3

Digital Control Design Using Pole Placement

In this chapter, we shall present some numerical techniques for designing proportional-integral-derivative algorithm control, which is considered to be the classical control algorithm in automatic systems engineering, and the polynomial R (regulation), S (sensitivity) and T (tracking) (RST) control using a more flexible digital approach with two degrees of freedom, which is easy to integrate in real-time industrial applications. We have also developed a predictive control structure whose horizon is finite and unitary. The methods based on pole placement, which are the foundation for the development of these types of control algorithms, are used in the design of closed-loop systems, which ensure a high level of performance in driving industrial processes.

3.1. Digital proportional-integral-derivative algorithm control

For the design of proportional-integral-derivative (PID) algorithms in control systems, the required performances in the closed-loop system must be specified and the dynamic model of the process must be known (control and disturbance model).

Starting with this remark, the goal is to determine the structure and the parameters of the control algorithm.

This takes us back to the classic diagram of the continuous PID regulator, with filtering on the derivative component, presented in the literature [LAN 93, BIT 93, POP 00, POP 01, DAF 04].

Comment [User1]: AQ: Term "(PID)" has been set as "PID". Please check.

$$G_{PID}(s) \overset{\Delta}{=} k\left[1+\frac{1}{T_i s}+\frac{T_d s}{\frac{T_d}{\lambda}s+1}\right]. \qquad [3.1]$$

The discrete version of relation [3.1], which uses a sampling period h, expressed in terms of the z transform is the following:

$$G_{PID}(z) = k\left[1 + \frac{\frac{h}{T_i}z}{z-1} + \frac{\frac{\lambda T_d}{T_d + \lambda h}(z-1)}{z - \frac{T_d}{T_d + \lambda h}}\right], \qquad [3.2]$$

and in terms of z^{-1}

$$G_{PID}(z^{-1}) = k\left[1 + \frac{\frac{h}{T_i}}{1-z^{-1}} + \frac{\frac{\lambda T_d}{T_d + \lambda h}(1-z^{-1})}{1 - \frac{T_d}{T_d + \lambda h}z^{-1}}\right] = \frac{R(z^{-1})}{S(z^{-1})}, \qquad [3.3]$$

where

$$\begin{cases} R(z^{-1}) \stackrel{\text{def}}{=} k(1-z^{-1})\left(1 - \dfrac{T_d}{T_d + \lambda h}z^{-1}\right) + \dfrac{kh}{T_i}\left(1 - \dfrac{T_d}{T_d + \lambda h}z^{-1}\right) + \\ \qquad + \dfrac{k\lambda T_d}{T_d + \lambda h}(1-z^{-1})^2 \\ S(z^{-1}) \stackrel{\text{def}}{=} (1-z^{-1})\left(1 - \dfrac{T_d}{T_d + \lambda h}z^{-1}\right). \end{cases} \qquad [3.4]$$

Polynomials R, S are, therefore, defined by the following expressions:

$$\begin{cases} R\left(z^{-1}\right) \cancel{= T\left(z^{-1}\right)} = r_0 + r_1 z^{-1} + r_2 z^{-2} \\ S\left(z^{-1}\right) = \left(1 - z^{-1}\right)\left(1 + s_1 z^{-1}\right) \end{cases}. \qquad [3.5]$$

Note that the PID controller is characterized by the four parameters, r_0, r_1, r_2 and s_1, and that the block diagram of the closed-loop system, shown in Figure 3.1, has its transfer function expressed as follows:

$$G_{RST}\left(z^{-1}\right) = \frac{\hat{b}\left(z^{-1}\right)R\left(z^{-1}\right)}{\hat{a}\left(z^{-1}\right)S\left(z^{-1}\right) + \hat{b}\left(z^{-1}\right)R\left(z^{-1}\right)}. \qquad [3.6]$$

In the polynomial $S(z^{-1})$, the term $(1-z^{-1})$ ensures the effect of digital integration, while the term $(1+s_1 z^{-1})$ causes a digital filtering effect.

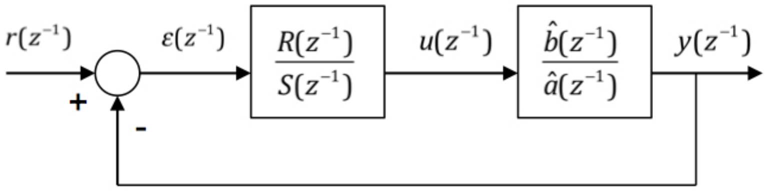

Figure 3.1. *PID-type control system in polynomial form*

The performance level of the system is specified by the desired poles for the system in the closed-loop system, the roots of the characteristic polynomial $P_i(z^{-1})$ (the denominator of [3.6]) being written as follows:

$$P_i\left(z^{-1}\right) \stackrel{\Delta}{=} \hat{a}\left(z^{-1}\right) S\left(z^{-1}\right) + \hat{b}\left(z^{-1}\right) R\left(z^{-1}\right). \qquad [3.7]$$

In practice, $P_i(z^{-1})$, which is recommended for the structure of the PID controller, is chosen in the following form:

$$P_i\left(z^{-1}\right) = 1 + p_1 z^{-1} + p_2 z^{-2}, \qquad [3.8]$$

which represents the discretized equivalent of a second-order continuous model.

As a result, calculation of parameters (R, S) of the PID controller assumes the resolution of the polynomial equation:

$$\hat{a}\left(z^{-1}\right)\left(1-z^{-1}\right)\left(1+s_1 z^{-1}\right) + \hat{b}\left(z^{-1}\right)\left(r_0 + r_1 z^{-1} + r_2 z^{-2}\right) \equiv P_i\left(z^{-1}\right), \qquad [3.9]$$

where $\hat{a}(z^{-1})$, $\hat{b}(z^{-1})$ and $P_i(z^{-1})$ are known polynomials. The solution to this equation sets restrictions on the model of the process (of the second order at the most with or without a time delay). It can be seen that the product $\hat{b}(z^{-1})R(z^{-1})$ defines the zeros in the closed-loop system; therefore, the PID controller introduces additional zeros through $R(z^{-1})$; the PID controller does not simplify the zeros of the process and can therefore be used for the regulation of processes whose model have unstable zeros; for the PID control system, performances are specified for regulation or tracking, and it is impossible to ensure independent dynamics for both tacking and regulation due to the same calculated PID control algorithm.

3.2. Digital polynomial RST control

In order to overturn the limitations of the PID algorithm, the structure in Figure 3.1 can be enriched using an additional tracking precompensating polynomial $T(z^{-1})$. The digital structure of the closed loop system is illustrated in Figure 3.2.

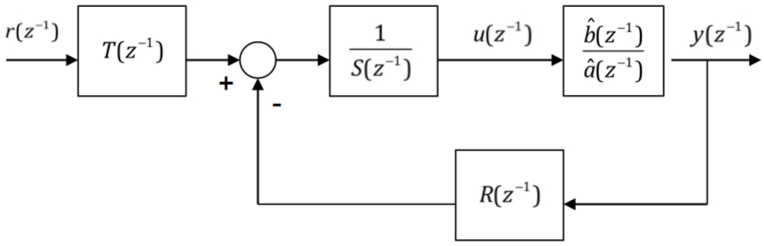

Figure 3.2. *RST control system*

The general algorithm of the digital control is expressed as follows:

$$u[k] = \frac{1}{s_0}\left[-\sum_{i=1}^{ns} s_i u[k-i] - \sum_{i=0}^{nr} r_i y[k-i] + \sum_{i=0}^{nt} t_i r[k-i]\right], \ \forall k \in \mathbb{N}\,. \qquad [3.10]$$

In this relation, $r[k]$ is the setpoint, $u[k]$ is the digital control, $y[k]$ is the system output and $\{s_i\}, i = 0..ns$; $\{r_i\}, i = 0..nr$; $\{t_i\}, i = 0..nt$ are the parameters of the digital controller.

The number of coefficients provides the complexity and the memory of the algorithms, which are imposed depending on the size of the process and on the levels of performance required.

Starting with expression [3.10] above, it is possible to build the canonical (R (regulation), S (sensitivity) and T (tracking) [RST]) polynomial structure, which allows for the implementation of digital control systems in many industrial applications [BOR 93, BOR 97, GEN 97, LAN 97, POP 00, OGA 01, DAU 04].

The control algorithm can easily be expressed in an equivalent manner in z^{-1}:

$$u\left[z^{-1}\right]=\frac{T\left(z^{-1}\right)}{S\left(z^{-1}\right)}r\left[z^{-1}\right]-\frac{R\left(z^{-1}\right)}{S\left(z^{-1}\right)}y\left[z^{-1}\right], \qquad [3.11]$$

where

$$\begin{bmatrix} R\left(z^{-1}\right)=r_0+r_1z^{-1}+...+r_{nr}z^{-nr} \\ S\left(z^{-1}\right)=s_0+s_1z^{-1}+...+s_{ns}z^{-ns} \\ T\left(z^{-1}\right)=t_0+t_1z^{-1}+...+t_{nt}z^{-nt}. \end{bmatrix} \qquad [3.12]$$

The mathematical model of the controlled process is of the ARX type, expressed using the polynomials $\hat{a}(z^{-1})$ and $\hat{b}(z^{-1})$:

$$\begin{bmatrix} \hat{b}\left(z^{-1}\right)=b_1z^{-1}+b_2z^{-2}+...+b_{nb}z^{-nb} \\ \hat{a}\left(z^{-1}\right)=1+a_1z^{-1}+...+a_{na}z^{-na} \end{bmatrix}. \qquad [3.13]$$

The transfer function of the system represented in Figure 3.2 is the following:

$$G_{RST}\left(z^{-1}\right)=\frac{\hat{b}\left(z^{-1}\right)T\left(z^{-1}\right)}{\hat{a}\left(z^{-1}\right)S\left(z^{-1}\right)+\hat{b}\left(z^{-1}\right)R\left(z^{-1}\right)}. \qquad [3.14]$$

The configuration of the previous structure can still be improved using a trajectory generator in order to build a system that provides high levels of performance during tracking (setpoint change) and during regulation (disturbance rejection).

In this case, a reference trajectory $y^*[k]$ is built using the tracking model $\hat{b}_m(z^{-1})/\hat{a}_m(z^{-1})$ as shown in Figure 3.3, where:

$$\begin{bmatrix} \hat{b}_m\left(z^{-1}\right)=b_{m1}z^{-1}+b_{m2}z^{-2}+...+b_{ms}z^{-ms} \\ \hat{a}_m\left(z^{-1}\right)=1+a_{m1}z^{-1}+...+a_{mr}z^{-mr}. \end{bmatrix} \qquad [3.15]$$

The structure shown in Figure 3.3 specifies the dynamic behavior during tracking. This time the global transfer function is obtained using relation [3.14] (which expresses the effect of the regulation) by considering the additional action of the tracking model:

$$G^{*}_{RST}\left(z^{-1}\right)=\frac{\hat{b}_m\left(z^{-1}\right)}{\hat{a}_m\left(z^{-1}\right)}G_{RST}\left(z^{-1}\right)= \\ =\frac{\hat{b}_m\left(z^{-1}\right)\hat{b}\left(z^{-1}\right)T\left(z^{-1}\right)}{\hat{a}_m\left(z^{-1}\right)\left[\hat{a}\left(z^{-1}\right)S\left(z^{-1}\right)+\hat{b}\left(z^{-1}\right)R\left(z^{-1}\right)\right]} \quad [3.16]$$

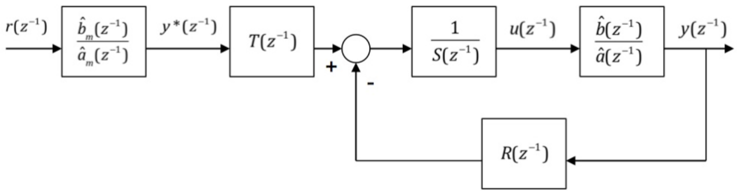

Figure 3.3. *RST structure with independent objectives in tracking and regulation*

3.3. RST control by pole placement

The placement of poles as shown is a strategy that allows a polynomial controller to be calculated without any restrictions on the degrees of the polynomials $\hat{a}(z^{-1})$ and $\hat{b}(z^{-1})$, no restriction on the delay of the process or on the zeros of the process (whether stable or unstable) [GOD 84, BOR 93, AST 97, LAN 97, DAU 04, POP 06].

The process is characterized by the irreducible model below:

$$G(z^{-1})=\frac{\hat{b}(z^{-1})}{\hat{a}(z^{-1})}, \quad [3.17]$$

where

$$\begin{cases}\hat{a}(z^{-1})=1+a_1z^{-1}+\ldots+a_{na}z^{-na}\\ \hat{b}(z^{-1})=b_1z^{-1}+\ldots+b_{nb}z^{-nb}\end{cases}. \quad [3.18]$$

In this case, the global transfer function of the closed-loop system is expressed as follows:

$$G_{RST}\left(z^{-1}\right)=\frac{T\left(z^{-1}\right)\hat{b}\left(z^{-1}\right)}{\hat{a}\left(z^{-1}\right)S\left(z^{-1}\right)+\hat{b}\left(z^{-1}\right)R\left(z^{-1}\right)}. \qquad [3.19]$$

3.3.1. *RST control for regulation dynamics*

Consider the polynomial that defines the level of performance regulation (disturbance rejection):

$$P_i\left(z^{-1}\right)\equiv 1+p_1 z^{-1}+p_2 z^{-2}+\ldots+p_n z^{-np}. \qquad [3.20]$$

From the previous relations, we can write that:

$$\hat{a}\left(z^{-1}\right)S\left(z^{-1}\right)+\hat{b}\left(z^{-1}\right)R\left(z^{-1}\right)\equiv P_i\left(z^{-1}\right). \qquad [3.21]$$

The polynomial equation [3.21] has a single solution in the following conditions:

$$\begin{cases} rank(\hat{a}(z^{-1}))=na \\ rank(\hat{b}(z^{-1}))=nb \\ rank(P(z^{-1}))=np\leq na+nb. \\ rank(S(z^{-1}))=ns=nb \\ rank(R(z^{-1}))=nr=na \end{cases} \qquad [3.22]$$

To solve equation [3.21], we use the following matricial form, $Mx=p$

where:

$$\begin{cases} \boldsymbol{x}^{\mathrm{T}}\overset{\Delta}{=}[1\, s_1\, s_2 \ldots s_{nb}\, r_0\, r_1 \ldots r_{na-1}] \\ \boldsymbol{p}^{\mathrm{T}}\underset{def}{=}[1\, p_1\, p_2 \cdots p_{na+nb}]. \end{cases} \qquad [3.23]$$

M is the Sylvester matrix (of dimensions $(na+nb+1)\times(na+nb+1)$) associated with the polynomials $\hat{a}(z^{-1})$ and $\hat{b}(z^{-1})$ (for the polynomials $\hat{a}(z^{-1})$ and $\hat{b}(z^{-1})$ prime to one another, the Sylvester matrix is invertible).

Vector x that contains the unknown coefficients of polynomials $S(z^{-1})$ and $R(z^{-1})$ is then obtained simply by inversion of the matrix M:

$$x = M^{-1} p. \qquad [3.24]$$

3.3.2. *RST polynomial control for tracking dynamics (setpoint change)*

In practice, the system output $y[k]$ needs to follow a trajectory that has been imposed that is $y^*[k]$. This trajectory can be constructed using a trajectory generator, as shown in Figure 3.3.

The transfer function of this generator is defined by the dynamic model:

$$G_m\left(z^{-1}\right) \stackrel{\text{def}}{=} \frac{\hat{b}_m\left(z^{-1}\right)}{\hat{a}_m\left(z^{-1}\right)}, \qquad [3.25]$$

where the polynomials $\hat{a}_m(z^{-1})$ and $\hat{b}_m(z^{-1})$ are introduced by the relations of [3.15]. In general, this transfer function is determined from the desired tracking performances, for example through a standard continuous second-order model.

Therefore, in practice, the discretized transfer function of the trajectory generator is represented with the imposed poles:

$$G_m\left(z^{-1}\right) = \frac{z^{-1}\left(b_{m0} + b_{m1} z^{-1}\right)}{1 + a_{m1} z^{-1} + a_{m2} z^{-2}}. \qquad [3.26]$$

It is precisely why the controller finally must execute between $y^*[k]$ and the output $y[k]$, this transfer function. In the moment, this demand is not possible, as the zeros of the process (meaning the polynomial $b(z^{-1})$) are preserved.

The final objective will then be to reach the reference trajectory, which in this case is defined as follows:

$$y^*\left[z^{-1}\right] \stackrel{\text{def}}{=} \frac{\hat{b}_m\left(z^{-1}\right)}{\hat{a}_m\left(z^{-1}\right)} r\left[z^{-1}\right], \qquad [3.27]$$

Polynomial $T(z^{-1})$ is chosen to ensure a unitary study-state gain between $y^*[k]$ and $y[k]$ (to compensate the polynomial $P_i(z^{-1})$).

This leads to the following choice for polynomial $T(z^{-1})$:

$$T\left(z^{-1}\right) = GP_i\left(z^{-1}\right), \qquad [3.28]$$

where G is a constant gain, defined by:

$$G \stackrel{\Delta}{=} \begin{cases} \dfrac{1}{\hat{b}(1)}, & \hat{b}(1) \neq 0 \\ 1, & \hat{b}(1) = 0\,. \end{cases} \qquad [3.29]$$

which leads to the following output:

$$y[z^{-1}] = G_{RST}(z^{-1})\,y^*[z^{-1}] = G\hat{b}(z^{-1})\,y^*[z^{-1}]\,. \qquad [3.30]$$

3.3.3. *RST control with independent objectives in tracking and regulation*

This approach results in the desired behavior of the system during tracking (setpoint change), independently from the behavior imposed during regulation (disturbance rejection), using the same control algorithm $u[z^{-1}]$.

Contrarily to the method of pole placement from section 3.3.2, this approach leads to the simplification of the stable zeros of the discrete model of the process, resulting in the desired performances during tracking and during regulation. ~~If all of the zeros.~~

Comment [User2]: AQ: Sentence "If all of the zeros…" seems incomplete. Please check.

The structure of the closed loop system in question, as represented in Figure 3.3. s stable; once the transition is over, the output follows $y^*[z^{-1}]$ precisely.

Comment [User3]: AQ: We have modified this sentence. Is it correct?

The control strategy enables a digital (*RST*) controller to be calculated for stable or unstable processes, without any restriction on the degrees of the polynomials $\hat{a}(z^{-1})$ and $\hat{b}(z^{-1})$ of the discrete transfer function of the process. However, due to the simplification of the zeros, it can only be applied to models with stable zeros.

The output of the tracking model $\hat{b}_m(z^{-1})/\hat{a}_m(z^{-1})$ specifies the desired trajectory $y^*[z^{-1}]$ that the output of the closed-loop system must follow. As a result, $T(z^{-1})$ contributes to the calculation of the value of $y^*[z^{-1}] = y[z^{-1}]$.

The poles in a closed-loop system are defined by the desired characteristic polynomial $P_i\,(z^{-1})$.

Calculation of the polynomials $R(z^{-1})$, $S(z^{-1})$ and $T(z^{-1})$ takes place in two steps. At first, using $R(z^{-1})$ and $S(z^{-1})$, the poles specified by $P_i(z^{-1})$ (regulation objective) are placed in a closed-loop system and the stable zeros of the discrete model of the process are simplified. In the second step, polynomial $T(z^{-1})$ is determined in order to find the reference trajectory $y^*[z^{-1}]$ at the output of the global control system.

3.3.3.1. *Performance in regulation (disturbance rejection)*

By eliminating the tracking part and the precompensator of the diagram in Figure 3.3, the transfer function in the closed-loop system with stable zeros is expressed as follows:

$$G_{RS}\left(z^{-1}\right)=\frac{\hat{b}\left(z^{-1}\right)}{\hat{a}\left(z^{-1}\right)S\left(z^{-1}\right)+\hat{b}\left(z^{-1}\right)R\left(z^{-1}\right)}. \qquad [3.31]$$

Comment [User4]: AQ: Equations have been renumbered. Please check.

Next, the following condition is imposed:

$$G_{RS}(z^{-1})=\frac{1}{P_i(z^{-1})}. \qquad [3.32]$$

As polynomial $P_i(z^{-1})$ is specified *a priori*, this results in the following equation:

$$\hat{a}\left(z^{-1}\right)S\left(z^{-1}\right)+\hat{b}\left(z^{-1}\right)R\left(z^{-1}\right)=\hat{b}\left(z^{-1}\right)P_i(z^{-1}) \qquad [3.33]$$

which allows $R(z^{-1})$ and $S(z^{-1})$ to be calculated.

Practically speaking, equation [3.33] demonstrates the manner in which the denominator of the transfer function $G_{RS}(z^{-1})$ must be factorized by $\hat{b}\left(z^{-1}\right)$. This operation leads to a natural factorization for $S(z^{-1})$:

$$S\left(z^{-1}\right)=\hat{b}\left(z^{-1}\right)S'\left(z^{-1}\right). \qquad [3.34]$$

This results in a simpler expression of equation [3.33]:

$$\hat{a}\left(z^{-1}\right)S'\left(z^{-1}\right)+R\left(z^{-1}\right)\equiv P_i\left(z^{-1}\right). \qquad [3.35]$$

Equation [3.35] has a single solution for:

$$\begin{cases} rank(\hat{a}(z^{-1}))=na \\ rank(P_i(z^{-1}))=np\leq na+ns' \\ rank(S'(z^{-1}))=ns' \\ rank(R(z^{-1}))=nr=na+ns' \end{cases}. \qquad [3.36]$$

Equation [3.35] can be put into a matricial form $Mx = p$, where M is the Sylvester matrix, with dimensions of $(na + ns') \times (na + ns')$. Clearly:

$$\begin{cases} x^T \overset{\Delta}{=} [1 \quad s_1 \quad s_2 \quad \dots \quad s_{ns'} \quad , \quad r_0 \quad r_1 \quad \dots \quad r_{na+ns'-1}] \\ p^T \overset{\Delta}{=} [1 \quad p_1 \quad p_2 \quad \dots \quad p_{na+ns'-1}] \end{cases}. \qquad [3.37]$$

Here, the p_i are coefficients of the characteristic polynomial $P_i(z^{-1})$ from [3.35]. The solution for the polynomials from [3.35] is then obtained by inversion of matrix M:

$$x = M^{-1} p \ . \qquad [3.38]$$

3.3.3.2. *Tracking performance (setpoint change)*

Polynomial $T(z^{-1})$ is calculated by imposing the specified condition that the global structure if Figure 3.3 (obtained by adding the tacking part and the precompensator) has the same behavior as the tracking system:

$$G^*_{RST}\left(z^{-1}\right) \overset{\text{def}}{=} \frac{\hat{b}_m\left(z^{-1}\right)}{\hat{a}_m\left(z^{-1}\right)} T\left(z^{-1}\right) G_{RS}\left(z^{-1}\right) = \frac{\hat{b}_m\left(z^{-1}\right)}{\hat{a}_m\left(z^{-1}\right)} . \qquad [3.39]$$

We have already established that:

$$G_{RS}\left(z^{-1}\right) = \frac{1}{P_i\left(z^{-1}\right)} \qquad [3.40]$$

which leads to the identification of $T(z^{-1})$ with P_i: $T(z^{-1}) = P_i(z^{-1})$.

The regulator is then described by the following equation:

$$\begin{aligned} &S\left(z^{-1}\right) u\left[z^{-1}\right] + R\left(z^{-1}\right) y\left[z^{-1}\right] = P_i\left(z^{-1}\right) y^*\left[z^{-1}\right] \Leftrightarrow \\ &\Leftrightarrow S\left(z^{-1}\right) u\left[z^{-1}\right] + R\left(z^{-1}\right) y^*\left[z^{-1}\right] = P_i\left(z^{-1}\right) y^*\left[z^{-1}\right]. \end{aligned} \qquad [3.41]$$

If we take into consideration the form of $S(z^{-1})$:

$$S\left(z^{-1}\right) = s_0 + s_1 z^{-1} + \dots + s_{ns} z^{-ns} = \hat{b}\left(z^{-1}\right) S'\left(z^{-1}\right), \qquad [3.42]$$

we can deduce that $s_0 = b_1$ and therefore:

$$S\left(z^{-1}\right) = b_1 + z^{-1}S^*\left(z^{-1}\right). \qquad [3.43]$$

The control algorithm (equation of the controller) for implementation is then:

$$u[z^{-1}] = \frac{1}{b_1}(P_i(z^{-1})y[z^{-1}] - S^{'}(z^{-1})u[z^{-1}] - R(z^{-1})y^*[z^{-1}]). \qquad [3.44]$$

3.4. Predictive RST control

This section presents the technique of predictive control, which is often used in advanced digital control systems. At the beginning, we will present the key ideas behind its design and the purpose of this approach. Next, we will highlight the various algorithmic steps of predictive control. Finally, we describe the most representative methods ("dead beat") for predictions in the range of a sample period [FLA 94, MAC 01, POP 04].

The calculation of predictive control is characterized by the following steps:

1) at each instant *kh* (where *h* is the sampling period), a prediction of the evolution of the process is carried out for a sequence of future actions on the input;

2) the digital control sequence is determined by optimizing a criterion based on the future values; the calculation takes into account the difference between the predicted value and the setpoint as well as the variations of the input;

3) only the first control of the sequence is effectively applied and a new calculation begins, taking into account the evolution of the process in real time.

At an instant $t = kh$, the next output of the process is predicted over a time horizon from a mathematical model of the process dynamics. This depends on the past evolution of the process and the control scenario suggested for the next period.

Depending on various solutions, we choose the control sequence that takes the output of the process to the value of the setpoint in an "optimal" manner in relation to an objective defined beforehand. The calculated control is applied to the real process at the instant *kh* up to the instant $(k + 1)h$.

The advantage of this approach lies in the fact that predictive control is better than PID control for the delayed process when the setpoint changes are known or programmed ahead. Predictive control is considered to be robust for a rough approximation of the controlled process.

An important parameter of predictive control is the *horizon*, which has several representations, as follows:

– *the model horizon* (*hM*) is the length of time for the response of the process in an open loop to reach 99% of its final value;

– *the control horizon* (*hC*) is the length of application time of the successive actions on the input of the process needed in order to bring the output to the desired value;

– *the prediction horizon* (*hP*) is the length time over which the output of the process is predicted and used in the recurrent calculation of the control in order to anticipate the setpoint.

We can consider the particular case corresponding to a predictive control for a horizon reduced to a single step. The model for the prediction of the output is given by identification from the data measured at the input and the output of the process. The calculation of the predictive control algorithm therefore involves deducing value $u[k]$, which simply ensures that the next output $y[k+1]$ is equal to the current setpoint $r[k]$. In this case, a prediction horizon is then equal to only one sampling period ($hP = h$).

In most situations a unitary horizon is sufficient, and we shall show that in such a case, predictive control is simple to determine and easy to implement.

3.4.1. *Finite horizon predictive control*

The algorithms of predictive control are based on the prediction of the output of the process. To calculate the predicted value, a model of the process is used. The easiest one to obtain is the model built from a step response, for example.

Let us consider a step response for a finite series of values a_i ($a_1, a_2,\ldots,a_{hM}$) taken at the instants (1, 2, 3,..., hM), which represents a convolution model. This model can be used to calculate the response of the system following a variation at the input.

We assume $y[0] = 0$ and that the input is a series of variations of the input (Δu_0 at $k = 0$ and Δu_i at $k = i$). The output of the system is then given by the following model:

$$y[k] = \sum_{i=1}^{hM} a_i \Delta u_{k-1}, \quad \forall k \in \mathbb{N}, \qquad [3.45]$$

where hM is the model horizon.

The weighted model can be defined as the derivative of the step response. In this case, the output of the system is described by a similar model as follows:

$$y[k]=\sum_{i=1}^{hM} h_i \Delta u_{k-1}, \quad \forall k \in \mathbb{N}, \tag{3.46}$$

where $h_i = a_i - a_{i-1}, i \in \overline{1, hM}$. For a given possible delay d, the first coefficients are equal to zero: $h_i = 0$ ($i \in \overline{1, d}$).

The prediction of the output on the horizon hC ($y_p[1]$, $y_p[2]$,...,y_p [hC]), for a sequence of hM actions (a_1, a_2,...,a_{hM}), based on convolution response model, can be written in the matricial form:

$$\underbrace{\begin{bmatrix} y_p[k+1] \\ y_p[k+2] \\ \vdots \\ y_p[k+hC] \end{bmatrix}}_{Y_p[k]} \overset{def}{=} \underbrace{\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ a_2 & a_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{hC} & a_{hC-1} & \cdots & \cdots & a_{hC-hM-1} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+hM-1} \end{bmatrix}}_{\Delta U[k]} . \tag{3.47}$$

If the control horizon hC is greater than the model horizon hM, we state that $a_i = a_{hM}$ for $i > hM$ and we write:

$$Y_p[k] = A\Delta U[k], (\forall) k \in \mathbb{N}. \tag{3.48}$$

The matrix A is obtained from the model of the step response. Relation [3.48] allows us to calculate the control variation $\Delta U[k]$. The control objective is to force the system output to follow the reference trajectory $r[k + 1]$, $r[k + 2]$,..., $r[k + hC]$. To do this, the minimization of a quadratic criterion is imposed:

$$J[k] = \sum_{i=1}^{hC} (r[k+i] - y_p[k+i])^2 + \lambda \sum_{i=1}^{hM} \Delta u^2[k+i]. \tag{3.49}$$

The vector $\Delta U[k]$ that minimizes the criterion $J[k]$ by the method of least squares is given in the following relation:

$$\Delta U[k] = (A^T A + \lambda^2 I)^{-1} (R[k] - Y_p[k]), \tag{3.50}$$

where

$$R^{\mathrm{T}}[k] \overset{def}{=} \left[r[k+1] \ldots r[k+hC] \right], \quad \forall k \in \mathbb{N}. \qquad [3.51]$$

Only the first element of $\Delta U[k]$ is used to update the control:

$$u[k] = u[k-1] + \Delta u[k], \ \forall k \in \mathbb{N}, \qquad [3.52]$$

which guarantees the desired evolution of the output $y[k]$.

3.4.2. *Predictive control with unitary horizon*

As stated above, a predictive control developed over a unitary horizon is sufficient in many cases. In this case, the prediction horizon is equal to a sampling period, while the model horizon hM and control horizon hC are also reduced to a single sampling period.

The process model, which specifies the predicted output, is assumed to have been known through an identification operation.

In Figure 3.4, we present the temporal evolution of the process output, of the desired output and the corresponding predictive control, for a unitary prediction horizon. These representations show the continual evolution of the corresponding values between the two successive sampling instants.

The structure of the closed-loop system is specified in Figure 3.2. We can consider that the model of the process is of an ARX type system, whose transfer function is the following:

$$G(z^{-1}) = \frac{\hat{b}(z^{-1})}{\hat{a}(z^{-1})}, \qquad [3.53]$$

where

$$\begin{cases} \hat{a}(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_{na} z^{-na} \\ \hat{b}(z^{-1}) = b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{nb} z^{-nb} \end{cases}. \qquad [3.54]$$

Relations [3.53] and [3.54] allow $y[k]$ to be expressed as follows:

$$y[k] = -\sum_{i=1}^{na} a_i y[k-i] + \sum_{i=1}^{nb} b_i u[k-i], \quad \forall k \in \mathbb{N}. \tag{3.55}$$

The predictor of the process is determined simply by the progress of a sampling period:

$$y[k+1] = \sum_{i=1}^{na} a_i y[k-i+1] + \sum_{i=1}^{nb} b_i u[k-i+1], \quad \forall k \in \mathbb{N}. \tag{3.56}$$

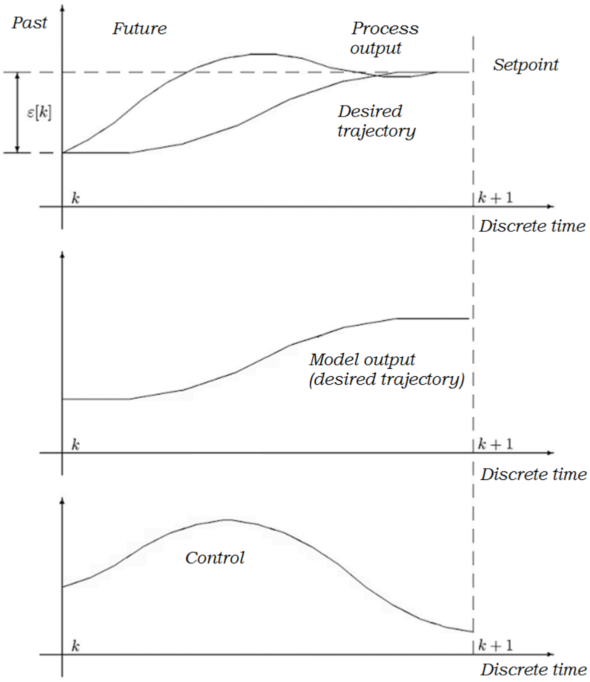

Figure 3.4. *Principle of predictive regulation with a unitary horizon*

The first type of predictive strategy is one that allows the control $u[k]$ to be chosen in such a way that $y[k+1]$ coincides exactly with the setpoint $r[k]$. We must, therefore, ensure that there is no error at the next sampling instant. This control technique is called dead-beat. It corresponds to the cancelling of the error in one go. Following this condition and equation [3.55], we can write:

$$r[k] = -\sum_{i=1}^{na} a_i y[k-i+1] + \sum_{i=1}^{nb} b_i u[k-i+1], \quad \forall k \in \mathbb{N} \,, \tag{3.57}$$

which can be expressed in a compact manner as:

$$\hat{b}(z^{-1})u[z^{-1}] = r[z^{-1}] + \hat{a}(z^{-1})y[z^{-1}]\,. \tag{3.58}$$

The predictive control $u[z^{-1}]$ can be calculated following the relation below:

$$u[z^{-1}] = \frac{1}{\hat{b}(z^{-1})} r[z^{-1}] + \frac{\hat{a}(z^{-1})}{\hat{b}(z^{-1})} y[z^{-1}]\,. \tag{3.59}$$

Expression [3.60] corresponds to the diagram represented in Figure 3.5.

In the figure, we can identify polynomials R, S, T, with respect to relation [3.60]:

$$T(z^{-1}) = 1, \;\; S(z^{-1}) = \hat{b}(z^{-1}), \;\; R(z^{-1}) = \hat{a}(z^{-1})\,. \tag{3.60}$$

For this type of algorithm, we can see that the following conditions are met: finite stabilization time, minimal rise time and stationary error equal to zero at every sampling instant.

The objective of the control is to adjust the current output $y[k]$ to the value of the reference $r[k]$ in a single step. This objective is not always realistic, as restrictions on the amplitude of the control are not considered. Strong oscillations appear in-between sampling periods, despite an accurate output at those exact periods.

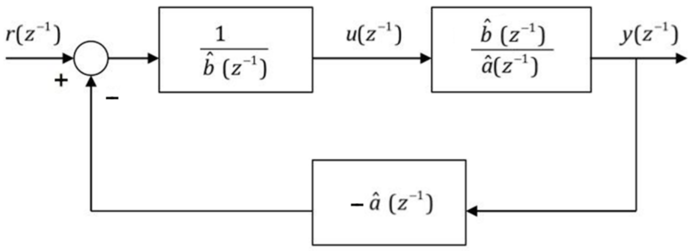

Figure 3.5. *Diagram of predictive dead-beat control*

A more realistic approach involves imposing tracking of the setpoint (seen as the reference trajectory) by the output, following a predetermined transient regime:

$$y[k+1] = y[k] + \alpha\big(r[k] - y[k]\big) = (1-\alpha)\,y[k] + \alpha r[k], \quad \forall\, k \in \mathbb{N}\,. \qquad [3.61]$$

If $\alpha \in (0,1)$, then the future output belongs to the segment linking $y[k]$ and $r[k]$. For $\alpha = 1$, we obtain the previous algorithm. As a result, in [3.61] if we replace the prediction value calculated using [3.56], we get the following recurrent relation:

$$(1-\alpha)y[k] + \alpha r[k] = -\sum_{i=1}^{na} a_i y[k-i+1] + \sum_{i=1}^{nb} b_i u[k-i+1]\,. \qquad [3.62]$$

Using relation [3.61], we can build the current control algorithm:

$$u[z^{-1}] = \frac{\alpha}{\hat{b}(z^{-1})} r[z^{-1}] + \frac{(1-\alpha) + \hat{a}(z^{-1})}{\hat{b}(z^{-1})} y[z^{-1}]\,. \qquad [3.63]$$

We therefore consider that the polynomials of the (RST) canonical structure are defined as follows:

$$\begin{cases} R(z^{-1}) = (\alpha - 1) - \hat{a}(z^{-1}) \\ S(z^{-1}) = \hat{b}(z^{-1}) \\ T(z^{-1}) = \alpha \end{cases}. \qquad [3.64]$$

The corresponding implementation diagram is presented in Figure 3.6.

Note the absence of the integral component on the direct path of the predictive structure, which does not guarantee that the stationary error will be equal to zero or that the coefficients of the model will be evaluated accurately.

A predictor expressed in an incremental form eliminates these disadvantages. The incremental predictor is defined from equations [3.55] and [3.56], by subtraction, as follows:

$$\begin{aligned} y[k+1] - y[k] &\overset{def}{=} \sum_{i=1}^{na} a_i y[k-1] - \sum_{i=1}^{na} a_i y[k-i+1] + \\ &+ \sum_{i=1}^{nb} b_i u[k-i+1] - \sum_{i=1}^{nb} b_i u[k-i] = \\ &= (1-z^{-1})\left[\sum_{i=1}^{nb} b_i u[k-i+1] - \sum_{i=1}^{na} a_i y[k-i+1]\right] = \\ &= (1-z^{-1})[z^{-d}\hat{b}^*(z^{-1})u[k] - \hat{a}^*(z^{-1})y[k]], \quad \forall k \in \mathbb{N}\,. \end{aligned} \qquad [3.65]$$

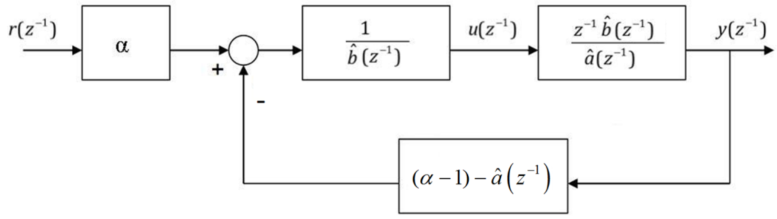

Figure 3.6. *Improved diagram for predictive dead-beat control*

On the other hand, condition [3.63] shows that the practical control algorithm for incremental predictive control is the following:

$$u[z^{-1}] = \frac{\alpha}{(1-z^{-1})\hat{b}(z^{-1})} r[z^{-1}] + \frac{(1-z^{-1})\hat{a}(z^{-1})-\alpha}{(1-z^{-1})\hat{b}(z^{-1})} y[z^{-1}]. \qquad [3.66]$$

This leads to the following polynomials:

$$\begin{cases} R(z^{-1}) = \alpha - (1-z^{-1})\hat{a}(z^{-1}) \\ S(z^{-1}) = (1-z^{-1})\hat{b}(z^{-1}) \\ T(z^{-1}) = \alpha \end{cases}. \qquad [3.67]$$

The diagram corresponding to relation [3.66] is represented in Figure 3.7. In a stationary regime, polynomial $R(z^{-1})$ is reduced to the value α of polynomial $T(z^{-1})$, meaning that the steady-state error has been cancelled out. Moreover, because of the integration effect, this diagram does not require the system parameters to be determined with any great precision as before; in addition, the implementation of this algorithm can be carried out independently of the estimation precision for the parameters of the process model [DAU 04, POP 06].

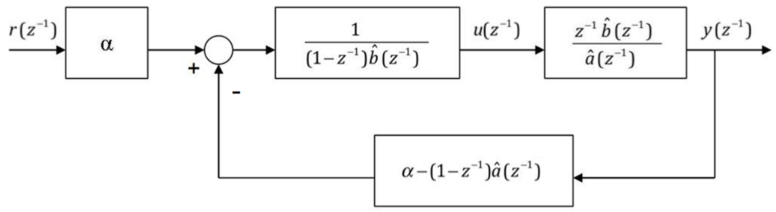

Figure 3.7. *Diagram of incremental predictive control*