Disclaimer

This summary is part of the lecture “Electrodynamics” by Prof. Dr. L. Novotny (FS19). It is based on the lecture.

Please report errors to huettern@student.ethz.ch such that others can benefit as well.

The upstream repository can be found at https://github.com/noah95/formulasheets

ETH Electrodynamics

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1 Conventions

V	Volume
dV	infinitesimal volume elements
A	surface
da	infinitesimal surface elements
ds	infinitesimal line element
∂V	closed surface of the volume V
∂A	circumference of area A
$\mathbf{n}$	unit vector normal to suface / circumference
ρ	Charge density
$\mathbf{j}$	Current density
$\mathbf{E}$	Electric field
$\mathbf{H}$	Magnetic field
$\mathbf{D}$	Displacement
$\mathbf{M}$	Magnetization
ϕ	Electric potential

2 Mathematics

2.1 Linear Algebra

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Rotation

The cross product of the nabla operator and the vector field $\vec{F}$

$$\nabla \times \vec{F} = \det \begin{bmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{F}_x & \vec{F}_y & \vec{F}_z \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial y}\vec{F}_z - \frac{\partial}{\partial z}\vec{F}_y \\ -\frac{\partial}{\partial x}\vec{F}_z + \frac{\partial}{\partial z}\vec{F}_x \\ \frac{\partial}{\partial x}\vec{F}_y - \frac{\partial}{\partial y}\vec{F}_x \end{bmatrix}$$

Divergence

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z = \operatorname{div}(\vec{F})$$

Combination

Always solve right to left.

$$\nabla \times \nabla \phi(\vec{r}) = \nabla \times (\nabla \phi(\vec{r})) = \cdots = 0$$

$$\nabla \cdot \nabla \times \vec{F} = \nabla \cdot (\nabla \times \vec{F}) = \cdots = 0$$

Rotation of rotation:

$$\nabla \times \nabla \times \vec{F} = \nabla \nabla \cdot \vec{F} - \nabla^2 \vec{F}$$

2.2 Integrals

Line integral inside Vector Field

1. Parametrize curve with t. Split integral if necessary for differenc parametrizations

$$x : f(t) \qquad y : f(t) \qquad z : f(t)$$

2. Calcualate derivative of normal vector along curve

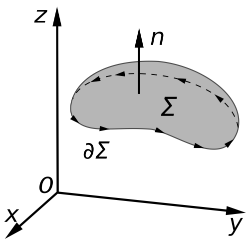
$$\frac{d\vec{s}}{dt} = x'(t)\vec{e}_x + y'(t)\vec{e}_y + z'(t)\vec{e}_z$$

$$d\vec{s} = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} dt$$

3. Solve integral

$$\int_{\partial A} F(\vec{r})\, d\vec{s} = \int_a^b F(x(t), y(t), z(t)) \cdot \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} dt$$

2.3 Stokes' Therorem

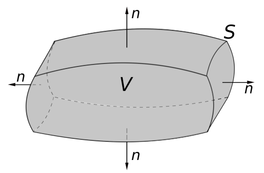


$$\int_{\partial A} F(\vec{r})\, d\vec{s} = \int_A [\nabla \times F(\vec{r})] \cdot \vec{n}\, da$$

The sum of flux along the contour ∂A in contour direction is the same as the sum of curl $\nabla \times \vec{F}$ in normal direction $\vec{n}$ on the area A.

2.4 Gauss Therorem

Also known as divergence theorem.



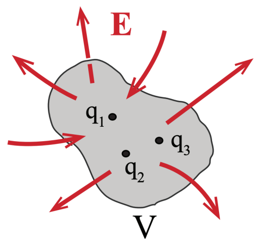
$$\int_{\partial V} \vec{F}(\vec{r}) \cdot \vec{n}\, da = \int_V \nabla \vec{F}(\vec{r})\, dV$$

Sum of flux across surface ∂A in normal direction $\vec{n}$ is the same as the sum of divergence inside the region V.

3 Pre-Maxwellian Electrodynamics

Gauss' Law

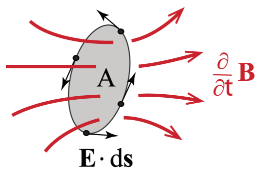
The net flux through a surface is equal to $1/\epsilon_0$ times the net electric charge within that surface.



$$\int_{\partial V} \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{n}\, da = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}, t)\, dV$$

Faraday's Law

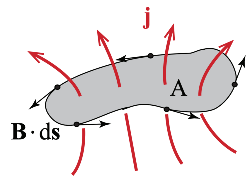
The electromotive force around a path is equal to the negative change in time of the magnetic flux enclosed by the path.



$$\int_{\partial A} \mathbf{E}(\mathbf{r}, t) \cdot\, ds = -\frac{\partial}{\partial t} \int_A \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{n}\, da$$

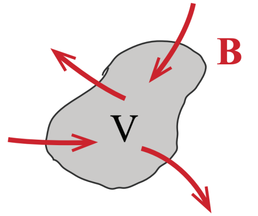
Ampere's Law

The magnetic field created by an electric current is proportional to the size of that electric current.



$$\int_{\partial A} \mathbf{B}(\mathbf{r}, t) \cdot\, ds = \mu_0 \int_A \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{n}\, da$$

Non-existence of magnetic charges



$$\int_{\partial V} \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{n}\, da = 0$$

Kirchhoff

Reducing Apere's law to any closed surfece states that the flux of current through any closed surface is zero: What flows in has to flow out.

$$\int_{\partial V} \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{n}\, da = 0$$

From Faraday's law if no time-varying magnetic fields are present follows the second Kirchhoff law (Knotenregel):

$$\int_{\partial A} \mathbf{E}(\mathbf{r}, t) \cdot\, ds = 0$$

The two Kirchhoff laws form the basis for circuit theory and electronic design.

4 Maxwell's Equations

The pre-Maxwellian equations summarize the electromagnetism before Maxwell. In 1873 however, Maxwell introduced a critical modification.

4.1 Displacement Current

The law that the net flux through a closed surface is zero is flawed. For example: Identical charges released will speed out because of Coulomb repulsion and there will be a net outward current. Kirchhoffs first law has to be corrected as follows:

$$\int_{\partial V} \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{n}\, da$$

`continuity equation`: The outward current is balanced by the decrease of charge inside the surface.

4.2 Maxwell's Equations in Integral Form

$$\int_{\partial V} \mathbf{D}(\mathbf{r},t)\cdot\mathbf{n}\,\mathrm{d}a = \int_V \rho(\mathbf{r},t)\,\mathrm{d}V$$
$$\int_{\partial A} \mathbf{E}(\mathbf{r},t)\cdot\,\mathrm{d}s = -\frac{\partial}{\partial t}\int_A \mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}\,\mathrm{d}a$$
$$\int_{\partial A} \mathbf{H}(\mathbf{r},t)\cdot\,\mathrm{d}s = \int_A \left[\mathbf{j}(\mathbf{r},t)+\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)\right]\cdot\mathbf{n}\,\mathrm{d}a$$
$$\int_{\partial V} \mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}\,\mathrm{d}a = 0$$

The displacement D and the magnetic field H account for secondary sources through

$$\mathbf{D}(\mathbf{r},t) = \epsilon_0\mathbf{E}(\mathbf{r},t)+\mathbf{P}(\mathbf{r},t)$$
$$\mathbf{B}(\mathbf{r},t) = \mu_0[\mathbf{H}(\mathbf{r},t)+\mathbf{M}(\mathbf{r},t)]$$

4.3 Maxwell's Equations in Differential Form

$$\nabla\cdot\mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t)$$
$$\nabla\times\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)$$
$$\nabla\times\mathbf{H}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)+\mathbf{j}(\mathbf{r},t)$$
$$\nabla\cdot\mathbf{B}(\mathbf{r},t) = 0$$

5 Electrostatics

We modify Faraday's law by setting the change of magnetic flux to zero.

$$\int_{\partial A} \mathbf{E}(\mathbf{r},t)\cdot\,\mathrm{d}s = 0$$

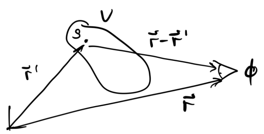
By applying the theorems of Gauss and Stokes we get the following identities:

$$\nabla\cdot\mathbf{E}(\vec{r}) = \frac{1}{\epsilon_0\rho(\vec{r})} \qquad \nabla\times\mathbf{E}(\vec{r}) = 0$$
$$\mathbf{E} = -\nabla\phi \qquad \nabla^2\phi(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r})$$

Combining them we can write the poisson equation:

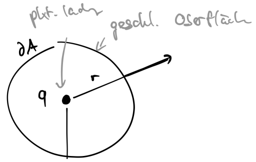
$$\nabla^2\Phi(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r}) \qquad \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}V'$$

Where $\phi(\vec{r})$ is the electric potential at point $\vec{r}$.



5.1 Point Charge

E-field of a point charge:



$$\int_{\partial A}\vec{E}\vec{n}\,\mathrm{d}a = E\cdot 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\vec{n}_r \qquad \vec{E} = -\nabla\phi \qquad \phi = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$