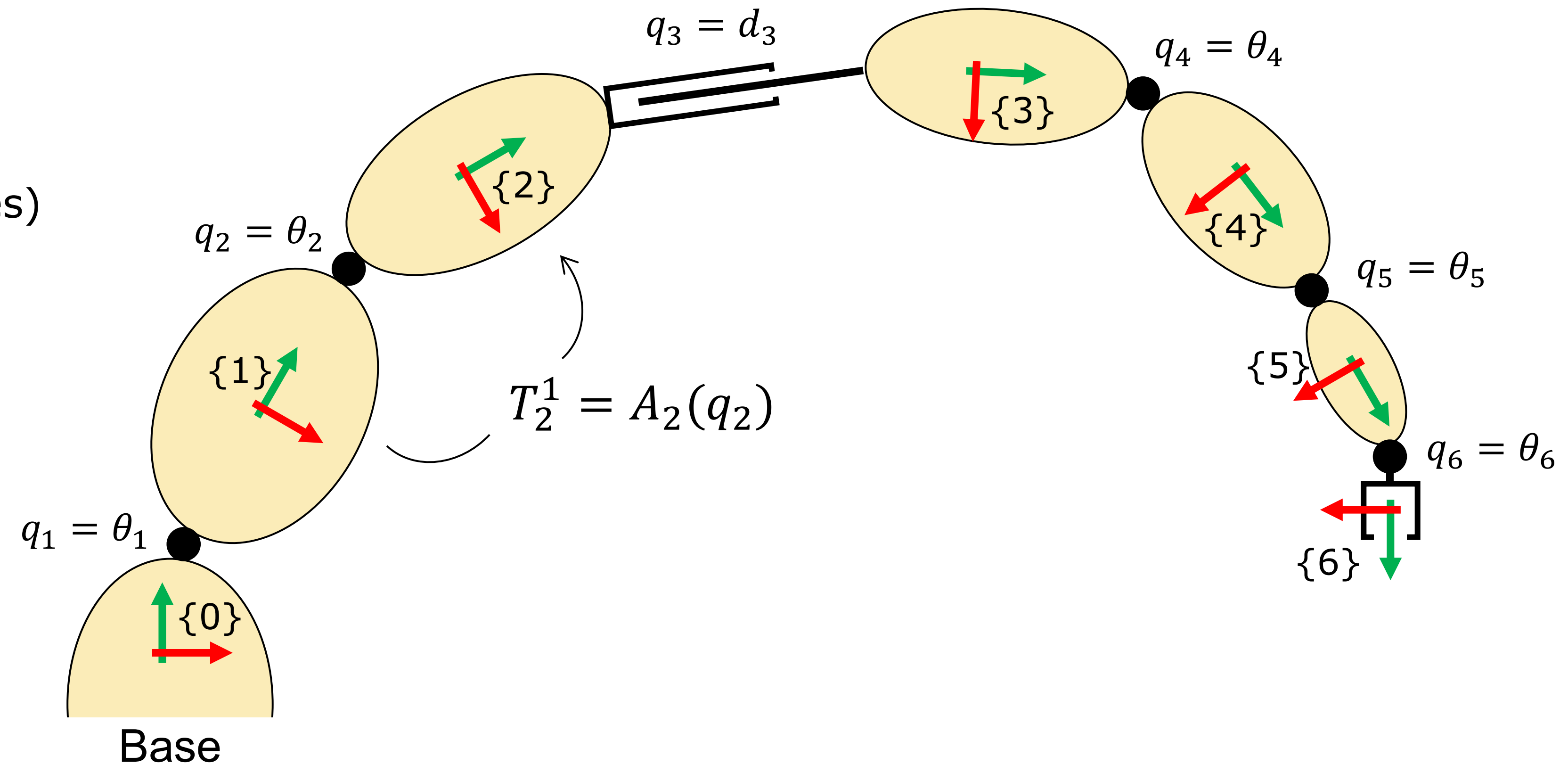
Kinematic Chains

Kinematic Chains

$n = 6$ joints/dofs

$n + 1 = 7$ links (bodies)

$n + 1 = 7$ frames



Note:
$A_i = T_i^{i-1}$

$$T_6^0 = A_1(q_1)\ A_2(q_2)\ A_3(q_3)\ A_4(q_4)\ A_5(q_5)\ A_6(q_6)$$

… but each homogeneous transformation matrix $A_j(q_j)$ is a rather complex function

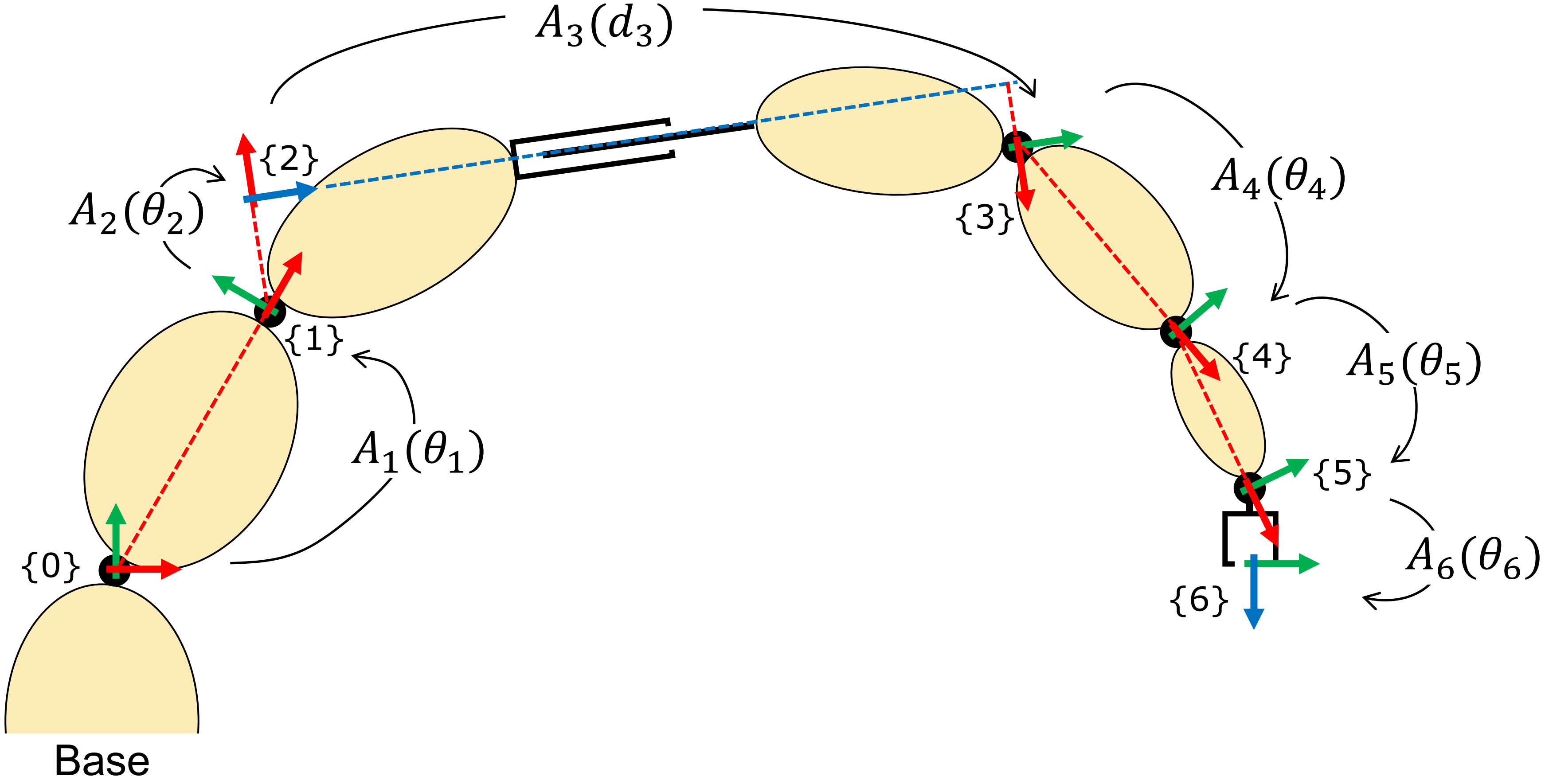
Kinematic Chains

$n = 6$ joints/dofs

$n + 1 = 7$ links (bodies)

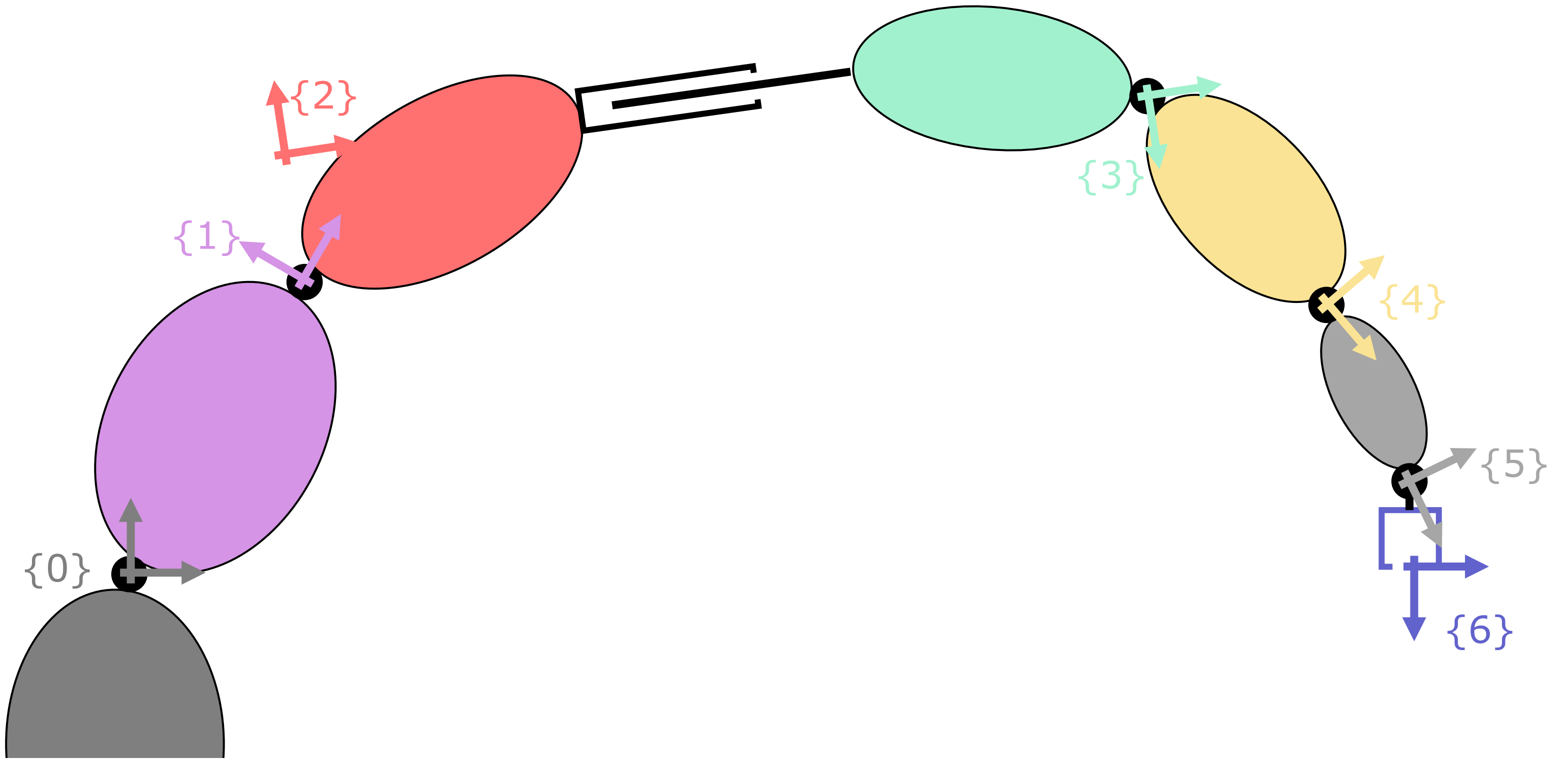
$n + 1 = 6 + 1$ frames

Last frame is special



$$T_6^0 = A_1(\theta_1)\ A_2(\theta_2)\ A_3(d_3)\ A_4(\theta_4)\ A_5(\theta_5)\ A_6(\theta_6)$$

Kinematic Chains



Denavit–Hartenberg Convention

- The coordinate frame {i} is determined from the frame {i-1} through a homogeneous transformation in the form:

$$A_i = Rot_{z,\theta_i}\, Trans_{z,d_i}\, Trans_{x,a_i}\, Rot_{x,\alpha_i}$$

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Where:
  - θ_i : joint angle
  - d_i : link offset
  - a_i : link length
  - α_i : link twist

Only 4 parameters instead of 6.

Why?

Denavit–Hartenberg Convention

- The coordinate frame {i} is determined from the frame {i-1} through a homogeneous transformation in the form:

$$A_i = Rot_{z,\theta_i}\, Trans_{z,d_i}\, Trans_{x,a_i}\, Rot_{x,\alpha_i}$$

- Where: θ_i : joint angle
  d_i : link offset
  a_i : link length
  α_i : link twist

Only 4 parameters instead of 6.

Why?

- New coordinate frame with z-axis aligned with the **next** joint axis
- Position along and rotation around the axis **not free** to choose anymore

Denavit–Hartenberg Convention

- Where: θ_i : joint angle
  d_i : link offset
  a_i : link length
  α_i : link twist

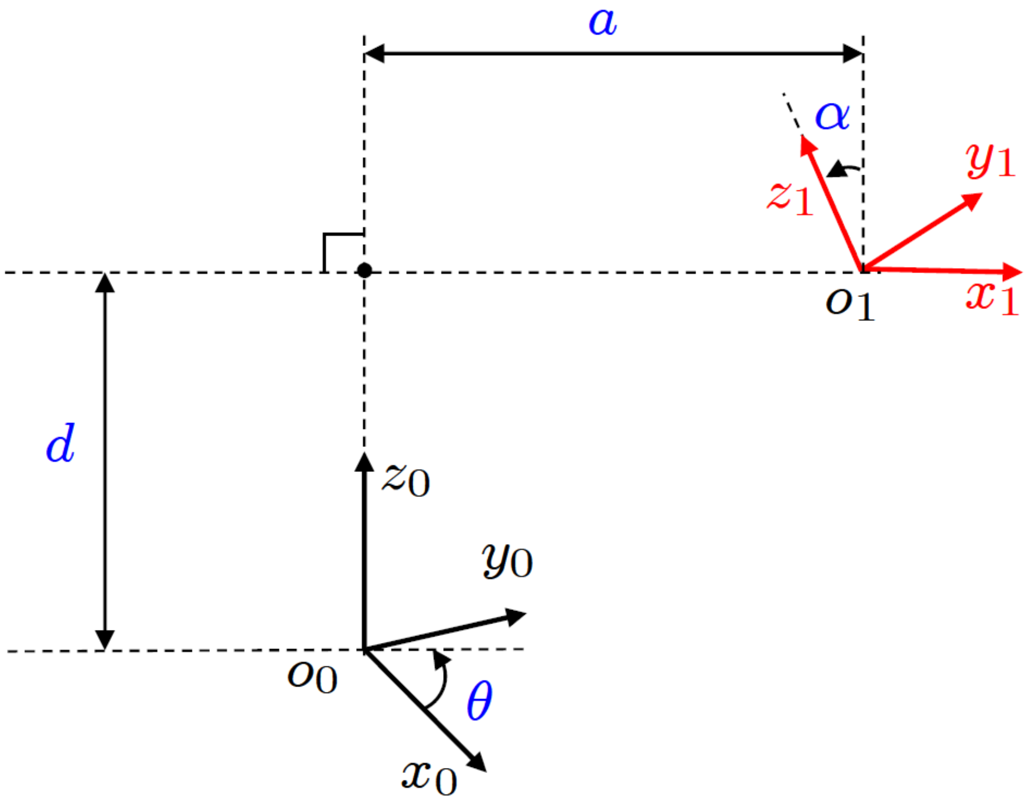
Only 4 parameters instead of 6

Why?

- The missing two parameters are due to the two Denavit–Hartenberg conditions:

DH1: The axis x_i is perpendicular to the axis z_{i-1}

DH2: The axis x_i intersects the axis z_{i-1}



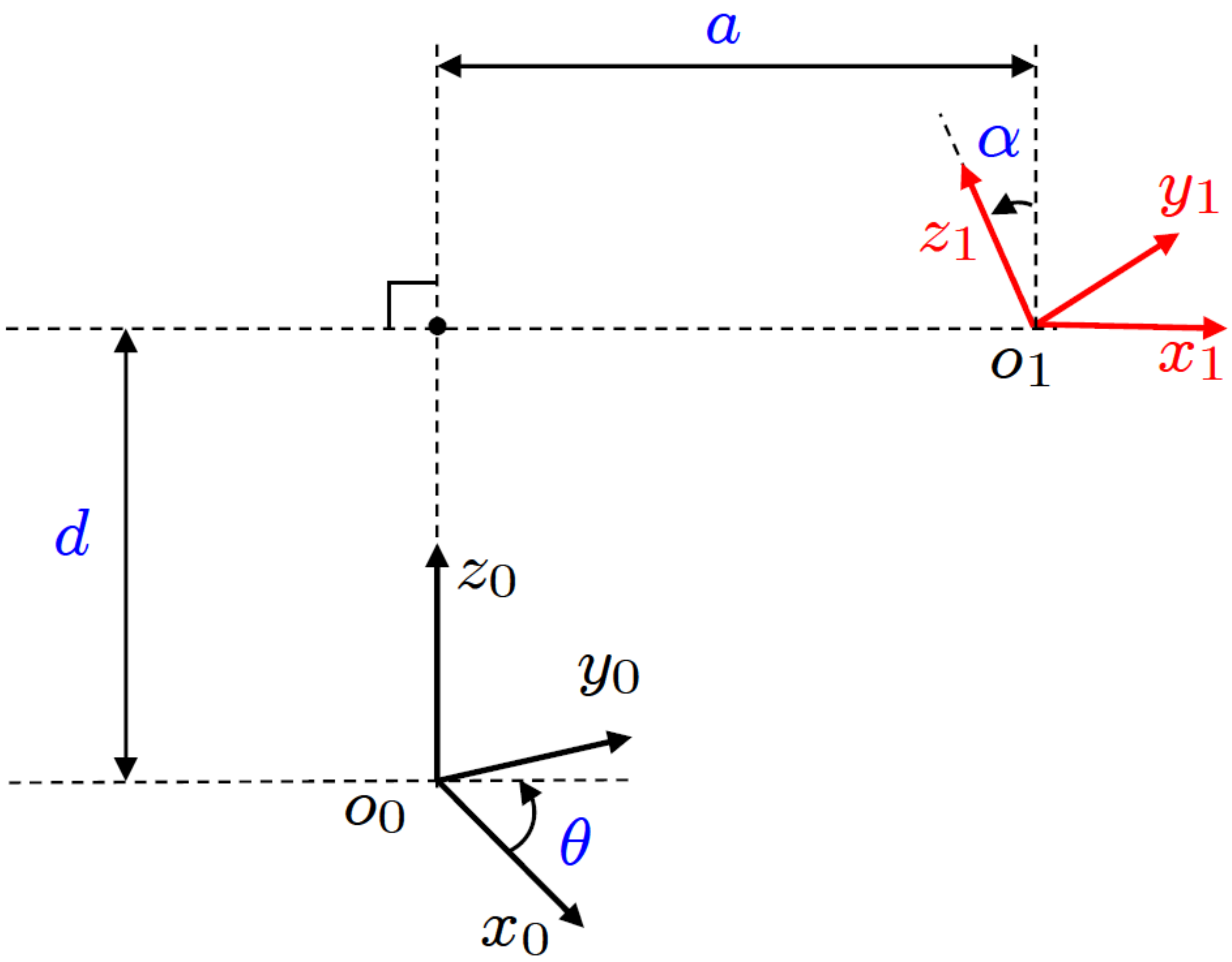
Physical Interpretation of θ, d, a, α

θ_1 : angle from x_0 to x_1 about z_0

d_1 : distance from o_0 to x_1 (along z_0)

a_1 : distance from z_0 to o_1 (along x_1)

α_1 : angle from z_0 to z_1 about x_1



Mnemonic:

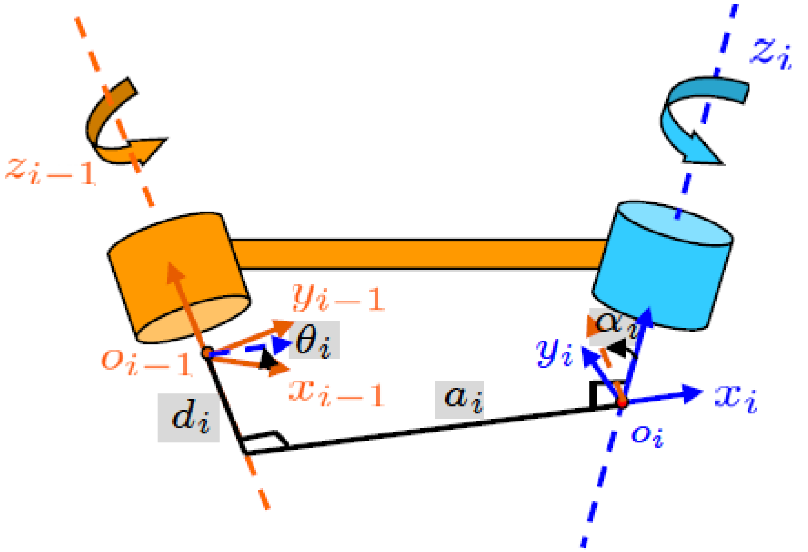
θ - d - a - α

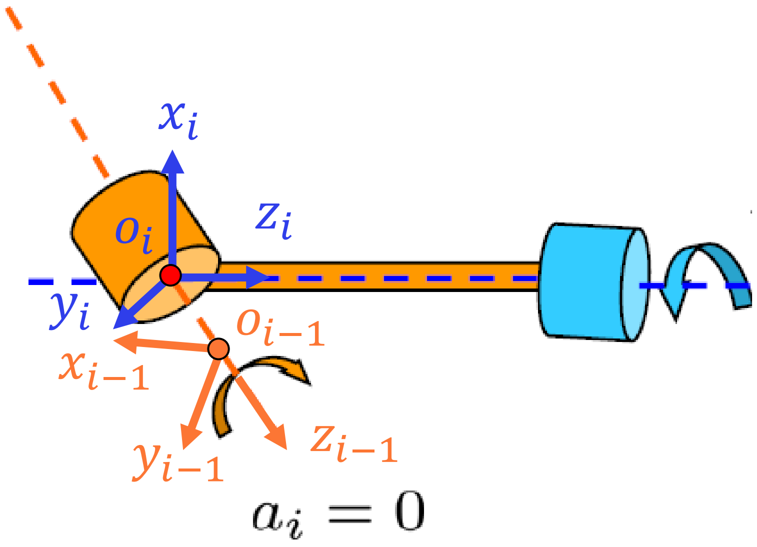
about-along - along-about

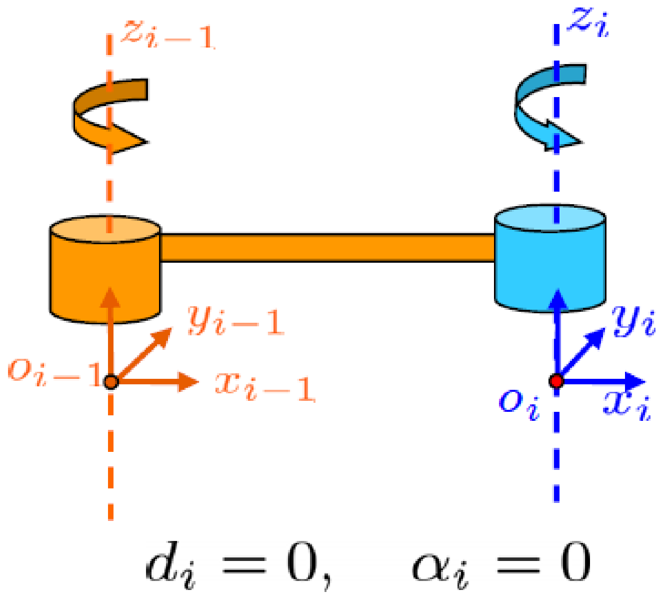
z_{i-1} x_i

Physical Interpretation of θ, d, a, α

θ_1	: angle from x_0 to x_1 about z_0	}	one of both is a variable: θ_1 for revolute, d_1 for prismatic
d_1	: distance from o_0 to x_1 (along z_0)		
a_1	: distance from z_0 to o_1 (along x_1)	}	always constant characteristic of the manipulator
α_1	: angle from z_0 to z_1 about x_1		

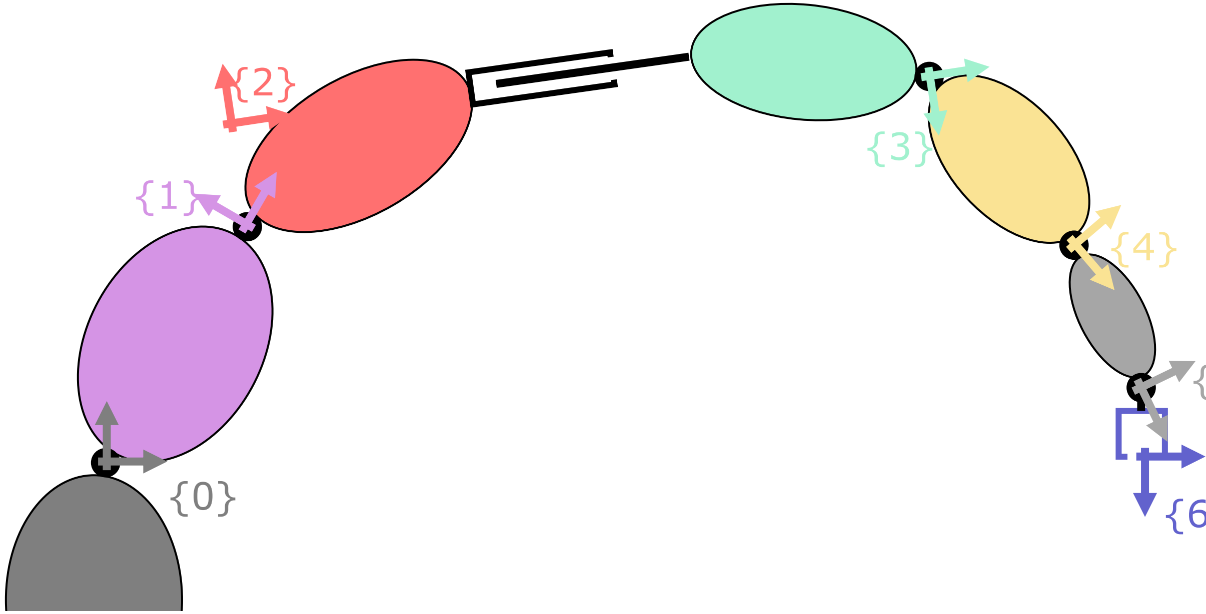


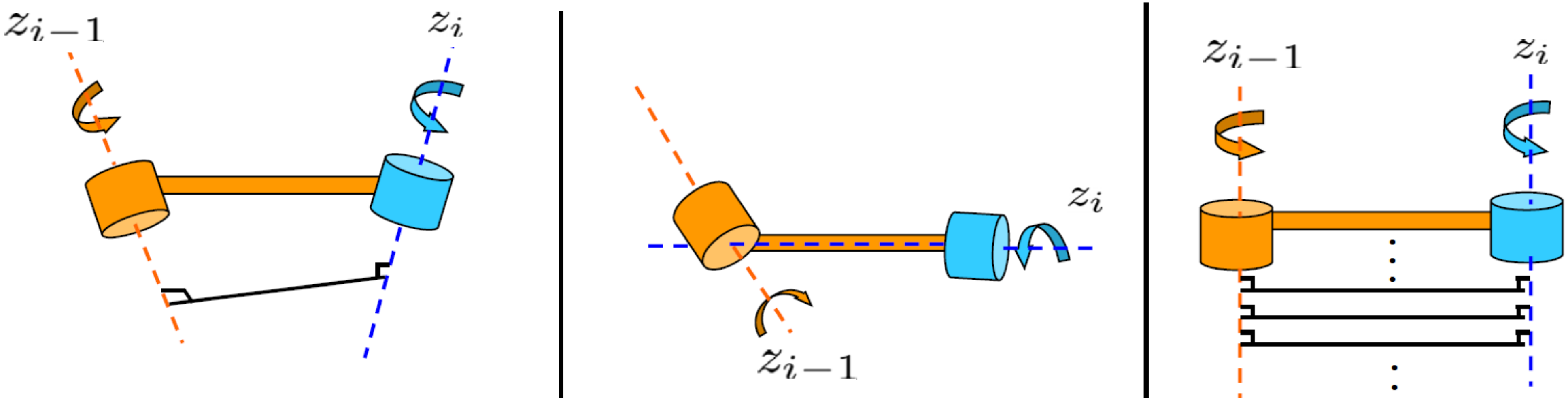




Assignment of Coordinate Frames

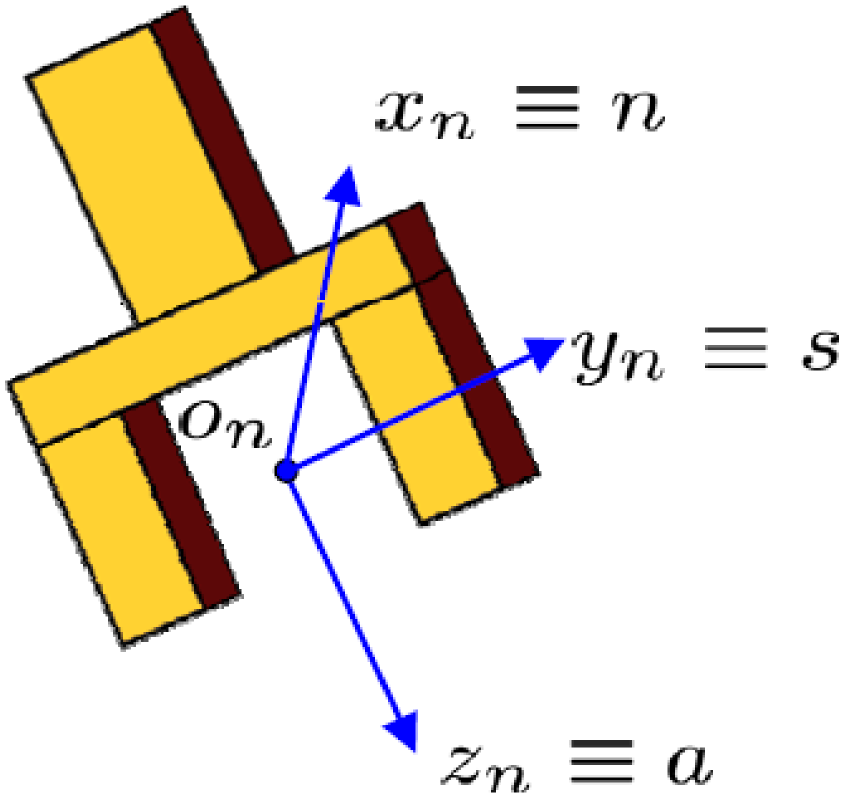
- z_i axis along the $i+1$ joint axis
- x_i axis parallel to $z_i \times z_{i-1}$
- y_i axis parallel to $z_i \times x_i$
- Origin o_i along z_i at the point of shortest distance to z_{i-1}





End Effector Coordinate Frame

- Origin o_n in the middle between the fingers
- z_n axis parallel to the fingers, also denoted as a (approach)
- y_n axis parallel to the closing direction of the fingers, also denoted as s (sliding)
- x_n axis parallel to $y_n \times z_n$, also denoted as n (normal)



Examples

Three-Link Cylindrical Robot

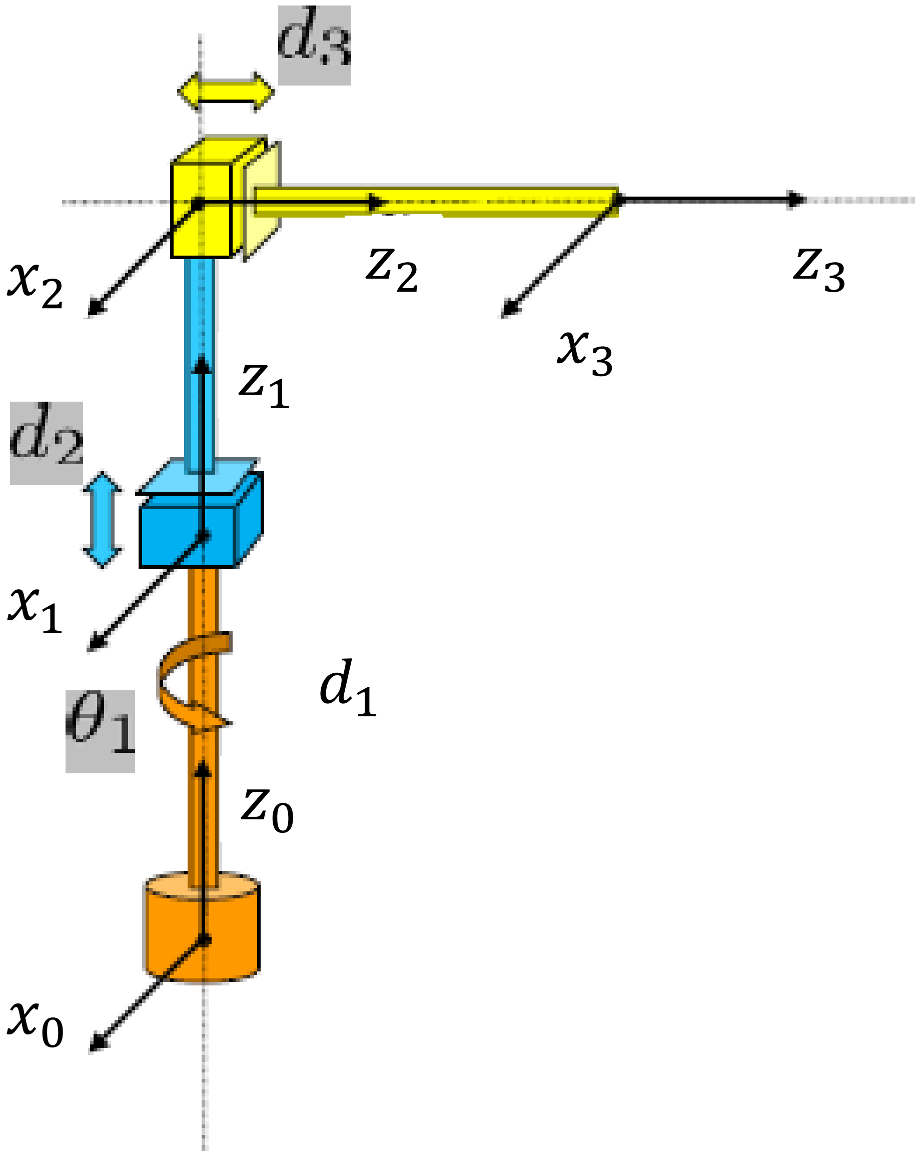
θ_i : angle from x_{i-1} to x_i about z_{i-1}

d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Joint i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	θ_1^*	d_1	0	0
2	0	d_2^*	0	-90
3	0	d_3^*	0	0



SCARA Manipulator

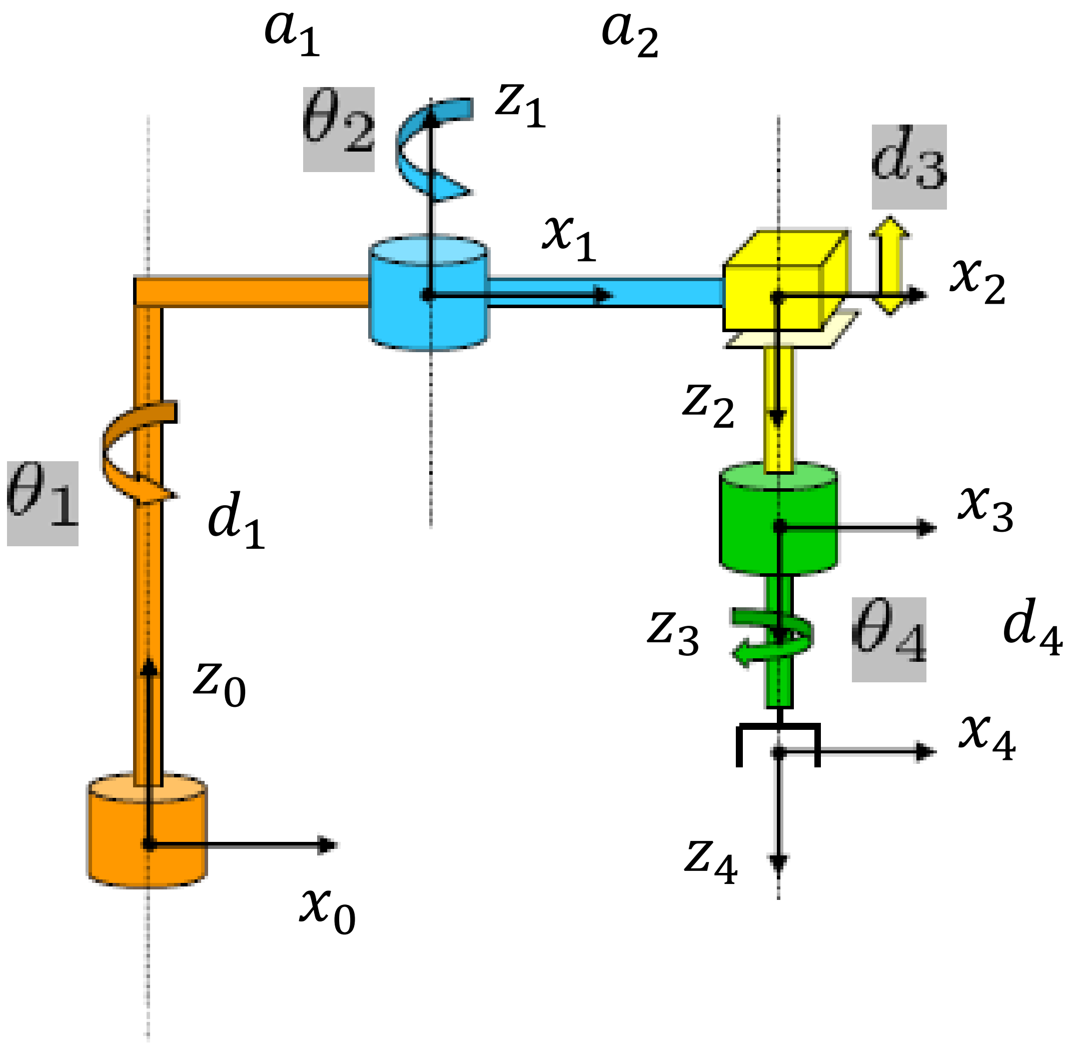
θ_i : angle from x_{i-1} to x_i about z_{i-1}

d_i : distance from o_{i-1} to x_i (along z_{i-1})

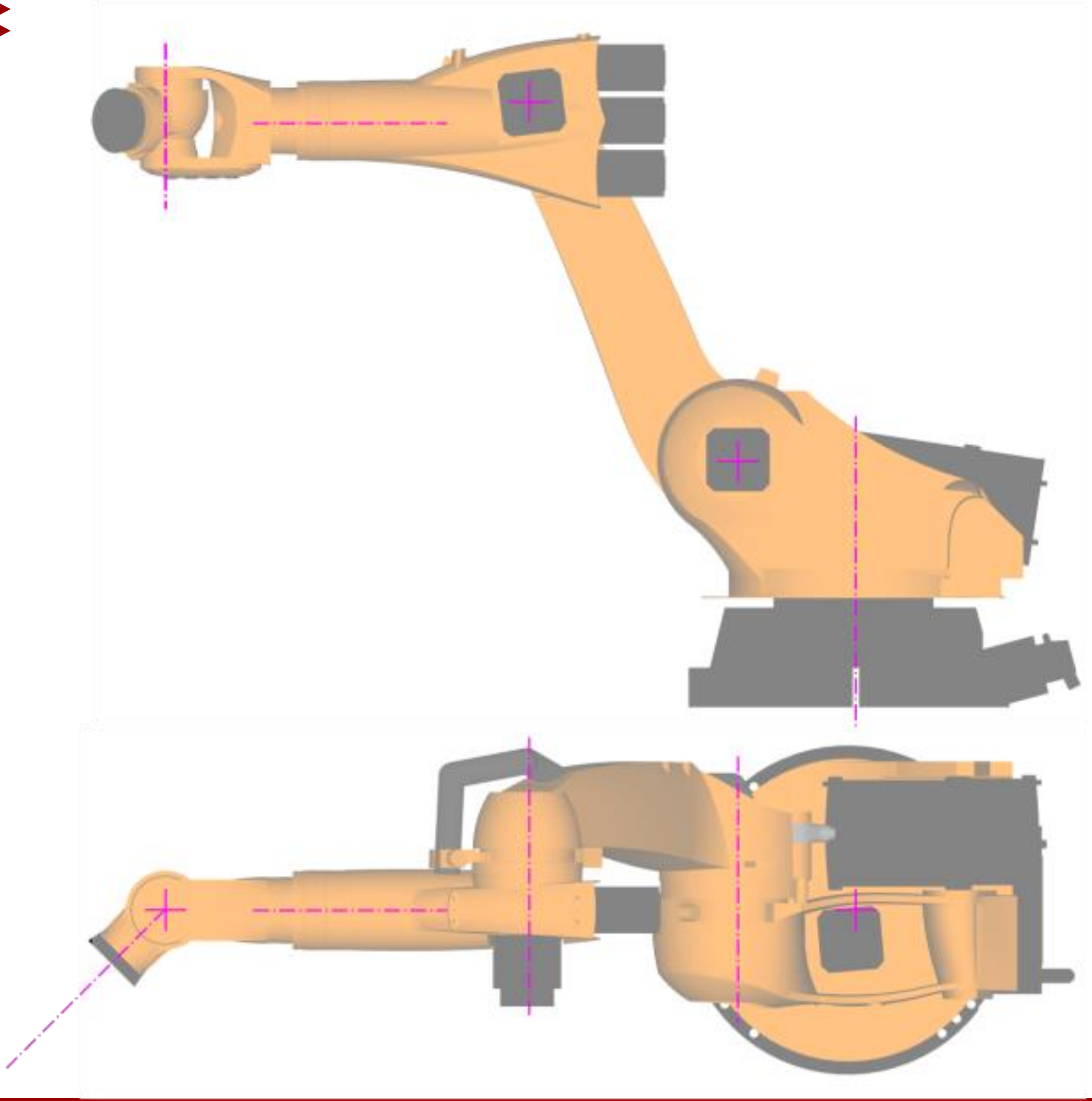
a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Joint i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	θ_1^*	d_1	a_1	0
2	θ_2^*	0	a_2	180
3	0	d_3^*	0	0
4	θ_4^*	d_4	0	0

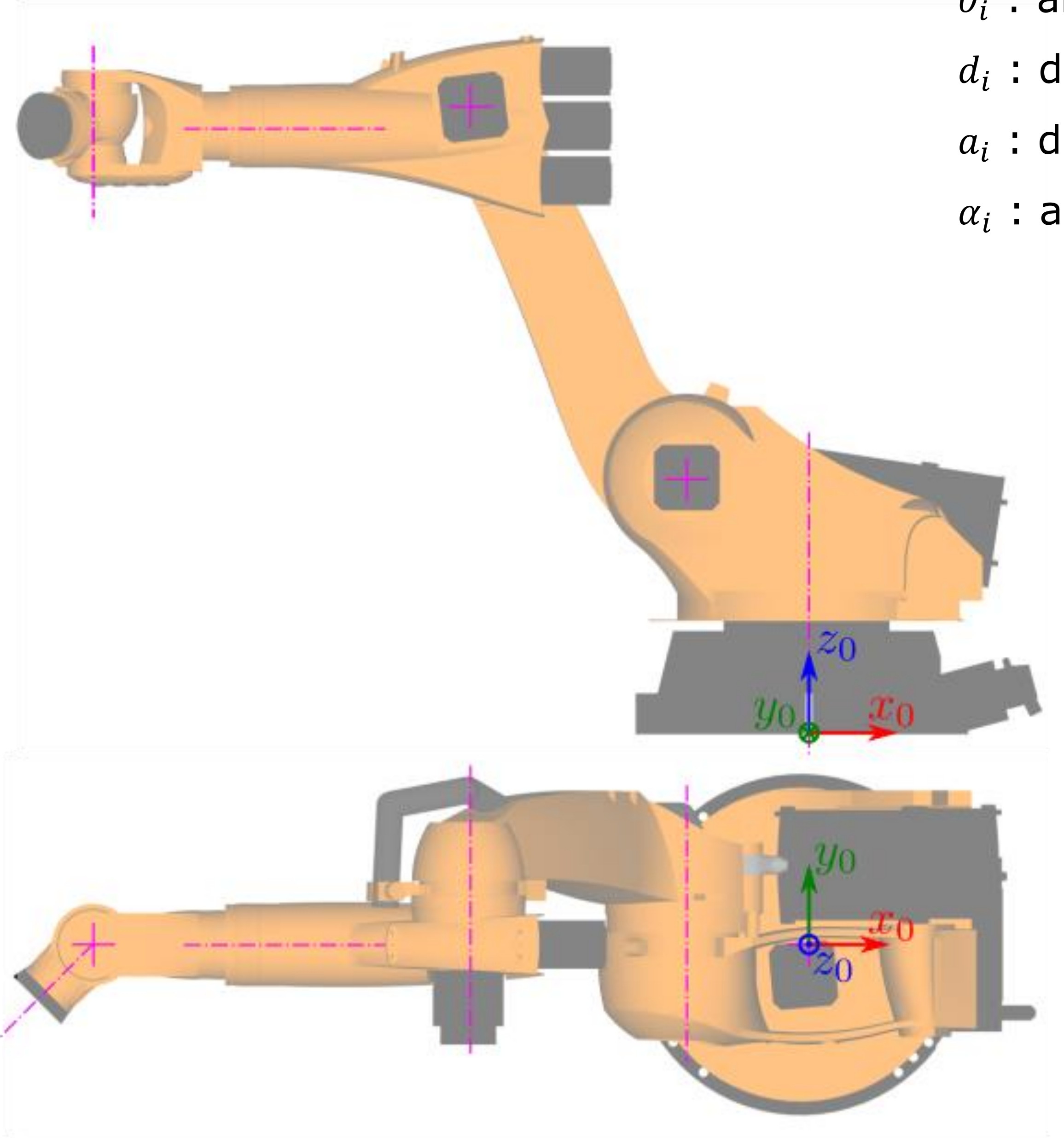


KUKA KR 210



- Number the joints
- Establish base frame
- Establish joint axes z_i
- Locate origin o_i
- Establish x_i and y_i

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

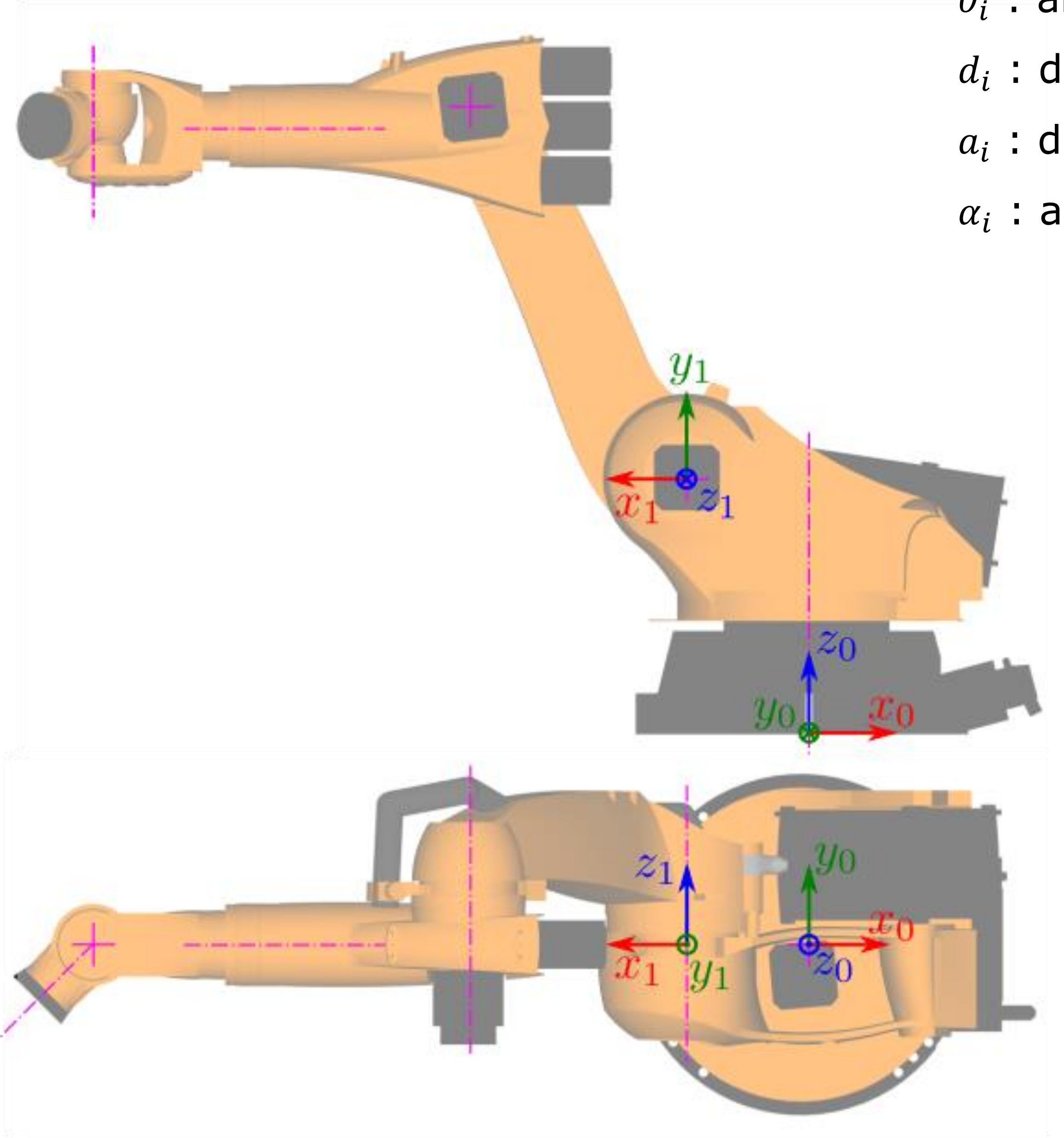
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	θ_i	d_i	a_i	α_i
1				
2				
3				
4				
5				
6				

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

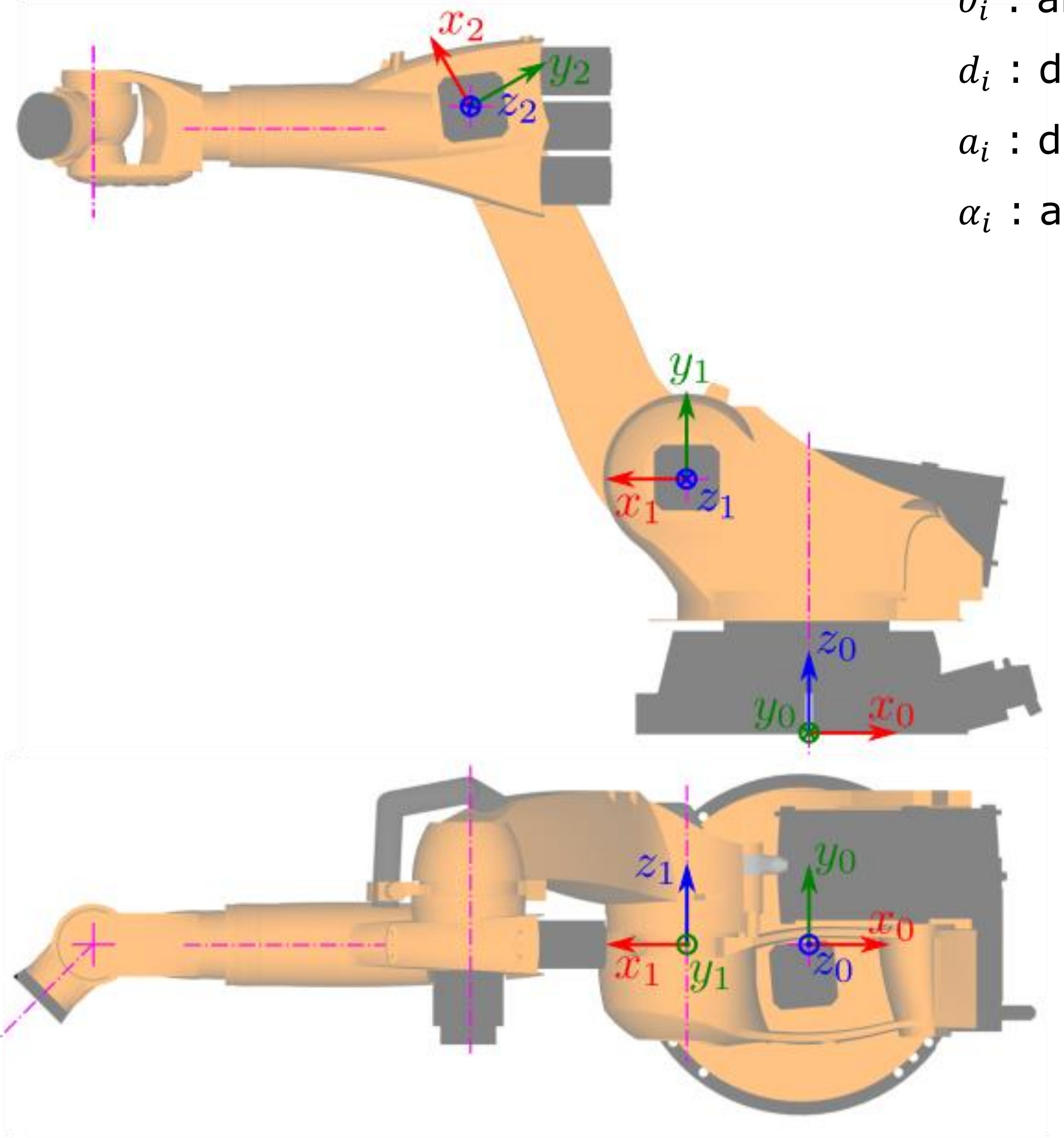
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	θ_i	d_i	a_i	α_i
1	$\theta_1^* = 180$	200	100	90
2				
3				
4				
5				
6				

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

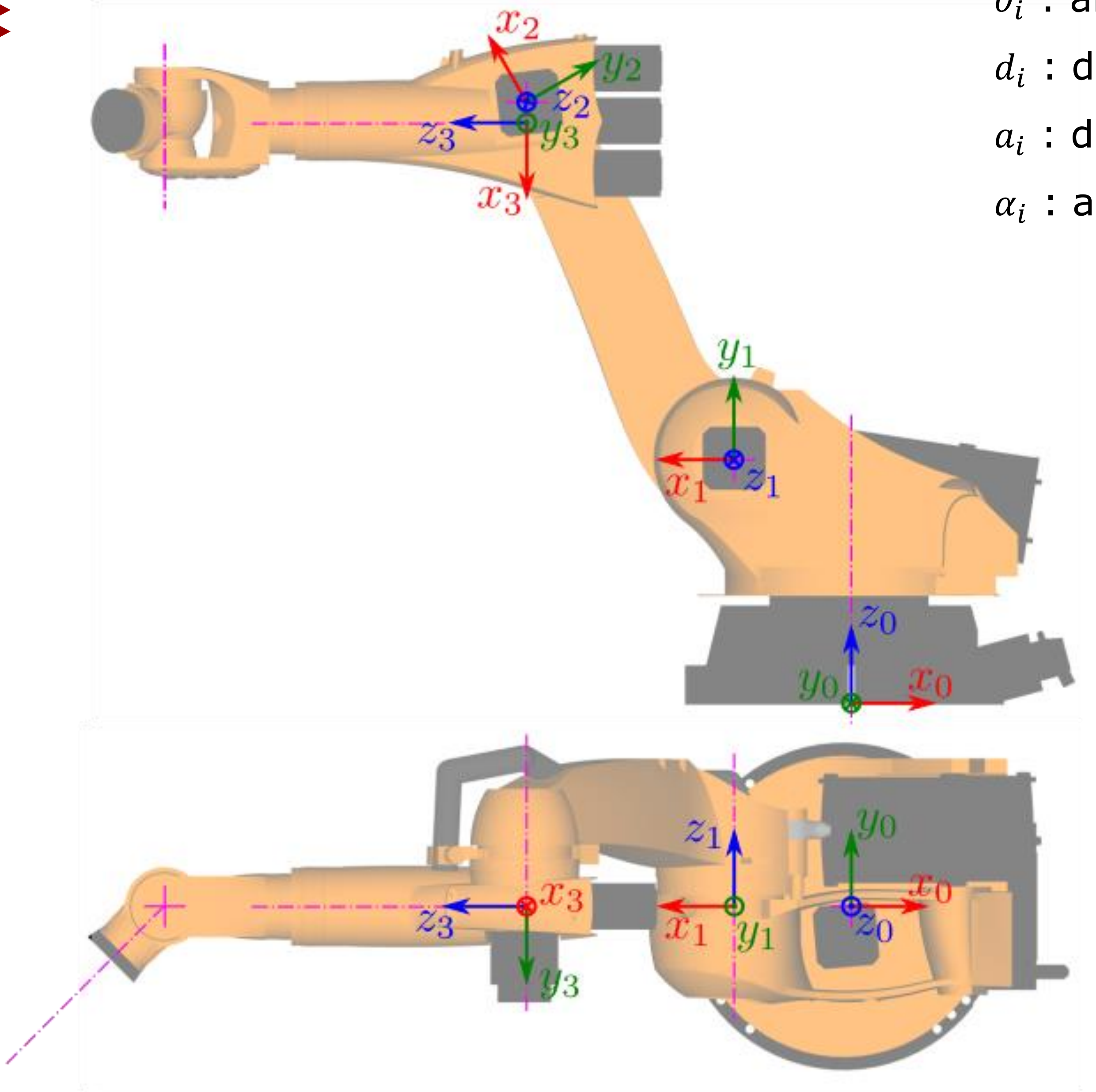
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	$\theta_1^* = 180$	200	100	90
2	$\theta_2^* = 60$	0	300	0
3				
4				
5				
6				

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

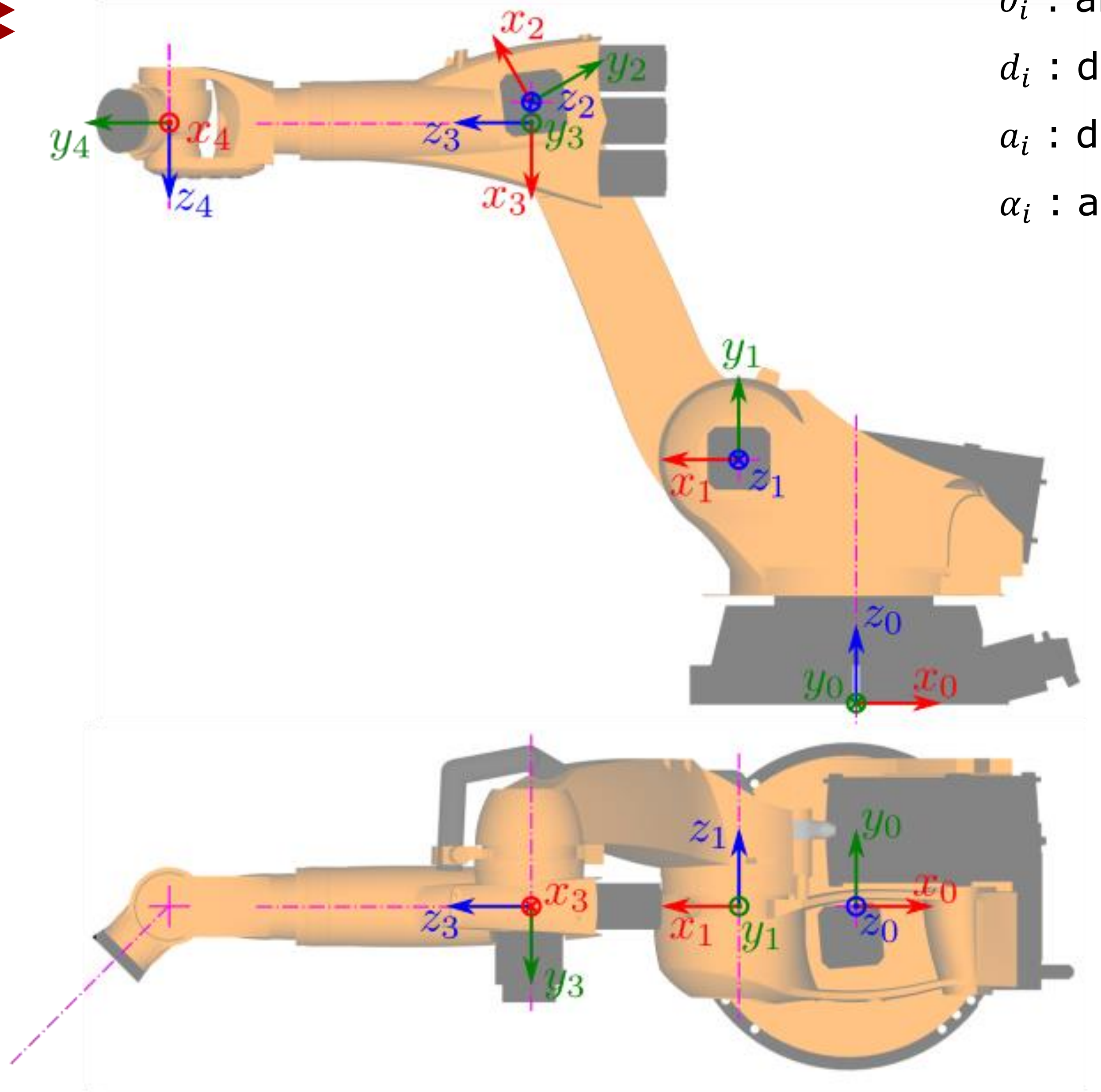
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	θ_i	d_i	a_i	α_i
1	$\theta_1^* = 180$	200	100	90
2	$\theta_2^* = 60$	0	300	0
3	$\theta_3^* = -150$	0	20	-90
4				
5				
6				

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

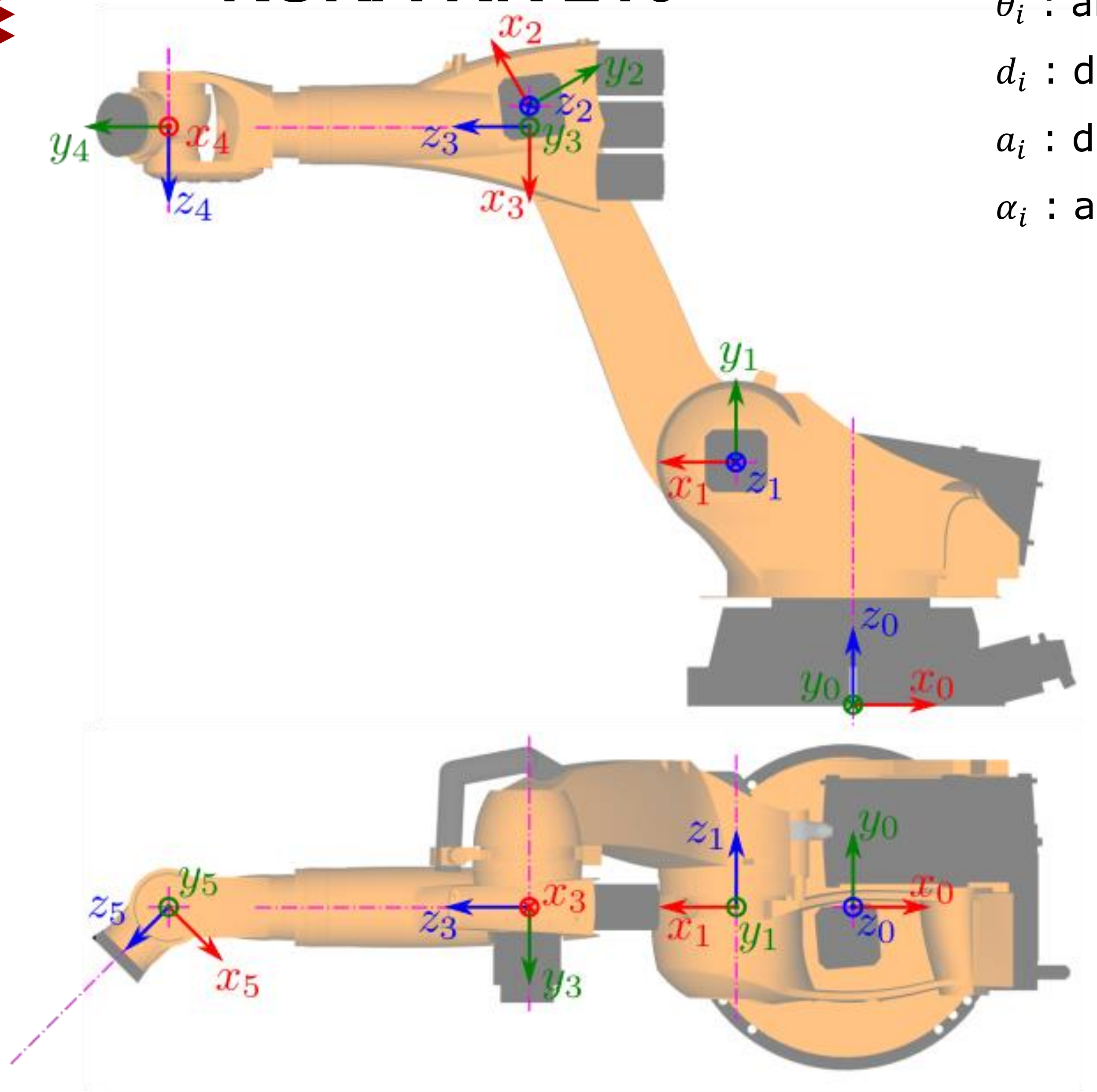
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	$\theta_1^* = 180$	200	100	90
2	$\theta_2^* = 60$	0	300	0
3	$\theta_3^* = -150$	0	20	-90
4	90	$d_4^* = 300$	0	90
5				
6				

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

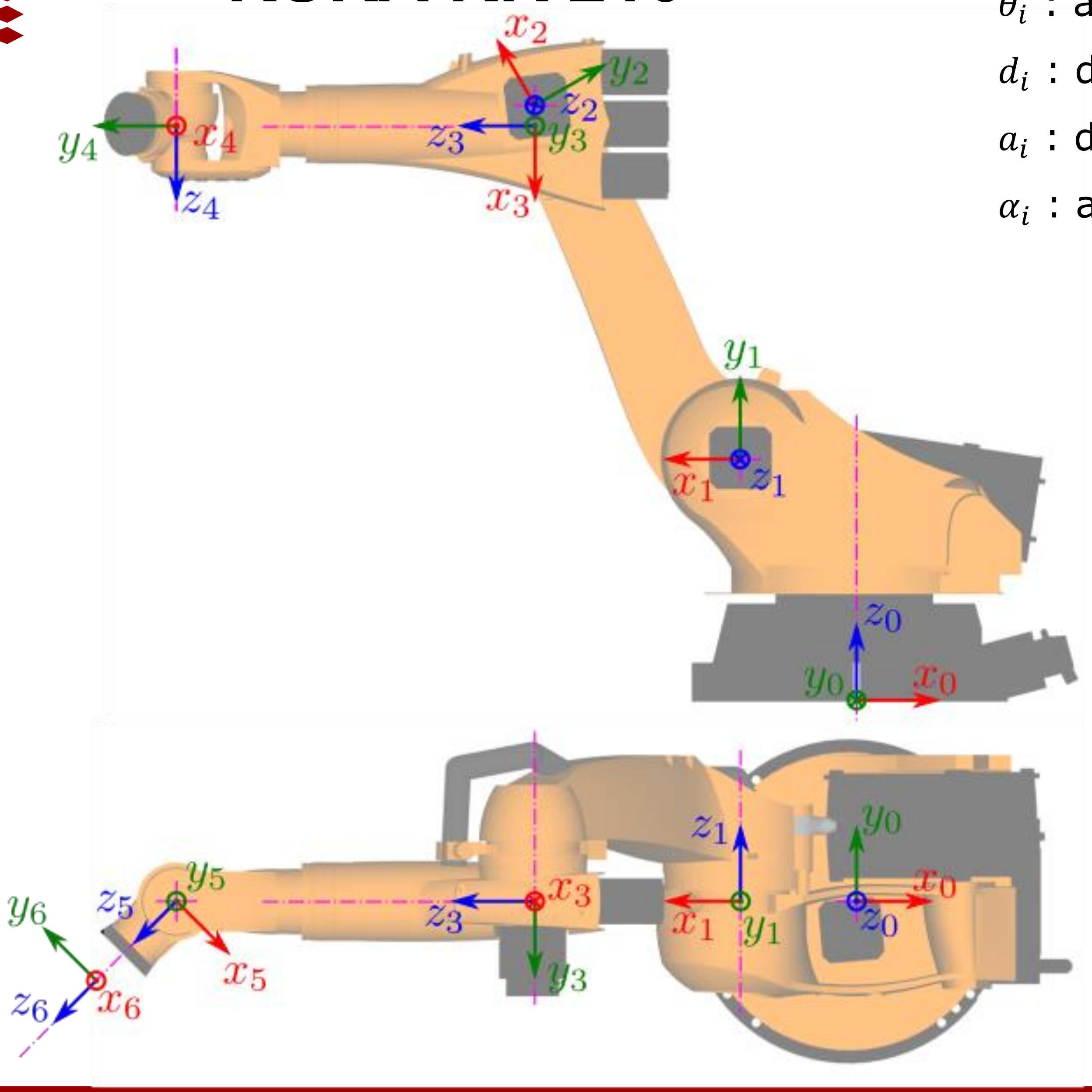
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	$\theta_1^* = 180$	200	100	90
2	$\theta_2^* = 60$	0	300	0
3	$\theta_3^* = -150$	0	20	-90
4	90	$d_4^* = 300$	0	90
5	$\theta_5^* = -45$	0	0	-90
6				

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

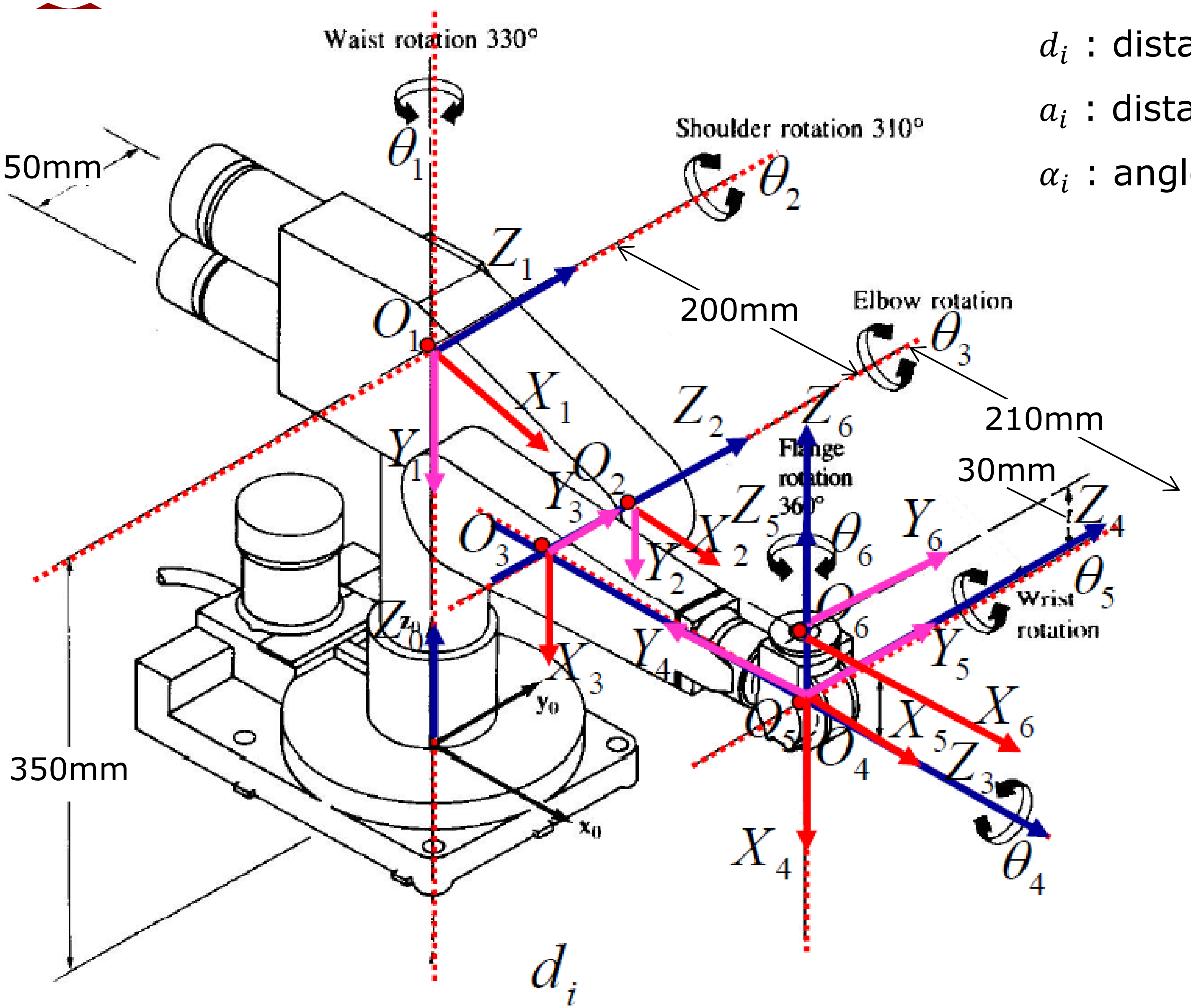
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	$\theta_1^* = 180$	200	100	90
2	$\theta_2^* = 60$	0	300	0
3	$\theta_3^* = -150$	0	20	-90
4	90	$d_4^* = 300$	0	90
5	$\theta_5^* = -45$	0	0	-90
6	$\theta_6^* = 90$	50	0	0

PUMA 260



θ_i : angle from x_{i-1} to x_i about z_{i-1}

d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	θ_1^*	350	0	-90
2	θ_2^*	0	200	0
3	θ_3^*	-50	0	0
4	θ_4^*	210	0	90
5	θ_5^*	0	0	90
6	θ_6^*	30	0	0

Inverse Kinematics

Inverse Kinematics

- **Forward** kinematics:

  Known joint degrees of freedom $q_1, q_2, \dots q_n$

  → find the pose of the end effector as:

$$H = T_n^0 = A_1(q_1)\, A_2(q_2)\, A_3(q_3) \dots A_n(q_n)$$

- **Inverse** kinematics:

  Known homogeneous transformation H for end effector

  → solve the nonlinear system of equations

$$A_1(q_1)\, A_2(q_2)\, A_3(q_3) \dots A_n(q_n) = H$$

  for $q_1, q_2, \dots q_n$

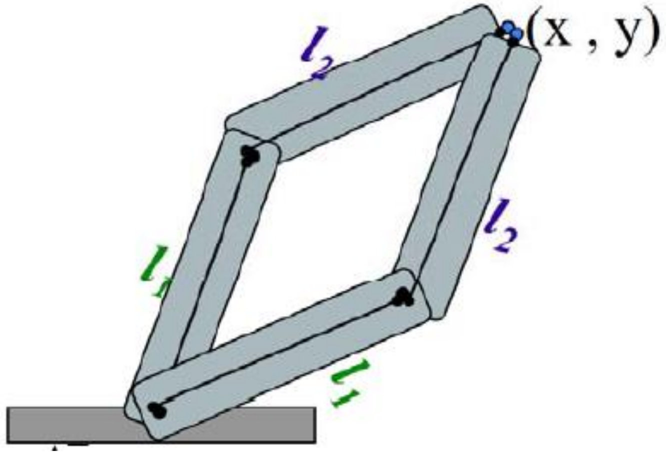
  → 12 nonlinear equations → too difficult to find analytical solutions for

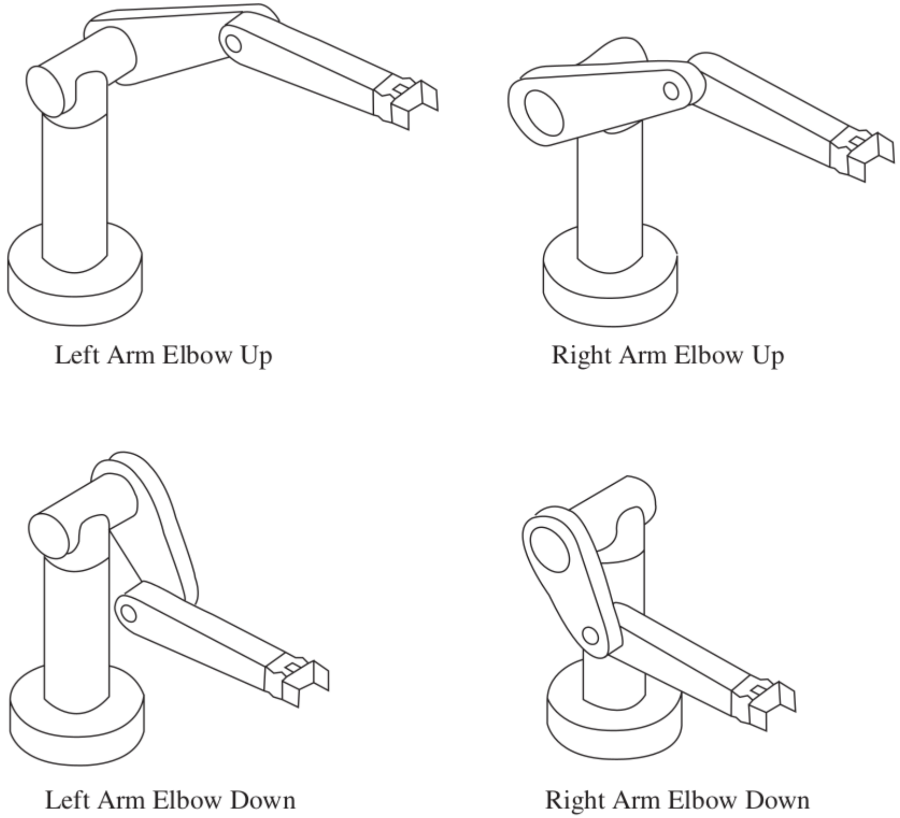
Inverse Kinematics

- 12 nonlinear equations
  → too difficult to find analytical solutions

- Not unique solutions
  – Redundant manipulator
  – Elbow-up/elbow-down solutions

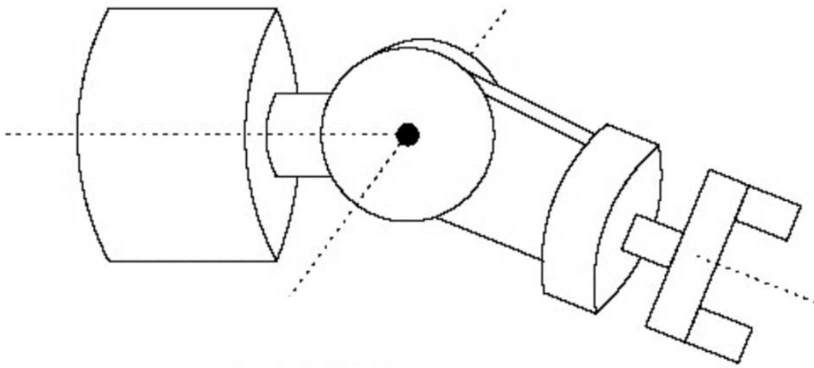
- Kinematic decoupling
  – Inverse position: geometric approach
  – Inverse orientation Euler angles





Inverse Kinematics – Kinematic Decoupling

- Decoupling of position and orientation
  - Assume a manipulator where the last 3 dofs correspond to a spherical wrist
  - First solve for the position of the wrist center
  - Then solve for the orientation of the end-effector

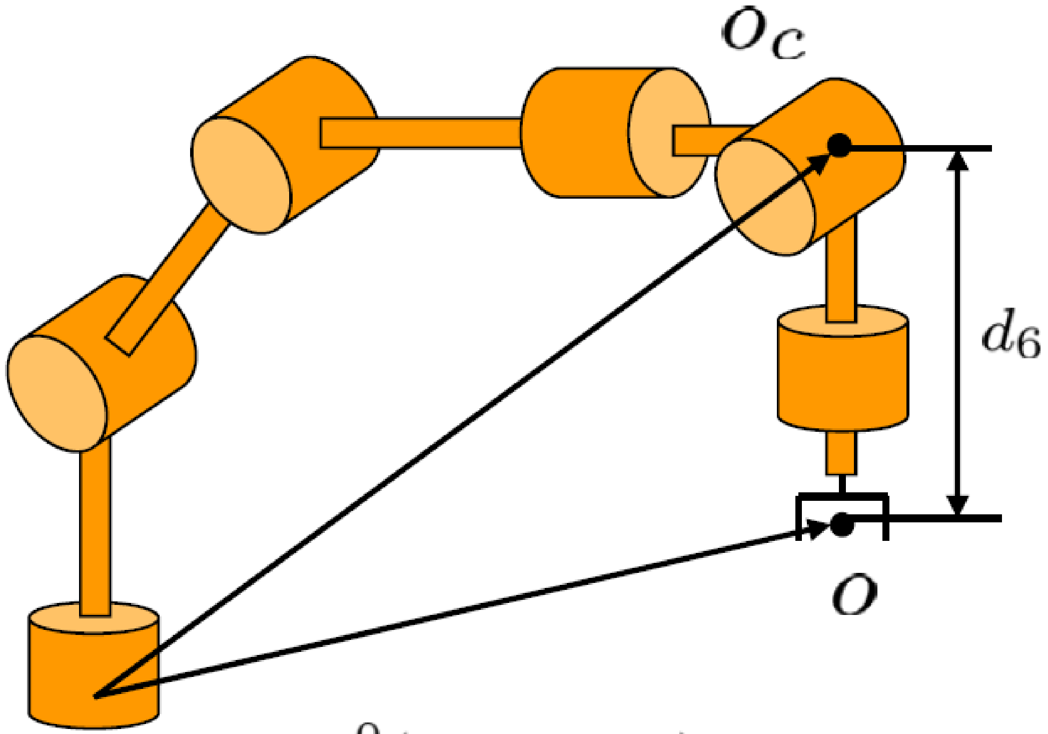


Inverse Kinematics – Kinematic Decoupling

Kinematic decoupling for 6-DoF manipulator with spherical wrist:

- Inverse position kinematics
  → wrist center
- Inverse orientation kinematics
  → wrist orientation

Axes z_3, z_4, z_5 intersect at o_c:
their rotations will not affect the position of o_c



$$\begin{cases} R_6^0(q_1, \ldots, q_6) = R \\ o_6^0(q_1, \ldots, q_6) = o \end{cases}$$

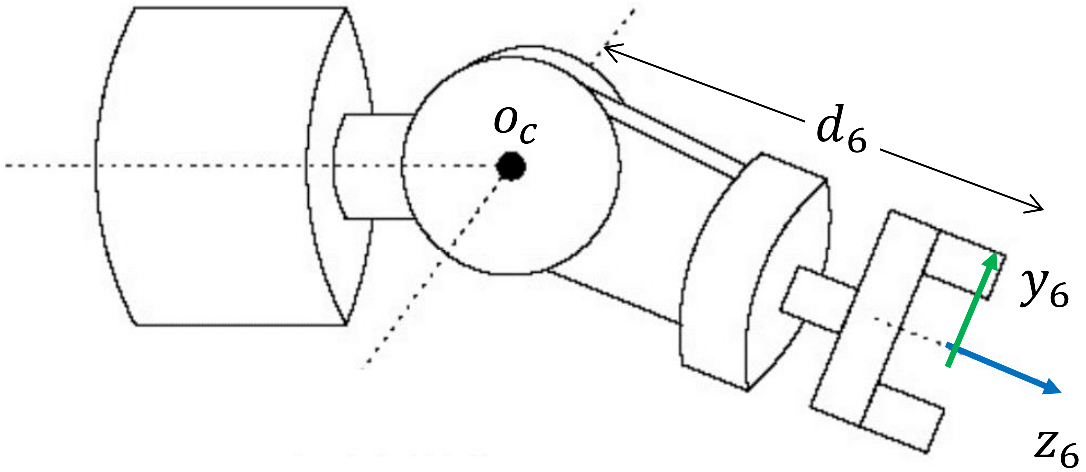
Kinematic Decoupling

- Intended homogeneous transformation matrix:

$$H = A_1A_2A_3A_4A_5A_6 = T_6^0 = \begin{bmatrix} R_6^0 & o_6^0 \\ 0_{1\times3} & 1 \end{bmatrix}$$

Based on the wrist configuration we know:

$$o_c^0 = o_6^0 - d_6R_6^0\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix} o_x - d_6r_{13} \\ o_y - d_6r_{23} \\ o_z - d_6r_{33} \end{bmatrix}$$



- We find q_1, q_2, q_3 that satisfy o_c^0
- We apply forward kinematics to calculate

$$R_3^0 = R_1^0(q_1)R_2^1(q_2)R_3^2(q_3)$$

- We find q_4, q_5, q_6 from solving:

$$R_6^0 = R_3^0R_6^3(q_4,q_5,q_6) \quad \Rightarrow \quad R_6^3(q_4,q_5,q_6) = (R_3^0)^TR_6^0$$

Kinematic Decoupling

Kinematic decoupling for 6-DoF manipulator with spherical wrist:

Axes z_3, z_4, z_5 intersect at o_c:
their rotations will not affect the position of o_c

Inverse position

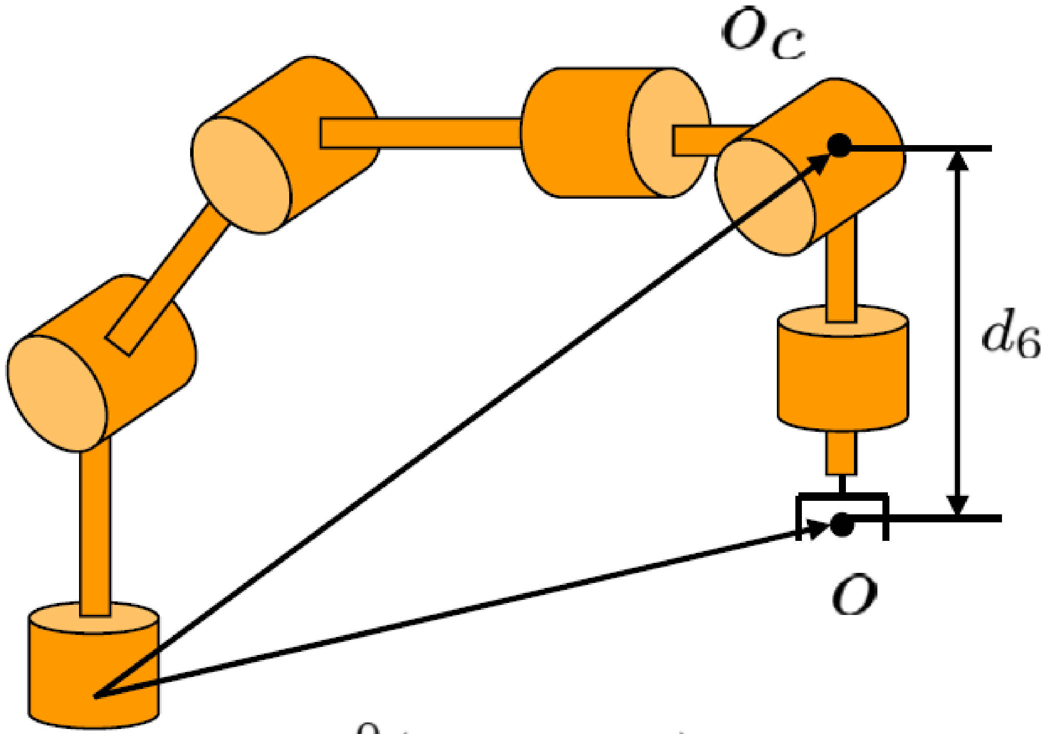
$$o_c^0(q_1, q_2, q_3) = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

$\rightarrow q_1, q_2, q_3$

Inverse orientation

$$R_6^3(q_4, q_5, q_6) = (R_3^0(q_1, q_2, q_3))^T R_6^0$$

$\rightarrow q_4, q_5, q_6$



$$\begin{cases} R_6^0(q_1, \ldots, q_6) = R \\ o_6^0(q_1, \ldots, q_6) = o \end{cases}$$

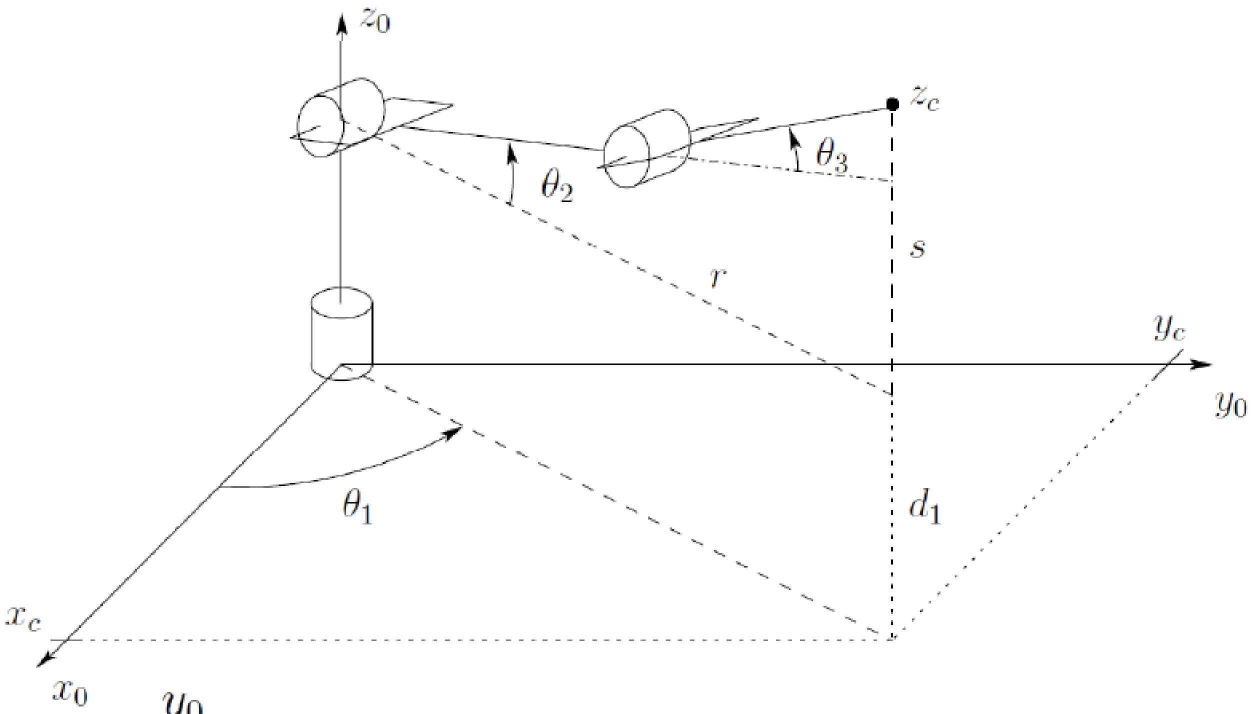
Inverse position

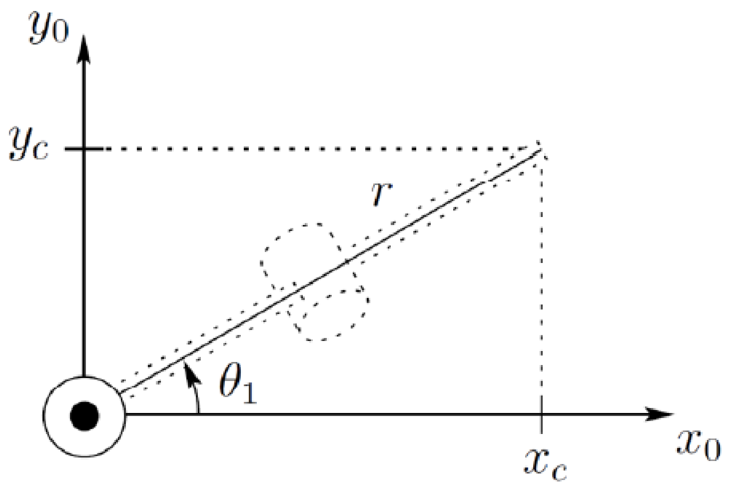
- We find q_1, q_2, q_3 that satisfy:

$$o_c^0(q_1, q_2, q_3) = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

- But How?
- By a graphical method:
  - Projecting the manipulator onto the xy-plane of a link frame
  - Applying trigonometry on the projected geometry

Inverse Kinematics of Articulated Manipulator



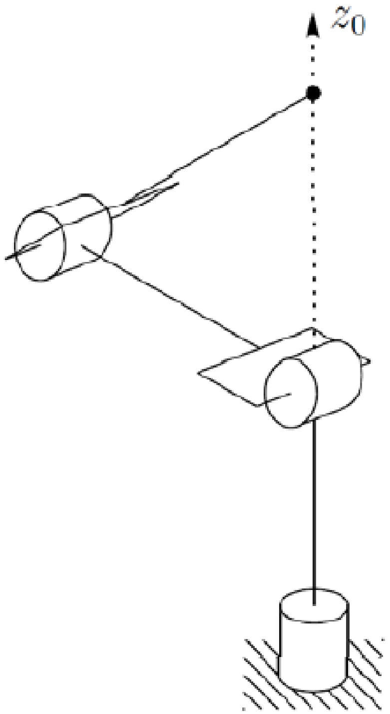


The angle θ_1 is easy to determine:

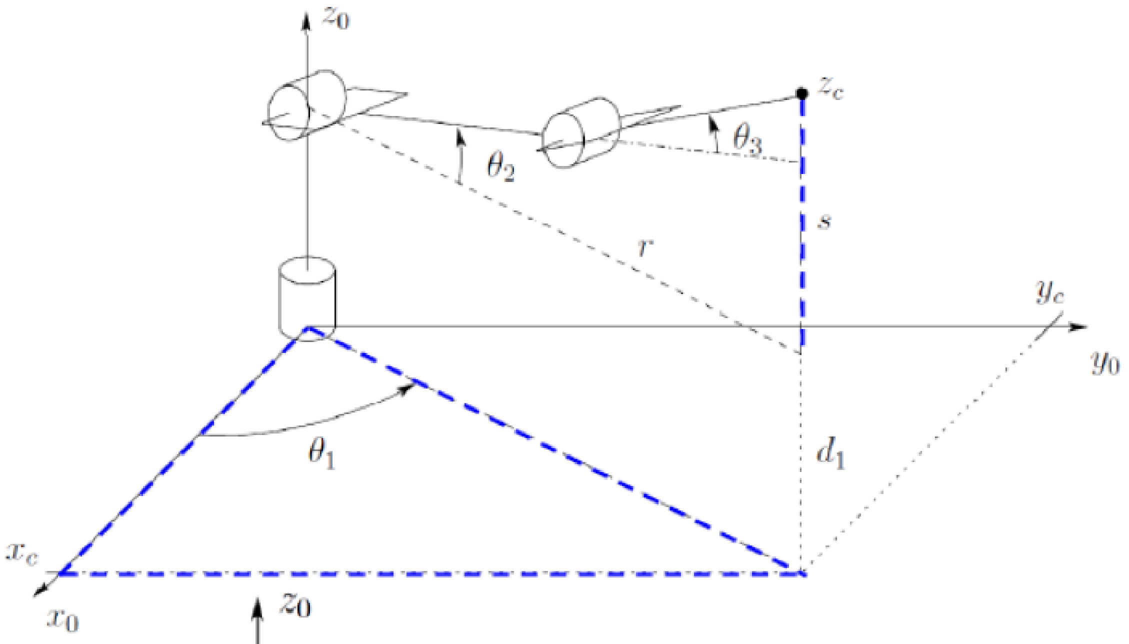
$$\theta_1 = \text{Atan2}(x_c, y_c)$$

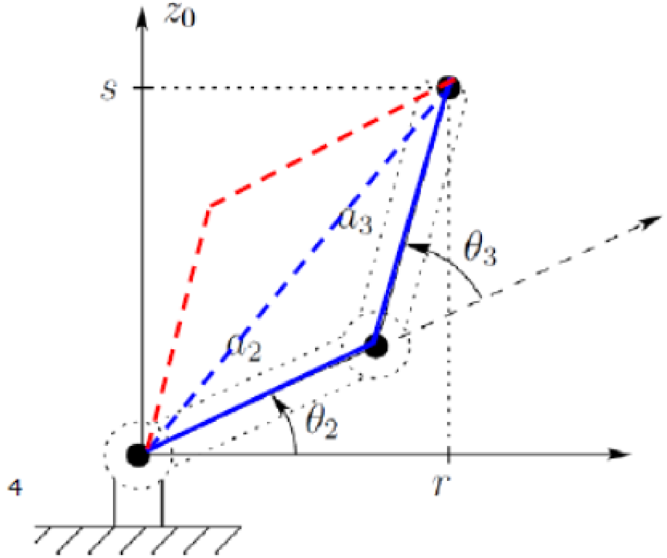
when $(x_c, y_c) \neq (0,0)$

Otherwise for $(x_c, y_c) = (0,0)$ there is a singularity w.r.t. determining θ_1:



Inverse Kinematics of Articulated Manipulator





From the law of cosines:

$$\cos(\theta_3) = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

where:

$$r^2 = x_c^2 + y_c^2$$
$$s = z_c - d_1$$

There are two solutions:

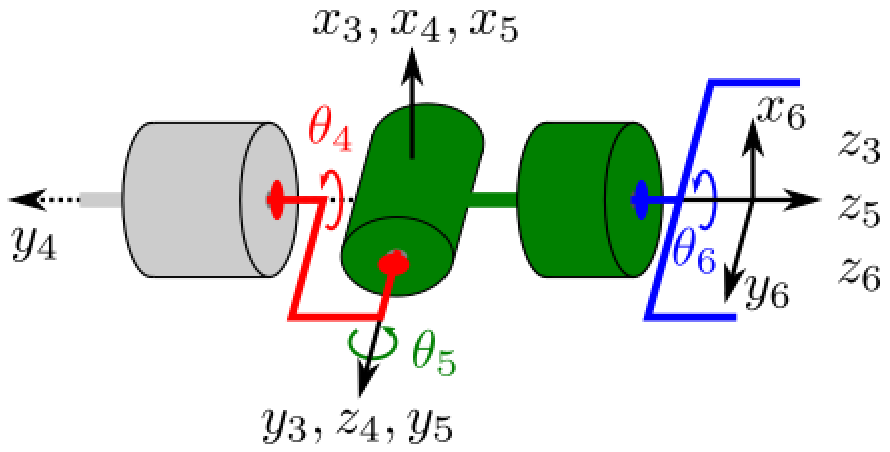
$$\theta_3 = \text{Atan2}\left(c_3, \pm\sqrt{1 - c_3^2}\right)$$

(elbow-down or elbow-up)

Then one can also find:

$$\theta_2 = \text{Atan2}(r, s) - \text{Atan2}(a_2 + a_3c_3, a_3s_3)$$

Inverse Orientation



- Find θ_4, θ_5, θ_6 that satisfy

$$R_6^3(\theta_4, \theta_5, \theta_6) = (R_3^0)^T R_6^0$$

$$\begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix} = (R_3^0)^T R_6^0$$

→Same problem as finding the Euler angles in Lecture 2:

$$\theta_5 = \theta = \text{Atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right)$$

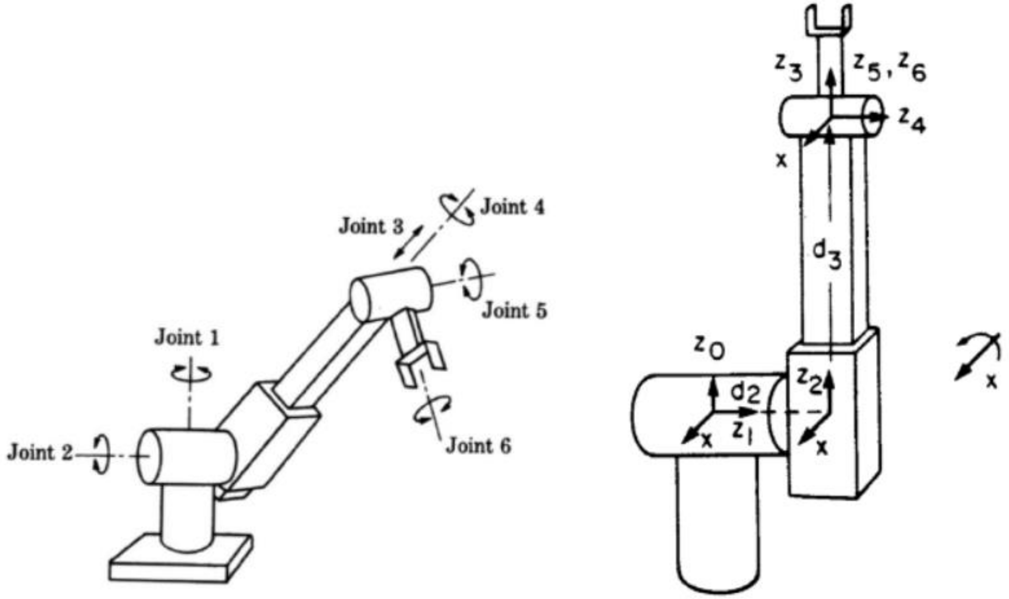
$$\theta_4 = \phi = \text{Atan2}(r_{13}, r_{23})$$

$$\theta_6 = \psi = \text{Atan2}(-r_{31}, r_{32})$$

Exercises

Problem 1

Given the Stanford arm in the figure below, with $d_2 = 0.1$ m, answer the following questions.



Question 1

Find the link parameters for the robotic arm (d_3 is a prismatic joint variable, other joints are rotational joints, the link coordinate frames have been established as shown in the figure). Hint: not all 6 joints are visible in the schematic, you have to deduct the existence of some joints from the corresponding z-axes in the model.

Joint i	θ_i	d_i	a_i	α_i
1				
2				
3				
4				
5				
6				

Question 2

Find the forward kinematic model for the arm and represent it in homogeneous matrix form.

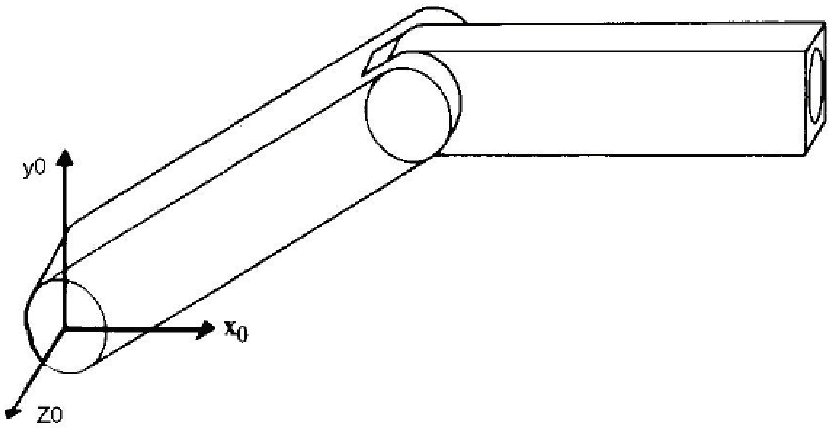
Question 3

Represent the orientation of the end-effector with Yaw-Pitch-Roll angles.

Exercises

Problem 2

A two degree-of-freedom manipulator is shown in the figure below. Given that the length of each link is $1\,\mathrm{m}$, establish its link coordinate frames and find T_1^0, T_2^1 and the kinematics matrix. For coordinate frame 2, assume a revolute joint at the tip of the robotic arm with its axis parallel to z_0.



Question 1

Find the forward kinematics solution for this manipulator, i.e. the homogeneous transformation matrix for the end-effector as a function of the joint angles.

Question 2

Find the inverse kinematics solution for this manipulator assuming the position of the robot tip is known, i.e. elements r_{14} and r_{24} in the homogeneous transformation matrix. (Hint: use trigonometry and the law of cosines)