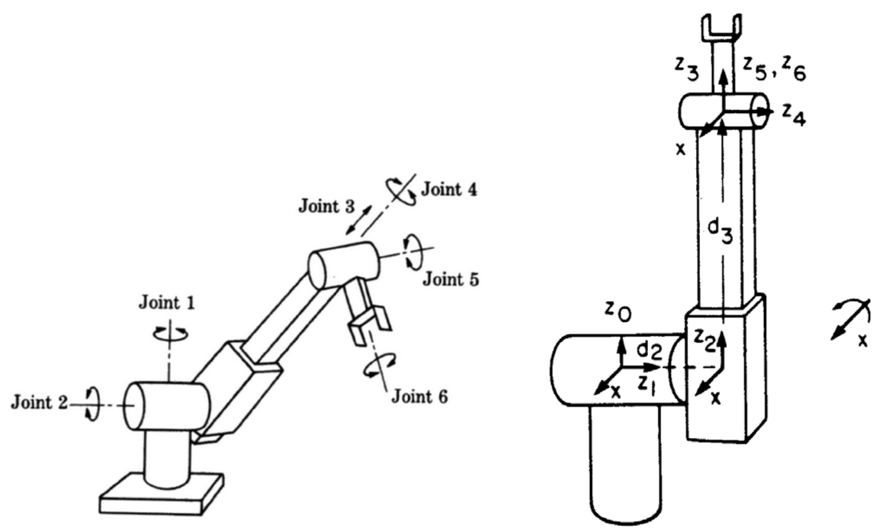
Robotics 34753

Exercise solutions for week 3

September 2023

Problem 1

Given the Stanford arm in the figure below, with $d_2 = 0.1\,\mathrm{m}$, answer the following questions.



Question 1

Find the link parameters for the robotic arm (d_3 is a prismatic joint variable, other joints are rotational joints, the link coordinate frames have been established as shown in the figure). Hint: not all 6 joints are visible in the schematic, you have to deduct the existence of some joints from the corresponding z-axes in the model.

Solution

Joint i	θ_i	d_i	a_i	α_i
1	θ_1^* (0°)	0	0	−90°
2	θ_2^* (0°)	0.1 m	0	90°
3	0°	d_3^*	0	0°
4	θ_4^* (0°)	0	0	−90°
5	θ_5^* (0°)	0	0	90°
6	θ_6^* (0°)	0	0	0°

Question 2

Find the forward kinematic model for the arm and represent it in homogeneous matrix form.

Solution

$$T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; T_2^1 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; T_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4^3 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; T_5^4 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; T_6^5 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^0 = T_1^0 T_2^1 T_3^2 T_4^3 T_5^4 T_6^5 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & c_1 d_3 s_2 - 0.1 s_1 \\ r_{21} & r_{22} & r_{23} & d_3 s_1 s_2 + 0.1 c_1 \\ r_{31} & r_{32} & r_{33} & c_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$r_{11} = c_1\left(c_2\left(c_4c_5c_6 - s_4s_6\right) - c_6s_2s_5\right) - s_1\left(c_4s_6 + c_5c_6s_4\right)$
$r_{12} = c_1\left(c_2\left(-c_4c_5s_6 - c_6s_4\right) + s_2s_5s_6\right) - s_1\left(c_4c_6 - c_5s_4s_6\right)$
$r_{13} = c_1\left(c_2c_4s_5 + c_5s_2\right) - s_1s_4s_5$
$r_{21} = s_1\left(c_2\left(c_4c_5c_6 - s_4s_6\right) - c_6s_2s_5\right) + c_1\left(c_4s_6 + c_5c_6s_4\right)$
$r_{22} = s_1\left(c_2\left(-c_4c_5s_6 - c_6s_4\right) + s_2s_5s_6\right) + c_1\left(c_4c_6 - c_5s_4s_6\right)$
$r_{23} = s_1\left(c_2c_4s_5 + c_5s_2\right) + c_1s_4s_5$
$r_{31} = -s_2\left(c_4c_5c_6 - s_4s_6\right) - c_2c_6s_5$
$r_{32} = c_2s_5s_6 - s_2\left(-c_4c_5s_6 - c_6s_4\right)$
$r_{33} = c_2c_5 - c_4s_2s_5$

Question 3

Represent the orientation of the end-effector with Yaw-Pitch-Roll angles.

Solution

The rotation matrix of the end-effector with respect to the initial frame can be written as a function of the Yaw-Pitch-Roll angles ψ, θ, ϕ

$$R_6^0 = R_{z,\phi} R_{y,\theta} R_{x,\psi} = \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

If $c_\theta \neq 0$, then

$$\theta = \mathrm{Atan2}\left(\pm\sqrt{r_{11}^2 + r_{21}^2}, -r_{31}\right)$$

$$\psi = \mathrm{Atan2}\left(\frac{r_{33}}{c_\theta}, \frac{r_{32}}{c_\theta}\right)$$

$$\phi = \mathrm{Atan2}\left(\frac{r_{11}}{c_\theta}, \frac{r_{21}}{c_\theta}\right)$$

If $c_\theta = 0$ and $s_\theta = 1$, i.e. $\theta = 90°$, then

$$R_6^0 = \begin{bmatrix} 0 & \sin(\psi - \phi) & \cos(\psi - \phi) \\ 0 & \cos(\psi - \phi) & -\sin(\psi - \phi) \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ -1 & 0 & 0 \end{bmatrix}$$

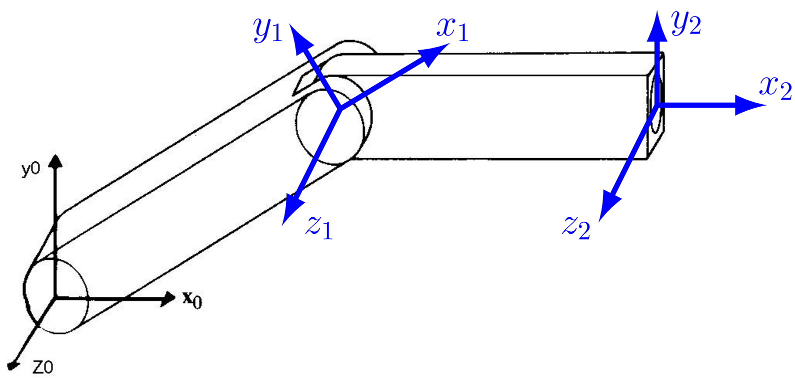
hence $\psi - \phi = \mathrm{Atan2}(r_{22}, r_{12})$
If $c_\theta = 0$ and $s_\theta = -1$, i.e. $\theta = 270°$, then

$$R_6^0 = \begin{bmatrix} 0 & -\sin(\psi + \phi) & -\cos(\psi + \phi) \\ 0 & \cos(\psi + \phi) & -\sin(\psi + \phi) \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 1 & 0 & 0 \end{bmatrix}$$

hence $\psi + \phi = \mathrm{Atan2}(r_{22}, -r_{12})$
In the last two cases, one of the two angles ϕ, ψ can be chosen arbitrarily.

Problem 2

A two degree-of-freedom manipulator is shown in the figure below. Given that the length of each link is 1 m, establish its link coordinate frames and find T_1^0, T_2^1 and the kinematics matrix. Define the z_2-axis of coordinate frame 2 as if there was a revolute joint at the tip of the robotic arm with its axis parallel to z_0.



Question 1

Find the forward kinematics solution for this manipulator, i.e. the homogeneous transformation matrix for the end-effector as a function of the joint angles.

Solution

The coordinate frames 1 and 2 are respectively defined according to the Denavit-Hartenberg convention as shown in the figure above.
For frame 1, the Denavit-Hartenberg parameters are θ_1^*, $d_1 = 0$, $a_1 = 1$ m and $\alpha_1 = 0°$. These lead to the following transformation matrix

$$T_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 & c_1 \\ s_1 & c_1 & 0 & s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For frame 2, the Denavit-Hartenberg parameters are θ_2^*, $d_2 = 0$, $a_2 = 1$ m and $\alpha_2 = 0°$. These lead to the following transformation matrix

$$T_2^1 = \begin{bmatrix} c_2 & -s_2 & 0 & c_2 \\ s_2 & c_2 & 0 & s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In total, the transformation matrix for the end-effector is

$$T_2^0 = T_1^0 T_2^1 = \begin{bmatrix} \cos(\theta_1^* + \theta_2^*) & -\sin(\theta_1^* + \theta_2^*) & 0 & \cos(\theta_1^*) + \cos(\theta_1^* + \theta_2^*) \\ \sin(\theta_1^* + \theta_2^*) & \cos(\theta_1^* + \theta_2^*) & 0 & \sin(\theta_1^*) + \sin(\theta_1^* + \theta_2^*) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix defines the forward kinematics of the robotic arm.
If sufficient elements of the total transformation matrix

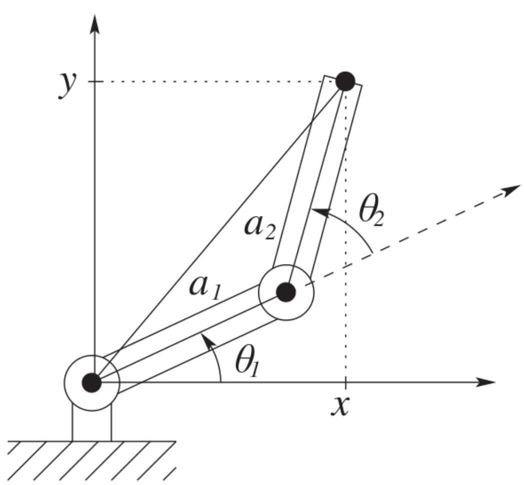
$$T_2^0 = T_1^0 T_2^1 = \begin{bmatrix} r_{11} & r_{12} & 0 & r_{14} \\ r_{21} & r_{22} & 0 & r_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are known, the inverse kinematics problem can be solved for the angles θ_1^* and θ_2^*.

Question 2

Find the inverse kinematics solution for this manipulator assuming the position of the robot tip is known, i.e. elements r_{14} and r_{24} in the homogeneous transformation matrix. (Hint: use trigonometry and the law of cosines)

Solution



The distance from the base origin to the tip is $\sqrt{x^2 + y^2} = \sqrt{r_{14}^2 + r_{24}^2}$, while both links have length $a_1 = a_2 = 1$.

Using the law of cosines and assuming the elbow down position, i.e. $\theta_2^* > 0$:

$$\cos(\pi - \theta_2^*) = \frac{a_1^2 + a_2^2 - r_{14}^2 - r_{24}^2}{2a_1a_2} \quad \Rightarrow \quad -\cos(\theta_2^*) = \frac{2 - r_{14}^2 - r_{24}^2}{2} \quad \Rightarrow$$

$$\theta_2^* = \mathrm{Atan2}\,(c_2, s_2) = \mathrm{Atan2}\left(\frac{r_{14}^2 + r_{24}^2}{2} - 1, \sqrt{1 - \left(\frac{r_{14}^2 + r_{24}^2}{2} - 1\right)^2}\right)$$

or using the law of cosines and assuming the elbow up position i.e. $\theta_2^* < 0$:

$$\cos(\pi + \theta_2^*) = \frac{a_1^2 + a_2^2 - r_{14}^2 - r_{24}^2}{2a_1a_2} \quad \Rightarrow \quad -\cos(\theta_2^*) = \frac{2 - r_{14}^2 - r_{24}^2}{2} \quad \Rightarrow$$

$$\theta_2^* = \mathrm{Atan2}\,(c_2, s_2) = \mathrm{Atan2}\left(\frac{r_{14}^2 + r_{24}^2}{2} - 1, -\sqrt{1 - \left(\frac{r_{14}^2 + r_{24}^2}{2} - 1\right)^2}\right)$$

Then θ_1^* can be found from the equation

$$\theta_1^* + \mathrm{Atan2}\,(a_1 + a_2c_2, a_2s_2) = \mathrm{Atan2}\,(r_{14}, r_{24}) \quad \Rightarrow$$

$$\theta_1^* = \mathrm{Atan2}\,(r_{14}, r_{24}) - \mathrm{Atan2}\,(1 + c_2, s_2)$$

where $c_2 = \cos(\theta_2^*)$ and $s_2 = \sin(\theta_2^*)$ are now known.